Sigma Models on Supercosets

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Vladimir Mitev
aus Sofia, Bulgarien

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Gutachter der Dissertation:	Prof. Dr. V. Schomerus
	Prof. Dr. K. Fredenhagen
	Prof. Dr. A. Ludwig
Gutachter der Disputation:	Prof. Dr. V. Schomerus
	Dr. J. Teschner
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Vorsitzende des Prüfungsausschusses:	Prof. Dr. J. Bartels
Vorsitzender des Promotionsausschusses:	Prof. Dr. J. Bartels
Dekan der Fakultät MIN:	Prof. Dr. H. Graener

Abstract

The purpose of this thesis is to deepen our understanding of the fundamental properties and defining features of non-linear sigma models on superspaces. We begin by presenting the major concepts that we have used in our investigation, namely Lie superalgebras and supergroups, non-linear sigma models and two dimensional conformal field theory. We then exhibit a method, called cohomological reduction, that makes use of the target space supersymmetry of non-linear sigma models to compute certain correlation functions. We then show how the target space supersymmetry of Ricci flat Lie supergroups simplifies the perturbation theory of suitably deformed Wess-Zumino-Witten models, making it possible to compute boundary conformal weights to all orders. This is then applied to the OSP (2S+2|2S) Gross-Neveu Model, leading to a dual description in terms of the sigma model on the supersphere $S^{2S+1|2S}$. With this results in mind, we then turn to the similar, yet more intricate, theory of the non-linear sigma model on the complex projective superspaces $\mathbb{CP}^{N-1|N}$. The cohomological reduction allows us to compute several important quantities non-perturbatively with the help of the system of symplectic fermions. Combining this with partial perturbative results for the whole theory, together with numerical computations, we propose a conjecture for the exact evolution of boundary conformal weights for symmetry preserving boundary conditions.

Zusammenfassung

Das Ziel dieser Dissertation ist es, unser Verständnis der fundamentalen Eigenschaften und definierenden Merkmale nicht-linearer Sigmamodelle zu vertiefen. Wir stellen zuerst die Hauptkonzepte vor, die wir in unseren Untersuchungen verwendet haben, nämlich Lie-Superalgebren und -gruppen, nicht-lineare Sigmamodelle und zweidimensionale konforme Feldtheorien. Wir stellen dann die sogenannte Methode der kohomologischen Reduktion vor, welche die Zielraumsupersymmetrie nicht-linearer Sigmamodelle ausnutzt, um gewisse Korrelationsfunktionen zu berechnen. Wir zeigen dann, wie die Zielraumsupersymmetrie von Ricci-flachen Lie-Supergruppen die Störungstheorie von geeignet deformierten Wess-Zumino-Witten Modellen vereinfacht, was uns erlaubt, konforme Gewichte in Randtheorie in allen Ordnungen auszurechnen. Dies wenden wir dann auf das OSP (2S+2|2S) Gross-Neveu-Modell an, was auf eine duale Beschreibung mit Hilfe des Sigmamodells auf der Supersphäre $S^{2S+1|2S}$ führt. Nach diesen Ergebnissen konzentrieren wir uns auf das ähnliche, jedoch kompliziertere Sigmamodell auf den komplexen projektiven Räumen $\mathbb{CP}^{N-1|N}$. Die kohomologische Reduktion erlaubt uns, viele wichtige Grössen mit Hilfe eines Systems symplektischer Fermionen auszurechnen. Indem wir diese mit partiellen störungstheoretischen Ergebnissen und numerischen Rechnungen kombinieren, können wir eine Vermutung fuer die exakte Entwicklung konformer Gewichte in Randtheorien mit symmetrieerhaltenden Randbedingungen aufstellen.

Contents

Chapter 1

Introduction

Today, string theory is considered a major candidate for a theoretical description of quantum gravity and for the unification of all known fundamental interactions. In its original formulation, recalled in [1], string theory had a more limited ambition, for it was designed to model the strong nuclear interactions. After several setbacks and the subsequent development and experimental confirmation of quantum chromodynamics, or QCD, this line of research was left aside. It now appears however, that this abandonment may have been a premature decision. Indeed, if one looks at a close relative of QCD, namely maximally supersymmetric, that is $\mathcal{N} = 4$, super Yang-Mills in four dimensions, one finds supporting evidence for the conjecture of [2, 3] that this model can be alternatively described in terms of a superstring theory on the ten dimensional background $AdS_5 \times S^5$. The proposed duality is of the strong-weak coupling type, in the sense that it maps the weakly coupled region of the gauge theory to the strongly coupled one of the string theory and vice versa, as depicted in figure 1.1.

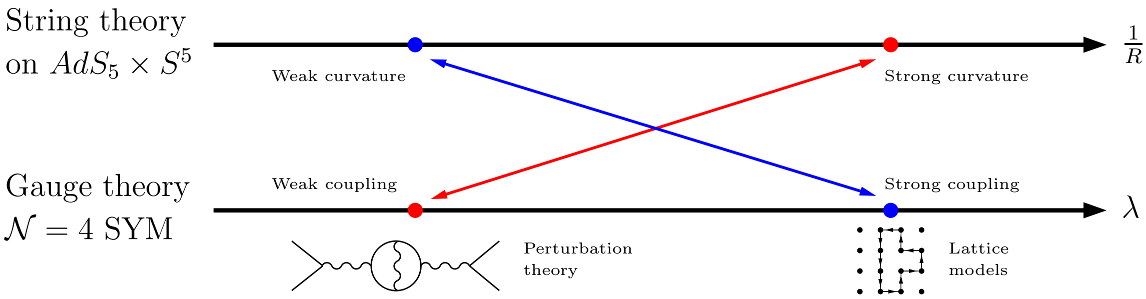

Figure 1.1: The hypothetical duality between four dimensional $\mathcal{N} = 4$ super Yang-Mills and string theory on $AdS_5 \times S^5$, plotted as a function of the radius R of $AdS_5 \times S^5$, or equivalently the coupling constant λ of the gauge theory. The lines connect equivalent regions on both sides. The full story is more complex since both theories depend on two parameters, namely the number of colours and the gauge coupling. Furthermore, since lattice models are non-perturbative, they can be used for all values of λ.

Thus, the quantities that can be easily computed in a perturbative expansion in the coupling constant in one theory become very difficult to calculate in the other. This is on one hand a blessing, since accepting the duality as a working hypothesis makes possible the analytic investigations of such gauge theory phenomena as quark confinement, that are usually only treatable via computer simulations, but on the other hand a problem, because it makes a proof of the conjectured duality all that harder. An improvement over the present situation, illustrated in figure 1.2, could involve the construction of another string theory model that would be a strong-weak coupling dual to the already well studied $AdS_5 \times S^5$ one. This would allow for perturbative comparisons to be made order by order between this hypothetical theory and $\mathcal{N} = 4$ super Yang-Mills.

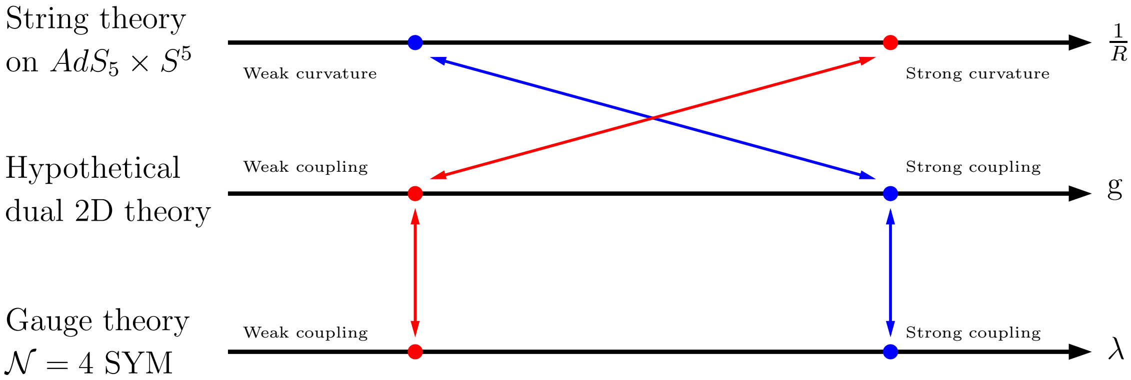

Figure 1.2: We can look for a two dimensional gauge theory that is dual to string theory on $AdS_5 \times S^5$. This would allow a perturbative comparison at low coupling to be made.

Formulating such a goal is one thing, finding such a dual description an entirely different matter altogether. The first step in the right direction passes by a better understanding of superstring theory on Anti de Sitter spaces, which entails gaining a better insight into the building blocks of these models, namely non-linear sigma models.

The term *sigma models* stands for theories whose fundamental fields are interpreted as the coordinates of some manifold called target space. Thus, they constitute an embedding of the space-time, or *worldsheet*, on which they are defined into the target space. Physically, sigma models can be understood as describing the motion of membranes, of a shape determined by the world sheet, in the embedding manifold. In this thesis, we will concern ourselves exclusively with nonlinear sigma models with target space, or internal, supersymmetry. Besides their importance in the field of superstring theory in Anti de Sitter backgrounds as explained in [4–9], they appear in relationships with dense polymers in two dimensions [10, 11], the quantum Hall plateau transitions [12] or disordered electron systems [13].

Non-linear sigma models on target superspaces possess a number of surprising properties which are gradually being uncovered. In particular, there exists several basic series of models which give rise to families of conformal field theories with continuously varying exponents, including the supergroup manifolds PSL (N|N), OSP (2S+2|2N) and a number of quotients thereof [12, 14–17]. Conformal symmetry in these models allows, but does not require, the addition of a Wess-Zumino term. A purely bosonic example of a theory with such properties is the two dimensional sigma model with S^1 as target space, illustrated in figure 1.3. This is however a free theory, so that conformal invariance is not a surprising feature.

Solving conformal field theories with such properties requires developing entirely new techniques which go far beyond the conventional algebraic methods. Numerical

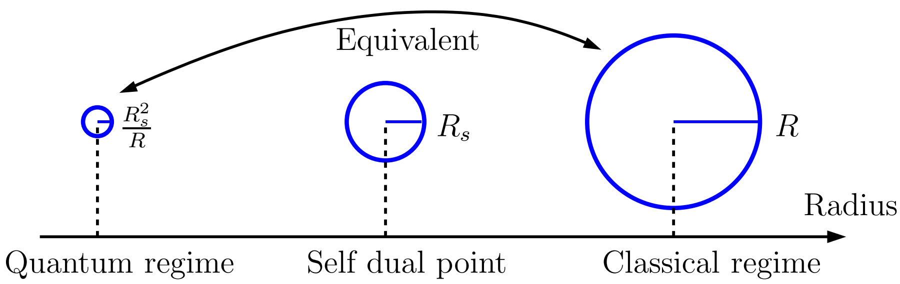

Figure 1.3: The moduli space of the two dimensional sigma model with S^1 as target space. This one parameter family of conformal field theories has operators with continuously varying dimension.

and algebraic studies of lattice discretizations [16, 18, 19] and supersymmetry aided all-order perturbative computations of spectra [20, 21] have been applied with astonishing results. In some cases is was possible to determine exact formulae for all boundary conformal weights as a function of the continuous couplings, or *moduli*, of the models.

While many different target spaces shall be considered in the course of this work, the main focus shall be on *symmetric* superspaces. These spaces can be understood as quotients of two supergroups in which the denominator one is fixed by an order two automorphism. Sigma models on bosonic symmetric spaces have been studied extensively because of their numerous applications in many different branches of physics. While they are well-known to possess an infinite number of classically conserved quantities, as described initially in [22–27] and later for example in [28, 29], quantum effects spoil integrability in many cases, as shown in [30, 31]. And even in those examples for which this does not happen, finding explicit formulae for the partition functions and the correlators is a difficult problem that has only been solved for a small set of models. Target space supersymmetry changes many aspects and leads to a number of remarkable properties, like for instance quantum conformal invariance. Yet, even in these very symmetric cases, finding explicit solutions is still rather difficult and requires developing new techniques, as for instance the ones pioneered by [32, 33]. Some remarkable recent advances, most importantly the results of [34] and [18, 19], seem to bring at least some partial solutions within reach. One of our aims here is to initiate and explore new solution strategies that incorporate target space supersymmetry as an essential feature.

The plan of this thesis can be outlined as follows.

In the second chapter, we present the main mathematical tools needed for our investigations. This starts with general Lie superalgebra and supergroup theory and their representations. We then move on to the description of coset superspaces, which provide the main ingredient for our definition of non-linear sigma models. From there, we sketch the basic elements of conformal field theory, both in the boundary and in the bulk, and present a short introduction to the important Wess-Zumino-Novikov-Witten and Gross-Neveu models.

The third chapter's main goal is to use the target space supersymmetry of the non-linear sigma models under consideration in order to identify simpler subsectors within the theory. The main goal is to show how the correlation functions of operators in those subsectors can be computed by using non-linear sigma models on smaller target spaces that are easier to work with. Presenting this method, for which we have coined the name of *cohomological reduction*, requires us to first spend some time explaining the implications of the fact that for a given nilpotent fermionic element of a Lie superalgebra, its cohomology in the Lie superalgebra itself can be endowed with a Lie superalgebra structure. We then extend these results to the computation of cohomologies of nilpotent fermionic symmetry generators on the space of functions and on the space of tensor forms of coset superspaces. Combining these results allows us to compute the cohomology of the Lagrangian itself and to show that it can be identified with the Lagrangian of another non-linear sigma model, thus enabling us to calculate the correlation functions of a specific class of operators. We then spend some time investigating the sigma models on homogeneous superspaces, before ending the chapter with a short note on the extension of the method of cohomological reduction to Wess-Zumino-Witten and Gross-Neveu models.

In the fourth chapter, we examine an important series of sigma models, namely those whose target spaces are the superspheres $S^{2M+1|2M}$. We start by presenting the prototype of these models, the circle, which helps us illustrate several principles and approaches that we use later on. We go on to argue that the $S^{2M+1|2M}$ models all possess two dimensional worldsheet conformal symmetry and concentrate on the computation of the boundary spectrum for a very specific boundary condition in $S^{3|2}$, the simplest non-trivial model. We first calculate in a combinatorial fashion the spectrum in the non-interacting limit of infinite volume. The symmetry of the model is then used to decompose said spectrum in characters of representations of the target space symmetry group. After these computations are done, we turn to the Gross-Neveu theories that we claim are strong-weak duals to the $S^{2M+1|2M}$ sigma models, thus generalizing the famous duality between the compactified free boson and the massless Thirring model. To support this hypothesis, we compute the partition function for a particular boundary condition as a function of the coupling constant. Thanks to symmetry arguments, this is achieved to all orders in perturbation theory, with the result that the change of the conformal weights depends only on the symmetry. Taking this spectrum in the limit of infinite coupling constant then reproduces the spectrum that we computed in the non-interacting point of the $S^{3|2}$ sigma model. We then close the chapter by sketching how the extension to world sheet supersymmetric sigma models is to be done.

Emboldened by the results concerning the superspheres, in the sixth chapter we turn to another family of sigma models, namely those whose target space is given by the complex projective superspaces of the kind $\mathbb{CP}^{S-1|S}$. While these spaces share many characteristics with the superspheres, they also have some special traits in that they are complex Ricci-flat spaces and thus constitute the simplest examples of Calabi-Yau supermanifolds. In addition, unlike the superspheres, they allow for the presence of a non-trivial antisymmetric term in the Lagrangian, so that the space of moduli becomes two dimensional. We start our treatment by classifying the possible symmetry preserving boundary conditions in the sigma model, whose boundary spectra we then compute in the particle and in the infinite volume limit. Using some perturbative computations that we back up with results obtained via the technique of cohomological reduction, we are then able to present a conjecture for the spectra for all values of the coupling constants. In the second part of the chapter, we present a spin chain description of the sigma models that gives numerical support for our claim. Unfortunately, unlike for the superspheres, no dual description could be found for the $\mathbb{CP}^{S-1|S}$ sigma models.

In the last part, we present some possible further applications of the methods and results gained in this thesis and close with an outlook of possible future research.

Chapter 2

Preliminaries

In this chapter, we will present the basic concepts necessary for the understanding of the computations and results to follow. We start with a discussion of the basic symmetry concepts that are important to us, namely Lie superalgebras and supergroups. From there, we continue with a presentation of homogeneous superspaces and of the physical models thereon, namely non-linear sigma models. The last part of this chapter then focuses on conformally invariant theories in two dimensions and presents some results concerning the perturbation theory of Wess-Zumino-Witten models that will be of use in the later parts of this work.

2.1 Lie superalgebras

For Lie groups, one usually starts by describing the groups as smooth manifolds endowed with a group operation and then defines the Lie algebras as being the tangent spaces to the identity element. In the case of Lie superalgebras and supergroups however, it is preferable to proceed in the opposite direction.

Definition 2.1.1. A *Lie superalgebra* $\mathfrak{g}$ over a field[1] $\mathbb{F}$ is a direct sum of two $\mathbb{F}$ vectorspaces: the even $\mathfrak{g}_{\bar{0}}$ and the odd $\mathfrak{g}_{\bar{1}}$, together with a gradation function $|\cdot|$, such that

$$|X| = \begin{cases} 0 \\ 1 \end{cases} \text{for } X \in \begin{cases} \mathfrak{g}_{\bar{0}} \\ \mathfrak{g}_{\bar{1}} \end{cases},$$

and a bilinear bracket $[\cdot\,,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is called the *supercommutator*. The bracket must

- be *graded anti-symmetric*, meaning

$$[X\,,Y] = (-1)^{|X||Y|}\,[Y\,,X]\ . \tag{2.1.1}$$

- be *compatible* with the gradation function

$$[\mathfrak{g}_{\bar{\imath}}\,,\mathfrak{g}_{\bar{\jmath}}] \subset \mathfrak{g}_{\bar{\imath}+\bar{\jmath} \bmod 2}\ . \tag{2.1.2}$$

- obey the *graded Jacobi identity*

$$[X\,,[Y\,,Z]] = [[X\,,Y]\,,Z] + (-1)^{|X||Y|}\,[Y\,,[X\,,Z]]\ . \tag{2.1.3}$$

[1]We will restrict ourselves to $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Corollary 2.1.1. The definition of $\mathfrak{g}$ implies in particular that $\mathfrak{g}_{\bar{0}}$ is an usual Lie algebra and that $\mathfrak{g}_{\bar{1}}$ transforms in a representation thereof.

Definition 2.1.2. We will call a bilinear form $(\cdot\,,\,\cdot) \,:\, \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ a metric, if it is:

- graded symmetric
$$(X\,,\,Y) = (-1)^{|X||Y|}\,(Y\,,\,X) \ . \tag{2.1.4}$$
- consistent
$$(X\,,\,Y) = 0 \qquad \text{for } X \in \mathfrak{g}_{\bar{0}},\, Y \in \mathfrak{g}_{\bar{1}} \ . \tag{2.1.5}$$
- invariant
$$([X\,,\,Y]\,,\,Z) = (X\,,\,[Y\,,\,Z]) \ . \tag{2.1.6}$$
- non-degenerate
$$(X\,,\,Y) = 0 \ \forall Y \in \mathfrak{g} \Longrightarrow X = 0 \ . \tag{2.1.7}$$

While for a given superalgebra $\mathfrak{g}$ the last requirement cannot always be fulfilled, a simple way of obtaining a form that satisfies the first three is to take a representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathsf{V})$ and to set

$$(X\,,\,Y) := \mathrm{str}\left(\rho(X)\rho(Y)\right) \ , \tag{2.1.8}$$

where the supertrace str is defined as the trace weighed appropriately with the gradation function, see (2.1.14) below. If the representation is the adjoint one ρ_{ad}, the resulting metric for $\mathfrak{g}$ is called the *Killing form*. Choosing a basis $\left\{T^A : A = 1, \ldots, \dim \mathfrak{g}\right\}$ of $\mathfrak{g}$ and setting the gradation for the indices to be the same as the one for the basis elements, i.e. $|A| := |T^A|$, we define the *structure constants* ${f^{AB}}_C$ as

$$\left[T^A\,,\,T^B\right] := {f^{AB}}_C T^C \ . \tag{2.1.9}$$

We can then express the Killing form using the structure constants:

$$\mathrm{K}^{AB} := \mathrm{str}\left(\rho_{\mathrm{ad}}(T^A)\rho_{\mathrm{ad}}(T^B)\right) = \sum_{C,D=1}^{\dim \mathfrak{g}} {f^{AD}}_C {f^{BC}}_D (-1)^{|C|} \ . \tag{2.1.10}$$

The structure constants and the corresponding Killing form can also be represented graphically as in figure 2.1. This will turn out to be useful later on.

Definition 2.1.3. A non-degenerate metric can be used to obtain another very important element of the Lie superalgebra. For a given metric $(\cdot\,,\,\cdot)$, that is expressed as a tensor $\kappa^{AB} = \left(T^A\,,\,T^B\right)$, we define the *quadratic Casimir* as

$$\mathsf{Cas} := \sum_{A,B=1}^{\dim \mathfrak{g}} \left(\kappa^{-1}\right)_{AB} T^A T^B \ . \tag{2.1.11}$$

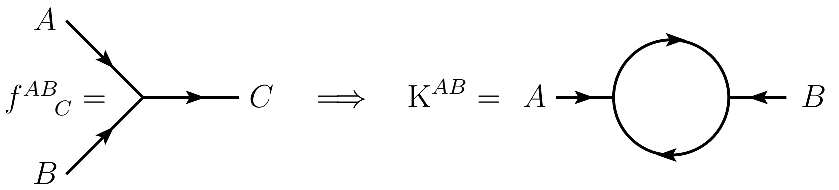

Figure 2.1: A graphical representation of the structure constants and of the Killing form.

The quadratic Casimir is to be understood as being part of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Using the symmetries of the structure constants as well as the properties of the metric, it is easy to see that Cas is a central element in $\mathfrak{U}(\mathfrak{g})$, which means that it commutes with all T^C. It is important to note that, unlike in the case of Lie algebras, the quadratic Casimir of Lie superalgebras cannot always be diagonalized and will in general possess non-trivial Jordan blocks. We provide more information concerning the quadratic Casimir elements in appendix B, especially in section B.4.

Definition 2.1.4. *Simple* Lie superalgebras are defined as being the ones lacking non trivial ideals. Semisimple Lie superalgebras are then direct sums of simple ones. The complete and lengthy classification of simple Lie superalgebras was presented in [35]. Here, we are merely interested in *classical Lie superalgebras* for which the even part $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Whether simple or not, they fall into two *types*:

I) those for which the odd part $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ decomposes into a direct sum of two simple modules of the even subalgebra $\mathfrak{g}_{\bar{0}}$. Among others, this is the case for $\mathrm{gl}\,(m|n)$, $\mathrm{sl}\,(m|n)$, $\mathrm{pgl}\,(n|n)$, $\mathrm{psl}\,(n|n)$ and $\mathrm{osp}\,(2|2n)$.

II) those for which $\mathfrak{g}_{\bar{1}}$ is an irreducible representation of $\mathfrak{g}_{\bar{0}}$. This category includes the superalgebras $\mathrm{osp}\,(m|2n)$ for $m \neq 2$.

We will not be interested in all the classical Lie superalgebras whether of type I or of type II. Those that are of importance to us are the following:

- The *general linear superalgebra* $\mathrm{gl}\,(m|n)$, defined as the set of matrices X

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) , \tag{2.1.12}$$

  where A, B, C and D are matrices of size $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively. The even part $\mathrm{gl}\,(m|n)_{\bar{0}}$ is spanned by those matrices whose B and C parts vanish, whereas for the odd part one takes only the matrices with zero A and D terms. The supercommutator is defined as the usual commutator if at least one of the elements if even, as the anticommutator if both are odd and is then extended to the full superalgebra by linearity. For $m = n$, we can define the algebra $\mathrm{pgl}\,(n|n)$ by quotienting out the ideal spanned by the identity, that is

$$\mathrm{pgl}\,(n|n) := \mathrm{gl}\,(n|n)\,/\mathrm{span}\{\mathbb{1}\} . \tag{2.1.13}$$

  These two kinds of algebras are quite useful, although none of them is simple.

- The *supertrace* of a matrix of the kind appearing in (2.1.12) is defined in [35] as the difference
$$\mathrm{str}(X) := \mathrm{tr}(A) - \mathrm{tr}(D)\ . \tag{2.1.14}$$
This leads to the *special linear superalgebra* $\mathrm{sl}\,(m|n)$, which is the subset of matrices X in $\mathrm{gl}\,(m|n)$ with vanishing supertrace, i.e.
$$\mathrm{sl}\,(m|n) := \{X \in \mathrm{gl}\,(m|n) : \mathrm{str}(X) = 0\}\ . \tag{2.1.15}$$
These superalgebras are simple as long as m is not equal to n, but have an abelian ideal spanned by the identity matrix otherwise. In that case, we can define the simple superalgebra $\mathrm{psl}\,(n|n)$ as the quotient
$$\mathrm{psl}\,(n|n) := \mathrm{sl}\,(n|n)\,/\mathrm{span}\{\mathbb{1}\}\ . \tag{2.1.16}$$

- The *orthosymplectic superalgebras* $\mathrm{osp}\,(m|2n)$ defined as the set of matrices in $\mathrm{gl}\,(m|2n)$ subject to the constraint
$$-X^{st} = \mathsf{J}_{m,2n} X \mathsf{J}^t_{m,2n}\ , \tag{2.1.17}$$
where
$$X^{st} = \left(\begin{array}{c|c} A^t & -C^t \\ \hline B^t & D^t \end{array}\right) \qquad \text{and } \mathsf{J}_{m,2n} = \left(\begin{array}{c|cc} \mathbb{1}_m & & \\ \hline & & \mathbb{1}_n \\ & -\mathbb{1}_n & \end{array}\right)\ . \tag{2.1.18}$$
These superalgebras are simple.

Definition 2.1.5. A *Cartan subalgebra* g_0 is a maximal abelian subalgebra of $\mathfrak{g}$. For the algebras that interest us, g_0 turn out to be the Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, i.e. of the even part of the Lie superalgebra in question. The dimension of g_0 is called the *rank* of $\mathfrak{g}$. Choosing a basis $\{H_i\}_{i=1}^{\mathrm{rank}(\mathfrak{g})}$ of g_0, we can perform the *root decomposition* of the superalgebra:
$$\mathfrak{g} = \mathsf{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathsf{g}_\alpha\ , \tag{2.1.19}$$
where the *root subspace* g_α is defined as
$$\mathsf{g}_\alpha := \{X \in \mathfrak{g} : [H_i, X] = \alpha(H_i)X = a_i X\}\ , \tag{2.1.20}$$
and Δ is the set of all *roots* α, counting multiplicities. In this notation, the roots α are of course to be understood as elements of the dual vector space $\mathsf{g}_0^\star$. The set of even, respectively odd roots $\Delta_{\bar{0}}$, $\Delta_{\bar{1}}$ contains all α for which the corresponding g_α is purely bosonic, respectively purely fermionic. The roots of the complex superalgebras that we are interested in are listed in table 2.1. Given a metric on $\mathfrak{g}$ and a root α, we define $H_\alpha \in \mathsf{g}_0$ by $\alpha(H) = (H_\alpha\,, H)$. Setting $(\alpha\,, \beta) := (H_\alpha\,, H_\beta)$ then provides us with a metric on the dual space $\mathsf{g}_0^\star$.

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\Delta_{\bar{0}}$	$\Delta_{\bar{1}}$
$\mathrm{gl}\,(M\|N)$, $\mathrm{sl}\,(M\|N)$	gl (M) $\oplus$ gl (N) , sl (M) $\oplus$ sl (N) $\oplus\, \mathbb{C}$	$\epsilon_i - \epsilon_j$ $\delta_k - \delta_l$	$\pm\epsilon_i \mp \delta_k$
$\mathrm{osp}\,(2M\|2N)$	so (2M) $\oplus$ sp (2N)	$\pm\epsilon_i \pm \epsilon_j$ $\pm\delta_k \pm \delta_l$ $\pm 2\delta_k$	$\pm\epsilon_i \pm \delta_k$
$\mathrm{osp}\,(2M+1\|2N)$	so (2M+1) $\oplus$ sp (2M)	$\pm\epsilon_i \pm \epsilon_j$ $\pm\epsilon_i$ $\pm\delta_k \pm \delta_l$ $\pm 2\delta_k$	$\pm\epsilon_i \pm \delta_k$ $\pm\delta_k$

Table 2.1: The root systems of gl, sl and osp type superalgebras in the standard basis $\epsilon_1, \ldots, \epsilon_M, \delta_1, \ldots, \delta_N$. We note that we require $i \neq j$ and $k \neq l$ in the third column. If H_i is a basis of the Cartan subalgebra of the first direct summand of $\mathfrak{g}_{\bar{0}}$ and $\tilde{H}_i$ a basis of the second one, then $\epsilon_i(H_j) = \delta_{ij}$, $\delta_i(\tilde{H}_j) = \delta_{ij}$ and $(\epsilon_i\,,\,\epsilon_j) = -\,(\delta_i\,,\,\delta_j) = \delta_{ij}$. We recommend [35] for more details.

Automorphisms of Lie superalgebras are defined as maps $\Omega : \mathfrak{g} \to \mathfrak{g}$ that preserve the algebra structure, i.e.

$$[\Omega(X)\,,\,\Omega(Y)] = [X\,,\,Y] \quad \forall\, X, Y \in \mathfrak{g}\;. \tag{2.1.21}$$

They are called *metric preserving* if $(\Omega(X)\,,\,\Omega(Y)) = (X\,,\,Y)$. For our purposes, we shall assume that this is always the case. If an automorphism $\dagger$ of a complex Lie superalgebra $\mathfrak{g}$ is involutive and anti-linear, it defines a *real form* thereof by setting $\mathfrak{g}_\dagger$ to be the set of elements it leaves invariant. We call such a real form *compact*, if its bosonic part is a direct sum of compact Lie algebras.

2.2 Representation theory

A representation ρ of the Lie superalgebra $\mathfrak{g}$ is a map $\rho : \mathfrak{g} \to \mathrm{End}(\mathsf{V})$ that preserves the supercommutation structure. Since $\mathfrak{g}$ is graded, so are V and $\mathrm{End}(\mathsf{V})$. We can choose a basis of V so that the elements of the Cartan subalgebra $\mathfrak{g}_0$ are represented by diagonal matrices. Thus the representation decomposes into subspaces characterized by the eigenvalues of the basis elements of $\mathfrak{g}_0$, or alternatively by a *weight* $\Lambda \in \mathfrak{g}_0^\star$, whose evaluation on the basis elements provides these eigenvalues. In particular, the roots of the superalgebra are weights of the adjoint representation. We can impose a *positivity relation* on the roots, $\Delta = \Delta^+ \cup \Delta^-$, denoting certain roots as positive and others as negative, calling the corresponding root generators raising, respectively lowering operators. This translates to a positivity relation for all the weights in $\mathfrak{g}_0^\star$.

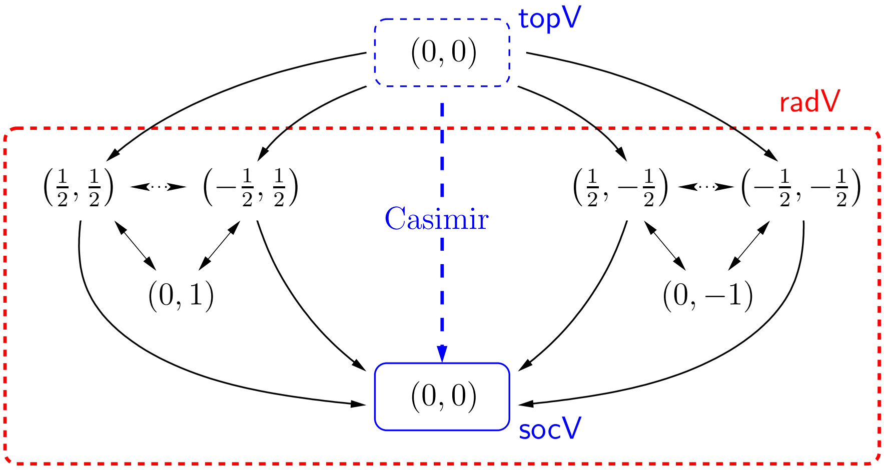

Figure 2.2: The eight dimensional indecomposable symmetric representation of $\mathrm{sl}\,(2|1) \cong \mathrm{osp}\,(2|2)$. The labels refer to the eigenvalues of the Cartan generators H and Z as defined in [36]. The full/dotted arrowlines represent the action of fermionic/bosonic generators. The quadratic Casimir is not diagonalizable. All its eigenvalues are zero, with $\mathsf{top}\,\mathsf{V}$ and $\mathsf{soc}\,\mathsf{V}$ building a two dimensional Jordan block. One easily reads that $\mathsf{soc}\,\mathsf{V} = \mathsf{rad}^2\,\mathsf{V}$ and that $\mathsf{rad}\,\mathsf{V}/\,\mathsf{soc}\,\mathsf{V}$ is the direct sum of two simple modules.

Therefore, the weights of V are ordered and we will mostly concentrate on the case in which there is a *highest weight*.

A subspace $\mathsf{U} \subset \mathsf{V}$ is *invariant* if

$$\rho(X) \cdot \mathsf{U} \subset \mathsf{U} \qquad \forall\, X \in \mathfrak{g} \tag{2.2.1}$$

and *irreducible* if there are no further subspaces $\mathsf{U}' \subset \mathsf{U}$ different from $\{0\}$ and U that are invariant as well. An invariant subspace $\mathsf{U} \subseteq \mathsf{V}$ defines a subrepresentation, or submodule, of $\mathfrak{g}$ which we call *maximal* if the quotient V/U is irreducible. A representation V is called *indecomposable* if it cannot be written as a direct sum of two or more irreducible submodules. Representations that are irreducible are indecomposable, but the converse is in general not true. We can now define the *radical* of a representation as

$$\mathsf{rad}\,\mathsf{V} := \{\, \cap \mathsf{U}_i : \mathsf{U}_i \subset \mathsf{V} \text{ is a maximal submodule} \,\} \ . \tag{2.2.2}$$

We can use this to build a chain of embedded radicals, the last on of which is zero, that is :

$$\mathsf{V} \subset \mathsf{rad}\,\mathsf{V} \subset \mathsf{rad}\,\mathsf{rad}\,\mathsf{V} \subset \cdots \subset \mathsf{rad}^n\,\mathsf{V} \subset \mathsf{rad}^{n+1}\,\mathsf{V} = \{0\} \ . \tag{2.2.3}$$

The last non trivial element in this chain is $\mathsf{rad}^n\,\mathsf{V}$ and is called the *socle*, or $\mathsf{soc}\,\mathsf{V}$, of the representation. We furthermore define the *top* of V to be the quotient $\mathsf{top}\,\mathsf{V} := \mathsf{V}/\,\mathsf{rad}\,\mathsf{V}$. We illustrate these concepts in figure 2.2 with an example taken from the representation theory of $\mathrm{sl}\,(2|1)$. We refer to [36] for more details.

If $\rho_{\mathsf{U},\mathsf{V}} : \mathfrak{g} \to \mathrm{End}\,\mathsf{U}, \mathrm{End}\,\mathsf{V}$ are two finite dimensional representations and f is a homogeneous element of the graded space of complex homomorphisms $\mathsf{Hom}_{\mathbb{C}}(\mathsf{V},\mathsf{U}) \cong$

$\mathsf{U} \otimes \mathsf{V}^\star$, then the action of $X \in \mathfrak{g}$ on f is defined as

$$[X \cdot f](v) := \rho_{\mathsf{U}}(X) \cdot f(v) - (-1)^{|f||X|} f\left(\rho_{\mathsf{V}}(X) \cdot v\right) \ . \tag{2.2.4}$$

This definition is extended to non homogeneous elements by linearity. In particular, if $\mathsf{U} \cong \mathbb{C}$ is the trivial representation, this defines the *dual representation* of V. The socle of the dual module is isomorphic to the top of the original one and vice versa, which implies that self dual representations have socles and tops that are isomorphic to each other.

Irreducible representations of the classical Lie superalgebras defined in the previous section are best obtained if one starts with a representation of their bosonic subalgebras first. For this construction, we need to make a distinction between type I and type II superalgebras. We quote without proof the results of [35].

In the type I case, we have by definition a decomposition of the odd part $\mathfrak{g}_{\bar{1}}$ into two simple $\mathfrak{g}_{\bar{0}}$-modules. Denote arbitrarily one of them as $\mathfrak{g}_1$ and the other as $\mathfrak{g}_{-1}$. As representations $\mathfrak{g}_{\pm 1}$ are dual to each other. If we set $\mathfrak{g}_0 \equiv \mathfrak{g}_{\bar{0}}$, we get a decomposition

$$\mathfrak{g} = \bigoplus_{i=-1}^{1} \mathfrak{g}_i \quad \text{with } [\mathfrak{g}_i\,,\,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \text{and } (\mathfrak{g}_i\,,\,\mathfrak{g}_j) = 0 \text{ if } i+j \neq 0 \ . \tag{2.2.5}$$

We note here that $\mathfrak{g}_0$ should not be confused with the Cartan subalgebra $\mathfrak{g}_0$. Equation (2.2.5) of course also implies that we can decompose the set of fermionic roots $\Delta_{\bar{1}}$ in $\Delta_1 \cup \Delta_{-1}$, depending on whether the root vector of a given fermionic root lies in $\mathfrak{g}_1$ or $\mathfrak{g}_{-1}$.

If we let D_Λ be a finite dimensional simple module of $\mathfrak{g}_0$ with highest weight Λ, we can extend it to a representation of $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by setting the action of the elements of $\mathfrak{g}_1$ to zero. From this, we finally arrive at a representation K_Λ of the full superalgebra, by letting[2]

$$\mathsf{K}_\Lambda := \mathsf{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathsf{D}_\Lambda \cong \mathsf{Alt}\left(\mathfrak{g}_{-1}\right) \otimes \mathsf{D}_\Lambda \ , \tag{2.2.6}$$

where the last part of the above equation is true since $[\mathfrak{g}_{-1}\,,\,\mathfrak{g}_{-1}] = 0$ and implies that $\dim \mathsf{K}_\Lambda = 2^{\dim \mathfrak{g}_1} \dim \mathsf{D}_\Lambda$. These representations are called *Kac modules* and are generically irreducible, except if

$$(\Lambda + \rho, \alpha) = 0 \quad \text{for some } \alpha \in \Delta_1 \ , \tag{2.2.7}$$

where ρ is the Weyl vector defined as

$$\rho = \rho_{\bar{0}} - \rho_{\bar{1}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\bar{0}}^+} \alpha - \frac{1}{2} \sum_{\beta \in \Delta_{\bar{1}}^+} \beta \ . \tag{2.2.8}$$

Definition 2.2.1. The number of independent roots for which (2.2.7) holds is called the *degree of atypicality* of the weight Λ. If there are no $\alpha \in \Delta_1$ that satisfy (2.2.7), then Λ is called *typical.* By extension, we refer to the representation with highest weight Λ as typical, respectively atypical if the highest weight itself is typical, respectively atypical.

[2] Alt denotes the exterior product.

We can now define the irreducible highest weight module S_Λ as being the top of K_Λ. If the Kac module is typical, then of course $\mathsf{S}_\Lambda = \mathsf{K}_\Lambda$. An important observation to make is that the superdimension of Kac modules for type I superalgebras is by construction *always zero.* We refer the reader to [36,37] for a step by step discussion of the representations of some particular type I superalgebras.

The case of type II superalgebras is more subtle. Here we will concentrate exclusively on the algebras $\mathfrak{g} = \mathrm{osp}\,(m|2n)$ with $m \neq 2$. Since the fermionic part transforms in an irreducible representation of $\mathfrak{g}_{\bar{0}}$, we can no longer perform the same decomposition as before. The solution to this conundrum requires splitting up the bosonic subalgebra into three parts. Using table 2.1, we define the two $n(n+1)/2$ dimensional vectorspaces

$$\mathfrak{g}_{\pm 2} := \{X_\alpha : \alpha = \pm(\delta_k + \delta_l) \text{ with } k, l = 1, \dots n\} \ . \qquad (2.2.9)$$

where X_α is the root generator associated to the root α. Let then $\mathfrak{g}_0 \cong \mathrm{so}\,(\mathrm{m}) \oplus \mathrm{gl}\,(\mathrm{n})$ be the subalgebra that contains the Cartan subalgebra together with all bosonic root generators not present in $\mathfrak{g}_{\pm 2}$. Under the action of $\mathfrak{g}_0$, the fermionic part of the superalgebra decomposes into two irreducible representations, namely $\mathfrak{g}_{\pm 1}$. We thus obtain a decomposition similar to before

$$\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}_i \quad \text{with } [\mathfrak{g}_i\,,\,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \text{and } (\mathfrak{g}_i\,,\,\mathfrak{g}_j) = 0 \text{ if } i+j \neq 0 \ . \qquad (2.2.10)$$

To construct the actual representations, we start as in the type I case with an irreducible integral highest weight representation D_Λ of $\mathfrak{g}_0$, which however in this case is not a representation of the full bosonic subalgebra. Let its highest weight vector be denoted by v_Λ. We extend D_Λ to a representation of $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ by letting the extra generators act trivially. We now define

$$b := 2\frac{(\Lambda\,,\,2\delta_n)}{(2\delta_n\,,\,2\delta_n)} \quad \text{and} \quad \mathsf{M}_\Lambda := \mathfrak{U}(\mathfrak{g}) X_{-2\delta_n}^{b+1} v_\Lambda \ , \qquad (2.2.11)$$

where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. The space M_Λ is an invariant subspace of $\mathsf{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\mathsf{D}_\Lambda$, which is now no longer finite dimensional, since $[\mathfrak{g}_{-1}\,,\,\mathfrak{g}_{-1}] \subset \mathfrak{g}_{-2}$. The Kac modules are defined as the finite dimensional quotient

$$\mathsf{K}_\Lambda := \mathsf{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\mathsf{D}_\Lambda / \mathsf{M}_\Lambda \ . \qquad (2.2.12)$$

The atypicality conditions (2.2.7) remain the same as before and we define the irreducible representations S_Λ as the top of K_Λ. In appendix B.1 we apply this general construction method to the specific case of osp (4|2). Unlike in the case of type I superalgebras, the superdimension of Kac modules defined via (2.2.12) is only zero[3] for the *typical* ones.

[3]See [35].

Example 2.2.1. Here, we present the very simple case of osp (1|2), which is built out of the bosonic elements $H, E_\pm$ together with the fermionic ones $F_\pm$, subject to the relations

$$\begin{array}{lll} [H\,,\,E_\pm] = \pm 2E_\pm & [H\,,\,F_\pm] = \pm F_\pm & [E_+\,,\,E_-] = H \\ [E_\pm\,,\,F_\mp] = F_\pm & [F_\pm\,,\,F_\pm] = \mp 2E_\pm & [F_+\,,\,F_-] = H\;. \end{array} \tag{2.2.13}$$

We proceed as in (2.2.10) and decompose the algebra as

$$\mathfrak{g}_0 = \text{span}\{H\} \qquad \mathfrak{g}_{\pm 1} = \text{span}\{F_\pm\} \qquad \mathfrak{g}_{\pm 2} = \text{span}\{E_\pm\}\;. \tag{2.2.14}$$

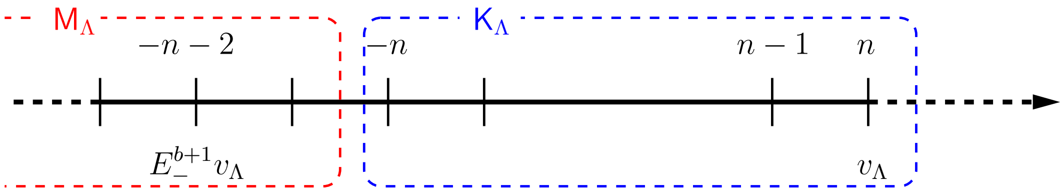

Figure 2.3: An illustration of the construction of the finite dimensional irreducible modules of osp (1|2). The vector $F_+(E_-)^{b+1}v_\Lambda$ is singular and is the highest weight vector in M_Λ.

The weights that will lead to finite dimensional modules are $\Lambda = n\delta_1$ with n a non-negative integer, so that $b = n$. The space $\mathsf{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\mathsf{D}_\Lambda$ is infinite dimensional and contains all vectors with weights $m\delta_1$, $m \leq n$ with multiplicity one. The reader is left with the task to see that all the vectors with weight smaller than $-n\delta_1$ belong to M_Λ. Thus the Kac modules are of dimension $2n + 1$ and are made out of two irreducible sp (2) representations. The Kac modules for osp (1|2) are always typical and thus always irreducible. The situation is pictured in figure 2.3.

A very important notion in the representation theory of Lie superalgebras is that of projective modules, which have the property that their tensor product with any representation is also a projective module.

Definition 2.2.2. A module $\mathsf{P}(\Lambda)$ is called a *projective cover* of the finite dimensional simple module $\mathsf{S}(\Lambda)$ if $\mathsf{top}\,\mathsf{P}(\Lambda) \cong \mathsf{S}(\Lambda)$ and if $\mathsf{P}(\Lambda)$ is *projective* in the category of finite modules. Projectivity here means that $\mathsf{P}(\Lambda)$ is a direct summand of a free module, or alternatively that for every surjective $\mathfrak{g}$-homomorphism $f : \mathsf{U} \to \mathsf{V}$ and every $\mathfrak{g}$-homomorphism $g : \mathsf{P}(\Lambda) \to \mathsf{V}$ there exists a, not necessarily unique, $\mathfrak{g}$-homomorphism $h : \mathsf{P}(\Lambda) \to \mathsf{U}$ such that $f \cdot h = g$. The projective covers form an ideal in the space of finite modules under tensor products and we will make great use of them in the later chapters.

Last, but not least in our discussion of the representation theory of Lie superalgebras is the concept of *characters* of representations. Given first a formal variable x, we can define a formal power series in x as the Laurent expansion

$$\chi(x) = \sum_{n\in\mathbb{Z}} a_n x^n \qquad \text{where } a_n \in \mathbb{C}\ . \tag{2.2.15}$$

These power series are not subject to the notion of radius of convergence and are well defined as long as the coefficients a_n are finite. Given a representation V of a Lie superalgebra $\mathfrak{g}$ with Cartan generators $\{H_i\}_{i=1}^r$, where $r \equiv \operatorname{rank}\mathfrak{g}$, we can define the *character* of V as the formal power series in r variables

$$\chi_{\mathsf{V}}(x_1,\ldots,x_r) := \operatorname{tr}_{\mathsf{V}}\left(\prod_{i=1}^r x_i^{H_i}\right) = \sum_{\Lambda\in\Lambda(\mathsf{V})} \operatorname{mult}_\Lambda \prod_{i=1}^r x_i^{\Lambda(H_i)}\ , \tag{2.2.16}$$

where $\Lambda(\mathsf{V})$ is the set of weights in V and $\operatorname{mult}_\Lambda$ their respective multiplicity. Thus, the characters are well defined as long as no multiplicity becomes infinite.

2.3 Lie supergroups

In order to define a Lie supergroup, we first have to set the stage by introducing some additional concepts. A finite dimensional *Grassmann algebra* $\mathfrak{G}$ of a field $\mathbb{F}$ is the 2^N dimensional vector space generated by N elements θ_i that anticommute with each other, $\{\theta_i\,,\,\theta_j\} = 0$. The number N is a priori a free parameter and one obtains different groups for different values of it, but here we will restrict ourselves to the case $N = \dim\mathfrak{g}_{\bar{1}}$. The Grassman algebra naturally decomposes into a direct sum of an even, $\mathfrak{G}_{\bar{0}}$, and an odd vectorspace, $\mathfrak{G}_{\bar{1}}$. Given a Lie superalgebra $\mathfrak{g}$, we take $\{t^a_{\bar{\imath}}\}$ to be a basis of the homogeneous subspaces $\mathfrak{g}_{\bar{\imath}}$ and define the *Grassmann envelope* $\mathfrak{G}(\mathfrak{g})$ to be the *Lie algebra* spanned by the linear combinations

$$X = \sum_{a=1}^{\dim\mathfrak{g}_{\bar{0}}} x_a t^a_{\bar{0}} + \sum_{i=1}^{\dim\mathfrak{g}_{\bar{1}}} y_i t^i_{\bar{1}} \qquad \text{where } x_a \in \mathfrak{G}_{\bar{0}},\ y_i \in \mathfrak{G}_{\bar{1}}\ , \tag{2.3.1}$$

whose commutator is defined as

$$X^i = \sum_a x^i_a T^a \Longrightarrow \left[X^1\,,\,X^2\right] = \sum_{a,b} x^1_a x^2_b \left[T^a\,,\,T^b\right]\ , \tag{2.3.2}$$

where the right hand side uses the supercommutator of $\mathfrak{g}$.

Definition 2.3.1. A Lie supergroup is the group generated by the elements e^X where X is an element in the Grassmann envelope $\mathfrak{G}(\mathfrak{g})$ of $\mathfrak{g}$. This definition is only strictly valid if $\mathfrak{g}$ is a real compact Lie superalgebra, for otherwise the whole group will not be covered. Generally, Lie supergroups can be defined categorically as supermanifolds with a group structure, as for instance in the book of Berezin [38], but for our applications, the first definition is sufficient.

2.4 Sigma models on coset superspaces G/G'

The general setup for non-linear sigma models involves a worldsheet Σ and a target space $\mathcal{M}$ as shown in figure 2.4. The basic fields of the theory depend on the worldsheet variables and are interpreted as coordinates in the target space. Thus the model describes the movement of Σ in a background given by $\mathcal{M}$.

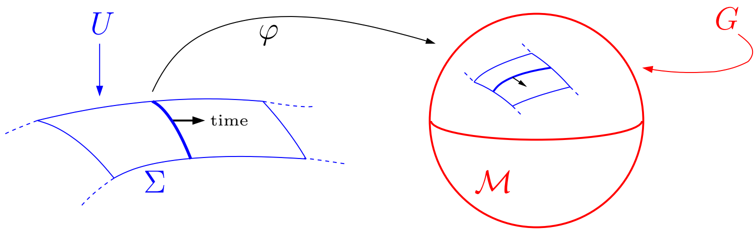

Figure 2.4: The basic elements of a nonlinear sigma model: a d dimensional world sheet Σ, a D dimensional target space $\mathcal{M}$ and a map φ between the two. The theory is assumed invariant under the action of a supergroup G that maps $\mathcal{M}$ onto itself and of a set U of transformations of the worldsheet.

In this section we give two equivalent formulations for the non-linear sigma models on right-coset superspaces of the form G/G', where G is some supergroup with non-degenerate metric and G' is a sub-supergroup. While the first is a priory easier to grasp and uses coordinates and tensor fields, the second is more algebraic in nature and will be of much use in the later parts of the chapter. In the beginning, we will make no assumptions concerning the structure of G', but later on fermionic elements of G' shall play a key role and we shall mostly concentrate on models for which $G' \equiv G^{\mathbb{Z}_n}$ is a subgroup invariant under some finite order automorphism.

2.4.1 General coset superspaces G/G'

We want to consider non-linear sigma models on homogeneous superspaces G/G', where the quotient is defined as the set of right cosets of G' in G through the identification

$$g \sim gg' \quad \text{for all} \quad g' \in G' \subset G \quad . \tag{2.4.1}$$

Example 2.4.1. Let us take $G = \mathrm{SU}(2) \cong S^3$ and $G' = \mathrm{U}(1) \cong S^1$. We parametrize these groups as

$$G = \left\{ g = \begin{pmatrix} e^{i(\alpha+\beta)}\cos\theta & e^{i(\alpha-\beta)}\sin\theta \\ -e^{-i(\alpha-\beta)}\sin\theta & e^{-i(\alpha+\beta)}\cos\theta \end{pmatrix} \right\} \quad G' = \left\{ g' = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right\} \ . \tag{2.4.2}$$

The relation between SU(2) and S^3 is clear if one sets $x_1 + ix_2 = e^{i(\alpha+\beta)}\cos\theta$ and $x_3 + ix_4 = e^{i(\alpha-\beta)}\sin\theta$, so that the sum of the squares of x_i equals one. Now, multiplying g by g' amounts to shifting β by t. Thus we can choose a representative of each coset such that x_1 is zero and we are left with x_2, x_3, x_4 that obey the equation $\sum_{i=2}^4 x_i^3 = 1$,

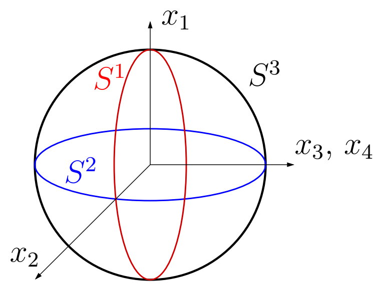

Figure 2.5: The sphere S^3 with the embedding of S^1 drawn in red. The quotient can then be represented as the equator.

i.e. the defining equation of S^2. Therefore, the right quotient $\mathrm{SU}(2)/\mathrm{U}(1)$ is the two dimensional sphere.

Back to the general case, let $\mathfrak{g}$ be the Lie superalgebra associated to G and assume that comes equipped with a metric $(\ ,\ )$ as in definition 2.1.2. This includes $\mathrm{gl}(m|n)$, $\mathrm{sl}(m|n)$[4], $\mathrm{psl}(n|n)$ or $\mathrm{osp}(m|2n)$. Similarly, let $\mathfrak{g}'$ be the Lie superalgebra associated to G'. We assume that the restriction of $(\ ,\ )$ to $\mathfrak{g}'$ is non-degenerate. In this case, the orthogonal complement $\mathfrak{m}$ of $\mathfrak{g}'$ in $\mathfrak{g}$ is a $\mathfrak{g}'$-module and one can write the following $\mathfrak{g}'$-module decomposition: $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$. In particular, this means that there are projectors P' onto $\mathfrak{g}'$ and $P = \mathbb{1} - P'$ onto $\mathfrak{m}$ which commute with the action of $\mathfrak{g}'$.

With the above requirements, the quotient G/G' can be endowed with a G-invariant metric g. This metric is by no means unique and generally depends on some number of continuous parameters which we shall call radii. The square root of the superdeterminant of g provides in the standard way a G-invariant measure μ on G/G', which is unique up to a multiplicative constant that depends on the radii of g. These two structures allow us to write down a purely kinetic Lagrangian for the sigma model on G/G' and quantize it in the path integral formalism. Inclusion of θ-terms, WZW terms or B-fields requires a better understanding of the geometry of the G/G' superspace. In fact, the θ and WZW terms are associated to G-invariant closed but not exact 2- and 3-forms, respectively. B-fields, on the other hand, are written in terms of G-invariant exact 2-forms. Every such linearly independent form comes with its own coupling constant. We shall only consider Lagrangians with a kinetic term and a B-field. Let b be some general linear combination of G-invariant exact 2-forms. Then the most general Lagrangian we consider can be written in the form

$$\mathcal{L} = \eta^{\mu\nu}\mathsf{g}(\partial_\mu, \partial_\nu) + \epsilon^{\mu\nu}\mathsf{b}(\partial_\mu, \partial_\nu) \ , \tag{2.4.3}$$

where $\eta^{\mu\nu}$ is the constant world sheet metric and $\epsilon^{\mu\nu}$ the antisymmetric tensor with $\epsilon^{01} = 1$. The Lagrangian is obviously evaluated on maps from the worldsheet Σ to the superspace G/G' and to every one of such maps one can associate a vector field ∂_μ on G/G', which appears in eq. (2.4.3) in a coordinate free notation.

[4]We exclude $\mathrm{sl}(n|n)$ and $\mathrm{pgl}(n|n)$, since they do not have a non-degenerate metric

There is a different way to formulate the sigma model on G/G', which makes its coset nature manifest and allows to explicitly construct the metric g and the B-field b in equation (2.4.3). For that purpose, instead of maps from the worldsheet to the target space G/G', we consider more general maps $g : \Sigma \to G$ from the world sheet to the Lie supergroup G. A basis set of 1-forms on G which are invariant under the global left G-action is provided by the so called Maurer-Cartan forms

$$J_\mu(x) \ := \ g^{-1}(x)\partial_\mu g(x) \ . \tag{2.4.4}$$

Higher G-invariant tensors may be built out of the Maurer-Cartan forms by taking tensor products. There is a subspace of such tensors which are also invariant with respect to the *local* right G'-action. These may be specified by their values on the coset superspace G/G'. We use this idea in order to build explicitly the G-invariant tensors g and b that enter the Lagrangian (2.4.3).

Under right G'-gauge transformations $g' : \Sigma \mapsto G'$ the Maurer-Cartan forms J_μ transform as

$$g(x) \ \mapsto \ g(x)g'(x) \qquad J_\mu(x) \ \mapsto \ (g'(x))^{-1}J_\mu(x)g'(x) + (g'(x))^{-1}\partial_\mu g'(x) \ . \tag{2.4.5}$$

Since the projection P on $\mathfrak{m}$ commutes with the action of $\mathfrak{g}'$, the projected forms $P(J_\mu)$ transforms by conjugation with g'. To build right G'-gauge invariant 2-forms we introduce the $\mathfrak{g}'$-intertwiners

$$\mathsf{G} \ \in \ \mathrm{End}_{\mathfrak{g}'}(\mathfrak{m} \circ \mathfrak{m}, \mathbb{C}) \qquad \text{and} \qquad \mathsf{B} \ \in \ \mathrm{End}_{\mathfrak{g}'}(\mathfrak{m} \wedge \mathfrak{m}, \mathbb{C}) \tag{2.4.6}$$

from the symmetric, respectively antisymmetric tensor product of $\mathfrak{m}$ with itself to the trivial representation. In terms of these intertwiners the Lagrangian (2.4.3) takes the explicit form

$$\mathcal{L} \ = \ \eta^{\mu\nu}\mathsf{G}\left(P(J_\mu), P(J_\nu)\right) + \epsilon^{\mu\nu}\mathsf{B}\left(P(J_\mu), P(J_\nu)\right) \ . \tag{2.4.7}$$

The choice of G and B, subject to some reality constraints, parametrizes the moduli space of the sigma model on G/G' with a kinetic term and a B-field only. Global left G-invariance of the Lagrangian (2.4.7) is automatic since Maurer-Cartan forms $J_\mu(x)$ are left G-invariant by construction. Right G'-gauge invariance, on the other hand, follows easily from the transformation properties of $P(J_\mu)$ and the def. (2.4.6) of G and B as invariant bilinear forms on the $\mathfrak{g}'$-module $\mathfrak{m} \otimes \mathfrak{m}$.

Example 2.4.2. To illustrate how the number of free parameters in a sigma model can be determined, let $G = \mathrm{SU}(\mathrm{N})$ for $N \geq 3$ and $G' = \mathrm{SU}(2)$, respectively $\mathrm{SO}(3)$ if N is even, respectively odd. We take the fundamental representation of G and the spin l representation of G':

$$\rho_\square : \, G \to \mathrm{End}\left(\mathbb{C}^N\right) \qquad \rho_l : \, G' \to \mathrm{End}\left(\mathbb{C}^N\right) \ , \tag{2.4.8}$$

where $l = \frac{N-1}{2}$. Using these faithfull representations, we can define an embedding of the group G' into G and thus a quotient. Denoting the spin k representation of $\mathfrak{g}'$ by (k), we see that the Lie algebra $\mathfrak{g}$ transforms as $((l)\otimes(l))/(0) \cong \bigoplus_{k=1}^{2l}(k)$. Since $\mathfrak{g}'$ itself transforms in the spin one representation, we see that

$$\mathfrak{m} \cong \bigoplus_{k=2}^{2l}(k)\ . \tag{2.4.9}$$

It is not hard to see, since for simple modules

$$(k)\circ(k) \cong \bigoplus_{n=0}^{\lfloor k\rfloor}(2k-2n) \qquad (k)\wedge(k) \cong \bigoplus_{n=0}^{\lfloor k-\frac{1}{2}\rfloor}(2k-2n-1)\ , \tag{2.4.10}$$

that $\mathfrak{m}\circ\mathfrak{m}$ contains $(N-2)$ $\mathfrak{g}'$ invariants, while the antisymmetric tensor product has none. Thus, the space G/G' defines a sigma model with a purely kinetic term that has $N-2$ free parameters. For $N=3$, this a symmetric space.

2.4.2 $G/G^{\mathbb{Z}_N}$ coset superspaces

In the previous subsection we have described the most general action with a kinetic term and a B-field for the G-invariant sigma model with target space G/G'. The formulation includes sigma models on symmetric spaces and certain generalizations that appear in the context of AdS compactifications. In fact, for many cases of interest, the Lie sub-superalgebra $\mathfrak{g}'$ in $\mathfrak{g}$ consists of elements that are invariant under some finite order automorphism $\Omega : \mathfrak{g} \mapsto \mathfrak{g}$. An automorphism of order N defines a decomposition

$$\mathfrak{g} \ =\ \mathfrak{g}'\oplus\bigoplus_{i=1}^{N-1}\mathfrak{m}_i \qquad , \quad \Omega|_{\mathfrak{g}'} \ =\ \mathbb{1} \quad , \quad \Omega(\mathfrak{m}_k) \ =\ e^{\frac{2\pi i k}{N}}\mathfrak{m}_k \tag{2.4.11}$$

of the superalgebra $\mathfrak{g}$ into eigenspaces of Ω. Extending our previous notation, we denote by P_i the projection maps onto $\mathfrak{m}_i$. Thanks to the properties of the Ω, we find

$$[\mathfrak{m}_i,\mathfrak{m}_j] \ \subset\ \mathfrak{m}_{i+j \text{ mod } N} \qquad (\mathfrak{m}_i,\mathfrak{m}_j) \ =\ 0 \quad \text{if} \quad i+j \ \neq\ 0 \text{ mod } N\ , \tag{2.4.12}$$

where we have set $\mathfrak{m}_0 \equiv \mathfrak{g}'$. Consequently, the subalgebra $\mathfrak{g}'$ acts on the Ω-eigenspaces $\mathfrak{m}_i$. Note that the spaces $\mathfrak{m}_i$ need not be indecomposable under $\mathfrak{g}'$ in which case the decomposition into $\mathfrak{g}'$-modules is finer than the decomposition (2.4.11) into eigenspaces of Ω.

Whenever a coset superspaces G/G' is defined by an automorphism Ω of order N we shall use the alternative notation $G/G^{\mathbb{Z}_N}$. The cases when the grading induced by Ω is compatible with the $\mathbb{Z}_2$ superalgebra grading, that is $\mathfrak{m}_{2i}\in\mathfrak{g}_{\bar{0}}$ and $\mathfrak{m}_{2i-1}\in\mathfrak{g}_{\bar{1}}$, were considered by Kagan and Young in [39]. They restricted to a family of Lagrangians for which G and B take the following special form

$$\mathsf{G}(X,Y) \ =\ \sum_{i=1}^{N-1} p_i\,(P_i(X),P_{N-i}(Y)) \quad , \quad \mathsf{B}(X,Y) \ =\ \sum_{i=1}^{N-1} q_i\,(P_i(X),P_{N-i}(Y))\ , \tag{2.4.13}$$

where the p_i and q_i are constants obeying the additional constraints

$$p_i = p_{N-i} \qquad q_i = -q_{N-i} \ . \tag{2.4.14}$$

We have a few comments to make. First, the forms of G and B in equation (2.4.13) do not give rise to the most general Lagrangian for coset superspaces G/G'. As an example consider the $\mathbb{Z}_2$ quotient

$$\frac{\mathrm{U}\,(2|2)}{\mathrm{U}\,(1) \times \mathrm{U}\,(1|2)} \ .$$

Since $\mathfrak{m} \cong \square \oplus \square^\star$ decomposes into a direct sum of the fundamental of $\mathrm{u}\,(1|2) \subset \mathfrak{g}'$ and of its dual and since the tensor product of $\square$ with $\square^\star$ contains the trivial representation of $\mathfrak{g}'$, the most general sigma model on the space in (2.4.15) will depend on two parameters. On the other hand, taking the Ansatz (2.4.14) leads to a one parameter space of Lagrangians, since the antisymmetric part in (2.4.14) must vanish for symmetric spaces.

Second, the properties of the theory defined by equations (2.4.13) certainly depend on the precise choice of the parameters p_i and q_i and it was shown in [39, 40] that one loop conformal invariance requires

$$p_i \ = \ 1 \qquad q_i \ = \ 1 - \frac{2i}{N} \qquad \text{for } i \ \neq \ 0 \ , \tag{2.4.15}$$

for all even N. We believe, however, that in most cases these conditions are not sufficient to guarantee the vanishing of the full beta function.

The last comment concerns the treatment of coset superspaces G/G' in which the denominator group G' has a non-trivial centralizer $Z \subset G$. For such coset superspaces, there exists a residual symmetry by right multiplications with elements of Z. In an equivalent formulation one can make all symmetries of G/G' to act from the left. For that we rewrite $G/G' = G \times Z / G' \times Z$ where the factor Z in the denominator is embedded diagonally into the numerator.

Example 2.4.3. To make the associated reformulation in our last comment a bit more explicit, we focus on the sigma model with target space given by a supergroup U. Without any further thought one might be tempted to describe this model through $G = U$ and $G' = \{e\}$. But as our introductory comments suggest, we prefer to rewrite the group manifold U as a coset superspace $U = U \times U/U$ and hence to set

$$G \ = \ \{(x, y) : x, y \in U\} \qquad , \qquad G' \ = \ \{(x, x) : x \in U\} \ . \tag{2.4.16}$$

The left and right action of G on itself is given by componentwise multiplication. The right coset superspace $G/G' \cong U$ is considered as the space of equivalence classes under the equivalence relation $(x, y) \sim (xz, yz)$, for all $z \in U$. In particular, $(xy^{-1}, 1)$ is the canonical representative of the equivalence class of (x, y). Hence, the currents J_μ and the projection map $P : \mathfrak{g} \to \mathfrak{m}$ are given by

$$J_\mu \ = \ \big(x^{-1}\partial_\mu x, y^{-1}\partial_\mu y\big) \qquad , \qquad P : (v, w) \ \mapsto \ \left(\frac{v - w}{2}, -\frac{v - w}{2}\right) \ . \tag{2.4.17}$$

If (,) is the invariant form on the Lie superalgebra of U and we take G to be given by

$$\mathsf{G}\left((v_1, w_1) \circ (v_2, w_2)\right) \;=\; (v_1, v_2) + (w_1, w_2) \tag{2.4.18}$$

we obtain the usual principal chiral model for U. In fact, one may easily show that

$$\mathsf{G}\left(P\left(J_\mu\right), P\left(J_\nu\right)\right)\eta^{\mu\nu} \;=\; \frac{1}{2}\left(u^{-1}\partial_\mu u, u^{-1}\partial_\nu u\right)\eta^{\mu\nu} \quad ,$$

where $u = xy^{-1} \in U$. Thereby we have established the standard geometric results that allows us to treat the sigma chiral model on U as a G/G' coset superspace model. The advantage of the seemingly more complicated coset description will become apparent later on.

2.4.3 Observables and correlators

Here, we give a description of the observables of non-linear sigma models and of their correlation functions. Let us denote by $\mathcal{G}$ and $\mathcal{G}'$ the space of all continuous maps from the world-sheet Σ to the supergroups G and G', respectively. Obviously, $\mathcal{G}'$ acts on $\mathcal{G}$ by point-wise (on Σ) right multiplication. Local observables of the G/G' quotient model are defined as some well behaved class of maps $\mathcal{O} : \mathcal{G} \times \Sigma \mapsto \mathbb{C}$ invariant under this right $\mathcal{G}'$ action

$$\mathcal{F}_{G/G'} \;=\; \{\; \mathcal{O} : \mathcal{G} \times \Sigma \mapsto \mathbb{C} \mid \mathcal{O}(g, x) \;=\; \mathcal{O}(g \cdot g', x) \text{ for all } g' \in \mathcal{G}'\} \;, \tag{2.4.19}$$

where we have denoted $\mathcal{O}(g, x) := \mathcal{O}(g(x))$.

One class of observables is obtained by restricting smooth right G'-invariant functions $f : G \mapsto \mathbb{C}$ to the image of an arbitrary map $g : \Sigma \mapsto G$. Existence of the 2-point function for this observable $f(g(x))$ requires that $f \in L_2(G/G')$. These are the tachyonic fields.

Similarly, all other observables can be obtained from smooth right G'-invariant tensor forms t of rank k on G by restricting them to the image of some arbitrary map $g : \Sigma \mapsto G$ and evaluating them on the set of vector fields $\partial_{\mu_1}, \ldots, \partial_{\mu_k}$. Existence of correlation functions for the observables $t_{g(x)}(\partial_{\mu_1}, \ldots, \partial_{\mu_k})$ imposes some further constraints. As an example, let us consider the Maurer-Cartan forms J_μ we have introduced in equation (2.4.4). Their components do not give rise to observables of the quotient model because there are not right G'-gauge invariant. Nevertheless, recalling their behavior (2.4.5) under right G'-gauge transformations, one can build the following observables

$$j_\mu \;:=\; g\left[\sum_{i=1}^{N-1}\left(p_i\eta^{\mu\nu} - q_i\epsilon^{\mu\nu}\right)P_i(J_\nu)\right]g^{-1} \;\in\; \mathcal{F}_{G/G'} \;. \tag{2.4.20}$$

These are the Noether currents for the global symmetry G of the G/G' sigma model, if the Lagrangian is of the form (2.4.13). The equations of motion imply that they are conserved, i.e. $\partial_\mu j^\mu = 0$.

In the following we shall denote by $\mathcal{O}(x)$ the restriction of the local observable $\mathcal{O}$ to the point x of the world-sheet. Given any set $\mathcal{O}_i \in \mathcal{F}_{G/G'}$ of such local observables we define their unnormalized *correlation functions* through

$$\boxed{\left\langle \prod_{i=1}^{N} \mathcal{O}_i(x_i) \right\rangle_{G/G'} = \int_{\mathcal{G}} [\mathrm{d}\mu_G]\, e^{-\mathcal{S}} \prod_{i=1}^{N} \mathcal{O}_i(x_i)} \,. \tag{2.4.21}$$

Here, $\mathcal{S} = \int_\Sigma d^2x\, \mathcal{L}$ is the action corresponding to the Lagrangian (2.4.7) of our model. Our definition of correlation functions involves an integration over elements of $\mathcal{G}$ with some left $\mathcal{G}$-invariant measure

$$[d\mu_G(g)] = \prod_{x\in\Sigma} d\mu_G(g(x)) \ , \tag{2.4.22}$$

where $d\mu_G$ is the unique (up to normalization) Haar measure on G. In eq. (2.4.21), the integration over G at every point of the worldsheet yields a factor which is the volume of G'. Strictly speaking, this makes sense only if G' is compact. We assume that the contribution of such factors can be properly regularized and renormalized by replacing the worldsheet Σ with a lattice, as in figure 2.6, and shall not dwell on such aspects.

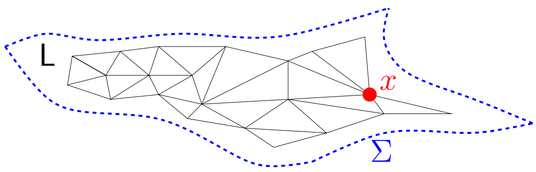

Figure 2.6: A lattice regularization of the world sheet Σ. At each point $x \in \mathsf{L}$ we place a copy of $L_2(G/G')$.

The reader might be curious about why we insist on integrating over maps $\mathcal{G}$ from the worldsheet to the group G rather then maps from the worldsheet to the quotient G/G'. In other words, why we do not fix the right G'-gauge invariance? As we shall see, keeping this symmetry explicit in the quantum theory simplifies the cohomology calculations on tensor fields.

2.5 Conformal field theory

In this section, we state the most important properties of two dimensional quantum conformal theories in the bulk and in the boundary case. It is unfortunately out of the scope of this work to present a self-contained introduction to conformal field theory. Whatever formulae we may present are shown only so as to establish our conventions and we refer to the review articles [41] and mostly to [42] for the rest.

2.5.1 Conformal transformations

If M is a d dimensional manifold with g as metric, then the conformal group $C(M, g)$ is made out of the diffeomorphisms $\varphi : M \to M$, that leave shapes unchanged, but are allowed to modify lengths by a scale factor Ξ:

$$x \mapsto x' \equiv \varphi(x) \Longrightarrow g_{\mu\nu} \mapsto g'(x') = \frac{\partial x^\alpha}{\partial x'_\mu} \frac{\partial x^\beta}{\partial x'_\nu} g_{\alpha\beta}(x) \stackrel{!}{=} \Xi(x) g_{\mu\nu}(x) \ . \tag{2.5.1}$$

If we specialize to the case $M = \mathbb{R}^d$ and $g_{\mu\nu} = \eta_{\mu\nu}$, the diagonal metric of signature $(p, d-p)$, we can ask ourselves which infinitesimal maps $x \mapsto x + \epsilon$ satisfy (2.5.1) to first order in ϵ. The resulting equation is

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \left(\partial_\alpha \epsilon^\alpha\right) \eta_{\mu\nu} \ . \tag{2.5.2}$$

As explained in [41], the solutions of this equation can be integrated to global transformations of $\mathbb{R}^d$, leading to the following set of conformal mappings independently of the dimension:

- Translations

$$x \mapsto x + a,\ a \in \mathbb{R}^d \qquad \Xi = 1 \tag{2.5.3}$$

- Rotations

$$x \mapsto \mathrm{R} \cdot x,\ \mathrm{R} \in \mathrm{SO\,(p,d\text{-}p)} \qquad \Xi = 1 \tag{2.5.4}$$

- Dilatations

$$x \mapsto \Lambda x,\ \Lambda \neq 0 \qquad \Xi = \frac{1}{\Lambda^2} \tag{2.5.5}$$

- Special conformal transformations

$$x \mapsto \frac{x - b|x|^2}{1 - b \cdot x + |b|^2},\ b \in \mathbb{R}^d \qquad \Xi = (1 + 2b \cdot x + |b|^2 |x|^2)^2 \tag{2.5.6}$$

These global transformations build the group SO (p+1, d-p+1) of dimension $\frac{(d+2)(d+1)}{2}$.

In two dimensions and for an Euclidean metric, we see that (2.5.1) reduces to the Cauchy-Riemann equations, so that locally all holomorphic or antiholomorphic mapping of the complex plane are angle preserving, as illustrated in figure 2.7. The number of globally conformal mappings remains however the same. Infinitesimally, conformal maps in $\mathbb{C}$ can be seen as shifts of the coordinates z, respectively $\bar{z}$ by small holomorphic $\epsilon(z)$, respectively antiholomorphic $\bar{\epsilon}(\bar{z})$ functions. A basis for these infinitesimal transformations is obtained by taking $\epsilon(z) = -\varepsilon z^{n+1}$, $\bar{\epsilon}(\bar{z}) = -\bar{\varepsilon} \bar{z}^{n+1}$, where n is an integer and ε, $\bar{\varepsilon}$ are vanishingly small. The generators of these maps can then be written as

$$l_n = -z^{n+1} \partial \qquad \bar{l}_n = -\bar{z}^{n+1} \bar{\partial} \ , \tag{2.5.7}$$

of which in particular l_{-1}, l_0, l_1 and $\bar{l}_{-1}, \bar{l}_0, \bar{l}_1$ generate the global transformations. One easily sees that the generators obey the commutation relations

$$[l_m\,,\, l_n] = (m-n) l_{m+n} \qquad \left[\bar{l}_m\,,\, \bar{l}_n\right] = (m-n) \bar{l}_{m+n} \qquad \left[l_m\,,\, \bar{l}_n\right] = 0 \ . \tag{2.5.8}$$

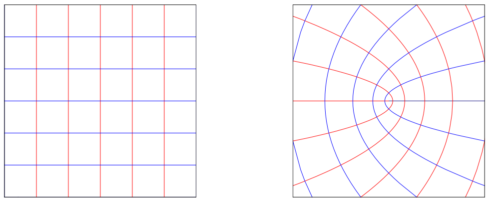

Figure 2.7: The transformation of an orthogonal grid under the conformal map $z \mapsto z^2$.

These relations define the *Witt algebra*. Classical conformal field theories in two dimensions have to be invariant under the action of the generators in equation (2.5.7), however this symmetry is in most cases completely destroyed by quantum anomalies. In those *few* fortunate situations in which conformal invariance survives, the algebra (2.5.8) has to be extended by the addition of a central element c called the central charge. We will have more to say about that in the next section.

2.5.2 The energy-momentum tensor

We require that the theories that we work with be provided with a conserved symmetric energy-momentum tensor $T_{\mu\nu}$. The tensor can be derived from the space-time transformations invariance of the Lagrangian of the model by using the standard Noether method for conserved currents. Dilatation invariance of the theory forces the current $d^\mu = T^\mu{}_\nu x^\nu$ to be conserved, which implies that T is traceless, i.e. $T^\mu{}_\mu = 0$. Passing to complex coordinates in two dimensions and using the symmetry and tracelessness of the energy momentum tensor leads to the following analytic requirements for its components:

$$\bar{\partial} T = 0 \qquad \partial \bar{T} = 0 \ , \tag{2.5.9}$$

where we have set

$$T \equiv T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right) \qquad \bar{T} \equiv T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) \ . \tag{2.5.10}$$

In complex coordinates, tracelessness is equivalent to the vanishing of the $T_{z\bar{z}} = \frac{1}{4} T^\mu{}_\mu$ component.

Quantization of a two dimensional field theory that is classically conformal will in general break the invariance outright. Even in those situations in which the quantum theory is conformally invariant, the Witt algebra has to be modified. To be more precise, if we expand the stress-energy tensor in modes as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2} \ , \tag{2.5.11}$$

then the classical dynamics of the conformally invariant theory will force the modes to obey the Witt algebra relations (2.5.8) with the commutators replaced by Poisson brackets. One then naively expects the quantum analogues of the modes to be operators that obey the Witt algebra with the Poisson brackets replaced by commutators. It turns out however that, since the symmetry algebra admits a non-trivial *central extension*, that in most cases the commutation relations will be slightly modified:

$$\begin{aligned} [L_m\,,\,L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \\ \left[\bar{L}_m\,,\,\bar{L}_n\right] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \\ \left[L_m\,,\,\bar{L}_n\right] &= 0\ . \end{aligned} \tag{2.5.12}$$

Here $c \in \mathbb{R}$ is the *central charge*, or conformal anomaly, of the theory. We thus have two commuting copies, spanned respectively by of the L_n or $\bar{L}_n$, of the *Virasoro algebra* with central charge c. While for our applications the central charge shall not play much of a role, it is important to note that for theories that require two dimensional reparametrization invariance, such as string theory, the number c has to be made to vanish by the addition of some extra sectors, for otherwise severe quantum anomalies appear.

Definition 2.5.1. The action of L_0 and $\bar{L}_0$ on the space of states of the physical theory cannot always be diagonalized, but their eigenvalues h, $\bar{h}$ are nevertheless an useful concept and are referred to as the *conformal weights*, or *conformal dimensions*, of the states. In the absence of boundaries on the worldsheet, the energy of a state is given by the sum $h + \bar{h}$ and its spin by the difference $h - \bar{h}$.

2.5.3 WZW and GN Models

In this section, we shall present two very important examples of 2d quantum conformal field theories, namely the Wess-Zumino-Witten models and the Gross-Neveu models. Both share the characteristic of having at least some Lie supergroup symmetry.

Let $g : \Sigma \to G$ be a map from a worldsheet into a given supergroup, whose superalgebra is $\mathfrak{g}$ with a metric $(\cdot, \cdot)$. We have to extend the map g to a G-valued map on a three dimensional manifold B whose boundary is Σ, that is $\partial B = \Sigma$. For a given basis $\left\{T^A\right\}_{A=1}^{\dim \mathfrak{g}}$ of the Lie superalgebra, we define as before

$$\kappa^{AB} := \left(T^A\,,\,T^B\right) \qquad f^{ABC} := \left(T^A\,,\,\left[T^B\,,\,T^C\right]\right)\ , \tag{2.5.13}$$

which are respectively graded symmetric and graded antisymmetric. The left invariant currents are written as

$$J^\alpha_A := \left(g^{-1}\partial^\alpha g\,,\,T^B\right)\kappa^{-1}_{BA}\ , \tag{2.5.14}$$

so that we can decompose $g^{-1}\partial^\alpha g = J^\alpha_A T^A$. The nonlinear sigma model on the group G

is then defined as

$$\begin{aligned}
\mathcal{S}_\sigma &:= \frac{1}{f^2}\mathcal{S}_{\rm kin} + k\mathcal{S}_{\rm WZ} \qquad \text{with} \\
\mathcal{S}_{\rm kin} &:= -\frac{i}{4\pi}\int_\Sigma dz\wedge d\bar z\,\left(g^{-1}\partial g\,,\,g^{-1}\bar\partial g\right) = -\frac{i}{4\pi}\int_\Sigma dz\wedge d\bar z\,\kappa^{AB}J_A\bar J_B \\
\mathcal{S}_{\rm WZ} &:= -\frac{i}{24\pi}\int_B \left(g^{-1}{\rm d}g,\left[g^{-1}{\rm d}g, g^{-1}{\rm d}g\right]\right) = -\frac{i}{24\pi}\int_B d^3x\,\epsilon_{\alpha\beta\gamma}f^{ABC}\,J_A^\alpha J_B^\beta J_C^\gamma\;,
\end{aligned} \tag{2.5.15}$$

where $\epsilon_{\alpha\beta\gamma}$ is the three dimensional antisymmetric tensor with $\epsilon_{123}=1$.

Definition 2.5.2. The sigma model on the supergroup G defined via $\mathcal{S}_\sigma$ of (2.5.15) is called a *principal chiral model*, respectively a *Wess-Zumino-Witten model* if the couplings constants obey $k=0$, respectively $k=\frac{1}{f^2}$. For WZW theories, the constant k is called the *level*.

We have a few remarks to make:

- If the bosonic base of G is a compact semisimple group, then we must require that the constant k be an integer, for otherwise the path integral would not be properly defined.
- For bosonic groups that are not abelian, quantum conformal invariance of the model requires that one takes the model at the Wess-Zumino-Witten point, that is with $\frac{1}{f^2}=k$. This requirement is however waived for the supergroups PSU (N|N) and OSP (2S+2|2S), who thus have for each value of the level a one parameter family of conformal theories parametrized by f. One can consider these models as special deformations of WZW models, as described in [34, 43].
- The sigma model of (2.5.15) has a left-right $\mathfrak{g}$ symmetry algebra, which becomes enhanced to a $\widehat{\mathfrak{g}}\oplus\widehat{\mathfrak{g}}$ affine algebra at level k at the Wess-Zumino-Witten point.

Let us now turn to a very different theory, namely the osp $(m|2n)$ *Gross-Neveu* model for m free real fermions and n pairs of bosonic ghosts. The free part of the action is determined through

$$\mathcal{S}^{\rm GN}_{\rm free} = \frac{1}{2\pi}\int_\Sigma d^2z\left[\sum_{i=1}^m\left(\psi_i\bar\partial\psi_i+\bar\psi_i\partial\bar\psi_i\right)+2\sum_{a=1}^n\left(\beta_a\bar\partial\gamma_a+\bar\beta_a\partial\bar\gamma_a\right)\right]\,. \tag{2.5.16}$$

This action defines a conformal field theory with central charge $c=\frac{m-2n}{2}$ and both left and right $\widehat{\rm osp}(m|2n)$ current symmetry at level $k=1$. The conformal dimension of the fundamental fields $(\psi_i,\beta_a,\gamma_b)$ is $h=\frac{1}{2}$ and they transform in the fundamental representation of the left horizontal osp $(m|2n)$ algebra. The interaction term for this theory is

$$\mathcal{S}^{\rm GN}_{\rm int} = \frac{g^2}{2\pi}\int_\Sigma d^2z\left[\sum_{i=1}^m\psi_i\bar\psi_i+\sum_{a=1}^n(\gamma_a\bar\beta_a-\beta_a\bar\gamma_a)\right]^2\,. \tag{2.5.17}$$

The interaction breaks the affine left-right $\widehat{\mathrm{osp}}(m|2n)$ symmetry to a single osp $(m|2n)$. An alternative way of understanding this model, is to think of the free part of the theory as a free field representation of the Wess-Zumino-Witten model of OSP (m|2n) at level one and of the interacting model as being a current-current perturbation thereof. For the special case of $m = 2$ and $n = 0$, the model is better known under the name of massless Thirring model.

2.5.4 Boundary Perturbation Theory

So far, we have only discussed the cases in which the two dimensional world sheet Σ has no boundaries. In the presence of boundaries, suitable conditions have to be set for the fundamental fields of the theory, for otherwise conformal invariance will be irredeemably broken. If we assume for the sake of simplicity that the world sheet is the upper half $\mathbb{H}$ of the complex plane, then we require that

$$T(z) = \bar{T}\,(\bar{z}) \qquad \text{for } z = \bar{z}\ . \tag{2.5.18}$$

Finding out all boundary conditions that are compatible with this requirement is a daunting task in general. Here, we will only concern ourselves with those boundary conditions that preserve the whole algebra $\mathcal{W}$ of chiral, i.e. holomorphic or anti-holomorphic, fields. Thus, we require that there be an automorphism ω of $\mathcal{W}$ such that at the boundary

$$W(z) = \omega(\bar{W})(\bar{z}) \qquad \text{for } W, \bar{W} \in \mathcal{W} \text{ and } z = \bar{z}\ . \tag{2.5.19}$$

This implies in particular that $\omega(\bar{T}) = T$ on the boundary. Relation (2.5.19) tells us that in the boundary theory the correlators of $\omega(\bar{W})$ are those of W if we continue them analytically in the lower half plane, see [44].

In the second part of chapter 4, we will deform the OSP (2S+2|2S) Wess-Zumino-Witten models at level one by a current-current perturbation. Under the deformation, conformal invariance will be preserved but the conformal dimensions of boundary fields will change. It was shown in [34] that one can use the supergroup symmetry to sum up the perturbative expansion of the conformal dimensions to all orders. In what follows, we will give a brief sketch of the arguments.

We want to consider the Wess-Zumino-Witten models on the special supergroups PSU (N|N) or OSP (2S+2|2S). These supergroups have the property that their Killing form vanishes, which from (2.1.10) implies that all double contractions of the structure constants must be zero, so that in particular all invariants that can be built out of the structure constants alone vanish as well. Using the pictorial representation introduced in figure 2.1, this means for example, that the invariants represented in figure 2.8 are zero.

This property is actually more general, for these groups do not posses non-zero invariants that result from a contraction involving at least one structure constant f^{ABC}.

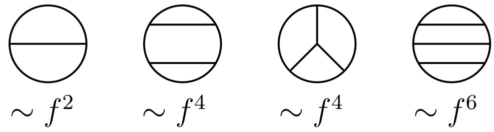

Figure 2.8: A few examples of vanishing invariants built out of the structure constants.

Therefore, all invariants of the type that contract the structure constants with some 3-tensor t, must vanish, i.e.

$$\sum_{A,B,C=1}^{\dim(\mathfrak{g})} f^{ABC} t_{ABC} \equiv 0 \quad \forall t \ . \tag{2.5.20}$$

Remembering that the current-current operator product expansions in WZW theories are

$$J^A(z)J^B(w) \sim \frac{f^{AB}{}_C J^C(w)}{z-w} + \frac{k\kappa^{AB}}{(z-w)^2} \ , \tag{2.5.21}$$

we see that the vanishing of the Killing form will be of great use if we have to contract many currents with each other.

The theories we are interested in are not the WZW models per se. We wish to perturb them by adding the current-current operator

$$\mathcal{S}_{\text{int}} = g \int_{\mathbb{H}} dz d\bar{z}\, \Psi(z,\bar{z}) := g \int_{\mathbb{H}} dz d\bar{z}\, \kappa^{-1}_{AB} J^A(z) \bar{J}^B(w) \ . \tag{2.5.22}$$

The correlation functions of boundary operators $\phi_i \equiv \phi_i(x_i)$ in the deformed theory are given by the expression:

$$\begin{aligned} \langle\, \phi_1 \cdots \phi_N \,\rangle_{g,L,\epsilon} &= \frac{1}{\mathsf{Z}} \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int_{\mathbb{H}^n_{L,\epsilon}} d^n z d^n \bar{z} \left\langle \phi_1 \cdots \phi_N \prod_{a=1}^{n} \Psi(z_a,\bar{z}_a) \right\rangle \\ \text{with } \mathsf{Z} &:= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int_{\mathbb{H}^n_{L,\epsilon}} d^n z d^n \bar{z} \left\langle \prod_{a=1}^{n} \Psi(z_a,\bar{z}_a) \right\rangle \ , \end{aligned} \tag{2.5.23}$$

where we have introduced an infrared regulator L as well as an ultraviolet one ϵ in the integration domain:

$$\mathbb{H}^n_{L,\epsilon} = \{(z_1,\ldots,z_n) : \text{Im}(z_a) > \epsilon, |z_a - z_b| > \epsilon, |z_a| < L\} \ . \tag{2.5.24}$$

The correlation functions on the right hand side of (2.5.23) are to be computed in the undeformed WZW model. When computing $\langle\, \phi_1 \cdots \phi_N \,\rangle_{L,\epsilon}$, we can ignore terms on the right hand side of (2.5.23) that contain loops of contracted currents, that is for which there exist a subset of the z_a so that all fields depending on z_a and z_a^* are contracted only among themselves. These infrared divergent contributions are cancelled when we divide by Z.

We want to concentrate on two point functions for the fields ϕ_{U}, ϕ_{V}, where U, V indicate the representation they transform in. As we shall argue in sectin 3.3.4, the perturbation (2.5.22) does not break conformal invariance, so that we expect

$$\langle\, \phi_{\mathsf{U}}(x)\phi_{\mathsf{V}}(y)\,\rangle_g = \frac{\mathfrak{I}_{\mathsf{UV}}(g)}{(x-y)^{D_{\mathsf{U}}(g)+D_{\mathsf{V}}(g)}}\ , \tag{2.5.25}$$

where $\mathfrak{I}_{\mathsf{UV}}$ is an intertwiner from the tensor product $\mathsf{U}\otimes\mathsf{V}$ to the trivial representation and the D_i are the restrictions of the L_0 mode. Here, the D_i need not be diagonalizable. Their general form is $D_i(g) = h_i(g)\mathbb{1} + \delta(g)$, where h_i is the conformal dimension and δ is a nilpotent operator that vanishes for theories in which L_0 can be diagonalized.

It was argued in [34], that in order to obtain the $h_i(g)$, one does not need to compute the full two point correlation function. In fact, the vanishing of any double contraction of the structure constants among themselves implies that in the computation of (2.5.25) using (2.5.23), one can drop the term proportional to ${f^{AB}}_C$ from (2.5.21). Thus we are left with a perturbation series that is abelian, hence the name of *quasi-abelian deformations* for these models. We quote the final result of [34] for the change in the conformal dimensions of the fields in the representation U

$$\boxed{h_{\mathsf{U}}(g) - h_{\mathsf{U}}(0) = -\frac{g}{k(k+g)}\,\mathsf{Cas}(\mathsf{U})}\ , \tag{2.5.26}$$

where $\mathsf{Cas}(\mathsf{U})$ is the value of the eigenvalues of the quadratic Casimir in the representation U. We shall make great use of this formula in chapter 4.

Chapter 3

Cohomological Reduction

In this chapter we present a method for the computation of certain correlation functions in non-linear sigma models with target space supersymmetry. We take our inspiration partially from the study of conformal field theories with $\mathcal{N} = (2, 2)$ world-sheet supersymmetry. For such models, a very conventional trick that one exploits through the so-called topological twists, is to identify special subsectors whose dependence on the couplings can be brought under control. The idea is to employ a fermionic world-sheet symmetry generator as a BRST operator and to select its cohomology as the relevant subsector. If the action of the model is trivial in the cohomology, then the correlation functions of subsector operators do not depend on the coupling constants of the theory. Such correlators can then be calculated in the classical limit, as described for example in [45]. Most of the contents of this chapter were published in the article [46], which was a collaboration with C. Candu, T. Creutzig and V. Schomerus.

The models we are interested in possess target space rather than world-sheet supersymmetry. A natural idea then is to promote an internal nilpotent symmetry to a BRST operator. In following this lead, we shall uncover a rather remarkable structure. Suppose we are starting with a sigma model on the quotient G/G', defined as the set of right G' cosets in G, with G' being some sub-supergroup of G. Let then Q be some fermionic generator in the superalgebra $\mathfrak{g}' \subset \mathfrak{g}$ such that $Q^2 = 0$ and note that such a Q is a symmetry of the G/G' sigma model. Through its cohomology, Q determines a subsector, which, quite remarkably, turns out to form the state space of another sigma model, one that is defined on the coset superspace H/H' with a new pair of supergroups $H' \subset H$. The target space H/H' has smaller dimension than G/G' and the symmetry algebra $\mathfrak{h}$ of the reduced theory is contained in the symmetry algebra $\mathfrak{g}$. In many cases, further reduction is possible until the procedure terminates because the remaining symmetry algebra does not contain any further nilpotent generators. Thereby, we obtain a chain of models $\{\mathcal{M}_\alpha\}_{\alpha\in A}$ which is parametrized by elements α of some partially ordered set A. The model $\mathcal{M}_\alpha$ is a subsector of $\mathcal{M}_\beta$, i.e. $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$, whenever $\alpha < \beta$. Let us give just one example here. It is provided by the following family of symmetric superspaces

$$\mathcal{M}^{\mathrm{U}/\mathrm{U}^2}_{(\alpha_1,\alpha_2)}(R,S) \;\cong\; \frac{\mathrm{U}\left(\mathrm{R}{+}\alpha_1|\alpha_1\right)}{\mathrm{U}\left(\mathrm{S}{+}\alpha_2|\alpha_2\right)\times \mathrm{U}\left(\alpha_1\text{-}\alpha_2|\mathrm{R}\text{-}\mathrm{S}{+}\alpha_1\text{-}\alpha_2\right)} \tag{3.0.1}$$

where $R, S, R-S$ and $\alpha_1, \alpha_2, \alpha_1-\alpha_2$ are all taken to be non-negative integers. The family (3.0.1) includes the complex projective spaces $\mathbb{CP}^{\mathrm{R}+\alpha_1-1|\alpha_1}$ for $S = 1$ and $\alpha_2 = 0$.

One of our goals is to find candidate target spaces for conformally invariant sigma models. A necessary condition for a theory to be conformal is that all of its subsectors form conformally invariant theories as well. As it was argued for instance in [17], vanishing of the one loop beta function requires that $R = 0$, so that the only candidates for conformal quotients in (3.0.1) are to be found in the families $\mathcal{M}_\alpha(0, S)$. The smallest subsector in these families is obtained for $\alpha_1 = S$ and $\alpha_2 = 0$, so that it takes the simple form U (S|S) /U (S) × U (S). For $S = 1$, this subsector is the theory of free symplectic fermions, but in all other cases it is a massive theory. Hence the only candidates for conformal quotients one can find within the list (3.0.1) are of the form

$$\mathcal{C}^{\mathrm{U/U}^2}_{(\alpha_1,\alpha_2)} \;\equiv\; \mathcal{M}^{\mathrm{U/U}^2}_{(\alpha_1,\alpha_2)}(0,1) \;\cong\; \frac{\mathrm{U}\,(\alpha_1|\alpha_1)}{\mathrm{U}\,(1{+}\alpha_2|\alpha_2)\times \mathrm{U}\,(\alpha_1{\text{-}}\alpha_2|\alpha_1{\text{-}}\alpha_2{\text{-}}1)} \tag{3.0.2}$$

with $\alpha_1 > \alpha_2 \geq 0$. Later we shall argue that the converse is also true: symmetric superspaces that possess a non-trivial conformal subsector with central charge $c \neq 0$ are actually conformal. Since the theory of free symplectic fermions has central charge $c = -2$, all the models in the list (3.0.2) give rise to conformal sigma models. The list includes the complex projective superspaces $\mathbb{CP}^{\alpha_1-1|\alpha_1}$ for which conformal invariance has been established before (see e.g. [16,47]). We shall extend this discussion to arbitrary compact symmetric superspaces in subsection 3.3.2, allowing us to, within this class, recover the complete classification of conformal models from [48].

But our approach is more general, for it also applies to all coset superspaces G/G' *without any additional assumption on the denominator subgroup* G'. In subsection 3.3.3 we look at examples for which G' is fixed under the action of some automorphism of order four, which we denote as $G/G^{\mathbb{Z}_4}$. Such generalized symmetric spaces have become popular through the investigation of strings in Anti de Sitter backgrounds. While we are not aiming at an exhaustive investigation of quotients within this class, we shall exhibit a few interesting examples, including the family

$$\mathcal{M}^{\mathrm{U/OSP}^2}_{(\alpha_1,\alpha_2)}(S) \;\cong\; \frac{\mathrm{PSU}\,(2\alpha_1|2\alpha_1)}{\mathrm{OSP}\,(2(\mathrm{S}{+}\alpha_2)|2\alpha_2)\times \mathrm{OSP}\,(2(\alpha_1{\text{-}}\alpha_2)|2(\alpha_1{\text{-}}\alpha_2{\text{-}}S))} \tag{3.0.3}$$

with some obvious restrictions on the choice of α_i and S such that all supergroups are well-defined. Note that, provided the α_i are large enough, the parameter S may now assume any integer value, meaning that it can also be negative. The minimal non-trivial subsector of these theories depends significantly on the parameter S. It is given by

$$\mathcal{R}^{\mathrm{PSU/OSP}^2}(S) \;\cong\; \frac{\mathrm{PSU}\,(2\mathrm{S}|2\mathrm{S})}{\mathrm{SO}(2\mathrm{S})\times \mathrm{SO}(2\mathrm{S})} \qquad \text{for} \quad S > 0 \ , \tag{3.0.4}$$

$$\mathcal{R}^{\mathrm{PSU/OSP}^2}(0) \;\cong\; \text{symplectic fermions} \qquad \text{for} \quad S = 0 \ , \tag{3.0.5}$$

$$\mathcal{R}^{\mathrm{PSU/OSP}^2}(S) \;\cong\; \frac{\mathrm{PSU}\,(\text{-}2\mathrm{S}|\text{-}2\mathrm{S})}{\mathrm{SP}(\text{-}2\mathrm{S})\times \mathrm{SP}(\text{-}2\mathrm{S})} \qquad \text{for} \quad S < 0 \ . \tag{3.0.6}$$

These are not conformal for $S \neq 0$ and reduce to a free theory for $S = 0$. The smallest interacting theory for $S = 0$ is obtained for $\alpha_1 = 1$, $\alpha_2 = 0$ and is the complex projective superspace

$$\frac{\mathrm{PSU}\,(2|2)}{\mathrm{OSP}\,(2|2)} \cong \mathbb{CP}^{1|2} \ . \tag{3.0.7}$$

For higher values of α_i however, the superspaces are not of the complex projective type. It would be interesting to understand whether the family (3.0.3) with $S = 0$ is conformally invariant, but we have little to say about this issue for now.

The series (3.0.3) contains a few other interesting minimal subsectors. In fact, for the $S = 1$, the minimal subsector is given in equation (3.0.4). After an appropriate change in the choice of reality conditions, we obtain the coset geometry for $AdS_2 \times S^2$ as defined in [49]. Similarly, if we set $S = -2$ and perform again the appropriate change of the real form, we find the quotient that appears in the description of $AdS_5 \times S^5$. Throughout most of this text, we shall consider sigma models without Wess-Zumino terms, mostly in order not to clutter the presentation too much. We shall comment on the possible inclusion of Wess-Zumino terms and the application to other 2-dimensional field theories in the concluding section.

We finish this introduction with a short guide for the subsequent sections. Subsections 3.1.2 to 3.1.6 present the main mathematical tools at our disposal. Since these parts are a bit technical, we included a non-technical summary in subsection 3.1.1, so that the impatient reader may, at least upon first reading, skip subsections 3.1.2 to 3.1.6. The mathematical background from section 3.1 is then used in section 3.2 to prove the main results of this chapter. In section 3.3, we shall illustrate how the cohomological reduction works for symmetric superspaces. Once this is understood, we venture into generalized symmetric spaces. Our conclusion contains a few more comments on possible applications to more types of models and to AdS backgrounds in string theory.

3.1 Reduction in representation theory

The following section contains most of the mathematical results we shall need below. Since several of our statements seem to be new, we decided to present and prove them in a rather mathematical style. For pedagogical reasons, however, we shall begin with a short overview of the most relevant notations and results. This should enable impatient readers to skip over subsections 3.1.2 – 3.1.6, at least upon first reading.

3.1.1 Overview over results

We assume $\mathfrak{g}$ to be a Lie superalgebra with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let us pick some fermionic element $Q \in \mathfrak{g}$ that squares to zero, i.e.

$$[Q, Q] = 2Q^2 = 0 \ . \tag{3.1.1}$$

Such elements exist for most Lie superalgebras of interest, with the exception of the series osp $(1|2N)$. The element Q defines a decomposition of $\mathfrak{g}$ into three Lie sub-superalgebras $\mathfrak{h}, \mathfrak{e}, \mathfrak{f}$,

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{e} \oplus \mathfrak{f} \qquad \text{such that} \\ \mathfrak{e} &= \mathsf{Im}_Q\, \mathfrak{g} \quad \text{and} \quad \mathfrak{h} \oplus \mathfrak{e} = \mathsf{Ker}_Q\, \mathfrak{g} \ . \end{aligned}$$

Under our assumptions, the bilinear form (,) restricts to a non-degenerate form on $\mathfrak{h} \subset \mathfrak{g}$, while the Lie sub-superalgebras $\mathfrak{e}$ and $\mathfrak{f}$, are isotropic, i.e. $(\mathfrak{e}, \mathfrak{e}) = 0 = (\mathfrak{f}, \mathfrak{f})$. Furthermore, we note that $\mathfrak{e}$ and $\mathfrak{f}$ both carry an action of the Lie superalgebra $\mathfrak{h}$, meaning that they are $\mathfrak{h}$-modules.

In subsection 3.1.2, we shall compute the Lie superalgebra $\mathfrak{h}$ for various choices of $\mathfrak{g}$ and any $Q \in \mathfrak{g}$, with the following results:

$$\mathfrak{h}(\mathrm{gl}\,(M|N)) \cong \mathrm{gl}\,(M - r_Q|N - r_Q)\ , \tag{3.1.2}$$

$$\mathfrak{h}(\mathrm{sl}\,(M|N)) \cong \mathrm{sl}\,(M - r_Q|N - r_Q)\ , \tag{3.1.3}$$

$$\mathfrak{h}(\mathrm{osp}\,(R|2N)) \cong \mathrm{osp}\,(R - 2r_Q|2N - 2r_Q)\ . \tag{3.1.4}$$

The answer depends on Q only through an integer $\mathrm{rank}\,(Q) \equiv r_Q \geq 1$ that will be defined in subsection 3.1.2. In all three cases we listed above, there exist elements Q with minimal rank $r_Q = 1$.

The element Q acts in any representation V of $\mathfrak{g}$ and defines the following cohomology classes

$$\mathsf{H}_Q(\mathsf{V}) \ := \ \mathsf{Ker}_Q\,\mathsf{V}/\,\mathsf{Im}_Q\,\mathsf{V}\ .$$

The linear space $\mathsf{H}_Q(\mathsf{V})$ comes equipped with an action of the Lie sub-superalgebra $\mathfrak{h} \subset \mathfrak{g}$. It is not difficult to see[1] that $\mathsf{V} \to \mathsf{H}_Q(\mathsf{V})$ is functorial, i.e. it is consistent with forming tensor products, direct sums and conjugation in the category of $\mathfrak{h}$-representations.

Though $\mathsf{H}_Q(\mathsf{V})$ is zero for many V, it can certainly lead to non-trivial representations. Note, for example, that the cohomology of the adjoint $\mathfrak{g}$-module $\mathsf{V} \cong \mathfrak{g}$ is given by $\mathsf{H}_Q(\mathfrak{g}) \cong \mathfrak{h}$. Furthermore, in the case of finite dimensional representations, one may actually show that V and $\mathsf{H}_Q(\mathsf{V})$ *possess the same super-dimension*. Hence, all representations V with non-vanishing super-dimension $\mathrm{sdim}\,\mathsf{V} = \dim \mathsf{V}_{\bar{0}}$ - $\dim \mathsf{V}_{\bar{1}}$ must give rise to $\mathsf{H}_Q(\mathsf{V}) \neq 0$. The condition $\mathrm{sdim}\,\mathsf{V} \neq 0$ is often satisfied for short multiplets[2]. For long, that is *typical*, irreducible multiplets V, on the other hand, the cohomology $\mathsf{H}_Q(\mathsf{V})$ is always trivial. More generally, we will see that $\mathsf{H}_Q(\mathsf{V}) = 0$ for all finite dimensional projective covers.

Let us now consider a Lie superalgebra $\mathfrak{g}$ along with a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ and denote the corresponding Lie supergroups by G and G', respectively. As before, we want to pick some fermionic element $Q \in \mathfrak{g}$ with $Q^2 = 0$. Let us now assume that Q is contained in the subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ so that its cohomology defines two Lie sub-superalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}' \subset \mathfrak{g}'$ with $\mathfrak{h}' \subset \mathfrak{h}$, whose associated Lie supergroups we denote by H and H', respectively. Note that the space of functions on the coset superspace G/G' carries an action of $\mathfrak{g}$ and that, the element Q acts on it and gives rise to some cohomology. One of our central claims is that the cohomology of a given geometric object defined on the coset superspace G/G', whether smooth function, tensor form or square integrable function, is equivalent to a similar object defined on H/H'. This gives rise to isomorphisms of

[1] See subsection 3.1.3

[2] Also known as atypical irreducible representations

the type

$$\mathsf{H}_Q(L_2(G/G')) \;\cong\; L_2(H/H') \;\;, \tag{3.1.5}$$

meaning that the cohomology of Q in the space of square integrable functions on G/G' may be interpreted as a space of square integrable functions on the coset superspace H/H'. We note that $L_2(H/H')$ carries an action of the Lie superalgebra $\mathfrak{h} = \mathsf{H}_Q(\mathfrak{g}) \subset \mathfrak{g}$, so that relation (3.1.5) is an isomorphism of $\mathfrak{h}$ modules.

The derivation of equation (3.1.5) is a bit involved and one must wait for subsection 3.1.6 to get a full-fledged proof. Thus, we shall content ourselves with some more qualitative arguments here. By its very construction, the space $\mathsf{H}_Q(L_2(G/G'))$ is a commutative algebra and hence it can be considered as an algebra of functions on some space X, that is acted upon by the supergroup H with Lie superalgebra $\mathsf{H}_Q(\mathfrak{g}) = \mathfrak{h}$. Since the action of G on G/G' is transitive, it suffices to understand the reduction from G/G' to X locally, near the image $\mathsf{e}G' \in G/G'$ of the group unit $\mathsf{e} \in G$. The tangent space at this point of the coset supermanifold is given by $\mathfrak{g}/\mathfrak{g}' \equiv \mathfrak{m}$ and its cohomology is

$$\mathsf{H}_Q(\mathfrak{m}) \;=\; \mathsf{H}_Q(\mathfrak{g}/\mathfrak{g}') \;=\; \mathsf{H}_Q(\mathfrak{g})/\,\mathsf{H}_Q(\mathfrak{g}') \;=\; \mathfrak{h}/\mathfrak{h}' \;, \tag{3.1.6}$$

meaning that the tangent vectors to the reduced space X lie in $\mathfrak{h}/\mathfrak{h}'$. Thereby we conclude that $X = H/H'$. Now, let $\langle\;,\;\rangle_{G/G'}$ be the G-invariant scalar product of $L_2(G/G')$. It is very easy to see that $\langle\;,\;\rangle_{G/G'}$ has a non-trivial representative in the cohomology of Q, so that the space $\mathsf{H}_Q(L_2(G/G'))$ of functions inherits an L_2 structure from $L_2(G/G')$. We shall denote it by $\langle\;,\;\rangle_{H/H'}$. Its H-invariance follows immediately from the G-invariance of $\langle\;,\;\rangle_{G/G'}$ and the inclusion $\mathfrak{h} \subset \mathsf{Ker}_Q\,\mathfrak{g}$. General results on measure theory [50] then imply that the scalar product $\langle\;,\;\rangle_{H/H'}$ arises from a measure on H/H', which is unique up to a normalization factor by H-invariance. Hence, we have established equation. (3.1.5).

Example 3.1.1. Let us discuss the Lie superalgebra $\mathfrak{g} = \mathrm{gl}\,(2|2)$ that will be of great importance in chapter 5. For Q we pick the supermatrix that contains a single entry in the upper right corner. It is then easy to check that

$$\mathsf{Ker}_Q\,\mathfrak{g} \;=\; \mathfrak{h}\oplus\mathfrak{e} \;\ni\; \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{22} & b_{21} & b_{22} \\ \hline 0 & c_{12} & d_{11} & d_{12} \\ 0 & 0 & 0 & a_{11} \end{array}\right) \;,\quad \mathsf{Im}_Q\,\mathfrak{g} \;=\; \mathfrak{e} \;\ni\; \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{22} \\ \hline 0 & 0 & 0 & d_{12} \\ 0 & 0 & 0 & a_{11} \end{array}\right) \;.$$

Consequently, $\mathsf{H}_Q(\mathfrak{g}) \cong \mathfrak{h} = \mathrm{gl}(1|1)$ consists of all those supermatrices in which a_{22}, b_{21}, c_{12} and d_{11} are the only non-zero entries. Let us furthermore specify the Lie sub-superalgebra $\mathfrak{g}'$ to consist of all elements in $\mathfrak{g}$ with vanishing entries $b_{11} = b_{21} = d_{12} = d_{21} = c_{11} = c_{12} = 0$. Hence, $\mathfrak{g}' \cong \mathrm{gl}\,(2|1) \oplus \mathrm{gl}\,(1)$. The cohomology $\mathsf{H}_Q(\mathfrak{g}') \cong \mathfrak{h}' = \mathrm{gl}\,(1) \oplus \mathrm{gl}\,(1)$ of $\mathfrak{g}'$ can be read off easily.

In our example, the quotient G/G' is the complex projective superspace $\mathbb{CP}^{1|2}$, which can *vaguely* be thought of as the product of S^2 with the space spanned by four Grassmann generators. The space of functions thereon may be decomposed into finite dimensional representations gl (2|2) as follows

$$L_2(\mathbb{CP}^{1|2}) \;\cong\; \bigoplus_{k=1}^{\infty} \; t(k,k) \quad .$$

The modules $t(k,k)$ are defined as the tensor product of the k-fold symmetric tensor product of the fundamental module with the k-fold symmetric tensor product of the dual fundamental module, where for $k \geq 2$ we need to remove all contractions between the covariant and contravariant indices. Their dimension is $16(2k-1)$. One can understand them as being generated from the spherical harmonics on the bosonic 2-sphere by application of four fermionic generators. For $k \neq 1$, the modules $t(k,k)$ turn out to be projective, so for them we get $\mathsf{H}_Q(t(k,k)) \cong 0$. The only non-vanishing cohomology comes from the 16-dimensional module $t(1,1)$, which is built from three atypical irreducibles[3], namely two copies of the trivial representation and one 14-dimensional module that can be identified with adjoint representation of psl (2|2). Each of these pieces contributes to cohomology, with the final result being a four dimensional cohomology

$$\mathsf{H}_Q(L_2(G/G')) \;=\; \mathsf{H}_Q(L_2(\mathbb{CP}^{1|2})) \;=\; \mathsf{H}_Q([0,0]) \;=\; \mathbb{R}^{2|2} \quad .$$

To me more precise, we note that the linear space $\mathbb{R}^{2|2}$ carries the 4-dimensional projective cover of gl (1|1). According to our general statement, the cohomology should agree with the space of functions on the quotient $H/H' = \mathrm{GL}\,(1|1)\,/\mathrm{GL}\,(1) \times \mathrm{GL}\,(1)$. The quotient possesses two fermionic coordinates and hence gives rise to a 4-dimensional algebra of functions over it,

$$L_2(H/H') \;\cong\; \mathbb{R}^{2|2} \quad .$$

It indeed agrees with the cohomology in the space of functions over $\mathbb{CP}^{1|2}$, as it was claimed in equation (3.1.5).

3.1.2 Reduction of Lie superalgebras

After this lengthy introduction, we are ready to delve full speed into the heart of the matter. As in the previous subsection, let $\mathfrak{g}$ stand for a Lie superalgebra and let Q be a fermionic element of $\mathfrak{g}$ whose bracket with itself vanishes, that is $[Q,Q] = 2Q^2 = 0$.

Lemma 3.1.1. The element $Q \in \mathfrak{g}$ gives rise to a linear map $Q : \mathfrak{g} \to \mathfrak{g}$, defined by $Q(X) = [Q, X]$ for all $X \in \mathfrak{g}$. One can show that

1) the subspaces $\mathsf{Ker}_Q\,\mathfrak{g}$ and $\mathsf{Im}_Q\,\mathfrak{g}$ are subalgebras of $\mathfrak{g}$,

2) the subalgebra $\mathsf{Im}_Q\,\mathfrak{g}$ is an ideal of $\mathsf{Ker}_Q\,\mathfrak{g}$,

3) the quotient space $\mathsf{H}_Q(\mathfrak{g})$ is a Lie superalgebra.

[3] See appendix D for more details.

All these assertions are easily established using nothing more that that the graded Jacobi identity of (2.1.3). The Lie bracket on the quotient space $\mathsf{H}_Q(\mathfrak{g})$ is induced from the Lie bracket of $\mathfrak{g}$ through

$$[x + \mathsf{Im}_Q\,\mathfrak{g}, y + \mathsf{Im}_Q\,\mathfrak{g}] \;=\; [x,y] + \mathsf{Im}_Q\,\mathfrak{g}, \qquad x, y \in \mathsf{Ker}_Q\,\mathfrak{g}\;. \tag{3.1.7}$$

We shall often refer to the space $\mathsf{H}_Q(\mathfrak{g})$ as the *cohomological reduction* of the Lie superalgebra $\mathfrak{g}$ with respect to Q. In discussing concrete examples, we shall restrict to the superalgebras presented in section 2.1, namely to $\mathrm{osp}\,(m|2n)$, $\mathrm{gl}\,(m|n)$, $\mathrm{sl}\,(m|n)$ for $n \neq m$ or $\mathrm{psl}\,(n|n)$. All these Lie superalgebras possess a metric $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ according to definition (2.1.2) of the previous chapter.

The adjoint action of Q can be represented by a matrix in Jordan normal form by choosing a basis $\{h_a\} \cup \{e_i, f_i\}$ of $\mathfrak{g}$ such that

$$[Q, h_a] \;=\; 0 \quad \text{and} \quad [Q, f_i] \;=\; e_i\;. \tag{3.1.8}$$

Using the invariance of the metric, we see that

$$(h_a, e_i) \;=\; 0\;, \qquad (e_i, e_j) \;=\; 0\;. \tag{3.1.9}$$

If follows from the non-degeneracy of the metric that the matrix $D_{ij} := (e_i, f_j)$ must be invertible. Thus, defining

$$h'_a \;:=\; h_a - \sum_{ij} (h_a, f_j)\left(D^{-1}\right)^{j,i} e_i\;, \tag{3.1.10}$$

$$f'_i \;:=\; f_i - \frac{1}{2}\sum_{j,k} (f_i, f_j)\left(D^{-1}\right)^{jk} e_k\;, \tag{3.1.11}$$

we see that

$$(h'_a, f'_i) \;=\; 0\;, \qquad \left(f'_i, f'_j\right) \;=\; 0\;. \tag{3.1.12}$$

To prove the second assertion in equation (3.1.12) we have used the following symmetry property of the matrix D

$$D_{ij} \;=\; ([Q,f_i], f_j) \;=\; -(-1)^{|f_i|}\,(f_i, [Q, f_j]) \;=\; -(-1)^{|f_i|}\,(f_i, e_j) \;=\; -D_{ji}, \tag{3.1.13}$$

where the last equality in the chain uses the consistency of the metric.

Let us denote by $\mathfrak{h}$, $\mathfrak{e}$ and $\mathfrak{f}$ the span of h'_a, e_i and f'_i, respectively. Notice that Q still remains in a Jordan normal form with respect to the new basis h'_a, e_i, f'_j. From the equations (3.1.9 and 3.1.12), we deduce the orthogonality conditions

$$(\mathfrak{h}, \mathfrak{e}) \;=\; (\mathfrak{h}, \mathfrak{f}) \;=\; (\mathfrak{e}, \mathfrak{e}) \;=\; (\mathfrak{f}, \mathfrak{f}) \;=\; 0\;. \tag{3.1.14}$$

If we use the invariance of the metric once more, it is not hard to derive the following features of the Lie bracket on $\mathfrak{g}$,

$$\begin{aligned} &[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad &&[\mathfrak{h},\mathfrak{e}] \subset \mathfrak{e}, \qquad &&[\mathfrak{h},\mathfrak{f}] \subset \mathfrak{f}, \\ &[\mathfrak{e},\mathfrak{e}] \subset \mathfrak{e}, &&[\mathfrak{f},\mathfrak{f}] \subset \mathfrak{f}, &&[\mathfrak{e},\mathfrak{f}] \subset \mathfrak{g}\,. \end{aligned} \tag{3.1.15}$$

Notice in particular, that both $\mathfrak{e}$ and $\mathfrak{f}$ provide some representation for the Lie superalgebra $\mathfrak{h}$. Furthermore, we observe that $\mathfrak{g}$ and $\mathfrak{h}$ possess the same cohomology under Q, that is $\mathsf{H}_Q(\mathfrak{g}) \cong \mathsf{H}_Q(\mathfrak{h})$. Next, let us define the projection map $p_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ through

$$p_{\mathfrak{h}}(x) \ := \ (x, h'_a)(B^{-1})^{ab} h'_b \ , \tag{3.1.16}$$

where $x \in \mathsf{Ker}_Q$ and $B_{ab} = (h'_a, h'_b)$. Since the kernel of $p_{\mathfrak{h}}$ is exactly $\mathfrak{e}$, the map $p_{\mathfrak{h}}$ is effectively defined on $\mathsf{H}_Q(\mathfrak{g})$. Taking into account equations (3.1.15), we see that $p_{\mathfrak{h}}$ provides the following algebra isomorphism

$$\mathfrak{h} \ \cong \ \mathsf{H}_Q(\mathfrak{g}) \ . \tag{3.1.17}$$

In the same spirit, one can define the $\mathfrak{h}$-module projection homomorphisms $p_{\mathfrak{e}}$ and $p_{\mathfrak{f}}$ from $\mathfrak{g}$ to $\mathfrak{e}$ and $\mathfrak{f}$, respectively,

$$\begin{aligned} p_{\mathfrak{e}}(x) &:= \ (x, f'_i)(D^{-1})^{ij} e_j \\ p_{\mathfrak{f}}(x) &:= \ x - p_{\mathfrak{h}}(x) - p_{\mathfrak{e}}(x) \ . \end{aligned} \tag{3.1.18}$$

These provide us with the following direct sum decomposition of $\mathfrak{g}$,

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{e} \oplus \mathfrak{f} \ , \tag{3.1.19}$$

that respects the action of $\mathfrak{h}$, meaning that it is an isomorphism of $\mathfrak{h}$ modules.

As described in section 2.1, the superalgebras we consider are characterized by a Cartan subalgebra $\mathfrak{g}_0$ and a set of roots Δ. If $\rho_\Box : \mathfrak{g} \to \mathrm{End}\,\mathsf{V}$ is the fundamental representation, then the Cartan subalgebra $\mathfrak{g}_0$ can be represented through diagonal matrices of $\mathrm{End}\,\mathsf{V}$, while Δ is a subset of the root system of the superalgebra $\mathrm{gl}(\mathsf{V})$.

Let us now perform the cohomological reduction for the Lie superalgebra $\mathfrak{g}$ for the case Q is a nilpotent root generator of the root q. In particular, this implies that $q \in \Delta_{\bar{1}}$ is such that $2q \notin \Delta_{\bar{0}}$. Examining the root decomposition of $\mathfrak{g}$

$$\mathfrak{g} \ = \ \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \ , \tag{3.1.20}$$

the superalgebras $\mathfrak{e}$ and $\mathfrak{f}$ can be easily evaluated

$$\mathfrak{e} = \ \mathbb{C}H_q \oplus \bigoplus_{\alpha - q \in \Delta} \mathfrak{g}_\alpha \ , \tag{3.1.21}$$

$$\mathfrak{f} = \ \mathfrak{g}_0 / \mathsf{Ker}\, q \oplus \bigoplus_{\alpha + q \in \Delta} \mathfrak{g}_\alpha \ , \tag{3.1.22}$$

where for any weight Λ one denotes by H_Λ the Cartan generator constructed via the identification

$$\Lambda(H) \ = \ (H_\Lambda, H) \ . \tag{3.1.23}$$

Therefore, we can write the cohomology of $\mathfrak{g}$ in the form

$$\mathsf{H}_q(\mathfrak{g}) := \mathsf{H}_Q(\mathfrak{g}) \cong \mathfrak{h} = \mathsf{Ker}\, q/\mathbb{C}H_q \oplus \bigoplus_{\alpha\pm q\notin\Delta} \mathfrak{g}_\alpha \ . \tag{3.1.24}$$

We can make use of this general result to compute the cohomological reduction of the superalgebras $\mathrm{gl}\,(M|N)$, $\mathrm{sl}\,(M|N)$ and $\mathrm{osp}\,(R|2N)$, whose root systems, we have listed in table 2.1.

Let us begin with the superalgebra $\mathrm{gl}\,(M|N)$, for which we let Q be a root generator for $q = \epsilon_r - \delta_s$. The requirement $\alpha \pm q \notin \Delta$ is satisfied for the following set of roots

$$\epsilon_i - \epsilon_j,\ \epsilon_i - \delta_k, \quad \text{with} \quad \delta_k - \delta_l, \quad i,j \neq r,\ k,l \neq s \ . \tag{3.1.25}$$

These give rise to the root system of a $\mathrm{gl}\,(M-1|N-1)$ subalgebra. As a basis of the Cartan subalgebra one may choose the Cartan generators H_{ϵ_i}, H_{δ_k} which are defined through eq. (3.1.23). Evaluating

$$\mathsf{Ker}(\epsilon_r - \delta_s)/\mathbb{C}H_{\epsilon_r-\delta_s} = \mathsf{Ker}(\epsilon_r - \delta_s) \cap \mathsf{Ker}(\epsilon_r + \delta_s) = \mathsf{Ker}\,\epsilon_r \cap \mathsf{Ker}\,\delta_s \ , \tag{3.1.26}$$

we deduce with the help of equation (3.1.24) that

$$\mathsf{H}_{\epsilon_r-\delta_s}(\mathrm{gl}\,(M|N)) \cong \mathrm{gl}\,(M-1|N-1) \ . \tag{3.1.27}$$

The cohomological reduction of $\mathrm{sl}\,(M|N)$ is only slightly different. As the roots of $\mathrm{sl}\,(M|N)$ and $\mathrm{gl}\,(M|N)$ are the same, the analysis of (3.1.25) remains unchanged. The Cartan algebra of $\mathrm{sl}\,(M|N)$ can be viewed as the subalgebra of the Cartan algebra of $\mathrm{gl}\,(M|N)$ defined by the kernel of str, where we have introduced the supertrace operator $\mathrm{str} := \sum \epsilon_i - \sum \delta_k$. Therefore, equation (3.1.26) has to be replaced by

$$\begin{aligned} \mathsf{Ker}\,\mathrm{str} \cap \mathsf{Ker}(\epsilon_r - \delta_s)/\mathbb{C}H_{\epsilon_r-\delta_s} &= \mathsf{Ker}\,\mathrm{str} \cap \mathsf{Ker}(\epsilon_r - \delta_s) \cap \mathsf{Ker}(\epsilon_r + \delta_s) \\ &= \mathsf{Ker}(\sum_{i\neq r} \epsilon_i - \sum_{k\neq s} \delta_k) \cap \mathsf{Ker}\,\epsilon_r \cap \mathsf{Ker}\,\delta_s \ , \end{aligned}$$

which leads to the Cartan subalgebra of $\mathrm{sl}\,(M-1|N-1)$. Therefore we obtain

$$\mathsf{H}_{\epsilon_r-\delta_s}(\mathrm{sl}\,(M|N)) \cong \mathrm{sl}\,(M-1|N-1) \ . \tag{3.1.28}$$

A similar analysis may be performed for the osp type superalgebras. If we choose $q = \epsilon_r \pm \delta_s$ then $\alpha \pm q$ is not a root for all α of the following list

$$\pm\epsilon_i \pm \epsilon_j, \quad \pm\epsilon_i \pm \delta_k, \quad \pm\delta_k \pm \delta_l, \qquad i \neq j, \quad i,j \neq r, \quad k,l \neq s \ , \tag{3.1.29}$$

in the case of $\mathrm{osp}\,(2M|2N)$ and

$$\pm\epsilon_i \pm \epsilon_j, \quad \pm\epsilon_i \pm \epsilon_i \pm \delta_k, \quad \pm\delta_k \pm \delta_l, \qquad i \neq j, \quad i,j \neq r, \quad k,l \neq s \ , \tag{3.1.30}$$

in the case of osp $(2M+1|2N)$. Those in equation (3.1.29) correspond to the root system of an osp $(2M-2|2N-2)$ subalgebra, while the roots in (3.1.30) are associated with an osp $(2M-1|2N-2)$ subalgebra. Again, one may take the Cartan generators H_{ϵ_i}, H_{δ_k} as a basis of the Cartan subalgebra. The cohomological reduction of the Cartan subalgebra proceeds exactly as in equation (3.1.26):

$$\mathsf{Ker}(\epsilon_r \pm \epsilon_r)/\mathbb{C}H_{\epsilon_r \pm \delta_s} \;=\; \mathsf{Ker}(\epsilon_r \pm \epsilon_r) \cap \mathsf{Ker}(\epsilon_r \mp \epsilon_r) \;=\; \mathsf{Ker}_{\epsilon_r} \cap \mathsf{Ker}_{\delta_s} \;, \tag{3.1.31}$$

Leading us to conclude that

$$\mathsf{H}_{\epsilon_r \pm \delta_s}(\text{osp}\,(R|2N)) \;\cong\; \text{osp}\,(R-2|2N-2) \;, \tag{3.1.32}$$

for any choice of R. At this point we have determined the cohomology $H_Q(\mathfrak{g})$ for all elements Q that belong to the Cartan eigenspace $\mathfrak{g}_q$ of an isotropic root q.

From equations (3.1.27, 3.1.28 and 3.1.32), we may infer that, up to isomorphism, the cohomological reduction of $\mathfrak{g}$ with respect to Q does not depend on the choice of the isotropic root q. This gives rise to the question:

How can we characterize Qs that give rise to different Lie superalgebras $\mathsf{H}_Q(\mathfrak{g})$?

In the following we want to prove that the isomorphism class of the cohomological reduction *depends only on the rank of the matrix Q in the fundamental representation.* To begin with we observe that an automorphism γ of $\mathfrak{g}$ induces an automorphism of the cohomology, i.e.

$$\mathsf{H}_Q(\mathfrak{g}) \;\cong\; \mathsf{H}_{\gamma(Q)}(\mathfrak{g}) \;. \tag{3.1.33}$$

The main idea is to use the group of inner automorphisms provided by the even subalgebra of $\mathfrak{g}$ in order to bring a general Q with vanishing self-bracket to some simpler form.

- Consider the Lie superalgebra gl $(M|N)$ first. Let V, V_M and V_N be the fundamental gl $(M|N)$, gl (M) and gl (M) modules, respectively. To bring Q to some simpler form, we shall use the following gl $(M|N)_{\bar{0}} \cong$ gl (M) $\oplus$ gl (N) module isomorphism

$$\text{gl}\,(M|N)_{\bar{1}} \;\cong\; \mathsf{V}_M \otimes_{\mathbb{C}} \mathsf{V}_N^\star \oplus \mathsf{V}_N \otimes_{\mathbb{C}} \mathsf{V}_M^\star \;, \tag{3.1.34}$$

  where $\mathsf{V}^\star$ denotes the dual representation. The module isomorphism (3.1.34) is provided by the invertible linear map

$$\begin{aligned} \varphi(v \otimes \alpha)(a) &= v\alpha(a), & v \otimes \alpha &\in \mathsf{V}_M \otimes_{\mathbb{C}} \mathsf{V}_N^\star \\ \varphi(v \otimes \alpha)(u) &= 0, & u &\in \mathsf{V}_M \\ \varphi(a \otimes \omega)(v) &= a\omega(v), & a \otimes \omega &\in \mathsf{V}_N \otimes_{\mathbb{C}} \mathsf{V}_M^\star \\ \varphi(a \otimes \omega)(b) &= 0, & b &\in \mathsf{V}_N \;. \end{aligned} \tag{3.1.35}$$

We say that Q has rank (k, l) if it can be represented as

$$\varphi^{-1}(Q) \; = \; \sum_{i=1}^{k} v_i \otimes \alpha^i + \sum_{i=1}^{l} a_i \otimes \omega^i \ , \qquad (3.1.36)$$

where all v's, a's, α's and ω's are linearly independent among themselves. Clearly $k, l \leq \min(M, N)$. Let $b_1, \ldots, b_M$ denote a basis of V_M and $f_1, \ldots, f_N$ be a basis of V_N. Denote by b^i, f^k the dual bases. Then, from the definition of the general linear group, there are elements $A' \in \mathrm{GL}\,(\mathrm{M})$, $B' \in \mathrm{GL}\,(\mathrm{N})$ such that

$$v_i \; = \; A' \cdot b_i, \quad \alpha^i \; = \; B' \cdot f^i, \qquad i = 1, \ldots, k \ . \qquad (3.1.37)$$

Moreover, the group elements A', B' are not unique, for their action on the remaining basis vectors $b_{k+1}, \ldots, b_M$ and $f^{k+1}, \ldots, f^N$ is not fixed. Choosing an inner automorphism $\gamma' = \mathrm{Ad}\, A'^{-1} \circ \mathrm{Ad}\, B'^{-1}$ we see that one can bring Q to the simpler form

$$\varphi^{-1}(\gamma'(Q)) \; = \; \sum_{i=1}^{k} b_i \otimes f^i + \sum_{i=1}^{l} a'_i \otimes \omega'^i \ , \qquad (3.1.38)$$

where $a'_i = B'^{-1} \cdot a_i$ and $\omega'^i = A'^{-1} \cdot \omega^i$. The condition $Q^2 = 0$ is equivalent to the following constraints on the vectors a'_i, ω'^i in eq. (3.1.38)

$$f^j(a'_i) \; = \; 0, \quad \omega'^i(b_j) \; = \; 0 \ ,$$

where $i = 1, \ldots, l$ and $j = 1, \ldots, k$. This implies that the vectors a'_i lie entirely in the subspace of V_N spanned by the basis vectors $f_{k+1}, \ldots, f_N$, while the form ω'^i lies in the subspace of $\mathsf{V}^\star_M$ that is spanned by the basis forms $b^{k+1}, \ldots, b^M$. Therefore, the linear independence of a'_i, ω'^i imposes an additional restriction on the rank (k, l) of Q

$$k + l \; \leq \; \min(M, N) \ . \qquad (3.1.39)$$

The existence of the group elements $A'' \in \mathrm{GL}\,(\mathrm{M})$ and $B'' \in \mathrm{GL}\,(\mathrm{N})$ satisfying

$$A'' \cdot b_i \; = \; b_i, \quad B'' \cdot f^i \; = \; f^i \ , \qquad (3.1.40)$$

for $i = 1, \ldots, k$ and

$$a'_m \; = \; A'' \cdot f_m, \quad \omega'^n \; = \; B'' \cdot b^n \ , \qquad (3.1.41)$$

for $m = k+1, \ldots, k+l$ and $n = k+1, \ldots, k+l$ is ensured by equation (3.1.39). Defining $\gamma'' = \mathrm{Ad}\, A''^{-1} \circ \mathrm{Ad}\, B''^{-1}$ we see that Q can be brought into a standard form which depends only on its rank (k, l)

$$\varphi^{-1}((\gamma'' \circ \gamma')(Q)) \; = \; \sum_{i=1}^{k} b_i \otimes f^i + \sum_{i=k+1}^{k+l} f_i \otimes b^i \ . \qquad (3.1.42)$$

We can now perform the cohomological reduction of $\mathrm{gl}(M|N)$ with respect to the fermionic generators

$$\varphi\left(\sum_{i=1}^{k} b_i \otimes f^i + \sum_{i=k+1}^{k+l} f_i \otimes b^i\right) \tag{3.1.43}$$

by a lengthy but straightforward calculation, that leads to the following statement

$$\mathsf{H}_Q(\mathrm{gl}(M|N)) \;\cong\; \mathrm{gl}(M - \mathrm{rank}(Q)|N - \mathrm{rank}(Q)) \ , \tag{3.1.44}$$

where the total rank of Q is defined as $\mathrm{rank}(Q) = k + l \leq \min(M, N)$.

- The generalization to the superalgebras $\mathrm{sl}(M|N)$ is straightforward. The procedure to bring Q to the canonical form (3.1.43) is identical with the one described in the $\mathrm{gl}(M|N)$ case. The cohomological reduction of $\mathrm{sl}(M|N)$ with respect to this canonical form of Q may be performed explicitly and leads to the expected result

$$\mathsf{H}_Q(\mathrm{sl}(M|N)) \;\cong\; \mathrm{sl}(M - \mathrm{rank}(Q)|N - \mathrm{rank}(Q)) \ . \tag{3.1.45}$$

- Finally, let us also deal with the Lie superalgebras $\mathrm{osp}(R|2N)$, where $R = 2M$ or $R = 2M + 1$. Denote by V, V_R and V_{2N} the fundamental $\mathrm{osp}(R|2N)$, so (R) and sp (2N) modules, respectively. Furthermore, let $(\ ,\ )$ be the symmetric invariant scalar product in V_R and $\langle\ ,\ \rangle$ be the antisymmetric invariant scalar product in V_{2N}. For $R = 2M$ we shall consider a basis $b_1, \ldots, b_{2M}$ such that the matrix elements of the scalar product $S_{ij} = (b_i, b_j)$ take the form

$$S \;=\; \begin{pmatrix} 0_{M\times M} & 1_{M\times M} \\ 1_{M\times M} & 0_{M\times M} \end{pmatrix} \ , \tag{3.1.46}$$

while for $R = 2M + 1$ we shall consider a basis $b_1, \ldots, b_{2M+1}$ such that the matrix elements of the scalar product $S_{ij} = (b_i, b_j)$ take the form

$$S \;=\; \begin{pmatrix} 0_{M\times M} & 1_{M\times M} & 0_{M\times 1} \\ 1_{M\times M} & 0_{M\times M} & 0_{M\times 1} \\ 0_{1\times M} & 0_{1\times M} & 1 \end{pmatrix} \ . \tag{3.1.47}$$

We also consider a basis $f_1, \ldots, f_{2N}$ such that the matrix elements of the scalar product $A_{ij} = \langle f_i, f_j \rangle$ take the form

$$A \;=\; \begin{pmatrix} 0_{N\times N} & -1_{N\times N} \\ 1_{N\times N} & 0_{N\times N} \end{pmatrix} \ . \tag{3.1.48}$$

With respect to the decomposition $\mathsf{V} \cong \mathsf{V}_R \oplus \mathsf{V}_{2N}$, the invariant scalar product in V is $G := S \oplus A$.

To bring Q into some simpler form, we shall use the following $\mathrm{osp}(R|2N)_{\bar{0}} \cong$ so (R) $\oplus$ sp (2N) module isomorphism

$$\mathrm{osp}(R|2N)_{\bar{1}} \;\cong\; \mathsf{V}_R \otimes_{\mathbb{C}} \mathsf{V}_{2N} \ , \tag{3.1.49}$$

which is provided by the invertible linear map

$$\begin{aligned} \chi(s\otimes a)(b) = &\ s\langle a,b\rangle, \qquad s\otimes a \in \mathsf{V}_R \otimes_{\mathbb{C}} \mathsf{V}_{2N} \\ \chi(s\otimes a)(t) = &\ a(s,t), \qquad t\in \mathsf{V}_R,\ b\in \mathsf{V}_{2N}\ . \end{aligned} \tag{3.1.50}$$

We say that Q has rank k if it can be represented as

$$\chi^{-1}(Q) \ = \ \sum_{i=1}^{k} s_i \otimes a_i\ , \tag{3.1.51}$$

where the s's and a's are linearly independent among themselves, subjecting Q to the condition $k \leq \min(R, 2N)$. The requirement that Q be nilpotent, i.e. $Q^2 = 0$, can be worked out from equations (3.1.50) to be equivalent to the following set of constraints on the vectors s_i, a_i

$$(s_i, s_j) \ = \ 0, \qquad \langle a_i, a_j\rangle \ = \ 0\ , \tag{3.1.52}$$

for $i, j = 1, \ldots, k$. These conditions are compatible with the linear independence of the s_i and a_i if and only if

$$k \ \leq \ M, \qquad k \ \leq \ N\ . \tag{3.1.53}$$

This restriction on the rank k allows us to define some linearly independent vectors $s_{k+1}, .., s_R$ and $a_{k+1}, \ldots, a_{2N}$ such that the matrix elements (s_i, s_j), for $i, j = 1, \ldots, R$ and $\langle a_i, a_j\rangle$, for $i, j = 1, \ldots, 2N$ take the form in equations (3.1.46), (3.1.47)) and in equation (3.1.48), respectively. Therefore, from the definition of the SO (R) and SP (2N) groups, there exist elements $A \in$ SO (R) and $B \in$ SP (2N) such that

$$s_i \ = \ A\cdot b_i, \qquad a_j \ = \ B\cdot f_j\ , \tag{3.1.54}$$

for $i = 1, \ldots, R$ and $j = 1, \ldots, 2N$. We see that Q can be brought to a simple standard form depending only on its rank k

$$\chi^{-1}(\gamma(Q)) \ = \ \sum_{i=1}^{k} b_i \otimes f_i \tag{3.1.55}$$

by acting with the inner automorphism $\gamma = \mathrm{Ad}\, A^{-1} \circ \mathrm{Ad}\, B^{-1}$. We perform the cohomological reduction of $\mathrm{osp}\,(R|2N)$ with respect to the fermionic generators

$$\chi\left(\sum_{i=1}^{k} b_i \otimes f_i\right) \tag{3.1.56}$$

by an explicit calculation. Thereby, we end up with the following statement

$$\mathsf{H}_Q(\mathrm{osp}\,(R|2N)) \ \cong \ \mathrm{osp}\,(R - 2\,\mathrm{rank}\,(Q)|2N - 2\,\mathrm{rank}\,(Q))\ , \tag{3.1.57}$$

where $\mathrm{rank}\,(Q) = k \leq \min\,([R/2], N)$.

3.1.3 Reduction of modules

Let $\mathfrak{g}$ be one of the superalgebras considered in section 3.1.2 and Q be an odd element of $\mathfrak{g}$ with vanishing self-bracket. As we have shown in section 3.1.2, there is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the following isomorphism holds: $\mathsf{H}_Q(\mathfrak{g}) \cong \mathfrak{h}$.

We notice that there is a $\mathfrak{h}$-stable filtration of modules

$$\mathsf{V} \supset \mathsf{Ker}_Q \mathsf{V} \supset \mathsf{Im}_Q \mathsf{V} . \tag{3.1.58}$$

The above holds since V is a $\mathfrak{h}$-submodule by restriction, while $\mathsf{Ker}_Q \mathsf{V}$ and $\mathsf{Im}_Q \mathsf{V}$ are $\mathfrak{h}$-submodules because $\mathfrak{h} \subset \mathsf{Ker}_Q \mathfrak{g}$. Finally, $\mathsf{Ker}_Q \mathsf{V} \supset \mathsf{Im}_Q \mathsf{V}$ follows from $Q^2 = 0$.

The existence of the $\mathfrak{h}$-stable filtration (3.1.58) means that $\mathsf{H}_Q(\mathsf{V})$ is generally a quotient of a submodule of the restriction of V to $\mathfrak{h}$. However, if V is self-dual, meaning that V has an invariant non-degenerate scalar product, then one can repeat the steps (3.1.8 – 3.1.14, 3.1.18 – 3.1.19) and prove a similar $\mathfrak{h}$-module direct sum decomposition for V

$$\mathsf{V} \cong \mathsf{W} \oplus \mathsf{E} \oplus \mathsf{F} , \tag{3.1.59}$$

where $\mathsf{W} \cong \mathsf{H}_Q(\mathsf{V})$ and $\mathsf{E} = \mathsf{Im}_Q \mathsf{V}$. We list some of the properties of the subquotients $\mathsf{H}_Q(\mathsf{V})$ that will prove useful for the following.

Lemma 3.1.2. Let U and V be $\mathfrak{g}$ modules. Then the following $\mathfrak{h}$-module isomorphisms hold

a) $\mathsf{H}_Q(\mathsf{U} \oplus \mathsf{V}) \cong \mathsf{H}_Q(\mathsf{U}) \oplus \mathsf{H}_Q(\mathsf{V})$

b) $\mathsf{H}_Q(\mathsf{V}^\star) \cong \mathsf{H}_Q(\mathsf{V})^\star$

c) $\mathsf{H}_Q(\mathsf{U} \otimes \mathsf{V}) \cong \mathsf{H}_Q(\mathsf{U}) \otimes \mathsf{H}_Q(\mathsf{V})$, if U, V are finite dimensional.

Proof. a) The direct sum of the modules U and V means that there are orthogonal idempotents $\mathfrak{i}_\mathsf{U}$ and $\mathfrak{i}_\mathsf{V}$ such that they commute with the action of $\mathfrak{g}$ and $\mathfrak{i}_\mathsf{U}\mathsf{U} = \mathsf{U}$, $\mathfrak{i}_\mathsf{V}\mathsf{V} = \mathsf{V}$. One thus has

$$\begin{aligned} \mathfrak{i}_\mathsf{U} \mathsf{Ker}_Q(\mathsf{U} \oplus \mathsf{V}) &= \mathsf{Ker}_Q(\mathfrak{i}_\mathsf{U}\mathsf{U} \oplus \mathfrak{i}_\mathsf{U}\mathsf{V}) = \mathsf{Ker}_Q \mathsf{U} \\ \mathfrak{i}_\mathsf{U} \mathsf{Im}_Q(\mathsf{U} \oplus \mathsf{V}) &= \mathsf{Im}_Q(\mathfrak{i}_\mathsf{U}\mathsf{U} \oplus \mathfrak{i}_\mathsf{U}\mathsf{V}) = \mathsf{Im}_Q \mathsf{U} \end{aligned}$$

and therefore $\mathfrak{i}_\mathsf{U} \mathsf{H}_Q(\mathsf{U} \oplus \mathsf{V}) = \mathsf{H}_Q(\mathsf{U})$. Similarly, $\mathfrak{i}_\mathsf{V} \mathsf{H}_Q(\mathsf{U} \oplus \mathsf{V}) = \mathsf{H}_Q(\mathsf{V})$, which completes the proof of a).

b) The elements of $\mathsf{H}_Q(\mathsf{V}^\star)$ are equivalence classes $\pi(\mu) = \mu + Q \cdot \mathsf{V}^\star$ of forms $\mu \in \mathsf{Ker}_Q \mathsf{V}^\star$, that is $\pi(\mu)$ is the equivalence class of forms that have the same restriction on $\mathsf{Ker}_Q \mathsf{V}$ as μ. Therefore the projection map π is actually the restriction to $\mathsf{Ker}_Q \mathsf{V}$. Moreover, the condition that $\mu \in \mathsf{Ker}_Q \mathsf{V}^\star$ is equivalent to the requirement that μ vanishes on $\mathsf{Im}_Q \mathsf{V}$, that is $\mathsf{Ker}_Q \mathsf{V}^\star \cong (\mathsf{V}/\mathsf{Im}_Q \mathsf{V})^\star$. These two observation lead to b)

$$\begin{aligned} \mathsf{H}_Q(\mathsf{V}^\star) = \pi(\mathsf{Ker}_Q \mathsf{V}^\star) = \mathsf{Ker}_Q \mathsf{V}^\star\big|_{\mathsf{Ker}_Q \mathsf{V}} &\cong (\mathsf{V}/\mathsf{Im}_Q \mathsf{V})^\star\big|_{\mathsf{Ker}_Q \mathsf{V}} \\ &= (\mathsf{Ker}_Q \mathsf{V}/\mathsf{Im}_Q \mathsf{V})^\star = \mathsf{H}_Q(\mathsf{V})^\star \end{aligned}$$

c) There exist bases h'_a, e'_i, f'_i of U and h''_b, e''_j, f''_j of V that bring the action of Q to a Jordan normal form

$$\begin{aligned} Q\cdot h'_a &= 0, & Q\cdot e'_i &= 0, & Q\cdot f'_i &= e'_i \\ Q\cdot h''_b &= 0, & Q\cdot e''_j &= 0, & Q\cdot f''_j &= e''_j\ . \end{aligned}$$

Computing the action of Q in the corresponding tensor basis of $\mathsf{U}\otimes\mathsf{V}$ we get that $\mathsf{Ker}_Q(\mathsf{U}\otimes\mathsf{V})$ is spanned by

$$h'_a\otimes h''_b,\quad h'_a\otimes e''_j,\quad e'_i\otimes h''_b,\quad e'_i\otimes f''_j-(-1)^{|e'_i|}f'_i\otimes e''_j$$

and $\mathsf{Im}_Q(\mathsf{U}\otimes\mathsf{V})$ is spanned by

$$h'_a\otimes e''_j,\quad e'_i\otimes h''_b,\quad e'_i\otimes f''_j-(-1)^{|e'_i|}f'_i\otimes e''_j\ ,$$

where $|\cdot|$ denotes the grading function. Thus, $\mathsf{H}_Q(\mathsf{U}\otimes\mathsf{V})$ is spanned by $h'_a\otimes h''_b$. Finally we notice that h'_a spans $\mathsf{H}_Q(\mathsf{U})$ and h''_b spans $\mathsf{H}_Q(\mathsf{V})$, which proves c). ☐

Corollary 3.1.1. For a finite dimensional $\mathfrak{g}$-module V, we observe that

$$\text{sdim}\ \mathsf{H}_Q(\mathsf{V})\ =\ \text{sdim}\,\mathsf{V}\ . \tag{3.1.60}$$

The statement follows from the existence of a Jordan normal form for the representation of Q in V.

The vanishing of the superdimension of a module V is a *necessary* constraint for the triviality of the cohomological reduction $\mathsf{H}_Q(\mathsf{V})$. Atypical simple modules do not generally satisfy this constraint, while projective modules do as shown in [51].

Lemma 3.1.3. If V is a *finite dimensional projective* $\mathfrak{g}$-module, then $\mathsf{H}_Q(\mathsf{V})\cong 0$.

Proof. Let Γ^+ be the set of weights Λ parametrizing the finite dimensional simple $\mathfrak{g}$-modules $\mathsf{S}(\Lambda)$. Denote by $\mathsf{P}(\Lambda)$ the projective covers of $\mathsf{S}(\Lambda)$ in the sense of definition 2.2.2. The projective module V can then be represented as

$$\mathsf{V}\ \cong\ \bigoplus_{\Lambda\in\Gamma^+} d_\Lambda(\mathsf{V})\mathsf{P}(\Lambda)\ , \tag{3.1.61}$$

where only a finite number of multiplicities $d_\Lambda(\mathsf{V})$ do not vanish. Proving lemma 3.1.3 becomes equivalent to proving that $\mathsf{H}_Q(\mathsf{P}(\Lambda))=0$ for any $\Lambda\in\Gamma^+$. We show in the following that this task is equivalent to yet another one. We start by defining the induced modules

$$\mathsf{B}(\Lambda)\ :=\ \text{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}\text{Res}_{\mathfrak{g}_{\bar 0}}\mathsf{S}(\Lambda)\ \cong\ \mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{g}_{\bar 0}}\mathsf{S}(\Lambda) \tag{3.1.62}$$

which are finite dimensional and, according to [52], are also projective in the category of finite dimensional $\mathfrak{g}$-modules. The surjective map $\Pi:\mathsf{B}(\Lambda)\to\mathsf{S}(\Lambda)$

$$\Pi(u\otimes_{\mathfrak{g}_{\bar 0}} s)\ =\ u\cdot s \tag{3.1.63}$$

defines a projective $\mathfrak{g}$-module homomorphism. By our definition in section 2.2, $\mathsf{top}(\mathsf{B}(\Lambda))$ is the direct sum of all quotients of $\mathsf{B}(\Lambda)$ by a maximal submodule. Since Π is surjective, we have $\mathsf{B}(\Lambda)/\,\mathsf{Ker}\,\Pi \cong \mathsf{S}(\Lambda)$ and since $\mathsf{S}(\Lambda)$ is a simple module, $\mathsf{Ker}\,\Pi$ must be a maximal submodule and therefore $\mathsf{S}(\Lambda) \subset \mathsf{top}\,\mathsf{B}(\Lambda)$. On the other hand, decomposing $\mathsf{B}(\Lambda)$ as in equation (3.1.61) we can explicitly compute

$$\mathsf{top}\,\mathsf{B}(\Lambda) \;=\; \bigoplus_{\Lambda'\in\Gamma^+} d_{\Lambda'}(\mathsf{B}(\Lambda))\,\mathsf{top}\,\mathsf{P}(\Lambda') \;=\; \bigoplus_{\Lambda'\in\Gamma^+} d_{\Lambda'}(\mathsf{B}(\Lambda))\mathsf{S}(\Lambda')\ . \tag{3.1.64}$$

which from $\mathsf{S}(\Lambda) \subset \mathsf{top}\,\mathsf{B}(\Lambda)$ implies that $\mathsf{P}(\Lambda)$ must be a direct summand of $\mathsf{B}(\Lambda)$. Thus, we see that proving $\mathsf{H}_Q(\mathsf{P}(\Lambda)) = 0$ for any $\Lambda \in \Gamma^+$ is equivalent to proving that $\mathsf{H}_Q(\mathsf{B}(\Lambda)) = 0$ for any $\Lambda \in \Gamma^+$.

To compute the cohomology $\mathsf{H}_Q(\mathsf{B}(\Lambda))$, we yet again construct a basis of $\mathsf{B}(\Lambda)$ which brings the action of Q to a Jordan normal form. Let $a_1, \ldots, a_B$ be a basis of $\mathfrak{g}_{\bar{0}}$ and $b_1, \ldots, b_F$ be a basis of $\mathfrak{g}_{\bar{1}}$. According to the Poincaré-Birkoff-Witt theorem, the elements of the form

$$b_{i_1} \cdots b_{i_k} a_1^{l_1} \cdots a_B^{l_B}, \qquad k, l_i \geq 0,\ i_1 < \cdots < i_k \tag{3.1.65}$$

are a basis of $\mathfrak{U}(\mathfrak{g})$. Given a basis s_α of $\mathsf{S}(\Lambda)$, the basis (3.1.65) of $\mathfrak{U}(\mathfrak{g})$ provides a basis

$$b_{i_1} \cdots b_{i_k} \otimes s_\alpha, \qquad k \geq 0,\ i_1 < \cdots < i_k \tag{3.1.66}$$

of $\mathsf{B}(\Lambda)$ by means of the def. (3.1.62). By choosing a basis such that $b_1 = Q$, we immediately bring the action of Q to a Jordan normal form, so that it becomes obvious that $\mathsf{Ker}_Q(\mathsf{B}(\Lambda)) = \mathsf{Im}_Q(\mathsf{B}(\Lambda))$ is spanned by the basis vectors (3.1.66) with $i_1 = 1$. □

3.1.4 Reduction of smooth functions on G/G'

We shall restrict to Lie superalgebras $\mathfrak{g}$ of the type considered in subsection 3.1.2. They all have an invariant, supersymmetric, consistent and non-degenerate bilinear form $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ that by definition 2.1.2 we call a metric. Consider a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $(\ ,\ )$ restricts to a metric on $\mathfrak{g}'$ and suppose there is an odd element $Q \in \mathfrak{g}'$ with vanishing self-bracket.

According to equations (3.1.17 and 3.1.19), $\mathsf{H}_Q(\mathfrak{g})$ and $\mathsf{H}_Q(\mathfrak{g}')$ are isomorphic to some subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and, respectively, $\mathfrak{h}' \subset \mathfrak{g}'$, with the following direct sum decompositions

$$\begin{aligned} \mathfrak{g} &\cong \ \mathfrak{h} \oplus \mathfrak{e} \oplus \mathfrak{f} \\ \mathfrak{g}' &\cong \ \mathfrak{h}' \oplus \mathfrak{e}' \oplus \mathfrak{f}' \end{aligned} \tag{3.1.67}$$

as $\mathfrak{h}$ and $\mathfrak{h}'$-modules, respectively. Here $\mathfrak{e} := \mathsf{Im}_Q\,\mathfrak{g}$, $\mathfrak{e}' := \mathsf{Im}_Q\,\mathfrak{g}'$. Since by our assumption $Q \in \mathfrak{g}' \subset \mathfrak{g}$, we obtain the subalgebra inclusions:

$$\mathfrak{h}' \subset \mathfrak{h}, \qquad \mathfrak{e}' \subset \mathfrak{e}, \qquad \mathfrak{f}' \subset \mathfrak{f}\ .$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{g}'$ in $\mathfrak{g}$ with respect to $(\ ,\ )$. The assumption on the non-degeneracy of the metric and of its restriction to $\mathfrak{g}'$ implies the following facts for $\mathfrak{m}$:

a) $\mathfrak{m}$ is an $\mathfrak{g}'$-module

b) $(\ ,\ )|_{\mathfrak{m}\times\mathfrak{m}}$ is an $\mathfrak{g}'$-invariant non-degenerate scalar product

c) viewed as as an $\mathfrak{g}'$-module by restriction, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} \;\cong\; \mathfrak{g}' \oplus \mathfrak{m}\ . \tag{3.1.68}$$

Statements a) and b) are rather straightforward to prove, while c) results from the construction of a projection on $\mathfrak{g}'$ with the inverted metric $(\ ,\ )|_{\mathfrak{g}'\times\mathfrak{g}'}$, much like in eq. (3.1.18). From eq. (3.1.59) and point c), $\mathfrak{m}$ decomposes as an $\mathfrak{h}'$-module into the direct sum

$$\mathfrak{m} \;\cong\; \mathfrak{n} \oplus \mathfrak{p} \oplus \mathfrak{q}\ , \tag{3.1.69}$$

where $\mathfrak{n} \cong \mathsf{H}_Q(\mathfrak{m})$ and $\mathfrak{p} = \mathsf{Im}_Q\,\mathfrak{m}$. Computing the cohomology of the direct sum decomposition (3.1.68) with the help of property a) of lemma 3.1.2 together with equations (3.1.67) and (3.1.69), we are rewarded with the analogous result:

$$\mathfrak{h} \;\cong\; \mathfrak{h}' \oplus \mathfrak{n}\ . \tag{3.1.70}$$

One useful consequence of equations (3.1.68 – 3.1.70) is the following $\mathfrak{h}'$-module isomorphism

$$\mathsf{H}_Q(\mathfrak{g}/\mathfrak{g}') \;\cong\; \mathsf{H}_Q(\mathfrak{m}) \;\cong\; \mathfrak{n} \;\cong\; \mathfrak{h}/\mathfrak{h}'\ . \tag{3.1.71}$$

Let $\mathfrak{G}(\mathfrak{g})$ be the Grassmann envelope of $\mathfrak{g}$ with respect to some Grassmann algebra $\mathfrak{G}$ as defined in section 2.3. Furthermore, a given antilinear involutive automorphism of $\mathfrak{g}$ can be extended to an automorphism of $\mathfrak{G}(\mathfrak{g})$, thus defining a real form $\mathfrak{G}(\mathfrak{g})_\dagger$. Suppose now that G is a connected Lie supergroup with Lie algebra $\mathfrak{G}(\mathfrak{g})_\dagger$ and that G' is a connected subgroup of G with Lie algebra $\mathfrak{G}(\mathfrak{g}')_\dagger$. Let H denote the subgroup of G with Lie algebra $\mathfrak{G}(\mathfrak{h})_\dagger$ and H' the subgroup of G' with Lie algebra $\mathfrak{G}(\mathfrak{h}')_\dagger$. We want to perform the cohomological reduction of the space of smooth functions $\mathfrak{F}(G/G')$ with respect to Q and show that there is an H-module isomorphism

$$\mathsf{H}_Q(\mathfrak{F}(G/G')) \cong \mathfrak{F}(H/H')\ , \tag{3.1.72}$$

where $\mathfrak{F}(H/H')$ denotes the algebra of smooth functions on H/H'. Equation (3.1.71) was already used in section 3.1.1 to give a local argument for the isomorphism (3.1.72). In order to prove the claim (3.1.72), we shall identify $\mathfrak{F}(G/G')$ with the space $\mathfrak{F}(G)^{G'}$ of smooth functions on G invariant with respect to the right G'-action. We perform the same identification for $\mathfrak{F}(H/H') = \mathfrak{F}(H)^{H'}$.

Let us look closer at $\mathsf{Im}_Q\,\mathfrak{F}(G/G')$. The set of points of G/G' where all elements of $\mathsf{Im}_Q\,\mathfrak{F}(G/G')$ vanish are precisely those points of G/G' which are invariant with respect to the action of $e^{\eta Q}$, where η is an odd Grassmann number. We denote this subset by $(G/G')^Q$. Let G^Q and $(G')^Q$ denote the subgroup of G and, respectively, G' invariant with respect to the adjoint action of $e^{\eta Q}$. These are the subgroups on which the vector field $\mathcal{D}(Q)$ corresponding to the adjoint action of Q vanishes. This means that $\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$ is the subset of smooth functions on G vanishing on G^Q.

Lemma 3.1.4. The following equivalence of supermanifolds holds

$$(G/G')^Q \;=\; G^Q/(G')^Q \tag{3.1.73}$$

Proof. In the neighborhood of $\mathsf{e}G'$, where e is the identity of G, the distinct equivalence classes of G/G' can be parametrized as

$$e^v G' \;, \tag{3.1.74}$$

where $v \in \mathfrak{G}(\mathfrak{m})_\dagger$ is small enough. If we denote by v the coordinate of the point (3.1.74) then we get the geodesic system of coordinates at $\mathsf{e}G'$. Indeed, the coordinate space $\mathfrak{G}(\mathfrak{m})_\dagger$ can be identified with the tangent space at the point $\mathsf{e}G'$ with coordinates $v = 0$

$$(\mathcal{L}(v)f)(0) \;=\; \frac{d}{dt}(e^{tv}\cdot f)(0)\big|_{t=0} \;=\; \frac{d}{dt}f(-tv)\big|_{t=0} \;=\; -(v(f))(0)\;,$$

where $\mathcal{L}$ denotes the Lie derivative. The exponential mapping

$$v \mapsto e^v G' \tag{3.1.75}$$

can be extended to the whole tangent space $\mathfrak{G}(\mathfrak{m})_\dagger$. This extension is in general no longer injective, i.e./ it ceases to be a system of coordinates. However, assuming Hopf-Rinow theorem can be generalized to supermanifolds [53], the map (3.1.75) must be surjective, that is any group element $g \in G$ can be represented in the form

$$g \;=\; e^v g'$$

for some $v \in \mathfrak{G}(\mathfrak{m})_\dagger$ and $g' \in G'$. Using this global representation, one can easily see that $(G/G')^Q$ is the image of the exponential mapping (3.1.75) restricted to $\mathsf{Ker}_Q\,\mathfrak{G}(\mathfrak{m})_\dagger$. It follows that G^Q has a transitive action on $(G/G')^Q$, with its stabilizer at $\mathsf{e}G' \in (G/G')^Q$ with respect to the left action on G^Q being $(G')^Q = G^Q \cap G'$. This completes the proof of claim (3.1.73). □

Corollary 3.1.2. Let $L(Q)$ denote the vector field corresponding to the left action of Q. Then one has

$$\mathsf{Im}_Q\,\mathfrak{F}(G/G') \;=\; \mathsf{Im}_{L(Q)}\,\mathfrak{F}(G)^{G'} \;=\; \mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)^{G'} \;=\; (\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G))^{G'} \tag{3.1.76}$$

Proof. The first equality results from the identification $\mathfrak{F}(G/G') = \mathfrak{F}(G)^{G'}$ while the second equality is a consequence of $Q \in \mathfrak{g}'$. To prove the last equality notice that $\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$ is composed of functions on G vanishing on G^Q. Then $(\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G))^{G'}$ becomes the space of functions on G/G' vanishing on the submanifold G^Q/G'. Notice that $G^Q/G' = G^Q/(G')^Q$, because both supermanifolds are G^Q-transitive and have the same stabilizer $(G')^Q = G^Q \cap G'$. Therefore, according to equation (3.1.73), $(\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G))^{G'}$ can be seen as the space of functions on G/G' that vanish on $(G/G')^Q$, which coincides with the definition of $\mathsf{Im}_Q\,\mathfrak{F}(G/G')$. □

We also have obvious analogous equalities for the kernel of Q:

$$\mathsf{Ker}_Q\,\mathfrak{F}(G/G') \;=\; \mathsf{Ker}_{L(Q)}\,\mathfrak{F}(G)^{G'} \;=\; \mathsf{Ker}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)^{G'} \;=\; (\mathsf{Ker}_{\mathcal{D}(Q)}\,\mathfrak{F}(G))^{G'}\ . \quad (3.1.77)$$

Combining the equations 3.1.76 and 3.1.77, we get the following prescription for computing the cohomology

$$\mathsf{H}_Q(\mathfrak{F}(G/G')) \;=\; (\mathsf{H}_{\mathcal{D}(Q)}(\mathfrak{F}(G)))^{G'}\ . \quad (3.1.78)$$

Let us now concentrate on computing the right hand side of the above equation, starting with $\mathsf{H}_{\mathcal{D}(Q)}(\mathfrak{F}(G))$. The image of a function f under the projection map $\pi : \mathfrak{F}(G) \to \mathfrak{F}(G)/\,\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$ given by

$$\pi(f) \;:=\; f + \mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G) \quad (3.1.79)$$

is the equivalence class of functions which have *the same restriction on G^Q as f*, that is

$$\pi(f) \;=\; f|_{G^Q}\ , \quad (3.1.80)$$

since a function in $\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$ vanishes on the set of points left invariant under Q. In particular, any function whose restriction to G^Q vanishes must be in the image of Q. We further notice that the left and the right G-actions on $\mathfrak{F}(G)$ induce corresponding left and right G^Q-action on the quotient space $\mathfrak{F}(G)/\,\mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$

$$L(X)\pi(f) \;:=\; \pi(L(X)f), \qquad R(X)\pi(f) \;:=\; \pi(R(X)f)\ .$$

Lemma 3.1.5. The following isomorphism of H-modules and commutative algebras holds

$$\mathsf{H}_{\mathcal{D}(Q)}(\mathfrak{F}(G))^{G'} \;\cong\; \mathfrak{F}(H/H')\ . \quad (3.1.81)$$

Proof. If $X \in \mathfrak{g}$ and $f \in \mathsf{Ker}_{L(Q)}\,\mathfrak{F}(G)^{G'}$, then

$$L([Q,X])\pi(f) \;=\; \pi(L[Q,X]f) \;=\; \pi(L(Q)L(X)f) \;=\; \pi(\mathcal{D}(Q)L(X)f) \;=\; 0\ ,$$

because the left and right $\mathfrak{g}$ actions on $\mathfrak{F}(G)$ commute and $\mathsf{Ker}\,\pi = \mathsf{Im}_{\mathcal{D}(Q)}\,\mathfrak{F}(G)$. This shows that the space of functions $\mathsf{H}_{\mathcal{D}(Q)}(\mathfrak{F}(G))^{G'}$ is left invariant with respect to the action of $\mathfrak{e}$. Denote by N the subgroup of G with the Lie superalgebra $\mathfrak{e}$. The latter being an ideal of $\mathsf{Ker}_Q\,\mathfrak{g}$, N is a normal subgroup of G^Q with $H = N\backslash G^Q$. Then equation (3.1.81) claims that $\mathsf{H}_{\mathcal{D}(Q)}(\mathfrak{F}(G))^{G'}$ is a space of functions on $N\backslash G^Q/G' = H/G' = H/H'$. The last equality comes from the fact that both H/G' and H/H' are H-transitive and have the same stabilizer $H' = G' \cap H$. □

In conclusion we wee that the cohomology of a smooth function on G/G' is computed by restricting it to $H/H' \subset G/G'$. *Let us denote this restriction map by ρ.*

3.1.5 Reduction of smooth tensor forms on G/G'

Let $\mathsf{T}_k(G/G')$ be the space of smooth tensor forms of rank k on G/G'. We argue that equation (3.1.72) can be generalized to

$$\mathsf{H}_Q(\mathsf{T}_k(G/G')) \;\cong\; \mathsf{T}_k(H/H')\ , \tag{3.1.82}$$

where $\mathsf{T}_k(H/H')$ is the space of smooth tensor forms of rank k on H/H'. We shall only give a local argument in favor of this claim. Introducing the geodesic coordinates (3.1.74), one can perform the following identification in the neighborhood of the point $\mathsf{e}G' \in G/G'$

$$\mathsf{T}_k(G/G') \;\cong\; \mathfrak{F}(G/G') \otimes \mathfrak{m}^{\otimes k}\ .$$

This local trivialization extends to an isomorphism of G'-modules. Using the property c) of lemma 3.1.2, we get

$$\mathsf{H}_Q(\mathsf{T}_k(G/G')) \;\cong\; \mathsf{H}_Q(\mathfrak{F}(G/G')) \otimes \mathsf{H}_Q(\mathfrak{m})^{\otimes k} \cong \mathfrak{F}(H/H') \otimes \mathfrak{n}^{\otimes k} \cong \mathsf{T}_k(H/H')\ . \tag{3.1.83}$$

Most probably, one can give a global argument for the claim (3.1.82) by introducing the frame bundle

$$\mathsf{T}_k(G/G') \cong (\mathfrak{F}(G) \otimes F(G)^{\otimes k})^{G'}\ ,$$

where $F(G)$ is the moving frame attached to every point of G, which is built out of the components of the Maurer-Cartan form.

In conclusion, the cohomology of a tensor form on G/G' is computed, as can be seen from equation (3.1.83), by restricting it first to the submanifold H/H' and second to the tensor space of H/H'. The second step is equivalent to throwing out all components of the tensor not lying in the tensor space of H/H' seen as a submanifold of G/G'. *We denote this restriction map by ρ again.*

3.1.6 Reduction of $L_2(G/G')$

We want to refine the argument of (3.1.72), so that it may apply not only to smooth, but also to square integrable functions, i.e. we want to show that the elements of $\mathsf{H}_Q(L_2(G/G'))$ are square integrable with respect to some H-invariant measure on H/H', leading to

$$\mathsf{H}_Q(L_2(G/G')) \;\cong\; L_2(H/H')\ . \tag{3.1.84}$$

In order to do so, let us again make us of the geodesic coordinates v of equation (3.1.74). Let $v_\mathfrak{n}$, $v_\mathfrak{p}$ and $v_\mathfrak{q}$ denote the projection of v onto the real Grassmann envelope of the direct summand $\mathfrak{n}$, $\mathfrak{p}$ and, respectively, $\mathfrak{q}$ in eq. (3.1.69). We then embed $\mathfrak{F}(H/H')$ into $\mathfrak{F}(G/G')$ by means of the injection map

$$i(f)(v) \;=\; f(v_\mathfrak{n})e^{\alpha(v_\mathfrak{p},v_\mathfrak{q})}\ , \tag{3.1.85}$$

where v is small enough and α is, for the moment, an arbitrary number. Notice that eq. (3.1.85) defines the function $i(f)$ globally. Indeed, the definition (3.1.85) allows to compute the action of the enveloping Lie superalgebra $\mathfrak{U}(\mathfrak{g})$ on $i(f)$. The latter can be extended to the action of the group G, whose knowledge is enough to define the values of $i(f)$ at any point of G/G'.

The most important property of the injection map (3.1.85) is

$$\pi \circ i \;=\; \mathbb{1} \;, \tag{3.1.86}$$

where π is the projection of equations (3.1.79) and (3.1.80). As a consequence, any element of $\mathsf{Ker}_Q\, \mathfrak{F}(G/G')$ can be represented in the form

$$i(f) + \mathcal{L}(Q)h \;, \tag{3.1.87}$$

where $\mathcal{L}(Q)$ denotes the Lie derivative with respect to Q.

We now prove (3.1.84) by showing that for a proper choice of α in eq. (3.1.85) one has

$$\langle i(f_1), i(f_2)\rangle_{G/G'} \;=\; \langle f_1, f_2\rangle_{H/H'} \;. \tag{3.1.88}$$

The equation should be understood as follows: i) the existence of one side implies the existence of the other side and ii) for a G-invariant scalar product $\langle\ ,\ \rangle_{G/G'}$ on $L_2(G/G')$ induced by the G-invariant measure on G/G' there is a corresponding H-invariant scalar product $\langle\ ,\ \rangle_{H/H'}$ on $L_2(H/H')$ induced by the H-invariant measure on H/H'.

Indeed, let the measure on G/G' be given locally by $d\mu_G(v) = w(v)dv$. Suppose $i(f)$ is L_2 normalizable. Then its norm can be written in the form

$$\int_{H/H'} d\mu_H f^2 \;,$$

where $d\mu_H$ is a measure on H/H' locally defined by a weight function $w'(v_{\mathfrak{n}})$ obtained by integrating

$$w(v)e^{2\alpha(v_{\mathfrak{p}},v_{\mathfrak{q}})} \;,$$

with respect to the coordinates $v_{\mathfrak{p}}$ and $v_{\mathfrak{q}}$. Notice that there is always a choice of α such that w' exists even for non-compact homogeneous spaces G/G'. Of course, in order to perform the integration yielding the explicit form of w' one must work with an atlas of G/G'. However, the only thing that matters to us is its H-invariance or, equivalently, the H-invariance of the scalar product $\langle\ ,\ \rangle_{H/H'}$ associated to it by eq. (3.1.88). We thus check

$$\langle i(\mathcal{L}(X)f_1), i(f_2)\rangle_{G/G'} + \langle i(f_1), i(\mathcal{L}(X)f_2)\rangle_{G/G'} \;=\; 0, \qquad X \in \mathfrak{h} \;. \tag{3.1.89}$$

Notice that $(v_{\mathfrak{p}}, v_{\mathfrak{q}})$ is Q-exact because its restriction to $v_{\mathfrak{q}} = 0$ vanishes. Therefore $\mathcal{L}(X)(v_{\mathfrak{p}}, v_{\mathfrak{q}})$ is also Q-exact, because $[Q, X] = 0$. Finally,

$$(\mathcal{L}(X)i(f))(v) - i(\mathcal{L}(X)f)(v) \;=\; f(v_{\mathfrak{n}})\alpha e^{\alpha(v_{\mathfrak{p}},v_{\mathfrak{q}})}\mathcal{L}(X)(v_{\mathfrak{p}}, v_{\mathfrak{q}}) \tag{3.1.90}$$

is Q-exact as well, because $f(v_{\mathfrak{n}})e^{\alpha(v_{\mathfrak{p}},v_{\mathfrak{q}})}$ is Q-invariant. We then use the exactness of the expression (3.1.90) to commute the Lie derivative $\mathcal{L}(X)$ with the injection i in eq. (3.1.89).

We conclude this section by noticing that eq. (3.1.88) can be written in an equivalent way as

$$\langle f_1, f_2\rangle_{G/G'} \;=\; \langle \rho(f_1), \rho(f_2)\rangle_{H/H'} \;, \tag{3.1.91}$$

where $f_1, f_2 \in \mathsf{Ker}_Q\, L_2(G/G')$. This is the localization phenomenon.

3.2 Cohomological reduction in the field theory

We are now prepared to revisit the sigma models on G/G'. We have shown in sec. 2.4.3 how the local observables of the sigma model on G/G' can be constructed from functions on $L_2(G/G')$ and (some well behaved subspace of the space of smooth) tensor forms on G/G'. The results of sec. (3.1.4–3.1.6) straightforwardly imply that the cohomological reduction of the space of local observables in the sigma model on G/G' coincides precisely with the space of local observables in the sigma model on H/H', that is

$$\boxed{\mathsf{H}_Q(\mathcal{F}_{G/G'}) \;\cong\; \mathcal{F}_{H/H'}}\;. \tag{3.2.1}$$

Let us now look at Q-invariant correlation functions of local fields $\mathcal{O}$. As the results of the previous section suggest, we shall demonstrate that any correlation function of such fields can be computed in the H/H' coset superspace theory.

First we need to compute the cohomological reduction of the action $\mathcal{S}_{G/G'}$ associated to the Lagrangian in equation (2.4.3). Since the Lagrangian is entirely fixed by a G-invariant metric and an exact G-invariant 2-form, we can apply the results of sec. 3.1.5 in order to compute their cohomology class. The classes of the two tensor forms are computed by restricting them to the points of the submanifold H/H' and to its tensor space respectively. As a result we obviously get an H-invariant metric and an exact H-invariant 2-form on H/H'. Employing the restriction map ρ of sections 3.1.4 and 3.1.5, we conclude that

$$\rho(\mathcal{S}_{G/G'}) \;=\; \mathcal{S}_{H/H'} \tag{3.2.2}$$

is an action for the sigma model on H/H' with a similar kinetic term and B-field structure as $\mathcal{S}_{G/G'}$. The pullback of eq. (3.2.2) takes a more familiar form to usual cohomological calculations in field theory

$$\mathcal{S}_{G/G'} = \mathcal{S}_{H/H'} + \mathcal{L}(Q)R \;,$$

where $\mathcal{L}(Q)$ denotes the Lie derivative with respect to Q and R is some residual functional, obviously non G-invariant. The possibility of constructing G-invariant terms $\mathcal{L}(Q)R$ out of non G-invariant terms R is a special feature of the supergroup symmetry. According to one of the main ideas behind cohomological reduction, the Q-exact term in the action does not contribute to the calculation of correlation functions of Q-invariant local fields.

To make things more precise, notice that the localization formula (3.1.91) for the computation of the scalar product of Q-invariant functions can be generalized to the integral of any Q-invariant object. Therefore, we trivially obtain from eq. (2.4.21)

$$\begin{aligned} \left\langle \prod_{i=1}^{N} \mathcal{O}_i(x_i) \right\rangle_{G/G'} &= \int_{\mathcal{H}} \mathrm{d}\mu_H e^{-\rho(\mathcal{S}_{G/G'})} \prod_{i=1}^{N} \rho\left(\mathcal{O}_i\right)(x_i) \\ &= \left\langle \prod_{i=1}^{N} \rho\left(\mathcal{O}_i\right)(x_i) \right\rangle_{H/H'} . \end{aligned} \tag{3.2.3}$$

where we have used eq. (3.2.2). Consequently, the subsector of the sigma model on G/G' which we obtain through cohomological reduction is composed of the localized observables $\rho(\mathcal{O}_i)$. Finally, using the central statement (3.2.3), we conclude that this subsector is exactly identified with the local observables of the sigma model on H/H'.

3.3 Applications

In the first subsection we discuss applications of cohomological reduction to conformal field theory. In the second subsection we present a general treatment of sigma models on supercoset spaces $G/G^{\mathbb{Z}_2}$ defined by a degree two automorphism, that is on symmetric superspaces. The last subsection deals with some specific examples involving automorphisms of degree four.

3.3.1 Conformal field theory

The cohomological reduction we have described in the previous two subsections allows us to identify certain simple subsectors of the parent theory in which all correlation functions can be computed explicitly through the reduced model. The latter is often much simpler than the original theory. In fact, we shall find many examples below in which the subsector is a free or even topological field theory. The existence of such simple subsectors may signal very special features of the parent model. In particular, it can imply its scale invariance.

In order to make a more precise statement we need a bit of preparation. Let us recall that the coset G/G' gives rise to a family of sigma models which is parametrized by the metric G and the B-field B. Invariance of the action under G determines the two background fields up to a finite number of parameters, which, upon quantization, may get renormalized. This renormalization of G and B can affect the properties of our theory and in particular of its stress-energy tensor.

Let us now consider the quantized G/G' model that comes with some fixed choice of G and B. The associated stress tensor T_G is conserved and symmetric. On the other hand, the trace of T_G may be non-zero due to quantum effects. The components of T_G are G-invariant, i.e. they commute with all generators $X \in \mathfrak{g}$. In general, T_G can be decomposed into a sum $T_G = \sum_i T_G^{(i)}$ of terms where each of the summands $T_G^{(i)}$

belongs to a single indecomposable representation of $\mathfrak{g}$. We say that T_G is a true G-invariant if every summand $T_G^{(i)}$ is a direct summand. This must be distinguished from more generic cases for which some of the summands $T_G^{(i)}$, although transforming in the trivial representation of $\mathfrak{g}$, are coupled to other fields through the action of a nilpotent symmetry generator N from the center of the enveloping Lie superalgebra $\mathfrak{U}(\mathfrak{g})$. In this case, $T_G^{(i)} = N t_G^{(i)}$ for some field $t_G^{(i)}$, which is called a logarithmic partner of $T_G^{(i)}$.

Let us now assume that the tensor T_G is a true G-invariant in the sense we have described above. Suppose furthermore that the theory contains a *conformal* subsector H/H' with a non-vanishing stress tensor T_H. According to our assumption, T_H is conserved, symmetric and traceless. Consequently, the stress tensor of the original theory must be conserved, symmetric and traceless up to some Q-exact terms. Since we assumed T_G to be a true invariant, though, none of its components — and in particular the trace of T_G — can be obtained by acting with an element of $\mathfrak{U}(\mathfrak{g})$ on some other fields. Hence, T_G must be traceless and hence the G/G' model is conformal.

Let us stress again that our assumption on T_G to be a true invariant is rather strong. We are not prepared to state precise conditions under which this assumption is actually satisfied in general. However, when the superspaces G/G' have *at most one degree of freedom* in the choice of G and B one can get a simple constraint for the conformality of the parent theory from the conformality of the cohomological subsector theory: *the sigma model G/G' is conformal if H/H' is conformal and its central charge is non-zero.* Indeed, in this case G and B is either proportional to i) a single $\mathfrak{g}$ true invariant or to ii) a single invariant socle of a $\mathfrak{g}$-indecomposable module. If H/H' is the conformally invariant maximal cohomological reduction with a non-zero central charge, then T_G cannot be an invariant socle. Otherwise we would get a contradiction, because its 2-point function would vanish and the 2-point function of T_G must coincide with the 2-point function of T_H. The latter, however, cannot vanish because the central charge of the conformal H/H' sigma model is non-zero.

3.3.2 Sigma models on symmetric superspaces

In this section, we want to present a classification of the cohomological reductions of $\mathbb{Z}_2$ cosets, i.e. of symmetric superspaces. These supermanifolds G/G' have the property that G' is a direct product of supergroups of which at most two are simple. For each simple factor whose superalgebra contains nilpotent fermionic elements, we can perform the cohomological reduction. Reductions performed with Q operators that come from different simple factors commute with each other. As an example, consider the coset space $\mathfrak{g}/\mathfrak{g}' = \mathrm{gl}\,(M+m|N+n)\,/\mathrm{gl}\,(M|N) \oplus \mathrm{gl}\,(m|n)$. The denominator has two simple factors, so that we can reduce in two ways as outlined in figure 3.1.

In table 3.1 below, we describe the different cohomological sectors of all possible sigma models on symmetric superspaces. We only write down the complex case, but different reality conditions can then easily be taken into consideration.

Some of the minimal subsectors are topological. This occurs when the whole Lagrangian is in the image of Q, which is the case whenever the right side of table 3.1

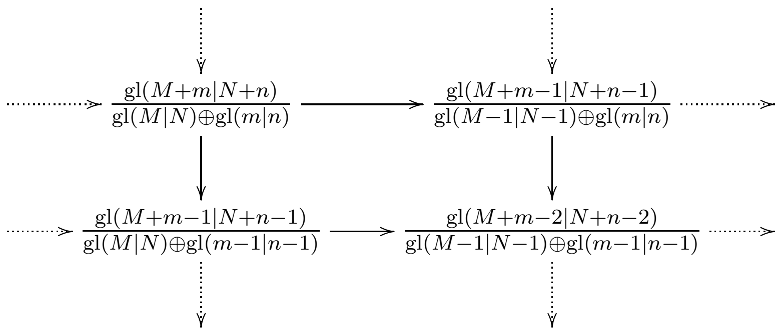

Figure 3.1: Possible cohomological reductions of $\mathrm{gl}\,(M+m|N+n)\,/\mathrm{gl}\,(M|N)\oplus\mathrm{gl}\,(m|n)$.

can be brought to the form $\mathfrak{g}/\mathfrak{g}$. This happens for the GL (N|N), OSP (2N+1|2N) and OSP (2N|2N) principal chiral models as well as for the cosets

$$\frac{\mathrm{GL\,(N{+}p|N{+}q)}}{\mathrm{GL\,(N|N)\times GL\,(p|q)}}\quad \frac{\mathrm{GL\,(2N|2N)}}{\mathrm{OSP\,(2N|2N)}}\quad \frac{\mathrm{OSP\,(2N{+}p|2N{+}2q)}}{\mathrm{OSP\,(2N|2N)\times OSP\,(p|2q)}}\quad \frac{\mathrm{OSP\,(2N|2N)}}{\mathrm{GL\,(N|N)}}\ .$$

On the other hand, some cohomological reductions lead to free conformal field theories, for which there are only two possibilities. Either they reduce to the $c=1$ free boson model to the $c=-2$ theory of a pair of symplectic fermions. The former case occurs for the OSP (2N+2|2N) principal chiral model and the real Grassmannians

$$\frac{\mathrm{OSP\,(2{+}2m{+}2n|2m{+}2n)}}{\mathrm{OSP\,(1{+}2m|2m)\times OSP\,(1{+}2n|2n)}}\ , \tag{3.3.1}$$

whereas the latter occurs for the PSL (N|N) principal chiral model as well as for the cosets

$$\frac{\mathrm{GL\,(m{+}n{+}1|m{+}n{+}1)}}{\mathrm{GL\,(m{+}1|m)\times GL\,(n|n{+}1)}}\quad \text{and}\quad \frac{\mathrm{PSL\,(2N|2N)}}{\mathrm{OSP\,(2N|2N)}}\ . \tag{3.3.2}$$

As was shown in [48] by direct computation of the all loop β function, these are the only sigma models on symmetric spaces that are conformally invariant. The superspaces G/G' in (3.3.1) and (3.3.2) have only one radius and no G-invariant B-field. We thus see that the argument of section 3.3.1 leads to the same classification of conformally invariant sigma models, while this time being non-perturbative in nature. We devote the whole of chapters 4 and 5 to the study of the two of these families of conformal field theories, namely the non-linear sigma models on the superspheres $S^{2m+1|2m}$, obtained from (3.3.1) with $n=0$, and the complex projective superspaces $\mathbb{CP}^{S-1|S}$, that we get from the first term of (3.3.2) with $n=0$ and $S\equiv m+1$.

$\mathcal{R}$	$\mathcal{M}$	Comments
$\frac{\mathrm{psl}(1\|1)\oplus\mathrm{psl}(1\|1)}{\mathrm{psl}(1\|1)}$	$\frac{\mathrm{psl}(1+\alpha\|1+\alpha)\oplus\mathrm{psl}(1+\alpha\|1+\alpha)}{\mathrm{psl}(1+\alpha\|1+\alpha)}$	C
$\frac{\mathrm{gl}(1\|1)\oplus\mathrm{gl}(1\|1)}{\mathrm{gl}(1\|1)}$	$\frac{\mathrm{gl}(1+\alpha\|1+\alpha)\oplus\mathrm{gl}(1+\alpha\|1+\alpha)}{\mathrm{gl}(1+\alpha\|1+\alpha)}$	T
$\frac{\mathrm{sl(R)}\oplus\mathrm{sl(R)}}{\mathrm{sl(R)}}$	$\frac{\mathrm{sl}(R+\alpha\|\alpha)\oplus\mathrm{sl}(R+\alpha\|\alpha)}{\mathrm{sl}(R+\alpha\|\alpha)}$	
$\frac{\mathrm{gl(R+S)}}{\mathrm{gl(R)}\oplus\mathrm{gl(S)}}$	$\frac{\mathrm{gl}(R+S+\alpha+\beta\|\alpha+\beta)}{\mathrm{gl}(R+\alpha\|\alpha)\oplus\mathrm{gl}(S+\beta\|\beta)}$	T for $R=0$ or $S=0$
$\frac{\mathrm{gl}(R\|S)}{\mathrm{gl(R)}\oplus\mathrm{gl(S)}}$	$\frac{\mathrm{gl}(R+\alpha+\beta\|S+\alpha+\beta)}{\mathrm{gl}(R+\alpha\|\alpha)\oplus\mathrm{gl}(\beta\|S+\beta)}$	C for $R=S=1$ T for $R=0$ or $S=0$
$\frac{\mathrm{psl}(1\|1)\oplus\mathrm{psl}(1\|1)}{\mathrm{psl}(1\|1)}$	$\frac{\mathrm{psl}(2\alpha\|2\alpha)}{\mathrm{osp}(2\alpha\|2\alpha)}$	C
$\frac{\mathrm{gl}(2\|2)}{\mathrm{osp}(2\|2)}$	$\frac{\mathrm{gl}(2+2\alpha\|2+2\alpha)}{\mathrm{osp}(2+2\alpha\|2+2\alpha)}$	T
$\frac{\mathrm{sl(R)}}{\mathrm{so(R)}}$	$\frac{\mathrm{sl}(R+2\alpha\|2\alpha)}{\mathrm{osp}(R+2\alpha\|2\alpha)}$	T for $R=1$
$\frac{\mathrm{sl(2R)}}{\mathrm{sp(2R)}}$	$\frac{\mathrm{sl}(2\alpha\|2R+2\alpha)}{\mathrm{osp}(2\alpha\|2R+2\alpha)}$	
$\frac{\mathrm{sl}(1\|2R)}{\mathrm{osp}(1\|2R)}$	$\frac{\mathrm{sl}(1+2\alpha\|2R+2\alpha)}{\mathrm{osp}(1+2\alpha\|2R+2\alpha)}$	
$\frac{\mathrm{so(R)}\oplus\mathrm{so(R)}}{\mathrm{so(R)}}$	$\frac{\mathrm{osp}(R+2\alpha\|2\alpha)\oplus\mathrm{osp}(R+2\alpha\|2\alpha)}{\mathrm{osp}(R+2\alpha\|2\alpha)}$	C for $R=2$ T for $R=0$ or $R=1$
$\frac{\mathrm{sp(2R)}\oplus\mathrm{sp(2R)}}{\mathrm{sp(2R)}}$	$\frac{\mathrm{osp}(2\alpha\|2R+2\alpha)\oplus\mathrm{osp}(2\alpha\|2R+2\alpha)}{\mathrm{osp}(2\alpha\|2R+2\alpha)}$	T for $R=0$
$\frac{\mathrm{osp}(1\|2R)\oplus\mathrm{osp}(1\|2R)}{\mathrm{osp}(1\|2R)}$	$\frac{\mathrm{osp}(1+2\alpha\|2R+2\alpha)\oplus\mathrm{osp}(1+2\alpha\|2R+2\alpha)}{\mathrm{osp}(1+2\alpha\|2R+2\alpha)}$	
$\frac{\mathrm{so(R+S)}}{\mathrm{so(R)}\oplus\mathrm{so(S)}}$	$\frac{\mathrm{osp}(R+S+2\alpha+2\beta\|2\alpha+2\beta)}{\mathrm{osp}(R+2\alpha\|2\alpha)\oplus\mathrm{osp}(S+2\beta\|2\beta)}$	C for $R=S=1$ T for $R=0$ or $S=0$
$\frac{\mathrm{osp}(R\|2S)}{\mathrm{so(R)}\oplus\mathrm{sp(2S)}}$	$\frac{\mathrm{osp}(R+2\alpha+2\beta\|2S+2\alpha+2\beta)}{\mathrm{osp}(R+2\alpha\|2\alpha)\oplus\mathrm{osp}(2\beta\|2S+2\beta)}$	T for $R=0$ or $S=0$
$\frac{\mathrm{sp(2R+2S)}}{\mathrm{sp(2R)}\oplus\mathrm{sp(2S)}}$	$\frac{\mathrm{osp}(2\alpha+2\beta\|2R+2S+2\alpha+2\beta)}{\mathrm{osp}(2\alpha\|2R+2\alpha)\oplus\mathrm{osp}(2\beta\|2S+2\beta)}$	T for $R=0$ or $S=0$
$\frac{\mathrm{osp}(2\|2R+2S)}{\mathrm{osp}(1\|2R)\oplus\mathrm{osp}(1\|2S)}$	$\frac{\mathrm{osp}(2+2\alpha+2\beta\|2R+2S+2\alpha+2\beta)}{\mathrm{osp}(1+2\alpha\|2R+2\alpha)\oplus\mathrm{osp}(1+2\beta\|2S+2\beta)}$	
$\frac{\mathrm{so(2R)}}{\mathrm{gl(R)}}$	$\frac{\mathrm{osp}(2R+2\alpha\|2\alpha)}{\mathrm{gl}(R+\alpha\|\alpha)}$	T for $R=0,1$
$\frac{\mathrm{sp(2R)}}{\mathrm{gl(R)}}$	$\frac{\mathrm{osp}(2\alpha\|2R+2\alpha)}{\mathrm{gl}(\alpha\|R+\alpha)}$	T for $R=0$

Table 3.1: The left column presents the possible minimal non-trivial sectors labelled by R, S and the right one the chain of models to which they belong. We denote by T the models that have a topological subsector and by C those models that are conformally invariant.

3.3.3 Examples involving generalized symmetric spaces

We will now turn our attention to a few generalized symmetric spaces in which the denominator supergroup G' is left invariant under the action of some automorphism

$\Omega : G \mapsto G$ of order four. We are not attempting to provide a classification of such cosets, but restrict our discussion to three interesting examples. The first series of models contains theories whose minimal subsector is given by the sigma model for $AdS_5 \times S^5$ and $AdS_2 \times S^2$ spaces. The second and third example extend the construction of superspheres and complex projective spaces, respectively. In all three families of models we shall identify previously unknown candidates for conformal cosets, see equations (3.3.7), (3.3.12) and (3.3.15).

Example 3.3.1. Our first case involves the coset superspaces

$$\mathfrak{g}/\mathfrak{g}' = \frac{\mathrm{psu}\,(2(M+m)|2(N+n))}{\mathrm{osp}\,(2m|2n) \oplus \mathrm{osp}\,(2N|2M)} \tag{3.3.3}$$

defined for $M+m = N+n$ by the following automorphism of order four: $\Omega = -st \circ \mathrm{Ad}_X \circ \mathrm{Ad}_Y$ with

$$X = \left(\begin{array}{cc|cc} & \mathbb{1}_{M+m} & & \\ -\mathbb{1}_{M+m} & & & \\ \hline & & & \mathbb{1}_{N+n} \\ & & -\mathbb{1}_{N+n} & \end{array}\right) \qquad Y = \mathrm{diag}\,(\mathbb{1}_m, -\mathbb{1}_{2M+m}, \mathbb{1}_{N+2n}, -\mathbb{1}_N)\ . \tag{3.3.4}$$

Here, in order to properly define the automorphism, one has to embed the superalgebra psu $(2(M+m)|2(N+n))$ in the fundamental representation of su $(2(M+m)|2(N+n))$. The invariant subalgebra $\mathfrak{g}'$ is a direct sum for which the grading of the second summand is opposite that of the first one. In order to know the number of free parameters in the metric and B field defining the model, we have to know how the Ω eigenspaces transform under the action of $\mathfrak{g}'$. The result is

$$\mathfrak{m}_1 \cong \square \otimes \square \qquad \mathfrak{m}_2 \cong \left(\varnothing \otimes \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \varnothing\right) \qquad \mathfrak{m}_3 \cong \square \otimes \square\ . \tag{3.3.5}$$

Here, as well as in the following examples, $\varnothing$ denotes the trivial representation, $\square$ the fundamental representation and $\square^\star$ its dual. Tensor products of the fundamental representation and of its dual that possess certain permutation symmetry are denoted by the appropriate Young tableaux. We live it as an exercise to the reader to show that

$$\mathfrak{m} \circ \mathfrak{m} \cong 4\varnothing \oplus \cdots \qquad \mathfrak{m} \wedge \mathfrak{m} \cong \varnothing \oplus \cdots\ , \tag{3.3.6}$$

so that the Lagrangian that we can write down for these models has five coupling constants, four of which parametrize the kinetic term.

We want to mention three special cases for these cosets

- Without loss of generality, we choose Q to lie only in the second direct summand of $\mathfrak{g}'$. Assuming that $M = N$ and thus $m = n$, we see that the maximal reduction in this case leads to the sigma model on the $\mathbb{Z}_2$ coset PSU (2m|2m) /OSP (2m|2m), which is conformal. We thus arrive at the conclusion that the sigma models on the $\mathbb{Z}_4$ coset spaces

$$\mathcal{C}_{(N,n)} \;\cong\; \frac{\text{PSU}\,(2(\text{N+n})|2(\text{N+n}))}{\text{OSP}\,(2\text{n}|2\text{n}) \times \text{OSP}\,(2\text{N}|2\text{N})} \tag{3.3.7}$$

are promising candidates for conformal sigma models for all non negative values of N and n.

- If we specialize to $M = n = 2$, $m = N = 0$ and change the reality conditions appropriately, we obtain the well known $\mathbb{Z}_4$ coset space $\text{PSU}\,(2,2|4)\,/\text{SO}\,(4{,}1) \times \text{SO}\,(5)$ whose bosonic base is $AdS_5 \times S^5$. This model cannot be reduced any further, since $\mathfrak{g}'$ is purely bosonic. It constitutes the maximal reduction of the two parameter discrete family of models

$$\mathcal{M}_{(m,n)} = \frac{\text{PSU}\,(2m+2n+2, 2|2m+2n+4)}{\text{OSP}\,(2\text{m}|\ 2\text{m+2},\ 2) \times \text{OSP}\,(2\text{n}|2\text{n+4})}\ . \tag{3.3.8}$$

- Setting $M = n = 0$, $m = N = 1$ and again taking the appropriate boundary conditions, leads to the space $\text{PSU}\,(1,1|2)\,/\text{SO}\,(2) \times \text{SO}\,(2)$ whose bosonic base is $AdS_2 \times S^2$. This case is the maximal reduction of the family of sigma models with $\mathfrak{g} = \text{psu}\,(2(m+n+1)|2(m+n+1))$ and $\mathfrak{g}' = \text{osp}\,(2m+2|2m) \oplus \text{osp}\,(2n+2|2n)$, subject to a certain reality conditions.

Example 3.3.2. We are interested in the $\mathbb{Z}_4$ coset

$$\mathfrak{g}/\mathfrak{g}' = \frac{\text{osp}\,(M+2m|2N+2n)}{\text{osp}\,(p|2q) \oplus \text{osp}\,(M-p|2(N-q)) \oplus \text{u}\,(m|n)}\ . \tag{3.3.9}$$

The corresponding automorphism is $\Omega = Ad_X$ with

$$X = \left(\begin{array}{cc|ccc} I^p_M & & & & \\ & J_{2m} & & & \\ \hline & & I^q_N & & \\ & & & I^q_N & \\ & & & & J_{2n} \end{array}\right) \quad \text{where} \quad \begin{array}{l} I^p_n = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_{n-p} \end{pmatrix} \\[2ex] J_{2n} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}\ . \end{array} \tag{3.3.10}$$

Under the action of $\mathfrak{g}'$, the Ω eigenspaces transform as

$$\begin{aligned} \mathfrak{m}_1 &\cong (\square \otimes \varnothing \otimes \square) \oplus (\varnothing \otimes \square \otimes \square) \\ \mathfrak{m}_2 &\cong \left(\varnothing \otimes \varnothing \otimes \begin{array}{c}\square\\[-4pt]\square\end{array}\right) \oplus \left(\varnothing \otimes \varnothing \otimes \begin{array}{c}\square\\[-4pt]\square\end{array}^{\star}\right) \oplus (\square \otimes \square \otimes \varnothing) \\ \mathfrak{m}_3 &\cong (\square \otimes \varnothing \otimes \square^{\star}) \oplus (\varnothing \otimes \square \otimes \square^{\star})\ , \end{aligned} \tag{3.3.11}$$

where by $\mathsf{U} \otimes \mathsf{V} \otimes \mathsf{W}$ we understand a module defined as the tensor product of the $\mathsf{U}, \mathsf{V}, \mathsf{W}$ representations of respectively $\mathrm{osp}\,(p|2q)$, $\mathrm{osp}\,(M-p|2(N-q))$ and $\mathrm{u}\,(m|n)$. When selecting the fermionic operator $Q \in \mathfrak{g}'$, we choose it to be fully contained in one of the direct summands of $\mathfrak{g}'$. Since the first two lead, after suitable choice of the parameters M, N, p, q, to the same reduction, we will assume, that Q is either in $\mathrm{osp}\,(M-p|2(N-q))$ or in $\mathrm{u}\,(m|n)$.

- If now we have $M = 2N$ and $p = 2q$, then we can pursue the reduction of the first type until we get rid of the orthosymplectic parts in $\mathfrak{g}'$ to arrive at the sigma model on the symmetric space OSP (2m|2n) /U (m|n) which is not a conformal theory.
- If on the other hand $m = n$, then taking the second type of reduction can be used to remove the unitary part of $\mathfrak{g}'$ so as to obtain the sigma model on the symmetric space OSP (M|2N) /OSP (p|2q) × OSP (M-p|2(N-q)), which is a conformal field theory for $p = 1$, $q = 0$ and $M = 2N + 2$. We therefore come to the conclusion that for all $N, n \in \mathbb{N}$ the sigma models on the homogeneous spaces

$$\mathcal{C}_{(N,n)} \;\cong\; \frac{\mathrm{OSP\,(2N{+}2{+}2n|2N{+}2n)}}{\mathrm{OSP\,(2N{+}1|2N)} \times \mathrm{U\,(n|n)}} \tag{3.3.12}$$

  are candidates for conformally invariant sigma models. For $n = 0$ they reduce to the symmetric spaces $S^{2N+1|2N}$, i.e. the superspheres, whereas for $N = 0$ they remain a $\mathbb{Z}_4$ homogeneous space.

Example 3.3.3. The last case under consideration is the $\mathbb{Z}_4$ coset

$$\mathfrak{g}/\mathfrak{g}' = \frac{\mathrm{u}\,(M+2m|N+2n)}{\mathrm{u}\,(p|q) \oplus \mathrm{u}\,(M-p|N-q) \oplus \mathrm{u}\,(m|n) \oplus \mathrm{u}\,(m|n)} \tag{3.3.13}$$

defined by the automorphism $\Omega = Ad_Y$, where

$$Y = \left(\begin{array}{cc|cc} I_M^p & & & \\ & J_{2m} & & \\ \hline & & I_N^q & \\ & & & J_{2n} \end{array}\right) . \tag{3.3.14}$$

We need not spell out the decomposition of $\mathfrak{m}_i$ in modules of $\mathfrak{g}'$, it suffices to say that the only representations that appear are of the kind $A \otimes B \otimes C \otimes D$, where A, B, C, D are either the trivial, fundamental or dual fundamental of respectively $\mathrm{u}\,(p|q)$, $\mathrm{u}\,(M-p|N-q)$, the first $\mathrm{u}\,(m|n)$ and the second $\mathrm{u}\,(m|n)$. We choose Q to be diagonally embedded in the $\mathrm{u}\,(m|n) \oplus \mathrm{u}\,(m|n)$ part of $\mathfrak{g}'$, so that the reduction procedure sends the parameters m and n to $m-1$ and $n-1$. If $m = n$, then the reduction terminates with the symmetric space U (M|N) /U (p|q) × U (M-p|N-q). The sigma models with this target spaces are conformal for $M = N$ and $p = q \pm 1$, with the special case $p = 1$ and

$q = 0$ corresponds to the complex symmetric superspaces $\mathbb{CP}^{S-1|S}N$. In conclusion, we can state that the sigma models on the homogeneous spaces

$$\mathcal{C}_{(M,N,n)} \cong \frac{\mathrm{U}\left(\mathrm{M+N+2n|M+N+2n}\right)}{\mathrm{U}\left(\mathrm{M+1|M}\right) \times \mathrm{U}\left(\mathrm{N\text{-}1|N}\right) \times \mathrm{U}\left(\mathrm{n|n}\right) \times \mathrm{U}\left(\mathrm{n|n}\right)}\,, \tag{3.3.15}$$

are expected to be conformal for values of $M, N, n \in \mathbb{N}$ with $N > 0$.

3.3.4 Extensions of the cohomological reduction

In this section, we want to expand the technique of cohomological reduction to encompass Wess-Zumino-Witten and Gross-Neveu models.

- The Wess-Zumino term on the supergroup G with the superalgebra $\mathfrak{g}$ was written down in (2.5.15). We see that the Wess-Zumino term $\mathcal{S}_{WZ}$ is a trilinear combination of the left invariant currents J and can be cohomologically reduced in a similar fashion as the bilinear $\mathcal{S}_{kin}$ kinetic part.

  A straightforward if lengthy computation shows that, choosing a fermionic operator $Q \in \mathfrak{g}$ that squares to zero, the cohomologically reduced model is the Wess-Zumino-Witten model on the supergroup H whose superalgebra is $\mathsf{H}_Q(\mathfrak{g})$. Thus, the Wess-Zumino-Witten models on a supergroup reduce in the same way as the principal chiral models on the same supergroup. It can even happen that a principal chiral model and a WZW model both reduce to the same theory. An interesting example of that is furnished by the PSU (N|N) WZW and principal chiral models, the maximal reduction of which is the $c = -2$ free theory of a single pair of symplectic fermions.

- In the case of the Gross-Neveu models written down in (2.5.16) and (2.5.17), we can, at the free point, take the fermionic nilpotent generator

  $$Q = \frac{1}{4\pi i}\left\{\oint_0 dz\,(\psi_1 + i\psi_2)\,\beta_1 - \oint_0 d\bar{z}\,\left(\bar{\psi}_1 + i\bar{\psi}_2\right)\bar{\beta}_1\right\}\,, \tag{3.3.16}$$

  and compute that

  $$\mathcal{L}^{GN}_{\text{int}} = \left[\sum_{i=3}^{m}\psi_i\bar{\psi}_i + \sum_{a=2}^{n}(\gamma_a\bar{\beta}_a - \beta_a\bar{\gamma}_a)\right]^2 + Q\cdot B\,. \tag{3.3.17}$$

  It is furthermore not hard to see that a field is in the cohomology of Q if and only if it does not contain any contribution from the fields $\psi_1, \psi_2, \beta_1, \gamma_1$. The cohomologically reduced model is therefore the osp $(m-2|2(n-1))$ Gross-Neveu model. An interesting example is obtained if we set $m = 2n+2$, in which case the maximal reduction of the Gross Neveu model is the massless Thirring model of two real fermions, which defines a *conformal* field theory that is dual to the theory of a compactified free boson. On the other hand, the compactified free boson provides the endpoint of the cohomological reduction of the sigma models on the superspheres $S^{2n+1|2n}$. In [19], it was proposed that there exists a duality between the osp $(2n+2|2n)$ Gross-Neveu models and the sigma models on $S^{2n+1|2n}$ and what we see here supports this claim. We will spend more time in the next chapter on this hypothesis.

Chapter 4

The supersphere sigma model

The goal of this chapter is to investigate the non-linear sigma models on the superspheres $S^{2M+1|2M}$ with a special emphasis on the case $M = 1$. In the process, we present a dual description of the theory in the form of a Gross-Neveu model. The majority of this chapter was first presented in [20], which was written in collaboration with T. Quella and V. Schomerus.

4.1 General considerations

In this section, we will investigate the non-linear sigma models with target space provided by odd-dimensional superspheres $S^{2M+1|2M}$ with superdimension one. They can be viewed as direct generalizations of the more familiar bosonic spheres and admit at least three different descriptions that will be somewhat useful for us below. First, we can think of them as supermanifolds in $\mathbb{R}^{2M+2|2M}$ defined by the equation

$$\sum_{i=1}^{2M+2} x_i^2 + 2\sum_{a=1}^{M} \eta_a \eta_{a+M} \;=\; R^2 \quad . \tag{4.1.1}$$

Here, $x_i, i = 1, \ldots, 2M+2$, and $\eta_j, j = 1, \ldots, 2M$, are the bosonic and fermionic coordinates of $\mathbb{R}^{2M+2|2M}$, respectively. If we group these coordinates in a single vector X and use the matrix $\mathsf{J} \equiv \mathsf{J}_{2M+2,2M}$ defined in (2.1.18), we can write this equation simply as

$$\sum_{a=1}^{4M+2} X_a \mathsf{J}^{ab} X_b = R^2 \quad \text{with } X_a = \begin{cases} x_a \text{ for } a = 1, \ldots, 2M+2 \\ \eta_{a-2M-2} \text{ for } a = 2M+3, \ldots, 4M+2 \end{cases} . \tag{4.1.2}$$

Combining this description of the supersphere with the definition (2.1.17), it becomes evident that $S^{2M+1|2M}$ comes equipped with an osp$(2M+2|2M)$ action. In fact, the Lie superalgebra osp$(2M+2|2M)$ acts on the embedding space $\mathbb{R}^{2M+2|2M}$ through its fundamental representation, which respects relation (4.1.2) by the very definition of OSP$(2M+2|2M)$. It is furthermore not hard to see that the stabilizer of any one given point on the supersphere is isomorphic to the subsupergroup OSP (2M+1|2M) $\subset$ OSP$(2M+2|2M)$. Hence, we arrive at a second description of $S^{2M+1|2M}$ as a symmetric space:

$$S^{2M+1|2M} \;=\; \frac{\text{OSP}(2M+2|2M)}{\text{OSP (2M+1|2M)}} \quad . \tag{4.1.3}$$

From the discussion of the previous chapter, we immediately see that the cohomological reduction of minimal rank defined in section 3.1.2 will relate the sigma model on $S^{2M+1|2M}$ to the one on $S^{2M-1|2M-2}$. The minimal sector being of course the bosonic circle $S^{1|0}$, we infer that the spectrum of the non-linear sigma models on any of the superspheres possesses a subset of states isomorphic to the state space of the free boson. We will make use of this fact later on.

The final way of defining the superspheres is to solve the constraint (4.1.2) explicitly by parametrizing the supersphere $S^{2M+1|2M}$ through $2M+1$ angular coordinates φ_j and $2M$ fermionic variables η_j. In the case of the supersphere $S^{3|2}$, for example, the line element takes the following form

$$ds^2 \;=\; 2R^2(1-\eta_1\eta_2)d\eta_1 d\eta_2 + R^2(1-2\eta_1\eta_2)d\Omega_3 \tag{4.1.4}$$

where

$$d\Omega_3 \;=\; d\varphi_1^2 + \cos^2\varphi_1\, d\varphi_2^2 + \sin^2\varphi_1\, d\varphi_3^2 \tag{4.1.5}$$

is the usual line element of the 3-dimensional unit sphere and we have rescaled the fermions by a factor R. All three descriptions of the supersphere $S^{2M+1|2M}$ will be used frequently throughout the rest of this work.

Next we turn to the non-linear sigma model on the superspheres. Once more, there are different ways to introduce this theory. The most basic one is to think of it as a linear sigma model for the fields x_i and η_j with a non-linear constraint (4.1.1) on the field configurations. Another possibility is to introduce angle variables that solve (4.1.1) and to write the action directly with them. In the case of the 3-dimensional supersphere the latter takes the form

$$\begin{aligned}\mathcal{S}^\sigma \;=\; \frac{R^2}{2\pi}\int_\Sigma d^2z\, \Big[& 2(1-\eta_1\eta_2)\left(\partial\eta_1\bar\partial\eta_2 - \partial\eta_2\bar\partial\eta_1\right) \\ & + (1-2\eta_1\eta_2)\left(\partial\varphi_1\bar\partial\varphi_1 + \cos^2\varphi_1\,\partial\varphi_2\bar\partial\varphi_2 + \sin^2\varphi_1\,\partial\varphi_3\bar\partial\varphi_3\right)\Big]\ .\end{aligned} \tag{4.1.6}$$

The coupling constant in front of the action is determined by the radius R of $S^{3|2}$. For the sigma model on the purely bosonic three dimensional sphere the coupling R runs and in order for the flow to end in a non-trivial fixed-point one must add a WZ term as shown in [54], but the presence of the two fermionic directions changes the situation drastically. As was proven in [16], the β-function of the non-linear sigma model on $S^{2M+1|2M}$ is identical to the one on a sphere S^d, whose dimension $d = 2M+1-2M = 1$ is given by the difference between the number of bosonic and fermionic coordinates. Consequently, the β-function vanishes for the theory on $S^{2M+1|2M}$, implying that the model (4.1.6) and its higher dimensional generalizations define a family of conformal field theories at central charge $c = 1$ with continuously varying exponents. This confirms *independently* what was argued in subsection 3.3.1 of the previous chapter, namely that the sigma models on $S^{2M+1|2M}$ must be conformally invariant since they depend on *only* *one* coupling constant and since their minimal sector via cohomological reduction is a free theory.

Of course, unlike the sigma model on $S^1 \cong \mathrm{U}(1)$, the theory defined by the action (4.1.6) is not free. For large radii, the model is weakly coupled and its properties may by studied perturbatively. But as we pass to a more strongly curved background, computing quantities as a function of the radius R may seem like a very daunting task, made even harder by the lack of symmetry to work with. As a conformal field theory, the sigma model on the supersphere $S^{3|2}$ possesses the usual chiral Virasoro symmetries. But for a model with multiple bosonic coordinates the two sets of chiral Virasoro generators are not sufficient to make the theory rational. In addition, there is a single set of global $\mathrm{osp}(4|2)$ generators, whose Noether currents however, fail to be chiral for generic points in the moduli space, so that we lack affine $\widehat{\mathrm{osp}}(4|2)$. Deprived of the protection of current algebra symmetries, we can make no use of the usual algebraic tools of conformal field theory and so have to undertake a rather different route.

Years of experience with sigma models have show that they often possess interesting dual descriptions. The simplest such duality is the one between the free compactified boson and the massless Thirring model. Let us recall that the latter involves two real fermions ψ_1 and ψ_2 and the following action

$$\mathcal{S}^{\mathrm{Th}}_{m=0} = \frac{1}{2\pi}\int_\Sigma d^2z \left[\sum_{i=1}^{2}\left(\psi_i\bar\partial\psi_i + \bar\psi_i\partial\bar\psi_i\right) + g^2\left(\psi_1\bar\psi_2 - \psi_2\bar\psi_1\right)^2\right] \tag{4.1.7}$$

where the compactification radius R of the free boson is related to the coupling g through $R^2 = 1 + g^2$. By analogy, one may hope to uncover a dual description of the model on the supersphere $S^{2M+1|2M}$ that becomes weakly coupled around some finite value of the radius, deep in the strongly curved regime. Indeed, such a dual description was proposed recently, by Candu and Saleur in [19], where they claimed that there exists one special value R_0 of the radius at which the non-linear sigma model on $S^{2M+1|2M}$ can be described as a non-interacting Gross-Neveu model involving $2M+2$ real fermions ψ_i along with M bosonic $\beta\gamma$ systems γ_a and β_a,

$$\mathcal{S}^{\mathrm{GN}}_{\mathrm{free}} = \frac{1}{2\pi}\int_\Sigma d^2z \Bigg[\sum_{i=1}^{2M+2}\left(\psi_i\bar\partial\psi_i + \bar\psi_i\partial\bar\psi_i\right) + 2\sum_{a=1}^{M}\left(\beta_a\bar\partial\gamma_a + \bar\beta_a\partial\bar\gamma_a\right)\Bigg] . \tag{4.1.8}$$

All the fields appearing in this theory possess conformal weight $h_i = h_a = 1/2$ so that the central charge is $c = \frac{1}{2}(2M+2) - M = 1$. At this point in the moduli space, the theory possesses two commuting sets of chiral $\mathrm{osp}(4|2)$ currents $J^\mu \equiv J^\mu(z)$ and $\bar J^\mu \equiv \bar J^\mu(\bar z)$. The affine symmetry is broken down to a global $\mathrm{osp}(4|2)$ symmetry by the following $\mathrm{osp}(4|2)$ invariant marginal deformation

$$\begin{aligned}\mathcal{S}^{\mathrm{GN}}_{\mathrm{int}} &= \frac{g^2}{2\pi}\int_\Sigma d^2z J_\mu(z)\Omega(\bar J^\mu(\bar z)) \\ &= \frac{g^2}{2\pi}\int_\Sigma d^2z \left[\sum_{i=1}^{2M+2}\varpi_i\psi_i\bar\psi_i + \sum_{a=1}^{M}(\gamma_a\bar\beta_a - \beta_a\bar\gamma_a)\right]^2 .\end{aligned} \tag{4.1.9}$$

Here, Ω is a particular automorphism of the $\mathrm{osp}\,(2M+2|2M)$ current algebra which leaves a subalgebra $\mathrm{osp}\,(2M+1|2M)$ invariant. It will be spelled out explicitly below. The numbers ϖ_i are given by[1] $\varpi_1 = -1$ and $\varpi_i = 1$ for $i \neq 1$. The full theory $\mathcal{S}^{\mathrm{GN}} = \mathcal{S}^{\mathrm{GN}}_{\mathrm{free}} + \mathcal{S}^{\mathrm{GN}}_{\mathrm{int}}$ is claimed to be equivalent to the supersphere sigma model with the two coupling constants related by $R^2 = 1 + g^2$. The equivalence is a strong-weak coupling duality since $\mathcal{S}^{\mathrm{GN}}$ becomes weakly coupled as the radius approaches $R_0 = 1$. Note that this duality is a direct generalization of the relation between the compactified free field and the massless Thirring model. The prescription relates one real fermion to each bosonic coordinate of the embedding space $\mathbb{R}^{2M+2|2M}$ and one pair of $\beta\gamma$ bosonic ghosts to each pair of fermionic directions. Observe, however, that the duality between the supersphere sigma models and the Gross-Neveu models is one between interacting conformal field theories, which makes it is much less trivial than its purely bosonic counterpart.

The plan of this chapter goes as follows. First, we shall look at the sigma model on the circle, which will provide us some insights as to the general features of the model. Then we shall study the next simplest case, namely the sigma model (4.1.6) for the supersphere $S^{3|2}$ and determine its exact spectrum in the limit of infinite radius. For simplicity, we shall also restrict to the partition function on a strip with Neumann boundary conditions imposed along both boundaries. After a detailed discussion of the low lying states, we present a closed formula for the full partition function in (4.3.21). The latter is then decomposed explicitly into the contributions coming from states which transform in the same representation Λ under the global $\mathrm{osp}\,(4|2)$. We then devote time to the theory (4.1.8) and its deformation by the term (4.1.9). In particular, we study the bulk and boundary spectrum of the free field theory. One of the resulting boundary partition functions is then expanded explicitly in terms of $\mathrm{osp}\,(4|2)$ characters. This allows us to compare with the spectrum of the sigma model at finite radius, using some of the tools developed in [34]. In the last part of the chapter we spend some time investigating the $\mathcal{N} = 1$ worldsheet extension of the supersphere models and compute the boundary spectra of volume filling branes in the infinite volume limit.

4.2 The case of the circle

It is illuminating to first look the the simplest prototype of the superspheres $S^{2M+1|2M}$, namely the circle which we obtain by setting $M = 0$. The sigma model action for the two constrained bosonic coordinates is

$$\mathcal{S} = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2 z\, \left(\partial X_1 \bar{\partial} X_1 + \partial X_2 \bar{\partial} X_2\right) = \frac{R^2}{4\pi}\int_\Sigma \mathrm{d}^2 z\, \partial\varphi\bar{\partial}\varphi\ , \tag{4.2.1}$$

[1]Let us note that the signs ϖ_i in the iteraction term are directly linked to the automorphism Ω. These signs were missing in the original formulation of the conjecture by Candu and Saleur [19]. They are irrelevant for $M = 0$ but play a certain role when $M \geq 1$.

where we have solved the constraints by introducing $X_1 = R\cos(\varphi)$ and $X_2 = R\sin(\varphi)$. Thus, the free boson φ is compactified at radius 2π irrespective of the radius R of the circle, so that the spectrum of the theory does not become continuous in the limit of infinite volume. The operator product expansion of φ is

$$\varphi(z,\bar{z})\varphi(w,\bar{w}) \sim -\frac{1}{R^2}\log|z-w|^2 , \tag{4.2.2}$$

so that the properly normalized holomorphic part of the energy momentum tensor becomes

$$T(z) = -\frac{R^2}{2}\left(\partial\varphi\partial\varphi\right)(z) . \tag{4.2.3}$$

Because of the constant compactification radius of the free boson, the vertex operators are $e^{in\varphi}$, where n is an integer. Their conformal dimension is $\frac{n^2}{2R^2}$ and they have a charge n under the $U(1)$ current $J = iR^2\partial\varphi$. Imposing Neumann boundary conditions, one obtains the following partition function

$$\mathsf{Z}^{\sigma}_{\mathfrak{N},0}(q,z) := \mathrm{tr}_{\mathcal{H}}\left(z^{J_0}q^{L_0-\frac{c}{24}}\right) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2R^2}} . \tag{4.2.4}$$

Here the subscript $\mathfrak{N}$ denotes Neumann boundary conditions and 0 refers to the fact that we are dealing with the $M=0$ case of supersphere $S^{2M+1|2M}$. Furthermore the superscript σ is there to distinguish this case from the Gross-Neveu one that we consider later on. The action (4.2.1) has the further symmetry $X_1 \mapsto X_1$, $X_2 \mapsto -X_2$, which, once the constraints have been resolved, corresponds to $\varphi \mapsto -\varphi$. The full symmetry in the boundary is thus O(2) and the partition function decomposed in characters of O(2) simple modules becomes

$$\begin{aligned}\mathsf{Z}^{\sigma}_{\mathfrak{N},0}(q,z) = q^{-\frac{1}{24}}\Bigg[&\frac{1}{2}\left(\frac{1}{\phi(q)}+\frac{1}{\phi_\star(q)}\right)\chi_{\varnothing} + \frac{1}{2}\left(\frac{1}{\phi(q)}-\frac{1}{\phi_\star(q)}\right)\chi_{\Box\wedge\Box}\\ &+\sum_{n=1}^{\infty}\frac{q^{\frac{n^2}{2R^2}}}{\phi(q)}\chi_n\Bigg] ,\end{aligned} \tag{4.2.5}$$

where the characters of the symmetric traceless tensor product of the fundamental representation $\Box$ are $\chi_n := z^n + z^{-n}$. The only difference between the trivial $\varnothing$ and the adjoint $\Box\wedge\Box$ representation of O(2) is the action of the reflection group element which is 1 in the first case and -1 in the second. We refer to C.1.1 for details concerning the functions ϕ and $\phi_\star$.

In our computation so far, we have suppressed the notation of Planck's constant. In the path integral formulation, $\hbar$ appears in the measure that weighs the different paths, namely as

$$\mathsf{Z} = \int [d\mu] e^{\frac{1}{\hbar}S_c} , \tag{4.2.6}$$

where S_c is the classical action. One can take the classical limit in the path integral approach by letting $\hbar$ tend to zero, which has the effect of cancelling out the contributions of all paths, except for the ones that are solutions of the equations of motion and thus extremize the action. In the case of the superspheres, as one sees from the equations (4.2.1) and (4.1.6), the coupling constant, or equivalently the radius, can be put in front of the action and redefined so as to absorb $\hbar$. If we do so, we see that taking the limit $R \to \infty$ produces equivalent results to sending $\hbar$ to zero, so that *the limit of infinite volume corresponds to the semi-classical regime of the theory.*

If we take this limit in equation (4.2.5), we see that the number of states with vanishing energy becomes infinite since the conformal dimension of the vertex operators $e^{in\varphi}$ tends to zero. The partition function then becomes

$$\mathsf{Z}^{\sigma}_{\mathfrak{N},0}(q,z) \xrightarrow{R\to\infty} \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} z^n = \frac{1}{\eta(q)} \left(1 + \sum_{n=1}^{\infty} \chi_n \right) . \tag{4.2.7}$$

The vertex operators can be identified with functions on the circle and as such they provide a basis for the space $L^2(S^1)$. It is a *generic* feature of sigma models that the space of ground states of a *volume filling* brane in the semi-classical limit can be identified with the space of functions on the target space. This is easily understood qualitatively, since as the volume grows, the curvature of the space diminishes and since the energy of the vertex operators scales roughly with the curvature, we see the degeneracy appearing in the limit. More details can be found in [19]. The partition function (4.2.7) can also be derived in a way that illustrates some of the methods that we will use in section 4.3. As we argued, the states of zero energy can be identified with the functions $e^{in\varphi}$ on the circle. The excited states can then be obtained by acting with ∂, the worldsheet derivative along the brane, on the $e^{in\varphi}$, where each application of the derivative increases the conformal dimension by one. Thus, a state of energy N and U(1) charge n will be of the form $e^{in\varphi} \times \left(c_1\partial^N\varphi + c_2\partial^{N-1}\varphi\partial\varphi + c_3\partial^{N-2}\varphi\partial\varphi\partial\varphi + \cdots\right)$ for some constants c_i. Therefore, it is easy to see that the number of fields of energy N and U(1) charge n will be independent of n and equal to the number of integer partitions of the number N. Since the generating function of the partitions of integers is $\phi(q)^{-1}$, we see that equation (4.2.7) is obtained by multiplying the partition function $\sum_n z^n$ of the states of zero energy by $\eta(q)^{-1}$, which implements the counting of the number of derivatives acting on the functions. This way of constructing the partition function will be very useful in the next section.

Besides the infinite volume limit, another important approximation we can take is called the *particle limit.* In this case, we change not the target space, but the world sheet. More specifically, if Σ is a two dimensional cylinder for which the spacial coordinate σ is compactified, then we let the radius of compactification tend to zero. Expanding our fields in Fourier series in σ, we see that the smaller this radius gets, the higher the energy of the non trivial Fourier modes becomes, so that in the limit of zero radius only the constant term in the expansion survives. Thus, as illustrated in figure 4.1, we are effectively left with a one-dimensional worldsheet and the sigma model becomes a

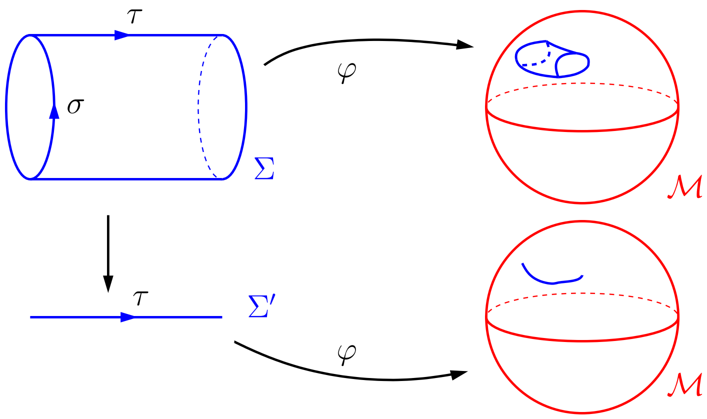

Figure 4.1: The particle limit is obtained by replacing the two dimensional worldsheet by a line.

quantum mechanical system whose Hamiltonian is *proportional to the Laplacian* of the target space. The particle limit has the obvious effect of removing all string excitations, leaving only the vertex operators in the theory, meaning that the partition function becomes

$$\mathsf{Z}^{\sigma}_{\mathfrak{N},0}(q,z) \longrightarrow q^{-\frac{1}{24}} \sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2R^2}} \ . \tag{4.2.8}$$

We can of course take both limits at the same time, as shown in figure 4.2.

$$\begin{array}{ccc}
\frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2R^2}} & \xrightarrow{\infty-\text{volume limit}} & \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} z^n \\
\Big\downarrow \text{particle limit} & & \Big\downarrow \text{particle limit} \\
q^{-\frac{1}{24}} \sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2R^2}} & \xrightarrow{\infty-\text{volume limit}} & q^{-\frac{1}{24}} \sum_{n\in\mathbb{Z}} z^n
\end{array}$$

Figure 4.2: Various limits of the partition function for Neumann boundary conditions of the S^1 sigma model.

As a preparation for some of the arguments that we will make for the $S^{3|2}$ sigma model, we read from equation (4.2.4) that the conformal dimensions of the boundary fields can be expressed

$$h_n(R) = \frac{n^2}{2R^2} = h_0(R_0) + \mathsf{Cas}(\Lambda_n)\frac{1}{2}\left(\frac{1}{R^2} - \frac{1}{R_0^2}\right) \ , \tag{4.2.9}$$

where, $h_0(R_0) = \frac{n^2}{2R_0^2}$, Λ_n is the weight of the n-fold symmetric traceless representation and $\mathsf{Cas}(\Lambda_n) = n^2$ is the normalized quadratic Casimir of O(2). We thus see that, just as in formula (2.5.26) for WZW models, the change of the conformal dimensions away from a given radius R_0 is completely determined by the transformation properties of the fields and is given by the value of the quadratic Casimir times some *universal function*. We will spend most of the remaining chapter arguing that the same behavior is present for all $S^{2M+1|2M}$ sigma models.

Thus ends our short presentation of the S^1 sigma model.

4.3 Semi-classical limit for $S^{3|2}$

In this section we shall focus on the sigma model for the supersphere $S^{3|2}$ with large radius R. At the point $R = \infty$ we can compute the partition function on a strip with Neumann boundary conditions on both sides. The two main ingredients are the exact analysis of the minisuperspace spectrum on $S^{3|2}$ in subsection 4.3.1 and a good control of the combinatorics that determine the field theoretic spectrum at $R = \infty$. The latter will be explained in subsection 4.3.2. The complete spectrum is finally decomposed into finite dimensional representations of the global symmetry algebra osp (4|2) in the third subsection.

4.3.1 Particle on the supersphere $S^{3|2}$

As state before, analyzing the particle limit of a non-linear sigma model amounts to understanding the behavior of the Laplacian on the target space. The Laplacian on the supersphere $S^{3|2}$ was analyzed in full detail by Candu and Saleur [19], but we shall provide a new derivation that is particularly well suited for the discussion in the following subsections.

As a warm-up exercise, let us recall the spectrum of the Laplacian on a three dimensional sphere S^3, whose space of functions carries an action of so (4) ≅sl(2)⊕sl(2). Therefore, eigenfunctions of the Laplacian on S^3 are organized in finite dimensional multiplets of sl(2)⊕sl(2). According to the Peter-Weyl theory for SU(2) ≅ S^3, there is one such multiplet φ_m for each integer $m \in \mathbb{N}$. It has dimension $(m+1)^2$ and transforms in the representation $(\frac{m}{2}, \frac{m}{2})$. The eigenvalues of the Laplacian on the multiplet φ_m is given by $m(m+2)$. For the supersphere $S^{3|2}$ we expect very similar results, with the exception that the multiplicities should roughly exceed those of the bosonic model by a factor of four, since we have two additional fermionic degrees of freedom.

Before we extend these thoughts to the supersphere, however, let us mention a few facts on the Lie superalgebra osp (4|2). Its bosonic subalgebra is 9-dimensional and it consists of three commuting copies of sl(2). This implies that irreducible representations $[j_1, j_2, j_3]$ of osp (4|2) are labeled by three spins j_i. In these representations the quadratic Casimir element takes the value

$$\mathsf{Cas}(j_1, j_2, j_3) \;=\; -4j_1(j_1 - 1) + 2j_2(j_2 + 1) + 2j_3(j_3 + 1) \;\; . \tag{4.3.1}$$

A generic, that is *typical*[2], representation possesses dimension

$$\dim[j_1, j_2, j_3] \;=\; 16(2j_1+1)(2j_2+1)(2j_3+1) \;\; . \tag{4.3.2}$$

The representations of osp (4|2) that appear in the spectrum of the Laplacian on the supersphere $S^{3|2}$ are however not generic. On the supersphere, wave functions are organized in osp (4|2) multiplets ϕ_m with $m \in \mathbb{N}$. The first multiplet ϕ_0 consists of a single function, namely the constant $\phi_0 = 1$, which transforms in the trivial one dimensional representation $[0, 0, 0]$. For positive values of m, the multiplet ϕ_m transforms in the irreducible representation $[\frac{1}{2}, \frac{m-1}{2}, \frac{m-1}{2}]$ of osp (4|2). Consequently, the space $\mathcal{H}_0$ of square integrable functions on the supersphere $S^{3|2}$ decomposes as follows,

$$\mathcal{H}_0 \;\cong\; [0,0,0] \;\oplus\; \bigoplus_{m=1}^{\infty} \left[\frac{1}{2}, \frac{m-1}{2}, \frac{m-1}{2}\right] \;\;=\; \bigoplus_{m=0}^{\infty} [\Lambda_{m,0}] \;. \tag{4.3.3}$$

Here we have also introduced the symbol $[\Lambda_{m,0}]$ such that

$$[\Lambda_{0,0}] = [0,0,0] \qquad [\Lambda_{m+1,0}] = \left[\frac{1}{2}, \frac{m}{2}, \frac{m}{2}\right] \;. \tag{4.3.4}$$

In particular $[\Lambda_{1,0}] = \Box$ is the fundamental representation and the $[\Lambda_{n,0}]$ are the symmetric traceless tensor powers of it. According to equation (4.3.1), the Laplacian takes the values m^2 on $[\Lambda_{m,0}]$. The quadratic dependence on m is similar to the one we find on the bosonic sphere, while on the other hand the degeneracies here are much larger. In fact, upon restriction to the bosonic subalgebra, the eigenspaces of the Laplacian decompose according to

$$\left[\frac{1}{2}, \frac{k}{2}, \frac{k}{2}\right]\Big|_{\mathrm{sl}(2)\oplus\mathrm{sl}(2)\oplus\mathrm{sl}(2)} \;\cong\; \left(\frac{1}{2}, \frac{k}{2}, \frac{k}{2}\right) \oplus \left(0, \frac{k+1}{2}, \frac{k+1}{2}\right) \oplus \left(0, \frac{k-1}{2}, \frac{k-1}{2}\right)$$

for $k = m - 1 \geq 1$. When $k = 0$, the last term must be omitted. The formula implies that the dimension of the representation $[\Lambda_{k,0}]$ is given by $\dim[\Lambda_{k,0}] = 4k^2 + 2$ for $k \geq 1$. This is roughly four times as large as the dimension of the eigenspaces on the bosonic sphere S^3, as one would expect.

As an instructive exercise, we shall prove the decomposition (4.3.3). To this end, let us collect the bosonic coordinate functions $X_i := x_i, i = 1, \ldots, 4$ and the fermionic generators $X_{4+i} := \eta_i$ into a single multiplet X. We recall that the six functions X_i are subject to the constraint (4.1.1), which may be recast into the more covariant form $X_a X_b \mathsf{J}^{ab} = R^2$. The multiplet X transforms in the fundamental representation $[\Lambda_{1,0}] = \left[\frac{1}{2}, 0, 0\right]$ of osp (4|2). When we restrict from osp (4|2) to its bosonic subalgebra, X splits into a 4-dimensional multiplet in the $(\frac{1}{2}, \frac{1}{2})$ representation of so (4) $\cong$ sl(2)$\oplus$sl(2) and a 2-dimensional multiplet in the $(\frac{1}{2})$ representation of sp(2) $\cong$ sl(2). While the former is

[2]See appendix B

spanned by the bosonic coordinate functions x_i, the latter consists of the odd elements η_i.

The algebra $\mathcal{H}_0$ of functions on $S^{3|2}$ is generated by the six coordinates X_i, meaning that every square integrable function can be arbitrarily well approximated by a polynomial in X_i. The space of polynomials comes with an integer grading given by the degree of homogeneity. Since the homogeneous polynomials transform in the graded symmetric tensor product of the fundamental representation $[\Lambda_{1,0}]$, one might be inclined to identify the direct sum $\mathsf{Sym}[\Lambda_{1,0}] = \bigoplus_n [\Lambda_{1,0}]^{\circ n}$ of all graded symmetric tensor powers of the fundamental representation with the space $\mathcal{H}_0$. Such an identification, however, would disregard the defining equation (4.1.1) of the supersphere. The constraint (4.1.1) generates an ideal in the symmetric tensor algebra $\mathsf{Sym}[\Lambda_{1,0}]$ that has to be divided out in order to avoid overcounting of states. The two-fold symmetric tensor power of the fundamental representation, for example, is given by $[\Lambda_{1,0}]^{\circ 2} = [0,0,0] \oplus [\Lambda_{2,0}]$. The constraint (4.1.1) identifies the multiplet $[0,0,0]$ with the constant function. The latter has been counted already by the very first term $[\Lambda_{1,0}]^{\circ 0} = [0,0,0]$. Consequently, when considering the space of homogeneous polynomials in X_i up to degree m, we have to quotient out the subspace of polynomials that contain the factor $X_a X_b \mathsf{J}^{ab}$, which is isomorphic to the space of homogeneous polynomials of degree less or equal to $m-2$. Thereby we are led to the following expression for $\mathcal{H}_0$,

$$\begin{aligned}\mathcal{H}_0 &= \lim_{N\to\infty} \left(\bigoplus_{m=0}^{N} [\Lambda_{1,0}]^{\circ m} \right) \Big/ \left(\bigoplus_{m=0}^{N-2} [\Lambda_{1,0}]^{\circ m} \right) = \bigoplus_{m=0}^{\infty} [\Lambda_{m,0}] \\ &= [0,0,0] \oplus \bigoplus_{k=0}^{\infty} \left[\frac{1}{2}, \frac{k}{2}, \frac{k}{2} \right] \end{aligned} \quad (4.3.5)$$

where we have used the tensor product decomposition[3] $[\Lambda_{1,0}]^{\circ m} \cong \bigoplus_{i=0}^{\lfloor m/2 \rfloor} [\Lambda_{m-2i,0}]$.

Before we move on, let us construct the partition function for the non-linear sigma model on the supersphere in the *combined particle and infinite volume* limit, that is the quantity:

$$\mathsf{Z}_0 \equiv \mathsf{Z}_0(z_1, z_2, z_3) := \mathrm{tr}_{\mathcal{H}_0}(z_1^{H^1} z_2^{H^2} z_3^{H^3}) , \quad (4.3.6)$$

where H^i are the three Cartan generators and the trace is taken evaluated in the space $\mathcal{H}_0$ of square integrable functions on the supersphere $S^{3|2}$. The results we sketched in the previous paragraphs imply that

$$\mathsf{Z}_0 = 1 + \sum_{m=0}^{\infty} \chi_{[\frac{1}{2},\frac{m}{2},\frac{m}{2}]}(z_1, z_2, z_3) \quad (4.3.7)$$

$$\text{where} \quad \chi_{[\frac{1}{2},\frac{m}{2},\frac{m}{2}]}(z_1, z_2, z_3) = \chi_{(\frac{1}{2},\frac{m}{2},\frac{m}{2})} + \chi_{(0,\frac{m+1}{2},\frac{m+1}{2})} + \chi_{(0,\frac{m-1}{2},\frac{m-1}{2})} \ . \quad (4.3.8)$$

In the second line the last term should be omitted for $m=0$ and the character $\chi_{(j_1,j_2,j_3)} = \prod_i \chi_{j_i}(z_i)$ denotes a product of bosonic sl(2) characters. The partition function Z_0 can

[3] By $\lfloor x \rfloor$ we mean the floor function of x.

be written in a different form that mimics our proof of the formula (4.3.3). To this end, let us consider the module $\mathsf{Sym}[\Lambda_{1,0}]$, defined as the direct sum of all symmetric tensor powers of $[\Lambda_{1,0}]$. We think of it as being generated by four bosonic coordinates in the $(\frac{1}{2},\frac{1}{2})$ representation of sl(2)⊕sl(2) ≅ so(4) along with the two fermionic ones in the $(\frac{1}{2})$ representation of sl(2) ≅ sp(2). On $\mathsf{Sym}[\Lambda_{1,0}]$ we introduce the number operator N that counts the number of bosonic and fermionic coordinate functions in a given monomial. Since there are no non-trivial relations in $\mathsf{Sym}[\Lambda_{1,0}]$ we can easily compute

$$\mathsf{Z}^S(t) \;=\; \mathrm{tr}_{\mathsf{Sym}[\Lambda_{1,0}]}(t^N z_1^{H^1} z_2^{H^2} z_3^{H^3}) \;=\; \frac{(1+z_1^{\frac{1}{2}}t)(1+z_1^{-\frac{1}{2}}t)}{(1-z_2^{\frac{1}{2}}z_3^{\frac{1}{2}}t)(1-z_2^{\frac{1}{2}}z_3^{-\frac{1}{2}}t)(1-z_2^{-\frac{1}{2}}z_3^{\frac{1}{2}}t)(1-z_2^{-\frac{1}{2}}z_3^{-\frac{1}{2}}t)}\;.$$

Multiplying this quantity with $(1-t^2)$ implements the constraint (4.1.1) on the level of generating functions. We can then remove t by sending it to $t \to 1$. The result is a rather elegant new formula for the partition function Z_0,

$$\mathsf{Z}_0(z_1,z_2,z_3) \;=\; \lim_{t\to 1}\left[(1-t^2)\mathsf{Z}^S(t;z_1,z_2,z_3)\right]\;. \tag{4.3.9}$$

If the quotient is expanded in a Taylor series and expressions are reorganized into characters of osp (4|2) we recover our previous result (4.3.7).

4.3.2 The complete boundary spectrum

Moving away from the particle limit, but keeping the volume infinite, we can generalize the results of the previous section to account for the string excitations. At the infinite radius point, the fields of the theory are easy to list and their weights agree with their classical values, as we have argued in our section concerning the circle. For simplicity, we shall study the boundary spectrum of a volume filling brane, i.e. with *Neumann boundary conditions* imposed on all fields of the model. This makes it sufficient to consider only the derivative ∂_u along the boundary, rather than two world-sheet derivatives ∂ and $\bar{\partial}$. From now on, the letters $x_i = x_i(u), \eta_a = \eta_a(u)$ and $X_i = X_i(u)$ shall denote boundary fields rather than coordinate functions.

Beginning to analyze the space $\mathcal{H}$ of boundary fields we quickly realize that this space is spanned by monomials Φ of the form

$$\Phi \;=\; \prod_{i_0} X_{i_0} \prod_{i_1} \partial X_{i_1} \prod_{i_2} \partial^2 X_{i_2} \cdots \;. \tag{4.3.10}$$

The number of factors involving no, one, two, etc. derivatives $\partial = \partial_u$ of the fundamental fields is arbitrary. Let us stress at this point already that the defining relation (4.1.1) of the supersphere imposes many relations between monomials of the form (4.3.10). The space $\mathcal{H}$, comes equipped with an integer grading, i.e. $\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_n$ is spanned by monomials Φ with a total number n of derivatives. The expression $X_a\partial X_b\partial^4 X_c$, for example, is an element of $\mathcal{H}_5$.

Associated with the integer grading of the state space $\mathcal{H}$ there is a corresponding decomposition of the partition function

$$\mathsf{Z}(q, z_1, z_2, z_3) \;=\; \mathrm{tr}_{\mathcal{H}}(q^{L_0-\frac{c}{24}} z_1^{H^1} z_2^{H^2} z_3^{H^3}) \;=\; q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \mathsf{Z}_n(z_1, z_2, z_3)\, q^n \;. \qquad (4.3.11)$$

The coefficients $\mathsf{Z}_n \equiv \mathsf{Z}_n(z_i)$ are infinite linear combinations of osp (4|2) characters. The formula (4.3.9) for Z_0 that was discussed in the previous subsection, can in the present context be seen as encoding all information on the osp (4|2) transformation law of fields with conformal weight $h = 0$. These states are in one-to-one correspondence with functions on the supersphere $S^{3|2}$ (recall that we are working at $R = \infty$).

Let us now turn to states involving a single derivative ∂. Since $\mathcal{H}_1$ is built from fields of the form $\phi_n(X_i)\partial X_i$, where $\phi_n \in \mathcal{H}_0$ and since the ∂X_i transform in $[\Lambda_{1,0}]$, one might at first sight suspect that $\mathsf{Z}_1' := \mathsf{Z}_0 \times \chi_{\Lambda_{1,0}}$ coincides with Z_1. But this is not true since it actually counts many fields twice. So far, we have not accounted for the derivative of the supersphere relation (4.1.1). Taking the derivative of this constraint we find

$$\sum_{i,j} X_i \partial X_j \mathsf{J}^{ij} \;=\; 0 \;.$$

This additional condition tells us to subtract Z_0 from Z_1'. Hence we are led to the identification $\mathsf{Z}_1 = \mathsf{Z}_0(\chi_{\Lambda_{1,0}} - \chi_{\Lambda_{0,0}})$ and a simple computer program can decompose this product into characters of osp (4|2), leading to

$$\mathsf{Z}_1 \;=\; \sum_{k=0}^{\infty} \Big(\chi_{[1,\frac{k}{2},\frac{k}{2}]} + \chi_{[\frac{1}{2},\frac{k}{2},\frac{k}{2}]} \Big) \quad . \qquad (4.3.12)$$

In order to gain some more familiarity with the state counting we invite the reader to construct the contribution Z_2 of fields with two derivatives to the total partition function. The answer is given by

$$\begin{aligned} \mathsf{Z}_2 \quad = \quad & \chi_{[0,0,0]} + 2 \sum_{k=0}^{\infty} \chi_{[\frac{1}{2},\frac{k}{2},\frac{k}{2}]} + \chi_{[1,0,0]} \\ + \quad & \sum_{k=1}^{\infty} \Big(\chi_{[1,\frac{k+1}{2},\frac{k-1}{2}]} + \chi_{[1,\frac{k-1}{2},\frac{k+1}{2}]} + 2\chi_{[\frac{1}{2},\frac{k}{2},\frac{k}{2}]} + 2\chi_{[1,\frac{k}{2},\frac{k}{2}]} \Big) \quad . \end{aligned} \qquad (4.3.13)$$

Instead of explaining this formula we shall turn to the higher subtraces Z_i right away. To begin with, let us enumerate expressions in which no field appears without derivative and where the total degree of the derivatives adds up to n. There are $p(n)$ of these terms, where $p(n)$ is the *number of partitions of the integer* n. We shall denote the set of partitions by $P(n)$ and think of their elements as sequences

$$\mu := (\mu_i, i = 1, 2, 3, \ldots) \quad \text{such that} \quad \sum_{i=1}^{\infty} i\mu_i = n \;. \qquad (4.3.14)$$

With $n = 3$, for example, we have to consider terms involving $\partial^3 X_i$, $\partial^2 X_i \partial X_j$ and $\partial X_i \partial X_j \partial X_k$ corresponding to the sequences $(\mu_1, \mu_2, \mu_3) = (0,0,1), (1,1,0)$ and $(3,0,0)$, respectively. In our notations we shall suppress the infinite number of zero entries to the right of the last non-zero one. To each partition $\mu \in P(n)$, we associate the trace $\chi_{\Lambda_{1,0}^{\otimes\mu}}$ over the space $[\Lambda_{1,0}]^{\circ\mu_1} \otimes [\Lambda_{1,0}]^{\circ\mu_2} \cdots$,

$$\chi_{\Lambda_{1,0}^{\otimes\mu}}(z_1, z_2, z_3) \;:=\; \prod_{i=1}^{\infty} \chi_{\Lambda_{1,0}^{\circ\mu_i}}(z_1, z_2, z_3) \quad . \tag{4.3.15}$$

The factors on the right hand side involve traces over the μ_i^{th} symmetric tensor product of the fundamental representation $[\Lambda_{1,0}]$. Such factors arise from the product of μ_i derivatives of order i of the fundamental field multiplet. Let us now set

$$\mathsf{Z}'_n := \mathsf{Z}_0 \sum_{\mu \in P(n)} \chi_{\Lambda_{1,0}^{\otimes\mu}} \tag{4.3.16}$$

to be Z_0 multiplied with the sum of the $p(n)$ traces (4.3.15). Clearly, Z'_n is not the same as Z_n. In fact, we still have to correct for some overcounting, since we have to subtract all possible derivatives of degree up to n of the supersphere relations (4.1.1). Each one of the $p(n)$ partitions $\mu \in P(n)$ has to be investigated on its own in order to understand which relations apply to it. Suppose that for a given partition μ, the entry μ_j does not vanish. This means that the corresponding fields contain a factor $\partial^j X_a$. Hence, there exist relations between such fields that arise from the j^{th} derivative of the supersphere relation (4.1.1). These must be removed. We may formalize this prescription by introducing the special partitions ϵ^i which have a single entry $\epsilon^i_i = 1$ in the i^{th} position and are zero otherwise. The sequence ϵ^i is an element of $P(i)$. Let us also denote by $\mu - \epsilon^i$ the partition from $P(n-i)$ that is obtained by subtracting the entries. If the resulting sequence contains a negative entry, i.e. if $\mu_i = 0$, then we set $\chi_{\Lambda_{1,0}^{\otimes(\mu-\epsilon^i)}} = 0$. With these notations, we can now formalize our resolution for the issue of overcounting. Taking into account the constraints imposed by the i^{th} derivative of (4.1.1) amounts to subtracting from Z'_n all functions of the form $\mathsf{Z}_0 \chi_{\Lambda_{1,0}^{\otimes(\mu-\epsilon^i)}}$. Here, $\mu \in P(n)$ and i runs through all integers $i = 1, 2, \ldots$ such that $\mu_i \neq 0$. After removing all these terms from Z'_n we realize that we actually overdid things with our correction. In fact we have deleted those expressions for which two ore more relations are simultaneously fulfilled, so that we need to put them back in. Thus, we must add all the terms $\mathsf{Z}_0 \chi_{\Lambda_{1,0}^{\otimes(\mu-\epsilon^i-\epsilon^j)}}$ with $i < j$, as represented for a simple example in figure 4.3. The resulting expression overcounts those polynomials that obey three different relations, etc. A simple induction leads to the following expression for Z_n

$$\mathsf{Z}_n \;=\; \mathsf{Z}_0 \sum_{\mu \in P(n)} \left(\chi_{[\frac{1}{2},0,0]^{\otimes\mu}} - \sum_{i=1}^{n} \chi_{[\frac{1}{2},0,0]^{\otimes(\mu-\epsilon^i)}} + \sum_{i<j=1}^{n} \chi_{[\frac{1}{2},0,0]^{\otimes(\mu-\epsilon^i-\epsilon^j)}} - \cdots \right) \tag{4.3.17}$$

All notations that are used in this expression have been introduced in the preceding paragraph. We have placed the subscript $[\Lambda_{1,0}] = [\frac{1}{2}, 0, 0]$ back on the symbol χ to

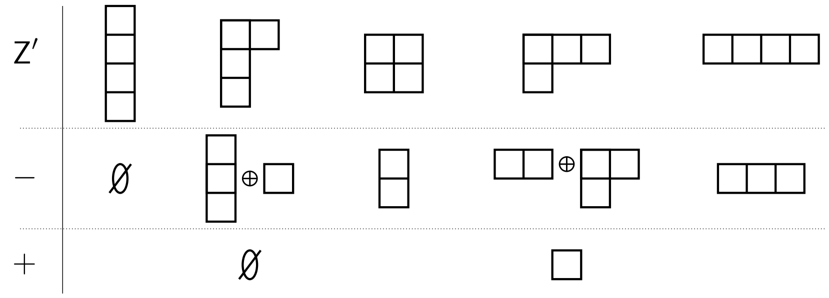

Figure 4.3: The different partitions for $n = 4$ that make up Z' together with the corresponding subtractions and corrections. We remind that $\square$ denotes the fundamental representation and $\varnothing$ the trivial one.

emphasize the relation to the fundamental multiplet. The reader is invited to check that our general formula for Z_n reproduces the previous expressions (4.3.7,4.3.12,4.3.13) for Z_n when $n \leq 2$.

Having found a formula for Z_n, we can insert it into our general prescription (4.3.11) with the result that,

$$\mathsf{Z} = q^{-\frac{1}{24}} \mathsf{Z}_0 \sum_{n=0}^{\infty} q^n \sum_{\mu \in P(n)} \left(\chi_{[\frac{1}{2},0,0]^{\otimes \mu}} - \sum_{i=1}^{n} \chi_{[\frac{1}{2},0,0]^{\otimes(\mu-\epsilon^i)}} + \sum_{i<j=1}^{n} \chi_{[\frac{1}{2},0,0]^{\otimes(\mu-\epsilon^i-\epsilon^j)}} - \cdots \right) . \tag{4.3.18}$$

Now, since $\mu - \epsilon^j$ is a partition in $P(n-j)$, we are led to the idea of combining in the above alternating sum all those terms that belong to partitions of the same size. Denoting by $p_d(x; y)$ the function that counts the number of distinct, i.e. whose elements are all different, partitions of x with exactly y elements, we leave to the reader the combinatorial homework to deduce

$$\begin{aligned}
\mathsf{Z} &= q^{-\frac{1}{24}} \mathsf{Z}_0 \sum_{n=0}^{\infty} q^n \left(\sum_{j=0}^{n} \underbrace{\left(\sum_{k=0}^{j} (-1)^k p_d(j;k) \right)}_{=:c_j} \sum_{\mu \in P(n-j)} \chi_{[\frac{1}{2},0,0]^{\otimes \mu}} \right) \\
&= q^{-\frac{1}{24}} \mathsf{Z}_0 \sum_{n,j=0}^{\infty} q^n c_j \sum_{\mu \in P(n-j)} \chi_{[\frac{1}{2},0,0]^{\otimes \mu}} = q^{-\frac{1}{24}} \mathsf{Z}_0 \left(\sum_{j=0}^{\infty} c_j q^j \right) \sum_{n=0}^{\infty} q^n \sum_{\mu \in P(n)} \chi_{[\frac{1}{2},0,0]^{\otimes \mu}} \\
&= q^{-\frac{1}{24}} \mathsf{Z}_0 \, \phi(q) \sum_{n=0}^{\infty} q^n \sum_{\mu \in P(n)} \chi_{[\frac{1}{2},0,0]^{\otimes \mu}} .
\end{aligned} \tag{4.3.19}$$

We refer the reader to figure 4.4 for an example of a distinct partition.

The numbers c_j can easily be recognized as the coefficients in the Taylor expansion of the Euler ϕ-function. In fact the generating function for distinct partitions of a number

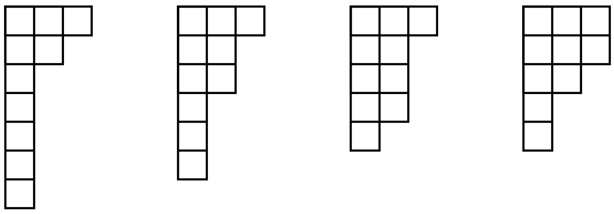

Figure 4.4: The distinct partitions of 10 with exactly 3 summands. Thus $p_d(10;3) = 4$.

n into precisely l distinct numbers is given by

$$\prod_{k=1}^{\infty}(1+zq^k) \;=\; \sum_{n=0}^{\infty}\sum_{l=0}^{n} p_d(n;l)\, z^l\, q^n \quad . \tag{4.3.20}$$

For $z = -1$ the left hand side reduces to the Euler function $\phi(q)$ while the right hand side gives the sum $\sum_{n=0}^{\infty} c_n q^n$. Note that during the resummation in the second line of equation (4.3.19) we could drop a number of terms since $P(n)$ is empty for $n < 0$. The result (4.3.19) has a rather surprising interpretation, since it tells us that we may at first discard all the derivatives of the supersphere relations for the computation of subtraces Z_i. Derivatives of equation (4.1.1) may then simply be taken into account by *multiplying the result with the Euler function* $\phi(q)$.

The conclusion of the previous discussion may now be employed to derive a much simpler formula for the partition function which generalizes the expression (4.3.9) for Z_0. Without paying respect to the supersphere relations, it is straightforward to enumerate derivative fields. Recall that the four fundamental bosonic fields carry charges $(0, \pm\frac{1}{2}, \pm\frac{1}{2})$ under the three Cartan generators (H^1, H^2, H^3). Similarly, the two fundamental fermionic fields are only charged under the first Cartan generator H^1 such that their charges are $(\pm\frac{1}{2}, 0, 0)$. Hence, the partition function can now be represented in the form

$$\mathsf{Z} \;=\; q^{-\frac{1}{24}}\mathsf{Z}_0\,\phi(q)\prod_{n=1}^{\infty}\frac{(1+z_1^{\frac{1}{2}}q^n)(1+z_1^{-\frac{1}{2}}q^n)}{(1-z_2^{\frac{1}{2}}z_3^{\frac{1}{2}}q^n)(1-z_2^{\frac{1}{2}}z_3^{-\frac{1}{2}}q^n)(1-z_2^{-\frac{1}{2}}z_3^{\frac{1}{2}}q^n)(1-z_2^{-\frac{1}{2}}z_3^{-\frac{1}{2}}q^n)} \quad . \tag{4.3.21}$$

The infinite product enumerates all states in the unconstrained state space. According to our previous discussion, the derivatives of the supersphere constraints can be implemented through a simple multiplication with the Euler function $\phi(q)$. Our final formula for the partition function of a volume filling brane in the non-linear sigma model at $R = \infty$ is indeed very simple.

4.3.3 Casimir decomposition of the boundary spectrum

The goal of this section is to expand the partition sum (4.3.11) of the volume filling brane in terms of osp (4|2) characters. To be more concrete, we would like to derive explicit formulae for the branching functions $\psi^{\mathsf{K}}_\Lambda(q)$ in the decomposition

$$\mathsf{Z}(q,z_1,z_2,z_3) \;=\; \sum\nolimits_\Lambda \chi^{\mathsf{K}}_\Lambda(z_1,z_2,z_3)\;\psi^{\mathsf{K}}_\Lambda(q) \quad , \tag{4.3.22}$$

where $\Lambda = (j_1, j_2, j_3)$ runs over the set (B.1.2) of integral highest weights for osp (4|2) subject to the further condition that j_2+j_3 be an integer. Here, the functions $\chi^{\mathsf{K}}_\Lambda(z_1,z_2,z_3)$ are characters of the Kac modules[4] K_Λ of osp (4|2). The latter form a *basis* in the space of all characters so that the expansion coefficients are uniquely determined. Finding an explicit formula for the branching functions $\psi^{\mathsf{K}}_\Lambda(q)$ is the main result of this section. The final expression takes the following form

$$\begin{aligned}\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) = \;& \frac{q^{2j_1(j_1-1)-j_2(j_2+1)-j_3(j_3+1)}}{\eta(q)\phi(q)^3}\sum_{n,m=0}^{\infty}(-1)^{m+n}q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1}\\ &\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)\ .\end{aligned} \tag{4.3.23}$$

Let us add a couple of remarks here. To begin with, the decomposition (4.3.22) of the supersphere partition function has also been considered in the work of Candu and Saleur [18,19]. In their context, the branching functions ψ^{K} are related to representation spaces of the so-called Brauer algebra. The connection has interesting implications, but it does not provide explicit formulae for ψ^{K}. Our formula (4.3.23) has not appeared in the literature before. In addition, we would want to stress that the decomposition of the partition function into characters of *Kac modules* is a somewhat formal procedure that does not fully capture the representation content of the spectrum, at least not for the atypical sector of the theory. One may notice, for example, that some of the expansion coefficients $C^{(n)}_\Lambda$ in $\psi^{\mathsf{K}}_\Lambda(q) = \sum C^{(n)}_\Lambda q^n$ are negative. Only for typical labels Λ will the numbers $C^{(n)}_\Lambda$ be positive. For atypical representations Λ, on the other hand, the characters $\chi^{\mathsf{K}}_\Lambda$ of the Kac modules have to be decomposed into characters of irreducible atypical representations χ_Λ as described in (B.2.11) in order to obtain branching functions with non-negative integral multiplicities. We will have more to say in subsection 4.3.4 on the actual transformation properties of the fields.

Proof. We will show equation (4.3.23) in several steps. To begin with, we shall decompose the partition function into representations of the bosonic subalgebra of osp (4|2). Our second step then is to recombine bosonic characters into the characters of full osp (4|2) multiplets. Once this is achieved, the resulting expressions still require some resummation in order to bring them into a more appealing form.

In our computation, we shall split the full partition function into three different parts and decompose them separately before putting all this together. We shall start with the fermionic contributions in the numerator of the partition function (4.3.21). Apart from the factors that arise from derivative fields, there are also two terms in Z_0 that account for fermionic zero modes. We may simply set the parameter t to $t = 1$ in those two

[4] Again, see Appendix B.

factors and combine them with the q-dependent terms in the numerators of eq. (4.3.21) to obtain

$$\begin{aligned}
\mathsf{Z}^{\mathrm{F}}(q,z_1) &:= \prod_{n=0}^{\infty}(1+z_1^{\frac{1}{2}}q^n)\,(1+z_1^{-\frac{1}{2}}q^n) \;=\; (1+z_1^{\frac{1}{2}})\prod_{n=0}^{\infty}(1+z_1^{\frac{1}{2}}q^{n+1})\,(1+z_1^{-\frac{1}{2}}q^n) \\
&= q^{-\frac{1}{8}}\left(z_1^{-\frac{1}{4}}+z_1^{\frac{1}{4}}\right)\frac{1}{\phi(q)}\,\theta_2(z_1^{\frac{1}{2}}|q) \;=\; \frac{1}{\phi(q)}\sum_{n\in\mathbb{Z}} z_1^{\frac{n}{2}}\left(q^{\frac{n(n+1)}{2}}+q^{\frac{n(n-1)}{2}}\right) \\
&= \frac{1}{\phi(q)}\sum_{n\in\frac{\mathbb{N}}{2}}\left(q^{n(2n+1)}+q^{n(2n-1)}-q^{(n+1)(2n+3)}-q^{(n+1)(2n+1)}\right)\chi_n(z_1) \quad .
\end{aligned}$$

Along the way we have used a number of simple identities[5] for θ-functions. As a result, all the fermionic contributions to the partition function have been decomposed explicitly into multiplets of the even part of osp$\,(4|2)$. Note that the two fermions transform non-trivially only under the first subalgebra sl(2) and hence there is no dependence on z_2 and z_3 this time.

The second piece of the partition function (4.3.21) that we would like to split off concerns the bosonic zero modes, i.e. the denominator of the minisuperspace partition function Z_0. Its decomposition into bosonic representations is straightforward

$$\lim_{t\to 1}\frac{1-t^2}{(1-z_2^{\frac{1}{2}}z_3^{\frac{1}{2}}t)(1-z_2^{\frac{1}{2}}z_3^{-\frac{1}{2}}t)(1-z_2^{-\frac{1}{2}}z_3^{\frac{1}{2}}t)(1-z_2^{-\frac{1}{2}}z_3^{-\frac{1}{2}}t)} = \sum_{n\in\frac{\mathbb{N}}{2}}\chi_n(z_2)\chi_n(z_3) \quad (4.3.24)$$

Note that the sum of characters on the left hand side encodes the well-known spectrum of a bosonic 3-sphere $S^3 \cong \mathrm{SU}(2)$. Therefore we can just state this equality without any detailed calculation. The commuting left and right invariant vector fields are generated by the second and third copy of sl(2) within the even part of osp$\,(4|2)$. Hence, there is no dependence on the parameter z_1.

It remains to analyze the q-dependent factors in the denominator of the partition

[5] See equation (C.1.3).

function (4.3.21). Their contribution may be expanded as follows

$$
\begin{aligned}
&\prod_{n=1}^{\infty}\Big((1-z_2^{\frac12}z_3^{\frac12}q^n)(1-z_2^{\frac12}z_3^{-\frac12}q^n)(1-z_2^{-\frac12}z_3^{\frac12}q^n)(1-z_2^{-\frac12}z_3^{-\frac12}q^n)\Big)^{-1}\\
&= \left(\sum_{n\in\mathbb{Z}}\frac{z_2^{\frac n2}z_3^{\frac n2}}{\phi(q)^2}\sum_{m=0}^{\infty}(-1)^m\left(q^{\frac m2(m+2n+1)}-q^{\frac m2(m+2n-1)}\right)\right)\times\left(\quad z_3\longrightarrow z_3^{-1}\quad\right)\\
&= \sum_{\substack{k,l\in\mathbb{Z}\\ k+l\in2\mathbb{Z}}}\frac{z_2^{\frac k2}z_3^{\frac l2}}{\phi(q)^4}\sum_{n,m=1}^{\infty}(-1)^{n+m}q^{k\frac{n+m}{2}+l\frac{n-m}{2}}\left(q^{\frac{n(n+1)}{2}}-q^{\frac{n(n-1)}{2}}\right)\left(q^{\frac{m(m+1)}{2}}-q^{\frac{m(m-1)}{2}}\right)\\
&= \frac{1}{\phi(q)^4}\sum_{\substack{k,l\in\mathbb{N}\\ k+l\in2\mathbb{N}}}\sum_{n,m=1}^{\infty}(-1)^{n+m}\frac{(1-q^n)(1-q^m)(1-q^{n+m})(1-q^{n-m})}{q^{-(k(n+m)+l(n-m)+n(n-1)+m(m-1))/2}}\chi_{\frac k2}(z_2)\chi_{\frac l2}(z_3)\ .
\end{aligned}
$$

In the first line of the above computation we have used the lemma (C.1.1). Since all the contributions being captured by this computation are associated with bosonic fields, characters with a non-trivial z_1 dependence do not arise.

In order to obtain the decomposition of Z into characters of $\mathrm{osp}\,(4|2)_{\bar 0}\cong \mathrm{sl}(2)\oplus\mathrm{sl}(2)\oplus\mathrm{sl}(2)$, we need to put the results from the preceding three computations together into one expression. The answer contains products of characters which depend on the same variables z_2 and z_3. These products can be re-expanded with the help of the following auxiliary formula

$$
\begin{aligned}
&\sum_{p=0}^{\infty}\chi_{\frac p2}(z_2)\chi_{\frac p2}(z_3)\sum_{\substack{k,l\in\mathbb{N}\\ k+l\in2\mathbb{N}}}a_{k,l}\chi_{\frac k2}(z_2)\chi_{\frac l2}(z_3)\\
&\qquad=\sum_{\substack{k,l\in\mathbb{N}\\ k+l\in2\mathbb{N}}}\chi_{\frac k2}(z_2)\chi_{\frac l2}(z_3)\left(\sum_{p=0}^{\infty}\sum_{r=0}^{\min\{k,p\}}\sum_{s=0}^{\min\{l,p\}}a_{|k-p|+2r,|l-p|+2s}\right)
\end{aligned}
\tag{4.3.25}
$$

which holds for an arbitrary set of numbers $a_{k,l}$. When applied to the case at hand, we find

$$
\begin{aligned}
\mathsf{Z}=\ &\frac{1}{\phi(q)^3\eta(q)}\,\mathsf{Z}^{\mathrm{F}}(q,z_1)\sum_{\substack{j_2,j_3\in\frac12\mathbb{N}\\ j_2+j_3\in\mathbb{N}}}\chi_{j_2}(z_2)\chi_{j_3}(z_3)\sum_{m,n=1}^{\infty}(-1)^{m+n}q^{\frac{n(n-1)}{2}+\frac{m(m-1)}{2}}\\
&\times\ (1-q^{n+m})\big(q^{(n-m)(j_2-j_3)}-q^{(n-m)(j_2+j_3+1)}\big)
\end{aligned}
\tag{4.3.26}
$$

Thereby, we completed out first task, namely to decompose the full partition function Z into irreducible representations of the bosonic subalgebra of $\mathrm{osp}\,(4|2)$.

Our next issue is to combine bosonic characters back into the characters of Kac modules of osp (4|2). Since the even part of osp (4|2) is a subalgebra of osp (4|2), it is clear that the characters of osp (4|2) Kac modules, possess a decomposition into characters of the bosonic subalgebra. These decomposition formulae may be inverted such that bosonic characters can be written as infinite linear combinations of osp (4|2) characters. All necessary details are provided in Appendix B. The resulting expression for the partition function Z is of the form (4.3.22) with

$$\begin{aligned}
\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) \;&=\; \frac{1}{\eta(q)\phi(q)^3}\sum_{k=0}^{\infty}\sum_{m,n=1}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+n+k}q^{2j_1(j_1+k+2l)}q^{\frac{n(n-1)}{2}+\frac{m(m-1)}{2}}\\
&\quad\times\sum_{r,s=0}^{k}q^{(n-m)(r-s)}(1-q^{n+m})(q^{(n-m)(j_2-j_3)}-q^{(n-m)(j_2+j_3+1)})\\
&\quad\times\Big[q^{j_1+\frac{k+2l}{2}(k+2l+1)}+q^{-j_1+\frac{k+2l}{2}(k+2l-1)}-q^{5j_1+3+\frac{k+2l}{2}(k+2l+5)}-q^{3j_1+\frac{k+2l}{2}(k+2l+3)}\Big]\\
&=\; \frac{q^{2j_1(j_1-1)}}{\eta(q)\phi(q)^3}\sum_{m,n=1}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^k\sum_{l=0}^{\infty}q^{j_1(2|k|+4l+1)+\frac{|k|}{2}(|k|-1)+l(2l+2|k|-1)}(1-q^{|k|+2l+2j_1})\\
&\quad\times(-1)^{m+n}q^{\frac{n(n-1)}{2}+\frac{m(m-1)}{2}}q^{(n-m)k}(1-q^{n+m})(q^{(n-m)(j_2-j_3)}-q^{(n-m)(j_2+j_3+1)})\ .
\end{aligned}$$

We will now make several transformations and resummations in order to cast this unwieldy expression into the form (4.3.23) we have spelled out above. Making the substitution $n+m=r+2, n-m=s$ with $r\in\mathbb{N}$ and $s=-r,-r+2,\ldots,r$, using the trick (C.1.4) and then substituting $r\to r+1$ gives the result

$$\begin{aligned}
\psi^{\mathsf{K}}(q) = \frac{q^{2j_1(j_1-1)}}{\eta(q)\phi(q)^3}&\sum_{k=-\infty}^{\infty}\sum_{r,l=0}^{\infty}(-1)^{r+k}q^{j_1(2|k|+1)+\frac{|k|(|k|-1)}{2}+l(2l+2|k|+4j_1-1)}\left(q^{|k|+2l+2j_1}-1\right)\\
\times q^{\frac{(r+2)(r+1)}{2}}&\left(q^{(r+1)(j_2-j_3+k)}+q^{(r+1)(-j_2+j_3-k)}-q^{(r+1)(j_2+j_3+1+k)}-q^{(r+1)(-j_2-j_3-1-k)}\right).
\end{aligned}$$

In order to simplify the sum over r, we now need to split the summation over k into three parts, according to whether it is positive, zero or negative. We then recombine the summations over positive and negative k into a single sum and employ another auxiliary formula (C.1.5) from appendix C.1 to find

$$\begin{aligned}
\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) \;&=\; q^{2j_1(j_1-1)-j_2(j_2+1)-j_3(j_3+1)}\frac{1}{\eta(q)\phi(q)^3}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\,q^{\frac{r}{2}+j_1}\\
&\times\left(q^{(j_2-\frac{r}{2})^2}-q^{(j_2+\frac{r}{2}+1)^2}\right)\left(q^{(j_3-\frac{r}{2})^2}-q^{(j_3+\frac{r}{2}+1)^2}\right)\Big[q^{l(2l+4j_1-1)}(1+q^{2l+2j_1})\\
&+\sum_{k=1}^{\infty}(-1)^kq^{j_1(2k+1)+\frac{k(k-1)}{2}+l(2l+2k+4j_1-1)}(1-q^{r+1})(q^{(r+1)(k-1)}+q^{-(r+1)k})\Big]\ .
\end{aligned}$$

Once again we need to rearrange the sum over k. Terms can be combined into a single summation if we let l run over half-integers rather than integers. Making the substitutions $l \to 2m$ and $r \to n$, leads to the formula

$$\begin{aligned}\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) &= \frac{q^{2j_1(j_1-1)-j_2(j_2+1)-j_3(j_3+1)}}{\eta(q)\phi(q)^3}\sum_{n,m=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^{m+n+k}q^{\frac{m}{2}(m+4j_1-1)+\frac{n}{2}+j_1}\\ &\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)q^{|k|(2j_1+m)+\frac{|k|(|k|-1)}{2}+(n+1)k}\ .\end{aligned}$$

It is advantageous to split the summation over k again depending on whether k is negative or non-negative. Then we substitute r for the sum $r = m + k$ and s for the difference $s = m - k$. After some rather trivial but tedious steps we can thereby bring ψ^{K} into the form

$$\begin{aligned}\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) &= \frac{q^{2j_1(j_1-1)-j_2(j_2+1)-j_3(j_3+1)}}{\eta(q)\phi(q)^3}\sum_{n,m=0}^{\infty}(-1)^{m+n}q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1}\\ &\times\ \left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\ \left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)\ \sum_{s=0}^{2m}q^{-s(n+1)}\ .\end{aligned}$$

It is left to the reader to use lemma (C.1.2) in order to show that this is equal to the formula (4.3.23) we spelled out at the beginning of this section. □

In the end, we are have one important remark left to make, namely that the decomposition (4.3.22) in no ways implies that the states of the theory transform in Kac modules of osp (4|2). While this is certainly so for typical representations, we have a priori no way of knowing how the atypical ones combine to form possibly quite complicated indecomposable structures. For us, the characters of Kac modules were simply a convenient basis to use in the partition function.

4.3.4 Cohomological reduction

In the last section we arrived at the conclusion that one can write the partition function of the volume filling brane in $S^{3|2}$ at the infinite volume limit as

$$\mathsf{Z}^{\sigma}_{\mathfrak{N},M=1} = \sum_{\Lambda\in\Gamma^+}\chi^{\mathsf{K}}_{\Lambda}\psi^{\mathsf{K}}_{\Lambda}(q) = \sum_{\Lambda\in\Gamma^+}\chi_{\Lambda}\psi_{\Lambda}(q)\ , \tag{4.3.27}$$

where $\Lambda \in \Gamma^+$ runs over all osp (4|2) weights allowed by (B.1.2) for which the sum $j_2 + j_3$ is an entire number. Here, χ_Λ and $\chi^{\mathsf{K}}_\Lambda$ are characters of irreducible, respectively Kac modules and the character functions $\psi^{\mathsf{K}}_\Lambda(q)$ are shown in (4.3.23). As we argued before, the decomposition in Kac modules is correct only at the level of characters and has the effect of causing the branching functions for atypical weights to have some negative coefficients in their q-expansion. It was shown in [19] that the partition function is properly decomposed in the characters of the trivial $\Lambda_{0,0}$, adjoint $\Lambda_{0,1}$, symmetric traceless tensors of the fundamental $\Lambda_{n,0}$ and projective covers. We claim that Z can be rewritten as

$$\begin{aligned}\mathsf{Z}^{\sigma}_{\mathfrak{N},M=1} &= q^{-\frac{1}{24}}\left[\frac{1}{2}\left(\frac{1}{\phi(q)}+\frac{1}{\phi_\star(q)}\right)\chi_{\Lambda_{0,0}}+\frac{1}{2}\left(\frac{1}{\phi(q)}-\frac{1}{\phi_\star(q)}\right)\chi_{\Lambda_{0,1}}+\frac{1}{\phi(q)}\sum_{n=1}^{\infty}\chi_{\Lambda_{n,0}}\right]\\ &\quad+\sum_{\Lambda\in\Gamma^+}\chi^{\mathsf{P}}_{\Lambda}\psi^{\mathsf{P}}_{\Lambda}(q)\ , \end{aligned}\tag{4.3.28}$$

where $\chi^{\mathsf{P}}_{\Lambda}$ are the characters of the projective modules associated to the weight Λ and the character functions $\psi^{\mathsf{P}}_{\Lambda}(q)$ have non negative coefficients. Using the formula (B.1.12) given in appendix B, we arrive at the following expressions for the character functions of the projective covers for atypical weights for which the quadratic Casimir is zero:

$$\begin{aligned}\psi^{\mathsf{P}}_{\Lambda_{0,0}}(q) &= \frac{1}{2}\left[\psi^{\mathsf{K}}_{\Lambda_{0,0}}(q)+\sum_{n=2}^{\infty}(-1)^n\psi^{\mathsf{K}}_{\Lambda_{0,n}}(q)-\frac{q^{-\frac{1}{24}}}{2}\left(\frac{1}{\phi(q)}+\frac{1}{\phi_{\star(q)}}\right)\right]\\ \psi^{\mathsf{P}}_{\Lambda_{0,1}}(q) &= \frac{1}{2}\left[\psi^{\mathsf{K}}_{\Lambda_{0,1}}(q)+\sum_{n=2}^{\infty}(-1)^n\psi^{\mathsf{K}}_{\Lambda_{0,n}}(q)-\frac{q^{-\frac{1}{24}}}{2}\left(\frac{1}{\phi(q)}-\frac{1}{\phi_{\star(q)}}\right)\right]\\ \psi^{\mathsf{P}}_{\Lambda_{0,l}}(q) &= \sum_{n=l+1}^{\infty}(-1)^{n+l+1}\psi^{\mathsf{K}}_{\Lambda_{0,n}}(q)\quad \text{for } l\geq 2\ .\end{aligned}\tag{4.3.29}$$

For the other atypical weights, the expressions are as follows,

$$\begin{aligned}\psi^{\mathsf{P}}_{\Lambda_{k,0}}(q) &= \sum_{n=1}^{\infty}(-1)^{n+1}\psi^{\mathsf{K}}_{\Lambda_{k,n}}(q)=\sum_{n=1}^{\infty}(-1)^{n+1}\psi^{\mathsf{K}}_{\Lambda_{k,-n}}(q)\quad \text{for } k\geq 1\\ \psi^{\mathsf{P}}_{\Lambda_{k,l}}(q) &= \sum_{n=l+1}^{\infty}(-1)^{n+l+1}\psi^{\mathsf{K}}_{\Lambda_{k,n}}(q)\quad \text{for } k\geq 1, l\geq 1\\ \psi^{\mathsf{P}}_{\Lambda_{k,-l}}(q) &= \sum_{n=l+1}^{\infty}(-1)^{n+l+1}\psi^{\mathsf{K}}_{\Lambda_{k,-n}}(q)\quad \text{for } k\geq 1, l\geq 1\ ,\end{aligned}\tag{4.3.30}$$

while for typical weights, we of course have $\psi^{\mathsf{P}}_{\Lambda}=\psi^{\mathsf{K}}_{\Lambda}$. In chapter 3, we argued that the cohomological reduction of projective covers is null. Furthermore, due to (3.1.60), the reduction of the atypical irreducible modules $[\Lambda_{n,0}]$, $[\Lambda_{0,1}]$ cannot be zero, since their superdimension does not vanish. In fact, $[\Lambda_{n,0}]$ reduce to the traceless symmetric representations of rang n of O(2), whereas $[\Lambda_{0,1}]$ reduces to the adjoint. Therefore, the cohomological reduction takes (4.3.28) and turns it into (4.2.5) for $R\to\infty$. Towards the end of this chapter, we will convince ourselves explicitly that the reduction is valid for all values of the radius.

4.4 The OSP(4|2) GN

The main point of study of this section is the osp $(2S+2|2S)$ Gross-Neveu models that we have conjectured to be dual to the supersphere sigma models. Our first stop is the free bulk theory defined by the action of (4.1.8). After a brief discussion of the bulk spectrum, that is the spectrum of worldsheets without boundaries, that is valid for all M, we specialize to $M = 1$ and express the bulk partition function through characters of the model's affine $\widehat{\mathrm{osp}}(4|2)$ symmetry at level $k = 1$.[6] Then, in section 4.4.2, we single out one particular symmetry preserving boundary condition and write down its spectrum, which we then decompose according to the action of the global osp $(4|2)$ symmetry. Once such a Casimir decomposition has been performed, we can apply the results of [34] and determine the boundary spectrum throughout the entire moduli space that is generated by the deformation (4.1.9). We show that as the coupling constant tends to infinity, we recover precisely the spectrum of the volume filling brane in the sigma model on the supersphere $S^{3|2}$. The principle of cohomological reduction, then allows us to spell out a relation between the Gross-Neveu theory coupling g and the radius R of the non-linear sigma model.

4.4.1 Construction of the free bulk theory

Before we plunge into the discussion of the spectrum and symmetries of the free Gross-Neveu model (4.1.8), it is useful to recall the case $M = 0$, which is the fermionic description of the free boson. As is well known, the compactified free boson at radius $R = 1$ is equivalent to an orbifold of the free field theory of two real fermions. Individually, each of the two fermionic fields is equivalent to a copy of the Ising model with central charge 1/2. In order to ensure that only sectors to contribute are those in which both fermions obey the same periodicity boundary conditions, the two factors need to be coupled by an orbifold construction. The details of this are discussed in the next few paragraphs.

We start with the critical, i.e. conformal, Ising model and recall that the Virasoro algebra with $c = 1/2$ possesses three simple sectors, which we shall label by the conformal weights of their ground states, i.e. through $[0], [1/2]$ and $[\sigma] = [1/16]$. The character functions of these sectors read as follows,

$$\widehat{\chi}_\epsilon(q) \;=\; \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} + (-1)^{2\epsilon}\sqrt{\frac{\theta_4}{\eta}}\right) \quad , \quad \widehat{\chi}_\sigma(q) \;=\; \sqrt{\frac{\theta_2}{2\eta}} \tag{4.4.1}$$

with the notation $\epsilon = 0, 1/2$. The product of two Ising models contains a special sector $\mathfrak{o} = [1/2, 1/2]$ of conformal weight one, that generates an abelian group $\mathfrak{O}_0 \cong \mathbb{Z}_2$ in the fusion ring. The two elements of this group are called *simple currents* since their fusion with an arbitrary representation *always* yields a single contribution. As recalled in [42],

[6] The discrepancy between our value $k = 1$ and the $k = -1/2$ that appears in the work of Candu and Saleur is entirely due to different conventions.

the corresponding simple current orbifold model is equivalent to the compactified free boson at radius one.

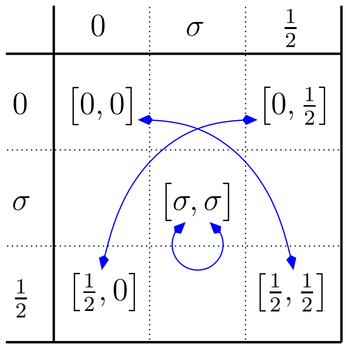

Figure 4.5: The action of the group of simple currents maps the sector 0 to $\frac{1}{2}$ and leaves σ invariant. Therefore, on the fields that have zero monodromy in the tensor product of two Ising models, the orbits will take the form depicted by the blue arrows above.

The construction of the simple current orbifold proceeds in several simple steps. First, we need to list all sectors $[J]$ of the theory which possess integer monodromy charge:

$$Q_J(\mathfrak{o}) := h_J + h_{\mathfrak{o}} - h_{\mathfrak{o}\times J} \overset{!}{\in} \mathbb{Z} \qquad \forall \mathfrak{o} \in \mathfrak{O}_0 \ . \tag{4.4.2}$$

These $[J]$ are then organized into orbits $\mathcal{O}_a$ under the action of the simple current group $\mathfrak{O}_0$, with each such orbit $\mathcal{O}_a$ contributing one term to the partition function of the orbifold model, with a coefficient $|\mathfrak{O}_0|/|\mathcal{O}_a|$ that is given by the ratio between the order $|\mathfrak{O}_0|$ of the orbifold group and the length $|\mathcal{O}_a|$ of the orbit[7]. Here, as depicted in figure 4.5, there exist five sectors that have integer monodromy charge, namely $[\epsilon_1, \epsilon_2]$ with $\epsilon_i \in \{0, \frac{1}{2}\}$ together with $[\sigma, \sigma]$. The action of $\mathfrak{O}_0$ organizes them into three orbits, two of length two and one that is left invariant by fusion with $\mathfrak{o}$. Consequently, the associated simple current orbifold invariant takes the form

$$\mathsf{Z}^{\text{orb}(\mathfrak{O}_0)}_{\text{Ising}^2}(q) \;=\; \mathsf{Z}^{\text{GN}}_{M=0}(q) \;=\; \left|\widehat{\chi}_{(0,0)} + \widehat{\chi}_{(1/2,1/2)}\right|^2 + \left|\widehat{\chi}_{(0,1/2)} + \widehat{\chi}_{(1/2,0)}\right|^2 + 2\left|\widehat{\chi}_{(\sigma,\sigma)}\right|^2 \ . \tag{4.4.3}$$

The characters on the right hand side are products of characters of the $c = 1/2$ Virasoro algebra, for example $\widehat{\chi}_{(0,1/2)}(q) := \widehat{\chi}_0(q)\widehat{\chi}_{1/2}(q)$. According to the claims we stated above, the simple current orbifold (4.4.3) agrees with the free boson compactified at radius $R = 1$,

$$\mathsf{Z}^{\text{GN}}_{M=0}(q) \;=\; \frac{1}{|\eta(q)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{1}{8}(n+2w)^2} \bar{q}^{\frac{1}{8}(n-2w)^2} \;=\; \mathsf{Z}^{R=1}(q) \ . \tag{4.4.4}$$

The interested reader can find the detailed proof of (4.4.4) in [41] or [42]. Our aim now is to extend equation (4.4.4) to the case $M > 0$.

[7]See for instance [55]

For $M > 0$, our theory (4.1.8) is built of $2M + 2$ real fermions and of M free $\beta\gamma$-systems[8] with central charge $c = -1$. In order to have osp$(2M + 2|2M)$ global symmetry, it is imperative that all these fields obey the same periodicity conditions on the boundary, i.e. they have to either be all periodic or all anti-periodic. Before we spell out the relevant bulk partition function, we need a bit more background on the $\beta\gamma$-systems.

As in the case of real fermions, we shall consider sectors which differ by the choice of boundary conditions on the fields β and γ. For that, we introduce a family of ground states $|\nu\rangle$ for $\nu \in \frac{1}{2}\mathbb{Z}$ that are characterized by the conditions

$$\beta_{r+\nu}|\nu\rangle = 0 \quad , \quad \gamma_{r-\nu}|\nu\rangle = 0 \quad \text{for} \quad r \in \mathbb{N} + \frac{1}{2} \,. \tag{4.4.5}$$

From these ground states we generate the corresponding sectors by application of raising operators. Assigning charges $q_\beta = 1/2$ and $q_\gamma = -1/2$ to the modes of the fields β and γ, respectively, and $q_\nu = \nu/2$ to the ground state $|\nu\rangle$, the generating function for the sector ν reads,

$$\widehat{\chi}^{(\nu)}(q,y) = q^{\frac{1}{24}-\frac{\nu^2}{2}}\, y^{\frac{\nu}{2}} \prod_{n=0}^{\infty} \frac{1}{(1-y^{\frac{1}{2}}q^{n+\frac{1}{2}-\nu})(1-y^{-\frac{1}{2}}q^{n+\frac{1}{2}+\nu})} = \frac{q^{-\nu^2/2}\, y^{\frac{\nu}{2}}\, \eta(q)}{\theta_4(q, y^{1/2}q^{-\nu})} \tag{4.4.6}$$

All the constructed sectors carry an action of an affine $\widehat{\mathrm{sl}}(2)$ current algebra at level $k = -1/2$. One can easily see this symmetry by constructing the currents with the fields β and γ as follows[9]:

$$E^1_+(z) = \frac{1}{2}\beta^2(z) \,, \quad H^1(z) = -\frac{1}{2}\,(\beta\gamma)(z) \,, \quad E^1_-(z) = -\frac{1}{2}\,\gamma^2(z) \;. \tag{4.4.7}$$

Consequently, we can decompose the generating functions (4.4.6) into characters of irreducible representations of $\widehat{\mathrm{sl}}(2)_{-1/2}$. In case of $\widehat{\chi}^{(0)}$, for example, the decomposition is given by

$$\widehat{\chi}^{(0)}(q,y) = \frac{\eta(q)}{\theta_4(q,y^{1/2})} = \widehat{\chi}_0^{k=-1/2}(q,y) + \widehat{\chi}_{1/2}^{k=-1/2}(q,y) \,. \tag{4.4.8}$$

The two characters on the right hand side belong to irreducible highest weight representations with lowest weight $h = \epsilon \in \{0, 1/2\}$,

$$\widehat{\chi}_\epsilon^{k=-1/2}(q,y) = \frac{\eta(q)}{2}\left[\frac{1}{\theta_4(q,y^{1/2})} + (-1)^{2\epsilon}\frac{1}{\theta_3(q,y^{1/2})}\right] \;. \tag{4.4.9}$$

Let us note that the ground states transform in representations of spin ϵ, which in this case is identical to the conformal weight of the primary fields. Similar decomposition

[8] See [56] for a detailed analysis of this rather unusual CFT in the context of our work.

[9] The superscript 1 will become clear later on

formulae exist for all the other functions (4.4.6), which are all related by the action of spectral flow automorphisms. In particular, for the $\nu = \frac{1}{2}$ case of (4.4.6) we have

$$\begin{aligned} \widehat{\chi}^{(1/2)} &= \widehat{\chi}^{k=-1/2}_{\sigma;+} + \widehat{\chi}^{k=-1/2}_{\sigma;-} \quad \text{with} \\ \widehat{\chi}^{k=-1/2}_{\sigma;\pm}(q,y) &= \frac{y^{1/4}\eta(q)}{2}\left[\frac{1}{i\theta_1(q,y^{-1/2})} \pm \frac{1}{\theta_2(q,y^{-1/2})}\right] \quad . \end{aligned} \tag{4.4.10}$$

The two characters on the left hand side belong to the two irreducible lowest weight representations of the current algebra with spins 1/4, respectively 3/4 and conformal weights $-1/8$. The subscript σ refers to the fact that despite appearances, these representations have a lot in common with the σ sector in the free fermion theory as we shall see later on.

As explained for instance in [56], the spectral flow automorphisms of the algebra $\widehat{\mathrm{sl}}(2)_{-1/2}$ act on the modes of the currents as follows

$$H^1_n \mapsto H^1_n - \frac{k}{2} w\delta_{n,0} \qquad E^1_{\pm,n} \mapsto E^1_{\pm,n\mp w} \ , \tag{4.4.11}$$

where w is an integer. These automorphisms map representations of $\widehat{\mathrm{sl}}(2)_{-1/2}$ onto each other, as illustrated quite plastically in [57].

We are now ready to discuss the relevant bulk modular invariant for the theory (4.1.8) for M greater than zero. Let us begin with the product of M $\beta\gamma$-systems and $2M+2$ real fermions. This theory contains a group $\mathfrak{O}_M$ of simple currents that consists of all elements $\mathfrak{o}$ of the form

$$\mathfrak{o} = [\epsilon_1, \ldots \epsilon_M; \epsilon_{M+1}, \ldots, \epsilon_{3M+2}] \quad \text{with} \quad \epsilon_i \in \{0, 1/2\} \quad \text{and} \quad \sum_{i=1}^{3M+2} \epsilon_i \equiv 0 \bmod 1 \ .$$

The first M entries of $\mathfrak{o}$ denote sectors of the $\beta\gamma$-system while the remaining ones are representing sectors in the Ising models. Together, the elements $\mathfrak{o}$ generate the abelian group $\mathfrak{O}_M \cong \mathbb{Z}_2^{3M+1}$.

Let us first deal with the sector involving representations with zero spectral flow, or $\nu = 0$. Under the action of $\mathfrak{O}_M$, the sectors with vanishing monodromy charge split into two orbits of maximal length. Hence we are led to the following contribution to the partition function,

$$\mathsf{Z}^{\mathrm{GN}}_{M,0}(q, y_1, \ldots, y_M) = \left|\sum\nolimits_{\mathfrak{o}\in\mathfrak{O}_M} \widehat{\chi}_{\mathfrak{o}\times[0,\ldots,0;0,\ldots,0]}\right|^2 + \left|\sum\nolimits_{\mathfrak{o}\in\mathfrak{O}_M} \widehat{\chi}_{\mathfrak{o}\times[0,\ldots,0;0,\ldots,0,1/2]}\right|^2 \ . \tag{4.4.12}$$

However, the total theory has to be invariant under the spectral flow symmetry, so that we must also add twisted contributions $\mathsf{Z}^{\mathrm{GN}}_{M,\nu}$. It was already mentioned above that all the bosonic ghosts and all the fermions have to have identical periodicity conditions in order not to spoil osp$(2M+2|2M)$ symmetry. Consequently the spectral flow must act diagonally, that is simultaneously on all sectors, by half-integer shifts.[10] In the fermionic factors, spectral flow by $\nu = 1/2$ brings us to σ-representations. Integer units of the spectral flow, however, do not give anything new. In the ghost sectors things works differently because the application of a diagonal spectral flow leads to an infinite number of new representations constructed from the ground states $|\nu\rangle$ for $\nu \in \frac{1}{2}\mathbb{Z}$. Since the orbits of the half-integer spectral flow representations possess a stabilizer subgroup of order 2^{2M+1} with respect to the action of $\mathfrak{O}_M$ we finally end up with the partition function

$$\begin{aligned} \mathsf{Z}^{\mathrm{GN}}_M(q,y_1,\ldots,y_M) &= \sum_{\nu\in\frac{1}{2}\mathbb{Z}} \mathsf{Z}^{\mathrm{GN}}_{M,\nu}(q,y_1,\ldots,y_M) \\ &= \sum_{\nu\in\mathbb{Z}}\left[\left|\sum\nolimits_{\mathfrak{o}\in\mathfrak{O}_M}\widehat{\chi}^{(\nu)}_{\mathfrak{o}\times[0,\ldots,0;0,\ldots,0]}\right|^2+\left|\sum\nolimits_{\mathfrak{o}\in\mathfrak{O}_M}\widehat{\chi}^{(\nu)}_{\mathfrak{o}\times[0,\ldots,0;0,\ldots,0,1/2]}\right|^2\right] \\ &\qquad + 2^{2M+1}\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}\left|\prod_{a=1}^{M}\widehat{\chi}^{(\nu)}(q,y_a)\big(\widehat{\chi}_\sigma(q)\big)^{2M+2}\right|^2 \ . \end{aligned}$$

Here, the superscript (ν) on a function $f(y_i)$ of M variables y_i is defined through the prescription $f^{(\nu)}(y_i) = q^{-M\nu^2/2}f(y_i q^{-2\nu})$.

After this set of general statements, we can specialize the rest of our analysis to the case $M = 1$. The state space of our orbifold theory can be equipped with the action of an affine $\widehat{\mathrm{osp}}(4|2)$ Lie superalgebra, whose bosonic part is the direct sum of three bosonic $\widehat{\mathrm{sl}}(2)$ algebras, two at level 1 and one at level $-\frac{1}{2}$. We have already spelled out expressions for the first set of sl(2) currents in equation (4.4.7) above. The currents associated with the other two copies if sl(2) take the form

$$E^2_\pm(z) = \frac{1}{2i}\left[(\psi_1\psi_3)-(\psi_2\psi_4)\pm i\big((\psi_1\psi_4)+(\psi_2\psi_3)\big)\right] , \tag{4.4.13}$$

$$H^2(z) = \frac{1}{2i}\big((\psi_3\psi_4)+(\psi_1\psi_2)\big) \quad , \quad H^3(z) = \frac{1}{2i}\big((\psi_3\psi_4)-(\psi_1\psi_2)\big) ,$$

$$E^3_\pm(z) = \frac{1}{2i}\left[(\psi_1\psi_3)+(\psi_2\psi_4)\pm i\big((\psi_1\psi_4)-(\psi_2\psi_3)\big)\right] . \tag{4.4.14}$$

They generate two commuting copies of the current algebra $\widehat{\mathrm{sl}}(2)_1$. In addition, we can introduce the eight fermionic currents through the following expressions

$$\begin{aligned} F^{+++}(z) &= i\beta\,(\psi_3+i\psi_4)\,(z)\ , & F^{+--}(z) &= i\beta\,(\psi_3-i\psi_4)\,(z)\ , \\ F^{++-}(z) &= i\beta\,(\psi_1+i\psi_2)\,(z)\ , & F^{+-+}(z) &= i\beta\,(\psi_1-i\psi_2)\,(z)\ , \end{aligned}$$

[10] It is worth mentioning that these diagonal spectral flow transformations are also the only ones which commute with the action of the orbifold group. Note also that half-integer spectral flow on ghosts and fermions implies integer spectral flow on the currents such as those defined in eq. (4.4.7) and below.

and similarly for $F^{-\pm\pm}(z)$ with the field β in the above formulae exchanged with γ. Observe that all terms that contribute to the seventeen currents are quadratic in the basic fields. Since by construction these basic fields are either all in the periodic or anti-periodic simultaneously, the currents must obey periodic boundary conditions on the entire state space. In order to rewrite the partition function of our bulk theory in terms of affine $\widehat{\text{osp}}(4|2)$ characters, we recall the following formulae for characters of an $\widehat{\text{sl}}(2)$ currents algebra at level $k = 1$,

$$\widehat{\chi}_0^{k=1}(q,z) \;=\; \frac{\theta_3(q^2,z)}{\eta(q)} \quad , \quad \widehat{\chi}_{1/2}^{k=1}(q,z) \;=\; \frac{\theta_2(q^2,z)}{\eta(q)} \quad .$$

The lower index $j = 0, 1/2$ now denotes the spin of representations of the $\widehat{\text{sl}}(2)$ current algebra. In terms of characters of the bosonic current algebras, the orbifold partition function reads

$$\begin{aligned} \mathsf{Z}^{\text{GN}}_{M=1}(q,z_i) \;=\; & \sum_{\nu=-\infty}^{\infty} \left| \widehat{\chi}^{(\nu)}_{(0;0,0)}(q,z_i) + \widehat{\chi}^{(\nu)}_{(\frac{1}{2};\frac{1}{2},\frac{1}{2})}(q,z_i) \right|^2 + \\ & + \sum_{\nu=-\infty}^{\infty} \left| \widehat{\chi}^{(\nu)}_{(0;\frac{1}{2},\frac{1}{2})}(q,z_i) + \widehat{\chi}^{(\nu)}_{(\frac{1}{2};0,0)}(q,z_i) \right|^2 \end{aligned} \tag{4.4.15}$$

where the action of the spectral flow involves the first variable $z_1 \equiv y$ only and we have defined

$$\widehat{\chi}_{(j_1;j_2,j_3)}(q,z_i) \;=\; \widehat{\chi}_{j_1}^{k=-\frac{1}{2}}(q,z_1)\; \widehat{\chi}_{j_2}^{k=1}(q,z_2)\; \widehat{\chi}_{j_3}^{k=1}(q,z_3) \quad .$$

To compare the formula (4.4.15) with our previous expression (4.4.13) one has to specialize to $z_2 = z_3 = 1$. Going one step further we can combine characters of the bosonic current algebra into $\widehat{\text{osp}}(4|2)_1$ characters according to,

$$\widehat{\chi}_{\{0\}}(q,z_i) \;=\; \widehat{\chi}_{(0;0,0)}(q,z_i) + \widehat{\chi}_{(\frac{1}{2};\frac{1}{2},\frac{1}{2})}(q,z_i)\;, \tag{4.4.16}$$

$$\widehat{\chi}_{\{1/2\}}(q,z_i) \;=\; \widehat{\chi}_{(0;\frac{1}{2},\frac{1}{2})}(q,z_i) + \widehat{\chi}_{(\frac{1}{2};0,0)}(q,z_i) \quad . \tag{4.4.17}$$

The primary fields in $\widehat{\chi}_{\{0\}}$, respectively $\widehat{\chi}_{\{\frac{1}{2}\}}$ transform in the trivial, respectively fundamental representation of the horizontal subalgebra osp$\,(4|2)$. Alter all this toil, the results of this section may be summarized succinctly via the following simple formula

$$\mathsf{Z}^{\text{GN}}_{M=1}(q,z_i) \;=\; \sum_{\nu=-\infty}^{\infty} \left| \widehat{\chi}^{(\nu)}_{\{0\}}(q,z_i) \right|^2 + \sum_{\nu=-\infty}^{\infty} \left| \widehat{\chi}^{(\nu)}_{\{1/2\}}(q,z_i) \right|^2 \quad , \tag{4.4.18}$$

i.e. the orbifold partition function is the charge conjugate modular invariant partition function for the sectors $\{0\}$ and $\{1/2\}$ of the $\widehat{\text{osp}}(4|2)_1$ current algebra. It is remarkable that spectral flow relates all the representations occurring here and that the fusion is purely abelian [56]. In contrast to some of the other WZNW theories on supergroups, as shown for instance in [43, 58–60], this guarantees the existence of an *irreducible* theory whose correlation functions have no logarithmic singularities. However, by fermionizing the $\beta\gamma$ systems and keeping additional zero-modes, however, one can, if one so wishes, construct a *logarithmic lift* of the theory as in [61] or [59].

4.4.2 Boundary conditions and their spectra

In the next step we wish to discuss boundary conditions in the bulk orbifold theory that we constructed in the preceding section. In doing so, we will focus on a particular brane, the choice of which could seem ad hoc initially. However, it will turn out later on that the spectrum of this brane will be deformed by the interaction (4.1.9) into the spectrum of space-filling brane of the sigma model. As before, we treat the cases $M = 0$ and $M = 1$ in some detail and postpone comments on higher values of M to the following section.

In the case of zero M we need to construct a brane in the orbifold (4.4.3) which corresponds to a Neumann brane in the free boson theory at large radius. Fortunately, in this case the deformation is well known. Proceeding backwards from the infinite radius to $R = 1$, we pass the self-dual radius where Neumann and Dirichlet branes cannot be distinguished and get exchanged by T-duality. Consequently, the brane we would like to describe in the free boson theory at $R = 1$ is the Dirichlet brane which has the spectrum

$$\mathsf{Z}_{\mathfrak{D}}(R=1,q) \;=\; \sum_{w\in\mathbb{Z}} \frac{q^{\frac{w^2}{2}}}{\eta(q)} \;=\; \frac{\theta_3(q)}{\eta(q)} \quad . \tag{4.4.19}$$

Our task now is to show how the same spectrum can be obtained from the orbifold model.

The Ising model is the simplest of the Virasoro minimal models, and it has precisely three different conformal boundary conditions, one for each of irreducible representations $[0]$, $[1/2]$ and $[\sigma] = [1/16]$. Here and in the following, we shall label boundary conditions and sectors by the same symbol. As shown in [62], the spectrum of excitations between any two of these boundary conditions is described by the respective fusion rules. In order to make contact with the bosonic description, let us rewrite the partition function (4.4.19) with the characters (4.4.1) of the two Ising models. Simple manipulations lead to

$$\mathsf{Z}_{\mathfrak{D}}(R=1,q) \;=\; \frac{\theta_3(q)}{\eta(q)} \;=\; \widehat{\chi}_{(0,0)} + \widehat{\chi}_{(1/2,1/2)} + \widehat{\chi}_{(0,1/2)} + \widehat{\chi}_{(1/2,0)} \quad . \tag{4.4.20}$$

This spectrum can be considered as the orbit of the sum $[0,0] \oplus [0,1/2]$ under the action of the orbifold group $\mathfrak{O}_0$. Since $[0,0] \oplus [0,1/2]$ is precisely the fusion product $[0,\sigma] \times [0,\sigma]$ we conclude that the desired point-like brane at $R = 1$ descends under the orbifold construction from the boundary condition $[0,\sigma]$ in the product of two Ising models. This conclusion is fully consistent with the free fermion construction of the bosonic current $J \sim \psi_1\psi_2$ of the $R = 1$ model. In fact, as is well known, the boundary label $[0,\sigma]$ corresponds to the gluing conditions

$$\psi_1(z) \;=\; -\bar{\psi}_1(\bar{z}) \qquad\qquad \psi_2(z) \;=\; \bar{\psi}_2(\bar{z}) \qquad\qquad (\text{for } z = \bar{z}) \tag{4.4.21}$$

in the underlying free fermion description. The sign in the gluing condition for the first fermionic field is associated with the non-trivial boundary label $[\sigma]$. It implies that the current $J \sim \psi_1\psi_2$ satisfies Dirichlet boundary conditions $J = -\bar{J}$ all along the boundary.

We can now turn our attention to the case $M = 1$, for which we focus on a brane which is associated with the twisted gluing conditions

$$J^1(z) \;=\; \bar{J}^1(\bar{z}) \;\;,\;\; J^2(z) \;=\; \bar{J}^3(\bar{z}) \;\;,\;\; J^3(z) \;=\; \bar{J}^2(\bar{z}) \tag{4.4.22}$$

for the bosonic currents $J^i = E^i_a t^a$ all along the boundary at $z = \bar{z}$. The underlying gluing automorphism Ω permutes the second and third copy of sl(2) in the bosonic subalgebra and it is easily shown that Ω extends to an involution on the entire superalgebra osp$\,(4|2)$. The corresponding gluing conditions for fermionic currents read,

$$F^{\xi\pm\pm}(z) \;=\; \bar{F}^{\xi\pm\pm}(\bar{z}) \qquad\qquad F^{\xi\pm\mp}(z) \;=\; \bar{F}^{\xi\mp\pm}(\bar{z}) \;\;. \tag{4.4.23}$$

A quick look back at the free field realization of the currents (4.4.13) suggests to implement the boundary conditions (4.4.22) and (4.4.23) through the following gluing prescription for the fundamental field multiplet,

$$\psi_1(z) \;=\; -\bar{\psi}_1(\bar{z})\,,\quad \psi_i(z) \;=\; \bar{\psi}_i(\bar{z}) \quad (i \neq 1)\,,\quad \beta_a(z) \;=\; \bar{\beta}_a(\bar{z})\,,\quad \gamma_a(z) \;=\; \bar{\gamma}_a(\bar{z}) \;\;. \tag{4.4.24}$$

Indeed, equations (4.4.24) reproduce the permutation of currents displayed in equations (4.4.22) and (4.4.23) upon insertion into the expression (4.4.13).

Just as in the case $M = 0$ above, having a non-trivial gluing condition for the fermion is associated with the occurrence of the brane label σ in the Ising model description. Hence we propose that the desired orbifold brane may be constructed from the brane

$$\mathfrak{B} := [0, 0; \sigma, 0, 0, 0] \tag{4.4.25}$$

in the covering theory. The spectrum for the latter is again given by fusion, and taking the orbit with respect to the orbifold group $\mathfrak{O}_1$ one easily arrives at

$$\mathsf{Z}^{\rm GN}_{\mathfrak{B};M=1} \;=\; \sum_{\gamma\in\mathfrak{O}_1} \left[\widehat{\chi}_{\gamma\times[0,0;0,0,0,0]} + \widehat{\chi}_{\gamma\times[0,0;0,1/2,0,0]}\right] \;\;. \tag{4.4.26}$$

For later convenience this result may also be rewritten in terms of irreducible characters of the underlying bosonic current algebra, leading to

$$\mathsf{Z}^{\rm GN}_{\mathfrak{B};M=1}(q, z_i) \;=\; \widehat{\chi}_{(0;0,0)} + \widehat{\chi}_{(0;\frac{1}{2},\frac{1}{2})} + \widehat{\chi}_{(\frac{1}{2};\frac{1}{2},\frac{1}{2})} + \widehat{\chi}_{(\frac{1}{2};0,0)} \;=\; \widehat{\chi}_{\{0\}} + \widehat{\chi}_{\{1/2\}} \;\;. \tag{4.4.27}$$

In the second step we have combined characters of the bosonic subalgebra into characters of the full $\widehat{\rm osp}(4|2)_1$, using the formulae (4.4.16) and (4.4.17). The spectrum of the orbifold brane preserves the affine Lie superalgebra, as desired. We also note that our partition function $\mathsf{Z}^{\rm GN}_{\mathfrak{B};M=1}(q)$ is identical to the one that appeared in the work of Candu and Saleur [18, 19]. We shall now see that it is related through a deformation to the partition function of the volume filling brane in the sigma model.

4.4.3 Casimir decomposition in the free GN model

Having found the full spectrum of an osp (4|2) symmetric brane in the free field theory (4.1.8), our next task is to expand it in terms of the Kac module characters $\chi^{\mathsf{K}}_{\Lambda}$. In other words, we need to find the branching functions $\psi^{\mathsf{K}}_{\Lambda}(q)$ in the decomposition,

$$\tilde{\mathsf{Z}} \;\equiv\; \mathsf{Z}^{\mathrm{GN}}_{\mathfrak{B};M=1}(q,z_i) \;=\; \sum\nolimits_{\Lambda} \chi^{\mathsf{K}}_{\Lambda}(z_1,z_2,z_3)\,\tilde{\psi}^{\mathsf{K}}_{\Lambda}(q) \quad . \tag{4.4.28}$$

In appendix C.2, we explicitly decompose this partition function in characters of simple modules of osp (4|2) up to order $q^{13/2}$. The range of $\Lambda = [j_1, j_2, j_3]$ is the set of allowed weight[11] in Γ^+ for which $j_2 + j_3 \in \mathbb{Z}$. The expansion (4.3.22) is of the same form as the one we found for the sigma model at $R = \infty$, with only the branching functions $\tilde{\psi}^{\mathsf{K}}$ being different. Our analysis will show that they read

$$\begin{aligned}\tilde{\psi}^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) \;&=\; \frac{1}{\eta(q)\phi^3(q)} \sum_{n,m=0}^{\infty} (-1)^{n+m} q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}} \\ &\quad\times\; \left(q^{(j_2-\frac{n}{2})^2} - q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2} - q^{(j_3+\frac{n}{2}+1)^2}\right) \quad .\end{aligned} \tag{4.4.29}$$

Before we derive this formula, we wish to comment on its implications. A short look back to formula (4.3.23) reveals a remarkable similarity between the two branching functions of the partition functions Z of the sigma model at infinite radius and $\tilde{\mathsf{Z}}$ of the free fields theory (4.1.8). In fact, they are identical up to an overall prefactor,

$$\boxed{\psi^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) \;=\; q^{-\frac{1}{2}\mathsf{Cas}(j_1,j_2,j_3)}\;\tilde{\psi}^{\mathsf{K}}_{[j_1,j_2,j_3]}(q)} \quad , \tag{4.4.30}$$

where the value of the quadratic Casimir was given in (4.3.1). Right now, one may be forgiven for thinking that this equation is simply an amusing observation about a fortunate similarity between the two Casimir decompositions. Our task is now to explain, why (4.4.30) is no coincidence and how it relates to the claim that the boundary spectrum for the sigma model at $R = \infty$ may be obtained by the current-current perturbation (4.1.9) from the free field theory (4.1.8).

Prior to that, we still need to prove (4.4.29). In order to calculate the branching functions $\tilde{\psi}^{\mathsf{K}}$ from the partition function $\tilde{\mathsf{Z}}$, we proceed as in subsection 4.3.3. In a first step we shall expand $\tilde{\mathsf{Z}}$ in terms of characters of the bosonic subalgebra osp $(4|2)_{\bar{0}}$. Then we combine the bosonic building blocks into characters of Kac modules for osp (4|2). The resulting expression for the branching function will require only very little additional analysis in order to cast them into the form (4.4.29).

Proof. The decomposition of $\tilde{\mathsf{Z}}$ into bosonic characters departs from the representation (4.4.27) of $\tilde{\mathsf{Z}}$ and then employs the following expansion formulae for $\widehat{\mathrm{sl}}(2)$ characters into

[11] See equation (B.1.2).

sums of characters of sl(2),

$$\widehat{\chi}_a^{k=-\frac{1}{2}}(q,u) = \frac{q^{\frac{1}{24}}}{\phi(q)^2} \sum_{k\in\mathbb{N}+a} \chi_k(z) \sum_{m=0}^{\infty}(-1)^m q^{\frac{m}{2}(m+4k+1)+k}\left(1-q^{2m+1}\right) \quad (4.4.31)$$

$$\widehat{\chi}_a^{k=1}(q,u) = \frac{1}{\eta(q)} \sum_{m\in\mathbb{N}+a} \chi_m(z) \left(q^{m^2} - q^{(m+1)^2}\right) \quad (4.4.32)$$

where $a \in \left\{0, \frac{1}{2}\right\}$. From the equality (4.4.27) and the two decomposition formulae (4.4.31) and (4.4.32) it is clear that $\tilde{\mathsf{Z}}$ can be written as

$$\tilde{\mathsf{Z}} = \sum_{\substack{(j_1,j_2,j_3)\in\frac{1}{2}\mathbb{N}^3 \\ j_2+j_3\in\mathbb{N}}} \chi_{(j_1,j_2,j_3)}(z_1,z_2,z_3)\, \tilde{\psi}^{\mathsf{B}}_{(j_1,j_2,j_3)}(q) \quad (4.4.33)$$

where $\chi_{(j_1,j_2,j_3)}$ are the characters of the irreducible representations of $\mathrm{osp}\,(4|2)_{\bar{0}}$, as before, and the branching functions $\tilde{\psi}^{\mathsf{B}}$ are given by

$$\tilde{\psi}^{\mathsf{B}}_{(j_1,j_2,j_3)}(q) = \frac{1}{\eta(q)\phi^3(q)} \sum_{m=0}^{\infty} (-1)^m\, q^{\frac{m}{2}(m+4j_1+1)+j_1}\,(1-q^{2m+1}) \\ \times\ (q^{j_2^2} - q^{(j_2+1)^2})\,(q^{j_3^2} - q^{(j_3+1)^2})\ . \quad (4.4.34)$$

Before we proceed let us note that the branching functions $\tilde{\psi}^{\mathsf{B}}_{\Lambda}$ possess the following important symmetry properties necessary for a proof in Appendix C,

$$\tilde{\psi}^{\mathsf{B}}_{(j_1,j_2,j_3)}(q) = -\tilde{\psi}^{\mathsf{B}}_{(-j_1-1,j_2,j_3)}(q) = -\tilde{\psi}^{\mathsf{B}}_{(j_1,-j_2-1,j_3)}(q) = -\tilde{\psi}^{\mathsf{B}}_{(j_1,j_2,-j_3-1)}(q)\ . \quad (4.4.35)$$

These imply in particular that $\psi^{\mathsf{B}}_{(j_1,j_2,j_3)}(q)$ vanishes identically if any of the spin labels j_a is equal to $j_a = -1/2$. As in our analysis of the sigma model's partition function Z in subsection 4.3.3, we can express all characters of representations of the bosonic subalgebra as infinite linear combinations of the characters of Kac modules. The required formulae can be found in Appendix B.2. With their help we now arrive at the following result for the branching functions $\tilde{\psi}^{\mathsf{K}}_{\Lambda}$:

$$\begin{aligned}
\tilde{\psi}^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) &= \frac{1}{\eta(q)\phi^3(q)} \sum_{n,m=0}^{\infty} (-1)^{n+m}\, q^{\frac{m}{2}(m+4j_1+1)+j_1+mn+\frac{n}{2}}\,(1-q^{2m+1}) \\
&\quad \times \sum_{k=0}^{[\frac{n}{2}]} (q^{(j_2-\frac{n}{2}+k)^2} - q^{(j_2+\frac{n}{2}-k+1)^2})\,(q^{(j_3-\frac{n}{2}+k)^2} - q^{(j_3+\frac{n}{2}-k+1)^2}) \\
&= \frac{1}{\eta(q)\phi^3(q)} \sum_{n,m=0}^{\infty} (-1)^{n+m}\, q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}}\,(1-q^{2m+1}) \\
&\quad \times (q^{(j_2-\frac{n}{2})^2} - q^{(j_2+\frac{n}{2}+1)^2})\,(q^{(j_3-\frac{n}{2})^2} - q^{(j_3+\frac{n}{2}+1)^2}) \sum_{k=0}^{\infty} q^{(2m+1)k}\ .
\end{aligned}$$

The sum over k at the end of this formula is a simple geometric series which cancels the last term in the first line, thereby recovering the expression (4.4.29). □

4.4.4 Deformation of the spectrum

The main result of our analysis so far was summarized concisely in eq. (4.4.30). In order to fully appreciate its content, let us review a few results from [34]. In that paper, the deformation of conformal weights was studied for the WZNW model on PSL(2|2). Many of the central results of [34], however, hold much more generally for models whose symmetries are described by an affine Lie superalgebra with vanishing dual Coxeter number.

To begin with, let us specify the bulk perturbation we would like to consider. As we shall argue momentarily, it is generated by the field,

$$\Phi \;=\; \sum \kappa_{\mu\nu} J^\mu(z)\Omega(\bar{J}^\nu(\bar{z})) \tag{4.4.36}$$

where the summation extends over all 17 bosonic and fermionic directions. The automorphism Ω we inserted here is the same as the gluing automorphism that was defined implicitly through our gluing conditions (4.4.22) and (4.4.23) in section 4.4.2. Note that the perturbing operator Φ breaks the global symmetry from $\mathrm{osp}\,(4|2) \otimes \mathrm{osp}\,(4|2)$ of the free GN model (4.1.8) to the twisted diagonal subalgebra. In other words, the symmetry transformations of the perturbed model are generated by elements of the form $X \otimes 1 + 1 \otimes \Omega(X)$. This means that any perturbing operator of the form Φ preserves half of the global bulk symmetries. What depends on the choice of the automorphism Ω is the precise set of transformations that is preserved. Similar statements can be made about boundary conditions. As we discussed in section 4.4.2, the boundary theory we put forward to compare with the boundary spectrum of the sigma model required to select a non-trivial gluing automorphism Ω. If this gluing automorphism would differ from the automorphism Ω in the definition of Φ, then the boundary condition and the deformation would preserve different sets of symmetry generators. Hence, the deformed boundary theory would no longer possess a global $\mathrm{osp}\,(4|2)$ symmetry. Such a theory could be conformal, but it cannot be equivalent to the boundary sigma model. Therefore, we know that the perturbing operator Φ must involve the same automorphism Ω that appeared in the gluing condition for currents at the boundary. An explicit formula for the operator Φ in terms of free fields is derived at the end of appendix D. The resulting expression agrees with the formula (4.1.9) for $\mathcal{S}^{\text{int}}$ that we anticipated in the introduction.

Having specified the deforming operator, we are now ready to discuss the properties of the deformation it generates. Here we shall closely follow the the recent analysis in [34] that we discussed in the last part of chapter 1. Everything we shall claim below is based on a rather simple mathematical result that was first formulated and exploited in the work of Bershadsky et. al. [15] for $\mathrm{psl}\,(N|N)$, but holds equally for $\mathrm{osp}(2M+2|2M)$.

To evaluate the change of conformal weights away from the free GN model, we perform a perturbative analysis of 2-point functions in our theory. In any such computation of perturbed correlators, the initial step is to remove all the current insertions through current algebra Ward identities. In the process, pairs of currents get contracted using

$$J^\mu(z)\, J^\nu(w) \;=\; \frac{f^{\mu\nu}{}_\sigma}{z-w}\, J^\sigma(w) + \frac{k\kappa^{\mu\nu}}{(z-w)^2} + \ldots \;\sim\; \frac{k\kappa^{\mu\nu}}{(z-w)^2} \quad . \qquad (4.4.37)$$

The first equality is the usual operator product for osp (4|2) currents. Since we are only interested in computing the invariants h, we can, as argued in 1, forget about all the terms that involve the structure constants f of the Lie superalgebra osp (4|2). This applies to the first term in the above operator product which distinguishes the non-abelian currents from the abelian algebra of flat target spaces. Here and in the following we shall use the symbol $\sim$ to mark equalities that are true up to terms involving structure constants. In conclusion, we have seen that, as far as the computation of conformal dimensions is concerned, we may neglect the non-abelian nature of the currents J^μ. The remaining non-zero terms correspond one by one to terms in the radius deformation of a compactified free boson. Hence, they can be regularized and summed as in the abelian case, which is well explained in [63].

In [34] several other statements were needed to study a deformation that preserved simultaneously both left and right global symmetries. The perturbation (4.1.9) we consider here, however, is of a much simpler type. We can therefore directly move on to evaluate the conformal dimension of boundary fields. Unlike in [34], the following arguments apply to all boundary conditions, as long as they preserve the affine $\widehat{\text{osp}}(4|2)$ symmetry. It does not require any further assumptions on the localization of the brane. Let Ψ be some multiplet of boundary fields transforming in a representation $[\Lambda]$ of osp (4|2). We denote by $h_0(\Psi)$ the conformal weight of Ψ at the WZ-point. Upon deformation with the field (4.4.36), the weight of Ψ behaves as a free boson state in (4.2.9), namely

$$h(\Psi) \;=\; h_0(\Psi) - \frac{1}{2}\frac{g^2}{1+g^2}\;\mathsf{Cas}(\Lambda) \;=\; h_0(\Psi) + \frac{1}{2}\left(\frac{1}{R^2} - 1\right)\mathsf{Cas}(\Lambda) \qquad (4.4.38)$$

where Cas is the quadratic Casimir element of the Lie superalgebra osp (4|2), as before. Here, just like in subsection 4.3.4, the cohomological reduction allows us to find the states that belong to the free boson sector, so that we obtain the identification $R^2 = 1 + g^2$. In figure 4.6, we illustrate how the Casimir evolution affects the lowest lying fields of the Gross-Neveu model.

Through the Casimir decomposition (4.4.28) of the boundary partition function $\tilde{\mathsf{Z}}$ we have separated all boundary fields according to their osp (4|2) transformation law. This now allows us to evaluate the shift of conformal weights for entire blocks rather than individual field multiplets. More concretely, the conformal weights of all fields that are counted by the branching function $\tilde{\psi}^{\mathsf{K}}_{[j_1,j_2,j_3]}$ undergo the same shift by[12]

$$\delta_g(h) \;=\; -\frac{1}{2}\frac{g^2}{1+g^2}\;\mathsf{Cas}[j_1,j_2,j_3] \;=\; \frac{g^2}{1+g^2}\,\big(2j_1(j_1-1) - j_2(j_2+1) - j_3(j_3+1)\big)$$

[12] Let us recall that all irreducible multiplets that can be tied together in an indecomposable representation must have identical Casimir eigenvalues, see appendix A.

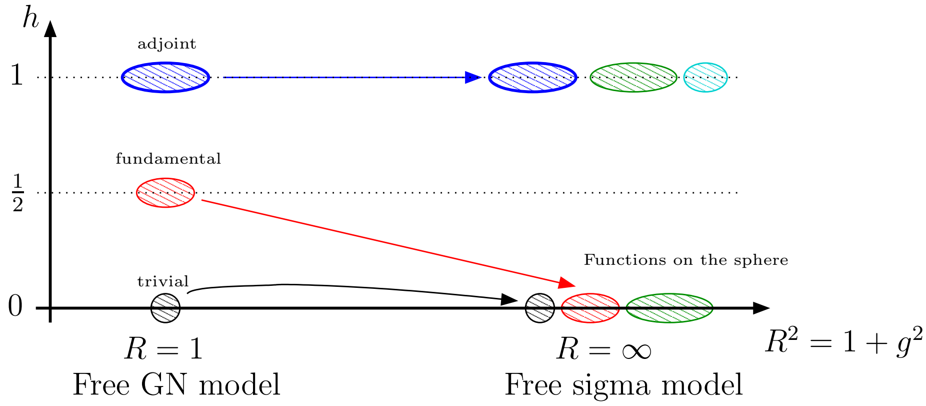

Figure 4.6: The basic fields $(\psi_i, \beta_a, \gamma_a)$ of the Gross Neveu model become under the deformation the fundamental fields $X^\mu = (x_i, \eta_a)$ of the supersphere σ model.

upon perturbation with Φ. Thereby, we can spell out the boundary spectrum of the perturbed model for any choice of $g^2 = R^2 - 1$,

$$\begin{aligned} \tilde{\mathsf{Z}}_R(q, z_i) = \; & q^{-\frac{1}{24}} \sum\nolimits_{j_i} \chi^{\mathsf{K}}_{[j_1,j_2,j_3]}(z_1, z_2, z_3) \times \\ & \times \; q^{\left(1-\frac{1}{R^2}\right)(2j_1(j_1-1)-j_2(j_2+1)-j_3(j_3+1))} \tilde{\psi}^{\mathsf{K}}_{[j_1,j_2,j_3]}(q) \; . \end{aligned} \tag{4.4.39}$$

For irrational values of the parameter R, the boundary spectrum is rather rich, containing irrational conformal weights. But as we reach the special value $R = \infty$, all conformal weights become integers. Equation (4.4.30) tells us even more: At this particular point, the perturbed boundary partition function coincides with the partition function Z of volume filling branes in the sigma model on the supersphere $S^{3|2}$ in the limit $R \to \infty$. For a few selected multiplets, the deformation from $R = \infty$ to $R = 1$ had been carried out in [19]. By performing the Casimir decompositions explicitly, we were able to extend such studies to the entire spectrum.

4.5 Generalization for higher dimensions

The aim of this section is to outline how the previous analysis may be extended to higher dimensional superspheres. We shall provide explicit formulae for the relevant boundary spectra of the sigma model at $R = \infty$ and for the free field theory (4.1.8). The latter are expressed in terms of characters of the affine $\widehat{\text{osp}}(2M+2|2M)$ superalgebra at $k = 1$. Note that the level does not depend on M. Since we have not attempted to construct the branching functions ψ_Λ and $\tilde{\psi}_\Lambda$ for the decomposition with respect to the global $\text{osp}(2M+2|2M)$ symmetry, we shall content ourselves with a few non-trivial tests. These are discussed in the second subsection. We believe that a full analysis, as in the case of $M = 1$, is possible but cumbersome.

4.5.1 Partition functions for superspheres at $R = 1, \infty$

The first task is to spell out the spectrum of the sigma model with Neumann boundary conditions at $R = \infty$. It turns out that our formula (4.3.21) for $M = 1$ admits the following straightforward generalization,

$$\mathsf{Z}^{\sigma}_{\mathfrak{N};M} \;=\; q^{-\frac{1}{24}}\mathsf{Z}_0^{(M)}\phi(q)\prod_{n=1}^{\infty}\frac{\prod_{m=1}^{M}(1+y_m q^n)(1+y_m^{-1}q^n)}{\prod_{k=1}^{M+1}(1-x_k q^n)(1-x_k^{-1}q^n)}\;. \tag{4.5.1}$$

Here again, the subscript $\mathfrak{N}$ stands for Neumann boundary conditions and the minisuperspace contribution is given by

$$\mathsf{Z}_0^{(M)} \;=\; \lim_{t\to 1}(1-t^2)\frac{\prod_{m=1}^{M}(1+y_m t)(1+y_m^{-1}t)}{\prod_{k=1}^{M+1}(1-x_k t)(1-x_k^{-1}t)}\;. \tag{4.5.2}$$

As before, the factor $\mathsf{Z}_0^{(M)}$ describes the space of functions on $S^{2M+1|2M}$. As mentioned above, we have not performed the analysis of subsection 4.3.3 for the more general partition function $\mathsf{Z}^{\sigma}_{\mathfrak{N};M}$, though this would surely be possible.

Next let us turn to the free GN model (4.1.8). Large parts of our analysis of the bulk spectrum were already performed for generic M. Once more, the theory possesses an affine $\widehat{\mathrm{osp}}(2M+2|2M)$ symmetry with level $k = 1$ (see appendix D for an explicit construction of the generators in terms of the basic fields). The bulk theory can be shown to possess a symmetry preserving boundary condition whose spectrum closely resembles equation (4.4.27). Before we are able to spell out the details, we shall quote from [42] the following expressions for characters of the affine Lie algebra $\widehat{\mathrm{so}}(2M+2)$ at level $k = 1$,

$$\begin{aligned}\widehat{\chi}^{\mathrm{so}}_{\varnothing}(q,x_i) &= \frac{1}{2\eta(q)^{M+1}}\left(\prod_{i=1}^{M+1}\theta_3(q,x_i)+\prod_{i=1}^{M+1}\theta_4(q,x_i)\right)\;,\\ \widehat{\chi}^{\mathrm{so}}_{\Box}(q,x_i) &= \frac{1}{2\eta(q)^{M+1}}\left(\prod_{i=1}^{M+1}\theta_3(q,x_i)-\prod_{i=1}^{M+1}\theta_4(q,x_i)\right)\;.\end{aligned} \tag{4.5.3}$$

Here, $\widehat{\mathrm{so}}(2M+2)_1$ is part of the bosonic subalgebra of $\widehat{\mathrm{osp}}(2M+2|2M)_1$. Similarly, we also need the corresponding characters of the affine $\widehat{\mathrm{sp}}(2M)$ at $k = -\frac{1}{2}$

$$\begin{aligned}\widehat{\chi}^{\mathrm{sp}}_{\varnothing}(q,y_i) &= \frac{\eta(q)^M}{2}\left(\frac{1}{\prod_{i=1}^{M}\theta_4(q,y_i)}+\frac{1}{\prod_{i=1}^{M}\theta_3(q,y_i)}\right)\;,\\ \widehat{\chi}^{\mathrm{sp}}_{\Box}(q,y_i) &= \frac{\eta(q)^M}{2}\left(\frac{1}{\prod_{i=1}^{M}\theta_4(q,y_i)}-\frac{1}{\prod_{i=1}^{M}\theta_3(q,y_i)}\right)\;.\end{aligned} \tag{4.5.4}$$

The characters we have just listed, furnish the basic building blocks for the relevant characters of our superalgebra $\widehat{\mathrm{osp}}(2M+2|2M)_1$ at level $k = 1$,

$$\begin{aligned}\widehat{\chi}^{\mathrm{osp}}_{\varnothing} &= \widehat{\chi}^{\mathrm{so}}_{\varnothing}\widehat{\chi}^{\mathrm{sp}}_{\varnothing}+\widehat{\chi}^{\mathrm{so}}_{\Box}\widehat{\chi}^{\mathrm{sp}}_{\Box}\;,\\ \widehat{\chi}^{\mathrm{osp}}_{\Box} &= \widehat{\chi}^{\mathrm{so}}_{\Box}\widehat{\chi}^{\mathrm{sp}}_{\varnothing}+\widehat{\chi}^{\mathrm{so}}_{\varnothing}\widehat{\chi}^{\mathrm{sp}}_{\Box}\;.\end{aligned} \tag{4.5.5}$$

For a particular choice of boundary conditions in the free field theory (4.1.8) the boundary partition function takes the following form

$$Z^{\mathrm{GN}}_{\mathfrak{B};M}(q,z_i) \;=\; \widehat{\chi}^{\mathrm{osp}}_{\varnothing}+\widehat{\chi}^{\mathrm{osp}}_{\square} = \frac{1}{\eta(q)}\frac{\prod_{i=1}^{M+1}\theta_3(q,x_i)}{\prod_{j=1}^{M}\theta_4(q,y_j)}\ , \tag{4.5.6}$$

where the first M variables $z_i = y_i$ are associated with the symplectic part while the remaining $M+1$ variables $z_{M+i} = x_i$ are affiliated with Cartan elements of the orthogonal subalgebra. Eq. (4.5.6) generalizes equation (4.4.27) to $M \geq 1$.

4.5.2 Test of the duality

As in the previous section, we would like to show that the two partition functions (4.5.1) and (4.5.6) are related to each other by deformation with the interaction term (4.1.9) or, equivalently, by deforming the radius R of the sigma model from $R = \infty$ all the way down to $R = 1$. In principle, this may be achieved by repeating our analysis in subsections 4.3.3 and 4.4.3 above. The first step is to decompose the partition function (4.5.6) of the sigma model at $R = \infty$ in terms of character functions for the global $\mathrm{osp}(2M+2|2M)$ symmetry,

$$Z^{\sigma}_{\mathfrak{N},M} \;=\; \sum_{\Lambda\in\Gamma^+}\chi^{\mathrm{osp}(2M+2|2M)}_{\Lambda}(x_i,y_j)\,\psi^{(M)}_{\Lambda}(q)\ , \tag{4.5.7}$$

where Γ^+ is the set of all integral dominant labels of $\mathrm{osp}(2M+2|2M)$ that are compatible with the consistency conditions of [35]. The existence of such a decomposition is guaranteed, but in case of $M > 1$ explicit formulae for the branching functions ψ would still need to be worked out.

The second step is to pass from $R = \infty$ to finite values of the radius. Since all the general results we outlined in subsection 4.4.4 hold for any value of M, the boundary partition function of the sigma model at radius R reads

$$Z^{\sigma}_{M}(R) \;=\; \sum_{\Lambda\in\Gamma^+}\chi^{\mathrm{osp}(2M+2|2M)}_{\Lambda}(x_i,y_j)\,\psi^{(M)}_{\Lambda}(q)\,q^{\frac{1}{2}\frac{1}{R^2}C(\Lambda)}\ . \tag{4.5.8}$$

Here we expressed the partition function through the branching functions ψ at $R = \infty$ rather than through the ones at $R = 1$, as in subsection 4.4.4. Therefore, the coefficient of the Casimir element had to be properly adjusted. Note also that we normalized the quadratic Casimir operator such that $\mathsf{Cas}(\square) = 1$ for all values of M.

For the sigma models on odd dimensional superspheres $S^{2M+1|2M}$ to be dual to the GN model, we would have to find

$$Z^{\sigma}_{M}(R=1) \;=\; Z^{\mathrm{GN}}_{\mathfrak{B};M}\ , \tag{4.5.9}$$

provided we have correctly identified the appropriate boundary condition in the free field theory (4.1.8). Throughout the last sections, we have checked relation (4.5.9) explicitly for $M = 1$. It is quite amusing to verify it also in the much simpler case of $M = 0$. When $M = 0$, the decomposition of the partition function at $R = \infty$ into characters of osp(2|0)≅ so(2), takes a particularly simple form,

$$\begin{aligned} \mathsf{Z}^{\sigma}_{\mathfrak{R},M=0} &= q^{-\frac{1}{24}}\phi(q)\sum_{n\in\mathbb{Z}} z^n \sum_{k\in\mathbb{Z}} \frac{z^k}{\phi(q)^2}\sum_{m=0}^{\infty}(-1)^m\left(q^{\frac{m+1}{2}(m+2|k|)} - q^{\frac{m+1}{2}(m+2(|k|+1))}\right) \\ &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} z^n = \sum_{n\in\mathbb{Z}} \chi^{\mathrm{so}(2)}_n(z)\psi^{(0)}_n(q) \;, \end{aligned} \tag{4.5.10}$$

with $\chi^{\mathrm{so}(2)}_n(z) = z^n$ and $\psi^{(0)}_n(q) = 1/\eta(q)$. Following our equation (4.5.8), the partition function for radius R becomes

$$\mathsf{Z}^{\sigma}_{M=0}(R) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} z^n\, q^{\frac{1}{2}\frac{1}{R^2}n^2} \;.$$

Therefore, at $R = 1$ we obtain

$$\mathsf{Z}^{\sigma}_{M=0}(R=1) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2}} = \mathsf{Z}^{\mathrm{GN}}_{\mathfrak{B};M=0}(q,z) \;, \tag{4.5.11}$$

in agreement with our general prediction (4.5.9).

Although we have not been able to find a conclusive proof of (4.5.9) for $M \geq 2$, we wish to give some additional supporting evidence. To this end, we need a few more details about representations of osp$(2M+2|2M)$ and the corresponding values of the quadratic Casimir element. The representations we are interested in are labeled by integral dominant highest weights Λ of the form

$$\begin{aligned} \Lambda &= a_1\delta_1 + a_2(\delta_1+\delta_2) + \cdots + a_M(\delta_1 + \cdots \delta_M) + a_{M+1}\epsilon_1 + \cdots + a_{2M-1}(\epsilon_1 + \cdots \epsilon_{M-1}) \\ &\quad + a_{2M}\frac{\epsilon_1 + \cdots + \epsilon_M - \epsilon_{M+1}}{2} + a_{2M+1}\frac{\epsilon_1 + \cdots + \epsilon_M + \epsilon_{M+1}}{2} \;, \end{aligned} \tag{4.5.12}$$

where δ_i and ϵ_j appear in the construction of the weight system of osp$(2M+2|2M)$ and obey $(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$. The numerical coefficients $a_i \in \mathbb{N}$ must moreover obey some additional consistency conditions that can be found in [35]. The value of the quadratic Casimir in the representation of weight Λ can now be expressed in terms of the coefficients a_i as,

$$\begin{aligned} \mathsf{Cas}(\Lambda) &= (\Lambda, \Lambda + 2\rho) = -\sum_{i=1}^{M}\left(\sum_{j=i}^{M} a_j - 2i\right)\sum_{k=i}^{M} a_k + \frac{(a_{2M} - a_{2M+1})^2}{4} \\ &+ \sum_{i=1}^{M}\left(\sum_{j=i}^{M-1} a_{M+j} + \frac{a_{2M}+a_{2M+1}}{2} + 2(M+1-i)\right)\left(\sum_{k=i}^{M-1} a_{M+k} + \frac{a_{2M}+a_{2M+1}}{2}\right) \;. \end{aligned}$$

The fundamental representation corresponds to $a_1 = 1$ and $a_i = 0$ for $i \neq 1$ so that $C_{\delta_1} = -(1-2) = 1$ for all M. The value of the quadratic Casimir does not only determine the deformation of conformal weights, see eq. (4.5.9). It is also needed to compute the conformal weight

$$h_\Lambda = \frac{\mathsf{Cas}(\Lambda)}{2k} \tag{4.5.13}$$

of fields that are primary with respect to the underlying affine superalgebra at level k. In our case, the level k must be set to $k=1$, as before.

After this preparation we can begin to test equation (4.5.9). Let us first try to recover the ground states of the free field theory at $R=1$. It is clear that the vacuum state at $R=1$ is obtained by deforming the unique $\mathrm{osp}(2M+2|2M)$ invariant field with weight $h=0$ at $R=\infty$. So, we can turn to the ground states in the second sector of eq. (4.5.6) right away. From (4.5.7) we infer that the boundary sigma model contains a single field multiplet that transforms in the fundamental representation with $\Lambda = \delta_1$ and has conformal weight $h=0$. Under the proposed deformation, the conformal weight of this multiplet is lifted from $h=0$ to $h=1/2$, since $\mathsf{Cas}(\delta_1)=1$. The latter value agrees precisely with the ground state energy of the corresponding affine representation when $k=1$ as given by (4.5.13).

We want to go a little further and recover states in the $R=1$ model whose weight is one above the ground states. Let us pick, for example, a multiplet that transforms on the representation $\Lambda = 3\delta_1$. In the large radius limit, this representation arises for the first time among the states of weight $h=3$, since in equation (4.5.1) terms containing y_1^3 are multiplied by q^3 or higher powers of q. Since $\mathsf{Cas}(3\delta_1)=3$, the proposal (4.5.9) tells us that the weight of this multiplet gets deformed to $h = 3 - \frac{3}{2} = \frac{3}{2}$. Hence, it should appear among the first descendants of the sector over the fundamental representation. Indeed, the irreducible representation with highest weight $3\delta_1$ is contained in the tensor product of the fundamental representation with the adjoint representation. Thus, $Z^{\mathrm{GN}}_{\mathfrak{B};M}$ contains this representation with $h=\frac{3}{2}$ exactly as predicted by eq. (4.5.9).

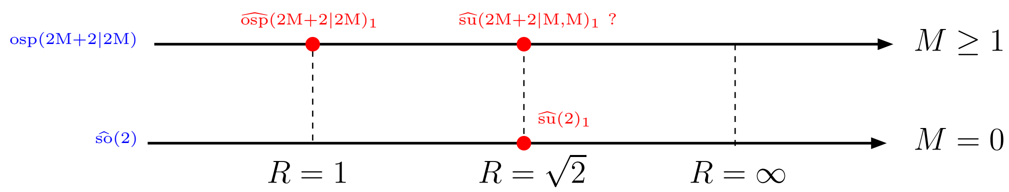

Figure 4.7: The symmetries of the model can vary considerably with the value of the radius. The red dots specify points in the moduli space where the symmetry becomes bigger and the generic symmetry algebras are written in blue.

Let us end this discussion with some remarks concerning the change of the symmetries of the model as we vary the radius, depicted in figure 4.7. For the free boson, it is well known that at the *self-dual* point given by $R=\sqrt{2}$, the model becomes equivalent to a $\widehat{\mathrm{su}}(2)_1$ WZW model, thus the symmetry gets enhanced. For $M>0$ something similar seems to happen, though we have not been able to give conclusive evidence.

We nevertheless see that for $R = \sqrt{2}$, the space of states with $h = 1$ increases and transforms in the adjoint representation of su$(2M+2|M,M)$, which would suggest the presence of affine symmetry.

4.6 $\mathcal{N} = 1$ extension of the model

In this section, we want to investigate an extension of the $S^{2M+1|2M}$ sigma models that includes worldsheet supersymmetry. Our long term goal is to apply the insight gained from the study of this worldsheet supersymmetric variant to the $\mathcal{N} = 2$ extension of the complex projective superspaces $\mathbb{CP}^{S-1|S}$ models, which could bring fresh light to the proposal of [64]. In particular, the next chapter will see us investigating the non-worldsheet supersymmetric versions of the $\mathbb{CP}^{S-1|S}$ sigma model quite thoroughly, though unfortunately a study of its $\mathcal{N} = 2$ analogue lies outside the scope of this work.

Just as in the non-worldsheet supersymmetric version of $S^{2M+1|2M}$, we want to compute the spectra of Neumann branes in the infinite volume limit. Unfortunately, the methods that we used before in section 4.3.2 turn out to be very hard to generalize. We must thus look for a new way of deriving formula (4.5.1) and hope that it can be easily extended. It turns out that by using a method in many ways reminiscent of usual BRST quantization[13] does the trick. For that we reformulate the original problem of counting functions and their derivatives on $S^{2M+1|2M}$ as one involving an infinite set of decoupled oscillators. We view the X_a and their derivatives as independent creation operators

$$X_{a,n} \equiv \partial^n X_a \qquad \text{for } n \in \mathbb{N},\, a \in \{0, \ldots, 4M+2\} \ . \tag{4.6.1}$$

All the $X_{a,n}$ commute with each other in the graded sense. Dual to these operators, we define the set of mutually commuting annihilation operators ∂^a_n by setting

$$[\partial^a_m \, , \, X_{b,n}] = \delta^a_b \delta_{m,n} \ . \tag{4.6.2}$$

We introduce a vacuum state $|\,0\rangle$, requiring that it be annihilated by all the ∂^a_n. Acting on it with the $X_{a,n}$ generates an infinite dimensional complex vector space. We can introduce the a set of number operators N_n by

$$N_n = \sum_{a=1}^{4M+2} X_{a,n} \partial^a_n \ , \tag{4.6.3}$$

which lead to a total number operator $N := \sum_{n=0}^{\infty} N_n$ that counts the number of oscillators applied to the vacuum and to an energy operator $E := \sum_{n=0}^{\infty} n N_n$.

We wish to impose the constraint $X_{a,0} \mathsf{J}^{ab} X_{b,0} = 1$ on the elements of this space. Defining the even operator $H_0 := \sum_{a,b} \mathsf{J}_{ba} \partial^a_0 \partial^b_0$, we see that the states in the kernel of

[13] The cohomological arguments shown in this section are only superficially related to those in chapter 3, i.e. we are *not* dealing with a cohomological reduction.

H_0 are precisely the ones in which the $X_{a,0}$ are organized in traceless symmetric tensors, i.e. exactly the ones we are interested in. For example

$$
\begin{aligned}
&H_0 \underbrace{|\,0\rangle}_{\varnothing} = 0 \qquad H_0 \underbrace{X_{a,0}\,|\,0\rangle}_{\Box} = 0 \\
&H_0 \underbrace{\left(X_{a,0}X_{b,0} - \frac{1}{2}\mathsf{J}_{ab}\mathsf{J}^{cd}X_{c,0}X_{d,0}\right)|\,0\rangle}_{\Box\Box} = 0 \ ,
\end{aligned}
\tag{4.6.4}
$$

so that we recover the symmetric traceless tensors that form a basis for the functions on the superspheres. Thus, imposing $X_{a,0}\mathsf{J}^{ab}X_{b,0} = 1$ is equivalent to requiring that the physical states be in the kernel of H_0. We now have to take into account all the derivatives of the supersphere constraint. For that, we define a bosonic operator D_1 by setting $[D_1\,,\,\partial^a_n] = \partial^a_{n+1}$. Using it, we set by recursion

$$
H_n := [D_1\,,\,H_{n-1}] = \sum_{k=0}^{n}\sum_{a,b=1}^{4M+2} \mathsf{J}_{ba} \begin{pmatrix} n \\ k \end{pmatrix} \partial^a_k \partial^b_{n-k} \ . \tag{4.6.5}
$$

Imposing the additional constraints is then easily seen to be equivalent to requiring that physical states be annihilated by all H_n. We now want to reformulate this requirements in the language of BRST quantization. We introduce a set of fermionic operators, called *ghosts*, $\{b_n,\,c_n\}_{n\in\mathbb{N}}$ with the anticommutation relations

$$
\{b_n\,,\,b_m\} = \{c_n\,,\,c_m\} = 0 \qquad \{b_n\,,\,c_m\} = \delta_{n,m} \ , \tag{4.6.6}
$$

and use them to define the anticommuting operators $Q_n := c_n H_n$. We require that the vacuum state be annihilated by all the b_n. To each set $\{b_n,\,c_n\}$ we associate a somewhat arbitrary charge operator T_n, requiring that

$$
[T_n\,,\,b_m] = \delta_{n,m} b_m \qquad [T_n\,,\,c_m] = -\delta_{n,m} c_m \ . \tag{4.6.7}
$$

Now, the states that are constructed only using the $X_{a,n}$, i.e. without the application of ghost creation operators, and that are annihilated by a given H_n can be understood as lying in the cohomology of the corresponding Q_n as presented in figure 4.8.

If we denote by $\mathcal{V}$ the Verma module obtained by letting the creation operators $X_{a,n}$ and c_m act freely on $|\,0\rangle$, we can introduce the partition function

$$
\begin{aligned}
\mathsf{Z}_{\mathcal{V}} &= \mathrm{tr}_{\mathcal{V}}\left[\prod_{i=1}^{M+1} x_i^{J_i} \prod_{j=1}^{M} y_j^{\tilde{J}_j} u^N q^{E-\frac{c}{24}} \prod_{n=0}^{\infty} t_n^{T_n}\right] \\
&= q^{-\frac{1}{24}} \prod_{m=0}^{\infty} \frac{\prod_{j=1}^{M}(1+y_j u q^m)(1+y_j^{-1} u q^m)}{\prod_{i=1}^{M+1}(1-x_i u q^m)(1-x_i^{-1} u q^m)} \prod_{n=0}^{\infty}(1+t_n^{-1}) \ ,
\end{aligned}
\tag{4.6.8}
$$

where by J_i, respectively $\tilde{J}_j$ we denote the Cartan generators of the orthogonal, respectively symplectic subalgebra of $\mathrm{osp}\,(2M+2|2M)$. We want to compute the same trace

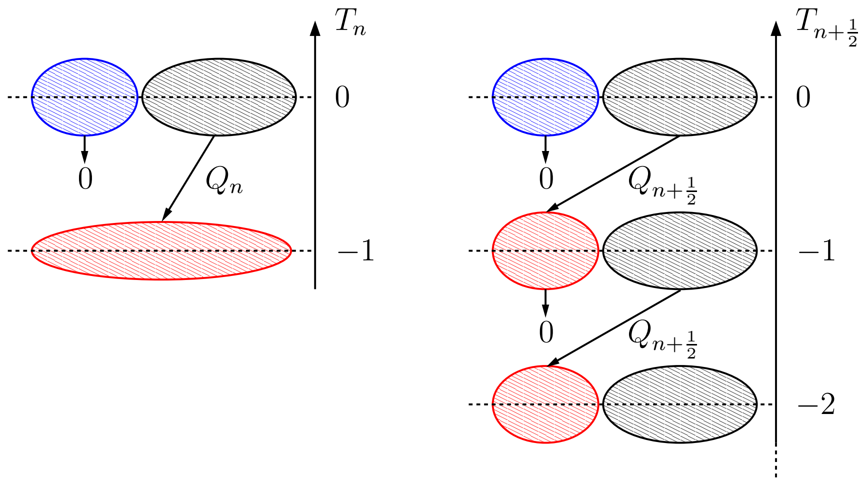

Figure 4.8: The computation of the cohomology of Q_n. The cohomology of Q_n is drawn blue, the image is red. The left side presents the integer mode case with fermionic ghosts, whereas the right side is concerned with the half integer case with bosonic ghosts.

over the space of physical states defined by the *combined cohomology* of all the operators Q_n. Our experience from chapter 3 can be of great help here. For a given operator Q_n we can decompose the partition function as

$$\mathsf{Z}_{\mathcal{V}} = \mathsf{Z}_{\mathcal{V}}^{\text{cohomology}} + \mathsf{Z}_{\mathcal{V}}^{\text{image}} + \mathsf{Z}_{\mathcal{V}}^{\text{preimage}} = \mathsf{Z}_{\mathcal{V}}^{\text{cohomology}} + \mathsf{Z}_{\mathcal{V}}^{\text{image}}(1 + t_n u^2 q^n) \ , \tag{4.6.9}$$

where we could express $\mathsf{Z}_{\mathcal{V}}^{\text{preimage}}$ with the help of $\mathsf{Z}_{\mathcal{V}}^{\text{image}}$ since

$$[T_m\, , \, Q_n] = -\delta_{mn} Q_n \quad [N\, , \, Q_n] = -2Q_n \quad [E\, , \, Q_n] = -nQ_n \ . \tag{4.6.10}$$

The variables t_n are not of any physical relevance, since they only count the number of ghost operators. Thus, we can set $t_n = -u^{-2}q^{-n}$ and obtain

$$\mathsf{Z}_{\mathcal{V}}(t_n = -u^{-2}q^{-n}) = \mathsf{Z}_{\mathcal{V}}^{\text{cohomology}}(t_n = -u^{-2}q^{-n}) = \mathsf{Z}_{\mathcal{V}}^{\text{cohomology}} \ , \tag{4.6.11}$$

where the last equality is valid since, as argued in figure 4.8, the cohomology lies exclusively in states on which T_n is zero. Since all the Q_n anticommute, we can perform this restriction for all the t_n simultaneously and thus obtain the partition function on the combined cohomology:

$$\mathsf{Z}^{\sigma}_{\mathfrak{N},M} = q^{-\frac{1}{24}} \prod_{m=0}^{\infty} \frac{\prod_{j=1}^{M}(1 + y_j u q^m)(1 + y_j^{-1} u q^m)}{\prod_{i=1}^{M+1}(1 - x_i u q^m)(1 - x_i^{-1} u q^m)} \prod_{n=0}^{\infty}(1 - u^2 q^n) \ . \tag{4.6.12}$$

Since we are not interested in counting the number of oscillators, the variable u is irrelevant and we can carefully take the limit $u \to 1$. By doing so, we obtain

$$\begin{aligned} \mathsf{Z}^{\sigma}_{\mathfrak{N},M} &= q^{-\frac{1}{24}} \underbrace{\left[\lim_{u\to 1} (1-u^2) \frac{\prod_{j=1}^{M}(1+y_j u)(1+y_j^{-1}u)}{\prod_{i=1}^{M+1}(1-x_i u)(1-x_i^{-1}u)} \right]}_{\equiv \mathsf{Z}_0} \times \\ & \quad \times \phi(q) \prod_{m=1}^{\infty} \frac{\prod_{j=1}^{M}(1+y_j q^m)(1+y_j^{-1}q^m)}{\prod_{i=1}^{M+1}(1-x_i q^m)(1-x_i^{-1}q^m)} \ , \end{aligned} \tag{4.6.13}$$

which coincides precisely with our previous results written in (4.5.1).

This new method can now applied to an extension of our model that includes $\mathcal{N}=1$ world sheet supersymmetry. We refer the reader to [65] for an introduction to the world-sheet supersymmetric non-linear sigma model on the bosonic spheres S^d. In order to formulate the theory, we extend the previous fields $X_a(z,\bar{z})$ to fields Φ_a on a superspace with two fermionic directions, with the expansion:

$$\Phi_a(z,\bar{z},\theta,\bar{\theta}) = X_a(z,\bar{z}) + i\theta\bar{\Psi}_a(z,\bar{z}) - i\bar{\theta}\Psi_a(z,\bar{z}) + i\theta\bar{\theta}F_a(z\bar{z}) \ . \tag{4.6.14}$$

The world sheet supersymmetric action that one then derives for these fields takes the form

$$\mathcal{S} = \frac{1}{2\pi}\int_{\Sigma} d^2z\, \mathsf{J}^{ab} \left(\partial X_a \bar{\partial} X_b + \frac{i}{2}\Psi_a\bar{\partial}\Psi_b + \frac{i}{2}\bar{\Psi}_a\partial\bar{\Psi}_b + \frac{1}{4}F_aF_b \right) \ . \tag{4.6.15}$$

It is easy to see that this action is invariant under the two anticommuting sets of superspace transformations

$$\begin{array}{llll} 1)\ X_a \mapsto -i\Psi_a & \Psi_a \mapsto 2\partial X_a & \bar{\Psi}_a \mapsto F_a & F_a \mapsto -2i\partial\bar{\Psi}_a \\ 2)\ X_a \mapsto i\bar{\Psi}_a & \Psi_a \mapsto F_a & \bar{\Psi}_a \mapsto -2\bar{\partial}X_a & F_a \mapsto -2i\bar{\partial}\Psi_a \end{array} \ , \tag{4.6.16}$$

which square to $-2i\partial$, respectively to $-2i\bar{\partial}$. The world sheet extension of the super-sphere constraint is then $\mathsf{J}^{ab}\Phi_a\Phi_b = R^2$, which by (4.6.16) translates to a set of three relations for the component fields:

$$\begin{aligned} X_a\mathsf{J}^{ab}X_b &= R^2 \\ \bar{\Psi}_a\mathsf{J}^{ab}X_b &= \Psi_a\mathsf{J}^{ab}X_b = 0 \\ F_a\mathsf{J}^{ab}X_b &= i(-1)^{|a|}\bar{\Psi}_a\mathsf{J}^{ab}\Psi_b \ . \end{aligned} \tag{4.6.17}$$

The last equation can be resolved by setting the auxiliary fields F_a equal to

$$F_a = i\frac{(-1)^{|b|}\bar{\Psi}_b\mathsf{J}^{bc}\Psi_c}{R^2}X_a \ , \tag{4.6.18}$$

which, when inserted into the Lagrangian, leads to

$$\mathcal{S} = \frac{1}{2\pi}\int_{\Sigma} d^2z \left[\mathsf{J}^{ab}\left(\partial X_a\bar{\partial}X_b + \frac{i}{2}\Psi_a\bar{\partial}\Psi_b + \frac{i}{2}\bar{\Psi}_a\partial\bar{\Psi}_b \right) - \frac{\left((-1)^{|a|}\bar{\Psi}_a\mathsf{J}^{ab}\Psi_b\right)^2}{4R^2} \right] \ , \tag{4.6.19}$$

subject to the constraints

$$X_a \mathsf{J}^{ab} X_b = R^2 \qquad \Psi_a \mathsf{J}^{ab} X_b = \bar{\Psi}_a \mathsf{J}^{ab} X_b = 0 \ . \tag{4.6.20}$$

A rescaling of the fermionic fields by multiplication with $\sqrt{2}e^{-i\pi/4}$ allows a comparison to be made between this action and the one of formulae (2.5.16) and (2.5.17). We are thus lead to the conclusion that we have obtained a Lagrangian that combines the supersphere and Gross-Neveu models into one, with the additional requirement that the coordinates X_a be orthogonal to their worldsheet superpartners and that the GN coupling be related to the supersphere radius by $g^2 = R^{-2}$.

If we take the limit of infinite radius, the fermionic interaction term drops out. Taking into account that one boson and one fermion are removed by the constraints, it becomes easy to read the value of the central charge:

$$c = \underbrace{(2M+1)\times 1 + M\times(-2)}_{X_a} + \underbrace{(2M+1)\times\frac{1}{2} + M\times(-1)}_{\Psi_a} = \frac{3}{2} \ . \tag{4.6.21}$$

To compute the boundary spectrum at this point, we take our previous construction and add a new set of operators

$$X_{a,n+\frac{1}{2}} \equiv \partial^n \Psi_a \ , \tag{4.6.22}$$

together with their corresponding $\partial^a_{n+\frac{1}{2}}$. Of course, the grading of these new fields is opposite to the grading of the $X_{n,a}$ for integer n. We define the odd operator $D_{\frac{1}{2}}$ by setting $\left[D_{\frac{1}{2}}\ ,\ \partial^a_n\right] = \partial^a_{n+\frac{1}{2}}$ and use it to extend the definition of H_n to half integer values of n by setting $H_n := \left[D_{\frac{1}{2}}\ ,\ H_{n-\frac{1}{2}}\right]$. To complete the procedure, new bosonic ghosts $\left\{b_{n+\frac{1}{2}}, c_{n+\frac{1}{2}}\right\}_{n\in\mathbb{N}}$ are needed. Thus (4.6.6) gets extended to half integer modes, with the anticommutator replaced by a graded commutator. The partition function of the Verma module generated by the $X_{a,n}$ and c_n for $n \in \frac{\mathbb{N}}{2}$ is now

$$\begin{aligned} \mathsf{Z}_{\mathcal{V}} &= q^{-\frac{1}{16}} \prod_{m=0}^{\infty} \frac{\prod_{j=1}^{M}(1+y_j u q^m)(1+y_j^{-1}uq^m)\prod_{i=1}^{M+1}(1+x_i u q^{m+\frac{1}{2}})(1+x_i^{-1}uq^{m+\frac{1}{2}})}{\prod_{i=1}^{M+1}(1-x_i u q^m)(1-x_i^{-1}uq^m)\prod_{j=1}^{M}(1-y_j u q^{m+\frac{1}{2}})(1-y_j^{-1}uq^{m+\frac{1}{2}})} \times \\ &\quad \times \prod_{n=0}^{\infty} \frac{1+t_n^{-1}}{1-t_{n+\frac{1}{2}}^{-1}} \ . \end{aligned} \tag{4.6.23}$$

As before, we introduce the nilpotent operators $Q_n := c_n H_n$ and define the physical states as being those that lie in their combined cohomology. Their cohomology in $\mathcal{V}$ is graphically sketched in figure 4.8. The considerations of (4.6.11) are still applicable, so that by setting $t_n = -u^{-2}q^{-n}$, we obtain

$$\begin{aligned}
\mathsf{Z}^{\sigma,\mathcal{N}=1}_{\mathfrak{N},M} &= q^{-\frac{1}{16}}\left[\lim_{u\to 1}(1-u^2)\frac{\prod_{j=1}^{M}(1+y_j u)(1+y_j^{-1}u)}{\prod_{i=1}^{M+1}(1-x_i u)(1-x_i^{-1}u)}\right]\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^{n+\frac{1}{2}}}\times \\
&\times\prod_{m=1}^{\infty}\prod_{i=1}^{M+1}\frac{(1+x_i q^{m-\frac{1}{2}})(1+x_i^{-1}q^{m-\frac{1}{2}})}{(1-x_i q^m)(1-x_i^{-1}q^m)}\times \\
&\times\prod_{m=1}^{\infty}\prod_{j=1}^{M}\frac{(1+y_j q^m)(1+y_j^{-1}q^m)}{(1-y_j q^{m-\frac{1}{2}})(1+y_j^{-1}q^{m-\frac{1}{2}})}\ . \qquad (4.6.24)
\end{aligned}$$

Thus, we have arrived at the expression for the partition function of a volume filing brane in the worldsheet supersymmetric extension of the superspheres $S^{2M+1|2M}$. As expected, the lowest lying states correspond to the functions on the superspheres and their worldsheet superpartners have energy $\frac{1}{2}$. Let us now look at the case of the world sheet supersymmetric free boson, which corresponds to setting $M = 0$ in (4.6.24). Then the partition function becomes

$$\begin{aligned}
\mathsf{Z}^{\sigma,\mathcal{N}=1}_{\mathfrak{N},M=0} &= q^{-\frac{1}{16}}\sum_{r\in\mathbb{Z}}x^n\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^{n+\frac{1}{2}}}\prod_{m=1}^{\infty}\frac{(1+xq^{m-\frac{1}{2}})(1+x^{-1}q^{m-\frac{1}{2}})}{(1-xq^m)(1-x^{-1}q^m)} \\
&= \frac{q^{-\frac{1}{16}}}{\phi(q)}\sum_{n=-\infty}^{\infty}x^n\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})\ . \qquad (4.6.25)
\end{aligned}$$

We see that this is precisely the boundary partition function of the free boson at the infinite volume limit combined with the partition function of one real free fermion. This theory is known for having $\mathcal{N} = 1$ worldsheet supersymmetry. We can also obtain formula (4.6.25) by working directly with the action (4.6.15). We namely see that the constraints (4.6.17) are solved for $M = 0$ if we set

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = R\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \quad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \eta\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \quad \begin{pmatrix} \bar{\Psi}_1 \\ \bar{\Psi}_2 \end{pmatrix} = \bar{\eta}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}\ , \qquad (4.6.26)$$

where φ is a free boson compactified on a circle of radius 2π and η, $\bar{\eta}$ are two free real fermions of conformal dimension $\frac{1}{2}$. Plugging this into (4.6.15), we see that the interaction term of the fermions vanishes, so that

$$\mathcal{S}_{M=0} = \frac{1}{2\pi}\int_{\Sigma} d^2z\left(R^2\partial\varphi\bar{\partial}\varphi + \frac{i}{2}\eta\bar{\partial}\eta + \frac{i}{2}\bar{\eta}\partial\bar{\eta}\right)\ . \qquad (4.6.27)$$

This is simply the action of a free boson combined with two uncoupled real free fermions that are uncharged under the global O(2) symmetry. Thus, the boundary spectrum of a Neumann brane in the infinite volume limit will be simply the product of the boundary spectrum for the free boson with the fermionic contributions:

$$\mathsf{Z}_F = q^{-\frac{1}{48}}\prod_{n\in\mathbb{N}+\frac{1}{2}}(1+q^n)\ . \qquad (4.6.28)$$

Therefore, we see that this method recovers (4.6.25).

We may now be tempted to perform the same analysis for the spectrum (4.6.24), as we did before for its non-supersymmetric analogue. The naive Ansatz would be to decompose the spectrum in representation of $\mathrm{osp}\,(2M+2|2M)$ and then change the conformal weights proportionally to the quadratic Casimir. It turns out that this is not possible for $M \neq 0$ due to instabilities. In fact, if the conformal dimensions of the fields in (4.6.24) are changed by a factor proportional to an universal function of the couplings times the quadratic Casimir, the energy spectrum becomes unbounded from below for $M \geq 1$, irrespectively of the possible form of this universal function. This is due to the appearance of fields in *typical* representations V_{Λ_n} whose energy in the partition function (4.6.24) grows linearly with n, but whose value of $\mathsf{Cas}(\Lambda_n)$ increases quadratically. The cohomological reduction removes this fields, so that the instability is not present for the supersymmetric variant of the free boson. This suggests that either the model becomes ill defined for $M \geq 1$, or that the naive Casimir evolution Ansatz fails to deliver the right result. So far, we have not been able to find the correct answer.

Chapter 5

Complex Projective Superspaces

The goal of this chapter is to extend the investigations of [18] and of the preceding chapter to the non-linear sigma models with target space given by the complex projective superspaces $\mathbb{CP}^{S-1|S}$. These models give rise to a 2-parameter family of conformal fields theories with central charge $c = -2$, since, in addition to the sigma model coupling g_σ, or radius R, one can also introduce a theta term with arbitrary coefficient θ. The results, a combined effort with C. Candu, T. Quella, H. Saleur and V. Schomerus, were published in [21].

In the first part we shall approach the $\mathbb{CP}^{S-1|S}$ model through its continuum formulation. Using the target space supersymmetry in the framework of cohomological reduction, together with perturbative computations in the limit of large volume, will allow us to come up with a very plausible hypothesis regarding the expressions for the conformal weights of boundary fields as a function of the radius R and theta angle θ.

5.1 The Sigma Model on Projective Superspaces

The aim of this section is to review some facts about the complex projective superspace $\mathbb{CP}^{S-1|S}$ and the non-linear sigma model thereon. In the first subsection we discuss two different formulations of the theory. The first one involves a constraint and it is manifestly $\mathrm{U}(S\,|S)$ invariant. There exists an alternative description, in which the constraint is solved at the expense of breaking the $\mathrm{U}(S\,|S)$ symmetry down to $\mathrm{U}(S-1|S)$. Both formulations will play some role in the subsequent analysis. The second subsection contains a comprehensive analysis of $\mathrm{U}(S\,|S)$ symmetric boundary conditions. We shall argue that there exists an infinite family of such boundary conditions, one for each integer M. They correspond to the choice of a complex line bundle in $\mathbb{CP}^{S-1|S}$ along with a connection one-form A_M. For $S = 2$ the connection one-form is a supersymmetric version of the gauge field produced by a Dirac monopole of charge M.

5.1.1 The sigma model on $\mathbb{CP}^{S-1|S}$

Complex projective superspaces $\mathbb{CP}^{S-1|S}$ are built in a way that resembles closely the construction of the superspheres in the previous chapter. Let us start with flat superspace $\mathbb{C}^{S|S}$, whose S complex bosonic, respectively fermionic coordinates are denoted by z_a, respectively ξ_a. Within this flat complex superspace we consider the real odd dimensional supersphere defined by the equation

$$\sum_{a=1}^{S} z_a z_a^* + \sum_{a=1}^{S} \xi_a \xi_a^* = 1 \quad . \tag{5.1.1}$$

The superspheres $S^{2S-1|2S}$ carry an action of U(1) by simultaneous phase rotations of all bosonic and fermionic coordinates,

$$z_a \longrightarrow e^{i\varpi} z_a \quad , \quad \xi_a \longrightarrow e^{i\varpi} \xi_a \quad . \tag{5.1.2}$$

Note that this transformation indeed leaves the constraint (5.1.1) invariant, thus allowing us to define the complex projective superspace $\mathbb{CP}^{S-1|S}$ as the quotient space $S^{2S-1|2S}/\mathrm{U}(1)$.

The algebra of functions on the supersphere $S^{2S-1|2S}$ carries an action of the the type I Lie supergroup U (S|S) ⊂ OSP (2S|2S). These transformations include the phase rotations (5.1.2) which act trivially on $\mathbb{CP}^{S-1|S}$. Hence, the stabilizer subgroup of a point on the projective superspace is given by $\mathrm{U}(1) \times \mathrm{U}(S-1|S)$ where the first factor corresponds to the action (5.1.2). We conclude that

$$\mathbb{CP}^{S-1|S} = \frac{\mathrm{U}\,(\mathrm{S}|\mathrm{S})}{\mathrm{U}(1) \times \mathrm{U}(S-1|S)} \quad . \tag{5.1.3}$$

Their simplest representative is $\mathbb{CP}^{0|1}$ i.e. the space with just two real fermionic coordinates. The sigma model with this target space is equivalent to the theory of two symplectic fermions, which has been extensively investigated, as for example in [66, 67]. Let us also recall that for $S = 2$, the bosonic base of $\mathbb{CP}^{1|2}$ is a 2-sphere, allowing us to view $\mathbb{CP}^{1|2}$as a bundle with fermionic complex 2-dimensional fibers. Just as in the case of their bosonic analogues, the second homology group $H_2(\mathbb{CP}^{S-1|S}) = \mathbb{Z}$ of complex projective superspaces is non-trivial. Consequently, $\mathbb{CP}^{S-1|S}$ supports line bundles whose second Chern-class is characterized by an integer $M \in \mathbb{Z}$. In the case of $\mathbb{CP}^{1|2}$, the expression for the corresponding connection one-form is well known from the theory of Dirac monopoles. We shall often refer to the integer M as the *monopole number*.

The construction of the sigma model on $\mathbb{CP}^{S-1|S}$ can be inferred from the geometric construction we outlined above. The model involves a field multiplet $Z_\alpha = Z_\alpha(z, \bar{z})$ with S bosonic components $Z_\alpha = z_\alpha, \alpha = 1, \ldots, S$, and the same number of fermionic fields $Z_\alpha = \xi_{\alpha-S}, \alpha = S+1, \ldots, 2S$. To distinguish between bosons and fermions we introduce from now on a grading function $|\cdot|$, which is 0 when evaluated on the labels of bosonic and 1 on the labels of fermionic quantities. In addition we also need a non-dynamical U(1) gauge field a. With this field content, the action takes the form

$$\mathcal{S} = \frac{1}{2g_\sigma^2} \int_\Sigma d^2z (\partial_\mu - i a_\mu) Z_\alpha^\dagger (\partial_\mu + i a_\mu) Z_\alpha - \frac{i\theta}{2\pi} \int_\Sigma d^2z \epsilon^{\mu\nu} \partial_\mu a_\nu \tag{5.1.4}$$

and the fields[1] Z_α are subject to the constraint $Z^\dagger_\alpha Z_\alpha = 1$. The integration over the abelian gauge field can be performed explicitly and it leads to the replacement

$$a_\mu \;=\; \frac{i}{2}\left[Z^\dagger_\alpha \partial_\mu Z_\alpha - (\partial_\mu Z^\dagger_\alpha) Z_\alpha\right] \quad . \tag{5.1.5}$$

The term multiplied by θ does not contribute to the equations of motion for a_μ. As its bosonic counterpart, the $\mathbb{CP}^{S-1|S}$ sigma model on a closed surface possesses instanton solutions. The corresponding instanton number is computed by the term that multiplies the parameter θ. Since it is integer valued, the parameter $\theta = \theta + 2\pi$ can be considered periodic as long as the world-sheet has no boundary.

In order to pass to our second formulation of the $\mathbb{CP}^{S-1|S}$ model we employ the gauge freedom to solve the constraint $Z^\dagger_\alpha Z_\alpha = 1$ as follows

$$Z_1 = Z^\dagger_1 \;=\; \frac{1}{\sqrt{1 + w^\dagger \cdot w}}, \quad Z_{i+1} \;=\; \frac{w^i}{\sqrt{1 + w^\dagger \cdot w}}, \quad Z^\dagger_{i+1} \;=\; \frac{w^{\bar\imath}}{\sqrt{1 + w^\dagger \cdot w}} \; . \tag{5.1.6}$$

Thereby we have parametrized the target space $\mathbb{CP}^{S-1|S}$ through a set of $S-1$ complex bosonic components $w_1, \ldots, w_{S-1}$ and a set of S complex fermionic ones $w_S, \ldots, w_{2S-1}$. Plugging this parametrization (5.1.6) back into the action (5.1.4) we obtain an unconstrained reformulation of the $\mathbb{CP}^{S-1|S}$ sigma model

$$\mathcal{S} \;=\; \frac{1}{2g_\sigma^2}\int_\Sigma d^2z\, g_{i\bar\jmath}\partial_\mu w^{\bar\jmath}\partial_\mu w^i + \frac{i\theta}{2\pi}\int_\Sigma d^2z\, \epsilon^{\mu\nu} i g_{i\bar\jmath}\partial_\nu w^{\bar\jmath}\partial_\mu w^i, \tag{5.1.7}$$

where $g_{i\bar\jmath}$ is the canonical Fubini-Study metric on $\mathbb{CP}^{S-1|S}$

$$g_{i\bar\jmath} \;=\; \frac{\delta_{ij}}{1 + w^\dagger \cdot w} - \frac{(-1)^{|j|} w^{\bar\imath} w^j}{(1 + w^\dagger \cdot w)^2} \; . \tag{5.1.8}$$

The disadvantage of this reformulation is the non-linear action of the $\mathrm{U}(S\,|S)$ supergroup on the projective coordinates $w, \bar w$. Let us recall in passing that the Fubini-Study metric on $\mathbb{CP}^{S-1|S}$ determines the following Kähler two-form

$$\mathsf{w} \;=\; d^2z\, \epsilon^{\mu\nu} i g_{i\bar\jmath}\partial_\nu w^{\bar\jmath}\partial_\mu w^i \;=\; -i g_{i\bar\jmath} dw^{\bar\jmath} \wedge dw^i \; . \tag{5.1.9}$$

The Kähler form is properly normalized and generates the second integral cohomology group of $\mathbb{CP}^{S-1|S}$, that is

$$\int \frac{\mathsf{w}}{2\pi} \;=\; 1 \; . \tag{5.1.10}$$

It follows, as stated before, that our bulk model is not affected if we shift θ by integer multiples of 2π, i.e. we can restrict the parameter θ to the interval $\theta \in [-\pi, \pi[$.

[1]Note that we eliminated the radius R of the complex projective space in favor of a coupling g_σ^{-2} entering the action in front of the metric. Equivalently, we can set $g_\sigma^2 = 1$ and work with a radius parameter R appearing in the modified constraint $Z^\dagger_\alpha Z_\alpha = 4R^2$.

5.1.2 Action of the boundary model

We are now going to discuss $\mathrm{U}(S\,|S)$ symmetric boundary conditions of the $\mathbb{CP}^{S-1|S}$ model. For readers used to the string theoretic concept of branes and the geometric classification of boundary conditions, the final outcome is not surprising. As noted, $\mathbb{CP}^{S-1|S}$ admits a natural left action of $\mathrm{U}(S\,|S)$and, since $\mathbb{CP}^{S-1|S}$ is homogeneous under this action, any $\mathrm{U}(S\,|S)$ symmetric brane must be volume filling. But branes are not simply (sub-)manifolds in target space. They also carry a bundle $\mathfrak{L}$ with connection A. In the case at hand, there is an infinite family of complex line bundles $\mathfrak{L}_M$ on $\mathbb{CP}^{S-1|S}$ which are parametrized by the integer $M \in \mathbb{Z}$. To ensure $\mathrm{U}(S\,|S)$ invariance, the connection A_M must have constant curvature Ω_M. Consequently, its curvature is proportional to the Kähler form w, i.e. $\Omega_M \sim M\mathsf{w}$. We shall now see how these geometric insights manifest themselves in the world-sheet description. Our presentation will not make any more reference to string theoretic notions.

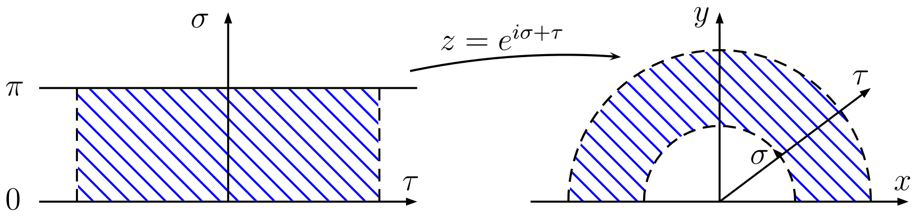

Figure 5.1: The transformation from the strip to the half plane.

We want to consider the $\mathbb{CP}^{S-1|S}$ sigma model on a world-sheet Σ with boundary. The choice we have in mind is a strip $\Sigma = [0,\pi] \times \mathbb{R}$ or, equivalently, the upper half of the complex plane $z = x + iy, y > 0$. We are looking for boundary conditions which arise from adding boundary terms of the form

$$\mathcal{S}_b \;=\; \int_{-\infty}^{0} dx\, \left(A^L_i(w,\bar{w})\partial_x w^i + A^L_{\bar{\imath}}(w,\bar{w})\partial_x w^{\bar{\imath}}\right) + \int_0^{\infty} dx\, (\,L \;\leftrightarrow\; R\,) \tag{5.1.11}$$

to the action (5.1.4, 5.1.7). Here, $A^{L(R)} = A^{L(R)}_i dw^i + A^{L(R)}_{\bar{\imath}} dw^{\bar{\imath}}$ are one-forms which are at least locally defined on $\mathbb{CP}^{S-1|S}$. When we map the half plane back to the strip, points with $z = x > 0$ are mapped to the right boundary while those with $z = x < 0$ end up on the left side. To find consistent boundary conditions we require invariance of the total action $\mathcal{S}_t = \mathcal{S} + \mathcal{S}_b$ with respect to arbitrary variations $\delta w^i(z,\bar{z})$. It follows that

$$g_{i\bar{\jmath}}\left(\frac{1}{2g_\sigma^2}\partial_y + \frac{\theta}{2\pi}\partial_x\right) w^{\bar{\jmath}} = \; 2\Omega^{L(R)}_{ij}\partial_x w^j + 2\Omega^{L(R)}_{i\bar{\jmath}}\partial_x w^{\bar{\jmath}} \tag{5.1.12}$$

$$g_{\bar{\imath}j}\left(\frac{1}{2g_\sigma^2}\partial_y - \frac{\theta}{2\pi}\partial_x\right) w^{j} = \; 2\Omega^{L(R)}_{\bar{\imath}\bar{\jmath}}\partial_x w^{\bar{\jmath}} + 2\Omega^{L(R)}_{\bar{\imath}j}\partial_x w^{j},$$

where $z = \bar{z} < 0\,(> 0)$ and $\Omega^{L(R)}$ is the curvature 2-form of the connection $A^{L(R)}$. It is globally defined on $\mathbb{CP}^{S-1|S}$ through

$$\Omega^{L(R)} = dA^{L(R)} \;=\; -\Omega^{L(R)}_{ij} dw^j \wedge dw^i - 2\Omega^{L(R)}_{i\bar{\jmath}} dw^{\bar{\jmath}} \wedge dw^i - \Omega^{L(R)}_{\bar{\imath}\bar{\jmath}} dw^{\bar{\jmath}} \wedge dw^{\bar{\imath}}. \quad (5.1.13)$$

Before imposing the conditions of U$(S\,|S)$ symmetry, we note that our boundary conditions (5.1.12) should preserve the complex structure of the $\mathbb{CP}^{S-1|S}$ supermanifold. Consequently, the complex conjugate of the first equation in (5.1.12) must yield the second equation without any additional constraint. While applying this constraint one must take into account the reality condition of a scalar field in Euclidean space-time

$$w^i(z,\bar{z})^* \;=\; w^{\bar{\imath}}(1/z^*, 1/\bar{z}^*). \quad (5.1.14)$$

Thus, we conclude that the two equations in (5.1.12) are compatible if and only if $\Omega^{L(R)}$ is *imaginary*.

Boundary conditions (5.1.12) are said to preserve the global U$(S\,|S)$ symmetry if they are invariant with respect to an infinitesimal action of the supergroup. To give a precise meaning to this statement, let us note that the set of equations (5.1.12) can be interpreted as the vanishing of some real vector field L on the left boundary, whose components read

$$L^i = \left(\frac{1}{2g_\sigma^2}\partial_y - \frac{\theta}{2\pi}\partial_x\right) w^i - 2g^{i\bar{k}}\Omega^L_{\bar{k}\bar{\jmath}}\partial_x w^{\bar{\jmath}} - 2g^{i\bar{k}}\Omega^L_{\bar{k}j}\partial_x w^j, \quad (5.1.15)$$

$$L^{\bar{\imath}} = \left(\frac{1}{2g_\sigma^2}\partial_y + \frac{\theta}{2\pi}\partial_x\right) w^{\bar{\imath}} - 2g^{\bar{\imath}k}\Omega^L_{kj}\partial_x w^j - 2g^{\bar{\imath}k}\Omega^L_{k\bar{\jmath}}\partial_x w^{\bar{\jmath}} \quad (5.1.16)$$

with a similar expression for another real vector field R on the right boundary. The global U$(S\,|S)$ invariance of the boundary conditions (5.1.12) is then equivalent to the invariance of the vector field L, R with respect to the infinitesimal action of the u$\,(S|S)$ Lie superalgebra. In other words, the Lie derivative of the vector field (5.1.15) with respect to any u$\,(S|S)$ Killing vector must vanish. As world-sheet translations and global symmetry transformations commute, it follows that the 2-forms $\Omega^{L(R)}$ must be invariant. On the other hand, on any irreducible complex symmetric superspace, there is only one invariant closed 2-form, namely the Kähler form w. Hence, invariance of the boundary conditions with respect to the global symmetry requires that

$$\Omega^L \;=\; iM\mathsf{w} \qquad , \qquad \Omega^R \;=\; iN\mathsf{w} \quad (5.1.17)$$

where w is the Kähler form (5.1.9). In the classical theory, the M, N can assume any real value. For the associated path integral to be well defined, however, they must be integers. Even though this sections deals with the classical action, we shall assume $M, N \in \mathbb{Z}$ from now on. For later use it is convenient to re-write L and R in an index free notation,

$$L \;=\; \frac{1}{2g_\sigma^2}\partial_y + iJ\left(\frac{\theta}{2\pi} + M\right)\partial_x \quad , \quad R \;=\; \frac{1}{2g_\sigma^2}\partial_y + iJ\left(\frac{\theta}{2\pi} + N\right)\partial_x\;. \quad (5.1.18)$$

Here we have introduced the (globally well defined) complex structure J on the tangent space of $\mathbb{CP}^{S-1|S}$. The components $L^i = dw^i(L)$ and $L^{\bar{\imath}} = dw^{\bar{\imath}}(L)$ are recovered from L with the help of the canonical basis dw^i and $dw^{\bar{\imath}}$ in the cotangent space. Note that L and R only contain a specific combination $\Theta_M = 2M + \theta/\pi$ and, respectively, $\Theta_N = 2N + \theta/\pi$. We conclude that the periodic variable θ of the bulk theory gets promoted to a real valued variable Θ in the boundary problem [68]. In the limit $g_\sigma \to 0$, the value of Θ is irrelevant. In other words, the boundary conditions are purely Neumann when we approach infinite radius.

Before we close this section, let us briefly write the boundary conditions in terms of the manifestly U$(S\,|S)$ covariant formulation (5.1.4) of our theory. In this case, the variations of the basic fields Z_α must be consistent with the constraint eq. (5.1.1),

$$(\delta Z_\alpha^\dagger) Z_\alpha + Z_\alpha^\dagger \delta Z_\alpha = 0 \quad .$$

In order for the boundary contributions to the variation of the action to vanish, we must impose the usual twisted Neumann boundary conditions of the type

$$\begin{aligned} (\partial_y + i a_y) Z_\alpha &= \Theta_M g_\sigma^2 (\partial_x + i a_x) Z_\alpha \ , \\ (\partial_y - i a_y) Z_\alpha^\dagger &= -\Theta_M g_\sigma^2 (\partial_x - i a_x) Z_\alpha^\dagger \end{aligned} \tag{5.1.19}$$

for $z = \bar{z} < 0$ and a similar condition with M replaced by N, i.e. Θ_M replaced by Θ_N, along the right half $z = \bar{z} > 0$ of the boundary. The parameters $\Theta_M = 2M + \theta/\pi$ and $\Theta_N = 2N + \theta/\pi$ are the same combination of the θ angle in the bulk and the monopole numbers M, N that appeared in eq. (5.1.18).

So far we have only discussed the classical theory. Understanding the detailed properties of the associated quantum field theories is the main aim of the following sections. For the time being let us just mention that the non-linear sigma models on $\mathbb{CP}^{S-1|S}$ have been argued to possess vanishing β function [16]. This means that they give rise to conformal quantum field theories for any choice of the two couplings g_σ and θ. The central charge of these models must agree with the central charge of the free field theory at $g_\sigma \sim 0$, i.e. all models of this type have $c = -2$.

5.2 Spectrum of the non-interacting sigma model

Our discussion of the quantum field theory will begin with the limiting case $g_\sigma = 0$ in which all the interactions are turned off. To keep things computationally manageable, we will restrict to the first non-trivial case with $S = 2$, though most of what we are about to describe generalizes quite easily to higher dimensional projective superspaces. Our goal is to investigate the spectrum of the $\mathbb{CP}^{1|2}$ model on the strip (or half-plane) with twisted Neumann boundary conditions imposed along the boundary. In string theory terms, this is equivalent to considering volume filling branes which wrap the bosonic base of $\mathbb{CP}^{1|2}$and are delocalized along the fermionic directions. In a first step we shall analyze the spectrum in the particle limit. Then, in the second step, we include derivative fields and construct a partition function for the theory in the limit of vanishing coupling g_σ.

5.2.1 Spectrum for a particle moving on $\mathbb{CP}^{1|2}$

As we argued in chapter 4, the particle, or minisuperspace, approximation amounts to considering the string as a point like object, thus neglecting the σ dependence of the fields $w^i(\tau, \sigma)$. Thereby, we reduce the field theory to a point particle problem. We shall discuss the quantization of this system in two different ways. In the first description we use the gauge fixed formulation of the theory in terms of variables $w^i, w^{\bar{\imath}}$. The spectrum of the associated Hamiltonian is known from [69] and the results allows us to formulate a quantization condition on the monopole numbers. For our second approach, we also take the infinite volume limit and employ a U (2|2) covariant formulation, just like in the previous chapter. While this does not lead to anything new in this limit, the covariant methodology is more easily extended to the full field theory. So, let us start from the action (5.1.7), while setting all σ-derivatives to zero. After integration of the transverse coordinate σ of the strip $\Sigma = [0, \pi] \times \mathbb{R}$ we get the following particle theory:

$$\mathcal{S} \;=\; \int_{-\infty}^{\infty} d\tau \left(\frac{\pi}{2g_\sigma^2} g_{i\bar{\jmath}} \dot{w}^{\bar{\jmath}} \dot{w}^i + A_i \dot{w}^i + A_{\bar{\imath}} \dot{w}^{\bar{\imath}} \right) \;, \tag{5.2.1}$$

where locally the connection one-form A is the difference of the two one-forms A^R and A^L, that is

$$A \;=\; A^R - A^L \;\; . \tag{5.2.2}$$

The classical Hamiltonian of this quantum mechanical system takes the following simple form

$$H \;=\; -\frac{2g_\sigma^2}{\pi} g^{i\bar{\jmath}} (\Pi_i - A_i)(\Pi_{\bar{\jmath}} - A_{\bar{\jmath}}) \;\; , \tag{5.2.3}$$

where the canonical momenta are given as usual by

$$\Pi_i \;=\; \frac{\pi}{2g_\sigma^2} g_{i\bar{\jmath}} \dot{w}^{\bar{\jmath}} + A_i \;, \quad \Pi_{\bar{\imath}} \;=\; \frac{\pi}{2g_\sigma^2} g_{\bar{\imath}j} \dot{w}^j + A_{\bar{\imath}} \;\; . \tag{5.2.4}$$

The standard canonical quantization of this model replaces the Poisson brackets of the basic fields and their conjugated momenta with commutators, so that

$$[w^i, \Pi_j] \;=\; [w^{\bar{\imath}}, \Pi_{\bar{\jmath}}] \;=\; \delta^i_j \;\; . \tag{5.2.5}$$

Note that the factor i of the usual commutation relations $[x^i, p_j] = i\delta^i_j$ is missing because we are formulating the theory in Euclidean time $\tau = it$. For the quantization procedure to make sense, the one-form A must be a connection on a complex line bundle over $\mathbb{CP}^{1|2}$, see [70]. This furnishes a quantization condition for the curvature of the connection,

$$dA \;=\; -il\mathsf{w} \tag{5.2.6}$$

with l any integer and w the Kähler form on $\mathbb{CP}^{1|2}$. The space of sections of such bundles may be realized explicitly as equivariant functions $f(w, \bar{w})$ on $\mathbb{CP}^{1|2}$ with the property

$$f(e^{i\alpha} w, e^{-i\alpha} \bar{w}) = e^{il\alpha} f(w, \bar{w}) \;. \tag{5.2.7}$$

Taking into account (5.2.2) we get the condition that $l = M - N$ must necessarily be an integer. Hence, if we admit e.g. $A^L = 0$ as a possible boundary conditions, mutual consistency requires $N \in \mathbb{Z}$. The quantized form of the classical Hamiltonian (5.2.3) is, up to a numerical prefactor, the Bochner-Laplacian $\Delta^{(l)}_{\mathbb{CP}^{S-1|S}}$ on the complex line bundle over $\mathbb{CP}^{1|2}$ with monopole charge $l \in \mathbb{Z}$

$$\hat{H}^{(l)} \;=\; -\frac{g_\sigma^2}{\pi}\Delta^{(l)}_{\mathbb{CP}^{S-1|S}} \;. \tag{5.2.8}$$

We spend most of appendix D, dissecting the Bochner Laplacian, whose eigenvalues were studied in [69]. For the Hamiltonian we obtain

$$h_l(k) \;=\; \frac{g_\sigma^2}{\pi}\left(2k^2 + (2k+|l|)(|l|-1) - l^2\right) \qquad \text{for} \;\; k \;=\; 0,1,2,\ldots \tag{5.2.9}$$

From the spectrum we can read off which u(2|2) multiplets are realized as sections of monopole bundles on $\mathbb{CP}^{1|2}$. We will list the corresponding representations of U(2|2) a bit later at the end of our second construction of the spectrum.

Let us now see how to reproduce the spectrum of the particle theory within the U(2|2) covariant formulation. As before, we depart from the space $\mathbb{C}^{2|2}$ with coordinates $Z = (z_1, z_2, \xi_1, \xi_2)$. The 4-tuple Z transforms in the fundamental representation V of u(2|2). On the projective superspace $\mathbb{CP}^{1|2}$, the multiplet Z and its conjugate $Z^\dagger$ obey the following constraint

$$Z^\dagger \cdot Z \;=\; 1 \qquad . \tag{5.2.10}$$

Note that $Z^\dagger$ transforms in the dual fundamental representation $Z^\dagger \in V^\star$ so that the above equation respects with the u(2|2) symmetry. Consequently, if we quotient the space of functions on $\mathbb{C}^{2|2}$ by the ideal that is generated from $Z^\dagger \cdot Z - 1$, we end up with some non-trivial u(2|2) module $\mathcal{B}$. The center of u(2|2) acts on $\mathcal{B}$ through the phase rotations (5.1.2), thereby defining a decomposition $\mathcal{B} = \bigoplus_l \mathcal{B}_l$ where $\mathcal{B}_l \subset \mathcal{B}$ consists of elements $f \in \mathcal{B}$ such that $f \to \exp(il\varpi)f$ under the map (5.1.2). The spaces $\mathcal{B}_l$ contain precisely all sections of the complex line bundle with monopole number l.

We want to determine the partition function of the particle limit, i.e. a function that counts sections in the monopole line bundles, or, equivalently, elements in the u(2|2) module $\mathcal{B}_l$. Before we construct this counting function, let us introduce the following basis in the 4-dimensional Cartan subalgebra,

$$J_x = \frac{1}{2}\left(\begin{array}{c|c}\sigma_3 & 0\\ \hline 0 & 0\end{array}\right) \quad J_y = \frac{1}{2}\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \sigma_3\end{array}\right) \quad J_z = \frac{1}{2}\left(\begin{array}{c|c}I_2 & 0\\ \hline 0 & -I_2\end{array}\right) \quad J_u = \frac{I_4}{2}\;. \tag{5.2.11}$$

Here I_n is the n-dimensional identity and σ_3 the Pauli matrix $\sigma_3 = \mathrm{diag}(1,-1)$. The partition function reads

$$\mathsf{Z}^{(0)}_{M,N}(x,y,z) \;=\; \mathrm{tr}_{\mathcal{B}_l}(x^{J_x}y^{J_y}z^{J_z}) = \lim_{t\to 1}\oint_{|u|=1} du\,\frac{1-t^2}{u^{l/2+1}}\prod_{\alpha,\beta=\pm\frac{1}{2}}\frac{(1+y^\alpha z^{-\beta}u^\beta t)}{(1-x^\alpha z^\beta u^\beta t)}\;, \tag{5.2.12}$$

where $l = M-N$ is the difference of the monopole numbers, as before. The trace is taken over all sections of line bundles on $\mathbb{CP}^{1|2}$ and the integral over u is to be understood in the formal sense, i.e. as a projector. The limit $t \to 1$ implements the constraint (5.2.10) just like in section 4.3, while the integral over the variable u selects those states that stretch between two line bundles with respective monopole numbers N and M, such that the difference $M-N$ is fixed. Of course, states within $\mathsf{Z}^{(0)}_{M,N}$ still carry a J_u charge, which takes the constant value $J_u = l/2$.

Our aim now is to decompose the partition function of the particle theory into characters of the symmetry u(2|2). In a first step we expand $\mathsf{Z}^{(0)}$ into characters of 8-dimensional bosonic subalgebra $\mathrm{u}(2|2)_{\bar{0}} \cong \mathrm{sl}_2 \oplus \mathrm{sl}_2 \oplus \mathrm{u}(1) \oplus \mathrm{u}(1)$. The latter are given by

$$\chi^{\mathsf{B}}_{(j_1,j_2,a,b)}(x,y,z,u) \;=\; \chi_{j_1}(x)\,\chi_{j_2}(y)\,z^a\,u^b\;, \tag{5.2.13}$$

where $j_1, j_2 \in \mathbb{N}/2$ and $a, b \in \mathbb{C}$. It is rather straightforward to compute the corresponding branching functions and we shall not spell out the results of this intermediate step here as it is very similar to the one we carried out in chapter 4. The next step then is to combine the characters of the bosonic subalgebra into characters of u(2|2). Two types of characters turn out to appear. The generic ones are the characters of Kac modules, whose relation to the characters of the bosonic subalgebra is given by

$$\chi^{\mathsf{K}}_{[j_1,j_2,a,b]} = \chi^{\mathsf{B}}_{(j_1,j_2,a,b)}\Big(1 + z^{-1}\chi_{(\frac{1}{2},\frac{1}{2})} + z^{-2}\left(\chi_{(1,0)} + \chi_{(0,1)}\right) + z^{-3}\chi_{(\frac{1}{2},\frac{1}{2})} + z^{-4}\Big)\,. \tag{5.2.14}$$

Here and in the following we abbreviate the products $\chi_{j_1}(x)\chi_{j_2}(y)$ of sl_2-characters as $\chi_{(j_1,j_2)}$. In this expression, the first factor is associated with the bosonic multiplet of ground states while the expression within brackets arises from the four fermionic lowering operators in a Kac module of u(2|2). In addition to the Kac modules, we also need formulae for characters of some special atypical irreducibles. According to [71], the characters of these atypicals are given by

$$\begin{aligned}
\chi_{[\frac{l}{2},0,\frac{l}{2},\frac{l}{2}]} &= \chi^{\mathsf{B}}_{(\frac{l}{2},0,\frac{l}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-1}{2},\frac{1}{2},\frac{l-2}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-2}{2},0,\frac{l-4}{2},\frac{l}{2})}\\
\chi_{[\frac{l-2}{2},0,\frac{4-l}{2},-\frac{l}{2}]} &= \chi^{\mathsf{B}}_{(\frac{l}{2},0,-\frac{l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-1}{2},\frac{1}{2},-\frac{2+l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-2}{2},0,\frac{4-l}{2},-\frac{l}{2})}\\
\chi_{[\frac{l+1}{2},\frac{1}{2},\frac{l+2}{2},\frac{l}{2}]} &= \chi^{\mathsf{B}}_{(\frac{l+1}{2},\frac{1}{2},\frac{l+2}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},0,\frac{l}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l+2}{2},0,\frac{l}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},1,\frac{l}{2},\frac{l}{2})}\\
&\quad + \chi^{\mathsf{B}}_{(\frac{l+1}{2},\frac{1}{2},\frac{l-2}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-1}{2},\frac{1}{2},\frac{l-2}{2},\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},0,\frac{l-4}{2},\frac{l}{2})}\\
\chi_{[\frac{l}{2},0,\frac{4-l}{2},-\frac{l}{2}]} &= \chi^{\mathsf{B}}_{(\frac{l+1}{2},\frac{1}{2},\frac{2-l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},0,-\frac{l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l+2}{2},0,-\frac{l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},1,-\frac{l}{2},-\frac{l}{2})}\\
&\quad + \chi^{\mathsf{B}}_{(\frac{l+1}{2},\frac{1}{2},\frac{2-l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l-1}{2},\frac{1}{2},\frac{2-l}{2},-\frac{l}{2})} + \chi^{\mathsf{B}}_{(\frac{l}{2},0,\frac{4-l}{2},-\frac{l}{2})}\;,
\end{aligned} \tag{5.2.15}$$

where $l \geq 0$ and the value $l = -1$ is admitted only in the third equation. It is understood that a bosonic character is to be omitted on the right hand side if one of its first two labels is negative. We also note that $\left[\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\right]$ and $\left[0,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]$ are associated with the fundamental representation and its dual. The formulae (5.2.14) and (5.2.15) are the only ones we need in order to obtain the expansion of the partition function in terms of characters of u(2|2)

$$\begin{aligned} \mathsf{Z}^{(0)}_{M,N}(x,y,z) \;&=\; (1+\delta_{l,0})\;\chi_{\left[\frac{|l|}{2},0,\frac{l+2-2\mathrm{sgn}(l)}{2},\frac{l}{2}\right]} + \\ &+\chi_{\left[\left|\frac{l}{2}+\frac{3}{4}\right|-\frac{1}{4},\frac{1+\mathrm{sgn}(l+1)}{4},\frac{l+3-\mathrm{sgn}(l+1)}{2},\frac{l}{2}\right]} + \sum_{k=2}^{\infty}\chi^{\mathsf{K}}_{\left[k-1+\frac{|l|}{2},0,\frac{|l|}{2}+2,\frac{l}{2}\right]}\,, \end{aligned} \tag{5.2.16}$$

where $l = M - N$ and $\mathrm{sgn}(x) = 1$ if $x \geq 0$, $\mathrm{sgn}(x) = -1$ otherwise. The first two summands in this formula involve characters of irreducible atypicals while all remaining ones are associated with full Kac modules. In the special case that $l = M - N = 0$, the partition function counts functions on $\mathbb{CP}^{1|2}$. The three characters of atypical representations appearing in (5.2.16) for $l = 0$, while counting the eigenvalues of the Cartan elements accurately, do not reflect the actual transformation property of the fields. They contain the 16 fields that transform in the tensor product of the fundamental representation with its dual, which is an indecomposable representation of Casimir zero. Just as in chapter 4 we could describe the space of functions as the multiplicity free direct sum of the symmetric traceless representations, so too can we here understand the decomposition (5.2.16) for $l = 0$ as the direct sum of supersymmetric, self-dual, traceless u(2|2) tensors $t(k,k)$ of rank $k > 1$ and the 16-dimensional indecomposable traceless but reducible tensor $t(1,1)$. Further details can be gathered in section 5.6 and appendix D. We remark further, that for values $l = M - N > 0$, the lowest value of j_1 in (5.2.16) is $j_1 = |l|/2$. Such a cutoff is a well known feature of sections in monopole bundles.

The result (5.2.16) agrees with our earlier description of the spectrum (5.2.9). To relate the two findings we note that in a representation $\Lambda = [j_1, j_2, a, b]$ of u(2|2) the quadratic Casimir elements take the value

$$\mathsf{Cas}_\alpha(\Lambda) \;=\; 2[j_1(j_1+1) - j_2(j_2+1) + b(a-2)] - 4\alpha b^2. \tag{5.2.17}$$

Since u(2|2) is not semisimple, there exists a one-parameter family of such Casimir elements. It is parametrized by the coefficient $-\alpha$ of E^2 where E denotes the central element of u(2|2). More details can be found in Appendix B.4. Plugging in the labels of representations from eq. (5.2.16) one recovers the spectrum (5.2.9) of the Bochner-Laplacian, provided the parameter α in the Casimir element[2] is set to $\alpha = 1$. This concludes our discussion of the particle limit.

5.2.2 Partition function at infinite radius

The partition function of the boundary conformal field theory in the limit of vanishing target space curvature can be constructed by extending our discussion of the particle

[2]See appendix D for details

limit to incorporate derivatives along the boundary. The main formula is

$$\begin{aligned} \mathsf{Z}_{M,N}(q,x,y,z) &= \phi(q)\, q^{\frac{1}{12}} \oint_{|u|=1} \frac{du}{u^{l/2+1}}\, \phi(q) \lim_{t\to 1}(1-t^2) \times \\ &\times \prod_{\alpha,\beta=\pm\frac{1}{2}} \frac{1+y^\alpha\,(zu^{-1})^\beta\, t}{1-x^\alpha\,(zu)^\beta\, t} \prod_{n=1}^{\infty} \prod_{\alpha,\beta=\pm\frac{1}{2}} \frac{1+y^\alpha\,(zu^{-1})^\beta\, q^n}{1-x^\alpha\,(zu)^\beta\, q^n} \ . \end{aligned} \tag{5.2.18}$$

The constraint $Z^\dagger \cdot Z = 1$ is implemented as in the particle limit by taking the limit of t to one. In the field theory, just as in chapter 4, we obtain an infinite tower of field identifications by taking derivatives of (5.2.10). The implementation of these constraints proceeds in the manner described in section 4.3.2 and leads to a multiplication of the counting function with $\phi(q)$. The formal integral over the variable u has to effect of projecting onto those states that transform in the same fashion under global gauge transformations. However, here too the field theory is more complicated that the particle limit, since local gauge degrees of freedom have to be implemented as well. The equations of motion for the gauge field (5.1.5), describe a completely in terms of the basic fields Z and $Z^\dagger$, removing the degrees of freedom of the gauge field itself. We are still left however with local gauge transformations $Z \to Ze^{i\varpi(z)}$. If our partition function is to count only physical states, we have to restrict ourselves to states invariant under local transformation that shift the gauge field. Looking at (5.1.5) again, we see that this implies the constraint

$$Z^\dagger \cdot \partial Z - \partial Z^\dagger \cdot Z \overset{!}{=} 0 \ , \tag{5.2.19}$$

as well as the vanishing of all of its derivatives. This set of constraints is linearly independent of the relations that we get from deriving $Z^\dagger \cdot Z = 1$ but can be implemented in the same way. Therefore, on the level of the partition function (5.2.18), the double counting of fields which are related by local gauge transformations is avoided by another multiplication with the Euler function $\phi(q)$.

Now that we understand the basic expression from the partition function of the model, let us decompose the field theory spectrum into representations of the global symmetry u(2|2). Since a similar computation was already painstakingly presented in chapter 4, we shall only sketch the main steps here. As in the particle limit, we expand into bosonic characters first,

$$\mathsf{Z}_{M,N}(q,x,y,z) = \sum \chi^{\mathsf{B}}_{(j_1,j_2,a,b)}(x,y,z)\, \psi^{\mathsf{B}}_{(j_1,j_2,a,b)}(q) \ ,$$

where $b = l/2$ and the sum runs over all $j_1, j_2 \in \frac{\mathbb{N}}{2}$, $a \in \frac{\mathbb{Z}}{2}$ for which $a + b \equiv 2j_1 \bmod 2$ and $a - b \equiv 2j_2 \bmod 2$. The characters χ of the even part $\mathrm{u}(2|2)_{\bar{0}}$ were displayed in equation (5.2.13) above. For the associated branching functions ψ^{B} one finds

$$\psi^{\mathsf{B}}_{(j_1,j_2,a,b)}(q) \;=\; \frac{q^{\frac{1}{12}}}{\phi(q)^4}\left(q^{-j_2} - q^{3j_2+2} + \left(q^{\frac{a-b}{2}} + q^{\frac{b-a}{2}}\right)\left(1-q^{2j_2+1}\right)\right) \times$$
$$\times\, q^{j_2^2+\left(\frac{a-b}{2}\right)^2} \sum_{\substack{l=\left|\frac{a+b}{2}\right| \\ j_1+l\in\mathbb{N}}}^{\infty} \sum_{m,n=1}^{\infty} (-1)^{m+n} \frac{\left(1-q^{m+n}\right)\left(q^{(m-n)(j_1-l)} - q^{(m-n)(j_1+l+1)}\right)}{q^{-\frac{m(m-1)+n(n-1)}{2}}},$$

where we require that a and b be such that

$$a+b \equiv 2j_1 \bmod 2 \qquad a-b \equiv 2j_2 \bmod 2\ . \tag{5.2.20}$$

The branching functions for the Kac modules of the full superalgebra u(2|2) can be obtained through the following infinite sums

$$\psi^{\mathsf{K}}_{[j_1,j_2,a,b]} = \sum_{n=0}^{\infty}(-1)^n \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{r,s=0}^{n-2m} \psi^{\mathsf{B}}_{\left(j_1+\frac{n}{2}-(m+r),\, j_2+\frac{n}{2}-(m+s),\, a+n,\, b\right)}\ . \tag{5.2.21}$$

Weights $[j_1, j_2, a, b]$ of u(2|2) are atypical when $b = \pm(j_1 - j_2)$ or $b = \pm(j_1 + j_2 + 1)$. Whenever the weights are atypical, our expressions for ψ^K must be summed further to obtain branching functions of irreducible representations. The necessary formulae are listed in Appendix B.5. Here, we shall simply display our results in terms of the branching functions ψ^{K},

$$\mathsf{Z}_{M,N}(q,x,y,z) \;=\; \sum \chi^{\mathsf{K}}_{[j_1,j_2,a,l/2]}(x,y,z)\, \psi^{\mathsf{K}}_{[j_1,j_2,a,l/2]}(q)\ . \tag{5.2.22}$$

The sum runs over all $j_1, j_2 \in \frac{\mathbb{N}}{2}$, $a \in \frac{\mathbb{Z}}{2}$ for which $a + l/2 \equiv 2j_1 \bmod 2$ and $a - l/2 \equiv 2j_2 \bmod 2$. For our purposes, the branching functions ψ^{K} are already good enough, since we are only interested in the values that the quadratic Casimir takes on the states of our theory and not in their precise transformation properties which, since indecomposable representations appear quite naturally, can be very complicated. We recall that the characters χ^{K} of u(2|2) Kac modules are given by eq. (5.2.14). For typical weights, the functions ψ^{K} are proper branching functions with non negative integer coefficients.

It is very instructive to apply the same combinatorial constructions to the simpler theory of symplectic fermions, i.e. for $S = 1$. The symmetry of this model is described by the superalgebra u(1|1). We select a particular basis J_z, J_u for the Cartan subalgebra by fixing the values in the fundamental representation according to

$$J_z \;=\; \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right) \qquad J_u \;=\; \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & 1 \end{array}\right)\ . \tag{5.2.23}$$

Just as in the case of the $\mathbb{CP}^{1|2}$ model, we construct the partition function in the limit $R \to \infty$ by taking tensor products of the fundamental representation of u(1|1) and its dual. After that we apply our constraint and gauge prescription. The partition function $\mathsf{Z} = \sum_l \mathsf{Z}_l u^{l/2}$ for all bundles is then given by the formula

$$\begin{aligned}\mathsf{Z}(q,z,u) &= q^{\frac{1}{12}}\phi(q)^2 \lim_{t\to 1}(1-t^2)\prod_{n=0}^{\infty}\frac{(1+z^{-\frac{1}{2}}u^{\frac{1}{2}}q^n)(1+z^{\frac{1}{2}}u^{-\frac{1}{2}}q^n)}{(1-z^{\frac{1}{2}}u^{\frac{1}{2}}q^n)(1-z^{-\frac{1}{2}}u^{-\frac{1}{2}}q^n)} \\ &= \frac{q^{\frac{1}{12}}}{\phi(q)}\sum_{\substack{a,b\in\mathbb{Z}/2\\ a+b\in\mathbb{Z}}} z^a\left(1+z^{-1}\right)u^b q^{\frac{(b-a)(b-a+1)}{2}} ,\end{aligned} \tag{5.2.24}$$

where in the product of the first line we are instructed to make the formal substitution $q^0 \to t$ before evaluating the limit $t \to 1$. Since Kac module characters for u(1|1) are defined by[3]

$$\chi^{\mathsf{K}}_{\langle a\,|\,b\rangle} = z^a\left(1+z^{-1}\right)u^b ,$$

we obtain the following expression for the branching functions

$$\psi^{\mathsf{K}}_{\langle a\,|\,b\rangle}(q) = \frac{q^{\frac{1}{12}}}{\phi(q)}q^{\frac{(b-a)(b-a+1)}{2}} \qquad \text{for } a,b\in\mathbb{Z}/2, a+b\in\mathbb{Z} . \tag{5.2.25}$$

The quadratic Casimir takes the value $2b(2a-1)-4b^2$ in the Kac module labeled by $\langle a\,|\,b\rangle$. For a given value of $b = l/2 = (M-N)/2$, there are four states of conformal weight $h = 0$ in the spectrum. More precisely, we find that

$$\mathsf{Z}^{(0)}_{M,N}(z,u) = \chi^{\mathsf{K}}_{\langle l/2\,|\,l/2\rangle} + \chi^{\mathsf{K}}_{\langle l/2+1\,|\,l/2\rangle} ,$$

where $l = M-N$. When $l \neq 0$, the two u(1|1) multiplets that appear in the decomposition of $\mathsf{Z}^{(0)}$ are typical. This changes only for $l = 0$. In that case, the two atypical multiplets $\langle 0\,|\,0\rangle$ and $\langle 1\,|\,0\rangle$ combine into a 4-dimensional projective indecomposable of u(1|1). Such boundary theories of the symplectic fermions with four ground states were first constructed in [72]. Let us also observe that the number of characters in the decomposition of $\mathsf{Z}^{(0)}$ agrees with the number of atypical characters in the corresponding decomposition (5.2.16) for the $\mathbb{CP}^{1|2}$ model. This was to be expected, since from our discussion in chapter 3 we know that the non-linear sigma models on the complex projective spaces can be related, via cohomological reduction, to the model of two symplectic fermions. It is precisely those states that transform in atypical irreducible representations in (5.2.18) that we find in the symplectic fermions theory.

5.3 Sigma model perturbation theory

Our aim here is to arrive at formulae for the boundary weights of fields of the $\mathbb{CP}^{1|2}$ model at finite couplings g_σ and θ. In the first subsection, we sketch the method of

[3] In our notations, the second label b refers to the value of the central element E of u(1|1). This differs from the notations that were used e.g. in [58].

background field expansion in complex supersymmetric target spaces and specialize it to the computation of conformal weights. Similarly to the case of the superspheres, the shift of the conformal weights turns out to be given by a particular quadratic Casimir of u(2|2). The results of the first subsection are then combined with our expression (5.2.22) for the free partition function, allowing us to arrive at a conjecture for the partition function of the $\mathbb{CP}^{1|2}$ model with Neumann-type boundary conditions.

5.3.1 Background field expansion and 2-point functions

Let us consider a sigma model on an arbitrary Kähler supermanifold of superdimension $2p|2q$. If we parametrize the supermanifold through real coordinates φ^i, its action takes the following form

$$\mathcal{S}[\varphi] \;=\; \frac{1}{2g_\sigma^2}\int_\Sigma d^2z\,\big(\partial_\mu\varphi(z),\partial_\mu\varphi(z)\big)_{\varphi(z)} + \frac{i\theta}{2\pi}\int_{\varphi(\Sigma)} \mathsf{w}, \tag{5.3.1}$$

where $(X,Y)_\varphi$ denotes the scalar product of two vector fields X,Y at the point φ of the supermanifold and w is the Kähler form. We assume the latter to be normalized such that $\int_{\phi(\Sigma)}\mathsf{w}$ is integer. For the path integral measure we use

$$\mathcal{D}[\varphi] \;=\; \prod_{x\in\Sigma} d\mu\big(\varphi(z)\big), \qquad d\mu(\varphi) = \sqrt{g(\varphi)}\,d\varphi^1\dots d\varphi^{2p+2q}\ .$$

The measure may be regularized by putting the theory on a square lattice with spacing a. To evaluate the scalar product we introduce a basis $e_i = \frac{\overleftarrow{\partial}}{\partial\varphi^i}$ of right derivatives. Expanding two vectors $X = e_iX^i$ and $Y = e_iY^i$, with respect to this basis, we obtain

$$(X,Y) \;=\; (-1)^{|i|}X^ig_{ij}Y^j = g_{ij}Y^jX^i\ . \tag{5.3.2}$$

Here, the order of factors does certainly matter. From the symmetry $(X,Y) = (Y,X)$ of the scalar product in the tangent space we derive the following symmetry of the metric tensor

$$g_{ij} \;=\; (-1)^{|i||j|}g_{ji}\ .$$

We are interested in computing perturbatively the partition function and the correlation functions by the steepest descent method around the *constant* classical solution $\varphi(z,\bar z) = \bar\varphi$. For arbitrary Riemannian manifolds, one can perform the perturbation theory in the background field method by switching to the geodesic coordinates as defined in [73]. When dealing with complex spaces, however, there exists more appropriate coordinates which keep the complex structure manifest. Let w^a be a set of holomorphic coordinates for the Kähler supermanifold and choose some point on it with fixed coordinates w_0^a. A set of holomorphic coordinates v^a for the complex supermanifold $\mathcal{M}$ is called a *normal system of coordinates* at w_0^a if the metric $g_{a\bar b}(v,\bar v|w_0,\bar w_0)$ is of the form

$$g_{a\bar b}(v,\bar v|w_0,\bar w_0) = g_{a\bar b}(w_0,\bar w_0) + \sum_{n=1}^{\infty} c_{a\bar b a_1\bar b_1\cdots a_n\bar b_n}(w_0,\bar w_0)\ v^{\bar b_n}v^{a_n}\dots v^{\bar b_1}v^{a_1}. \tag{5.3.3}$$

The holomorphic transition functions $w = c_{w_0}(v)$ between the set of holomorphic coordinates w and the normal coordinates v at w_0 are completely fixed by the required form of the metric (5.3.3). In fact, one can prove by induction that the transition functions $c_{w_0}(v)$ must possess the following power series expansion in v

$$w^\sigma = c^\sigma_{w_0}(v) = w^\sigma_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\nabla_v^{n-1} v\Big)\Big|_{w_0}(w_0) \tag{5.3.4}$$

$$= w^\sigma_0 + v^\sigma - \sum_{n=2}^{\infty} \frac{1}{n!}\Gamma^\sigma_{b_1 b_2;b_3;\dots;b_n}\Big|_{w_0} v^{b_n}\cdots v^{b_3}v^{b_2}v^{b_1} . \tag{5.3.5}$$

Here, ∇ denotes the covariant derivative on the Kähler manifold. It involves the Christoffel symbols which may be computed from the metric according to

$$\Gamma^i_{jk} = g^{il}\frac{\partial}{\partial w^k} g_{lj} .$$

In eq. (5.3.5) we have expressed the expansion coefficients through multiple covariant derivatives $\Gamma^\sigma_{b_1 b_2;\dots;b_n}$ of the Christoffel symbols $\Gamma^\sigma_{b_1 b_2}$. When evaluating these derivatives, we only treat the lower labels b_i as tensor indices, i.e. the covariant derivatives do not act on the label σ.

In order to actually compute the metric (5.3.3) we use a nice trick. Namely, we propose to consider some holomorphic mapping $w^a(\zeta)$ from a compact Riemann surface Σ, parametrized by the holomorphic coordinate ζ, to the complex symmetric space that is parametrized by the holomorphic coordinates w^a. Since the components of vector fields are known in any frame, the metric in normal coordinates v at w_0 may be derived from the equation

$$\big(\partial w(\zeta), \bar\partial \bar w(\bar\zeta)\big)_{w(\zeta)} = \big(\partial v(\zeta), \bar\partial \bar v(\bar\zeta)\big)_{\{v(z), w_0\}} . \tag{5.3.6}$$

The solution can be written as a power series in $v, \bar v$ with coefficients built out of the components of the curvature tensor at w_0. Indeed, it is not hard to check that

$$(\partial v(\zeta), \bar\partial \bar v(\bar\zeta))_{\{v(\zeta), w_0\}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}\partial\Big(Q^n\big(\bar v(\bar\zeta)\big)v(\zeta), \bar\partial \bar v(\bar\zeta)\Big)_{w_0}, \tag{5.3.7}$$

where we used the operator

$$Q(\bar Y)X = R(X, \bar Y)X$$

which is defined for arbitrary (anti-)holomorphic vectors $(\bar Y)X$ and R is the curvature tensor on our Kähler supermanifold. In the case of complex projective superspace $\mathbb{CP}^{S-1|S}$ the curvature tensor reads

$$R(X, \bar Y)Z = (X, \bar Y)Z + (Z, \bar Y)X . \tag{5.3.8}$$

Plugging this back in to the series (5.3.7), one may sum the expression to obtain

$$(X,\bar{Y})_{\{v,w_0\}} \;=\; \frac{(X,\bar{Y})_{w_0}}{1+(v,\bar{v})_{w_0}} - \frac{(X,\bar{v})_{w_0}(v,\bar{Y})_{w_0}}{\big(1+(v,\bar{v})_{w_0}\big)^2} \tag{5.3.9}$$

where $X(v)$ and $\bar{Y}(\bar{v})$ are arbitrary holomorphic and, respectively, anti-holomorphic vector fields and the scalar product $(\ ,\ )_{w_0}$ is computed with the Fubini-Study metric (5.1.8) at w_0.

In the background field method, the coordinates v and $\bar{v}$ are now promoted to fields $v(z,\bar{z})$ and $\bar{v}(z,\bar{z})$ on the world-sheet. The action (5.3.1) becomes

$$\mathcal{S}[v] \;=\; \int_\Sigma d^2z \left(\frac{1}{g_\sigma^2}+\frac{i\theta}{\pi}\right)\big(\partial v,\bar{\partial}\bar{v}\big)_{\{v,w_0\}} + \left(\frac{1}{g_\sigma^2}-\frac{i\theta}{\pi}\right)\big(\bar{\partial}v,\partial\bar{v}\big)_{\{v,w_0\}} \tag{5.3.10}$$

where the metric $(\ ,\ )_{\{v,w_0\}}$ in normal coordinates was computed in eq. (5.3.7) as a power series of matrix elements of the curvature tensor (5.3.8). For the applications we have in mind, the action (5.3.10) is formulated on a world-sheet with boundary.

Let us assume that the boundary conditions that are imposed along the boundary preserve the global supergroup symmetry. Then the path integration factorizes into two contributions. One of them is a finite dimensional integral along the value w_0 of the fundamental field $w(z_0)$ at one point z_0 of the world-sheet. The second is the path integral along its "deviation" $v(z)$. For the measure, this split takes the following form

$$\mathcal{D}[w,\bar{w}] \;=\; d\mu(w_0,\bar{w}_0)\,\mathcal{D}[v,\bar{v}]\ , \tag{5.3.11}$$

where

$$\mathcal{D}[v,\bar{v}] \;=\; \prod_{x\neq 0}\frac{i^{p+q}}{2^{p+q}}\sqrt{g\big(v(x),\bar{v}(x)|w_0,\bar{w}_0\big)}\,dv^1(x)\wedge d\bar{v}^1(x)\dots dv^{p+q}(x)\wedge d\bar{v}^{p+q}(x)\ . \tag{5.3.12}$$

One can check that the superdeterminant of the metric in normal coordinates does never depend on $v(x)$. For the Fubini-Study metric (5.1.8) on the complex projective superspace $\mathbb{CP}^{S-1|S}$ one even finds that

$$g\big(v(x),\bar{v}(x)|w_0,\bar{w}_0\big) \;=\; g(w_0,\bar{w}_0) \;=\; 1\ . \tag{5.3.13}$$

In conclusion, computations in the background field expansion for $\mathbb{CP}^{S-1|S}$ are performed with the standard path integral measure using the free field theory action

$$\mathcal{S}_0[v] \;=\; \int_\Sigma d^2z \left(\frac{1}{g_\sigma^2}+\frac{i\theta}{\pi}\right)\big(\partial v,\bar{\partial}\bar{v}\big)_{w_0} + \left(\frac{1}{g_\sigma^2}-\frac{i\theta}{\pi}\right)\big(\bar{\partial}v,\partial\bar{v}\big)_{w_0}\ . \tag{5.3.14}$$

The interaction terms are obtained by expanding the Fubini-Study metric (5.3.9) in the fluctuation field v. After this preparation we are now ready to compute some quantities in the sigma model on $\mathbb{CP}^{S-1|S}$.

As a warm-up example, let us calculate the index $J_{0,0}^{g_\sigma,\theta}(q) := \mathsf{Z}_{0,0}^{g_\sigma,\theta}(q,1,1,-1)$, i.e. the partition function of the boundary theory with $M = 0 = N$ specialized to the values $x = 1 = y$ and $z = -1$. It is easy to see from eq. (5.2.14) that the characters of Kac modules vanish at this special point, simply because the contributions from bosons and fermions cancel against each other. It follows from our eq. (5.2.22) that the index J vanishes at $g_\sigma = 0$. Our aim here is to show that it actually vanishes for all values of g_σ and θ. According to eq. (5.3.11), the perturbative partition function $J_{0,0}^{g_\sigma,\theta}$ of the sigma model eq. (5.3.10) can be written as

$$J_{0,0}^{g_\sigma,\theta}(q) \;=\; \int d\mu(w_0,\bar{w}_0) j_{0,0}^{g_\sigma,\theta}(w_0,\bar{w}_0). \tag{5.3.15}$$

We shall call $j_{0,0}^{g_\sigma,\theta}(w_0,\bar{w}_0)$ the local partition function. By carefully analyzing the perturbative expansion of the partial partition function one can prove that it receives no corrections from the interaction terms, that is

$$j_{0,0}^{g_\sigma,\theta}(w_0,\bar{w}_0) \;=\; j_{0,0}^{(0)}(w_0,\bar{w}_0), \tag{5.3.16}$$

where $j_{0,0}^{(0)}(w_0,\bar{w}_0)$ is the local partition function of the free theory (5.3.14). The equality (5.3.16) may be derived with the help of the property (5.3.7) of the metric in normal coordinates. It expresses the perturbative local index in terms of tensor powers of the curvature tensor on $\mathbb{CP}^{S-1|S}$. But all the corrections to the index vanish. In fact, one may show (see appendix B.6) that all scalars constructed from the tensor powers of the curvature tensor on $\mathbb{CP}^{S-1|S}$ are zero. This completes the proof of eq. (5.3.16). It remains to integrate the local index over the target space coordinates w_0. Since neither the measure nor the free action contain w_0, we infer that the local index itself must be constant. Using that the superspace $\mathbb{CP}^{S-1|S}$ has vanishing volume we can now conclude $J_{0,0}^{g_\sigma,\theta}(q) = 0$, as we had claimed before.

The main goal of this section is to compute 2-point functions and thereby to determine the conformal dimensions of boundary fields as a function of g_σ and θ. Let $\mathcal{O}[w](z)$ denote a (boundary) field of the sigma model on our Kähler manifold. After insertion of the change of coordinates formula (5.3.5), the fields become functionals of the (constant) background w_0 and the fluctuation field v. The correlation functions are then given by

$$\Big\langle \prod_\nu \mathcal{O}_\nu[w](z_\nu,\bar{z}_\nu)\Big\rangle \;= \int d\mu(w_0)\, \Big\langle \prod_\nu \mathcal{O}_\nu\big[c_{w_0}(v)\big](z_\nu,\bar{z}_\nu)\; e^{-\mathcal{S}^{\rm int}_{g_\sigma,\theta}[v]}\Big\rangle_{w_0}. \tag{5.3.17}$$

The prescription is to compute the quantity on the the right hand side by expanding in powers of v both the interaction *and* the fields $\mathcal{O}_\nu[c_{w_0}(v)]$. Here, the notation $\langle\ \ \rangle_{w_0}$ used in (5.3.17) means that the expression in brackets must be calculated in the free theory of (5.3.14) with fixed zero mode w_0.

Having described the setup, we can apply (5.3.17) to the computation of boundary 2-point functions for boundary condition changing fields with $M = N$ in the $\mathbb{CP}^{1|2}$ sigma model. For those fields corresponding to functions on $\mathbb{CP}^{1|2}$ appearing in (5.2.16) for $l = M-N = 0$ and that transform in a given highest weight representation $[\Lambda] = [j_1, j_2, a, b]$ of u(2|2), we get

$$\boxed{h^{g_\sigma,\theta}_{N,N}[\Lambda] \;=\; \frac{g_\sigma^2}{\pi}\left[1 - g_\sigma^4\left(\frac{\theta}{\pi}+2N\right)^2\right]\mathsf{Cas}_{\alpha=1}(\Lambda) + O(g_\sigma^8)}\,. \tag{5.3.18}$$

As an illustration, let us do a step by step computation of the first term in the above expansion. Let the U_Λ be eigenfunctions of the Laplacian on $\mathbb{CP}^{1|2}$ with

$$\int d\mu(w) U_{\Lambda_1}[w]U_{\Lambda_2}[w] = \delta_{\Lambda_1\Lambda_2} \quad \text{and} \quad \Delta U_\Lambda = \mathsf{Cas}_{\alpha=1}(\Lambda)U_\Lambda\ . \tag{5.3.19}$$

In the full quantum theory, we expect

$$\langle\, U_{\Lambda_1}(z_1,\bar z_1)U_{\Lambda_2}(z_2,\bar z_2)\,\rangle = \frac{\delta_{\Lambda_1\Lambda_2}}{|z_1-z_2|^{2h_{\Lambda_1}}} \approx \delta_{\Lambda_1\Lambda_2}\left(1 - 2h_{\Lambda_1}\log|z_1-z_2|^2\right)\ . \tag{5.3.20}$$

On the other hand, 5.3.17 leads to

$$\begin{aligned}&\langle\, U_{\Lambda_1}(z_1,\bar z_1)U_{\Lambda_2}(z_2,\bar z_2)\,\rangle\\ &\approx \int d\mu(w_0)\,\big\langle\, \big(U_{\Lambda_1}[w_0] + \partial_i U_{\Lambda_1}[w_0]v^i(z_1,\bar z_1)\big)\big(U_{\Lambda_2}[w_0] + \partial_j U_{\Lambda_2}[w_0]v^j(z_2,\bar z_2)\big)\,\big\rangle_{w_0}\ ,\end{aligned}$$

where we have made use of the fact that the interaction term $\mathcal{S}^{\text{int}}_{g_\sigma,\theta}$ does not contribute to the order g_σ^2 of the correlation functions. We thus get

$$\begin{aligned}&\langle\, U_{\Lambda_1}(z_1,\bar z_1)U_{\Lambda_2}(z_2,\bar z_2)\,\rangle\\ &\approx \int d\mu(w_0)\left(U_{\Lambda_1}[w_0]U_{\Lambda_2}[w_0] + \partial_i U_{\Lambda_1}[w_0]\partial_j U_{\Lambda_2}[w_0]\,\langle\, v^i(z_1,\bar z_1)v^j(z_2,\bar z_2)\,\rangle_{w_0}\right)\\ &= \delta_{\Lambda_1\Lambda_2} + \int d\mu(w_0)\partial_i U_{\Lambda_1}[w_0]\partial_j U_{\Lambda_2}[w_0]g^{ij}\frac{2g_\sigma^2}{\pi\left(1+g_\sigma^4\left(\frac{\theta}{\pi}+2N\right)^2\right)}\log|z_1-z_2|^2\\ &= \delta_{\Lambda_1\Lambda_2}\left(1 - \mathsf{Cas}_{\alpha=1}(\Lambda_1)\frac{2g_\sigma^2}{\pi\left(1+g_\sigma^4\left(\frac{\theta}{\pi}+2N\right)^2\right)}\log|z_1-z_2|^2\right)\ ,\end{aligned} \tag{5.3.21}$$

where in the last line we performed a partial integration and used the fact that U_{Λ_1} is an eigenfunction of the Laplacian. The expression for the free correlation function of the v's was taken from [63]. Comparison of (5.3.21) with (5.3.20) and expansion in g_σ^2 leads to the first term in (5.3.18). This computation shows that, on all manifolds, the correction of order g_σ^2 are proportional to the quadratic Casimir, since the precise form of $\mathcal{S}_{int}$ did not play a role. It does however enter in the analysis of the next terms that appear in the perturbative expansion. Here, the fact that the Ricci tensor[4]of the manifold that

[4]The Ricci tensor is proportional to the Killing form and thus null

we are working with is zero lead to the precise form of (5.3.18). At order g_σ^8 we expect to receive a contribution from a non-zero contraction of the Riemann tensor with itself. We do not yet know what form this contribution will take.

It is easy to see from [63] that conformal weights for boundary condition changing operators with $M = N$ depend on g_σ and θ only through the combination

$$\left(g_\sigma^{eff}\right)^2 \;=\; \frac{g_\sigma^2}{1+g_\sigma^4\left(\frac{\theta}{\pi}+2N\right)^2}\,, \tag{5.3.22}$$

which gives the dependence on g_σ and θ in the propagator of the quantum fields. The computation of the latter for boundary conditions of the type (5.1.18) with $M = N$ can be found in [74]. We have not managed to carry the computation of weights to higher orders. This is partly due to the fact that the background field expansion breaks the psl(2|2) symmetry down to sl(1|2) so that some of the simplifications that arise from special features of the Lie superalgebra psl(2|2) (see e.g. [15]) are not directly applicable. Nevertheless, we take eq. (5.3.18) as a strong indication that boundary weights of tachyonic vertex operators transforming in some representation Λ of u(2|2) behave as,

$$h^{g_\sigma,\theta}_{M,N}(\Lambda) \;=\; h^*_{M,N}(g_\sigma,\theta)+\frac{g_{M,N}(g_\sigma,\theta)}{4}\,\mathsf{Cas}_{\alpha=1}(\Lambda) \tag{5.3.23}$$

with some functions $h^*_{M,N}(g_\sigma,\theta)$ and $g_{M,N}(g_\sigma,\theta)$ that will be determined below. This conjectured behavior of the conformal weights will be one of the central ingredients in our formula for the boundary partition function of the $\mathbb{CP}^{1|2}$ model. It has also passed extensive numerical checks that we describe in the second part of this work.

5.3.2 Partition function at finite coupling

It is now time to spell out the central formula of this paper. We propose the following boundary partition function of the $\mathbb{CP}^{1|2}$ model with monopole bundle boundary conditions M, N imposed along the two boundaries of the strip,

$$\boxed{\mathsf{Z}^{g_\sigma,\theta}_{M,N}(q,x,y,z) \;=\; q^{\frac{\lambda_{M,N}(\lambda_{M,N}-1)}{2}}\sum_\Lambda \chi^{\mathsf{K}}_\Lambda(x,y,z)\,q^{\frac{1}{4}g_{M,N}\,\delta_l\mathsf{C}(\Lambda)}\;\psi^{\mathsf{K}}_\Lambda(q)}\;. \tag{5.3.24}$$

Here, the weights $\Lambda = [j_1, j_2, a, b]$ run over the same values as in (5.2.22) while the functions $\lambda_{M,N}(g_\sigma,\theta)$ and $g_{M,N}(g_\sigma,\theta)$ encode the complete dependence of the partition function on the coupling constants of the theory. These functions are universal, meaning that they do not depend on the representation the field transforms in. Below, in (5.3.27) and (5.3.28), we will provide explicit formulae below for them. The functions λ and g are model-independent, that is they turn out to be the same for all $\mathbb{CP}^{S-1|S}$ models, regardless of the value of S. Hence, the only model dependent inputs in the theory are the Kac module branching functions ψ^{K} and the difference $\delta_l\mathsf{C}$ of Casimir elements spelled out below in (5.3.25). For $S = 2$, the former were determined in section 5.2.2

through our analysis of the model at infinite volume, or $g_\sigma = 0$. The relevant Casimir element Cas_α was displayed in equation (5.2.17) before. What appears here in eq. (5.3.24) is the difference

$$\begin{aligned} \delta_l \mathsf{C}([j_1, j_2, a, l/2]) := &\ \mathsf{Cas}_{\alpha=1}([j_1, j_2, a, l/2]) - \mathsf{Cas}_{\alpha=1}(\mu_{0,l}) \\ = &\ 2[j_1(j_1+1) - j_2(j_2+1)] + l(a-2) - l^2 \ + |l| \end{aligned} \tag{5.3.25}$$

The weight $\mu_{0,l}$ corresponds to the representation of the ground state. The latter minimizes the value of $-\mathsf{Cas}_{\alpha=1}$ among all the representations of fix l that appear in the decomposition (5.2.22), see appendix D for details.

Let us now address the two functions λ and g in more detail. Obviously, the function λ determines the conformal weight of the ground state in the boundary theory, while on the other hand the function g encodes how conformal weights of the excited states change relative to the ground state as we vary the two bulk couplings g_σ and θ. We claim that both λ and g are independent of the integer S, i.e. they are the same for all projective superspaces $\mathbb{CP}^{S-1|S}$. Our argument is based first on the observation, made in [75], that all $\mathbb{CP}^{S-1|S}$ models contain a pair of symplectic fermions as a true subsector and second on our argument of chapter 3, that the embedding of the fields of the symplectic fermion model in the theory with target space $\mathbb{CP}^{S-1|S}$ preserves the correlation functions.

By the method of cohomological reduction, we can confidently state that for the $\mathbb{CP}^{1|2}$ model, those $h = 0$ states that are in the symplectic fermion subsector are to be found within the first two atypical multiplets in the decomposition (5.2.16) at $g_\sigma = 0$. *Since the knowledge of the conformal weights of theses two multiplets for $g_\sigma \neq 0$ suffices to determine the two functions λ and g uniquely, we can derive them both within the free field theory of symplectic fermions.*

Therefore, our first goal is to compute the functions $\lambda_{M,N}$ within the symplectic fermion model. To this end we look back at our formula (5.1.19) that describes the gluing condition of fields at the boundary in terms of the parameters N, M and θ. Comparing with [63], we see that these boundary conditions are of Neumann type, twisted by the presence of a nontrivial matrix W of the form

$$W(\Theta) = i g_\sigma^2 \begin{pmatrix} \Theta & 0 \\ 0 & -\Theta \end{pmatrix} . \tag{5.3.26}$$

The matrix W relates the derivatives along and perpendicular to the boundary of the world-sheet. Since $\Theta = 2N + \theta/\pi$, the matrix W may be written as a sum $W = B(\theta) + F(N)$ of a 'bulk magnetic field' $B = B(\theta)$ and the 'field strength' $F = F(N)$ of the monopole. If we choose different monopole numbers M, N on the two sides of the strip, the gluing conditions along the left and the right boundary are different. Consequently, the corresponding boundary condition changing fields must be in twisted sectors. In order to determine the twist parameter λ, we use to formalism of section 2.5.4 to reformulate the boundary condition in terms of a gluing automorphism ω that relates chiral fields rather than the derivatives ∂_x and ∂_y. The gluing automorphism is given by

$$\omega := \frac{1+W}{1-W} \quad .$$

Let us denote the two different values of Θ along the left and the right boundary by Θ_M and Θ_N. Similarly, we shall use the symbols $W_i = W(\Theta_i)$ and $\omega_i = \omega(\Theta_i)$ for the corresponding field strength W and the gluing automorphism ω along the two half-lines. It follows that the symplectic fermions possess monodromy

$$\omega_{MN} = \omega_M \omega_N^{-1} \;=\; \frac{1+g_\sigma^4\Theta_M\Theta_N + W(\Theta_M-\Theta_N)}{1+g_\sigma^4\Theta_M\Theta_N - W(\Theta_M-\Theta_N)}$$

when taken around a boundary field insertion. The trace of this monodromy matrix ω_{MN} determines the twist parameter of the symplectic fermions through $2\cos 2\pi\lambda = \mathrm{tr}\omega_{MN}$. Putting all this together we find

$$\boxed{\cos 2\pi\lambda_{M,N}(g_\sigma,\theta) \;=\; \frac{(1+g_\sigma^4\Theta_M\Theta_N)^2-(\Theta_M-\Theta_N)^2g_\sigma^4}{(1+g_\sigma^4\Theta_M\Theta_N)^2+(\Theta_M-\Theta_N)^2g_\sigma^4}}\,. \tag{5.3.27}$$

We remind that $\Theta_M = 2M + \theta/\pi$ and $\Theta_N = 2N + \theta/\pi$.

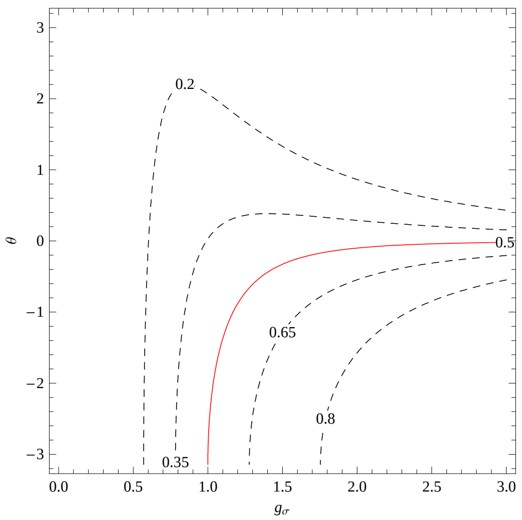

Figure 5.2: A representation of lines of constant $\lambda_{0,1}$ as a function of the coupling constant and of the θ angle, with the value of λ written on the lines. The red one denotes a change in the branch of arccos is (5.3.27). Keeping θ fixed and taking the coupling constant to infinity leads to $\lambda = 1$ or $\lambda = 0$ depending on whether one has crossed the red line or not.

There are a few special cases to be discussed. To begin with let us choose $M = N$. When the two boundary conditions on both sides of the interval are identical so that $\Theta_M = \Theta_N$, then $\cos 2\pi\lambda = 1$ and consequently the twist parameter vanishes. Similarly, we note that the twist parameter always vanishes in the limit of infinite radius, i.e. when

$g_\sigma = 0$. The boundary theory with vanishing twist parameter was constructed explicitly in [72], while the more general case was considered in [76].

It remains to find the second set of functions $g_{M,N}$. We shall see momentarily that they are very closely related to $\lambda_{M,N}$. As we have just argued, the ground states in our symplectic fermion model on the upper half-plane are twist fields with a twist parameter λ. The corresponding ground state conformal weight is

$$h_\lambda^{\rm gr} \;=\; \frac{1}{2}\lambda(\lambda-1) \quad .$$

Excited states in the symplectic fermion model are generated by acting with modes of the form $\chi_{-\lambda-n}, n \leq 0$. Hence, the first excitations above the ground states possess conformal weight $h^{\rm ex} = h_\lambda^{\rm gr} + \lambda$. These states of the symplectic fermions are embedded into the second term in the decomposition (5.2.16). Consequently, the two functions λ and g must be related by

$$\lambda_{M,N}(g_\sigma,\theta) \;=\; \frac{1}{4}\delta\mathsf{C}(\mu_{1,M-N})g_{M,N}(g_\sigma,\theta) \;=\; \frac{1}{2}|M-N|g_{M,N}(g_\sigma,\theta) \tag{5.3.28}$$

where the expressions for the weights $\mu_{0,l}$ and $\mu_{1,l}$ in terms of the labels $[j_1, j_2, a, b]$ can be found in Appendix. D. The equation determines $g_{M,N}$ in terms of the twist parameter $\lambda_{M,N}$, at least when $M \neq N$. When M equals N, the twist parameter vanishes, but, since the coefficient $|M-N|$ on the right hand side of equation (5.3.28) also goes to zero as $M \to N$, the function $g_{N,N}$ can be computed as

$$g_{N,N}(g_\sigma,\theta) \;=\; \lim_{M\to N}\left(\frac{2\lambda_{M,N}(g_\sigma,\theta)}{|M-N|}\right) \;=\; \frac{4g_\sigma^2}{\pi\big[1+g_\sigma^4(\frac{\theta}{\pi}+2N)^2\big]} \quad . \tag{5.3.29}$$

Hence, the universal function $g_{N,N}$ is related to the effective coupling g_σ^{eff} we found while analyzing the background field expansion in eq. (5.3.22),

$$g_{N,N}(g_\sigma,\theta) \;=\; \frac{4}{\pi}\left(g_\sigma^{eff}\right)^2 \quad . \tag{5.3.30}$$

Before we conclude this subsection let us spell out one more special case of our expression for λ, so as to prepare for our lattice analysis in the next section. In the second part, we will perform numerical calculations for nonzero values of the monopole charges M, N. Numerical simulations with $M = 0$ and $N = -1$ at the point $g_\sigma = 1$ will give the ground state energy $h_\lambda = -1/8$. This corresponds to the twist parameter $\lambda = 1/2$. To reproduce this values, we need

$$\cos 2\pi\lambda_{0,1}(g_\sigma^2 = 1,\theta) \;=\; \frac{\left(1+\frac{\theta}{\pi}\left(\frac{\theta}{\pi}-2\right)\right)^2-4}{\left(1+\frac{\theta}{\pi}\left(\frac{\theta}{\pi}-2\right)\right)^2+4} \;=\; -1 \quad .$$

We read off that the lattice model must flow to the continuum theory with $\theta = \pi$. It is interesting to note that the θ angle of the bulk theory may be determined from the behavior of boundary conformal weights.

5.4 Discretization and Numerics

In (5.3.24), we made a conjecture for the exact partition function of those boundary theories that leave the u(2|2) symmetry unbroken on the basis of two key observations. First, perturbative studies around the point of infinite volume, or $g_\sigma = 0$, point suggest that the conformal weights evolve with the quadratic Casimir element. Second, the method of cohomological reduction allowed us to show that the conformal dimensions of at least a subset of the fields changes in the desired way and to furthermore determine the universal functions $g_{M,N}$, $\lambda_{M,N}$ together with the ground state energies. While the embedding of symplectic fermions is a non-perturbative feature of the $\mathbb{CP}^{1|2}$ model, the Casimir evolution was only analyzed perturbatively in the coupling constant g_σ.

For an introduction to the basic theory of spin chains, we refer the reader to [77]. In the following we shall introduce a $\mathrm{u}(S\,|S)$ spin chain[5] built out of alternating tensor products $\big(V \otimes V^\star\big)^{\otimes L}$ of the fundamental $\mathrm{u}(S\,|S)$ representation V and its dual $V^\star$. The $\mathrm{u}(S\,|S)$ symmetry, combined with the requirements of *locality* and *criticality* will constrain the Hamiltonian to a linear combination of generators of the $\mathrm{u}(S\,|S)$ centralizer algebra of the spin chain. Restricting to *homogeneous interactions* and choosing a normalization, we shall reduce the number of parameters defining the discrete Hamiltonian of the spin chain to a single real number w.

We claim that the continuum limit of this homogeneous quantum chain provides a discretization of the $\mathbb{CP}^{S-1|S}$ sigma model. Since the sigma model depends on two variables g_σ and θ, and the spin chain model only on one, namely w, the description is only valid for a specific value of the angle, which turns out to be $\theta = \pi$. The evidence in favor of the claim consists of a set of comparisons of similar structures both on the sigma model and the spin chain side, such as symmetries, boundary conditions, spectra and various compatibility checks.

In particular, we check that there is a one to one correspondence between the $\mathrm{u}(S\,|S)$ symmetry preserving boundary conditions of the sigma model and those of the spin chain and that the coupling constant g_σ of the sigma model is determined by the parameter w independently of them. Finally, we identify a limit for w in which the spectrum of the spin chain is in perfect numerical agreement with the perturbative calculations in the sigma model for $g_\sigma \to 0$.

At a second step, we assume that the proposed $\mathrm{u}(S\,|S)$ alternating spin chain indeed provides a discretization for the $\mathbb{CP}^{S-1|S}$ sigma model. The discrete theory can then be studied numerically and the Casimir evolution for weights checked non-perturbatively. We shall find remarkable agreement between our analytical studies of the continuum model and the numerical results for the spin chain. The agreement suggests that our proposal for the partition functions of boundary theories is exact and that in particular, it does not receive non-perturbative corrections.

[5] We refer the curious reader to [18, 19] for a similar spin chain for the superspheres $S^{2M+1|2M}$.

5.5 Brauer algebra and alternating $\mathbf{u}(S\,|S)$ spin chain

In this section, we intend to establish the Hamiltonian

$$H \;=\; -\sum_{i=1}^{2L} \left[tE_i + wP_{i,i+2}\right] \,, \tag{5.5.1}$$

as a promising candidate for the description of a discrete variant of the bulk dynamics in the $\mathbb{CP}^{S-1|S}$ sigma models. This discussion requires some background in *walled Brauer algebras*, which we describe first.

The straightforward lattice studies of the two dimensional $\mathbb{CP}^S$ models involve the Monte Carlo simulation of a model with S dimensional complex unit vectors on the vertices and U(1) gauge fields on the edges of a square lattice, together with the proper discretization of the topological term[6], as described in [78,79]. Condensed matter physics has provided an alternative to this approach, whereby the fields can emerge dynamically as collective excitations of quantum spins. The conjecture by Haldane [80, 81] that the long distance properties of SU(2) spin chains is described by the O(3) sigma model at $\theta = 0$ for integer ($\theta = \pi$ for half integer) spin opened the way to studying the mapping of most general spin chains to sigma models [82].

The geometric quantization arguments of [13, 82] show that the simplest spin chain we could use to understand the $\mathbb{CP}^{S-1|S}$ model is based on alternating the fundamental representation[7] V of $\mathrm{u}(S\,|S)$ together with its dual $V^\star$.

Unfortunately, integrable spin chains for this choice of representations turn out to have a non generic continuum limit, described by a WZW model [59]. To see the physics of the $\mathbb{CP}^{S-1|S}$ model, we need to employ more generic interactions, but the ones we found to describe the physics of the continuum theory do not preserve integrability. Although we did not perform a systematic analysis on the existence of local integrals of motion, we believe it cannot happen due to the following reasons. Integrable spin chains with continuous physical parameters in the Hamiltonian are generically engineered from R-matrices of quantum supergroups. For the spin chains we considered there is no manifest quantum supergroup symmetry. In line with this general expectation, we did not find any relativistic $\mathrm{u}(S\,|S)$ symmetric solution to the Yang-Baxter equation, which depends on a continuous physical parameter, if the choice of the auxiliary space in the quantum inverse scattering method is restricted to V and $V^\star$.

Fortunately, a lot can still be understood analytically by studying the properties of the chains under the simultaneous action of the Lie superalgebra symmetry and its commutant [10]. In the present case, this commutant is given by the walled Brauer algebra. The algebraic approach that we are about to review has a number of appealing features, including the fact that many of its features are independent of the value of S.

Throughout the following subsections we denote the generators of the Brauer algebra $B_{2L}(0)$ by $E_1, P_1, \ldots, E_{2L-1}, P_{2L-1}$. In the symbol $B_{2L}(0)$, the subscript $2L$ is related

[6] this is somewhat less obvious of course, as there is no topology on the lattice

[7] For a description of these modules, see [16].

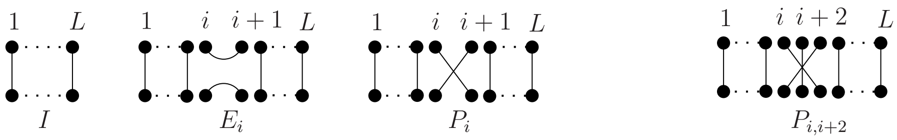

Figure 5.3: The identity I and the generators E_i, P_i of the Brauer algebra of dimension $(2L-1)!!$ are represented on the left; the walled Brauer algebra generator $P_{i,i+2} = P_iP_{i+1}P_i$ is represented on the right.

to the dimension of the Brauer algebra by $\dim B_{2L}(0) = (4L-1)!!$ and the parameter in parenthesis denotes the so-called fugacity for loops. The defining relations of $B_{2L}(0)$ can be found in [83]. The words of this Brauer algebra admit a representation as graphs on $4L$ labelled vertices with $2L$ edges connecting the vertices pairwise in all $(4L-1)!!$ possible ways, where crossings are allowed. The identity I of the Brauer algebra and the generators E_i, P_i are represented by the graphs on the left in figure 5.3. The algebraic structure of the Brauer algebra is obtained by stacking diagrams. More precisely, in order to multiply the diagrams one arranges the first $2L$ vertices horizontally with the remaining $2L$ vertices on top of the first ones. The product of a diagram d_1 with a diagram d_2 is the diagram d_1d_2 obtained by first placing the diagram d_1 on top of the diagram d_2, then identifying the top of the diagram d_2 with the bottom of the diagram d_1 and finally replacing every loop generated in this process by zero. The *periodic* Brauer algebra is an extension of the Brauer algebra by the two generators E_{2L} and P_{2L} which satisfy the same defining relation as the original ones if the index $i \equiv i + 2L$ is regarded as periodic. The words of the periodic Brauer algebra are diagrams with the top and the bottom being circles wrapped around a cylinder and carrying $2L$ vertices each, such that the latter are pairwise connected in all possible ways by $2L$ edges living on the surface of the cylinder. One can easily see that the periodic Brauer algebra has infinite dimension.

Definition 5.5.1. The elements E_i and $P_{i,i+2} := P_iP_{i+1}P_i$ freely generate a subalgebra called the *walled Brauer algebra.* The generators $P_{i,i+2}$ are represented on the right in figure 5.3.

This algebra is of central importance for the study of $\mathrm{u}\,(S|S)$-invariants as explained in the following. As stated before, let V denote the fundamental representation of $\mathrm{u}\,(S|S)$ and let $V^\star$ be its dual. Generalizing the well known statement for gl (S), Sergeev proved in [84] that there is a surjective homomorphism from the walled Brauer algebra to the invariants of the tensor module $(V \otimes\ V^\star)^{\otimes 2L}$ or, equivalently, to the $\mathrm{u}\,(S|S)$-centralizer of $(V \otimes\ V^\star)^{\otimes L}$. This means that the $\mathrm{u}\,(S|S)$-centralizer of $(V \otimes\ V^\star)^{\otimes L}$ can be viewed as some representation of the walled Brauer algebra. In particular, the most general $\mathrm{u}\,(S|S)$-symmetric spin chain Hamiltonian H one can write down must represent some element of the walled Brauer algebra. If we restrict to nearest neighbor interactions only, thus defining a $\mathrm{u}\,(S|S)$ version of the Heisenberg chain, we get a Hamiltonian of the form

$$H_{\mathrm{TL}} \;=\; -\sum_i t_i E_i \; . \tag{5.5.2}$$

This Hamiltonian as well as all of its powers and the corresponding evolution operator $e^{-\tau H_{\mathrm{TL}}}$ lie entirely in the Temperley-Lieb subalgebra of the walled Brauer algebra. Thus, by the double centralizer theorem, the symmetry of H_{TL} must be bigger then $\mathrm{u}\,(S|S)$. It was shown in [85] that the spectrum of low lying excitation of the homogeneous chain H_{TL} in the scaling limit can be obtained by the free field theory of a pair of free symplectic fermions,

$$\mathcal{S} \;\sim\; \int_\Sigma d^2z\, \partial_\mu\eta_1(z,\bar{z})\partial^\mu\eta_2(z,\bar{z}). \tag{5.5.3}$$

The degeneracies of the excitations of the lattice model must then be computed by employing independent representation theoretic tools developed in [10].

We are naturally interested in deformations of the Temperley-Lieb Hamiltonian (5.5.2) which break the symmetry all the way down to $\mathrm{u}\,(S|S)$ while preserving conformal invariance in the continuum limit. The simplest Ansatz for $\mathrm{u}\,(S|S)$-symmetric Hamiltonian is the sum of generators of the walled Brauer algebra. Since the generator $P_{i,i+2}$ corresponds to second nearest neighbor interactions on the spin chain $(V \otimes\; V^\star)^{\otimes L}$, it is natural to consider the following $\mathrm{u}\,(S|S)$-symmetric deformation of the Hamiltonian in (5.5.2)

$$H_{\mathrm{gen}} \;=\; -\sum_i \left[t_i E_i + w_i P_{i,i+2} + a_i E_i E_{i+1} + b_i E_{i+1} E_i\right] \; . \tag{5.5.4}$$

The eigenvalues of the Hamiltonian (5.5.4) are more easily computed by working in the adjoint - that is in the diagrammatic - representation of the walled Brauer algebra, rather than in the representation on $(V \otimes\; V^\star)^{\otimes L}$. However, when switching between the alternating spin chain and adjoint representations of the walled Brauer algebra one looses control of the degeneracies of eigenvalues. These can be recovered from representation theory by methods similar to those used in [18]. We shall call the Hamiltonian (5.5.4) algebraic when it is considered in the adjoint representation of the walled Brauer algebra. The actual spectrum of the $\mathrm{u}\,(S|S)$ alternating spin chain will be a subset of the spectrum of the algebraic Hamiltonian (5.5.4), that we will refer to as a $\mathrm{u}\,(S|S)$-sector of the algebraic Hamiltonian. With a little bit of representation theory of the walled Brauer algebra one can prove that the eigenvalues of the $\mathrm{u}\,(S-1|S-1)$ spin chain Hamiltonian are a *subset* of the eigenvalues of the $\mathrm{u}\,(S|S)$ spin chain Hamiltonian. This is done in essentially the same way as in [18], where a similar result was obtained for the $\mathrm{osp}\,(2M+2|2M)$-symmetric spin chains.

At a critical point, the space of states of the statistical model usually possesses some additional discrete symmetries. Without loss of generality one can impose these discrete symmetries on the Hamiltonian (5.5.4), thereby reducing the number of independent couplings t_i, w_i, a_i, b_i. The scale invariant vacuum in periodic boundary conditions is necessarily translation invariant. Consequently, we shall restrict to *homogeneous*

Hamiltonians in (5.5.4), i.e. to those that are invariant under the discrete shift automorphism

$$E_i \to E_{i+1}, \quad P_{i-1,i+1} \to P_{i,i+2}$$

of the periodic walled Brauer algebra. If we additionally assume invariance with respect to the reflection automorphism

$$E_i \to E_{2L-i+1}, \qquad P_{i,i+2} \to P_{2L-i,2L-i+2},$$

then the Hamiltonian becomes

$$H \;=\; -\sum_{i=1}^{2L} \left[tE_i + wP_{i,i+2} + a(E_iE_{i+1} + E_{i+1}E_i) \right] . \tag{5.5.5}$$

Furthermore, we shall restrict to real couplings t, w and a. We now need to gain further insight into the roles of the new couplings w and a. In particular we shall argue that w is an exactly marginal coupling which corresponds to the radius parameter R of the continuum theory, while on the other hand the coupling a seems to be irrelevant and will eventually be set to zero.

In order to interpret the couplings a and w we shall mostly work in the $\mathrm{u}\,(1|1)$ subsector, i.e. we will consider the Hamiltonian of (5.5.5) as an operator on the state space of the $\mathrm{u}\,(1|1)$ alternating spin chain. The resulting theory can be understood as a discrete version of the free theory of symplectic fermions, by introducing a set of $2L$ creation and annihilation fermionic operators

$$\{\varphi_i, \bar{\varphi}_j\} \;=\; \delta_{ij}, \qquad i,j = 1, \ldots, 2L\ . \tag{5.5.6}$$

These may be employed to represent the generators of the walled Brauer algebra through the following quadratic expressions

$$\begin{aligned} E_j =&\; (-1)^j(\bar{\varphi}_j - \bar{\varphi}_{j+1})(\varphi_j + \varphi_{j+1}) \\ P_{j-1,j+1} =&\; (-1)^j\left[1 - (\bar{\varphi}_{j-1} - \bar{\varphi}_{j+1})(\varphi_{j-1} - \varphi_{j+1})\right] . \end{aligned} \tag{5.5.7}$$

The continuum limit of the $\mathrm{u}\,(1|1)$ Hamiltonian (5.5.5) with $a = 0$ is described by an action of the type (5.5.3), the same we found for $w = 0$. In other words, when $a = 0$ and $S = 1$, the perturbation with $P_{i,i+2}$ is truly redundant for its only effect on the lattice is to renormalize the sound velocity

$$v_{\text{sound}} \;=\; 2t\sqrt{1+4w}\ .$$

Switching on the coupling $a \neq 0$ in the $\mathrm{u}\,(1|1)$ alternating spin chain (5.5.5) provides a quartic interaction in terms of the discrete fermions (5.5.6). The resulting model does not seem to be exactly solvable. One of the fourth order terms of the continuum theory,

$$\delta\mathcal{S} \sim \int_\Sigma d^2z\, \eta_1(z)\eta_2(z)\partial_\mu\eta_1(z)\partial_\mu\eta_2(z),$$

has been studied in detail in [86], where it was shown to be either marginally relevant or marginally irrelevant, depending on the sign of its coupling. In the continuum theory, adding a fourth order term in the fermions is actually inconsistent with the $\mathrm{u}(1|1)$ symmetry of the model.[8] Free symplectic fermions possess 16 bulk fields of weight $h = \bar{h} = 1$, eight of which are obtained by multiplying $1, \eta_1, \eta_2, \eta_1\eta_2$ with $\partial\eta_1$ or $\partial\eta_2$, while the for the remaining eight one uses similar expressions with$\bar{\partial}$ in place of ∂. Under the right (or left) action of $\mathrm{u}(1|1)$, these transform in four indecomposable projectives. A closer look reveals that only two of the 16 fields are true invariants, i.e. annihilated by all the $\mathrm{u}(1|1)$ generators, and that both are quadratic in the fermions. Hence, adding a fourth order term to the symplectic fermion model breaks the $\mathrm{u}(1|1)$ symmetry, leading us to to conclude that non-zero values of the parameter a in the lattice theory will not affect the continuum theory, at least not for small enough value of a.

We suggest that the above conclusions should remain essentially correct for $S > 1$. The numerical diagonalization of the algebraic Hamiltonian (5.5.5) for $a = 0$ indicates that its lowest eigenvalue lies in the $\mathrm{u}(1|1)$-sector. This means that one can compute this lowest eigenvalue by restricting the algebraic Hamiltonian (5.5.5) to the state space of the $\mathrm{u}(1|1)$ alternating spin chain. Hence, w should be *exactly marginal* even for $S > 1$, at least as long as $a = 0$. It is tempting to think that this conclusion remains valid for nonzero values of a and that a continues to be irrelevant. This concludes the set of arguments in favor of taking (5.5.1) as the Hamiltonian of the model.

To have a complete correspondence between the couplings of the $\mathbb{CP}^{S-1|S}$ sigma model and those of our lattice model we are still left with the problem of identifying a second lattice coupling that could implement the θ angle. Let us anticipate that the θ parameter corresponds to staggering the couplings of the lattice model, while postponing further details until the conclusion.

In the following we shall provide strong evidence for our claim that the spectrum of low lying excitations of the alternating $\mathrm{u}(S|S)$ spin chain with Hamiltonian given by (5.5.1) is described by the sigma model on the complex projective superspace $\mathbb{CP}^{S-1|S}$ with $\theta = \pi$. However, before we start gathering this evidence it is useful to sum up all our the arguments connecting the spin chain (5.5.5) to the $\mathbb{CP}^{S-1|S}$ sigma model in a diagram represented in figure. 5.4.

5.6 Open alternating $\mathrm{u}(S|S)$ spin chain

Following the outcome of our discussion in the last section, let us now work with the alternating $\mathrm{u}(S|S)$-spin chain on the space $(V \otimes V^\star)^{\otimes L}$ with Hamiltonian

$$H = -\sum_{i=1}^{2L-1} E_i - w\sum_{i=1}^{2L-2} P_{i,i+2} \,. \tag{5.6.1}$$

We remark that, since the number of V and $V^\star$ representations in this spin chain is the same, the symmetry is actually $\mathrm{psl}(S|S)$. Since in our investigation of the sigma

[8] We thank N. Read for a discussion of this point.

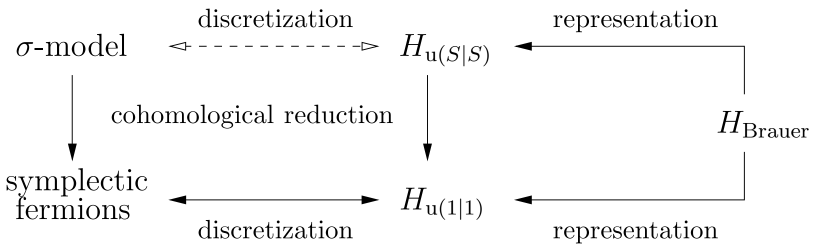

Figure 5.4: Connexion between the CFT of the $\mathbb{CP}^{S-1|S}$ sigma model and the symplectic fermions, on the one side, and various lattice models, on the other side. The only connexion in the diagram which is not yet fully understood is represented by a dashed line and will be supported in the following sections.

model we looked at boundary spectra, we now need to consider an appropriate *open chain* in order to be able to compare the numerical to the continuum results. Numerical evidence suggests that in the limit $w \to \infty$ and $L \to \infty$, the eigenvalues $E_h(L)$ of the Hamiltonian (5.6.1) become infinitely degenerated. *Therefore, we identify the point $w = \infty$ with the large volume limit of the sigma model on the complex projective superspace* $\mathbb{CP}^{S-1|S}$. A similar identification was successfully proposed in [18] for the $\mathrm{osp}(2S{+}2|2S)$-spin chain on $V^{\otimes L}$.

Without any additional algebraic guidance, the spectrum of the Hamiltonian (5.6.1) is rather difficult to analyze. In order to unravel some of the structure, it is useful to classify eigenvalues according to the representations of the walled Brauer algebra that appear in the decomposition of the state space $(V \otimes V^\star)^{\otimes L}$. If one is interested in states that transform according to some irreducible representation of the $\mathrm{u}(S|S)$ symmetry, it pays off to identify those representations of the walled Brauer algebra that are compatible with the required symmetry. The Hamiltonian (5.6.1) may then be restricted and diagonalized within each such building block.

We shall be mainly concerned with the numerical analysis of excitations of H whose eigenvalues vanish in the limit $w \to \infty$. On the sigma model side, these are the scaling dimensions of tachyonic fields, i.e. of those fields that can be built from square integrable functions on the complex projective superspace $\mathbb{CP}^{S-1|S}$. According to the results of [87] and of subsection 5.2.1, the space of tachyonic fields may be identified with the multiplicity free direct sum of supersymmetric, self-dual, traceless $\mathrm{u}(S|S)$ tensors $t(k,k)$ of rank $k > 1$ and the indecomposable traceless but reducible tensor $t(1,1) = V \otimes V^\star$ of which the trivial tensor $t(0,0)$ is a submodule. In our analysis of the $\mathbb{CP}^{1|2}$ model, these were denoted by $t(k,k) = \mu_{k,0}$ for $k \geq 2$. The space $t(1,1)$ contains the irreducible representations of weight $\mu_{1,0}$ once, and of weight $\mu_{0,0}$ twice, meaning that $\mathsf{top}\, t(1,1) \cong \mathsf{soc}\, t(1,1) \cong [\mu_{0,0}]$ and $\mathsf{rad}\, t(1,1)/\,\mathsf{soc}\, t(1,1) \cong [\mu_{1,0}]$. More details on these labels can be found in appendix D.

We now restrict the Hamiltonian (5.6.1) to the submodule of $(V \otimes V^\star)^{\otimes L}$ that contains all states in the $\mathrm{u}(S|S)$ representations $t(k,k)$, where $k = 0, \ldots, L$.

Definition 5.6.1. The vector space of all possible embeddings of $\mathrm{u}(S|S)$ tensors $t(k,k)$

into $(V \otimes V^\star)^{\otimes L}, k \neq 1$, can be endowed with an action of the walled Brauer algebra and it provides an indecomposable representation which we denote by $T_{L,L}(k,k)$.

Similarly, the vector space of all possible embeddings of the $\mathrm{u}(S|S)$ tensor $t(1,1) = V \otimes V^\star$ into $(V \otimes V^\star)^{\otimes L}$ can be endowed with an action of the walled Brauer algebra. In this case, the space gives rise to an indecomposable representation $I_{L,L}$. It is not difficult to see that the space $T_{L,L}(0,0)$ (which we defined previously) is a submodule of $I_{L,L}$. The corresponding quotient will be denoted by $T_{L,L}(1,1) = I_{L,L}/T_{L,L}(0,0)$. The space $T_{L,L}(1,1)$ is actually not irreducible either. In fact, it can be shown to possess the module $T_{L,L}(0,0)$ as a quotient. All these statements may be proved using the geometric, i.e. adjoint representation of the walled Brauer algebra.

Definition 5.6.2. Borrowing from the literature on self-avoiding walks, we shall call the lowest eigenvalue of the Hamiltonian (5.6.1) in the space $T_{L,L}(k,k)$ the *$2k$-legged water melon exponent $h_{0,0}(k)$*. Here the subscripts of $h_{0,0}(k)$ remind us that we are currently only dealing with the $M = N = 0$ boundary condition.

According to our discussion above, the degeneracy of $h_{0,0}(k)$ is $\dim t(k,k)$. The numerical results presented in figure 5.5 strongly suggest that the continuum limit of the watermelon exponents is given by the very simple expression

$$h_{0,0}(k) \;=\; \frac{g_{0,0}(w)\,\mathsf{Cas}(k)}{4} \;=\; \frac{g_{0,0}(w)k(k-1)}{2}\;, \tag{5.6.2}$$

where $\mathsf{Cas}(k)$ is the value of the quadratic Casimir[9] in the irreducible representations $t(k,k)$ for $k \neq 1$. For $k = 1$, $\mathsf{Cas}(k) = 0$ is the value of the quadratic Casimir in either the adjoint or the trivial representation of $\mathrm{u}(S|S)$. The degeneracy of the $h_{0,0}(1)$ watermelon exponent with the vacuum is due to the fact that, as we mentioned above, $T_{L,L}(0,0)$ is a quotient of $T_{L,L}(1,1)$.

The numerical results should be compared with our formulae (5.3.25) and (5.3.30) that determine the conformal weight $h = \delta\mathsf{C}g_{0,0}/4$ of boundary fields in the continuum model. Using the association of the k^{th} watermelon exponent with the weight $\mu_{k,0}$ and the dictionary at the end of appendix B, we conclude that

$$\delta_0\mathsf{C}([k-1,0,2,0]) \;=\; 2k(k-1)\;\;.$$

This is in perfect agreement with our continuum theory. Note that both on the lattice and in the continuum the ratio between the conformal weight and the value of the Casimir element is universal, i.e. it is independent of k. On the lattice, the universal function $g_{0,0} = g_{0,0}(w)$ depends on the lattice coupling w. The corresponding function $g_{0,0} = g_{0,0}(g_\sigma, \theta)$ is known explicitly, see equation (5.3.30). We will provide support below for the claim that $\theta = \pi$ in the continuum limit of our lattice theory. *Anticipating that, we can use the identification $g_{0,0}(w) = g_{0,0}(g_\sigma, \theta = \pi)$ to determine numerically the functional dependence $w = w(g_\sigma)$ of the lattice on the sigma model coupling g_σ.*

[9]For these representations, the value of the quadratic Casimir is independent of α, see (D.0.2).

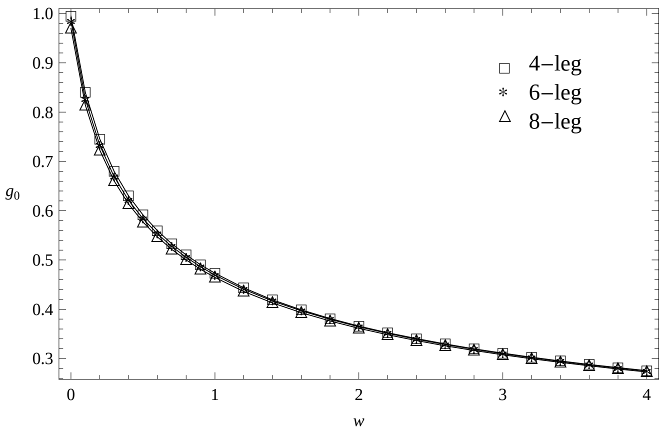

Figure 5.5: Plot of $g_{0,0}(w)$ extracted from the watermelon exponents $h_{0,0}(2)$, $h_{0,0}(3)$ and $h_{0,0}(4)$ computed at $L = 9$ with the help of eq. (5.6.2).

5.7 Twisted open alternating u $(S|S)$ spin chains

The numerical analysis performed in the previous section suggests that the spectrum of the open u $(S|S)$ spin chain is described in the continuum limit by the sigma model on $\mathbb{CP}^{S-1|S}$ subject to Neumann boundary conditions or modified Neumann boundary conditions in the presence of a θ-term. However, the sigma model on $\mathbb{CP}^{S-1|S}$ admits a much larger set of boundary conditions that do not break the global u $(S|S)$ symmetry, namely those described by the nontrivial complex line bundles over $\mathbb{CP}^{S-1|S}$. The complex line bundles can be different at the different ends of the string and we label them by two integers M and N called monopole charges. These bundles may be introduced by adding boundary terms to the action, that is integrals of locally defined 1-forms along the two boundaries. Each of these forms is then interpreted as a connection defining a complex line bundle. Naturally, if the two bundles are different, then so are the boundary conditions at the two boundaries of the world-sheet. Twisting of the spectrum should then be expected when $M \neq N$. In fact, as we showed in section 5.3.2, the u $(1|1)$ subsector of the $\mathbb{CP}^{S-1|S}$ sigma model is described by a pair of twisted free symplectic fermions with twisting parameter

$$\tan \pi\lambda_{M,N} = \frac{2lg_\sigma^2}{1 + \Theta_M \Theta_N g_\sigma^4}, \tag{5.7.1}$$

where $\Theta_M = \frac{\theta}{\pi} + 2M$, $\Theta_N = \frac{\theta}{\pi} + 2N$ and $l = M - N$. It is natural to ask *if one can associate a spin chain to each of these more general boundary conditions* and as we explain in the following, the answer to that question is *positive.* We shall describe the general setup in the following subsection, following that with a discussion of our numerical results, first for the u $(1|1)$ subsector and then for the watermelon exponents in the general twisted open chain.

5.7.1 Monopole boundary conditions

The space of sections in the non-trivial complex line bundles over $\mathbb{CP}^{S-1|S}$ is endowed with an action of $\mathrm{u}(S|S)$ rather than the $\mathrm{psl}(S|S)$ of (5.6.1). Therefore, in order to break the $\mathrm{psl}(S|S)$ symmetry one can proceed by considering the chain of section. 5.6 with some extra V's or some extra $V^\star$ glued to the ends of the chain. Depending on what we attach to either end of the chain, there are four cases to consider, that we list together with the Hamiltonians we chose to describe their dynamics:

$$\begin{array}{rlrl} V^{\otimes m} & \otimes (V\otimes V^\star)^{\otimes L}\otimes\ (V^\star)^{\otimes n}: & H^{VV^\star} & = H_L^V + H_B + H_R^{V^\star} \\ V^{\otimes m} & \otimes (V\otimes V^\star)^{\otimes L}\otimes\ V^{\otimes n}: & H^{VV} & = H_L^V + H_B + H_R^{V} \\ (V^\star)^{\otimes m} & \otimes (V\otimes V^\star)^{\otimes L}\otimes\ (V^\star)^{\otimes n}: & H^{V^\star V^\star} & = H_L^{V^\star} + H_B + H_R^{V^\star} \\ (V^\star)^{\otimes m} & \otimes (V\otimes V^\star)^{\otimes L}\otimes\ V^{\otimes n}: & H^{V^\star V} & = H_L^{V^\star} + H_B + H_R^{V}, \end{array} \tag{5.7.2}$$

where the *bulk* Hamiltonian is the same as in (5.6.1), i.e.

$$H_B = -\sum_{i=m+1}^{2L+m-1} E_i - w\sum_{i=m+1}^{2L+m-2} P_{i,i+2}\ , \tag{5.7.3}$$

while the boundary Hamiltonians are as follows

$$H_L^V = -\,u\sum_{i=1}^{m} P_{i,i+1} \qquad\qquad H_R^{V^\star} = -v\sum_{i=2L+m}^{2L+m+n-1} P_{i,i+1} \tag{5.7.4}$$

$$H_L^{V^\star} = -\,u\sum_{i=1}^{m-1} P_{i,i+1} - w' P_{m,m+2} - t' E_m \tag{5.7.5}$$

$$H_R^{V} = -\,t'' E_{2L+m} - w'' P_{2L+m-1,2L+m+1} - v\sum_{i=2L+m+1}^{2L+m+n-1} P_{i,i+1}\ . \tag{5.7.6}$$

Taking into account that the monopole charges M and N describing the boundary conditions of the $\mathbb{CP}^{S-1|S}$ sigma model can be both positive and negative, the existence of four types of chains (5.7.2) labelled by two *positive* integers m, n is quite suggestive of a possible identification. On the other hand, the boundary conditions in the $\mathbb{CP}^{S-1|S}$ sigma model and the bundles associated to the corresponding branes do not depend on the details of the connection, but only on their curvature, which is fixed by the monopole charge M or N. In view of the relation we are about to establish between the spectrum of the $\mathbb{CP}^{S-1|S}$ sigma model and that of the chains (5.7.2), the previous remarks raise the question as to how much the spectrum of the Hamiltonians (5.7.2) depend on the precise form of the boundary terms (5.7.4–5.7.5). We shall analyze this issue in the $\mathrm{u}(1|1)$ subsector first.

5.7.2 Numerics for the u $(1|1)$ subsector

To answer the question of universality and check the applicability of formula (5.7.1) to the chains (5.7.2), we first look at their u $(1|1)$ subsectors. In this subsector, we make use of the Brauer algebra representation (5.5.6) via discrete free fermions and extend to the twisted open spin chain. With the boundary interaction terms

$$\begin{aligned} P_{V\otimes\, V} = &\ -P_{V^\star\otimes\, V^\star} \ = \ [1-(\bar\varphi_1-\bar\varphi_2)(\varphi_1-\varphi_2)] \\ P_{V\otimes\, V^\star\otimes\, V} = &\ -P_{V^\star\otimes V\otimes\, V^\star} \ = \ [1-(\bar\varphi_1-\bar\varphi_3)(\varphi_1-\varphi_3)] \\ E_{V\otimes V^\star} = &\ -E_{V^\star\otimes V} \ = \ -(\bar\varphi_1-\bar\varphi_2)(\varphi_1+\varphi_2), \end{aligned}$$

we obtain a free system that can be studied numerically and with great efficiency. Let us anticipate the following three basic outcomes of the numerical analysis.

1. The u $(1|1)$ spin chains (5.7.2) flow to the free field theory of symplectic fermions with twisted boundary conditions of the form (5.1.19).

2. The twisting parameter λ does not depend on the boundary couplings u, t', w', t'', w'', v as long as t', t'', u and v are non-zero and the bulk length L of the chain is sufficiently large.

3. In the continuum limit, the dependence of the twisting parameter λ on m, n and w for all four chains (5.7.2) is reproduced by equation (5.3.27) for the $\mathbb{CP}^{S-1|S}$ sigma model with
$$\theta = \pi \tag{5.7.7}$$
provided the following identification between the monopole charges and the thickness of the boundaries of the chains is performed
$$V^{\otimes m}\otimes(V\otimes V^\star)^{\otimes L}\otimes(V^\star)^{\otimes n}: \qquad M = +m \quad N = +n \tag{5.7.8}$$
$$V^{\otimes m}\otimes(V\otimes V^\star)^{\otimes L}\otimes V^{\otimes n}: \qquad M = +m \quad N = -n \tag{5.7.9}$$
$$(V^\star)^{\otimes m}\otimes(V\otimes V^\star)^{\otimes L}\otimes(V^\star)^{\otimes n}: \qquad M = -m \quad N = +n \tag{5.7.10}$$
$$(V^\star)^{\otimes m}\otimes(V\otimes V^\star)^{\otimes L}\otimes V^{\otimes n}: \qquad M = -m \quad N = -n\,. \tag{5.7.11}$$

We now present the numerical evidence supporting these claims one by one.

The numerical calculations supporting claim 1) are presented in figure 5.6, in which we compare the conformal dimension h of the ground state of our spin chain with the expression
$$h \ = \ \frac{\lambda(\lambda-1)}{2} \tag{5.7.12}$$
which determined the conformal dimension of twist fields in terms of the twist parameter λ. For the lattice model, *the twist parameter is measured as the first excitation over the vacuum in the* $u\,(1|1)$ *subsector.*

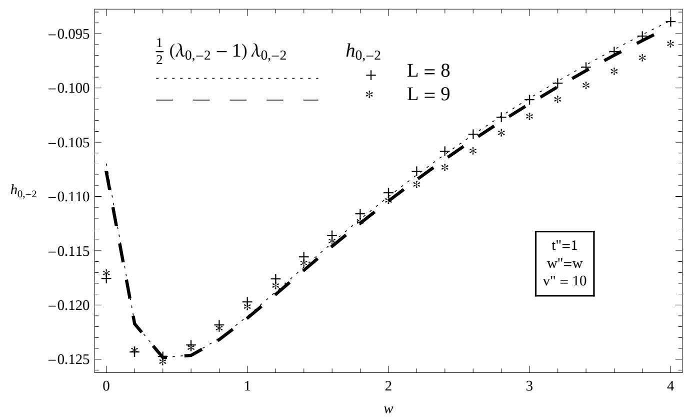

Figure 5.6: Conformal dimension of the ground state of one of the chains (5.7.2) compared to the prediction provided by a twisted spectrum.

Numerical evidence for the claim 2. on universality in the choice of the boundary terms (5.7.4–5.7.5) is presented in figure 5.7.

Combining our claims 1. and 2. we see that for generic boundary couplings u, t', w', t'', w'' and v the spectrum of the Hamiltonian (5.7.2), or at least of their $\mathrm{u}(1|1)$ subsectors, *depend only on the thickness m and n of the boundaries*. In conclusion, the number of relevant parameters in the four boundary terms (5.7.4–5.7.5) exactly matches the number of parameters needed to describe the set of boundary conditions preserving the global symmetry of the $\mathbb{CP}^{S-1|S}$ sigma model.

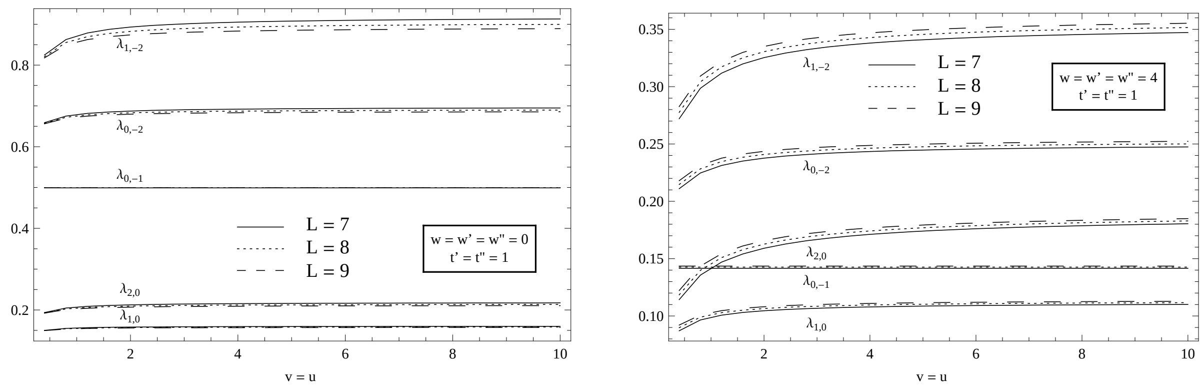

Figure 5.7: Universality of $\lambda_{M,N}$ for several chains at $w = 0$ and $w = 4$.

Finally, we present in figure 5.8 compelling evidence for the last claim 3. Using numerical data for $\lambda_{M,N}$ generated from chains with different values of M, N and w, we plotted on the same graph g_σ^2 expressed as a function of $\tan\pi\lambda_{M,N}$ from equation (5.7.1) with $\theta = \pi$. The appearance of a one to one correspondence between w and g_σ^2, which is independent of the chain we use, justifies the applicability of (5.7.1) to the spin chains, the correct value (5.7.7) of the θ-angle and the correct identification of the monopole charge (5.7.8–5.7.11).

This completes our analysis of the u $(1|1)$ subsector for the chains (5.7.2). So far, all our numerical results were in perfect agreement with the continuum $\mathbb{CP}^{0|1}$ sigma model. This supports our claim that the alternating u $(N|N)$ spin chain delivers a discrete formulation of the $\mathbb{CP}^{S-1|S}$ sigma model and it gives us sufficient confidence to address the watermelon exponents for twisted spin chains with $S > 1$.

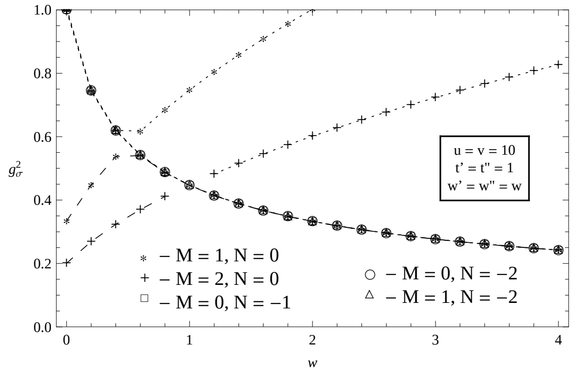

Figure 5.8: Numerical evaluation of the one-to-one correspondence between the $\mathbb{CP}^{S-1|S}$ sigma model coupling constant g_σ^2 and the bulk coupling constant w of the spin chains (5.7.2). For the chains $N = 0$ we represent both branches for g_σ^2 expressed as a function of $\tan \pi\lambda_{M,N}$. Calculations where made for $L = 800$.

5.7.3 Watermelon exponents for the twisted open chain

Our aim now is to generalize the discussion of section 5.6 to the case of general monopole boundary conditions with the precise goal of determining the conformal weight of tachyon vertex operators. For $l > 0$, the latter are associated with supersymmetric irreducible traceless tensors $t(k + l, k)$ of contravariant rank $k + l$ and covariant rank k, while for $l < 0$ these are the supersymmetric irreducible traceless tensors $t(k, k+|l|)$ of contravariant rank k and covariant rank $k + |l|$. In both case k is a non-negative integer, which for $\mathbb{CP}^{1|2}$ corresponds to the labels $\mu_{k,l}$ used before.

Let us restrict the algebraic Hamiltonians (5.7.2) to the representation of the walled Brauer algebra provided by the space of embeddings of the tensors $t(k+l, k)$ and $t(k, k+|l|)$ into the spin chains (5.7.8–5.7.11) with monopole numbers M and N. The lowest eigenvalue of the Hamiltonian in each of these sectors will be called the $(2k + |l|)$-legged watermelon exponent $h_{M,N}(k)$. As in the case of the chain in section 5.6, the watermelon exponents all vanish in the limit $w \to 0$, which we recall is the region that we associated with the large volume limit of the $\mathbb{CP}^{S-1|S}$ sigma model. The first two of these watermelon exponents are already contained in the u $(1|1)$ subsector of the model, both in the continuum and on the lattice. They are not degenerate, with $h_{M,N}(0)$ describing the twisted vacuum and $h_{M,N}(1)$ the first excitation. Their difference is

$$\lambda_{M,N} \;=\; h_{M,N}(1) - h_{M,N}(0)\ . \tag{5.7.13}$$

Another important observation coming from lattice calculations is the Casimir evolution for the *excitations* of the spin chains (5.7.2). Namely, numerical calculations provide compelling evidence that the following formula

$$\delta h_{M,N}(k) \;=\; h_{M,N}(k) - h_{M,N}(0) \;=\; g_{M,N}\frac{k(k+|l|-1)}{2} \tag{5.7.14}$$

holds for sufficiently large w and with an universal function $g_{M,N}$ that depends only on M, N and w. In order to compare with our continuum theory, we note that (5.3.25) gives

$$\delta_l \mathsf{C}\left[\frac{l}{2}+k-1, 0, \frac{l}{2}+2, \frac{l}{2}\right] - \delta_l \mathsf{C}\left[\frac{l}{2}, 0, \frac{l}{2}, \frac{l}{2}\right] \;=\; 2k(k+l-1) \tag{5.7.15}$$

for $l = M - N > 0$, while a similar result can be obtained when l is smaller than zero. We recall that the watermelon exponents $h_{M,N}(k)$ are associated with the label $\mu_{k,l}$ defined in appendix D. In conclusion, we see that our lattice observation (5.7.14) for the watermelon exponents agrees with their proposed continuum description in the $\mathbb{CP}^{1|2}$ model.

By analogy with section 5.6, the function $g_{M,N}$ should be interpreted as the effective tension of the string stretching between the bundle with monopole charge M and the bundle with monopole charge N. In the continuum theory, we related the function $g_{M,N}$ to the twist parameter $\lambda_{M,N}$ through the equation

$$\lambda_{M,N} \;=\; \frac{|M-N|}{2} g_{M,N}\ . \tag{5.7.16}$$

It is interesting to test the validity of this relation numerically. In figure 5.9 we represent the ratio $|l|g_{M,N}/2\lambda_{M,N}$ as a function of w. As before, we measure the function $g_{M,N}$ through the equation (5.7.14) for different excitations $h_{M,N}(k)$. If the Casimir evolution (5.7.14) holds true, then we should see a constant value of $|l|g_{M,N}/2\lambda_{M,N} = 1$ for the ratio, *independently* of the watermelon exponent that is used to measure $g_{M,N}$. While things work out remarkably well in the regime of large w, obvious discrepancies appear when w is close to $w \sim 0$, the possible interpretation of which are discussed in the next subsection.

5.7.4 Comments on the region of small w

There are actually several possibilities to interpret the failure of equation (5.7.16) near $w = 0$, two of which are the subject of our discussion below, while leaving the ultimate test of the correct explanation for future work. In confronting our numerical results with the proposed continuum description, we have tacitly *assumed that the spin chains (5.7.2)*

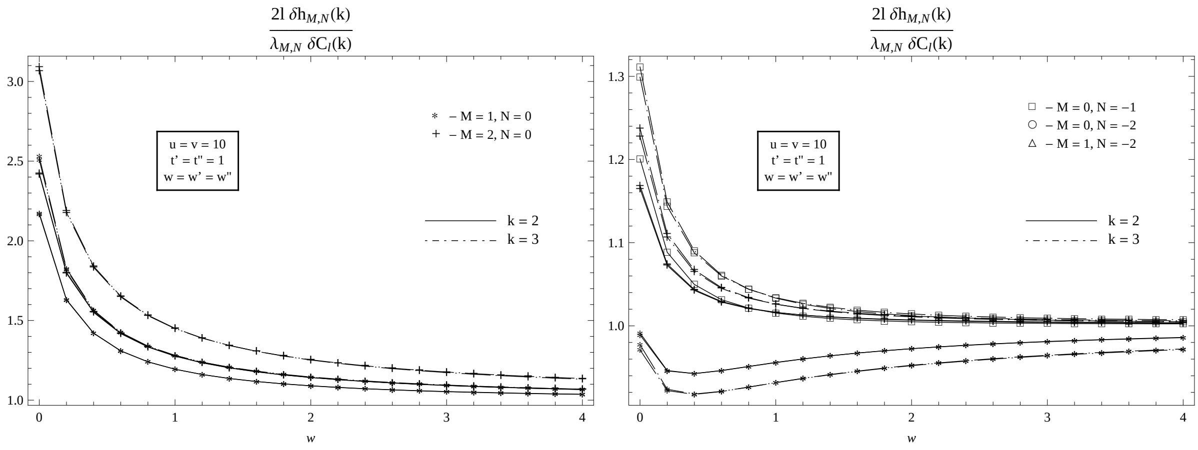

Figure 5.9: Test of eq. (5.7.16) following from the assumption of Casimir evolution (5.7.14). Calculations where made for spin chains (5.7.2) of bulk length $L = 7$ and $L = 8$ and the corresponding curves almost superpose.

at $w = 0$ *still describe a point in the moduli space of the* $\mathbb{CP}^{S-1|S}$ *sigma model.* This is a very strong assumption given that the symmetry of the bulk Hamiltonian (5.7.3) becomes much larger [10] than u $(S|S)$ at $w = 0$, essentially because the lines in the Brauer algebra representation are then prevented from crossing.

In assessing the meaning of the observed discrepancies, it is useful to recall that a similar issue has also appeared for the osp$(2M+2|2M)$ spin chain considered in [18], where the osp spin chain was proposed as a discretization of the $S^{2M+1|2M}$ supersphere sigma model. Generic features of the lattice spectrum were found to be in excellent agreement with the conjectured spectrum of the sigma model, as long as w was large. However, problems similar to the ones we described in the previous subsection were encountered at the point $w = 0$. Similarly to what we have found here for the u $(1|1)$ sector of the u $(S|S)$ spin chain, in the supersphere case, the discrepancy was only visible when looking at fields outside the O(2) subsector of osp$(2M+2|2M)$ theory. Looking at all these similarities, it seems likely that the discrepancies between lattice and continuum analysis in the u $(S|S)$ and osp $(2M+2|2M)$ model should have the same explanation.

In the case of the supersphere sigma model, however, the assumption of Casimir evolution for the whole spectrum stands on rather firm grounds. To begin with, the perturbative expansion for boundary conformal weights in the supersphere sigma model may be summed to all orders, as was done in [48], showing that terms that spoil the Casimir evolution vanish. Moreover, world-sheet instanton corrections cannot alter these findings, simply because they do not exist in this case. Finally, a conjectured duality between the supersphere sigma model and the osp $(2M+2|2M)$ Gross-Neveu model was shown in the past chapter to be perfectly consistent with the Casimir evolution of boundary conformal weights. All this makes it seem very likely that the conformal weights of the two investigated sigma models all evolve with the Casimir, as encoded in our formula (5.3.24).

Having argued that the discrepancies between our lattice and continuum results are unlikely to signal a breakdown of the Casimir evolution in the sigma model, we want to entertain a second logical possibility, namely that the continuum limit of the spin chains (5.7.2) is described by a $\mathbb{CP}^{S-1|S}$ sigma model only for $w > 0$, while at $w = 0$ it is not. If this were true, then the discrepancies observed in figure 5.9 would simply result from interchanging the thermodynamic limit $L \to \infty$ with the limit $w \to 0$. A similar non-commutativity of limiting procedures can also be numerically observed in the large volume limit $w \to \infty$ where the symmetry of the Hamiltonian is once more enhanced much beyond the generic $\mathrm{u}(S|S)$ transformations.

Support for our second explanation of the discrepancies comes from a closer inspection of the watermelon exponents. At $w = 0$, the lattice model is exactly solvable and we believe that the differences between water-melon exponents are given by the expression

$$\delta h_{M,N}(k) \;=\; \frac{k(k + 2\lambda_{M,N} - 1)}{2} \tag{5.7.17}$$

where $\lambda_{M,N}$ is again measured as the difference $\lambda_{M,N} = h_{M,N}(1) - h_{M,N}(0)$. The formula (5.7.17) can most certainly be derived analytically, but we simply justify it for now by observing that it fits the general pattern of boundary exponents of non-intersecting loop models as discussed in [88]. Indeed, it can be rewritten as

$$h_{M,N}(k) \;=\; h_{2\lambda_{M,N}-1,2\lambda_{M,N}-1+2k}$$

where on the right hand side we use the Kac formula at central charge $c = -2$:

$$h_{r,s} = \frac{(2r - s)^2 - 1}{8}$$

M	N	$\frac{2\delta h_{M,N}(k)}{k(k+2\lambda_{M,N}-1)}$		
		$k = 2$	$k = 3$	$k = 4$
1	0	1.050128	1.037253	1.010766
2	0	1.098296	1.094754	1.070405
0	-1	0.98817	0.969892	0.945022
0	-2	1.016252	1.006706	0.984296
1	-2	1.034566	1.033275	1.014131

Table 5.1: Numerical check of the proposed formula (5.7.17) for the watermelon exponents of the spin chains (5.7.2) at $w = 0$. Calculations where made for bulk length $L = 7$.

A verification of this formula is presented in table 5.1. The numbers in the grid should all go to unity in the scaling limit and we see that the agreement with (5.7.17) is quite impressive. On the other hand, the behavior of watermelon exponents in the chain with $w \neq 0$ is significantly different, supporting our claim that the continuum theory of the $w = 0$ lattice model does not belong to the continuous family of conformal field theories that is parametrized by $w > 0$.

5.7.5 Further comments

We want to end our discussion of the non-linear sigma model on $\mathbb{CP}^{1|2}$ by pointing out some further consequences of the Casimir evolution of the boundary conformal weights that could be checked in the limit of large volume. We first note that our perturbative computations for tachyonic vertex operators in (5.3.18) were only performed in the theory with equal boundary monopole charges $M = N$.

While the conjectured exact form (5.3.23) of watermelon exponents in the theory with arbitrary boundary monopole charges M, N passed several analytical and numerical tests, it could not be backed up by perturbative calculations *beyond the leading order* because we did not succeed to generalize the background field expansion to twisted boundary conditions. Nonetheless, we suspect that such a generalization exists and the watermelon exponents will most likely be computed again in terms of the *eigenvalues of some Laplacian* on the bundle with monopole charge $l = M - N$. The problem is that for $l \neq 0$ this Laplacian is not unique, as can be seen from the existence of an one-parameter family of $\mathrm{u}(S|S)$ Casimirs Cas_α in appendix B.4 and D. If we are to choose however

$$\alpha = 1 - \frac{g_{M,N}(g_\sigma, \theta)}{2}$$

then the conjectured form (5.3.23) for the watermelon exponents coincides exactly with a Casimir evolution type formula

$$h^{g_\sigma,\theta}_{M,N}(k) = \frac{g_{M,N}(g_\sigma, \theta)}{4}\,\mathsf{Cas}_\alpha(\mu_{k,l}),$$

which is most natural in the context of the background field method. On the other hand, these conjectured watermelon exponents possess the following simple expansion in the coupling g_σ,

$$h_{M,N}(k) \;=\; \frac{g_\sigma^2}{\pi}\,\mathsf{Cas}_{\alpha=1}(\mu_{k,l}) + \frac{2g_\sigma^4}{\pi^2} l^2 + O(g_\sigma^6)\ .$$

Here, $\mathsf{Cas}_{\alpha=1}(\mu_{k,l})$ are the eigenvalues of the Bochner-Laplacian of the complex line bundles over $\mathbb{CP}^{S-1|S}$ and, as we said, the first term can be reproduced by the semi-classical, i.e. large volume approximation. In the case $l \neq 0$ the first correction to the semi-classical result comes at order g_σ^4, which is an accessible non-trivial check to be performed once the perturbation theory for twisted boundary conditions has been ironed out.

Moving away from $\theta = \pi$ in the sigma model corresponds to *staggering the couplings*[10] of the spin chain. In the case $w = 0$, it is well known that staggering in fact does not

[10] See [16] for an introduction to this terminology

affect the spectrum at all, but for $w \neq 0$ we expect that staggering will renormalize the coupling constant to which the lattice model flows, turning $g_\sigma^2(w)$ into a function of both w and the staggering parameter, on top of affecting the value of θ in the formulae. Our continuum theory makes rather non-trivial predictions about this functional dependence that seem well worth further investigation.

Chapter 6

Conclusions

6.1 Looking back

The main goal of this thesis was to develop new tools for the investigation of non-linear sigma models and to apply them to specific examples so that an intuition of their behavior may be developed. Our major results can be summarized as follows.

We first presented the method of cohomological reduction that relates non-linear sigma models on different coset superspaces G/G' to each other, allowing us to gain insight into quite complicated theories and to compute some of their correlation functions. This is achieved thanks to an extensive use of the target space supersymmetry by choosing a BRST-like operator Q whose cohomology defines the space of physical states for the reduced theory. This correspondence preserves all correlators and can also be applied to other, non-geometric, theories such as for instance Gross-Neveu and Landau-Ginzburg models. An important application of this method was the classification of conformal symmetric superspaces in subsection 3.3.2 and the examples of conformal homogeneous superspaces in 3.3.3.

The sigma models on the superspheres $S^{2M+1|2M}$ had previously been studied using a loop model formulation in [18, 19], leading to a conjecture for a dual description of the model generalizing the well known free boson – Thirring model duality. We were able to delve further into this hypothesis and managed to obtain two central results. First we were able to compute the exact boundary spectrum of a volume filling brane in the non-interacting limit for all the superspheres $S^{2M+1|2M}$. More importantly, thanks to [48], we could in the case of $S^{3|2}$ compute this spectrum for all values of the curvature radius. The second result concerns the $\mathrm{osp}\,(2M+2|2M)$ Gross-Neveu model, for which we could construct the spectrum of a Dirichet D-brane and, thanks to [34], to deform it exactly as the coupling constant changes. The resulting identity between the sigma and Gross-Neveu models' respective spectra provided extremely strong additional support for the duality.

Encouraged by our results for the supersphere sigma models, we directed our attention to a similar problem and analyzed the boundary partition functions for all $\mathrm{u}(2|2)$ invariant boundary conditions of the sigma model on the projective superspace $\mathbb{CP}^{1|2}$. The dependence of this partition function on the bulk couplings and on the boundary monopole charges was displayed in equation (5.3.24). The main ingredients of the final result were found out to be the branching functions (5.2.21) of the model at infinite volume and two universal functions exhibited respectively in (5.3.27) and (5.3.28). The partition function encodes the dependence of boundary conformal weights on the various couplings and justifies and generalizes the results in [11]. To further our understanding of the $\mathbb{CP}^{1|2}$ sigma model, we introduced a lattice model on an alternating spin chain. Numerical studies of the latter revealed an excellent agreement with the predictions from the continuum theory, at least for sufficiently large values of the lattice coupling w. Furthermore, by appending layers of finite width at the extremities of the open spin chain in (5.7.2), we were able to implement all the boundary conditions of the continuum theory.

6.2 Unresolved issues

Despite the many successes we have had in studying the supersphere and complex projective superspaces sigma models, many open problems remain. The first and perhaps most important question concerns the bulk spectra of these theories. Let us concentrate on the Gross-Neveu dual of the supersphere for which we are able to say the most. While [34] focused on a bulk deformation preserving global left and right transformations simultaneously, the current-current perturbation (4.1.9) considered here for the free Gross-Neveu model is of a very different type for, since the deforming operator does not involve any tachyonic vertex operators, there is no mixing problem, neither in the boundary theories, nor in the bulk. On the other hand, the perturbation breaks the global bulk symmetry at the free point down to a single diagonal action of the symmetry algebra. Therefore, while it should be possible to deform bulk spectra, it becomes more difficult to identify the relevant $\mathrm{osp}(2M+2|2M)$ action as we deform from $R=1$ to $R=\infty$. Thus, the deformation of conformal dimensions away from the free points of the theory cannot follow a law as simple as in the boundary case. Several proposals have been made during the making of this thesis, the most promising of which used the left-right global symmetry of the free point, but no satisfactory answer has been found yet.

The second unsolved problem involves the incorporation of world sheet supersymmetry in our models. While we have been successfull in extending the supersphere sigma model to $\mathcal{N}=1$ supersymmetry in the infinite volume limit, we have no valid expressions for the interacting theory. We think that the dual description of the theory will be a Gross-Neveu model made worldsheet supersymmetric by the addition of $h=\frac{1}{2}$ fields transforming into the adjoint representation, but have found no way of providing conclusive evidence for this conjecture so far.

A further open question concerns the issue of conformal symmetry. For sigma models on *symmetric* superspaces $G/G^{\mathbb{Z}_2}$, we found a free subsector $H/H^{\mathbb{Z}_2}$ if and only if the original model was conformal. Furthermore, we also provided several examples of more general *homogeneous* coset superspaces G/G' that possess a free subsector. Though we are not prepared to argue that these coset sigma models are in fact conformal, we believe this to be the case, at least for an appropriate choice of the background fields G and B. In any case, this certainly deserves further investigation.

The fourth open problem lies within our study of the lattice model formulation of the

$\mathbb{CP}^{N-1|N}$ sigma model. We came to the striking conclusion that, like in the supersphere case, the chain without loop crossing in the Brauer formulation, or $w = 0$, seems to be in an universality class *not containing* the generic $w \neq 0$ case. Apparently, one cannot freely exchange the two limits of infinite spin chain length and zero coupling constant so that the perturbation induced by turning $w \neq 0$ on the lattice is *relevant*. The conformal field theory at the point $w = 0$ admits a very large, not yet fully explored symmetry, whose *bulk* spectrum should, in order to have consistency of the whole picture, contain a marginally relevant invariant operator that must be absent in the minimal $\mathrm{U}(1|1)$ subsector. As of the present moment, the existence of this elusive operator remains to be established.

6.3 Looking forward

As we have stated in the introduction, one of the main motivations for the study of superspace sigma models comes from the AdS/CFT correspondence. It is likely that the ideas of this work can be adjusted so as to apply to models that are relevant for the study of strings in Anti de Sitter geometries. In the case of euclidean AdS_3, for example, correlation functions of chiral primaries have been computed in the NSR formalism using the explicit solution of the WZNW model on the bosonic space $H_3^+ \times \mathrm{SU}(2)$. A closer look at the results of [89,90] shows that most of the intricate features of the full WZNW model cancel out from the correlation function of chiral primaries, leading to an answer that looks very much like a three point function in some free field theory. We hope to re-derive and extend these findings through a cohomological reduction, after re-phrasing the computation in the target space supersymmetric hybrid formalism [14]. A detailed analysis in currently being pursued.

Concerning the study of strings on AdS_5, concrete applications seem a little more speculative. Within the pure spinor approach, strings in $AdS_5 \times S^5$ may be described by coupling the superspace coset model $\mathrm{PSU}(2,2|4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ that we described in subsection 3.3.3, to the pure spinor ghost sector [5]. Since its denominator group is purely bosonic, we cannot apply our ideas, neither to the matter sector alone nor to the full theory. Actually, in this case any element $Q \in \mathrm{psl}(N|N)$ from the numerator Lie superalgebra satisfying $Q^2 = 0$ may be shown to possess trivial cohomology. A non-trivial subsector can only emerge after restricting to *physical states*, that is to the cohomology of the world-sheet BRST operator Q_{BRST} of the pure spinor theory. Generalizing our work to this setup would require a thorough analysis of the bi-complex that is generated by the BRST operator Q_{BRST} along with the space-time supersymmetry generator Q. A more direct application of the cohomological reduction to strings in AdS_5 might be possible within the light-cone gauge fixed[1] Green-Schwarz formulation. In this approach, the unbroken space-time symmetries are described by two copies of a centrally extended $\mathrm{psu}(2|2)$ algebra which share the same central elements and it might

[1]See for example [8]

be feasible to use some of the corresponding fermionic generators for a cohomological reduction.

A further possible direction for the investigation of string theory on Anti de Sitter spaces involves making use of the experience gained in chapters 4 and 5. The superspheres $S^{2M+1|2M}$ and the complex projective superspaces $\mathbb{CP}^{N-1|N}$ have been advocated previously in [32, 33] as good toy models for the world-sheet description of string theory on $AdS_5 \times S^5$. To begin with, it is certainly possible to determine the exact spectrum of the free sigma model on the string background $\mathrm{PSU}(2,2|4)/\mathrm{SO}(1,4) \times \mathrm{SO}(5)$ at the free point, much as this was done here. The deformation of the spectrum away from the infinite radius point cannot be as simple as in the supersphere case, since we know for sure that there are some operators whose anomalous dimensions do not possess such quasi-abelian dependence. Assuming nevertheless that such an exact deformation of the conformal dimensions is possible, we could start looking for special values of the radius at which the spectrum contains only half-integer or integer values for h. We know for sure that such a point exists, namely the radius R_0 for which the string model becomes dual to the free $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. One might hope that such a point is described by a free world-sheet theory, just as it is the case for the superspheres. In this sense, the dual of the free Yang-Mills theory would be the analogue of the free Gross-Neveu model. If one could find such strong-weak coupling duality within the world-sheet description of strings in AdS, it would reduce the AdS/CFT correspondence to a remaining weak-weak coupling duality, just as we argued in the introduction. Such candidates for world-sheet descriptions of weakly coupled gauge theory have appeared in the literature, as for instance in [91–93].

Assuming that we can iron out the issues that we have had with the worldsheet supersymmetric extension of the supersphere sigma models, we would like to investigate the $\mathcal{N} = 2$ version of the $\mathbb{CP}^{S-1|S}$ sigma models. If possible, this would shed further light on the proposal of [64] to use superstring theory on $\mathbb{CP}^{3|4}$ as a description of perturbative super Yang-Mills theory.

The last possible further reseach direction we wish to mention concerns the closely related non-compact sigma model on the coset space $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$ that was considered in a modified form in [12] because of its possible relevance for the theory of quantum Hall plateau transitions. The spin chain discussed in [12] involves infinite dimensional representations and a pure Heisenberg interaction.[2] It would be interesting, among other things, to study the role of next to nearest neighbor interactions in that case, and to analyze whether they allow fine tuning of the running coupling constant as in our model. It could also be of interest to interpret our bundle boundary conditions in terms of edge states in the Hall effect [68].

[2]This chain was proposed earlier in unpublished work by N. Read.

Acknowledgments

First and foremost, I would like to express my gratitude to my supervisor Volker Schomerus for his unfaltering council and support throughout the past three years. Furthermore, I would like to thank Constantin Candu, Thomas Creutzig, Thomas Quella and Hubert Saleur for their participation in our common projects and for the many insightfull discussions that we have had. I am especially in debt to Constantin Candu, Thomas Creutzig and Volker Schomerus for the many advices they have generously showered upon me during the drafting of this thesis.

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Най накрая, бих желал да се обърна към мойте родители, Маргарита и Валентин Митеви, с благодарност за всичко.

Appendix A

Notation

$\mathfrak{g}, G$	Lie superalgebra, Lie supergroup
$\mathfrak{g}_{\bar{0}}/\mathfrak{g}_{\bar{1}}$	Even/odd part of $\mathfrak{g}$
g_0	Cartan subalgebra of $\mathfrak{g}$
$\mathrm{rank}\,\mathfrak{g}$	Rank of $\mathfrak{g}$, equal to $\dim \mathsf{g}_0$
g_α	Root space for the root α
Δ	Set of roots of $\mathfrak{g}$
$\Delta_{\bar{0}}/\Delta_{\bar{1}}$	Even/ odd roots
Δ^+/Δ^-	Positive/ negative roots
Λ	Weight of $\mathfrak{g}$
Γ^+	Set of allowed integral highest weights
$\circ$, $\wedge$	Symmetric, antisymmetric tensor product
$\mathsf{soc}\,V, \mathsf{top}\,V, \mathsf{rad}\,V$	Socle, top and radical of V
$\mathsf{Ker}_Q, \mathsf{Im}_Q, \mathsf{H}_Q$	Kernel, image and cohomology of Q
$\mathsf{Sym}^n, \mathsf{Alt}^n$	$n-$ fold sym./ antisym. tensor product
$\eta^{\mu\nu}$	Constant space-time metric
$\epsilon^{\mu\nu}$	2d antisymmetric tensor with $\epsilon^{12}=1$
$f^{AB}{}_C$	Structure constants of $\mathfrak{g}$
K^{AB}	Killing form of $\mathfrak{g}$
Ω	Lie superalgebra automorphism
$\mathfrak{G}$	Grassmann algebra
$\mathfrak{G}(\mathfrak{g})$	Grassmann envelope of $\mathfrak{g}$
$J_\mu = g^{-1}\partial_\mu g$	Maurer-Cartan forms
$\mathcal{S}, \mathcal{L}$	Action, Lagrangian
Σ	Worldsheet
$\mathsf{g}/\,\mathsf{b}$	Metric/ form on G/G'
$\mathcal{O}$	Local observable
$\mathcal{F}_{G/G'}$	Space of local observables on G/G'
$\mathfrak{F}(G/G')$	Space of smooth functions on G/G'
λ	Symplectic fermions twist parameter
ω	Automorphism of boundary CFTs
$\mathcal{W}$	Space of chiral fields
$T/\,\bar{T}$	Hol./antihol. parts of the stress-energy tensor
$L_n, \bar{L}_n$	Modes of $T, \bar{T}$

Appendix B

Representation theory

We note that sections B.1, B.2 and B.3 were first published in [20] that was written together with T. Quella and V. Schomerus, whereas B.4, B.5 and B.6 originally appeared in [21], which was a collaboration with C. Candu, T. Quella, H. Saleur and V. Schomerus.

B.1 The special case of OSP(4|2)

This section contains a number of basic notations and results concerning the Lie superalgebra osp (4|2) that we use in the main text, mostly in subsections 4.4.3 and 4.3.3. The complex superalgebra $\mathfrak{g} := \mathrm{osp}\,(4|2)$ is a simple superalgebra of type II and may be realized in its fundamental representation as the following set of six by six matrices

$$\mathrm{osp}\,(4|2) := \left\{ \left(\begin{array}{c|c} A & B \\ \hline \mathsf{J}_2 B^t & D \end{array} \right) : \begin{array}{c} A^t = -A \\ D^t \mathsf{J}_2 = -\mathsf{J}_2 D \end{array} \right\} \text{ with } \mathsf{J}_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \ , \qquad \text{(B.1.1)}$$

where we use the standard definition of graded commutators. We have the usual separation of the superalgebra into a bosonic $\mathfrak{g}_{\bar{0}} = \mathrm{sp}(2) \oplus \mathrm{so}(4) \cong \mathrm{sl}(2) \oplus \mathrm{sl}(2) \oplus \mathrm{sl}(2)$ and a fermionic $\mathfrak{g}_{\bar{1}}$ subspace. In addition, the superalgebra has a $\mathbb{Z}$-grading that is compatible with its $\mathbb{Z}_2$ structure, i.e. $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the relation $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$ holds, with $\mathfrak{g}_0 \cong \mathrm{so}(4) \oplus \mathrm{gl}(1)$, $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Here $\mathfrak{g}_{\pm i}$ contain those generators that have a positive, respectively negative weight under the gl (1) part of $\mathfrak{g}_0$.

An integral dominant highest weight $\Lambda = (j_1, j_2, j_3)$ of $\mathfrak{g}_{\bar{0}}$ is also one for the full superalgebra $\mathfrak{g}$ if it obeys the following consistency conditions:

$$\text{If } j_1 = 0 \text{ then } j_2 = j_3 = 0 \quad , \qquad \text{If } j_1 = \frac{1}{2} \text{ then } j_2 = j_3 \qquad \text{(B.1.2)}$$

where the first spin is related to the symplectic subalgebra and the two others to the orthogonal one. In our notation, all the spins take integer or half-integer values. The finite dimensional irreducible representations $[\Lambda]$ of $\mathfrak{g}$ are then constructed as follows. Taking an irreducible highest weight representation (Λ) of $\mathfrak{g}_0 \cong \mathrm{so}(4) \oplus \mathrm{gl}(1)$ with highest weight $\Lambda = (j_1, j_2, j_3)$ associated to the highest weight vector v_Λ, we set

$$\mathsf{M}_\Lambda \ := \ \mathfrak{U}(\mathfrak{g})(E_1^-)^{2j_1+1} v_\Lambda \ \ , \qquad\qquad \mathsf{K}_\Lambda \ := \ \mathsf{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\Lambda)/\mathsf{M}_\Lambda \ ,$$

where E_1^- is the lowering operator of the symplectic subalgebra and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. In the above equation, we have considered the $\mathfrak{g}_0$-module (Λ) as a $\mathfrak{p}$-module by letting

$\mathfrak{g}_i, i = 1, 2$ act trivially on it. The finite dimensional representation K_Λ is called the Kac module of Λ and is generically irreducible. The set of Kac modules is divided into typical and atypical ones. If the Kac module K_Λ is typical, then it is guaranteed to be irreducible. In this case we define the simple module $[\Lambda]$ to be K_Λ. If, however, one or more of the following atypicality conditions

$$\begin{aligned} 2j_1 &= -j_2 - j_3 \ , \\ 2j_1 &= j_2 + j_3 + 2 \ , \\ 2j_1 &= \pm(j_2 - j_3) + 1 \end{aligned} \tag{B.1.3}$$

hold, then K_Λ is atypical and will generically contain a maximal invariant subspace I_Λ without being fully reducible, i.e. it will contain indecomposable submodules. In those cases, we set $[\Lambda] = \mathsf{K}_\Lambda/\mathsf{I}_\Lambda = \mathsf{top}\,\mathsf{K}_\Lambda$. We note, that I_Λ may be trivial even though K_Λ is atypical.

The eigenvalue of the quadratic Casimir in the simple module $[\Lambda]$ is given by the formula

$$\mathsf{Cas}(\Lambda) \ = \ -4j_1(j_1 - 1) + 2j_2(j_2 + 1) + 2j_3(j_3 + 1) \ . \tag{B.1.4}$$

In particular, $\mathsf{Cas}(\Lambda)$ is always a square, i.e. $\mathsf{Cas}(\Lambda) = k^2, k \in \mathbb{N}$, on atypical representations $[\Lambda]$, as one can easily check by using (B.1.3). The atypical weights $\Lambda = (j_1, j_2, j_3)$ can be divided into blocks Γ_k, such that weights in Γ_k possess the same eigenvalue $\mathsf{Cas}(\Lambda) = k^2$ of the quadratic Casimir element. The corresponding atypical labels can be listed explicitly [52],

$$\begin{aligned} \Gamma_0 &= \left\{ \Lambda_{0,0} = (0,0,0)\, ,\, \Lambda_{0,l} = \frac{1}{2}(l+1, l-1, l-1)\, ,\, l \geq 1 \right\} \\ \Gamma_k &= \{\Lambda_{k,l}\, ,\, l \in \mathbb{Z}\} \end{aligned} \tag{B.1.5}$$

where

$$\Lambda_{k,l} = \begin{cases} \frac{1}{2}(-l+2, -l-k, -l+k) & \text{if } l \leq -k \\ \frac{1}{2}(-l+1, l+k-1, -l+k-1) & \text{if } -k+1 \leq l \leq 0 \\ \frac{1}{2}(l+1, l+k-1, -l+k-1) & \text{if } 0 \leq l \leq k-1 \\ \frac{1}{2}(l+2, l+k, l-k) & \text{if } k \leq l \end{cases} \ . \tag{B.1.6}$$

One sees easily, that the weights $\Lambda_{k,-l}$ for $k \geq 1$ may be obtained from $\Lambda_{k,l}$ by simply exchanging the second and the third Dynkin label. Furthermore, it is possible to distinguish the weights $\Lambda_{k,l}$ according to the atypicality condition (B.1.3) they obey. The only weight to fulfill the first condition is $\Lambda_{0,0}$. The weights belonging to the second condition are $\Lambda_{0,l}$ for $l \geq 1$ and $\Lambda_{k,\pm l}$ for $l \geq k$. Finally, those the satisfy the last atypicality relation are the $\Lambda_{k,\pm l}$ for $l < k$.

The only atypical Kac modules $\mathsf{K}_{\Lambda_{k,l}}$ which are irreducible correspond to the weights $\Lambda_{k,0}$ for $k \geq 0$ and to $\Lambda_{0,1}$. The indecomposable structure of the remaining ones can be deciphered from the following diagram,

$$\begin{aligned} \mathsf{K}_{\Lambda_{0,2}}:&\ [\Lambda_{0,2}] \longrightarrow [\Lambda_{0,0}] \oplus [\Lambda_{0,1}] \\ \mathsf{K}_{\Lambda_{0,l}}:&\ [\Lambda_{0,l}] \longrightarrow [\Lambda_{0,l-1}] \text{ for } l \geq 3 \\ \mathsf{K}_{\Lambda_{k,l}}:&\ [\Lambda_{k,l}] \longrightarrow [\Lambda_{k,l-1}] \text{ for } l \geq 1 \\ \mathsf{K}_{\Lambda_{k,l}}:&\ [\Lambda_{k,l}] \longrightarrow [\Lambda_{k,l+1}] \text{ for } l \leq -1 \quad . \end{aligned} \tag{B.1.7}$$

These diagrams are to be understood to mean that the right hand side denotes the radical of K, whereas the left hand side designate the top. For example $\mathsf{rad}\,\mathsf{K}_{\Lambda_{0,2}} \cong [\Lambda_{0,0}] \oplus [\Lambda_{0,1}]$ and $\mathsf{top}\,\mathsf{K}_{\Lambda_{0,2}} \cong [\Lambda_{0,2}]$. The dimension of the typical Kac modules is

$$\dim[\mathsf{K}_{(j_1,j_2,j_3)}] = 16(2j_1-1)(2j_2+1)(2j_3+1) \tag{B.1.8}$$

whereas the dimension of the atypical ones may be inferred from their structure, together with the following formulas for the dimension of the irreducible representations,

$$\begin{aligned} \dim[\Lambda_{0,0}] &= 1\ , \qquad \dim[\Lambda_{0,1}] = 17\ , \qquad \dim[\Lambda_{k,0}] = 4k^2+2 \\ \dim[\Lambda_{0,l}] &= (2l+1)\left[(2l+1)^2-3\right] \text{ for } l \geq 2 \\ \dim[\Lambda_{k,l}] &= (2l+1)\left[4(k^2-1)-(2l+1)^2+7\right] \text{ for } l \leq k-1 \\ \dim[\Lambda_{k,l}] &= (2l+3)\left[(2l+3)^2-4(k^2-1)-7\right] \text{ for } l \geq k \end{aligned} \tag{B.1.9}$$

where, of course, $\dim[\Lambda_{k,-l}] = \dim[\Lambda_{k,l}]$. The decomposition of K_Λ for $j_1 \geq 1$, whether typical or not, into irreducible modules of the bosonic subalgebra has been computed in [94]. It takes the form

$$\begin{aligned} [\mathsf{K}_\Lambda]_{g_{\bar{0}}} \cong&\ (j_1,j_2,j_3) \bigoplus_{\alpha,\beta=\pm\frac{1}{2}} (j_1-\frac{1}{2}, j_2+\alpha, j_3+\beta) \\ & \bigoplus_{\alpha=\pm 1} \left[(j_1-1,j_2+\alpha,j_3) \oplus (j_1-1,j_2,j_3+\alpha)\right] \oplus 2(j_1-1,j_2,j_3) \\ & \oplus \bigoplus_{\alpha,\beta=\pm\frac{1}{2}} (j_1-\frac{3}{2}, j_2+\alpha, j_3+\beta) \oplus (j_1-2,j_2,j_3) \quad . \end{aligned} \tag{B.1.10}$$

There are a few special cases for which the decomposition is not generic. If $j_1 \leq 2, j_2 \leq 1$ or $j_3 \leq 1$ then the above decomposition formula must be truncated at the point where one ore more of the labels become negative. Moreover, there are two cases for which the multiplicity of the (j_1-1,j_2,j_3) submodule has to be changed. If $j_1 = 1, j_2 > 0, j_3 > 0$ or $j_1 > 1, j_2 = 0, j_3 > 0$ or $j_1 > 1, j_2 > 0, j_3 = 0$, then this block will appear only once and if both j_2 and j_3 are null, then it will not be present at all.

When $j_1 = \frac{1}{2}$, the Kac modules K_Λ with weight Λ obeying the consistency conditions (B.1.2) are equal to the irreducible modules $\left[\frac{1}{2},\frac{k}{2},\frac{k}{2}\right]$ and they possess the following structure

$$\left[\frac{1}{2},\frac{k}{2},\frac{k}{2}\right]_{|g_{\bar{0}}} \cong \left(\frac{1}{2},\frac{k}{2},\frac{k}{2}\right) \oplus \left(0,\frac{k+1}{2},\frac{k+1}{2}\right) \oplus \left(0,\frac{k-1}{2},\frac{k-1}{2}\right) \quad . \tag{B.1.11}$$

Finally, the Kac module $\mathsf{K}_{[0,0,0]}$ is trivial.

The last part of this section concerns the projective covers of the irreducible representations. For typical weights Λ the projective cover of the simple module is the simple module itself. In the atypical case, we get from [52] that:

$$\begin{aligned}
\mathsf{P}_{\Lambda_{0,0}} &: [\Lambda_{0,0}] \longrightarrow [\Lambda_{0,2}] \longrightarrow [\Lambda_{0,0}] \\
\mathsf{P}_{\Lambda_{0,1}} &: [\Lambda_{0,1}] \longrightarrow [\Lambda_{0,2}] \longrightarrow [\Lambda_{0,1}] \\
\mathsf{P}_{\Lambda_{0,2}} &: [\Lambda_{0,2}] \longrightarrow [\Lambda_{0,3}] \oplus [\Lambda_{0,1}] \oplus [\Lambda_{0,0}] \longrightarrow [\Lambda_{0,2}] \\
\mathsf{P}_{\Lambda_{0,l}} &: [\Lambda_{0,l}] \longrightarrow [\Lambda_{0,l+1}] \oplus [\Lambda_{0,l-1}] \longrightarrow [\Lambda_{0,l}] \quad \text{for } l \geq 3 \\
\mathsf{P}_{\Lambda_{k,l}} &: [\Lambda_{k,l}] \longrightarrow [\Lambda_{k,l+1}] \oplus [\Lambda_{k,l-1}] \longrightarrow [\Lambda_{k,l}] \ .
\end{aligned} \tag{B.1.12}$$

These diagrams are to be read in the same way as (B.1.7). For instance $\mathsf{top}\,\mathsf{P}_{\Lambda_{0,0}} \cong \mathsf{soc}\,\mathsf{P}_{\Lambda_{0,0}} \cong [\Lambda_{0,0}]$ and $\mathsf{rad}\,\mathsf{P}_{\Lambda_{0,0}}/\,\mathsf{soc}\,\mathsf{P}_{\Lambda_{0,0}} \cong [\Lambda_{0,2}]$.

B.2 Recombination of the bosonic characters

Let Z be a partition function with osp $(4|2)$ symmetry. If we denote the characters of the bosonic subalgebra by $\chi^{\mathsf{B}}_{(j_1,j_2,j_3)}(z_i) = \chi_{j_1}(z_1)\chi_{j_2}(z_2)\chi_{j_3}(z_3)$, we can write the partition function as

$$Z \;=\; \sum_{\Lambda\in\mathcal{J}} \chi^{\mathsf{B}}_{\Lambda}(z_1,z_2,z_3)\psi^{\mathsf{B}}_{\Lambda}(q) \;=\; \sum_{\Lambda\in\mathcal{J}'} \chi^{\mathsf{K}}_{\Lambda}(z_1,z_2,z_3)\psi^{\mathsf{K}}_{\Lambda}(q) \tag{B.2.1}$$

where $\mathcal{J}' \subset \mathcal{J}$ is the set of labels in $\mathcal{J} = \{(j_1,j_2,j_3); j_i \in \frac{\mathbb{N}}{2}\}$ that are compatible with the consistency conditions (B.1.2). Here, the first decomposition is in terms of bosonic characters while the second one is based on the characters of Kac modules. In order to find the relations between these two decompositions, we recall that the roots of the four fermionic lowering operators in $\mathfrak{g}_{-1} := \text{osp}(4|2)_{-1}$ are

$$\alpha_{1,2} = \left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right) \ \alpha_{3,4} = \left(-\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}\right) \ . \tag{B.2.2}$$

Let us first discuss the generic label $\Lambda = (j_1,j_2,j_3)$ where either $j_1 \geq \frac{3}{2}$, or $j_1 = 1$ and $(j_2,j_3) \neq (0,0)$. In such cases we can write the decomposition of the Kac module character $\chi^{\mathsf{K}}_{\Lambda}$ as

$$\chi^{\mathsf{K}}_{\Lambda} = \sum_{i=0}^{4} \sum_{\beta\in\mathsf{Alt}^i(\mathfrak{g}_{-1})} \chi^{\mathsf{B}}_{\Lambda+\beta} \ , \tag{B.2.3}$$

where β is any of the weights that appear in the i^{th} exterior product $\mathsf{Alt}^i(\mathfrak{g}_{-1})$ of $\mathfrak{g}_{-1}$. We also allow for negative spins using the formal prescription $\chi_j = -\chi_{-j-1}$. To treat the remaining cases with $j_1 \leq \frac{1}{2}$ we employ the formulas developed in appendix A. Inserting the decomposition of Kac modules into the partition function Z leads to a formula that expresses the bosonic branching functions $\psi^{\mathsf{B}}_{\Lambda}$ as sums of the branching functions ψ^{K}_{μ}. Our main aim is to invert this relation, i.e. to determine the branching functions ψ^{K} in terms of ψ^{B}. To this end let us state a few basic properties of ψ^{K} that will be checked afterwards, once we have an explicit formula,

$$\psi^{\mathsf{K}}_{[j_1,j_2,j_3]} \;=\; -\psi^{\mathsf{K}}_{[j_1,-j_2-1,j_3]} \;=\; -\psi^{\mathsf{K}}_{[j_1,j_2,-j_3-1]} \quad . \tag{B.2.4}$$

If we take this behavior of ψ^{K} for granted the decomposition formulas for the partition function Z and of χ^{K} in terms of bosonic characters imply,

$$\psi^{\mathsf{B}}_{\Lambda} = \sum_{i=0}^{4} \sum_{\beta\in\mathsf{Alt}^i(\mathfrak{g}_{-1})} \psi^{\mathsf{K}}_{\Lambda-\beta} \tag{B.2.5}$$

for all $\Lambda \in \mathcal{J}'$. Inverting this expression leads to the following result

$$\psi^{\mathsf{K}}_{\Lambda} \;=\; \sum_{n=0}^{\infty}(-1)^n \sum_{\beta\in\mathsf{Sym}^n(\mathfrak{g}_{-1})} \psi^{\mathsf{B}}_{\Lambda-\beta} \quad . \tag{B.2.6}$$

To establish formula (B.2.6) we plug (B.2.5) into (B.2.6). Thereby we obtain

$$\psi^{\mathsf{K}}_{\Lambda} \;=\; \sum_{i=0}^{\infty}(-1)^i \underbrace{\sum_{j=0}^{4}(-1)^j \sum_{\beta\in\mathsf{Sym}^{i-j}(\mathfrak{g}_{-1})} \sum_{\gamma\in\mathsf{Alt}^j(\mathfrak{g}_{-1})} \psi^{\mathsf{K}}_{\Lambda-\beta-\gamma}}_{=0 \text{ if } i\neq 0} = \psi^{\mathsf{K}}_{\Lambda} \quad , \tag{B.2.7}$$

thus showing that (B.2.6) inverts (B.2.5). In (B.2.7) we have set $\mathsf{Sym}^n(V) = \emptyset$ if $n < 0$ and used the identity:

$$\sum_{j=0}^{4}(-1)^j \sum_{\beta\in\mathsf{Sym}^{i-j}(V)} \sum_{\gamma\in\mathsf{Alt}^j(V)} c(\beta+\gamma) = 0 \quad , \tag{B.2.8}$$

which is true for every four dimensional vector space V and every function c as long as $i \geq 1$. To show (B.2.8), we introduce the symbol $\ominus$ which is to be understood as a sort of a negative of a direct sum as for example in $A \oplus B \ominus B = A$. Then (B.2.8) is equivalent to $\bigoplus_{j=0}^{4} \ominus^j \mathsf{Sym}^{i-j}(V) \otimes \mathsf{Alt}^j(V) = 0$ if $i \geq 1$, which can be shown using standard Young tableaux techniques. Denote a tableau consisting of one single row with m boxes by 1^m and a tableau with one single column of n boxes[1] by n^1 and compute that $1^m \otimes n^1 = 1^m n^1 \oplus 1^{m-1}(n+1)^1$ if $m \geq 1, n \geq 1, n \leq 4$. Thus

$$\begin{aligned}\bigoplus_{j=0}^{4} \ominus^j \mathsf{Sym}^{i-j}(V) \otimes \mathsf{Alt}^j(V) &= \bigoplus_{j=0}^{4} \ominus^j 1^{i-j} \otimes j^1 \\ &= 1^i \oplus \bigoplus_{j=1}^{3} \ominus^j \left[1^{i-j} j^1 \oplus 1^{i-(j+1)}(j+1)^i\right] \oplus 1^{i-4} \otimes 4^1 = 0\end{aligned} \tag{B.2.9}$$

[1]Since we work with a four-dimensional space V, $4^1 = 0^1$ must denote the trivial one-dimensional space.

if $i \geq 1$. Thereby we have established that our assumption (B.2.4) implies the result (B.2.6).

In order to complete our proof of equation (B.2.6) we still need to verify our assumption (B.2.4). Let us observe that the bosonic branching functions ψ^{B} possess the same symmetry property, because, since the bosonic characters χ^{B} are simply products of sl(2) characters $\chi_j = -\chi_{-j-1}$, the identity (B.2.4) holds trivially for ψ^{B} instead of ψ^{K}. We can use this fact to show that for $m = 2, 3$ we get

$$\begin{aligned} \psi^{\mathsf{K}}_{\omega_m(\Lambda)} &= \sum_{i=0}^{\infty}(-1)^i \sum_{\beta\in\mathsf{Sym}^i(\mathfrak{g}_{-1})} \psi^{\mathsf{B}}_{\omega_m(\Lambda)-\beta} = \sum_{i=0}^{\infty}(-1)^i \sum_{\beta\in\mathsf{Sym}^i(\mathfrak{g}_{-1})} \psi^{\mathsf{B}}_{\omega_m(\Lambda-\tilde{\omega}_m(\beta))} \\ &= -\sum_{i=0}^{\infty}(-1)^i \sum_{\beta\in\mathsf{Sym}^i(\mathfrak{g}_{-1})} \psi^{\mathsf{B}}_{\Lambda-\tilde{\omega}_m(\beta)} = -\sum_{i=0}^{\infty}(-1)^i \sum_{\beta\in\mathsf{Sym}^i(\mathfrak{g}_{-1})} \psi^{\mathsf{B}}_{\Lambda-\beta} \, . \end{aligned} \tag{B.2.10}$$

The labels $\omega_2(\Lambda)$ and $\tilde{\omega}_2(\Lambda)$ were introduced as $\omega_2(\Lambda) = (j_1, -j_2 - 1, j_3)$ and $\tilde{\omega}_2(\Lambda) = (j_1, -j_2, j_3)$ for all $\Lambda = (j_1, j_2, j_3)$. Similar conventions apply to ω_3 and $\tilde{\omega}_3$.

As we have noted before, the functions $\psi^{\mathsf{K}}_{\Lambda}$ can have Laurent expansions with negative coefficients. Such negative coefficients only appear in the atypical sector and they can be traced back to the fact that we expanded the partition function Z in terms of 'unphysical' characters of Kac modules rather than through those of irreducible representations. The relation between Kac modules and irreducible representation has direct implications on the corresponding branching functions. In fact, the branching functions ψ_Λ that are defined through a decomposition into characters of irreducible representations are related to the branching functions ψ^{K} by $\psi_{[j_1,j_2,j_3]}(q) = \sum_\Lambda \psi^{\mathsf{K}}_\Lambda(q)$. On the right hand side the summation extends over all those Kac modules K_Λ that contain the irreducible representation $[j_1, j_2, j_3]$ in their decomposition series. All relevant decomposition series were spelled out in eq. (B.1.7). This gives

$$\begin{aligned} \psi_{\Lambda_{0,0}}(q) &= \psi^{\mathsf{K}}_{\Lambda_{0,0}}(q) + \psi^{\mathsf{K}}_{\Lambda_{0,2}}(q) \\ \psi_{\Lambda_{0,l}}(q) &= \psi^{\mathsf{K}}_{\Lambda_{0,l}}(q) + \psi^{\mathsf{K}}_{\Lambda_{0,l+1}}(q) \qquad \forall\, l \geq 1 \\ \psi_{\Lambda_{k,0}}(q) &= \psi^{\mathsf{K}}_{\Lambda_{k,0}}(q) + \psi^{\mathsf{K}}_{\Lambda_{k,1}}(q) + \psi^{\mathsf{K}}_{\Lambda_{k,-1}}(q) \qquad \forall\, k \geq 1 \\ \psi_{\Lambda_{k,l}}(q) &= \psi^{\mathsf{K}}_{\Lambda_{k,l}}(q) + \psi^{\mathsf{K}}_{\Lambda_{k,l+1}}(q) \qquad \forall\, k \geq 1\,,\ l \geq 1 \\ \psi_{\Lambda_{k,l}}(q) &= \psi^{\mathsf{K}}_{\Lambda_{k,l}}(q) + \psi^{\mathsf{K}}_{\Lambda_{k,l-1}}(q) \qquad \forall\, k \geq 1\,,\ l \leq -1 \,. \end{aligned} \tag{B.2.11}$$

Let us stress that the branching functions $\psi_\Lambda(q)$ for irreducible representations of osp (4|2) are guaranteed to have non-negative integral coefficients.

B.3 A free field construction for $\widehat{\mathrm{osp}}(\mathrm{M}|2\mathrm{N})_1$

This appendix contains a free field construction of the affine osp(M|2N) algebra at level $k = 1$ in terms of free fermions and several bosonic ghost systems. Let us decompose all supermatrices $X \in \mathrm{osp}(M|2N)$ into blocks according to

$$X \;=\; \left(\begin{array}{c|cc} \mathcal{E} & \bar{\mathcal{T}} & \mathcal{T} \\ \hline -\mathcal{T}^t & \mathcal{F} & \mathcal{G} \\ \bar{\mathcal{T}}^t & \bar{\mathcal{G}} & -\mathcal{F}^t \end{array}\right) \tag{B.3.1}$$

where $\mathcal{E}$ is antisymmetric and $\mathcal{G}, \bar{\mathcal{G}}$ are symmetric. A basis for the various blocks in the supermatrix X is provided by

$$\begin{aligned}
\mathcal{E}_{ij} &= e_{ij} - e_{ji} \qquad 1 \le i < j \le M \\
\mathcal{F}_{ab} &= e_{ab} \qquad 1 \le a\,, b \le N \\
\mathcal{G}_{ab} &= \bar{\mathcal{G}}_{ab} \;=\; e_{ab} + e_{ba} \qquad 1 \le a \le b \le N \\
\mathcal{T}_{ia} &= \bar{\mathcal{T}}_{ia} \;=\; e_{ia} \qquad 1 \le i \le M\,,\, 1 \le a \le N
\end{aligned} \tag{B.3.2}$$

where e_{mn} are elementary matrices. The matrices we have just introduced describe the various blocks in the supermatrix X. We agree to denote by E_{ij} the supermatrix of the form (B.3.1) where $\mathcal{E}$ is given by $\mathcal{E}_{ij}$ and all other blocks vanish. The basis elements $F_{ab}, G_{ab}, \bar{G}_{ab}, T_{ia}, \bar{T}_{ia}$ are defined similarly.

Now let us introduce M free fermions ψ_i and $2N$ bosons β_a, γ_a with the following basic operator products,

$$\psi_i(z)\psi_j(w) \sim \frac{\delta_{ij}}{z-w}\,, \qquad \beta_a(z)\gamma_b(w) \sim -\gamma_a(z)\beta_b(w) \sim \frac{\delta_{ab}}{z-w}\,. \tag{B.3.3}$$

We can define the free field representation of the osp(M|2N) current algebra through

$$\begin{aligned}
E_{ij}(z) &= (\psi_i\psi_j)(z)\,, \qquad & F_{ab}(z) &= -(\beta_a\gamma_b)(z) \\
G_{ab}(z) &= (\beta_a\beta_b)(z)\,, \qquad & \bar{G}_{ab}(z) &= -(\gamma_a\gamma_b)(z) \\
T_{ia}(z) &= i(\psi_i\beta_a)(z)\,, \qquad & \bar{T}_{ia}(z) &= -i(\psi_i\gamma_a)(z)\;.
\end{aligned}$$

The invariant bilinear form for osp(M|2N) is $(X,Y) = \frac{1}{2}\mathrm{str}(XY)$. On the basis elements it takes the following from

$$\begin{aligned}
(E_{ij}, E_{kl}) &= -\delta_{ik}\delta_{jl} \quad i<j \text{ and } k<l \\
(F_{ab}, F_{cd}) &= -\delta_{ad}\delta_{bc} \\
(G_{ab}, \bar{G}_{cd}) &= -\delta_{ac}\delta_{bd} \quad \text{for } a \ne b \text{ and } c \ne d \quad (G_{aa}, \bar{G}_{bb}) = -2\delta_{ab} \\
(T_{ia}, \bar{T}_{jb}) &= \delta_{ij}\delta_{ab}\;.
\end{aligned} \tag{B.3.4}$$

With the help of this form and assuming that $M \ne 2N+1$, the holomorphic part of the energy momentum tensor is given by the Sugawara construction

$$
\begin{aligned}
T(z) = \frac{(J^\mu J_\mu)(z)}{2(k+g^\vee)} &= \frac{1}{2(k+g^\vee)}\Big[-\sum_{i<j=1}^{M}(E_{ij}^2) - \sum_{a,b=1}^{N}(F_{ab}F_{ba}) - \sum_{a<b=1}^{N}\big(\{G_{ab},\bar{G}_{ab}\}\big) \\
&\quad -\frac{1}{2}\sum_{a=1}^{N}\big(\{G_{aa},\bar{G}_{aa}\}\big) - \sum_{i=1}^{M}\sum_{a=1}^{N}\big([T_{ia},\bar{T}_{ia}]\big)\Big] \\
&= -\frac{1}{2}\sum_{i=1}^{M}(\psi_i\partial\psi_i) + \frac{1}{2}\sum_{a=1}^{N}\big((\beta_a\partial\gamma_a)-(\gamma_a\partial\beta_a)\big) \qquad \text{(B.3.5)}
\end{aligned}
$$

Here, the dual Coxeter number is given by $g^\vee = M-2N-2$ and the value of the level is $k=1$. The central charge of the system is easily seen to take the value $c = \frac{M}{2} - N$.

Let us now introduce the involutive automorphism Ω such that the fixed point set $\{X \in \mathrm{osp}(M|2N)|\Omega(X) = X\}$ is isomorphic to $\mathrm{osp}(M-1|2N)$. On the basis we introduced above, Ω acts non-trivially only on $E_{ij}, T_{ia}, \bar{T}_{ia}$. In fact, it multiplies all operators with $i=1$ by -1 and leaves the others invariant. If we denote the anti-holomorphic fields corresponding to $\psi_i, \beta_a, \gamma_a$ by $\bar{\psi}_i, \bar{\beta}_a, \bar{\gamma}_a$, the deformation operator $J^\mu\Omega(\bar{J}_\mu)$ can then be written as

$$
\begin{aligned}
J^\mu\Omega(\bar{J}_\mu) &= -\sum_{i<j=1}^{M}\varpi_i(\psi_i\psi_j)(\bar{\psi}_i\bar{\psi}_j) - \sum_{a,b=1}^{N}(\beta_a\gamma_b)(\bar{\beta}_b\bar{\gamma}_a) \\
&\quad +\sum_{a<b=1}^{N}\big[(\beta_a\beta_b)(\bar{\gamma}_a\bar{\gamma}_b)+(\gamma_a\gamma_b)(\bar{\beta}_a\bar{\beta}_b)\big] + \frac{1}{2}\sum_{a=1}^{N}\big[(\beta_a\beta_a)(\bar{\gamma}_a\bar{\gamma}_a)+(\gamma_a\gamma_a)(\bar{\beta}_a\bar{\beta}_a)\big] \\
&\quad -\sum_{i=1}^{M}\sum_{a=1}^{N}\varpi_i\big[(\psi_i\beta_a)(\bar{\psi}_i\bar{\gamma}_a)-(\psi_i\gamma_a)(\bar{\psi}_i\bar{\beta}_a)\big] \\
&= \frac{1}{2}\left[\sum_{i=1}^{M}\varpi_i\psi_i\bar{\psi}_i + \sum_{a=1}^{N}\big(\gamma_a\bar{\beta}_a - \beta_a\bar{\gamma}_a\big)\right]^2 \qquad \text{(B.3.6)}
\end{aligned}
$$

where $\varpi = (-1,1,\ldots,1)$. In order for the last line of (B.3.6) to make sense, we need to first expand the square and then bring all the fields in the standard normal ordering.

B.4 The quadratic Casimir elements

For a simple contragredient Lie superalgebra $\mathfrak{g}$ the invariant, supersymmetric, consistent, non-degenerate and bilinear form $\beta : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ exists and is defined uniquely up to a proportionality constant. Every such invariant form β defines a quadratic central element in the universal enveloping Lie superalgebra in the standard way. To be more precise, let T^A be a basis of $\mathfrak{g}$ and let T_A be the dual basis with respect to β, that is

$$\beta(T_A, T^B) \ = \ \delta_A^B. \tag{B.4.1}$$

Then the quadratic Casimir associated to the invariant form β is defined as

$$\mathsf{Cas} \ := \ \sum_{A=1}^{\dim \mathfrak{g}} T_A T^A \ . \tag{B.4.2}$$

It is not hard to verify that Cas is indeed central. The Lie superalgebra $\mathrm{u}(S\,|S)$ we are dealing with in parts of this work, however, is not simple. After a normalization has been fixed, it possesses a one parameter family of invariant, supersymmetric, consistent, non-degenerate and bilinear forms. Let $V \simeq V_{\bar{0}} \oplus V_{\bar{1}}$ denote the graded fundamental module of $\mathrm{u}(S\,|S)$ with even dimension $\dim V_{\bar{0}} = S$ and odd dimension $\dim V_{\bar{1}} = S$ and $\rho_V : \mathrm{u}(S\,|S) \to \mathrm{End}(V)$ be the corresponding representation. Then the one parameter space of invariant forms of $\mathrm{u}(S\,|S)$ is constructed by using the invariant supertrace

$$\beta(X,Y) \ = \ \mathrm{str}\rho_V(XY) + \alpha \mathrm{str}\rho_V(X)\mathrm{str}\rho_V(Y) \ . \tag{B.4.3}$$

Let now $E_i^{\ j}$ be the standard basis of $\mathrm{End}(V)$, that is the $2S \times 2S$ matrices with an entry 1 in the i-th row and j-th column and 0 entries everywhere else. According to the definition (B.4.1), the basis dual to $E_i^{\ j}$ with respect to the form (B.4.3) is given by

$$\left(E_i^{\ j}\right)^\star \ = \ (-1)^{|j|} E_j^{\ i} - \alpha \delta_j^i E \ ,$$

where we have denoted by E the identity matrix. The quadratic Casimir of a reductive Lie superalgebra is constructed in the same way as in equation (B.4.2). When the invariant forms are not unique, the same is true for the Casimir element. In particular, the quadratic Casimir element of $\mathrm{u}(S\,|S)$ that is associated to the form (B.4.3) becomes

$$\mathsf{Cas}_\alpha \ = \ E_i^{\ j} E_j^{\ i} (-1)^{|j|} - \alpha E^2 \ . \tag{B.4.4}$$

The eigenvalues of Cas_α in an irreducible representation with highest weight Λ can be evaluated by computing scalar products in the weight space $\mathfrak{h}^\star$ in exactly the same way as for simple Lie superalgebras. Let us see how this works. Choose the diagonal generators $D_1 = E_1^{\ 1}, \ldots, D_{2S} = E_{2S}^{\ 2S}$ as a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathrm{u}(S\,|S)$ and denote by $\epsilon_1, \ldots, \epsilon_S, \delta_1, \ldots, \delta_S$, respectively, the dual basis in $\mathfrak{h}$. The restriction of β to $\mathfrak{h}$ defines a natural isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}^\star$ by

$$\varphi(H')(H'') \ = \ \beta(H', H'') \tag{B.4.5}$$

and endows $\mathfrak{h}^\star$ with a scalar product

$$(\Lambda, \mu)_\alpha \ = \ \beta\big(\varphi^{-1}(\Lambda), \varphi^{-1}(\mu)\big) \ . \tag{B.4.6}$$

In the basis δ_i, ϵ_j of $\mathfrak{h}^\star$, the natural isomorphism (B.4.5) reduces to

$$\varphi(D_1) \;=\; \epsilon_1, \ldots, \varphi(D_{2S}) \;=\; \delta_S.$$

The matrix elements of the scalar product (B.4.6) in the weight space $\mathfrak{h}^\star$ of $\mathrm{u}(S\,|S)$ with respect to the basis ϵ_i, δ_j can easily be computed

$$(\epsilon_i, \epsilon_j)_\alpha \;=\; \delta_{ij} - \alpha, \quad (\delta_i, \delta_j)_\alpha = -\delta_{ij} - \alpha, \quad (\epsilon_i, \delta_j)_\alpha = -\alpha \; . \tag{B.4.7}$$

One natural way to parametrize the highest weight vectors Λ for irreducible representations of $\mathrm{u}(S\,|S)$ is by specifying the coordinates of Λ with respect to the basis ϵ_i, δ_j. Thus, if

$$\Lambda \;=\; \sum_{i=1}^{S} (\rho_i \delta_i + \sigma_i \epsilon_i) \tag{B.4.8}$$

is the highest weight of a highest weight representation, then

$$\sigma_i \;=\; \Lambda(D_i), \quad \rho_i \;=\; \Lambda(D_{S+i}), \qquad i = 1, \ldots, S \; . \tag{B.4.9}$$

The eigenvalues of the Casimir element do not depend on the conventions for positiveness in the weight space. To compute them, we shall use a non-standard, but convenient absolute ordering

$$\epsilon_1 \;>\; \ldots \;>\; \epsilon_S \;>\; \delta_1 \;>\; \ldots \;>\; \delta_S \tag{B.4.10}$$

which fixes the positive roots to

$$\epsilon_i - \epsilon_j, \quad \delta_k - \delta_l, \quad \epsilon_i - \delta_k \;,$$

where $i < j$ and $k < l$. Now if v_Λ is the highest weight vector of some highest weight representation, then the eigenvalue of the Casimir on that representation can be easily computed

$$\begin{aligned}\mathsf{Cas}_\alpha\, v_\Lambda = &\sum_{i=1}^{2S} (-1)^{|i|} D_i^2 v_\Lambda - \alpha E^2 v_\Lambda + \sum_{j=2}^{2S} \sum_{i=1}^{j-1} [E_i{}^j, E_j{}^i](-1)^{|j|} v_\Lambda \\ = &\left(\sum_{i=1}^{2S} (-1)^{|i|} \Lambda(D_i)^2 - \alpha \Lambda(E)^2 + \sum_{j=2}^{2S} \sum_{i=1}^{j-1} [(-1)^{|j|}\Lambda(D_i) - (-1)^{|i|}\Lambda(D_j)] \right) v_\Lambda \; .\end{aligned}$$

Using the equations (B.4.7), (B.4.8) and (B.4.9), one can derive the desired form for the eigenvalues $\mathsf{Cas}_\alpha(\Lambda)$ of the Casimir (B.4.4) in a highest weight representation with highest weight Λ, namely

$$\mathsf{Cas}_\alpha(\Lambda) \;=\; (\Lambda, \Lambda + 2\rho)_\alpha, \tag{B.4.11}$$

where ρ is the Weyl vector

$$2\rho \;=\; \sum_{1 \le i < j \le S} (\epsilon_i - \epsilon_j + \delta_i - \delta_j) - \sum_{i,j=1}^{S} (\epsilon_i - \delta_j)$$

with respect to the chosen absolute ordering (B.4.10). Keeping in mind that the Weyl vector is the half sum of all positive roots minus the half sum of all negative roots, formula (B.4.11) for the eigenvalues of the Casimir can be rendered *independent* of the definition of positiveness in the weight space.

In the main text we use another notation for the weights of u(2|2), which stems from a different choice (5.2.11) of basis for the Cartan subalgebra. With respect to this basis, a highest weight $\Lambda = [j_1, j_2, a, b]$ has the following components

$$\Lambda(J_x) \;=\; j_1\,, \quad \Lambda(J_y) \;=\; j_2, \quad \Lambda(J_z) \;=\; a\,, \quad \Lambda(J_u) \;=\; b\;. \tag{B.4.12}$$

The dictionary between the labels ρ_i, σ_j of eq. (B.4.8) and the labels j_1, j_2, a, b is easy to establish

$$\sigma_1 - \sigma_2 \;=\; 2j_1\,, \quad \rho_1 - \rho_2 \;=\; 2j_2\,, \quad \sigma_1 + \sigma_2 - \rho_1 - \rho_2 \;=\; 2a\,, \quad \sigma_1 + \sigma_2 + \rho_1 + \rho_2 \;=\; 2b\;. \tag{B.4.13}$$

Moreover, from eq. (B.4.11) we obtain our formula (5.2.17) for the value of the Casimir elements in the representations $[j_1, j_2, a, b]$ of u(2|2).

B.5 Atypical branching functions for U (2|2)

In this appendix we collect explicit formulas for the branching functions of atypical u(2|2) representations in terms of those for Kac-modules. As in chapter 5 of this thesis, finite dimensional representations of u(2|2) are labelled by four parameters $j_1, j_2 \in \mathbb{N}/2$ and $a, b \in \mathbb{R}$. There are five different kinds of atypicality conditions on these labels. For each of these we shall then list the atypical branching functions. All of them can be derived using the character formulas in [71].

- $b = j_1 - j_2 = 0$

$$\begin{aligned}
\psi_{[0,0,a,0]} &= \psi^{\mathsf{K}}_{[0,0,a,0]} + \psi^{\mathsf{K}}_{[0,0,a+4,0]} + \psi^{\mathsf{K}}_{[\frac{1}{2},\frac{1}{2},a+1,0]} + \psi^{\mathsf{K}}_{[\frac{1}{2},\frac{1}{2},a+3,0]} \\
\psi_{[\frac{1}{2},\frac{1}{2},a,0]} &= \psi^{\mathsf{K}}_{[\frac{1}{2},\frac{1}{2},a,0]} + \psi^{\mathsf{K}}_{[\frac{1}{2},\frac{1}{2},a+2,0]} + \psi^{\mathsf{K}}_{[0,0,a+1,0]} + \psi^{\mathsf{K}}_{[1,1,a+1,0]} \\
\psi_{[j,j,a,0]} &= \psi^{\mathsf{K}}_{[j,j,a,0]} + \psi^{\mathsf{K}}_{[j,j,a+2,0]} + \psi^{\mathsf{K}}_{[j-\frac{1}{2},j-\frac{1}{2},a+1,0]} + \psi^{\mathsf{K}}_{[j+\frac{1}{2},j+\frac{1}{2},a+1,0]} \quad \text{for } j \geq 1
\end{aligned} \tag{B.5.1}$$

- $b = j_1 - j_2 \neq 0$

$$\begin{aligned}
\psi_{[\frac{1}{2},0,a,\frac{1}{2}]} &= \psi^{\mathsf{K}}_{[\frac{1}{2},0,a,\frac{1}{2}]} + \psi^{\mathsf{K}}_{[0,\frac{1}{2},a+3,\frac{1}{2}]} \\
\psi_{[0,\frac{1}{2},a,-\frac{1}{2}]} &= \psi^{\mathsf{K}}_{[0,\frac{1}{2},0,a,-\frac{1}{2}]} + \psi^{\mathsf{K}}_{[\frac{1}{2},0,a+3,-\frac{1}{2}]} \\
\psi_{[j_1,0,a,j_1]} &= \psi^{\mathsf{K}}_{[j_1,0,a,j_1]} + \psi^{\mathsf{K}}_{[j_1-1,0,a+2,j_1]} \quad \text{for } j_1 \geq 1 \\
\psi_{[0,j_2,a,-j_2]} &= \psi^{\mathsf{K}}_{[0,j_2,a,-j_2]} + \psi^{\mathsf{K}}_{[0,j_2-1,a+2,-j_2]} \quad \text{for } j_2 \geq 1 \\
\psi_{[j_1,j_2,a,j_1-j_2]} &= \psi^{\mathsf{K}}_{[j_1,j_2,a,j_1-j_2]} + \psi^{\mathsf{K}}_{[j_1-\frac{1}{2},j_2-\frac{1}{2},a+1,j_1-j_2]} \quad \text{for } j_1 \text{ and } j_2 \geq 0
\end{aligned} \tag{B.5.2}$$

- $b = -j_1 + j_2 \neq 0$

$$\psi_{[j_1,j_2,a,-j_1+j_2]} = \psi^{\mathsf{K}}_{[j_1,j_2,a,-j_1+j_2]} + \psi^{\mathsf{K}}_{\left[j_1+\frac{1}{2},j_2+\frac{1}{2},a+1,-j_1+j_2\right]} \tag{B.5.3}$$

- $b = j_1 + j_2 + 1$

$$\psi_{[0,j_2,a,j_2+1]} = \psi^{\mathsf{K}}_{[0,j_2,a,j_2+1]} + \psi^{\mathsf{K}}_{[0,j_2+1,a+2,j_2+1]} \tag{B.5.4}$$

$$\psi_{[j_1,j_2,a,j_1+j_2+1]} = \psi^{\mathsf{K}}_{[j_1,j_2,a,j_1+j_2+1]} + \psi^{\mathsf{K}}_{\left[j_1-\frac{1}{2},j_2+\frac{1}{2},a+1,j_1+j_2+1\right]} \text{ for } j_1 \geq \frac{1}{2}$$

- $b = -j_1 - j_2 - 1$

$$\psi_{[j_1,0,a,-j_1-1]} = \psi^{\mathsf{K}}_{[j_1,0,a,-j_1-1]} + \psi^{\mathsf{K}}_{[j_1-1,0,a+2,-j_1-1]} \tag{B.5.5}$$

$$\psi_{[j_1,j_2,a,-j_1-j_2-1]} = \psi^{\mathsf{K}}_{[j_1,j_2,a,-j_1-j_2-1]} + \psi^{\mathsf{K}}_{\left[j_1+\frac{1}{2},j_2-\frac{1}{2},a+1,-j_1-j_2-1\right]} \text{ for } j_2 \geq \frac{1}{2}$$

Explicit expressions for the atypical branching functions are now obtained by plugging in our formula (5.2.21) for the branching functions of Kac modules. The coefficients of atypical branching functions turn out to be positive.

B.6 Vanishing invariants on $\mathbb{CP}^{S-1|S}$

We start by considering a general symmetric superspace G/H, where G is a Lie supergroup with an involutive automorphism σ such that H is the maximal compact subgroup of G fixed by σ. Let e be the identity of G and consider the point $o = \mathsf{e}H$. The Riemann structure on G/H is defined by the requirement that G is a supergroup of isometries. This means that the action of G defines the metric and the curvature tensor globally once their values are given at a single point, say o.

Let now $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie superalgebras of the Lie supergroups G and H respectively. Define the quotient vector space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. The commutation relations of $\mathfrak{g}$ split with respect to the involutive automorphism σ into the following three families

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}. \tag{B.6.1}$$

In particular, this means that $\mathfrak{m}$ is a representation of $\mathfrak{h}$, which we denote by $\rho : \mathfrak{h} \to \mathrm{gl}(\mathfrak{m})$.

The curvature tensor for symmetric spaces

$$R_o(X,Y)Z = [[X,Y],Z], \qquad X,Y,Z \in \mathfrak{m}, \tag{B.6.2}$$

was computed in [95]. We straightforwardly generalize this expression to symmetric superspaces, as long as X, Y, Z are even graded vectors. Let β be a $\mathfrak{g}$-invariant, non-degenerate, supersymmetric and consistent form on $\mathfrak{g} \times \mathfrak{g}$. If $\mathfrak{m}$ is an *irreducible real* representation of $\mathfrak{h}$, then the solution to the condition that H is an isometry group

$$(h \cdot X, h \cdot Y)_o = (X,Y)_o, \qquad X,Y \in \mathfrak{m}$$

is uniquely determined, up to a proportionality constant called the radius of G/H, by the restriction of β to $\mathfrak{m} \times \mathfrak{m}$

$$(X,Y)_o = \beta(X,Y). \tag{B.6.3}$$

Note that, in order to be compatible with the automorphism σ, the invariant $\mathfrak{g}$-form β must be block diagonal with respect to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Therefore, the non-degeneracy of β implies the non-degeneracy of $(\ ,\ )_o$ as defined in eq. (B.6.3).

The curvature tensor being covariantly constant, it commutes with the action of H at o. It will prove more comfortable to use instead of this commuting homomorphism

$$R_o \in \mathsf{Hom}_{\mathfrak{h}}\left(\wedge^2 \mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}\right)$$

the endomorphism $\Omega_o \in \mathrm{End}_{\mathfrak{h}}\, \mathfrak{m} \otimes \mathfrak{m}$ defined the following way

$$\big(Y \otimes W, \Omega_o(Z,X)\big)_o = \big(W, R_o(X,Y)Z\big)_o = \big([X,Y],[Z,W]\big)_o,$$

where the scalar product on $\mathfrak{m} \otimes \mathfrak{m}$ is defined as

$$(X \otimes Y, Z \otimes W)_o = (W,X)_o (Y,Z)_o.$$

Let T_i be a basis of $\mathfrak{m}$ and T_a be a basis of $\mathfrak{h}$. Again, because β is block diagonal with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, the restriction of β to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate. Denote by T^a the basis dual to T_a with respect to β, that is

$$\beta(T^a, T_b) = \delta^a_b.$$

We shall rise and lower the group indexes with the help of the form β and its inverse rather than with the Killing form of $\mathfrak{g}$, which might be degenerate even for simple Lie superalgebras. Because of equation (B.6.3), this is consistent with the rising and the lowering of tensor indexes at o with the metric $(\ ,\ )_o$ and its inverse. Using the equations (B.6.1) one can show that

$$\Omega_o(X,Y) = (-1)^{|a|}[T^a, X] \otimes [T_a, Y].$$

Put differently, the previous equation can be written as

$$\Omega_o = (-1)^{|a|}\rho(T^a) \otimes \rho(T_a) = \rho(T_a) \otimes \rho(T_b)\beta^{ab}\ ,$$

where

$$\beta_{ab} = \beta(T_a, T_b)$$

and β^{ab} is the inverse of β_{ab}. It becomes now obvious that a non-zero contraction in a tensor power of Ω_o

$$\Omega_o^{\otimes n} = \rho(T_{a_1}) \otimes \rho(T_{a_2}) \otimes \cdots \otimes \rho(T_{a_{2n-1}}) \otimes \rho(T_{a_{2n}})\beta^{a_2 a_1} \cdots \beta^{a_{2n} a_{2n-1}}$$

will result in a fusion of the type

$$\rho(T_{a_i}) \otimes \rho(T_{a_j}) \to \rho(T_{a_i} T_{a_j}).$$

In particular, subtracting all but one trace in $\Omega_o^{\otimes n}$ one gets an expression of the form

$$\rho(T_{a_1} \cdots T_{a_{2n}}) \beta^{a_{2n} \cdots a_1}, \tag{B.6.4}$$

where $\beta^{a_{2n} \cdots a_1}$ is one of the $(2n-1)!!$ $\mathfrak{h}$-invariant tensors that can be constructed by raising to the n-th tensor power the $\mathfrak{h}$-invariant tensors $\beta^{a_i a_j}$. Denote by $\mathcal{Z}(\mathfrak{h})$ the center of the universal enveloping superalgebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$. Then we see that the expression in equation (B.6.4) is an element of $\mathcal{Z}(\mathfrak{h})$ in the representation ρ. We arrive at the conclusion that all $\mathfrak{h}$-invariant rank 2 tensors built from the tensor powers of the curvature tensor R_o by tracing the appropriate number of times with the metric $(\ ,\ )_o$ can be interpreted as elements of $\mathcal{Z}(\mathfrak{h})$ in the representation ρ.

Consider now the case of complex projective superspaces

$$\mathbb{CP}^{S-1|S} = \mathrm{U}(S|S) / \mathrm{U}(S-1|S) \times \mathrm{U}(1).$$

Complexifying everything, we get that $\mathfrak{m}$ is the direct sum of the fundamental representation $\square_{S-1|S}$ of $\mathrm{sl}(S-1|S)$ and of its conjugate $\bar{\square}_{S-1|S}$, thus revealing the complex structure of the supermanifold. Moreover, $\mathfrak{h} = \mathrm{sl}(S-1|S) \oplus \mathfrak{z}$, where $\mathfrak{z}$ is a two dimensional center. Let β be the $\mathrm{u}\,(S|S)$-invariant, non-degenerate form provided by the supertrace in the fundamental representation. Then the restriction of β to $\mathfrak{h} \times \mathfrak{h}$ is block diagonal with respect to the direct sum decomposition $\mathfrak{h} = \mathrm{sl}(S-1|S) \oplus \mathfrak{z}$. One can choose as basis for $\mathfrak{z}$ the central element E of $\mathrm{u}\,(S|S)$ together with its dual N with respect to β. Recalling that the invariant tensor $\beta^{a_{2n} \cdots a_1}$ were built from tensor products of $\beta^{a_i a_j}$, we notice that E and N can only appear in equation (B.6.4) in pairs. Therefore, given that E is in the kernel of ρ, the invariant tensors in equation (B.6.4) are effectively in the ρ-image of $\mathcal{Z}\big(\mathrm{sl}(S-1|S)\big)$. Finally, all these must vanish because $\square_{S-1|S}$ and $\bar{\square}_{S-1|S}$ both belong to the block of the trivial representation of $\mathrm{sl}(S-1|S)$.

Appendix C

Special Identities

In this appendix we collect a few definitions and identities that we have employed to obtain the Casimir decompositions in the subsections 4.3.3 and 4.4.3. We also provide the first few terms in the Casimir decomposition of the partition function Z_B^{FF} for $S = 1$. Most of this appendix was part of [20], written together with T. Quella and V. Schomerus.

C.1 Identities used in the Casimir decomposition

We recall the definition of Euler's ϕ function and its associates

$$\begin{aligned} \phi(q) &= \prod_{n=1}^{\infty}(1-q^n) \;=\; \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(3n-1)n}{2}} \\ \phi_\star(q) &= \prod_{n=1}^{\infty}(1+q^n) \;=\; \sum_{n=0}^{\infty} p_d(n) q^n \;, \end{aligned} \tag{C.1.1}$$

where $p_d(n)$ is the number of distinct partitions of the integer n, i.e. the number of those partitions for which each summand is different. The Dedekind η function is then defined as $\eta(q) = q^{\frac{1}{24}}\phi(q)$. We have the following two identities

$$\begin{aligned} \frac{1}{\phi(q)} &= \sum_{n=0}^{\infty}\left[p_e(n)+p_o(n)\right] q^n \;=\; \sum_{n=0}^{\infty} p(n) q^n \\ \frac{1}{\phi_\star(q)} &= \sum_{n=0}^{\infty}\left[p_e(n)-p_o(n)\right] q^n \;, \end{aligned} \tag{C.1.2}$$

with $p_e(n)$, respectively $p_o(n)$ being the number of partitions of n for which the number of summands is even, respectively odd. We can now write down the definition of Jacobi's

θ functions, which in our conventions read

$$
\begin{aligned}
\theta_1(q|z) &= -i \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} z^r q^{\frac{r^2}{2}} = -i\, z^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1}) \\
\theta_2(q|z) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} z^r q^{\frac{r^2}{2}} = z^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) \\
\theta_3(q|z) &= \sum_{r\in\mathbb{Z}} z^r q^{\frac{r^2}{2}} = \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbb{N}+\frac{1}{2}} (1+zq^r)(1+z^{-1}q^r) \\
\theta_4(q|z) &= \sum_{r\in\mathbb{Z}} (-1)^r z^r q^{\frac{r^2}{2}} = \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbb{N}+\frac{1}{2}} (1-zq^r)(1-z^{-1}q^r)\ .
\end{aligned}
\tag{C.1.3}
$$

The following two lemmata contain auxiliary formulas that are needed to rewrite the partition function (4.3.21) in terms of characters of osp$\,(4|2)$.

Lemma C.1.1.

$$
\prod_{n=1}^{\infty} \frac{1}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n\in\mathbb{Z}} z^n \sum_{m=0}^{\infty} (-1)^m \frac{q^{\frac{m}{2}(m+2n+1)} - q^{\frac{m}{2}(m+2n-1)}}{\phi(q)^2}\ .
$$

Proof. We assume that $|q| < |z| < 1$, which is the relevant condition for the above expansion to make sense. We want to find the coefficients $f_l^N(q)$ in the relation

$$
\sum_{l\in\mathbb{Z}} f_l^N(q) z^l = \frac{1}{(1-z)\prod_{n=1}^{N}(1-zq^n)(1-z^{-1}q^n)}\ .
$$

To do this, we multiply both sides by z^{-k-1} and integrate them over z along a contour that surrounds zero in a counterclockwise direction. In order to stay within the region $|z| < 1$ it must cling to the unit circle on the inside. The left hand side of the previous equation gives us the coefficient $f_k^N(q)$. The right hand side is zero for $z = 0$ and the first order poles that are encircled by the contour are at $z = q^n$ for $n = 1, \ldots, N$. Their residues are given by

$$
\lim_{z\to q^n} \frac{z^{-k-1}(z-q^n)}{(1-z)\prod_{l=1}^{N}(1-zq^l)(1-z^{-1}q^l)} = \frac{(-1)^{n-1} q^{\frac{n}{2}(n-2k-1)}}{\prod_{l=1}^{N+n}(1-q^l)\prod_{l=1}^{N-n}(1-q^l)}\ .
$$

If we finally remove our cutoff N by sending $N \to \infty$, we arrive at

$$
\frac{1}{(1-z)\prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)} = \sum_{k\in\mathbb{Z}} z^k \sum_{n=0}^{\infty} \frac{(-1)^{n-1} q^{\frac{n}{2}(n-1-2k)}}{\phi(q)^2}\ .
$$

Multiplying both sides by $1 - z$ and using the lemma C.1.2 below to shuffle some minus signs around completes the proof. ∎

Lemma C.1.2.

$$\sum_{m=1}^{2n}(-1)^m q^{\frac{m(m-1)}{2}-mn} \;=\; 0 \qquad \text{for} \qquad n \geq 1$$

$$\sum_{m=1}^{\infty}\sum_{s=-r}^{r}(-1)^m q^{\frac{m(m-1)}{2}-m(n+s)}(1-q^m) \;=\; \sum_{m=1}^{\infty}\sum_{s=-r}^{r}(-1)^m q^{\frac{m(m-1)}{2}-m(-n+s)}(1-q^m)\;.$$

Proof. The first equation is shown to be true by splitting the sum in $\sum_{m=1}^{n}$ and $\sum_{m=n+1}^{2n}$ and showing that they are equal up to a sign. The second equation then follows easily from the first. □

There are a number of very simple auxiliary formulas that are needed for the Casimir decomposition in subsection 4.3.3. Let us only list two of them here:

- For a set of constants a_r, $r \in \mathbb{N}$, chosen so as to obtain convergence, we have

$$\sum_{r=0}^{\infty}(-1)^r q^{\frac{r(r+2)}{4}}(1-q^{r+2})a_r \;=\; \sum_{r=0}^{\infty}(-1)^r q^{\frac{r(r+2)}{4}}(a_r - a_{r-2})\;. \tag{C.1.4}$$

- The following simple rearrangement formula is used to rewrite the branching functions for the $S^{3|2}$ sigma model:

$$\begin{aligned}\left(q^{(j_2-\frac{r}{2})^2} - q^{(j_2+\frac{r}{2}+1)^2}\right)\left(q^{(j_3-\frac{r}{2})^2} - q^{(j_3+\frac{r}{2}+1)^2}\right) \;&=\; q^{j_2(j_2+1)+j_3(j_3+1)}q^{\frac{r^2}{2}+r+1} \\ &\times\;\left(q^{-(r+1)(j_2+j_3+1)} + q^{(r+1)(j_2+j_3+1)} - q^{(r+1)(j_2-j_3)} - q^{-(r+1)(j_2-j_3)}\right)\;.\end{aligned} \tag{C.1.5}$$

C.2 Casimir decomposition of $\mathsf{Z}^{\mathrm{GN}}_{\mathfrak{B},M=1}$

In section 4.4.3 we obtained the closed formulas (4.4.28) and (4.4.30) for the Casimir decomposition of the partition function $\mathsf{Z}^{\mathrm{GN}}_{\mathfrak{B},M=1}$. Since our expression for the branching functions is a bit complicated, let us reproduce the first few terms of the partition function explicitly,

$$
\begin{aligned}
&\mathsf{Z}^{\rm GN}_{\mathfrak{B},M=1}(q) = q^0\chi_{[0,0,0]} + q^{\frac12}\chi_{[\frac12,0,0]} + q^1\chi_{[1,0,0]} + q^{\frac32}\left(\chi_{[\frac32,0,0]}+\chi_{[\frac12,0,0]}\right)\\
&+q^2\left(\chi_{[2,0,0]}+\chi_{[1,0,0]}+\chi_{[\frac12,\frac12,\frac12]}+\chi_{[0,0,0]}\right)\\
&+q^{\frac52}\left(\chi_{[\frac52,0,0]}+\chi_{[\frac32,0,0]}+\chi_{[1,\frac12,\frac12]}+2\chi_{[\frac12,0,0]}\right)\\
&+q^3\left(\chi_{[3,0,0]}+\chi_{[2,0,0]}+\chi_{[\frac32,\frac12,\frac12]}+4\chi_{[1,0,0]}+\chi_{[\frac12,\frac12,\frac12]}+\chi_{[0,0,0]}\right)\\
&+q^{\frac72}\left(\chi_{[\frac72,0,0]}+\chi_{[\frac52,0,0]}+\chi_{[2,\frac12,\frac12]}+3\chi_{[\frac32,0,0]}+2\chi_{[1,\frac12,\frac12]}+3\chi_{[\frac12,0,0]}\right)\\
&+q^4\left(\chi_{[4,0,0]}+\chi_{[3,0,0]}+\chi_{[\frac52,\frac12,\frac12]}+3\chi_{[2,0,0]}+2\chi_{[\frac32,\frac12,\frac12]}+\chi_{[1,1,0]}+\chi_{[1,0,1]}\right.\\
&\left.+6\chi_{[1,0,0]}+4\chi_{[\frac12,\frac12,\frac12]}+3\chi_{[0,0,0]}\right)\\
&+q^{\frac92}\left(\chi_{[\frac92,0,0]}+\chi_{[\frac72,0,0]}+\chi_{[3,\frac12,\frac12]}+3\chi_{[\frac52,0,0]}+2\chi_{[2,\frac12,\frac12]}+\chi_{[\frac32,1,0]}\right.\\
&\left.+\chi_{[\frac32,0,1]}+5\chi_{[\frac32,0,0]}+4\chi_{[1,\frac12,\frac12]}+\chi_{[\frac12,1,1]}+7\chi_{[\frac12,0,0]}\right)\\
&+q^5\left(\chi_{[5,0,0]}+\chi_{[4,0,0]}+\chi_{[\frac72,\frac12,\frac12]}+3\chi_{[3,0,0]}+2\chi_{[\frac52,\frac12,\frac12]}+\chi_{[2,1,0]}+\chi_{[2,0,1]}\right.\\
&\left.+5\chi_{[2,0,0]}+5\chi_{[\frac32,\frac12,\frac12]}+\chi_{[1,1,1]}+\chi_{[1,1,0]}+\chi_{[1,0,1]}+14\chi_{[1,0,0]}+5\chi_{[\frac12,\frac12,\frac12]}+3\chi_{[0,0,0]}\right)\\
&+q^{\frac{11}{2}}\left(\chi_{[\frac{11}{2},0,0]}+\chi_{[\frac92,0,0]}+\chi_{[4,\frac12,\frac12]}+3\chi_{[\frac72,0,0]}+2\chi_{[3,\frac12,\frac12]}+\chi_{[\frac52,1,0]}\right.\\
&+\chi_{[\frac52,0,1]}+5\chi_{[\frac52,0,0]}+5\chi_{[2,\frac12,\frac12]}+10\chi_{[\frac32,0,0]}+2\chi_{[\frac32,1,0]}+2\chi_{[\frac32,0,1]}+\chi_{[\frac32,1,1]}\\
&\left.+8\chi_{[1,\frac12,\frac12]}+\chi_{[\frac12,1,1]}+11\chi_{[\frac12,0,0]}\right)\\
&+q^6\left(\chi_{[6,0,0]}+\chi_{[5,0,0]}+\chi_{[\frac92,\frac12,\frac12]}+3\chi_{[4,0,0]}+2\chi_{[\frac72,\frac12,\frac12]}+\chi_{[3,1,0]}\right.\\
&+\chi_{[3,0,1]}+5\chi_{[3,0,0]}+5\chi_{[\frac52,\frac12,\frac12]}+11\chi_{[2,0,0]}+2\chi_{[2,1,0]}+2\chi_{[2,0,1]}+\chi_{[2,1,1]}\\
&\left.+11\chi_{[\frac32,\frac12,\frac12]}+2\chi_{[1,1,1]}+4\chi_{[1,1,0]}+4\chi_{[1,0,1]}+22\chi_{[1,0,0]}+13\chi_{[\frac12,\frac12,\frac12]}+9\chi_{[0,0,0]}\right)\\
&+q^{\frac{13}{2}}\left(\chi_{[\frac{13}{2},0,0]}+\chi_{[\frac{11}{2},0,0]}+\chi_{[5,\frac12,\frac12]}+3\chi_{[\frac92,0,0]}+2\chi_{[4,\frac12,\frac12]}+\chi_{[\frac72,1,0]}\right.\\
&+\chi_{[\frac72,0,1]}+5\chi_{[\frac72,0,0]}+5\chi_{[3,\frac12,\frac12]}+11\chi_{[\frac52,0,0]}+2\chi_{[\frac52,1,0]}+2\chi_{[\frac52,0,1]}+\chi_{[\frac52,1,1]}\\
&+11\chi_{[2,\frac12,\frac12]}+2\chi_{[\frac32,1,1]}+5\chi_{[\frac32,1,0]}+5\chi_{[\frac32,0,1]}+16\chi_{[\frac32,0,0]}+15\chi_{[1,\frac12,\frac12]}+\chi_{[1,\frac32,\frac12]}\\
&\left.+\chi_{[1,\frac12,\frac32]}+4\chi_{[\frac12,1,1]}+21\chi_{[\frac12,0,0]}\right)+\dots\quad .
\end{aligned}
$$

One may deform this expression to values $R \neq 1$ by means of the formula (4.4.39) at the end of section 4.4.4.

Appendix D

Laplacian on complex line bundles over $\mathbb{CP}^{S-1|S}$

This section first appeared in the article [21]. Let g_{pq} be the matrix elements of the metric g on $\mathbb{CP}^{S-1|S}$ in some set of local real coordinates φ^p, g^{pq} be the matrix inverse to g_{pq}, ∇ be the Levi-Civita connection with respect to the metric g and $A = A_p(\varphi)d\varphi^p$ be the one-form monopole defining a complex line bundle over $\mathbb{CP}^{S-1|S}$. Then the Bochner-Laplacian on the complex line bundle over $\mathbb{CP}^{S-1|S}$ is defined by the following second order, u$(S\,|S)$-invariant differential operator

$$\Delta = g^{pq}(\nabla_p + A_q)(\nabla_p + A_q).$$

Existence theorems that are discussed in [96] ensure that a non-trivial complex line bundle exists and is unique if and only if the curvature $\Omega = dA$ of the connection A satisfies the following integrality condition

$$\int_{\mathbb{CP}^1} \frac{\Omega}{2\pi i} \in \mathbb{Z}.$$

Let w^i be a set of local holomorphic coordinates on $\mathbb{CP}^{S-1|S}$, allowing us to express the standard metric on $\mathbb{CP}^{S-1|S}$ in the Fubini-Study form:

$$g_{i\bar{\jmath}} = \frac{\delta_{ij}}{1 + w^\dagger \cdot w} - \frac{(-1)^{|j|} w^{\bar{\imath}} w^j}{(1 + w^\dagger \cdot w)^2},$$

where the sign conventions for the scalar product in the supereuclidean space $\mathbb{C}^{S-1|S}$ are $w^\dagger \cdot w = \delta_{ij} w^{\bar{\jmath}} w^i$. The metric form is

$$ds^2 = g_{pq} d\varphi^p d\varphi^q = 2 g_{i\bar{\jmath}} dw^{\bar{\jmath}} dw^i$$

and all the geodesics are closed and of fixed length $\sqrt{2}\pi$. The Kähler form

$$\mathsf{w} = -i g_{i\bar{\jmath}} dw^{\bar{\jmath}} \wedge dw^i$$

can be normalized to yield a generator for the second integral cohomology group. Indeed, from

$$\int_{\mathbb{CP}^1} \mathsf{w} = 2\pi,$$

the existence condition for the complex line bundle reduces to

$$\Omega = -il\mathsf{w},$$

where $l \in \mathbb{Z}$ is called the *monopole charge*.

By standard methods in the theory of complex line bundles, see [69], one can prove that the space of sections of the bundle with monopole charge l is isomorphic to the space of equivariant functions on $\mathbb{CP}^{S-1|S}$, that is the space of functions $f(w, \bar{w})$ with the property

$$f(e^{i\alpha}w, e^{-i\alpha}\bar{w}) = e^{i\alpha l} f(w, \bar{w}),$$

where α is real. This functional space can be constructed as a square integrable span of the monomials $Z^{i_1} \ldots Z^{i_{k+l}} \bar{Z}^{j_1} \ldots \bar{Z}^{j_k}$, where the Z^i are the components of some vector belonging to the $\mathrm{u}(S\,|S)$-fundamental representation $\Box$ satisfying $Z^\dagger \cdot Z = 1$ and k, l are integers such that $k \geq 0$, $k + l \geq 0$.

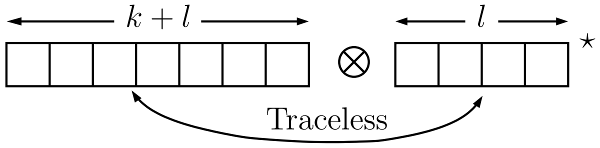

Figure D.1: The symmetric traceless tensors $t(k+l, k)$ are build by taking the tensor product between $\mathsf{Sym}^{k+l}\Box$ and $\mathsf{Sym}^{k}\Box^\star$ and removing all possible $\mathrm{u}(S\,|S)$-invariant contractions between the indices.

The harmonic decomposition of the space of equivariant functions with monopole charge $l \neq 0$ is a multiplicity free direct sum of $\mathrm{u}(S\,|S)$ supersymmetric traceless irreducible tensors $t(k+l, k)$ of contravariant rank $k + l \geq 0$ and covariant rank $k \geq 0$, shown in figure D.1. The highest weights of these tensors can be easily computed in the δ_i, ϵ_j basis of sec. B.4. If one chooses the absolute ordering (B.4.10) in the weight space of $\mathrm{u}(S\,|S)$ then the highest weight of the fundamental representation becomes ϵ_1, while of that of the dual representation $-\delta_S$. The weight of a supersymmetric tensor power of a vector follows immediately from the definition of the tensor action of the superalgebra. Thus, the highest weights of the supersymmetric irreducible traceless tensors $t(k+l, k)$, $l > 0$ are

$$\mu_{k,l} = \begin{cases} (k+l)\epsilon_1 - \delta_{S-k+1} - \cdots - \delta_S, & k \leq S \\ (k+l)\epsilon_1 - (k-S)\epsilon_S - \delta_1 - \cdots - \delta_S, & k > S \end{cases},$$

while those of the tensors $t(k'+l, k') = t(k, k+|l|)$, $l < 0$ are

$$\mu_{k,l} = \begin{cases} k\epsilon_1 - \delta_{S-k-|l|+1} - \cdots - \delta_S, & k + |l| \leq S \\ k\epsilon_1 - (k+|l|-S)\epsilon_1 - \delta_1 - \cdots - \delta_S, & k + |l| > S \end{cases},$$

where in both cases $k \geq 0$.

With this explicit construction of the complex line bundles at hand one can compute the spectrum of the Bochner-Laplacian, see [69]. The net result for the eigenvalues $e_l(k)$ of Δ is

$$e_l(k) = -2\left(k + \frac{|l|}{2}\right)\left(k + \frac{|l|}{2} - 1\right) + \frac{l^2}{2}, \tag{D.0.1}$$

where $k \geq 0$. Comparing this spectrum to the eigenvalues of the Casimir (B.4.4, B.4.11)

$$\mathsf{Cas}_\alpha(\mu_{k,l}) = 2k^2 + (2k + |l|)(|l| - 1) - \alpha l^2, \tag{D.0.2}$$

we see that

$$\Delta = -\,\mathsf{Cas}_{\alpha=1}\,.$$

In the end let us list the labels (B.4.12) of the highest weights $\mu_{k,l}$ of supersymmetric traceless irreducible u(2|2)-tensors $t(k + l, k)$ and $t(k, k + |l|)$. Using the dictionary (B.4.13) we get for $l \geq 0$

$$\mu_{0,l} = \left[\frac{l}{2}, 0, \frac{l}{2}, \frac{l}{2}\right], \quad \mu_{1,l} = \left[\frac{l+1}{2}, \frac{1}{2}, \frac{l}{2} + 1, \frac{l}{2}\right], \quad \mu_{k,l} = \left[\frac{l}{2} + k - 1, 0, \frac{l}{2} + 2, \frac{l}{2}\right],$$

for $k = 2, 3, \ldots$. When $l < 0$ we have

$$\mu_{0,-1} = \left[0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right], \quad \mu_{k,l} = \left[-\frac{l}{2} + k - 1, 0, \frac{l}{2} + 2, \frac{l}{2}\right], \quad k + |l| \geq 2.$$

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