Spectra of Conformal Sigma Models

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit den Spektren von konformen Sigma Modellen, die auf (verallgemeinerten) symmetrischen Räumen definiert sind. Die Räume, auf denen Sigma Modelle ohne Wess-Zumino-Term konform sind, sind Supermannigfaltigkeiten, also Mannigfaltigkeiten, die fermionische Richtungen aufweisen. Wir stellen die Konstruktion von Vertex Operatoren vor, gefolgt von der Hintergrundfeld-Entwicklung. Für semisymmetrische Räume berechnen wir anschließend die diagonalen Terme der anomalen Dimensionen dieser Operatoren in führender Ordnung. Das Resultat stimmt mit dem für symmetrische Räume überein, jedoch treten nichtdiagonale Terme auf, die hier nicht weiter betrachtet werden.

Anschließend präsentieren wir eine detaillierte Analyse des Spectrums des Supersphären $S^{3|2}$ Sigma Modells. Dies ist eins der einfachsten Beispiele für konforme Sigma Modelle auf symmetrischen Räumen und dient als Illustration für die Mächtigkeit der vorgestellten Methoden. Wir verwenden die erhaltenen Daten, um eine Dualität mit dem OSP$(4|2)$ Gross-Neveu Modell zu untersuchen, die von Candu und Saleur vorgeschlagen wurde. Wir verwenden dazu ein Resultat, welches die anomalen Dimensionen von $\frac{1}{2}$BPS Operatoren zu allen Ordnungen berechnet. Wir finden das gesamte Grundzustandsspektrum des Sigma Modells. Darüber hinaus legen wir dar, dass sowohl die Zwangsbedingungen als auch die Bewegungsgleichungen des Sigma Modells korrekt vom Gross-Neveu Modell implementiert werden. Die Dualität wird weiterhin durch ein neues exaktes Resultat für die anomalen Dimensionen der Grundzustände unterstützt. Andererseits beobachten wir für Operatoren mit mehreren Ableitungen Diskrepanzen. Es ist möglich, dass diese Diskrepanzen im Zusammenang mit einer bekannten Instabilität von Sigma Modellen stehen.

Die Instabilität von Sigma Modellen wird von Operatoren mit vielen Ableitungen verursacht, die bei beliebig kleiner Kopplung relevant werden. Diese Eigenschaft wurde bereits vor langer Zeit, zuerst im O(N)-Vektor-Modell, beobachtet. Gross-Neveu Modelle besitzen generisch eine ähnliche Instabilität. Ryu et al. haben beobachtet, dass solche Operatoren in $\mathfrak{psl}(N|N)$ Gross-Neveu Modellen möglicherweise nicht vorhanden sind. Die Beobachtung wurde für eine bestimmte Klasse von Operatoren in führender Ordnung bestätigt. Wir zeigen analytisch, dass im $\mathfrak{psl}(2|2)$ Modell in der Tat alle invarianten Operatoren irrelevant bleiben. Darüber hinaus bestimmen wir das Spektrum des BPS-Sektors für unendliche Kopplung. Wir finden keinen Hinweis auf einen Dualität mit dem $\mathbb{CP}^{1|2}$ Sigma Modell. Wir schließen mit einer Diskussion von marginalen Deformation von Kazama-Suzuki-Modellen.

Abstract

In this thesis the spectra of conformal sigma models defined on (generalized) symmetric spaces are analysed. The spaces where sigma models are conformal without the addition of a Wess-Zumino term are supermanifolds, in other words spaces that include fermionic directions. After a brief review of the general construction of vertex operators and the background field expansion, we compute the diagonal terms of the one-loop anomalous dimensions of sigma models on semi-symmetric spaces. We find that the results are formally identical to the symmetric case. However, unlike for sigma models on symmetric spaces, off diagonal terms that lead to operator mixing are also present. These are not computed here.

We then present a detailed analysis of the one-loop spectrum of the supersphere $S^{3|2}$ sigma model as one of the simplest examples. The analysis illustrates the power and simplicity of the construction. We use this data to revisit a duality with the OSP$(4|2)$ Gross-Neveu model that was proposed by Candu and Saleur. With the help of a recent all-loop result for the anomalous dimension of $\frac{1}{2}$BPS operators of Gross-Neveu models, we are able to recover the entire zero-mode spectrum of the supersphere model. We also argue that the sigma model constraints and its equations of motion are implemented correctly in the Gross-Neveu model, including the one-loop data. The duality is further supported by a new all-loop result for the anomalous dimension of the ground states of the sigma model. However, higher-gradient operators cannot be completely recovered. It is possible that this discrepancy is related to a known instability of the sigma model.

The instability of sigma models is due to symmetry preserving high-gradient operators that become relevant at arbitrarily small values of the coupling. This feature has been observed long ago in one-loop calculations of the O(N)-vector model and soon been realized to be a generic property of sigma models that persists to higher loop orders. A similar instability has been observed for Gross-Neveu models which can be seen as a certain deformation of WZNW models at level $k = 1$. Recently, Ryu et al. suggested that the $\mathfrak{psl}(N|N)$ Gross-Neveu models might be free of relevant high-gradient operators. They tested this proposal at one-loop level for a certain class of invariant operators. We extend the result to all invariant operators and all loops for the $\mathfrak{psl}(2|2)$ Gross-Neveu model. Additionally, we determine the spectrum of the BPS sector at infinite coupling and observe that all scaling weights become half-integer. Evidence for a proposed duality with the $\mathbb{C}\mathrm{P}^{1|2}$ sigma model is not found. We conclude with a brief discussion of marginal deformations of Kazama-Suzuki models.

This thesis is based on the following publications:

- A. Cagnazzo, V. Schomerus and V. Tlapák, *On the Spectrum of Superspheres*, arXiv:1408.6838, submitted to JHEP
- A. Cagnazzo, V. Schomerus and V. Tlapák, *High-Gradient Operators in the psl(2|2) Gross-Neveu Model*, arXiv:1410.4560

Contents

Chapter 1

Introduction

Non-linear sigma models (NLSM) play an important role in physics and mathematics. While in $d > 2$ dimensions they fail to be renormalisable, they are nevertheless still used as effective field theories and toy models to study phenomena such as chiral symmetry breaking in high-energy physics or the low energy behaviour of condensed matter models. When placed on a $d = 2$ dimensional world sheet, NLSM become renormalisable [1–3]. In string theory, 2d NLSM are the central ingredient in the perturbative expansion, see for example [4] for a review. Applications to condensed matter physics include the low energy effective description of spin chains and disordered fermions [5, 6].

A non-linear sigma model is a d-dimensional quantum field theory whose fields are interpreted as parametrizing a (pseudo) Riemannian (super)manifold $\mathcal{M}$, which is known as the target space. The properties of the NLSM depend on the target space. The presence of continuous symmetries simplifies the study of the theory. As such, homogeneous target spaces are particularly relevant. Homogeneous spaces can be represented as cosets of Lie (super)groups $\mathcal{M} \simeq G/H$. A symmetry is then given by left multiplication with elements of G and the subgroup $H \subset G$ is the stabilizer of a point on $\mathcal{M}$.

Homogeneous spaces where the subgroup H is left invariant by an automorphism $\Theta_M : G \to G$ of finite order M are known as (generalized) symmetric spaces or $\mathbb{Z}_M$ coset spaces. The case $M = 2$ corresponds to conventional symmetric spaces. It is commonly believed that NLSM on symmetric (super)spaces are quantum integrable, at least for certain choices of

H, see for example [7, 8] and references therein. Classical integrability can be established with much weaker assumptions on the denominator subgroup H and also for general M [9, 10]. Non-linear sigma models on symmetric spaces are of particular interest in the study of condensed matter systems. Supercosets defined by an automorphism of order four play an important role in the AdS/CFT correspondence since the motion of the superstring in the most prominent examples can be described as an NLSM on such cosets. Before introducing the questions addressed in the main body of this thesis, we describe in more detail the role of non-linear sigma models with supersymmetric target spaces in string theory. After that, we give a brief illustration of the appearance of supersymmetric models in the context of condensed matter systems.

String theory

In the perturbative definition of string theory two dimensional sigma models play a central role. The worldsheet theory is the non-linear sigma model given by the embedding of the worldsheet into the space in which the string propagates. The different choices of Riemannian surface as worldsheet then define the perturbative expansion. The standard Neveu-Schwarz-Ramond (NSR) approach to superstring theory is then to supersymmetrise the worldsheet theory by introducing fermionic fields that are spinors on the worldsheet manifold. This approach, while widely successful, does have its limitations. Target space supersymmetry is not manifest and has to be enforced through the GSO projection, which is also needed to remove the tachyon from the spectrum. On higher genus Riemann surfaces the spin structure is not unique and one has to sum over all possible choices in order to define a consistent perturbation theory. It is also not known how to consistently treat backgrounds with non-trivial Ramond-Ramond fluxes, which are represented by spin fields on the worldsheet, in the NSR description.

In order to circumvent these problems the alternative Green-Schwarz approach can be used. Here, worldsheet supersymmetry is traded for manifest supersymmetry of the target-space. The (pseudo) Riemannian manifold of the target-space is replaced by a supermanifold, meaning a space that includes fermionic directions. On the worldsheet, these fermions are scalar fields. This approach has the advantage that summation over spin structures is no longer required, and it allows the inclusion of non-trivial Ramond-Ramond fluxes. On the other hand, the Green-Schwarz action has a local fermionic symmetry, known as κ-symmetry, which needs to be gauge-fixed before quantization. This gauge-fixing breaks target-space covariance which in turn complicates calculations of, for example, scattering amplitudes.

The Green-Schwarz action has some unusual features. Even in flat space it is not free before light-cone gauge is fixed and it does not include a kinetic term for the fermions θ^A,

$$S_{\mathrm{GS}} = \frac{1}{2\pi\ell_S^2}\int_\Sigma d^2z\left(\partial x^\mu - \frac{i}{2}\theta^A\gamma^\mu\partial\theta_A\right)\left(\bar\partial x_\mu - \frac{i}{2}\theta^A\gamma_\mu\bar\partial\theta_A\right) + S_2\,, \quad (1.1)$$

where the coordinates of the bosonic base of the target space are denoted by x^μ, γ^μ are the Dirac matrices, and ℓ_S is the string length. Additional terms that make the action κ-symmetric are included in S_2. This form of the action corresponds to the choice of a degenerate metric in the fermionic directions. It is evident from the action (1.1) that both fermionic and bosonic fields have dimension zero. This essential for making target-space supersymmetry manifest.

A central motivation for considering the Green-Schwarz formalism is the AdS/CFT correspondence [11]. Conformal field theories are conjectured to be dual to strings moving in AdS backgrounds supported by non-trivial Ramond-Ramond fluxes. Many of these backgrounds, and in particular the most prominent ones, have the structure of a homogeneous space with a symmetric space as bosonic base. The Green-Schwarz string moving on these backgrounds can be described as an appropriate sigma model on a $\mathbb{Z}_4$ coset space [12–15], for example

$$\begin{aligned}\mathrm{AdS}_5\times S^5 &\simeq \left(\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}\right)_{\bar 0}\,,\\ \mathrm{AdS}_4\times\mathbb{CP}^3 &\simeq \left(\frac{\mathrm{OSP}(6|4)}{\mathrm{SO}(3,1)\times\mathrm{U}(3)}\right)_{\bar 0}\,.\end{aligned} \quad (1.2)$$

It is desirable to retain as much of the symmetry of the model as possible in a manifest manner in order to simplify calculations. Since κ-gauge-fixing breaks part of the symmetry, alternative approaches, specialized to the cases at hand, have been developed. The hybrid [16–18] and pure spinor [19, 20] formulations treat the fermionic coordinates on an equal footing with the bosonic ones by choosing a non-degenerate metric and thereby including a kinetic term for the fermions. To achieve a consistent formulation of string theory on these target spaces, the pure spinor and hybrid formalisms supplement the sigma model by pure spinor ghosts or additional matter sectors.

As we have seen, sigma models with geometric fermions appear naturally in approaches to string theory that deviate from the NSR description.

Among those, some of the most prominent ones are defined on $\mathbb{Z}_4$ cosets. Further applications include Witten's proposal of the $\mathbb{C}\mathrm{P}^{3|4}$ B model as a description of perturbative $\mathcal{N}=4$ super Yang-Mills theory [21], and their role as mirror duals of rigid Calabi-Yau manifolds [22–24].

Disordered fermions

In statistical models with disorder it is necessary to average Green's functions and their products with respect to a random potential which represents the disorder. One method of doing so uses the integral representation of the Green's functions, see [25]. Assume that for fixed disorder the model is gaussian, that is at most quadratic in the fields. The random Hamiltonian H can be split into two parts,

$$H = H_0 + V, \tag{1.3}$$

where H_0 is a fixed Hamiltonian and V is a random potential describing the disorder. Assume further that the potential is normally distributed so that its probability density function $\mathcal{P}[V]$ is given by

$$\mathcal{P}[V] = \exp\left(-\frac{1}{2\sigma}\mathrm{tr}\left(V^2\right)\right). \tag{1.4}$$

The trace is taken over any internal degrees of freedom and the parameter σ characterizes the strength of the disorder interaction. Observables of interest, such as longitudinal or transversal conductivities, can be expressed in terms of products of advanced and retarded Green's functions averaged over the disorder,

$$\overline{\left(\frac{1}{H-E-i\varepsilon}\right)\left(\frac{1}{H-E'+i\varepsilon}\right)\cdots}\,. \tag{1.5}$$

What makes computations difficult is the fact that the average is taken after finding the Green's functions. In order to do the averaging first, the Green's functions can be represented as gaussian integrals

$$\frac{1}{H-E-i\varepsilon} = \frac{1}{\det[H-E-i\varepsilon]}\int \mathcal{D}\psi\,\psi\psi^*\exp\bigl(\pm i\psi^*\left(H-E-i\varepsilon\right)\psi\bigr), \tag{1.6}$$

where ψ is a complex Grassmann field. The determinant prefactor can equally be rewritten as a gaussian integral over bosonic fields,

$$\frac{1}{\det[H-E-i\varepsilon]} = \int \mathcal{D}\varphi\exp\bigl(-i\varphi^*(H-E-i\varepsilon)\varphi\bigr). \tag{1.7}$$

The interaction term can be integrated out because it is quadratic. After integrating over V, the averaged Green's function is given by the expression

$$\overline{\left(\frac{1}{H-E-i\varepsilon}\right)} = \int \mathcal{D}\mathrm{V}\,\mathcal{D}\psi\,\mathcal{D}\varphi\,\psi\psi^*\exp(S_{\text{eff}})\,, \tag{1.8}$$

with the effective action

$$S_{\text{eff}} = i\psi^*(H_0-E-i\varepsilon)\psi + i\varphi^*(H_0-E-i\varepsilon)\varphi - \frac{\sigma}{2}\mathrm{tr}(\varphi^*\varphi+\psi^*\psi)^2\,. \tag{1.9}$$

The effective action (1.8) has a U(1|1) symmetry. Generalizing to higher products of Green's functions presents no difficulty as the random potential can always be integrated out first. These N-fold products are then described by analogous actions with U($N|N$) symmetry. Using real fermions and bosons instead would have resulted in an OSP(2|2) symmetric action.

The preceding discussion motivates the appearance of supersymmetric theories and in particular those where the fermions have the same dimension as the bosons in the study of condensed matter systems. For more detailed expositions see for example [26, 25].

Dualities

In every application of non-linear sigma models, a complete understanding of the theory is essential. Sigma models are interacting field theories and as such are only well understood in the small-coupling limit. From the point of view of string theory this corresponds to the point-particle limit, where the length of the string is small compared to the size of the target-space. Understanding sigma models when the string becomes large or, equivalently, for strongly curved backgrounds is of central importance. In the AdS/CFT correspondence, for example, the strong coupling limit of the string theory corresponds to the weak coupling regime of the dual gauge theory [11]. Thus, with good control over sigma models on AdS spaces at strong coupling we could gain new insights into the gauge/string duality, even without the help of integrability. At the same time, as we proceed to smaller radius R stringy effects start to dominate and the original geometric picture dissolves.

The sigma models discussed in this thesis are conformal for any value of the coupling but they fall outside the standard classification of rational CFTs. Instead, these theories are in general logarithmic and non-unitary. Therefore, many standard tools and results derived for rational, unitary CFTs naturally do not apply. The two properties – lack of unitarity and the presence of logarithmic singularities in correlation functions – are closely related. Lack of unitarity can be most easily seen by observing that the theories contain fermionic fields of dimension zero. These contribute with a negative sign to the central charge c of the theory. Then, c is no longer a good measure of the number of degrees of freedom and may even become negative. For example, the sigma model on $\mathrm{PSL}(N|N)$ has central charge $c = -2$ for all N. The presence of logarithmic singularities can be explained using representation theory of supergroups. Unlike in the purely bosonic case, finite dimensional representations of Lie supergroups are not always completely reducible but may form complicated indecomposable structures. Casimir operators can in general not be diagonalized on such indecomposable modules. The off-diagonal part then leads to logarithmic divergences in correlation functions. For a recent review of conformal sigma models and Wess-Zumino-Novikov-Witten (WZNW) models on coset superspaces see [27]. A comprehensive overview of the properties of the underlying Lie superalgebras can be found in [28].

For bosonic target spaces there are a few cases in which sigma models at small radius are well understood through a dual description. The simplest one is the free boson compactified on a circle of radius R. At $R = 1$ this theory possesses a description in terms of free fermions which can be understood through bosonization and is known as Coleman-Mandelstam duality [29, 30]. For interacting sigma models dualities are known as well. Sigma models on complete intersection Calabi-Yau spaces can often be described in terms of certain WZNW models when their couplings take special values which are known as Gepner points. These provide an exactly solvable description and are best understood through the use of linear sigma models [31].

Several dual descriptions of non-linear sigma models on supermanifolds in terms of deformed WZNW models on supergroups have been proposed [32–37]. Polyakov suggests a general pattern of dualities [32]. Candu and Saleur have suggested a duality between the odd-dimensional superspheres $S^{2S+1|2S}$ and the deformed WZNW model on $\mathrm{OSP}(2S + 2|2S)$, which is supported by data from lattice simulations [33, 34] and analysis of the boundary spectra [35]. It has also been argued that the NLSM on $\mathbb{C}\mathrm{P}^{1|2}$ should be dual to the deformed $\mathfrak{psl}(2|2)$ WZNW model [36, 37]. In this thesis, we use recent results on the bulk spectra of sigma [38] and WZNW [39] models to revisit these proposals.

Instability

Besides the strong-coupling description other fundamental questions about sigma models remain unanswered. It was observed in [40] that high-gradient operators become strongly relevant at arbitrarily small coupling. At infinite radius, or zero coupling, the scaling dimension of an operator is given by the number of derivatives. High-gradient operators, meaning operators with a large number of derivatives, are thus highly RG-irrelevant in the free theory. The usual assumption in perturbation theory is that corrections to the weight, the anomalous dimensions, are small as long as the coupling is small. However, it was shown in [40] that this assumption fails to hold for the O(N)-vector model at one-loop. High-gradient operators can in fact become relevant even at infinitesimal values of the coupling, which implies that there is an infinite number of additional couplings in the theory. Similar results have been found to hold for non-linear sigma models defined on a wide variety of compact and non-compact target (super)spaces, see [41] and references therein.

The generation of an infinite number of relevant operators presents a problem. One might hope that higher orders in perturbation theory cure this issue but it has actually been shown to become worse at two-loop order [42]. It is still unknown how to interpret the presence of these relevant operators. One possibility is to consider them to be signals of an instability of the UV fixed point or the presence of a new higher fixed point. On the other hand, lattice simulations have not shown signs of instabilities thus far. Some authors have related the relevance of the high-gradient operators to statistical fluctuations of the conductance of disordered metals at finite size [43–46], although the validity of those conclusions remains unclear [41]. Note that Polyakov has argued [32] that this problem may be a desirable feature which allows for a general pattern of dualities between non-linear sigma models and WZNW models.

Furthermore, it has also been observed that WZNW models, when perturbed by a current-current Gross-Neveu-like deformation, suffer from a similar issue [41]. This is true even if the perturbation preserves conformal symmetry as it does for a number of target supergroups. It was noted in [41] that the $\mathfrak{psl}(N|N)$ WZNW model at level $k = 1$ could be free of relevant high-gradient operators, at least to first order in perturbation theory. The idea was tested and confirmed for a certain class of invariant operators. For $\mathfrak{psl}(2|2)$ we extend these results to all orders and all invariant operators.

Organization of this thesis

- In Chapter 2 we provide a brief introduction to sigma models with a particular focus on homogeneous target spaces defined by an automorphism of finite order. Moreover, we discuss the related WZNW models and present a previously obtained all-order result for a particular deformation which is important in subsequent chapters.
- Chapter 3 then presents the method used to construct the spectra of coset sigma models. The general formula for the one-loop anomalous dimension of sigma models on symmetric cosets is presented and discussed. This review was previously published as part of [47]. We the begin to extend the formula for the anomalous dimension to the case of $\mathbb{Z}_4$ cosets, focussing on the conformal case. We compute the diagonal terms of the one-loop dilatation operator which turn out to be formally identical to the symmetric case. However, additional off-diagonal terms are present which are not yet under good control.
- In Chapter 4 we combine the results and methods presented in the previous chapters to study the spectrum of the supersphere $S^{3|2}$ sigma model and investigate a proposed duality with the $\mathfrak{osp}(4|2)$ Gross-Neveu model. We find some intriguing agreement but also some tension with the duality. These results were first published in [47]. Finally, we extend the computation of the anomalous dimension for the ground states of the sigma model to higher orders. We show that it vanishes at two- and three-loop order in the conformal case and argue that for the supersphere $S^{3|2}$ all higher order corrections vanish as well. These last results have not been previously published.
- In Chapter 5 we revisit a question about the stability of sigma models and related theories. The all-order result for deformed WZNW models reviewed in Chapter 2 is used to establish the absence of relevant high-gradient operators in the $\mathfrak{psl}(2|2)$ Gross-Neveu model. Furthermore, we obtain the first few levels of the spectrum at infinite coupling, which turns out to be half-integer valued. These results were first presented in [48].
- In Chapter 6 we comment on the existence of families of superspace Kazama-Suzuki models that possess at least one marginal deformation. These can be viewed as a generalization of the deformed WZNW models that we previously discussed.
- We conclude with Chapter 7 and highlight opportunities for future research directions. Technical details are collected in several appendices.

Chapter 2

Sigma Models with Target-Space Supersymmetry

In this chapter we introduce the main objects of study of this thesis, non-linear sigma models on (super)coset spaces. Since the theory of these spaces is based on the concept of a Lie superalgebra, we briefly introduce the most important notions from the theory of these algebras. We do however, assume a certain familiarity with the topic. For a comprehensive introduction, we refer to the literature. We then discuss non-linear sigma models in some generality before turning to the spaces on which we want to study them.

The related Wess-Zumino-Novikov-Witten models are introduced in Section 2.4. In Section 2.5 we briefly review an all-loop result from [39]. In the final section of this chapter we return to the representation theory of Lie superalgebras. We illustrate concepts mentioned in the first section by using representations of $\mathfrak{u}(1|1)$ as an example.

2.1 Lie Superalgebras

Lie superalgebras are a generalization of Lie algebras that includes a $\mathbb{Z}_2$-grading. While most concepts and constructions used in the theory of Lie algebras carry over to the superalgebra case, many important theorems fail to hold, or hold only in a limited sense. For example, simple Lie superalgebras can be classified using (generalized) Dynkin diagrams, most have a non-trivial Killing form, and non-trivial Casimir operators exist as well.

Instead of giving a complete introduction to the theory of Lie superalgebras and their representations, we will only introduce essential notation. We will illustrate an important point where the finite dimensional representation theory differs from that of conventional Lie algebras using a simple example. For a more self-contained introduction to the theory of Lie superalgebras see for example [49, 50]. A comprehensive collection of the properties of simple Lie superalgebras can be found in [28].

A Lie superalgebra $\mathfrak{g}$ is a direct sum of two vector spaces $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}}$ is a semi-simple Lie algebra which acts on $\mathfrak{g}_{\bar{1}}$. This decomposition defines a grading function $|\cdot|$ as

$$|t| = \begin{cases} 0 & \text{if} \quad t \in \mathfrak{g}_{\bar{0}} \\ 1 & \text{if} \quad t \in \mathfrak{g}_{\bar{1}}. \end{cases} \tag{2.1}$$

In order to generalize from Lie algebras to Lie superalgebras, one then replaces every property by its graded analogue. For example, the Lie bracket $[\cdot, \cdot]$ becomes graded anti-symmetric

$$[t, s] = (-1)^{|t||s|}[s, t] \qquad \forall s, t \in \mathfrak{g}. \tag{2.2}$$

Later, we will work with fields on supergroups where the generators of the Lie superalgebra are combined with Grassmann-valued fields such that the combinations behave as if they were elements of a regular Lie algebra.

While in concrete calculations one often does not need to consider the grading of the algebra, it has important consequences for the representation theory. Most importantly, it is no longer true that every finite dimensional representation is semi-simple, i.e. that it is the direct sum of irreducible representations. Instead, reducible but indecomposable representations can appear. One important property to keep in mind is that Casimir operators, while they still exist, cannot be diagonalized in an indecomposable representation. It is also no longer true that the tensor product of two irreducible (i.e. simple), finite dimensional representations is the direct sum of irreducible representations. For our purposes the most relevant indecomposable representations, or modules, are called projective. Projective modules $\mathcal{P}$ have the important property that they cannot appear as a submodule of a larger indecomposable. That means that, wherever they appear, they constitute direct summands.

The notion of a projective module generalizes that of a semi-simple module in the sense that the property of being projective is preserved under most operations, in particular under tensor products and restriction to subalgebras. Note that projective modules in this context are not related to representations where the commutation relations are satisfied only up to a phase, which are also known as projective representations. Projective modules play a central role in the harmonic analysis on supergroups. We will discuss this in more detail in chapter 3. Let us introduce some final pieces of terminology which we will be using throughout the text. A module that is both semi-simple and projective is called typical, while all other modules are known as atypical. In the physics literature these are also known as long and short multiplets, respectively, and atypical modules are further known as BPS representations.

2.2 General sigma models

A non-linear sigma model is a quantum field theory which is defined on a d-dimensional worldsheet Σ, where the fields take values in a Riemannian (super)manifold $(\mathcal{M}, g)$ and g is a Riemannian metric on the manifold $\mathcal{M}$. Note that the standard definition of a sigma model on a Riemannian manifold can be generalized without problem to include supermanifolds. The fields ϕ can be viewed as a map

$$\phi : \Sigma \to \mathcal{M} \tag{2.3}$$

that gives an embedding of the worldsheet into the target manifold. If the metric on the worldsheet is denoted $\eta^{\mu\nu}$ then the sigma model is defined by the action

$$S_{SM}[\phi(\sigma)] = \frac{1}{2}\int_\Sigma d^d\sigma\, \eta^{\mu\nu} g_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b, \tag{2.4}$$

where σ denotes the coordinates on the worldsheet. In $d > 2$ dimensions the action (2.4) is not renormalizable. It is, however, still used as an effective field theory to study, for example, chiral symmetry breaking.

In $d = 2$ dimensions the sigma model is renormalizable. The action (2.4) is not the most general one, as topological terms and a B-field can be added. Their existence and precise form depends on the target manifold in question. To first order in perturbation theory, the β-function of the sigma model (2.4) is proportional to the Ricci tensor of the target space [3],

$$\beta_{ab} = \frac{1}{2\pi}R_{ab}\,. \tag{2.5}$$

In this thesis we will consider the case where the target manifold is a (right) coset of Lie (super)groups. That is, $\mathcal{M} \simeq G/H$, with G a Lie (super)group and $H \subset G$ a sub(super)group. The quotient is defined through the identification

$$g \sim gh \qquad \forall h \in H \subset G \tag{2.6}$$

of elements $g \in G$. We assume further that the Lie (super)algebra $\mathfrak{g}$ of G is equipped with an invariant, non-degenerate bilinear form $(\cdot,\cdot)$ and that its restriction to the algebra $\mathfrak{h}$ of H is also non-degenerate. It follows that $\mathfrak{h}$ has an orthogonal complement $\mathfrak{m}$ in $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus_\perp \mathfrak{m}$, and that $\mathfrak{h}$ acts on $\mathfrak{m}$

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}. \tag{2.7}$$

In particular, the orthogonal projectors P' onto $\mathfrak{h}$ and $P = 1 - P'$ onto $\mathfrak{m}$ commute with the action of $\mathfrak{h}$.

Given the above assumptions, the coset G/H can be equipped with a G left-invariant and H right-invariant metric which is induced by the invariant bilinear form. In order to make the coset nature of sigma models manifest, we formulate them in terms of the Maurer-Cartan form J on G which is left-invariant under the global G-action. Its pullback to the worldsheet under the embedding map $g : \Sigma \to G$ is

$$J(\sigma) = g(\sigma)^{-1} dg(\sigma) =: J_\mu d\sigma^\mu. \tag{2.8}$$

Using these currents one can define the action

$$S = \frac{1}{2} \int_\Sigma d^2\sigma \Big(\eta^{\mu\nu} \big(P(J_\mu), P(J_\nu) \big) + \epsilon^{\mu\nu} B\big(P(J_\mu), P(J_\nu) \big) \Big) \tag{2.9}$$

where we have now also included the B-field, which is anti-symmetric in its arguments, and $\epsilon^{\mu\nu}$, the two-dimensional Levi-Civita tensor. The action (2.9) is still not the most general one. It is possible to deform the metric whenever $\mathfrak{m}$ is not irreducible as an $\mathfrak{h}$ representation. In that case, the metric can be scaled independently for each subrepresentation. The metric chosen in the action (2.9) is, however, the most symmetric one and will be the one we consider.

To see that the action (2.9) is well defined on the space G/H, consider the action of a local H right-transformation $h : \Sigma \to H$ on the Maurer-Cartan form J,

$$g(\sigma) \mapsto g(\sigma)h(\sigma) \quad \Rightarrow \quad J(\sigma) \mapsto h(\sigma)^{-1} J(\sigma) h(\sigma) + h(\sigma)^{-1} dh(\sigma)\,, \tag{2.10}$$

so that the projected currents transform by conjugation with $h(\sigma)$. Since the bilinear form is $\mathfrak{h}$ invariant, the action is independent of the choice of coset representative.

2.3 Generalized symmetric coset spaces

After describing the sigma model for general coset spaces, we now specialize to the case where the denominator group H is given as the invariant locus of an order $2N$ automorphism Θ of the numerator. Such spaces include symmetric spaces, corresponding to $N = 1$, as well as generalizations thereof which occur in the context of AdS compactifications. An automorphism of order $2N$ defines a direct sum decomposition of the algebra

$$\mathfrak{g} = \bigoplus_{A=0}^{2N-1} \mathfrak{m}_A, \qquad \text{with} \qquad \mathfrak{m}_0 = \mathfrak{h} \tag{2.11}$$

into the eigenspaces $\mathfrak{m}_A$ of Θ. The automorphism acts on these eigenspaces by

$$\text{and} \qquad \Theta(\mathfrak{m}_A) = e^{\frac{2\pi i A}{2N}} \mathfrak{m}_A \,. \tag{2.12}$$

Since Θ is an automorphism of $\mathfrak{g}$ one also has

$$\begin{aligned} &[\mathfrak{m}_A, \mathfrak{m}_B] \subset \mathfrak{m}_{A+B \bmod 2N} \quad \text{and} \\ &(\mathfrak{m}_A, \mathfrak{m}_B) = 0 \quad \text{if} \quad A + B \neq 0 \mod 2N \,. \end{aligned} \tag{2.13}$$

In particular, $\mathfrak{h} = \mathfrak{m}_0$ acts on the eigenspaces of the decomposition. Note that again the spaces $\mathfrak{m}_A$ need not be indecomposable under the action of $\mathfrak{h}$. In that case, the decomposition into $\mathfrak{h}$-modules is finer than the one given by the automorphism (2.11), but this will depend on the individual case.

2.3.1 Symmetric coset spaces

The case $N = 1$ corresponds to symmetric target spaces and has been extensively considered in the literature, see for example [51–54, 41, 37]. In this thesis we will be particularly interested in the case where the sigma model is conformal, in other words its β-function identically vanishes for all values of the coupling. The requirement that the β-function of the sigma model vanishes at one-loop level implies that the numerator group must have vanishing dual Coxeter number. This condition excludes the classical Lie groups and leads us to consider Lie supergroups. Among the basic simple supergroups only

$$\mathrm{OSP}(2P+2|2P), \qquad \mathrm{PSL}(P|P), \qquad \mathrm{D}(2,1;\alpha) \tag{2.14}$$

fulfill this requirement. Not all their symmetric cosets lead to conformal field theories beyond one-loop. In the absence of worldsheet supersymmetry, it was found in [50] that higher-loop contributions vanish if and only if

$$\begin{aligned} \mathbf{Cas}_{\mathfrak{g}}(\mathfrak{g}) &= 0 \\ \mathbf{Cas}_{\mathfrak{h}_i}(\mathfrak{m}) &= 0, \end{aligned} \tag{2.15}$$

where $\mathbf{Cas}_{\mathfrak{g}}(R)$ denotes the value of the quadratic Casimir operator of $\mathfrak{g}$ on the representation R. Here we have also split the algebra of the denominator group into its simple summands $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$. The first condition is again the statement that the dual Coxeter number $g^\vee = \mathbf{Cas}_{\mathfrak{g}}(\mathfrak{g})/2$ of the numerator group has to vanish. The second requirement restricts the possible choices of symmetric quotients to [50, 55, 8]

$$\frac{\mathrm{OSP}(2P+2Q+2|2P+2Q)}{\mathrm{OSP}(2P+1|2P)\times\mathrm{OSP}(2Q+1|2Q)}$$
$$\frac{\mathrm{PSL}(P+Q|P+Q)}{\mathrm{SL}(P+1|P)\times\mathrm{SL}(Q-1|Q)} \tag{2.16}$$
$$\frac{\mathrm{PSL}(2P|2P)}{\mathrm{OSP}(2P|2P)}.$$

Additionally, the principal chiral model, which is a non-linear sigma model with a group manifold as target space, on the supergroups (2.14) can be included in this list by observing that

$$G \simeq \frac{G\times G}{G}, \tag{2.17}$$

where the quotient is taken with respect to the diagonal right action. While equation (2.17) may seem trivial, it shows that every group can be thought of as a symmetric space. In this sense, the supergroups (2.14) satisfy the conditions (2.15) and the principal chiral model on these groups is conformal [8, 56, 18].

The first and second of the families (2.16) are real and complex super-Grassmannians. They include the odd dimensional superspheres $S^{2n+1|2n}$ and complex projective superspaces $\mathbb{C}\mathrm{P}^{n-1|n}$. These are the only spaces in the lists (2.14) and (2.16) that are simultaneously compact and Riemann.

2.3.2 Semi-symmetric cosets

The case $N = 2$ has received some attention as it includes AdS backgrounds that are of interest in the context of the AdS/CFT correspondence. Coset spaces defined by a $\mathbb{Z}_4$-automorphism are also referred to as semi-symmetric spaces. Conditions for the vanishing of the β-function at one-loop level were found in [18] and generalized to arbitrary N in [57]. For $N > 1$ we will assume that the $\mathbb{Z}_{2N}$ grading induced by Θ is consistent with the $\mathbb{Z}_2$ grading of the superalgebra, i.e. that $\mathfrak{m}_{2A} \subset \mathfrak{g}_{\bar 0}$ and $\mathfrak{m}_{2A+1} \subset \mathfrak{g}_{\bar 1}$. This condition is not necessary, and would indeed be too restrictive, for the symmetric case $N = 1$. For $N = 2$ it ensures that the bosonic base of the supermanifold is a symmetric space.

To exhibit the form of the action considered in [57], we generalize the notation for the projectors onto the eigenspaces of Θ, $P_A : \mathfrak{g} \to \mathfrak{m}_A$. The sigma model action is then given by

$$S_{G/H} = \frac{R^2}{2\pi} \int_\Sigma d^2z \sum_{A=1}^{2N-1} (p_A + iq_A)\big(P_A(J), P_{2N-A}(\bar{J})\big), \tag{2.18}$$

where we have introduced complex coordinates on the Euclidean worldsheet and scaled out an overall normalization. The overall coefficient R is the radius of the target space and plays the role of the inverse coupling. The coefficients p_A and q_A are required to obey the consistency conditions

$$p_A = p_{2N-A} \qquad \text{and} \qquad q_A = -q_{2N-A}. \tag{2.19}$$

The symmetric part corresponds to the metric, while the antisymmetric part corresponds to the B-field. In [57] it was found that the sigma model defined by the action (2.18) is conformal at one-loop level if

$$p_A = 1 \quad \forall A \qquad \text{and} \qquad q_A = 1 - \frac{A}{N} \tag{2.20}$$

and the dual Coxeter number of the numerator G vanishes. Note that the condition for one-loop conformal invariance of the sigma model (2.18) coincides with the condition for its classical integrability [10].

The action (2.18) with coefficients (2.20) is somewhat unusual as the kinetic terms for the fermions are of second order in the derivatives, as opposed to the more common first order. It is also not the action which is used in the Green-Schwarz formulation of string theory that we discussed in the introduction. The Green-Schwarz action (1.1) for $\mathbb{Z}_4$ cosets is obtained by setting $N = 2$ and $p_1 = 0$ which leaves no kinetic term for the fermions. The latter case corresponds to a degenerate metric on the supermanifold G/H and quantization requires gauge fixing of the resulting κ-symmetry.

2.4 WZNW models

Sigma models should not be discussed in isolation. Topological terms that respect the symmetry can be added to the action. The most well known case yields the Wess-Zumino-Novikov-Witten model on a (super)group by adding a Wess-Zumino (WZ) term to the action of the principal chiral model. If the structure constants of the (super)group G are f^{abc}, the Wess-Zumino term is

$$S_{\text{WZ}} = \int_B d^3\sigma \, \epsilon_{\mu\nu\rho} f^{abc} J_a^\mu J_b^\nu J_c^\rho, \tag{2.21}$$

where $\epsilon_{\mu\nu\rho}$ is the three-dimensional Levi-Civita symbol normalized as $\epsilon_{123} = 1$. The integration proceeds over an auxiliary three-dimensional manifold B such that $\partial B = \Sigma$. The extension of the fields to B may introduce an ambiguity into the definition of the action (2.21). In this case a quantization condition is imposed on the coupling constant in order to render the path integral well defined.

By construction, the principal chiral model and the Wess-Zumino term have a global $G \times G$ symmetry given by left and right multiplication. If the coupling constants of the principal chiral model and the Wess-Zumino term are fine-tuned, the symmetry of the model is enhanced to an affine $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ symmetry. In terms of the currents the affine $\hat{\mathfrak{g}}$-algebra is given as

$$J_a(z)J_b(w) = \frac{k(J_a, J_b)}{(z-w)^2} + \frac{f_{ab}{}^c J_c(w)}{z-w} + \text{non-sing.} \tag{2.22}$$

and the second copy is given in terms of the anti-holomorphic currents. The coefficient k is the level of the algebra. Note that in the case of the supergroups (2.14) the exact definition of the level in terms of the coupling constant in the action is somewhat subtle. Normally, the Killing form provides a canonical normalization of the metric. For the supergroups (2.14) the Killing form vanishes so that there is no canonical normalization. Since these subtleties are not relevant to our discussion, we suppress the coupling constant.

Apart from the aforementioned subtlety, WZNW models on supergroups can be treated in much the same way as on bosonic groups. In particular, the affine symmetry (2.22) implies a Virasoro symmetry given by the usual Sugawara construction. If $\mathfrak{g}$ is simple, the associated central charge c is

$$c = \frac{k\,\text{sdim}(\mathfrak{g})}{k + g^\vee}, \tag{2.23}$$

where $\text{sdim}(\mathfrak{g}) = \dim(\mathfrak{g}_{\bar{0}}) - \dim(\mathfrak{g}_{\bar{1}})$ is the superdimension of $\mathfrak{g}$ and $g^\vee$ is its dual Coxeter number. Observe that whenever $g^\vee = 0$ the central charge does not depend on the value of the level. This does not mean, however, that the field content of the theory is also independent of the level. Both the singular vectors of a representation as well as which representations occur in the first place do depend on the value of k, much as they do in the bosonic case. This can be easily seen by observing that the affine extension $\widehat{\mathfrak{g}_{\bar{0}}}$ of the bosonic subalgebra of $\mathfrak{g}$ is contained in $\hat{\mathfrak{g}}$. Here an additional peculiarity of supergroup WZNW models can be observed. Due to the indefinite signature of the metric, one part of $\widehat{\mathfrak{g}_{\bar{0}}}$ will generically have negative level. An affine algebra with negative level can be interpreted as arising from a WZNW model with non-compact target space.

Finally, principal chiral models on non-abelian bosonic groups are not conformal by themselves and need to be supplemented by the addition of the WZ term with an appropriately fine-tuned coefficient in order to cancel the anomaly. This is no longer true in the case of the groups (2.14).

2.5 An all-loop result for deformed WZNW models

In [39] current-current deformations of supergroup WZNW models were studied. In particular it was argued that the deformation by the operator

$$\omega(z,\bar{z}) = J^a(z)\bar{J}_a(\bar{z}) \tag{2.24}$$

is truly marginal, provided that the Lie supergroup has vanishing dual Coxeter number, i.e. that G is from the list (2.14). In the definition of ω the sum runs over all directions a in the Lie superalgebra $\mathfrak{g}$. The deformation breaks the affine symmetry. Since it does not even commute with the zero modes of the chiral currents, it also breaks the left and right $\mathfrak{g}$ symmetries. On the other hand, the sum of left and right zero modes does commute with the perturbing operator so that the deformed theory preserves the diagonal $\mathfrak{g}$ action.

Of course, under the perturbation with the operator (2.24) the conformal weight of fields can change, i.e. fields may develop an anomalous dimension which depends on the coupling g. In general, this anomalous dimension is difficult to compute, at least beyond the leading order in perturbation theory. Remarkably, for a special subset of fields, the authors of [39] managed to obtain an all order expression. In physics terminology, the fields for which this was possible are those that transform in maximally atypical, or $\frac{1}{2}$BPS, representations of the target space symmetry $\mathfrak{g}$. More precisely, the formulas of [39] hold for all indecomposable field multiplets of $\mathfrak{g}$ which contain a subrepresentation of non-zero superdimension. For such fields, the anomalous dimension reads

$$\delta_g^{(\infty)} h_{\mathrm{BPS}} = \frac{g}{2(1-k^2g^2)}\left[\mathbf{Cas}^D_{\mathfrak{g}}(\Lambda_{\mathrm{BPS}}) - (1-kg)\Big(\mathbf{Cas}^L_{\mathfrak{g}} + \mathbf{Cas}^R_{\mathfrak{g}}\Big)\right] . \tag{2.25}$$

Here $\mathbf{Cas}^{L/R}_{\mathfrak{g}}$ refers to the value of the quadratic Casimirs on the left and right representations in the unperturbed model, respectively. The superscript D means that the Casimir element is evaluated with respect to the diagonal action. We have placed the subscript 'BPS' on both sides of the equation to remind us that this formula should only be applied to fields that transform in maximally atypical representations Λ under the diagonal action. On the other hand, their transformation law with respect to the left- or right-action in the WZNW model is not constrained.

2.6 Intermezzo: Representations of $\mathfrak{u}(1|1)$

Before we proceed, we want to illustrate features that representations of Lie superalgebras exhibit. The reader who is familiar with the representation theory of Lie superalgebras may safely skip this section. It is intended to serve as a pedagogical example. As we mentioned in section 2.1, irreducible representations of Lie superalgebras may form indecomposables. To show that one cannot exclude these types of representations we work out an explicit example for the algebra $\mathfrak{u}(1|1)$.

The algebra $\mathfrak{u}(1|1)$ is generated by the even elements E and N and the odd elements $\psi^{\pm}$. The only non-trivial commutation relations are

$$[N, \psi^{\pm}] = \pm\psi^{\pm}, \qquad [\psi^{+}, \psi^{-}] = E. \tag{2.26}$$

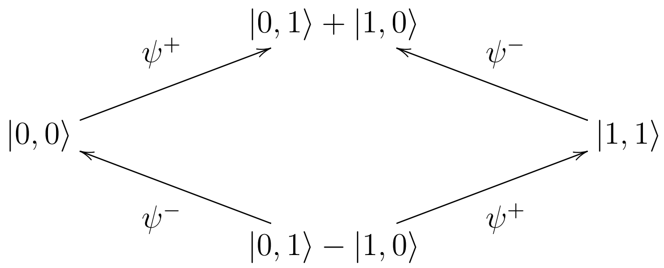
Recall that the bracket $[\cdot, \cdot]$ is graded anti-symmetric and we do not use different notation for the bracket when acting on even or odd generators.

Irreducible representations are labeled by the eigenvalues e and n of the bosonic generators on a lowest weight vector. For $e \neq 0$, we denote the irreducible representation $\langle e, n\rangle$. It is two-dimensional and spanned by the vectors $|0\rangle$ and $|1\rangle$ which are even and odd, respectively. The action of the generators is now defined as

$$\psi^{-}|0\rangle = 0, \quad \psi^{+}|0\rangle = |1\rangle \quad \text{and} \quad \psi^{-}|1\rangle = [\psi^{-}, \psi^{+}]|0\rangle = e|0\rangle. \tag{2.27}$$

Evidently, $\psi^{\pm}$ raise and lower the eigenvalue of N by 1. These representations are typical.

We now see why we excluded $e = 0$. If we set $e = 0$ in the definition (2.27) the representation becomes reducible, but indecomposable. The irreducible one dimensional subrepresentations are labeled by the eigenvalue of N and denoted $\langle n\rangle$. These are the irreducible atypical representations. To show that one cannot exclude indecomposalbe representations when dealing with Lie superalgebras, we take the tensor product of the two typical representations $\langle e, n\rangle$ and $\langle e', n'\rangle = \langle -e, n'\rangle$. Let us also introduce the shorthand $|a, b\rangle := |a\rangle \otimes |b\rangle$ with $a, b = 0, 1$. It is clear that $E|a, b\rangle = 0$. The action of the fermionic generators is given in the following diagram:



(2.28)

In following the action of the generators one needs to keep the grading of the states in mind. Since $E|a,b\rangle = 0$, one can only move along the arrows and not against them. Thus, the representation (2.28) is reducible but indecomposable. It has to be a projective representation since it was obtained from the tensor product of a typical representation, which is also projective. In this case, it is also obvious that the representation (2.28) cannot occur as a subrepresentation of a larger module since the odd generators square to zero. It is labeled $\mathcal{P}_{\langle n+n'+1\rangle}$. The projective representations $\mathcal{P}_{\langle n\rangle}$ are called projective covers. They are the smallest projective representations that contain the representation $\langle n\rangle$ as a subrepresentation. It is these projective covers that play a central role in the harmonic analysis on supergroups.

Chapter 3

The Spectrum of Coset Sigma Models

This chapter consists of three parts. In Section 3.1, we review the construction of vertex operators in sigma models on coset (super)spaces G/H that was presented in [38]. We will be making the same assumptions on the groups G and H as in Section 2.2. As was shown in [38], when the target-space is symmetric this choice of vertex operators diagonalizes the one-loop dilatation operator. The rest of this chapter will be devoted to the computation of the one-loop anomalous dimensions of vertex operators for $\mathbb{Z}_4$-coset spaces. Section 3.2 sets up the calculation by presenting the background field expansion. In Section 3.3 we then present the calculation of the diagonal terms of the one-loop dilatation operator. These turn out to be formally identical to the symmetric case. However, off-diagonal terms are also present. These are not computed here.

3.1 Construction of vertex operators

In this section we present the construction of vertex operators on homogeneous (super)spaces. We begin with a simple example in order to motivate the setup. While the construction of field operators reviewed in Subsection 3.1.2 holds for all coset models G/H, our discussion of the zero modes is limited to compact G.

3.1.1 Prologue: Vertex operators for flat targets

Let us motivate the prescription given [38] with a few comments on the usual vertex operators of a free boson, i.e. a sigma model on the coset space $S^1 = SO(2)/SO(1)$ with trivial denominator group $H = SO(1) = \{e\}$. As is well known, the space of such operators is spanned by

$$\Phi_{k;\mathbf{p},\overline{\mathbf{p}}}(z,\bar{z}) = e^{ik\theta(z,\bar{z})}\mathbf{p}_{\mathbf{m}}(j,\partial j,\dots)\overline{\mathbf{p}}_{\overline{\mathbf{m}}}(\bar{\jmath},\bar{\partial}\bar{\jmath},\dots)\,. \tag{3.1}$$

Here, $j = j(z)$ is the current $j = i\partial\theta$ and $\bar{\jmath}$ is of the same form but with a derivative $\bar{\partial}$ instead of ∂, i.e. $\bar{\jmath} = i\bar{\partial}\theta$. The object $\mathbf{p}_{\mathbf{m}}$ denotes the monomial

$$\mathbf{p}_{\mathbf{m}}(j,\partial j,\dots) = j^{m_1}(\partial j)^{m_2}\cdots$$

in j and its derivatives. The powers m_i are components of the multi-index $\mathbf{m} = (m_1, m_2, \dots)$ we have placed on $\mathbf{p}$. Of course, the definition of $\overline{\mathbf{p}}$ is similar, but with derivatives $\bar{\partial}$ instead of ∂. Note that the multi-index $\overline{\mathbf{m}}$ is independent of $\mathbf{m}$.

The operators $\exp(ik\theta)$ are associated to the zero modes of the free boson, i.e. there is one such operator for each function on the target space. For $\mathbf{m} = 0 = \overline{\mathbf{m}}$ we obtain the usual tachyon vertex operators. The choice $\mathbf{m} = (1,0,0,\dots) = \overline{\mathbf{m}}$ corresponds to the vertex operators for massless states etc.

3.1.2 Vertex operators for G/H

In generalizing this discussion to non-trivial coset models G/H we must address how to replace the currents j and $\bar{\jmath}$, the *tail* monomials $\mathbf{p}$ and $\overline{\mathbf{p}}$ and the *zero mode* contributions $\exp(ikX)$.

Let us begin with the fields j and $\bar{\jmath}$. One could imagine to simply take derivatives of coordinate fields θ_J that are associated with some choice of coordinates on G/H. While this works just fine for a flat target space, it is not the smartest choice for curved backgrounds. Instead, we shall adopt the definition

$$j_\alpha := E^J_\alpha(\theta)\partial\theta_J \quad , \quad \bar{\jmath}_\alpha := E^J_\alpha(\theta)\bar{\partial}\theta_J \tag{3.2}$$

where E^J_α is the Vielbein of the coset space. Equivalently, if we think of the points on G/H as being parametrized by orbits of group elements $g \in G$ under the right action of H, we can also construct j and $\bar{\jmath}$ as

$$j_\alpha = \left(g^{-1}\partial g, t_\alpha\right)\,, \qquad \bar{\jmath}_\alpha = \left(g^{-1}\bar{\partial} g, t_\alpha\right)\,. \tag{3.3}$$

Here, t_α runs through a basis in the quotient space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. Here we have made contact with the formulation presented in Chapter 2 and we have

$$j = j_\alpha t^\alpha = P(J). \tag{3.4}$$

As discussed in section 2.2, the space $\mathfrak{m}$ carries an action of the denominator Lie (super)algebra $\mathfrak{h}$. Its dimension coincides with the dimension of G/H. Note that there is one crucial difference with respect to the flat target S^1, namely our fields j and $\bar{\jmath}$ transform non-trivially under the action of the denominator algebra. Of course, physical fields of the coset model must be invariant. Hence, it will be important to keep track of how the composite fields we are about to construct transform under $\mathfrak{h}$.

A field can contain arbitrary products of j_α and $\bar{\jmath}_\alpha$ and their derivatives, just as for flat targets. Since the multiplets (j_α) and $(\bar{\jmath}_\alpha)$ transform in the representation $\mathfrak{m}$ of $\mathfrak{h}$, we can build tails in any subrepresentation $[\mu]$ that appears in some tensor power of $\mathfrak{m}$. More precisely, we can pick two multi-indices $\mathbf{m}$ and $\overline{\mathbf{m}}$ as in our discussion of the compactified free boson and then choose two intertwiners

$$\mathbf{P}_{\mu,\mathbf{m}} : \bigotimes_i \mathfrak{m}^{\odot m_i} \to [\mu]\,, \quad \overline{\mathbf{P}}_{\bar{\mu},\overline{\mathbf{m}}} : \bigotimes_i \mathfrak{m}^{\odot \bar{m}_i} \to [\bar{\mu}]. \tag{3.5}$$

Here, we used $\mathfrak{m}^{\odot m}$ to denote the m-fold (graded) symmetric tensor power of $\mathfrak{m}$. Given any such intertwiner, we construct the tail factor

$$\mathbf{p}_{\mu,\mathbf{m}}(j,\partial j,\dots) = \mathbf{P}_{\mu,\mathbf{m}}\left[j^{\odot m_1}\otimes(\partial j)^{\odot m_2}\otimes\cdots\right] = \mathbf{P}_{\mu,\mathbf{m}} j_{\mathbf{m}} \tag{3.6}$$

and similarly for the second contribution that involves $\bar{\jmath}$ and its derivatives with respect to $\bar{\partial}$. We have used tensor products and powers instead of ordinary ones to remind us that j is a multi-component object. Note that there is a finite number of intertwiners $\mathbf{P}_{\mu,\mathbf{m}}$ and $\overline{\mathbf{P}}_{\bar{\mu},\overline{\mathbf{m}}}$ for any given choice of $\mathbf{m}$ and $\overline{\mathbf{m}}$. This finite choice has no analogue in a flat background.

Having discussed the tail of our vertex operators, we also need to address the zero mode factors. In the compactified free boson the zero mode contribution was a function on the target space. Functions on the coset space G/H can be thought of as H-invariant functions on the group G. But since our tail factors transform non-trivially under H, it seems natural to admit zero mode contributions whose transformation behavior under the right action of H on G is non-trivial as well. More precisely, for any given representation S_λ of H on the carrier space $\mathcal{S}_\lambda$ let us consider the following space of $\mathcal{S}_\lambda$-valued functions on G,

$$\Gamma_\lambda = \Gamma_\lambda(G/H) = \{F \in L_2(G)\otimes\mathcal{S}_\lambda : F(gh) = S_\lambda(h^{-1})F(g)\ \forall h\in H\}\,. \tag{3.7}$$

Elements of the linear space Γ_λ may be considered as sections $\mathcal{D}_{\Lambda\lambda}$ in a homogeneous vector bundle on G/H [58]. The sections then transform in a representation Λ of G. We will analyse the structure of these vector bundles in the next subsection. We will denote the vertex operators that are associated to the sections

$$V_{\Lambda\lambda}(z,\bar{z}) := V[\mathcal{D}_{\Lambda\lambda}](z,\bar{z}). \tag{3.8}$$

At this point we have discussed three ingredients of our vertex operators, namely the tail factors $\mathbf{p}_{\mu,\mathbf{m}}$ and $\overline{\mathbf{p}}_{\overline{\mu},\overline{\mathbf{m}}}$ along with the zero mode contribution $\mathcal{D}_{\Lambda\lambda} \in \Gamma_\lambda$. These transform in the representations $\mu, \overline{\mu}$ and λ of the denominator algebra $\mathfrak{h}$. Obviously, a physical field in the coset model must be $\mathfrak{h}$ invariant. Hence, we must glue our three ingredients with an intertwiner

$$\mathbf{C}^{\lambda\mu\overline{\mu}} : [\lambda] \otimes [\mu] \otimes [\overline{\mu}] \ \rightarrow \ \mathbb{C} \tag{3.9}$$

from the triple tensor product between the representations $[\lambda], [\mu]$ and $[\overline{\mu}]$ of the denominator algebra $\mathfrak{h}$ to the complex numbers. Fields of the coset model now take the form

$$\Phi_{\mathbf{\Lambda}}(z,\bar{z}) = \mathbf{d}^{\lambda\mu\overline{\mu}}\big(V_{\Lambda\lambda} \otimes \jmath_{\mathbf{m}} \otimes \bar{\jmath}_{\overline{\mathbf{m}}}\big)(z,\bar{z}), \qquad \mathbf{\Lambda} := (\Lambda, \lambda, \mu, \overline{\mu}) \tag{3.10}$$

where we also defined $\mathbf{d}^{\lambda\mu\overline{\mu}} := \mathbf{C}^{\lambda\mu\overline{\mu}}\big(\mathbf{id}_\lambda \otimes \mathbf{P}_{\mu,\mathbf{m}} \otimes \overline{\mathbf{P}}_{\overline{\mu},\overline{\mathbf{m}}}\big)$. By construction, these fields are invariant under the action of the denominator group H. On the other hand, the action of the numerator group G is non-trivial. It is determined by the way the section $V_{\Lambda\lambda}$ transforms. The label Λ is the curved space analogue of the linear momentum k in a circular target S^1.

The labels $(\Lambda, \lambda, \mu, \overline{\mu})$ we have placed on the symbol Φ do not keep track of all the freedom we have in the construction of vertex operators. In order to count all possible fields of the coset model one needs to count the intertwiners $\mathbf{P}, \overline{\mathbf{P}}$ and $\mathbf{C}$ that were introduced in eqs. (3.5) and (3.9), respectively. In addition, there is often some freedom in the choice of the section $V_{\Lambda\lambda} \in \Gamma_\lambda$. While the number of intertwiners may be determined straightforwardly from the fusion rules of the Lie (super)algebra $\mathfrak{h}$, the space of sections in homogeneous vector bundles requires input from harmonic analysis. We will analyse the space Γ_λ in the next subsection. For $\mathrm{O}(N)$ vector models, i.e. the coset sigma models with target space $\mathrm{O}(N)/\mathrm{O}(N-1)$, the space of fields has been counted in [38] and the result was shown to agree with other descriptions of the field space for these models.

3.1.3 Homogeneous vector bundles on G/H

As we explained in the previous subsection, a good control over vertex operators of coset models requires some knowledge about sections in homogeneous vector bundles over G/H and their transformation behavior under the (left) action of G. Our main goal in this subsection is to explain the decomposition

$$\Gamma_\lambda \cong \sum_\Lambda n_{\Lambda\lambda}\,[\Lambda]\,. \tag{3.11}$$

Here, the linear space Γ_λ is considered as a representation of the numerator Lie (super)algebra $\mathfrak{g}$. The summation on the right hand side runs over irreducible representations $[\Lambda]$ of this algebra. Let us stress that for Lie superalgebras, the sum is not direct, at least not in general. We will return to this issue below.

In the expansion (3.11), each summand $[\Lambda]$ appears with some multiplicity $n_{\Lambda\lambda}$. Following standard mathematical notation, we shall also write

$$n_{\Lambda\lambda} = [\Gamma_\lambda : \mathcal{S}_\Lambda] \tag{3.12}$$

for the number of times a given irreducible representation $\mathcal{S}_\Lambda$ of $\mathfrak{g}$ appears in (the decomposition series of) the space Γ_λ of sections. It is a central result from harmonic analysis of compact supergroups that

$$[\,\Gamma_\lambda\,:\,\mathcal{S}_\Lambda\,] = [\,\mathcal{P}_\Lambda|_{\mathfrak{h}}\,:\,\mathcal{P}_\lambda\,]\,. \tag{3.13}$$

The objects $\mathcal{P}_\Lambda$ and $\mathcal{P}_\lambda$ denote representations of the Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. These particular representations are called projective covers, see e.g. [59, 27] for a precise definition and more background. They coincide with the irreducible representations $\mathcal{S}_\Lambda$ and $\mathcal{S}_\lambda$ when no shorting conditions are satisfied, i.e. when both Λ and λ are non-BPS. The case of BPS (or atypical) multiplets will be discussed in more detail below. After restriction to $\mathfrak{h} \subset \mathfrak{g}$, the representation $\mathcal{P}_\Lambda$ gives rise to a representation $\mathcal{P}_\Lambda|_{\mathfrak{h}}$ of $\mathfrak{h}$. The number on the right hand side of equation (3.13) denotes the number of times the representation $\mathcal{P}_\lambda$ appears in the representation $\mathcal{P}_\Lambda|_{\mathfrak{h}}$.

All this might seem a bit abstract at first. So, let us briefly illustrate the content of eq. (3.13) for the coset space $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. In this case, there exists an infinite set of complex line bundles which are parametrized by the monopole number $k \in \mathbb{Z}$. This number and hence the associated bundles are in one-to-one correspondence with irreducible representations $\mathcal{S}_k$ of the denominator group $H = \mathrm{U}(1)$. For monopole number $k = 0$ we are dealing with the trivial line bundle, i.e. with functions on S^2. Of course we know very well how the space of functions decomposes under the action of $\mathfrak{su}(2)$: Each integer spin representation appears with multiplicity one. We may recover this fact from our formula (3.13) as follows. The space of functions on S^2 is associated to the label $\lambda = 0$. We want to know how many times an irreducible representation $\mathcal{S}_\Lambda = \mathcal{S}_j$ of $\mathfrak{su}(2)$ appears in the decomposition of Γ_0. According to eq. (3.13), this number is given by

$$[\Gamma_0 : \mathcal{S}_j] = [\mathcal{S}_j|_{\mathrm{U}(1)} : \mathcal{S}_0] = \begin{cases} 1 & \text{for} \quad j \in \mathbb{N} \\ 0 & \text{for} \quad j \in \mathbb{N} + \frac{1}{2} \end{cases} . \tag{3.14}$$

Here $\mathcal{S}_0$ denotes the trivial representation of $\mathfrak{h}$. For bosonic Lie groups, we do not have to distinguish between projective covers $\mathcal{P}_j$ and irreducibles, i.e. $\mathcal{S}_j = \mathcal{P}_j$. The second equality follows from the fact that the spin j representation $\mathcal{S}_j$ contains exactly one state on which the generator J^3 of the $\mathfrak{u}(1) \subset \mathfrak{su}(2)$ has zero eigenvalue if and only if j is integer. For non-trivial monopole line bundles, the evaluation proceeds along the same lines. In this case the space Γ_k of sections contains each integer spin representation $\mathcal{S}_j$ satisfying $j \geq k$ with multiplicity one.

The only additional complication we have to deal with in applying eq. (3.13) to superspaces comes from the distinction between irreducibles and projective covers. For typical (long) multiplets $\mathcal{S}_\Lambda$ of a Lie superalgebra $\mathfrak{g}$, the projective cover $\mathcal{P}_\Lambda$ agrees with $\mathcal{S}_\Lambda = \mathcal{P}_\Lambda$. But if $\mathcal{S}_\Lambda$ is an atypical (short) multiplet then $\mathcal{P}_\Lambda \neq \mathcal{S}_\Lambda$ is an indecomposable representation. It should be considered as a very specific 'composite' representation that is built from several short multiplets. For the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(4|2)$ the projective covers are discussed explicitly in appendix B. Of course, short representations of the denominator algebra $\mathfrak{h}$ can also be combined into projective covers, see appendix C where the projective covers for $\mathfrak{osp}(3|2)$ are discussed. Let us finally mention that upon restriction from $\mathfrak{g}$ to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a projective cover $\mathcal{P}_\Lambda$ decomposes into a direct sum of projective covers $\mathcal{P}_\lambda$. Hence, the numbers on the right hand side of eq. (3.13) are well defined. We shall compute them for homogeneous vector bundles on the supersphere $S^{3|2}$ later on.

Let us briefly mention one simple example that can be used to illustrate how important the distinction between irreducibles and projective covers is. To this end we consider the homogeneous vector bundle Γ_{ad} on the supersphere $S^{3|2}$ that is associated with the adjoint representation of the denominator algebra $\mathfrak{osp}(3|2)$. It turns out that this bundle contains two multiplets of sections which transform in the adjoint representation $\mathcal{S}_{\mathrm{Ad}}$ of the numerator algebra $\mathfrak{osp}(4|2)$, i.e. $[\Gamma_{\mathrm{ad}} : \mathcal{S}_{\mathrm{Ad}}] = 2$. On the other hand, the adjoint representation of $\mathfrak{osp}(4|2)$ is 17-dimensional and that of $\mathfrak{osp}(3|2)$ is 12-dimensional. Hence, for dimensional reasons, the restriction of $\mathcal{S}_{\mathrm{Ad}}$ to $\mathfrak{osp}(3|2)$ contains $\mathcal{S}_{\mathrm{ad}}$ only once,

$$2 = [\mathcal{P}_{\mathrm{Ad}}|_{\mathfrak{h}}, \mathcal{P}_{\mathrm{ad}}] \neq [\mathcal{S}_{\mathrm{Ad}}|_{\mathfrak{h}}, \mathcal{S}_{\mathrm{ad}}] = 1. \tag{3.15}$$

This example demonstrates that harmonic analysis on superspaces requires a bit of extra care precisely because of the existence of BPS representations.

Before we conclude this subsection let us stress once more that formula (3.13) is restricted to compact (super)algebras. This does not mean that similar control of homogeneous vector bundles can not be achieved when G is non-compact. As long as H is compact, one can continue to derive results on the decomposition of homogeneous vector bundles from the harmonic analysis of G. So, if the latter is understood, homogeneous vector bundles pose no additional problems. When H in non-compact, however, normalizable sections of on G/H are no longer obtained from normalizable functions on G and hence cosets with non-compact denominator require an independent analysis. Nevertheless, the decomposition of homogeneous vector bundles is known in many concrete examples.

3.2 Background field expansion

In this section we define the background field expansion that is used in the calculations. We begin with the expansion of the action to first order before we present the expansion of the vertex operators.

3.2.1 One-loop action

For the one-loop computation of anomalous dimensions we need to expand the action to leading order in the background field expansion. In order to do so, we introduce the coordinates

$$\imath : G/H \to G \;, \qquad g_0 e^{\phi} H \mapsto g_0 e^{\phi} \;, \tag{3.16}$$

where $\phi \in \mathfrak{m}$. The expansion of the currents j in these coordinates is

$$j = Pe^{-\phi}\partial e^{\phi} = P\left[\partial\phi - \frac{1}{2}\left[\phi\,,\,\partial\phi\right] + \frac{1}{6}\left[\phi\,,\,\left[\phi\,,\,\partial\phi\right]\right]\right] + \cdots \tag{3.17}$$

and similarly for $\bar{\jmath}$. Let us introduce the notation $\phi_A := P_A\phi = t_i^A\phi_A^i$, where t_i^A with $i = 1, \ldots, \dim \mathfrak{g}_A$ denotes a basis of $\mathfrak{g}_A$. Note that the objects ϕ_A are Grassmann even by construction. Hence, in working with ϕ_A we do not have to worry about the grading.

The projected currents $P_A j$ that appear in the action can now be rewritten as

$$P_A j = \partial\phi_A - \frac{1}{2}\sum_{\substack{B+C\equiv A\\ B,C\neq 0}}\left[\phi_B\,,\,\partial\phi_C\right] + \frac{1}{6}\sum_{\substack{B+C+D\equiv A\\ B,C,D\neq 0}}\left[\phi_B\,,\,\left[\phi_C\,,\,\partial\phi_D\right]\right] + \cdots \tag{3.18}$$

Inserting this expression into the action (2.18) and expanding $S \sim S_0 + S_1$ up to the leading non-trivial order in the coupling we obtain

$$S_0 = \frac{R^2}{2} \int_\Sigma \frac{d^2 z}{\pi} \sum_{A=1}^{3} p_A \left(\partial\phi_A, \bar\partial\phi_{A'}\right) \tag{3.19}$$

for the tree-level (free) action and $S_1 = S_1^s + S_1^a$ where the symmetric part of the one-loop interaction is given by

$$\begin{aligned} S_1^s \;=\; & \frac{R^2}{2} \int_\Sigma \frac{d^2 z}{\pi} \Big[\sum_{A+B+C\equiv 0} \frac{p_A}{2} \big\{ - \big(\partial\phi_A, [\phi_B\,,\,\bar\partial\phi_C]\big) - \big([\phi_B\,,\,\partial\phi_C]\,,\bar\partial\phi_A\big) \big\} \\ & + \sum_{A+B+C+D\equiv 0} \frac{p_A}{6} \big\{ \big(\partial\phi_A, [\phi_B\,,\,[\phi_C\,,\,\bar\partial\phi_D]]\big) + \big([\phi_B\,,\,[\phi_C\,,\,\partial\phi_D]]\,,\bar\partial\phi_A\big) \big\} \\ & + \sum_A \sum_{\substack{B+C\equiv A \\ D+E\equiv A'}} \frac{p_A}{4} \big([\phi_B\,,\,\partial\phi_C]\,,[\phi_D\,,\,\bar\partial\phi_E]\big) \Big] \,, \end{aligned} \tag{3.20}$$

while the antisymmetric part takes the form

$$\begin{aligned} S_1^a \;=\; & \frac{R^2}{2} \int_\Sigma \frac{d^2 z}{\pi} \Big[\sum_{A+B+C\equiv 0} i\frac{q_A}{2} \big\{ - \big(\partial\phi_A, [\phi_B\,,\,\bar\partial\phi_C]\big) + \big([\phi_B\,,\,\partial\phi_C]\,,\bar\partial\phi_A\big) \big\} \\ & + \sum_{A+B+C+D\equiv 0} \frac{iq_A}{6} \big\{ \big(\partial\phi_A, [\phi_B\,,\,[\phi_C\,,\,\bar\partial\phi_D]]\big) - \big([\phi_B\,,\,[\phi_C\,,\,\partial\phi_D]]\,,\bar\partial\phi_A\big) \big\} \\ & - \sum_A \sum_{\substack{B+C\equiv A \\ D+E\equiv A'}} \frac{iq_A}{4} \big([\phi_B\,,\,\partial\phi_C]\,,[\phi_D\,,\,\bar\partial\phi_E]\big) \Big] \,. \end{aligned} \tag{3.21}$$

From the tree-level action S_0 we read off that the free 2-point correlation function is given by

$$\langle\, \phi_A(z,\bar z) \otimes \phi_{A'}(w,\bar w)\,\rangle_0 = -\frac{1}{R^2 p_A} \ln \left|\frac{z-w}{\epsilon}\right|^2 \sum_{i=1}^{\dim \mathfrak{g}_A} t_i^A \otimes t^{A',i} \,. \tag{3.22}$$

In our analysis below we shall split the one-loop terms of the interaction into three vertices and four vertices,

$$S_1 = \int \frac{d^2 z}{\pi} \left(\Omega_3(z,\bar z) + \Omega_4(z,\bar z)\right) \,. \tag{3.23}$$

Once again, we can then split these vertices into a symmetric and an anti-symmetric part, i.e. $\Omega_p = \Omega_p^s + \Omega_p^a$. If we consider the one-loop conformal case (2.20), the previous expressions simplify drastically. In particular the first row of (3.20) cancels so that

$$\Omega_3^s(z,\bar{z}) = 0 \tag{3.24}$$

and the first row of (3.21) takes the form

$$\begin{aligned}\Omega_3^a(z,\bar{z}) = R^2\frac{i}{2}\big\{ &-\frac{1}{2}(\partial\phi_1,[\phi_2,\bar{\partial}\phi_1]) + \frac{1}{2}(\partial\phi_3,[\phi_2,\bar{\partial}\phi_3]) \\ &-\frac{1}{4}(\partial\phi_2,[\phi_1,\bar{\partial}\phi_1]) + \frac{1}{4}(\partial\phi_2,[\phi_3,\bar{\partial}\phi_3]) \\ &-\frac{1}{4}(\partial\phi_1,[\phi_1,\bar{\partial}\phi_2]) + \frac{1}{4}(\partial\phi_3,[\phi_3,\bar{\partial}\phi_2])\big\}\,.\end{aligned} \tag{3.25}$$

For later convenience is important to notice that we can simplify the expression for the three vertex Ω_3 by adding total derivatives. One possibility is

$$\begin{aligned}\Omega_3^{a,(1)} &= \Omega_3(z,\bar{z})^a + R^2\frac{i}{8}\partial\big((\phi_2,[\phi_1,\bar{\partial}\phi_1]) - (\phi_2,[\phi_3,\bar{\partial}\phi_3])\big) \\ &\quad + R^2\frac{i}{8}\bar{\partial}\big((\partial\phi_1,[\phi_1,\phi_2]) - (\partial\phi_3,[\phi_3,\phi_2])\big) \\ &= -R^2\frac{i}{2}(\partial\phi_1,[\phi_2,\bar{\partial}\phi_1]) + R^2\frac{i}{2}(\partial\phi_3,[\phi_2,\bar{\partial}\phi_3])\,.\end{aligned} \tag{3.26}$$

Let us also briefly discuss the four vertex Ω_4, and in particular its symmetric part. It is convenient to further divide Ω_4^s into two parts, $\Omega_4 = \Omega_4^{s1} + \Omega_4^{s2}$. The first one contains the terms that appear in the second line of eq. (3.20) while in the second one we collect the terms from the third row. Considering that

$$\begin{aligned}(\partial\phi_A,[\phi_B,[\phi_C,\bar{\partial}\phi_D]]) &= -([\phi_B,\partial\phi_A],[\phi_C,\bar{\partial}\phi_D]) \\ &= +([\phi_C[\phi_B,\partial\phi_A]],\bar{\partial}\phi_D)\end{aligned} \tag{3.27}$$

and taking into account eq. (2.20), we can write Ω_4^{s1} as

$$\Omega_4^{s1} = -\frac{R^2}{6}\sum_{A+B+C+D\equiv 0}([\phi_B,\partial\phi_A],[\phi_C,\bar{\partial}\phi_D])\,. \tag{3.28}$$

After some partial integration it takes the form

$$\begin{aligned}\Omega_4^{s1} = -\frac{R^2}{6}\big[&\left([\phi_2\,,\,\partial\phi_2]\,,\,[\phi_2\,,\,\bar{\partial}\phi_2]\right) + 3\left([\phi_1\,,\,\partial\phi_3]\,,\,[\phi_3\,,\,\bar{\partial}\phi_1]\right) \\ &+3\left([\phi_3\,,\,\partial\phi_1]\,,\,[\phi_1\,,\,\bar{\partial}\phi_3]\right) + 3\left([\phi_1\,,\,\partial\phi_3]\,,\,[\phi_2\,,\,\bar{\partial}\phi_2]\right) \\ &+3\left([\phi_3\,,\,\partial\phi_1]\,,\,[\phi_2\,,\,\bar{\partial}\phi_2]\right) + \tilde{\Omega}_4^{s1}\big]\end{aligned} \tag{3.29}$$

where $\tilde{\Omega}_4^{s1}$ contains additional terms that do contribute to the diagonal part of the one-loop dilatation operator. In the conformal case, the second contribution Ω_4^{s2} to the symmetric part of the four vertex becomes

$$\Omega_4^{s2} = \frac{R^2}{8} \sum_{A \neq 0} \sum_{\substack{B+C \equiv A \\ D+E \equiv A'}} \left([\phi_B, \partial \phi_C], [\phi_D, \bar{\partial} \phi_E] \right) . \tag{3.30}$$

This concludes our discussion of the sigma model action and its background field expansion. We still need to take a look at the fields before we can compute the anomalous dimensions.

3.2.2 Expansion of vertex operators

The one-loop expansion of a general coset field around an arbitrary point $g_0 H$ may be written schematically as

$$\Phi_\Lambda(z, \bar{z} \,|\, g_0) = \mathsf{d} \circ \big(V^{(0)} + V^{(1)} \ldots \big) \otimes \big(j^{(0)}_{\mathbf{m}} + j^{(1)}_{\mathbf{m}} \ldots \big) \otimes \big(\bar{\jmath}^{(0)}_{\bar{\mathbf{m}}} + \bar{\jmath}^{(1)}_{\bar{\mathbf{m}}} \cdots \big) \ , \tag{3.31}$$

Let us spell out concrete expressions for the various terms in the expansion. For the zero mode contribution $V_{\Lambda\lambda} = V$ the leading terms in the background field expansion read

$$\begin{aligned} V &= \mathcal{D}_{\Lambda\lambda}(g_0 e^{t\phi(z,\bar{z})}) \big|_{t=1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} L_\Lambda^n \big(\mathrm{Ad}_{g_0} \phi(z, \bar{z}) \big) \mathcal{D}_{\Lambda\lambda}(g_0) \\ &= \mathcal{D}_{\Lambda\lambda}(g_0) + L_\Lambda \big(\mathrm{Ad}_{g_0} \phi(z, \bar{z}) \big) \mathcal{D}_{\Lambda\lambda}(g_0) + \cdots \\ &= V^{(0)} + V^{(1)} + \cdots \ . \end{aligned} \tag{3.32}$$

Here, $L_\Lambda(X)$ denotes the denotes the left-action of an element $X \in \mathfrak{g}$ on the section, which transforms in the representation Λ, and Ad is the adjoint action of the Lie (super)group on its algebra, see [38] for more details. By definition, $V^{(0)}$ is the constant term in the expansion around g_0. For later use note also that the left-action of an element $Y \in \mathfrak{h} \subset \mathfrak{g}$ is related to its right-action $R_\lambda(Y)$ by

$$L_\Lambda(\mathrm{Ad}_{g_0} Y) \mathcal{D}_{\Lambda\lambda}(g_0) = -R_\lambda(\mathrm{Ad}_{g_0} Y) \mathcal{D}_{\Lambda\lambda}(g_0) \, . \tag{3.33}$$

Similarly, we can also expand the tail factor. For the current j one finds that

$$j = \partial \phi - \frac{1}{2} \sum_{\substack{B+C \neq 0 \\ B,C \neq 0}} [\phi_B \, , \, \partial \phi_C] + \ldots \ . \tag{3.34}$$

Note that in the case of symmetric spaces, i.e. when the index A runs over $A = 0, 1$ only, the leading non-trivial term vanishes because the sum of $B = 1$ and $C = 1$ is $B + C = 0$ mod 2. Hence, while this term did not appear in [38], we need to consider it in dealing with semi-symmetric spaces. When the expansion of the current is inserted into expression (3.6) it gives

$$\jmath_{\mathbf{m}} := \jmath_{\mathbf{m}}^{(0)} + \jmath_{\mathbf{m}}^{(1)} + \cdots = \bigotimes_{\rho=1}^{r} \partial^{m_\rho}\phi + \\ - \sum_{k=1}^{r} \bigotimes_{\rho=1}^{k-1} \partial^{m_\rho}\phi \otimes \partial^{m_k - 1} \left[\frac{1}{2} \sum_{\substack{B+C\neq 0 \\ B,C\neq 0}} [\phi_B \, , \, \partial\phi_C] \right] \otimes \bigotimes_{\rho=k+1}^{r} \partial^{m_\rho}\phi \cdots \tag{3.35}$$

and similarly for the tail factors $\bar{\jmath}_{\overline{\mathbf{m}}}$ in which all the unbarred labels m are replaced by barred ones and anti-holomorphic derivatives $\bar{\partial}$ appear instead of the holomorphic ones.

3.3 One-loop anomalous dimensions

While our construction of vertex operators in coset sigma models was completely general and the property (3.13) holds for all homogeneous vector bundles on quotients G/H of a compact Lie (super)group G, the following results on the one-loop corrections to the spectrum of coset models had only been derived for symmetric (super)spaces. After reviewing these results we will begin with their extension to generalized symmetric spaces, including those relevant for the AdS/CFT correspondence.

3.3.1 The symmetric case

The computations carried out in [38] show that the one-loop anomalous dimensions depend only on the representation labels $\Lambda, \lambda, \mu, \overline{\mu}$ and not on the intertwiners $\mathbf{P}, \overline{\mathbf{P}}$ and $\mathbf{C}$ that enter the construction of fields (3.10) in the coset model. This is why we labeled our fields Φ by a subscript that makes no reference to the precise choice of intertwiners.

At zero sigma model coupling, i.e. for $R = \infty$, the sigma model fields possess their naive dimensions $(h_\infty, \bar{h}_\infty)$ that are given by the number of derivatives,

$$h_\infty = \sum_{j=1}^{\infty} j\, m_j \quad , \quad \bar{h}_\infty = \sum_{j=1}^{\infty} j\, \overline{m}_j \, , \tag{3.36}$$

where only finitely many of the m_j are non-zero. Once we turn on the interaction, these scaling weights are shifted by the so called anomalous dimension $\delta_R h$, i.e. at some finite value of the coupling R the scaling weights have the form

$$\big(h(R), \bar{h}(R)\big) = (h_\infty + \delta_R h, \bar{h}_\infty + \delta_R h)\,. \tag{3.37}$$

According to [38], the leading contribution to the anomalous dimension takes the form

$$\delta_R^{(1)} h = \frac{1}{2R^2}\big(\mathbf{Cas}_{\mathfrak{g}}(\Lambda) - \mathbf{Cas}_{\mathfrak{h}}(\mu) - \mathbf{Cas}_{\mathfrak{h}}(\overline{\mu})\big)\,. \tag{3.38}$$

In the derivation the result actually emerges as a sum of two different pieces that are associated with the zero mode factor and the tail of the vertex operator, respectively. Recall that the zero mode factor $V_{\Lambda\lambda}$ is a section in a homogeneous vector bundle Γ_λ. Such sections are acted upon by the Bochner Laplacian Δ_B, whose eigenvalues were expressed through the quadratic Casimir operators of $\mathfrak{g}$ and $\mathfrak{h}$ in [60],

$$\Delta_B V_{\Lambda\lambda}(\theta) = \big(\mathbf{Cas}_{\mathfrak{g}}(\Lambda) - \mathbf{Cas}_{\mathfrak{h}}(\lambda)\big) V_{\Lambda\lambda}(\theta)\,. \tag{3.39}$$

The contribution of the tail factors to the anomalous dimension can be written as a spin-spin interaction between fields j and $\bar{\jmath}$. It leads to a term of the form $\mathbf{Cas}_{\mathfrak{h}}(\lambda) - \mathbf{Cas}_{\mathfrak{h}}(\mu) - \mathbf{Cas}_{\mathfrak{h}}(\overline{\mu})$. Note that the first term in this combination cancels the constant shift $\mathbf{Cas}_{\mathfrak{h}}(\lambda)$ in the eigenvalues of the Bochner Laplacian so that we end up with the expression given in eq. (3.38).

Formula (3.38) is actually very general. It holds for *all* sigma models on symmetric superspaces with vanishing beta function. When properly interpreted, see [38], it can also be used for models with world-sheet supersymmetry, such as e.g. the $N = 2$ worldsheet supersymmetric sigma model on complex projective superspace $\mathbb{C}\mathrm{P}^{3|4}$ etc. In applications to non-conformal theories, such as e.g. the usual $\mathrm{O}(N)$ models, the formula for $\delta^{(1)} h$ requires a simple additional term,

$$\delta_R^{(1)} h = \frac{1}{2R^2}\Big(\mathbf{Cas}_{\mathfrak{g}}(\Lambda) - \mathbf{Cas}_{\mathfrak{h}}(\mu) - \mathbf{Cas}_{\mathfrak{h}}(\overline{\mu}) + \mathbf{Cas}_{\mathfrak{h}}(\mathfrak{m}) \sum_i (m_i + \overline{m}_i)\Big)\,. \tag{3.40}$$

Since vanishing of the one-loop beta function requires that $\mathbf{Cas}_{\mathfrak{h}}(\mathfrak{m}) = 0$ we recover the formula (3.38) for conformal sigma models. Our simple formula (3.38) or rather its generalization (3.40) summarizes and extends the results of many papers in which anomalous dimensions, mostly dealing with $\mathfrak{g}$-invariant fields, have been studied model by model, see e.g. [40, 53, 54, 61–64]. That all these computations may be captured by a single universal formula (3.40) is quite remarkable. Of course, this success is intimately tied to the construction (3.10) of vertex operators. We now see how well this construction was adapted to the computation of one-loop anomalous dimensions.

Much of the previous work on anomalous dimensions of (high-)gradient operators in sigma models was motivated by a somewhat puzzling instability that has first been observed in $\mathrm{O}(N)$-vector models [40] and later understood to be a rather generic feature of sigma model perturbation theory, see [41] and references therein. This instability arises because of the negative sign in front of the terms $\mathbf{Cas}_{\mathfrak{h}}(\mu)$ and $\mathbf{Cas}_{\mathfrak{h}}(\overline{\mu})$. Naively one might think that high gradient operators, i.e. operators (3.10) for which $\sum_j j(m_j + \overline{m}_j) = h_\infty + \bar{h}_\infty$ is large, are highly irrelevant. But it turns out that some of these operators acquire a very large negative anomalous dimension. More precisely, one can show that for every choice of the sigma model coupling R^{-2}, no matter how small, one can find a $\mathfrak{g}$-invariant high gradient operator $\mathcal{O} = \Phi_{0,\lambda,\mu,\overline{\mu}}$ such that

$$h_\infty(\mathcal{O}) + \bar{h}_\infty(\mathcal{O}) - \frac{1}{R^2}\big(\mathbf{Cas}_{\mathfrak{h}}(\mu) + \mathbf{Cas}_{\mathfrak{h}}(\overline{\mu})\big) < 2\,. \tag{3.41}$$

This is because $\mathbf{Cas}_{\mathfrak{h}}(\mu)$ grows quadratically with the weights of the representation μ and the maximal weight grows linearly with the number of currents j in the tail. On the other hand, the positive contribution $h_\infty(\mathcal{O})$ only grows linearly in the number of js. The argument shows that (infinitely many) high gradient operators become relevant for arbitrarily small sigma model coupling. One could have hoped that higher orders in perturbation theory correct the issue, but they turn out to make things even worse [42]. We would be ready to conclude that sigma models are inherently unstable if it were not for the many independent studies, e.g. through lattice discretisations, that display no pathologies. As far as we know, the problem has never been resolved but it is something to be kept in mind as we proceed.

3.3.2 Calculation for semi-symmetric spaces

Using the setup we presented in the previous sections, we can now begin the calculation of anomalous dimensions for sigma models on semi-symmetric spaces. In the computation we present below, we will restrict our attention to the diagonal terms of the one-loop dilatation operator with respect to the basis we defined above. Unlike in the symmetric case, the basis we have defined does not seem to diagonalise the one-loop dilatation operator. At this point, we do not have the off-diagonal terms under good control.

Outline of the computation

The one-loop anomalous dimension $\delta\mathbf{h} = \delta\mathbf{h}_\Phi$ of a field Φ appears in the coefficient of the logarithmic singularity of the two point function at one-loop, see e.g. [38] for a detailed discussion,

$$\begin{aligned}\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle_1 &= \\ &= \langle\, 2\delta\mathbf{h}\cdot \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle_0 \ln\left|\frac{\epsilon}{u-v}\right|^2 + \cdots .\end{aligned} \tag{3.42}$$

The correlation function on the right hand side is evaluated in the free theory, i.e. by performing Wick contractions with the propagator (3.22). By definition, the one-loop correlation function on the left hand side is obtained as the leading non-trivial term in

$$\begin{aligned}\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle &= \\ &= \int_{G/H} d\mu(g_0H)\, \left\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}\,|\,g_0) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v}\,|\,g_0) e^{-S_{\text{int}}}\,\right\rangle_{0,c} ,\end{aligned} \tag{3.43}$$

where the subscript c stands for 'connected'. When counting loops, recall that each propagator carries a factor $1/R^2$ and each insertion of the interaction produces a factor R^2. The one-loop contribution contains all terms that are suppressed by a factor $1/R^2$ relative to the tree-level.

Expanding the two point correlation function (3.43) to one-loop we have

$$\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle = \int_{G/H} d\mu(g_0H)\, \mathsf{d}\otimes\mathsf{d}\big(I_0 + I_1 + \dots\big) , \tag{3.44}$$

where

$$I_0 = \left\langle V^{(0)} \otimes j^{(0)}_{\mathbf{m}}(u) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}}}(\bar{u}) \otimes V^{(0)} \otimes j^{(0)}_{\mathbf{m}}(v) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}}}(\bar{v}) \right\rangle_0 . \tag{3.45}$$

Here we used the schematic representation (3.31) and the fact that derivatives of the fundamental field are (anti)holomorphic in the free theory. Before we describe the quantity I_1 it is useful to make a few comments on the tree-level contribution, as we restrict to those terms that are already non-vanishing at tree-level. This requires that the total $\mathbb{Z}_4$ grading of all currents j and their derivatives vanishes. The same condition must be satisfied for $\bar{\jmath}$ and all their derivatives. As mentioned above, off-diagonal terms also appear, but are not considered for now.

Let us now turn to I_1. As we noted above, the R^{-2} corrections to the correlation functions are collected in I_1. There are a variety of different terms. To begin with, there are three different terms in which no interaction term appears. In order to produce the desired factor $1/R^2$, these must involve one additional Wick contraction compared to the tree-level term. This contraction can either involve the two zero mode factors (case A), or two fields from the tails (case C) or one field from the head and one from the tail (case G). Next, there exist several terms that involve one interaction term. If the latter is given by the three vertex Ω_3, then we must expand either a zero mode factor V (case F) or a tail j (case D) to the leading non-trivial order. Terms involving a single interaction term Ω_4 contain two additional Wick contractions compared to the tree-level computation and hence also contain one factor $1/R^2$ (case B). Finally, we also have to consider one type of contributions with two interaction three-vertices Ω_3 since these contain three additional Wick contractions compared to tree-level (case E).

The quantity I_1 is obtained by summing all these different contributions, i.e. $I_1 = \sum I_K^\nu$ where

$$I_A = \left\langle V_u^{(1)} \otimes j^{(0)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(1)} \otimes j^{(0)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \right\rangle \tag{3.46}$$

$$I_B = \left\langle V_u^{(0)} \otimes j^{(0)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(0)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \; \mathcal{O}_4 \right\rangle \tag{3.47}$$

$$I_C^1 = \left\langle V_u^{(0)} \otimes j^{(1)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(1)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \right\rangle \tag{3.48}$$

$$I_D^1 = \left\langle V_u^{(0)} \otimes j^{(1)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(0)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \; \mathcal{O}_3 \right\rangle \tag{3.49}$$

$$I_E = \left\langle V_u^{(0)} \otimes j^{(0)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(0)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \; \frac{1}{2}\mathcal{O}_3^2 \right\rangle \tag{3.50}$$

$$I_F^1 = \left\langle V_u^{(1)} \otimes j^{(0)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(0)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \; \mathcal{O}_3 \right\rangle \tag{3.51}$$

$$I_G^1 = \left\langle V_u^{(1)} \otimes j^{(0)}_{\mathbf{m},u} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},u} \otimes V_v^{(0)} \otimes j^{(1)}_{\mathbf{m},v} \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}},v} \right\rangle \tag{3.52}$$

Here, the subscripts u and v label fields that are inserted at $(u, \bar{u})$ and $(v, \bar{v})$, respectively. We have introduced the following shorthand notation for integrated interaction vertices

$$\mathcal{O}_3 = -\int_{\mathbb{C}} \frac{d^2 z}{\pi} \; \Omega_3(z, \bar{z}) \,, \qquad \mathcal{O}_4 = -\int_{\mathbb{C}} \frac{d^2 z}{\pi} \; \Omega_4(z, \bar{z}) \,. \tag{3.53}$$

Note that we included the minus sign from the $\exp(-S)$ into our definition of $\mathcal{O}_{3,4}$. Similarly, we also put a factor 1/2 into our definition of I_E because this term arises from the second order term in the expansion of $\exp(-S)$. There is another term I_C^2 that resembles I_C^1 only that the tail factors $\bar{\jmath}$ are expanded to the next to leading order instead of j. Similarly, we also need to consider I_D^ν and I_G^ν with $\nu = 1,2,3,4$ that are obtained by expanding any of the four tail factors j_1, j_2 or $\bar{\jmath}_1, \bar{\jmath}_2$ beyond the leading order.

The first two cases (Case A,B) listed above also appear for symmetric spaces. In fact, these neither involve next to leading order tail factors nor an interaction three vertex Ω_3. Hence, the computation of case A,B is analogous to what was done in [38]. All other cases are new. Let us also note that the last two cases contribute off-diagonal terms. This is because $V^{(1)}$ contains a factor ϕ_A of grading $A \neq 0$. Hence, the other fields in the correlator must have total $\mathbb{Z}_4$ grading $A' = 1 - A \neq 0$. In the cases F and G this grading is carried by the tail of currents since Ω_3 is an element of grade $B = 0$. On the other hand, correlators in which all currents possess a total grading $A' \neq 0$ vanish at tree-level.

Calculating the various pieces

We now turn to the calculation of the one-loop corrections (3.46)–(3.52) that contribute diagonal terms to the dilatation operator. We will go through the list in order and point out parallels with the symmetric case.

Contributions from case A In this case the result is the same as for symmetric spaces. It is determined by the logarithmic contribution from I_A. Since the only logarithmic term arises from a contraction of the two dimension zero fields ϕ, without any derivative, the only expression that needs to be evaluated is

$$\begin{aligned}\int_{G/H} d\mu(g_0 H)\ \big\langle\, V^{(1)}(u,\bar{u}) \otimes V^{(1)}(v,\bar{v})\,\big\rangle = \\ = R^{-2} \ln \left|\frac{u-v}{\epsilon}\right|^2 \int_{G/H} d\mu(g_0 H)\ \left[\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\lambda}\right] V^{(0)} \otimes V^{(0)}\,.\end{aligned} \tag{3.54}$$

The details of the calculations can be found in subsection 3.2.1 of [38]. Note that it is important here that we chose the parameters $p_A = 1$. Beyond this conformal case, the results takes a more complicated form which cannot be written in terms of $\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\lambda}$. Hence, while it was very easy to compute the corrections to the scaling behavior in symmetric spaces beyond the conformal case, see [38], a simple extension to non-conformal semi-symmetric spaces does not exist.

Contributions from case B Let us now turn to the more interesting case B and analyse the following integral that is contained in it

$$\tilde{I}_B = -\int_{\mathbb{C}_\epsilon} d^2z \left\langle j^{(0)}_{\mathbf{m}}(u) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}}}(\bar{u}) \otimes j^{(0)}_{\mathbf{n}}(v) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{n}}}(\bar{v})\, \Omega_4(z,\bar{z}) \right\rangle_0 . \quad (3.55)$$

Compared to the original I_B we just dropped the group theoretic factor that is associated with the zero modes. It will later be reinstalled when we state the final answer.

While an integral similar to $\tilde{I}_B$ also appears for symmetric spaces and was computed for these in [38], we now have to pay attention to the grading and the coefficients, in particular the non-trivial q_A, in the action. As a result, while a subset of terms turns out to reproduce those found for symmetric spaces, we shall also find new contributions that have no counterpart in the previous computation. To begin with, we can rewrite the quantity (3.55) in the form,

$$\begin{aligned} \tilde{I}_B = -\Pi \cdot \Bigg[\sum_{\rho,\sigma=1}^{r} \sum_{\bar{\rho},\bar{\sigma}=1}^{\bar{r}} \left\langle j^{(0)}_{\mathbf{m}_\rho}(u) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{m}}_{\bar{\rho}}}(\bar{u}) \otimes j^{(0)}_{\mathbf{n}_\sigma}(v) \otimes \bar{\jmath}^{(0)}_{\overline{\mathbf{n}}_{\bar{\sigma}}}(\bar{v}) \right\rangle_0 \otimes \\ \otimes \int_{\mathbb{C}_\epsilon} \frac{d^2z}{\pi} \left\langle \partial^{m_\rho}\phi(u) \otimes \bar{\partial}^{\bar{m}_{\bar{\rho}}}\phi(\bar{u}) \otimes \partial^{n_\sigma}\phi(v) \otimes \bar{\partial}^{\bar{n}_{\bar{\sigma}}}\phi(\bar{v})\, \Omega_4(z,\bar{z}) \right\rangle_0 \Bigg] . \end{aligned} \quad (3.56)$$

Here, $j^{(0)}_{\mathbf{m}_\rho}$ denotes the tensor product (3.35) of currents with the ρ-th factor removed and we introduced a permutation Π that acts on a tensor power of $\mathfrak{m}$, see [38] for details.

Let us begin with a general statement and consider the integrand of eq. (3.56) with some definite choice of $\mathbb{Z}_4$ grading for the currents,

$$\left\langle \partial^m\phi_E(u) \otimes \bar{\partial}^{\bar{m}}\phi_F(\bar{u}) \otimes \partial^n\phi_G(v) \otimes \bar{\partial}^{\bar{n}}\phi_H(\bar{v}) \left([\phi_B, \partial\phi_A], [\phi_C, \bar{\partial}\phi_D]\right)(z,\bar{z}) \right\rangle_0 .$$

For the corresponding tree-level correlator to be non-zero, we must have $E = G'$ and $F = H'$. Carrying this a bit further, we conclude that a diagonal contribution to the anomalous dimension can only occur if either $A = B'$ and $C = D'$ or $A = D'$ and $C = B'$. This criterion excludes many terms from Ω_4^s. All relevant terms from Ω_4^{s1} have been listed in eq. (3.29). The first one we would like to consider in more detail is the one in which

$A = B = C = D = 2$. For the evaluation we make use of the following integral formula that we derive in Appendix A

$$\begin{aligned}\int_{\mathbb{C}_\epsilon} \frac{d^2z}{\pi} \frac{a!\,b!\,c!\,d!}{(z-u)^{a+1}(z-v)^{b+1}(\bar z-\bar u)^{c+1}(\bar z-\bar v)^{d+1}} = \\ = 2\ln\left|\frac{u-v}{\epsilon}\right|^2 \times \frac{(-1)^{a+c}(a+b)!(c+d)!}{(u-v)^{a+b+1}(\bar u-\bar v)^{c+d+1}} + \text{non-log.}\end{aligned} \tag{3.57}$$

As a simple consequence we find that

$$\begin{aligned}\frac{1}{6}\int_{\mathbb{C}_\epsilon} \frac{d^2z}{\pi} \langle \partial^m \phi_2(u) \otimes \bar\partial^{\bar m}\phi_2(\bar u) \otimes \partial^n \phi_2(v) \otimes \bar\partial^{\bar n}\phi_2(\bar v) \times \\ \times \left([\phi_2\,,\, \partial\phi_2]\,,\, [\phi_2\,,\, \bar\partial\phi_2]\right)(z,\bar z)\rangle_0 = \\ = 2\ln\left|\frac{u-v}{\epsilon}\right|^2 [t_{0\alpha}, t_{2i}] \otimes [t_0^\alpha, t_{2j}] \otimes t_2^i \otimes t_2^j \times \\ \times \frac{(-1)^{n+m}(m+n-1)!(\bar m+\bar n-1)!}{R^6 (u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}} + \text{non-log.}\end{aligned} \tag{3.58}$$

For a more detailed derivation we refer the reader to [38]. Note that in writing our result here we have made use of the fact that the indices i, j run only over a basis in $\mathfrak{g}_2$ while the index α runs over a basis in $\mathfrak{g}_0 = \mathfrak{h}$. Since we assumed that all elements of $\mathfrak{g}_A$, $A = 0, 2$, are bosonic, we conclude that $|i| = |j| = |\alpha| = 0$. This observation is crucial in comparing our result with the corresponding formula in the case of symmetric spaces.

The other terms in Ω_4^{s1} give similar contributions, but one has to be careful with the grading signs. Let us just state one more example

$$\begin{aligned}\frac{1}{6}\int_{\mathbb{C}_\epsilon} \frac{d^2z}{\pi} \langle \partial^m \phi_1(u) \otimes \bar\partial^{\bar m}\phi_1(\bar u) \otimes \partial^n \phi_3(v) \otimes \bar\partial^{\bar n}\phi_3(\bar v) \times \\ \times \left[3\left([\phi_3\,,\, \partial\phi_1]\,,\, [\phi_1\,,\, \bar\partial\phi_3]\right) + 3\left([\phi_1\,,\, \partial\phi_3]\,,\, [\phi_3\,,\, \bar\partial\phi_1]\right)\right](z,\bar z)\rangle_0 = \\ = -2\ln\left|\frac{u-v}{\epsilon}\right|^2 [t_{0\alpha}, t_{1i}] \otimes [t_0^\alpha, t_{1j}] \otimes t_3^i \otimes t_3^j \times \\ \times \frac{(-1)^{n+m}(m+n-1)!(\bar m+\bar n-1)!}{R^6 (u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}} + \text{non-log.}\end{aligned} \tag{3.59}$$

After computing all the logarithmic contributions from Ω_4^{s1} we need to sum them up and write them in terms of the tree-level correlation so that we can read off the anomalous dimension. Let us note that the relevant tree-level correlation function is given by

$$\Big\langle\, \partial^m\phi_a(u)\otimes\bar\partial^{\bar m}\phi_b(\bar u)\otimes\partial^n\phi_c(v)\otimes\bar\partial^{\bar n}\phi_d(\bar v)\,\Big\rangle_0 = \\ = (-1)^{m+\bar m+|b||c|}(m+n-1)!\,(\bar m+\bar n-1)!\,\frac{\delta_{c,a'}\delta_{d,b'}}{p_a p_b}\,\frac{t_a\otimes t_b\otimes t^{a'}\otimes t^{b'}}{(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\,. \tag{3.60}$$

With the help of this formula we arrive at

$$-\int_{\mathbb{C}_\epsilon}\frac{d^2z}{\pi}\Big\langle\, j^{(0)}_{\mathbf{m}}(u)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar u)\otimes j^{(0)}_{\mathbf{n}}(v)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{n}}}(\bar v)\,\Omega_4^{s1}(z,\bar z)\Big\rangle_0 = \\ = R^{-2}\ln\left|\frac{u-v}{\epsilon}\right|^2\times\Big\langle\, j^{(0)}_{\mathbf{m}}(u)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar u)\otimes \\ \otimes\Big[\big(\mathbf{Cas}^{\rm D}_{\mathfrak h}-\mathbf{Cas}^{\rm L}_{\mathfrak h}\otimes 1_{\rm R}-1_{\rm L}\otimes\mathbf{Cas}^{\rm R}_{\mathfrak h}\big)\,j^{(0)}_{\mathbf{n}}(v)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{n}}}(\bar v)\Big]\Big\rangle_0\,. \tag{3.61}$$

Now we have to insert the result back into the full I_B, i.e. we have put the zero mode contributions back in and then act with the intertwining operator $\mathsf{d}\otimes\mathsf{d}$. In evaluating the latter, we make use of the fact that

$$\mathsf{d}_{\lambda\mu\bar\mu}\,\big(1_\lambda\otimes\mathbf{Cas}^{\rm D}_{\mathfrak h}-1_\lambda\otimes\mathbf{Cas}^{\rm L}_{\mathfrak h}\otimes 1_{\rm R}-1_\lambda\otimes 1_{\rm L}\otimes\mathbf{Cas}^{\rm R}_{\mathfrak h}\big) = \\ = \big(\mathbf{Cas}^\lambda_{\mathfrak h}\otimes 1_\mu\otimes 1_{\bar\mu}-1_\lambda\otimes\mathbf{Cas}^\mu_{\mathfrak h}\otimes 1_{\bar\mu}-1_\lambda\otimes 1_\mu\otimes\mathbf{Cas}^{\bar\mu}_{\mathfrak h}\big)\,\mathsf{d}_{\lambda\mu\bar\mu}\,. \tag{3.62}$$

We obtain

$$\int_{G/H} d\mu(g_0H)\,\mathsf{d}\otimes\mathsf{d}\,I_B^{s1} = \\ = \frac{1}{R^2}\big(\,\mathbf{Cas}^\lambda_{\mathfrak h}-\mathbf{Cas}^\mu_{\mathfrak h}-\mathbf{Cas}^{\bar\mu}_{\mathfrak h}\big)\int_{G/H} d\mu(g_0H)\,\mathsf{d}\otimes\mathsf{d}\,I_0\,. \tag{3.63}$$

Here, the symbol I_B^{s1} denotes the contribution to I_B that comes with the insertion of Ω_4^{s1}. We still need to discuss Ω_4^{s2} and Ω_4^a.

So, let us consider Ω_4^{s2} next. The relevant correlator in this case is given by

$$
\begin{aligned}
-\frac{1}{8}\sum_{\substack{B+C\equiv A\\ D+E\equiv A'}}\int_{\mathbb{C}_\epsilon}\frac{d^2z}{\pi}\big\langle\partial^m\phi_F(u)\otimes\bar\partial^{\bar m}\phi_G(\bar u)\otimes\partial^n\phi_{F'}(v)\otimes\bar\partial^{\bar n}\phi_{G'}(\bar v)\times&\\
\times\left(\left[\phi_B\,,\,\partial\phi_C\right],\left[\phi_D\,,\,\bar\partial\phi_E\right]\right)(z,\bar z)\big\rangle_0 &=\\
=-\frac{1}{4}\ln\left|\frac{u-v}{\epsilon}\right|^2\frac{(-1)^{n+m}(m+n-1)!(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\times&\\
\times\sum_{B+C\equiv A}(-1)^{FG}t_{Fi}\otimes t_{Gj}\otimes t_{F'k}\otimes t_{G'l}\big[\delta_{F,C}\delta_{G,B}\left(\left[t^l_G\,,\,t^k_F\right],\left[t^i_{F'}\,,\,t^j_{G'}\right]\right)&\\
-(-)^{FG}\delta_{F,C}\delta_{G,B'}\left(\left[t^j_G\,,\,t^k_F\right],\left[t^i_{F'}\,,\,t^l_{G'}\right]\right)&\\
-(-)^{FG}\delta_{F,C'}\delta_{G,B}\left(\left[t^l_G\,,\,t^k_F\right],\left[t^i_{F'}\,,\,t^j_{G'}\right]\right)&\\
+\delta_{F,C'}\delta_{G,B'}\left(\left[t^l_G\,,\,t^k_F\right],\left[t^i_{F'}\,,\,t^j_{G'}\right]\right)\big]+\text{non log.}&
\end{aligned}
\tag{3.64}
$$

For later convenience let us spell out explicitly some of the terms that appear on the right hand side of eq. (3.64). For example, there appears a unique term that is multiplied by $t_{1i}\otimes t_{1j}\otimes t_{3k}\otimes t_{3l}$,

$$
\begin{gathered}
\frac{1}{2}\frac{(-1)^{n+m}\ln\left|\frac{u-v}{\varepsilon}\right|^2(m+n-1)!(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\times\\
t_{1i}\otimes t_{1j}\otimes t_{3k}\otimes t_{3l}\left(\left[t^l_1\,,\,t^k_1\right],\left[t^i_3\,,\,t^j_3\right]\right)
\end{gathered}
\tag{3.65}
$$

while there are two terms that contain the factor $t_{1i}\otimes t_{2j}\otimes t_{3k}\otimes t_{2l}$, namely

$$
\begin{gathered}
-\frac{1}{2}\frac{(-1)^{n+m}\ln\left|\frac{u-v}{\varepsilon}\right|^2(m+n-1)!(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\times\\
t_{1i}\otimes t_{2j}\otimes t_{3k}\otimes t_{2l}\left[\left(\left[t^l_2\,,\,t^k_1\right],\left[t^i_3\,,\,t^j_2\right]\right)-\left(\left[t^j_2\,,\,t^k_1\right],\left[t^i_3\,,\,t^l_2\right]\right)\right]\,.
\end{gathered}
\tag{3.66}
$$

We will not describe how to insert this result into I_B since the entire contribution turns out to cancel against an identical term from case E, see below.

Let us finally discuss the anti-symmetric part Ω^a_4. It is actually not difficult to see that its contributions to the anomalous dimension vanishes. Part of these cancellations can easily be read off from the result (3.59). A closer look reveals that the two terms in the second line give identical contributions and consequently the factor 2 in the final result. In Ω_a, the same terms appear but instead of being summed, they are subtracted so that the final result vanishes. The same is true for other contributions from Ω_4^a. In conclusion, we have shown that case B contains two non-vanishing contributions to the anomalous dimensions, namely those in eqs. (3.63) and (3.64). While the former is identical to the corresponding term for symmetric spaces, the latter will be shown to cancel contributions from case E.

Contributions from case C Next we need to compute the part of the one-loop correction that arises from the expansion of the currents. In our discussion we shall work with the component fields $\phi_i := (\phi, t_i)$. When written in terms of these component fields, the subleading part of the current becomes

$$j^{(1)} = \frac{1}{2} f^{ij}{}_a \phi_i \partial \phi_j t^a \,. \tag{3.67}$$

Here i, j and a run over a basis of the quotient space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. The two-point function is then

$$\begin{aligned}&\langle \partial^m j^{(1)}(u) \otimes \partial^n j^{(1)}(v) \rangle = \\ &= \frac{(-1)^{m+|a|}}{4R^4} \ln \left| \frac{u-v}{\varepsilon} \right|^2 \frac{(m+n+1)!}{(u-v)^{m+n+2}} f^{ij}{}_a f^{kl}{}_b \eta_{ik} \eta_{jl} t^a \otimes t^b + \mathcal{O}(\varepsilon).\end{aligned} \tag{3.68}$$

If we take into account that the Killing form of G vanishes by assumption and then compare with eq. (5.8) in [65] we can identify the combination of the structure constants that appears in the previous expression with the Ricci tensor of the coset space, i.e.

$$f^{ij}{}_a f^{kl}{}_b \eta_{ik} \eta_{jl} = 4R_{ab}(G/H) \,. \tag{3.69}$$

Below we shall see that a similar term involving the Ricci tensor also arises from case E. The latter actually cancels the contributions from case C. This is not an accident. The cancellation we see here in the computation of the anomalous dimension is really the same cancellation that is responsible for the vanishing of the one-loop beta-function.

Contributions from case D Our basic claim is that case D does not contribute to the anomalous dimension. While a similar statement for the cases F and G was very easy to prove, it requires more effort for case D. In the calculation we shall make use of the following integral formula

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)^{a+1}(\bar z-\bar x)^{b+1}(\bar z-\bar y)^{c+1}} &=\\ =\delta_{a,0}\frac{(-1)^{b+1}}{a!}\binom{b+c}{c}\frac{\ln\left|\frac{x-y}{\varepsilon}\right|^2}{(\bar x-\bar y)^{b+c+1}}&+\mathcal{O}(\varepsilon)\end{aligned}\tag{3.70}$$

This formula is derived in Appendix A. Let us point out that logarithmic singularities only exist when $a=0$. This implies most terms that appear in the evaluation of case D do not contain any logarithmic divergences.

After having made these observations, let us discuss the contributions from I_D. Recall that $\Omega_3=\Omega_3^a$ and that we can add a total derivative in order to bring Ω_3^a into the simple form $\Omega_3^{a,1}$. This is the form we shall use. We address the two terms of $\Omega_3^{a,1}$ separately. Given our introductory comments, the first term can only contribute through the following integral

$$\begin{aligned}&\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\langle\left[\phi_3\,,\,\partial^{m+1}\phi_2\right](u,\bar u)\otimes\partial^{n+1}\phi_3(v)(\partial\phi_1,[\phi_2,\bar\partial\phi_1])(z,\bar z)\rangle=\\ &-\frac{(-1)^{m+1}}{R^4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(m+n+1)!}{(u-v)^{m+n+2}}\left[t_{2i}\,,\,t_{3j}\right]\otimes\left[t_2^i\,,\,t_1^j\right]+\text{non-log.}\;.\end{aligned}\tag{3.71}$$

The second term in $\Omega_3^{a,1}$, on the other hand, contributes to the logarithmic singularity through

$$\begin{aligned}&\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\langle\partial^{n+1}\phi_1(u,\bar u)\otimes\left[\phi_1\,,\,\partial^{m+1}\phi_2\right](v)(\partial\phi_3,[\phi_2,\bar\partial\phi_3])(z,\bar z)\rangle\\ &-\frac{(-1)^{m+1}}{R^4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(m+n+1)!}{(u-v)^{m+n+2}}\left[t_{2i}\,,\,t_{3j}\right]\otimes\left[t_2^i\,,\,t_1^j\right]+\text{non-log.}\;,\end{aligned}\tag{3.72}$$

i.e. it is the same as the previous one. Since the two terms in $\Omega_3^{a,1}$ appear with opposite signs, we conclude that there are no contributions from case D to the anomalous dimension.

Contributions from case E Let us now address the final case with two insertions of the vertex Ω_3. Recall that in a theory with vanishing one-loop beta-function, the three vertex is purely antisymmetric, i.e. $\Omega_3=\Omega_3^a$. To ease the calculation we added a total derivative. Of course, we have to do this consistently. This means that we have to add the total derivative to the action and it has to be the same at all orders in the expansion of $\exp(-S)$. In particular we have to pay attention to use the same interaction as in case D, where we already made a particular choice, namely $\Omega_3^{a,1}$.

In the calculation we have two types of terms. The first type consists of contributions which involve two contractions between the two Ω_3 and two more contractions between the vertices and two currents. On the other hand, we can also have terms in which there is a single contraction between the two vertices and four contractions with the tails of currents. We shall argue that the first type cancels the contributions from case C while the second type chancels the contributions in eq. (3.64) of case B. The integral is evaluated to

$$\begin{aligned}\frac{1}{8}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2w}{\pi}\langle\partial^{m+1}\phi_2(u,\bar{u})\otimes\partial^{n+1}\phi_2(v)(\partial\phi_1,[\phi_2,\bar{\partial}\phi_1])(z,\bar{z})\times\\ \times(\partial\phi_3,[\phi_2,\bar{\partial}\phi_3])(w,\bar{w})\rangle_0 = \qquad (3.73)\\ = -\frac{(-1)^m}{8R^4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(m+n+1)!}{(u-v)^{m+n+2}}f^{ij}{}_a f^{kl}{}_b\eta_{ik}\eta_{jl}t_2^a\otimes t_2^b+\text{non-log.}\end{aligned}$$

Here we have inserted the first term of $\Omega_3^{a,1}$ at $(z,\bar{z})$ and the second term at $(w,\bar{w})$. The opposite choice turns out to give exactly the same result so that we get a numerical prefactor -1/4 in place of -1/8 after summing both contributions. We see that the result cancels against the contribution from case C for a and b labeling basis elements of $\mathfrak{g}_2$, i.e. when $|a|=|b|=0$. In evaluating the relevant integrals, we have used the formula

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)(w-y)(z-w)^2(\bar{z}-\bar{w})^2} = \\ = -\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)}\frac{1}{(z-y)^2}\frac{1}{(\bar{z}-\bar{y})} = \frac{\ln\left|\frac{u-v}{\varepsilon}\right|^2}{(x-y)^2(\bar{x}-\bar{y})^2}\end{aligned} \qquad (3.74)$$

and derivatives thereof. This can be derived using formulas in Appendix A. We can perform a similar analysis in case the currents from the tails possess odd grade rather than even. The result is

$$\begin{aligned}\frac{1}{8}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\langle\partial^{m+1}\phi_1(u,\bar{u})\otimes\partial^{n+1}\phi_3(v)(\partial\phi_1,[\phi_2,\bar{\partial}\phi_1])(z,\bar{z})\times\\ \times(\partial\phi_3,[\phi_2,\bar{\partial}\phi_3])(w,\bar{w})\rangle_0 = \qquad (3.75)\\ = \frac{(-1)^m}{8R^4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(m+n+1)!}{(u-v)^{m+n+2}}f^{ij}{}_a f^{kl}{}_b\eta_{ik}\eta_{jl}t_1^a\otimes t_3^b+\text{non-log.}\end{aligned}$$

In the process we have used the integral formula (A.11) from Appendix A. Once again, the result cancels the contribution from case C.

Let us then turn to the second type of terms in which we have four contractions between the vertices and the tail factor. As above, we illustrate the computations with a concrete example,

$$\begin{aligned}\frac{1}{8}\int_{\mathbb{C}_\epsilon}\frac{\mathrm{d}^2 z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2 z}{\pi}\langle\partial^m\phi_1(u)\otimes\bar\partial^{\bar m}\phi_2(\bar u)\otimes\partial^n\phi_3(v)\otimes\bar\partial^{\bar n}\phi_2(\bar v)\ \times \\ \left(\partial\phi_1,\left[\phi_2\,,\,\bar\partial\phi_1\right]\right)(z,\bar z)\left(\partial\phi_3,\left[\phi_2\,,\,\bar\partial\phi_3\right]\right)(w,\bar w)\rangle_0 = \\ =\frac{1}{4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(-1)^{n+m}(m+n-1)!\,(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\times \\ t_{1i}\otimes t_{2j}\otimes t_{3k}\otimes t_{2l}\left[\,\left(\left[t_2^l\,,\,t_1^k\right],\left[t_3^i\,,\,t_2^j\right]\right)-\left(\left[t_2^j\,,\,t_1^k\right],\left[t_3^i\,,\,t_2^l\right]\right)\,\right]+\text{non-log.}\end{aligned} \tag{3.76}$$

If we insert the second non-trivial term in $\Omega_3^{a,1}$ we obtain an identical contribution so that we just have to multiply the right hand side of the previous formula by a factor of two. We see that the new contributions cancel the term (3.66) from the right hand side of eq. (3.64).

Finally, there is another qualitatively somewhat different example we want to consider in which the only contraction between the two vertices is a contraction between non-derivative fields. This happens e.g. in

$$\begin{aligned}\frac{1}{8}\int_{\mathbb{C}_\epsilon}\frac{\mathrm{d}^2 z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2 z}{\pi}\langle\partial^m\phi_1(u)\otimes\bar\partial^{\bar m}\phi_1(\bar u)\otimes\partial^n\phi_3(v)\otimes\bar\partial^{\bar n}\phi_3(\bar v)\times \\ \left(\partial\phi_1,\left[\phi_2\,,\,\bar\partial\phi_1\right]\right)(z,\bar z)\left(\partial\phi_3,\left[\phi_2\,,\,\bar\partial\phi_3\right]\right)(w,\bar w)\rangle_0 = \\ =-\frac{1}{4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(-1)^{n+m}(m+n-1)!\,(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}}\times \\ t_{1i}\otimes t_{1j}\otimes t_{3k}\otimes t_{3l}\left(\left[t_1^l\,,\,t_1^k\right],\left[t_3^i\,,\,t_3^j\right]\right)+\text{non-log.}\end{aligned} \tag{3.77}$$

This and similar terms can be evaluated using the integral formula (A.10). Once we collect all such terms we see that these exactly cancel the terms contained in eq. (3.64).

Summing all the contributions

We have seen that most of the contributions on the diagonal that are new compared to the symmetric case cancel out. In the final result only contributions from case A, see eq. (3.54), and the piece (3.61) containing Ω_4^{s1} from case B are not canceled by other terms. These two contributions are identical to those that were found in the analysis of sigma models on symmetric spaces. Hence, in the basis that we have chosen, the diagonal part of the anomalous dimensions for semi-symmetric spaces agrees with that for symmetric spaces, i.e.

$$\delta_R^{(1,\text{diag})} h = \frac{1}{2R^2}\big(\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\lambda} + \mathbf{Cas}_{\mathfrak{h}}^{\lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\mu} - \mathbf{Cas}_{\mathfrak{h}}^{\overline{\mu}} \big) = $$
$$= \frac{1}{2R^2}\big(\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\mu} - \mathbf{Cas}_{\mathfrak{h}}^{\overline{\mu}}\big) \ .$$

It should be stressed once again that our analysis of semi-symmetric spaces does not include off-diagonal terms. Additionally, our computations relied on the choice (2.20) for the various coupling constants. The latter was motivated by the requirement that the one-loop beta-function vanishes. While the restriction to models with vanishing beta-function was easy to lift for symmetric spaces, see [38], our results for semi-symmetric spaces possess no simple generalization to models that do not satisfy the conditions (2.20).

Chapter 4

The Spectrum of Superspheres

In this chapter we apply the results and constructions reviewed in chapter 3 to the sigma model on the supersphere $S^{3|2}$. The supersphere is a symmetric space, $S^{3|2} \simeq \mathrm{OSP}(4|2)/\mathrm{OSP}(3|2)$, and one of the simplest examples from the list (2.16). Therefore, the sigma model on $S^{3|2}$ is conformal for all values of the radius R and the constructions and results outlined in the previous chapters apply. In the first section, we will discuss the construction of vertex operators and the computation of anomalous dimensions in great detail, thereby illustrating the power of the constructions of [38] that we reviewed in Chapter 3.

In the second section we will revisit a proposed dual description of the sigma model [33, 34]. It involves a current-current deformation of the WZNW model on OSP(4|2) at level $k = 1$ by the operator 2.24. Thus, we can exploit the all-loop result (2.25) for the anomalous dimensions of $\frac{1}{2}$BPS fields. For some special value of the deformation parameter, we are able to identify the low lying spectrum of the supersphere model. The identification includes the one-loop anomalous dimensions of the sigma model. For fields with two or more derivatives we find a few discrepancies. While these discrepancies have the potential to disprove the duality, there are some features of the perturbative results that seem to limit their applicability. We comment on these issues but will not settle them. On the other hand, our results exhibit several features that were anticipated by Polyakov [32]. In particular, we will show how the singular vectors of the WZNW model are related to the sigma model constraints and equations of motion. We conclude the chapter with a calculation of the anomalous dimensions of the ground states of conformal sigma models on symmetric spaces to higher orders. We find that the corrections vanish identically at two- and three-loop order and argue that for $\frac{1}{2}$BPS states all higher corrections must vanish as well. This last result further supports the proposed duality of the $S^{3|2}$ sigma model with the deformed OSP(4|2) WZNW model.

4.1 The spectrum of the supersphere $S^{3|2}$

The aim of this section is twofold. Partly, we would like to illustrate the general results we have reviewed in the previous chapter through the simplest non-trivial example of an interacting conformal sigma model, namely the theory with target space $S^{3|2}$. This supersphere can be considered as a quotient G/H of the compact supergroup $G = \mathrm{OSP}(4|2)$ by the subgroup $H = \mathrm{OSP}(3|2)$. Since the latter is fixed by an order two automorphism of the former, the supersphere $S^{3|2}$ is a compact symmetric superspace. Hence, all the results we outlined in Sections 3.1 and 3.3.1 of the previous chapter apply to this case. Our task is to work them out explicitly. This will require some input from the representation theory of $\mathfrak{osp}(4|2)$ and $\mathfrak{osp}(3|2)$ which can be found in Appendices B–D. The second purpose of this section is to gather some data about the supersphere sigma model that we can later use to test the conjectured duality with the OSP(4|2) Gross-Neveu model.

We will begin by describing several equivalent formulations of the supersphere sigma model. Concrete results on low-gradient operators and their anomalous dimensions are worked out in the second subsection. In the third subsection we describe the more conventional construction of (low-gradient) vertex operators in terms of the fundamental field of the non-linear sigma model. While this turns out to be significantly more cumbersome than the approach advocated in the previous subsection, it will allow us to understand the impact of symmetries and equations of motion.

4.1.1 The supersphere sigma model

The most basic description of the supersphere $S^{3|2}$ is as a co-dimension one supermanifold in the flat superspace $\mathbb{R}^{4|2}$ defined by the equation

$$X \cdot X := \sum_{j=1}^{4} x_j^2 + 2\eta_1\eta_2 \;=\; 1 \;. \tag{4.1}$$

Here, $x_j, j = 1, \ldots, 4$, and η_1, η_2, are the bosonic and fermionic coordinates of $\mathbb{R}^{4|2}$, respectively. We shall often combine these coordinates into a multiplet of supercoordinates $X = (X_A) = (x_j, \eta_1, \eta_2)$. For a pair X and Y in such multiplets the inner product $\cdot$ is defined as

$$X \cdot Y = \sum_j x_j y_j + \eta_1 \xi_2 - \eta_2 \xi_1 \,. \tag{4.2}$$

Here, we have denoted the fermionic coordinates of Y by ξ_1 and ξ_2. We can now write the action of the associated sigma model as

$$S^{\mathrm{SM}}[X, \rho] = \frac{R^2}{2\pi} \int d^2 z \big(\partial X \cdot \partial X - \rho(X \cdot X - 1)\big) \,. \tag{4.3}$$

Here ρ is a Lagrange multiplier that implements the supersphere constraint (4.1). The parameter R can be interpreted as the radius of the supersphere. In the regime where R is large, the sigma model is weakly coupled and perturbation theory should give reliable results. The equations of motion for the field multiplet X read

$$\partial \bar{\partial} X = (\partial X \cdot \bar{\partial} X) X \,. \tag{4.4}$$

From our description of the supersphere through equation (4.1) it is evident that $S^{3|2}$ comes equipped with an $\mathfrak{osp}(4|2)$ action. In fact, the Lie superalgebra $\mathfrak{osp}(4|2)$ acts on the embedding space $\mathbb{R}^{4|2}$ through its fundamental representation. By the very definition of $\mathfrak{osp}(4|2)$ this action respects the constraint (4.1). The supersphere $S^{3|2}$ can be obtained from the supergroup OSP(4|2) by taking the following quotient

$$S^{3|2} \;=\; \mathrm{OSP}(4|2)/\mathrm{OSP}(3|2) \tag{4.5}$$

with respect to the right action of the subsupergroup $\mathrm{OSP}(3|2) \subset \mathrm{OSP}(4|2)$. The latter appears as the stabilizer of a point $X = (X_A) = (1, 0, 0, \dots)$ on the supersphere. Since this stabilizer is left invariant by the reflection of the first coordinate, the quotient (4.5) is a symmetric superspace. In conclusion, we have shown that the sigma model (4.3) possesses all the properties that we assumed in the previous chapter.

In order to get a better feeling for how non-trivial the supersphere sigma model really is, we solve the constraint (4.1) explicitly. To this end, we parametrize $S^{3|2}$ through three angular coordinates ϑ_j and 2 fermionic variables η_b. The line element takes the following form

$$ds^2 \;=\; 2(1 - \eta_1\eta_2) d\eta_1 d\eta_2 + (1 - 2\eta_1\eta_2) d\Omega_3 \tag{4.6}$$

where

$$d\Omega_3 \;=\; d\vartheta_1^2 + \cos^2\vartheta_1 \; d\vartheta_2^2 + \sin^2\vartheta_1 \; d\vartheta_3^2$$

is the usual line element of the 3-dimensional unit sphere. In the sigma model, the coordinates are promoted to fields and the action reads

$$\begin{aligned} S^{\mathrm{SM}}[\vartheta,\eta] \; &= \frac{R^2}{2\pi}\int d^2z\Big(2(1-\eta_1\eta_2)(\partial\eta_1\bar\partial\eta_2-\partial\eta_2\bar\partial\eta_1) \\ &+(1-2\eta_1\eta_2)(\partial\vartheta_1\bar\partial\vartheta_1+\cos^2\vartheta_1\,\partial\vartheta_2\bar\partial\vartheta_2+\sin^2\vartheta_1\,\partial\vartheta_3\bar\partial\vartheta_3)\Big)\,. \end{aligned} \tag{4.7}$$

For the sigma model on the purely bosonic 3-sphere the coupling R runs and in order for the flow to end in a non-trivial fixed-point one must add a Wess-Zumino term [66]. But the presence of the two fermionic directions changes the situation profoundly. As shown in [67], the β-function of the sigma model on $S^{3|2}$ is the same as for a bosonic sigma model on a sphere S^d whose dimension $d = 3-2 = 1$ is given by the difference between the number of bosonic and fermionic coordinates. Consequently, the β-function vanishes for the sigma model on $S^{3|2}$, i.e. the model (4.7), defines a family of non-unitary interacting conformal field theories at central charge $c = 1$ with continuously varying exponents.

Before we apply the results reviewed in Chapter 3 to this model, let us note that the action (4.7) can be written very compactly if we factorize the metric with the help of the super-Vielbeins $E^J_\alpha(\vartheta,\eta)$,

$$g^{IJ}(\vartheta,\eta) \;:=\; \kappa^{\alpha\beta}E^I_\alpha(\vartheta,\eta)E^J_\beta(\vartheta,\eta)(-1)^{|\beta|(|I|+|\alpha|)} \tag{4.8}$$

where κ is the invariant form of $\mathfrak{osp}(4|2)$ and the indices α,β run over directions along the quotient $\mathfrak{m} = \mathfrak{osp}(4|2)/\mathfrak{osp}(3|2)$. We can now combine the Vielbeins with the derivatives of the coordinate fields $(\theta_J) = (\vartheta_j,\eta_a)$ as in eq. (3.2) to obtain

$$S^{\mathrm{SM}}[\theta] = \frac{R^2}{2\pi}\int d^2z\, g^{IJ}(\vartheta,\eta)\,\partial\theta_I\bar\partial\theta_J = \frac{R^2}{2\pi}\int d^2z\,\kappa^{\alpha\beta}j_\alpha(z,\bar z)\bar\jmath_\beta(z,\bar z)\,. \tag{4.9}$$

Of course, all the non-linearity of the action (4.7) is just hidden in the complicated structure of the fields j and $\bar\jmath$. Note that the latter transform in the fundamental representation of the stabilizer subgroup OSP(3|2). In the action the corresponding index α is contracted with the β so as to give an invariant.

Unlike the sigma model on $S^1 = \mathrm{U}(1)$, the theory defined by the action (4.7) is not free. For large radius R, the model is weakly coupled and its properties may by studied perturbatively. But as we pass to a more strongly curved background, computing quantities as a function of the radius R may seem like a very daunting task. This is even more so because there is very little symmetry to work with. As a conformal field theory, the sigma model on the supersphere possesses the usual chiral Virasoro symmetries. But for a model with multiple bosonic coordinates the two sets of chiral Virasoro generators are not sufficient to make the theory rational. Since there are no efficient tools to construct the theory at generic values of the radius parameter R, finding a dual description whose perturbative regime describes a strongly curved supersphere is of obvious interest.

4.1.2 Vertex operators and anomalous dimensions

Before we can begin constructing vertex operators for the supersphere sigma model we need a little bit of background on representations of both $\mathfrak{osp}(4|2)$ and $\mathfrak{osp}(3|2)$. A much more comprehensive discussion can be found in the appendices. It is heavily based on two papers by van der Jeugt [68, 69].

The bosonic subalgebra of the Lie superalgebra $\mathfrak{osp}(4|2)$ is $\mathfrak{so}(4)\oplus\mathfrak{sp}(2)$. Since this has rank $r = 3$, generic representations are labeled by triples of weights $[j_1, j_2, j_3]$. Atypical (or BPS) representations satisfy a single shortening condition. The possible conditions are listed in eq. (B.3). With one such condition relating the three weights j_i, atypical representations $\Lambda_{l,k}$ are labeled by two integers $l \geq 0$ and k. The precise relation between l, k and the weights j_i are given in eqs. (B.5) and (B.6). Let us only note that the label of the trivial representation is $\Lambda_{0,0}$ while that of the 17-dimensional adjoint is $\Lambda_{0,1}$. The representations $\Lambda_{l,0}$ on the other hand are associated with (graded) symmetric traceless tensors of $\mathfrak{osp}(4|2)$.

In the atypical representation $\Lambda_{l,k}$, the quadratic Casimir element $\mathbf{Cas}_\mathfrak{g}$ takes the value

$$\mathbf{Cas}_\mathfrak{g}(\Lambda_{l,k}) = l^2 \, . \tag{4.10}$$

We conclude that the Casimir element $\mathbf{Cas}_\mathfrak{g}$ is insensitive to the second label k of $\Lambda_{l,k}$. Atypical representations with the same value of the Casimir element are said to belong to the same block. Representations from the same block may appear within larger indecomposables, in particular they make up the projective covers $\mathcal{P}_{\Lambda_{l,k}}$. The composition series of these indecomposables are given in eqs. (B.12)–(B.15).

Let us turn our attention to the Lie superalgebra $\mathfrak{osp}(3|2)$. In this case, the bosonic subalgebra $\mathfrak{so}(3) \oplus \mathfrak{sp}(2)$ has rank two and hence generic representations are labeled by a pair $[q, p]$ of weights. The atypicals λ_0 and $\lambda_q = [q, 2q - 1]$, $q \geq 1/2$, form a 1-parameter family of representations that satisfy a single shortening condition. The label λ_0 is reserved for the trivial representation, $\lambda_{1/2}$ is the 5-dimensional fundamental. In the case of $\mathfrak{osp}(3|2)$, the adjoint is not atypical. Its label is $\lambda_{\text{ad}} = [1, 0]$.

In the representation $[q, p]$ the quadratic Casimir element $\mathbf{Cas}_{\mathfrak{h}}$ of the algebra $\mathfrak{osp}(3|2)$ takes the values

$$\mathbf{Cas}_{\mathfrak{h}}\big([q,p]\big) = (p+2q)(p-2q+1)\,. \tag{4.11}$$

We see that it vanishes for atypical representations λ_q. All these atypical representations belong to the unique single block from which indecomposable modules can be built. Once again, the most relevant indecomposables are the projective covers $\mathcal{P}_{\lambda_q}$ of atypicals. Their composition series are displayed in eqs. (B.12)–(B.15).

Before we proceed, let us remark that we are not interested in the indecomposable structure of the spectra. We are, effectively, computing the partition functions which are traces. Since traces are blind to the indecomposable structure we are only interested in the irreducible constituents. Therefore, we will often write + instead of ⊕, unless it is immediately obvious that a sum is indeed direct.

With these notations set up, we can begin to construct vertex operators. Our goal is to find all vertex operators with up to two derivatives that transform in $\frac{1}{2}$BPS representations $\Lambda_{l,k}$ of $\mathfrak{osp}(4|2)$. Let us start with the zero modes. By definition, these fields have vanishing scaling dimension at $R = \infty$ so they cannot contain any currents j or $\bar{\jmath}$. Consequently, the $\mathfrak{osp}(3|2)$ representations $\mu, \overline{\mu}$ and λ that label our vertex operators (3.10) are all trivial. Thus, the head must be taken from

$$\Gamma_0 = \Gamma_{\lambda_0} = \bigoplus_{l=0}^{\infty} \Lambda_{l,0}\,, \tag{4.12}$$

where $\Lambda_{l,0} = \frac{1}{2}[l+1, l-1, -l-1]$ for $l > 0$ and $\Lambda_{0,0}$ is the trivial representation. In order to find the decomposition displayed on the right hand side, we employed the decomposition formulas (D.1)–(D.3) along with the fundamental results (3.13). The summation is over all those representations Λ of $\mathfrak{osp}(4|2)$ for which the restriction of $\mathcal{P}_\Lambda$ to the subalgebra $\mathfrak{osp}(3|2)$ contains $\mathcal{P}_{\lambda_0}$. Our formulas in appendix D only list the decompositions for atypical representations $\Lambda = \Lambda_{l,k}$ but it is not difficult to see that typical (long) multiplets never contain $\mathcal{P}_{\lambda_0}$ in their decomposition. Hence, the formula (4.12) is exact, i.e. it accounts for all elements of Γ_0 not just for those that transform in $\frac{1}{2}$BPS representations. Of course, the space Γ_0 is nothing but the space of functions on the supersphere $S^{3|2}$. Aside from the trivial representation $\Lambda_{0,0}$ of $\mathfrak{osp}(4|2)$, which has vanishing Casimir, all other operators acquire a non-zero anomalous dimension,

$$\delta_R^{(1)} h(V_{\Lambda_{l,0},\lambda_0}) = \frac{1}{2R^2}\,\mathbf{Cas}_{\mathfrak{g}}(\Lambda_{l,0}) = \frac{l^2}{2R^2}\,. \tag{4.13}$$

The next set of operators we would like to look at are the operators of weight $(h_\infty, \bar{h}_\infty) = (1,0)$. Such operators contain a current j and hence have $\mu = \lambda = \lambda_{\frac{1}{2}}$ while $\overline{\mu} = \lambda_0$ is trivial. Hence, the head of the operators must be taken from sections in the bundle

$$\Gamma_{\lambda_{\frac{1}{2}}} = \Lambda_{0,1} \oplus \bigoplus_{l=1}^{\infty} \Lambda_{l,0} \oplus \text{typicals}\,. \tag{4.14}$$

The decomposition on the right hand side is obtained from the formulas in appendix D, just as in the previous example. We see that one $\frac{1}{2}$BPS section in the bundle of the fundamental representation $\lambda_{\frac{1}{2}}$ of $\mathfrak{osp}(3|2)$ is the adjoint multiplet of $\mathfrak{osp}(4|2)$. The corresponding fields are the Noether currents. According to our result (3.38) their one-loop anomalous dimension vanishes since both the Casimir of the fundamental $\lambda_{\frac{1}{2}}$ and the Casimir of the adjoint $\Lambda_{0,1}$ vanish. The remaining $\frac{1}{2}$BPS fields are derivatives of the zero modes. Their anomalous dimension is the same as for the zero modes themselves.

The $\frac{1}{2}$BPS spectrum of operators of weight $(h_\infty, \bar{h}_\infty) = (1,1)$ is a bit richer. In this case, our operators must contain j and $\bar{\jmath}$ so that $\mu = \lambda_{\frac{1}{2}} = \overline{\mu}$. In the tensor product of the two fundamental representations μ and $\overline{\mu}$ we find $\lambda = \lambda_0$, $[1,0]$, $[\frac{1}{2},1]$. Hence, the zero mode contributions can come from 3 different bundles. The decomposition of the bundle Γ_0 was described in eq. (4.12) already. So it remains to describe the two bundles

$$\Gamma_{[1,0]} = 2\Lambda_{0,1} + \Lambda_{0,2} + \text{typicals} \tag{4.15}$$

and

$$\Gamma_{[\frac{1}{2},1]} = \sum_{l=2}^{\infty} (2\Lambda_{l,0} + \Lambda_{l,1} + \Lambda_{l,-1}) + \text{typicals}\,. \tag{4.16}$$

If we sum up all the contributions from the three possible bundles, we find that the spectrum of operators of weight $(h_\infty, \bar{h}_\infty) = (1,1)$ decomposes into

$$\Gamma_{\lambda_{\frac{1}{2}} \otimes \lambda_{\frac{1}{2}}} \cong \Lambda_{0,0} + 2\Lambda_{0,1} + \Lambda_{0,2} + \Lambda_{1,0} + \sum_{l=2}^{\infty} (3\Lambda_{l,0} + \Lambda_{l,1} + \Lambda_{l,-1}) + \text{typicals} \tag{4.17}$$

The one-loop anomalous dimension of the corresponding operators is determined by the first label of the representation,

$$\delta_R^{(1)} h = \frac{1}{2R^2}\, \mathbf{Cas}_{\mathfrak{g}}(\Lambda_{l,k}) = \frac{l^2}{2R^2}\,. \tag{4.18}$$

We see in particular that our sigma model contains 145 operators with vanishing one-loop anomalous dimension. These sit in four different representations of $\mathfrak{osp}(4|2)$. There is one state in the trivial representation $\Lambda_{0,0}$.

This is the sigma model interaction that remains marginal at one-loop. It actually remains marginal at all-loops. In addition, there are two adjoint multiplets $\Lambda_{0,1}$ of dimension 17 each. The multiplicity two is actually a signature of the distinction between projective covers and irreducibles. As we explained above, one could have expected that the multiplicity of the adjoint $\mathfrak{osp}(4|2)$ section in the bundle associated to the adjoint representation $[1,0]$ of $\mathfrak{osp}(3|2)$ is given by the number of times the 12-dimensional $[1,0]$ appears in the decomposition of the 17-dimensional $\Lambda_{0,1}$. Clearly, this multiplicity is one which is not the correct answer for the number of $\Lambda_{0,1}$ multiplets in $\Gamma_{[1,0]}$. So indeed the example illustrates nicely how important it is to determine the multiplicity of short operators using decompositions of projective covers rather than irreducibles.

4.1.3 An alternative construction of vertex operators

In order to fully appreciate the results of the previous subsection and the elegance of their derivation, we would like to compare our findings with more conventional constructions of vertex operators from the fundamental field multiplet X. In doing so, we will have to struggle a little bit with the implications of the constraint (4.1) and the equations of motion (4.4) on counting coset fields. As a reward, we will understand e.g. that the number 145 of operators with vanishing one-loop anomalous dimension contains non-trivial information about the dynamics of the supersphere sigma model.

In building coset fields from the fundamental field multiplet X we shall start with the zero modes. For $h_\infty = \bar{h}_\infty = 0$ the relevant fields contain no derivatives and they are given by monomials $F_{l,0}(X)$ of order $l = 0, 1, 2, \dots$ in the components of X. Once we implement the constraint $X^2 = 1$ the components of $F_{l,0}(X)$ transform in the traceless symmetric tensor representations $\Lambda_{l,0}$. This agrees with our formula (4.12) above.

Let us now proceed to fields of weight $(h_\infty, \bar{h}_\infty) = (1, 0)$. These must be of the form

$$F_{l,0}(X)\,\partial X \tag{4.19}$$

for $l = 0, 1, 2, \dots$. The space of such objects transforms in the tensor product $\Gamma_0 \otimes \Lambda_{1,0}$ of symmetric traceless tensors with the fundamental $\Lambda_{1,0}$. But not all these fields are non-zero. In fact, by taking a derivative of the constraint $X^2 = 1$ we obtain

$$X \cdot \partial X = X_a \partial X^a = 0 \tag{4.20}$$

Consequently any field of the form $F_{l,0}(X) X \cdot \partial X$ vanishes. Such fields transform in the representation Γ_0. If we remove them from the list (4.19) we end up with a space of fields transforming in

$$\Gamma_0 \otimes \Lambda_{1,0} - \Gamma_0 = \Lambda_{0,1} + \sum_{l=1}^{\infty} \Lambda_{l,0} + \text{typicals} = \Gamma_{\lambda_{\frac{1}{2}}} \,. \qquad (4.21)$$

This agrees with our result (4.14). We have already interpreted the corresponding fields as the Noether currents and derivatives of the zero modes.

Let us now turn to the most interesting set of fields, those with weights $h = 1 = \bar{h}$. In this case, the counting will be affected by the equations of motion. The relevant fields can all be written in either of the following forms

$$F_{l,0}(X)\,\partial\bar{\partial}X \quad , \quad F_{l,0}(X)\,\partial X\bar{\partial}X \,. \qquad (4.22)$$

Our analysis of the space of these operators will proceed in two steps. First we shall fully implement the constraint $X^2 = 1$ and then we consider the equations of motion. By taking derivatives of the constraint $X^2 = 1$ we obtain the two equations

$$X \cdot \partial X = 0 = X \cdot \bar{\partial} X \,. \qquad (4.23)$$

We can multiply each of these two equations with one of the previously found operators of dimension $(h_\infty, \bar{h}_\infty) = (1, 0)$ or $(h_\infty, \bar{h}_\infty) = (0, 1)$, respectively. All such operators vanish. As we discussed above, they transform in $2\Gamma_{\lambda_{\frac{1}{2}}}$. Additionally, we also need to remove all operators created from the zero modes by multiplication with the operator $X \cdot \partial X X \cdot \bar{\partial} X$. These transform in Γ_0. This is not quite the end of story. In fact, there is another family of operators that vanishes because of the constraint $X^2 = 1$. To see this, we differentiate the constraint $X^2 = 1$ by $\partial\bar{\partial}$ and obtain

$$\partial X \cdot \bar{\partial} X = -X \cdot \partial\bar{\partial} X \,. \qquad (4.24)$$

This constraint allows us to remove all the operators of the form $F_{l,0}(X)\partial X \cdot \bar{\partial} X$. In other words when considering the second family in eq. (4.22), we can restrict to those operators for which $\partial X\bar{\partial}X$ transforms either in the representation $\Lambda_{2,0}$ (symmetric traceless) or in $\Lambda_{0,1}$ (antisymmetric). Putting all this together we find

$$\begin{aligned}
&\Gamma_0 \otimes \Lambda_{1,0} + \Gamma_0 \otimes (\Lambda_{2,0} + \Lambda_{0,1}) - 2\Gamma_{\lambda_{\frac{1}{2}}} - \Gamma_0 \\
&\qquad = \Lambda_0 + 3\Lambda_{0,1} + \Lambda_{0,2} + 2\Lambda_{1,0} + \sum_{l=2}^{\infty} (4\Lambda_{l,0} + \Lambda_{l,1} + \Lambda_{l,-1}) + \text{typicals} \\
&\qquad = \Gamma_{\lambda_{\frac{1}{2}} \otimes \lambda_{\frac{1}{2}}} + \Lambda_{0,1} + \sum_{l=1}^{\infty} \Lambda_{l,0} + \text{typicals} \,.
\end{aligned} \qquad (4.25)$$

A quick glance at eq. (4.17) shows that we obtained more than we expected. The reason is simple. While we have correctly implemented the constraint $X^2 = 1$, operators of weight $(h_\infty, \bar{h}_\infty) = (1,1)$ are the first ones to be sensitive to the equations of motion. The latter precisely remove the unwanted multiplets. In the block of the zero, for example, the operators

$$X_I \partial\bar{\partial} X_J - X_J \partial\bar{\partial} X_I \tag{4.26}$$

contribute one of the three $\Lambda_{0,1}$ in the decomposition we have listed. Once we insert the equations of motion, however, these operators are set to zero

$$X_I \partial\bar{\partial} X_J - X_J \partial\bar{\partial} X_I = \partial X \cdot \bar{\partial} X \; (X_I X_J - X_J X_I) = 0\,. \tag{4.27}$$

Hence, the fact that we found 145 operators of weight $(h_\infty, \bar{h}_\infty) = (1,1)$ with vanishing one-loop anomalous dimension is sensitive to the equations of motion. Without them there would be 17 additional such operators.

4.2 Duality with the $\mathfrak{osp}(4|2)$ Gross-Neveu model

One lesson which has been learned through past studies of sigma models is that they should not be considered as an isolated research topic. There exist other important constructions of 2D (conformal) field theories which are intimately tied to sigma models and sometimes can provide intriguing insights into the non-perturbative features of sigma models. We have already alluded to the example of sigma models on Calabi-Yau spaces which possess a dual description in terms of (products of) WZNW coset models. Another, more elementary, example is the compactified free boson which admits a dual description in terms of two Majorana fermions. The proposed duality between the sigma model on $S^{3|2}$ and the $\mathfrak{osp}(4|2)$ Gross-Neveu model that we described in the introduction is quite similar to the Coleman-Mandelstam duality between bosons and fermions only that the abelian symmetry $\mathfrak{u}(1) = \mathfrak{so}(2)$ is replaced by the non-abelian $\mathfrak{osp}(4|2)$.

In the first subsection we shall describe the $\mathfrak{osp}(4|2)$ Gross-Neveu model and some of its most basic features. Then we review a central all-loop result from [39] on the (target space) $\frac{1}{2}$BPS spectrum of perturbed supergroup WZNW models and explain how it applies to the $\mathfrak{osp}(4|2)$ Gross-Neveu model. In the third subsection we try to match the $\frac{1}{2}$BPS spectrum of the Gross-Neveu model for a certain value of the Gross-Neveu coupling to the one-loop spectrum of the supersphere sigma models. We will find perfect agreement for low lying states, but also some discrepancies that involve fields with more derivatives.

4.2.1 The $\mathfrak{osp}(4|2)$ Gross-Neveu model

The fundamental field multiplet $\Psi = (\Psi_A) = (\psi_j, \gamma_a)$ of the $\mathfrak{osp}(4|2)$ Gross-Neveu model consists of four Majorana fermions $\psi_j, j = 1, \ldots, 4$, and a bosonic $\beta\gamma$-system whose fields we shall denote by $\gamma_1 = \gamma$ and $\gamma_2 = \beta$. In addition, there is a second multiplet $\overline{\Psi} = (\overline{\psi}_j, \overline{\gamma}_a)$ of opposite chirality. All these six fields in Ψ possess conformal weight $h = 1/2$ and transform in the fundamental representations $\Lambda_{1,0}$ of $\mathfrak{osp}(4|2)$. The same applies to $\overline{\Psi}$. In terms of these field multiplets, the action of the Gross-Neveu model reads

$$S^{\mathrm{GN}}[\psi, \gamma, \overline{\psi}, \overline{\gamma}] = \frac{1}{2\pi} \int d^2 z \Big[\sum\nolimits_j \big(\psi_j \bar{\partial} \psi_j + \bar{\psi}_j \partial \bar{\psi}_j\big) + \big(\gamma_2 \bar{\partial} \gamma_1 + \bar{\gamma}_2 \partial \bar{\gamma}_1\big) \Big] + \frac{g}{2\pi} \int d^2 z \Big[\sum\nolimits_j \psi_j \bar{\psi}_j + (\gamma_1 \bar{\gamma}_2 - \gamma_2 \bar{\gamma}_1) \Big]^2 . \tag{4.28}$$

The $\mathfrak{osp}(4|2)$ invariance of this action is manifest since all indices are contracted with the $\mathfrak{osp}(4|2)$ invariant metric. When written in terms of Ψ and $\overline{\Psi}$, rather than its components, the action takes the same form as that of the massless Thirring model with its characteristic fourth order interaction term. When the coupling constant g is set to zero the model is free and scale invariant. It possesses a Virasoro symmetry with central charge $c = 1$. The latter receives a contribution $c_j = 1/2$ from each of the fermions ψ_j and $c_a = -1/2$ from the two components of the $\beta\gamma$-system. Switching on the coupling g introduces a very non-trivial action but it turns out to preserve conformal symmetry. In fact, the β-function for the coupling g is proportional to the dual Coxeter number $h^\vee = \mathbf{Cas}_{\mathfrak{g}}(\Lambda_{0,1})$ and hence vanishes for $\mathfrak{osp}(4|2)$. Therefore, the $\mathfrak{osp}(4|2)$ Gross-Neveu model defines a one-parameter family of interacting conformal field theories with central charge $c = 1$.

While the interaction in the $\mathfrak{osp}(4|2)$ Gross-Neveu model preserves the Virasoro and a global $\mathfrak{osp}(4|2)$ symmetry, the free field theory possesses additional current algebra symmetries that are broken when $g \neq 0$. In order to describe these symmetries, we recall that the components of the field multiplet Ψ obey the following operator product expansions

$$\psi_i(z)\psi_j(w) \sim \frac{\delta_{ij}}{z-w} + \ldots \quad , \quad \gamma_2(z)\gamma_1(w) \sim \frac{\delta_{ab}}{z-w} . \tag{4.29}$$

Using these operator product expansions between the fundamental constituents it is standard to show that the following quadratic combinations

$$J_{AB} = \Psi_A \Psi_B \quad \text{where} \quad (\Psi_A) = (\psi_i, \gamma_b) \tag{4.30}$$

obey the algebraic relations of an $\mathfrak{osp}(4|2)$ current algebra at level $k = 1$. Let us stress once again that this current algebra symmetry is broken as soon as we switch on the coupling.

The current algebra symmetry suggests interpreting the free theory at $g = 0$ as a Wess-Zumino-Novikov-Witten (WZNW) model. In addition, it is not difficult to verify that the fourth order interaction term of the Gross-Neveu model can be expressed in terms of the currents (4.30) as

$$\frac{g}{2\pi}\int d^2z \left[\sum\nolimits_i \psi_i\bar{\psi}_i + \gamma_1\gamma_2 - \gamma_2\bar{\gamma}_1\right]^2 \;=\; \frac{g}{2\pi}\int d^2z \sum_{AB} J_{AB}(z)\bar{J}^{AB}(\bar{z})\,. \tag{4.31}$$

Putting all this together we have shown that the Gross-Neveu model can be thought of as a deformed WZNW model at level $k = 1$,

$$S^{\mathrm{GN}} = S^{\mathrm{WZNW}}_{k=1} + \frac{g}{2\pi}\int d^2z \sum_{AB} J_{AB}(z)\bar{J}^{AB}(\bar{z}) \tag{4.32}$$

where the deformation is generated by an exactly marginal current-current interaction. This reformulation of the $\mathfrak{osp}(4|2)$ Gross-Neveu model will become important when we apply the powerful results of [39] to the Gross-Neveu model.

4.2.2 The BPS spectrum

In Section 2.5 we reviewed an all-loop result for deformed WZNW models. Let us now specialize the very general result (2.25) to the $\mathfrak{osp}(4|2)$ Gross-Neveu model or, equivalently, to the current-current deformation of the $\mathfrak{osp}(4|2)$ WZNW model at level $k = 1$. In this case our formula can be applied to all fields that transform in one of the atypical representations $\Lambda_{l,k}$ or any indecomposable composites formed from these. Let us recall that the value of the quadratic Casimir element assumes the value $\mathbf{Cas}_{\mathfrak{g}}(\Lambda_{l,k}) = l^2$ on such atypicals. Hence, our general formula (2.25) becomes

$$\delta_g^{(\infty)} h_{\mathrm{BPS}} = \frac{gl^2}{2\,(1-g^2)} - \frac{g}{2\,(1+g)}\left(\mathbf{Cas}^L_{\mathfrak{g}} + \mathbf{Cas}^R_{\mathfrak{g}}\right). \tag{4.33}$$

for fields transforming in $\Lambda = \Lambda_{l,k}$ with respect to the diagonal action of $\mathfrak{g}$. Note that the function $\delta_g^{(\infty)} h$ develops a singularity at $g = -1$, at least for a large number of states. This simple observation motivates the identification of the point $g = -1$ with the $R \to \infty$ limit of the $S^{3|2}$ sigma model. In fact, in the sigma model one expects that all winding states develop infinite energy when $R \to \infty$. So, if we want the sigma model to be dual to the

Gross-Neveu model, we are forced to identify $g = -1$ with the infinite radius limit. The precise relation between the coupling g and the radius R reads [39] [1]

$$g = \frac{4 - R^2}{4 + R^2}. \tag{4.34}$$

For a state to remain in the spectrum at the point $g = -1$, the anomalous dimension (4.33) has to remain finite. This is the case if

$$\mathbf{Cas}^L_{\mathfrak{g}} + \mathbf{Cas}^R_{\mathfrak{g}} = \frac{l^2}{2}. \tag{4.35}$$

We call eq. (4.35) the no-winding condition. For states that satisfy this condition, the anomalous dimension (4.33) simplifies to

$$\delta_g^{(\infty)} h_{\text{BPS}} = \frac{1}{4}\frac{gl^2}{1-g} = -\frac{l^2}{8} + \frac{l^2}{2R^2}\,. \tag{4.36}$$

Here we also inserted eq. (4.34) so that the anomalous dimension of the Gross-Neveu model fields is finally written in terms of the radius parameter R of the sigma model. We have now gathered all the ingredients we need in order to perform our first tests of the duality. Eq. (4.35) tells us which states of the free field theory make it into the spectrum at $g = -1$ and eq. (4.36) allows us to compute the corresponding conformal weight. We will now start to compare the resulting spectrum at $g = -1$ with the free supersphere sigma model.

In our discussion of the one-loop anomalous dimensions for coset sigma models we briefly commented on a puzzling instability that arises from high gradient operators. The same type of instabilities also appears in perturbed WZNW models, at least for generic choices of the target group and the level. To leading order in perturbation theory this was observed by Ryu et al. in [41]. With the help of formula (2.25) one may show that these instabilities persist to any order in perturbation theory. The authors of [41] also observed that no instabilities occur for $\mathfrak{psu}(N|N)$ WZNW models at level $k = 1$. This observation, however, does not carry over to our $\mathfrak{osp}(4|2)$ WZNW model at level $k = 1$. In fact, one can show that this theory contains instabilities arbitrarily close to the free field theory, much as it is the case for sigma models. For now, we shall close an eye on these issues but we will revisit them in Chapter 5.

[1] The cohomological methods developed in [55] imply that the relation is identical to the one that appears in the duality between a compactified free boson and the massless Thirring model.

4.2.3 Checking the proposed duality

We want to apply the results on the deformation of the $\frac{1}{2}$BPS spectrum in deformed supergroup WZNW models in order to test the proposed duality between the $\mathfrak{osp}(4|2)$ Gross-Neveu model and the supersphere sigma model. In the first subsection we shall show that the zero mode spectrum of the sigma model is recovered along with its one-loop deformation. This is a remarkable example of an emergent geometry. In the WZNW model, the fields that are associated with spherical harmonics of the supersphere possess very large scaling dimensions. These come down until they become zero modes, i.e. fields with vanishing scaling weight, in the sigma model limit. Let us anticipate that the singular vectors of the $\mathfrak{osp}(4|2)$ WZNW model at level $k = 1$ play an important role for this identification with the zero mode spectrum of the sigma model to work out. Then we turn to derivative fields of the sigma model. We will argue that the agreement continues to hold for fields of conformal weight $(h_\infty, \bar{h}_\infty) = (1, 0), (0, 1)$ in the sigma model. This may not come as a big surprise. Things become more interesting for the fields with conformal weight $(h_\infty, \bar{h}_\infty) = (1, 1)$ since these are sensitive to the equations of motion in the sigma model. Recall that in the sigma model we found 145 states with vanishing one-loop scaling dimension. This will be exactly matched by the deformed WZNW model. In the WZNW model, the scaling dimension of the corresponding 145 states is independent of the coupling so that the conjectured duality makes an interesting prediction: All higher-loop corrections to the scaling weight of the 145 states are actually zero. The match between the deformed WZNW model and the sigma model extends to many other fields with $(h_\infty, \bar{h}_\infty) = (1, 1)$. On the other hand, we will also find sigma model fields that cannot be reproduced within the deformed WZNW model.

Ground state spectrum

One key piece of evidence in support of the proposed duality is the observation that we can actually recover all the zero modes of the sigma model. Under the action of the global $\mathfrak{osp}(4|2)$ symmetry the space Γ_0 of functions on the supersphere decomposes into a sum of irreducible multiplets $\Lambda_{l,0}$, see eq. (4.12). Each of these multiplets appears with multiplicity one. Other atypical representations $\Lambda_{l,k}$, $k \neq 0$ do not occur.

As we have explained before, the states of the Gross-Neveu model can be constructed from a chiral multiplet $\Psi = \Psi^L$ that transforms in a 6-dimensional representation of $\mathfrak{osp}(4|2)$. The $\mathfrak{osp}(4|2)$ representation matrices are those known from the usual fundamental representation, but the grading rules are reversed so that the fermionic subspace is 4-dimensional while the bosonic has dimension 2. It is a remarkable fact that the conformal dimension h of all chiral operators $\mathcal{O}^L$ in the undeformed case is bounded from below by

$$h_0\big(\mathcal{O}^L_{[\Lambda]}\big) \geq \frac{1}{2}\,\mathbf{Cas}^L_{\mathfrak{g}}(\Lambda)\,. \tag{4.37}$$

for all $\mathcal{O}^L$ that transform in the representation $[\Lambda]$ with respect to the left $\mathfrak{osp}(4|2)$ action. Of course, the corresponding statement holds for all operators $\mathcal{O}^R$ that are constructed from the components of $\overline{\Psi} = \Psi^R$ and their derivatives. It is actually possible to establish the stronger lower bound

$$h_0\big(\mathcal{O}^L_{[\Lambda]}\big) \geq j_1 + j_2(j_2+1) + j_3(j_3+1) + |j_2 - j_3| \geq \frac{1}{2}\,\mathbf{Cas}^L_{\mathfrak{g}}(\Lambda) \tag{4.38}$$

which shows that the inequality (4.37) can only be saturated by very special multiplets, when $j_1 = 0, \frac{1}{2}$. It turns out that for each integer $l = 0, 1, 2, \ldots$ there is a unique field multiplet $\mathcal{O}^L_l$ such that

$$h_0(\mathcal{O}^L_l) = \frac{l^2}{2}. \tag{4.39}$$

The multiplet $\mathcal{O}^L_l$ is obtained as a graded symmetric component in the l-fold tensor product of the fundamental. Since our generating field multiplet Ψ is fermionic, i.e. its grading is reversed in comparison to the grading of the fundamental, the multiplet $\mathcal{O}^L_l$ must contain $l(l-1)/2$ derivatives. Hence, its conformal dimension $h(\mathcal{O}^L_l) = l/2 + l(l-1)/2 = l^2/2$.

Let us illustrate the construction of $\mathcal{O}^L_l$ with a few explicit examples. Of course, the operator $\mathcal{O}^L_0$ is just the identity field while $\mathcal{O}^L_1$ is the fundamental multiplet Ψ. The next multiplet $\mathcal{O}^L_2$ appears at $h(\mathcal{O}^L_2) = 2$,

$$\mathcal{O}^L_2 = \big(\psi_A \partial \psi_B + (-1)^{|A||B|}\psi_B \partial \psi_A\big)\ . \tag{4.40}$$

When we multiply the multiplet $\mathcal{O}^L_l$ with its anti-holomorphic partner $\mathcal{O}^R_l$ we obtain a set of bulk fields which transform in the product $\Lambda_{l,0} \otimes \Lambda_{l,0}$. The only component that can satisfy the no-winding condition is the one in the representation $\Lambda_{2l,0}$. Indeed,

$$\mathbf{Cas}_{\mathfrak{g}}(\Lambda_{2l,0}) = 4l^2 = 2\big(\mathbf{Cas}^L_{\mathfrak{g}}(\Lambda_{l,0}) + \mathbf{Cas}^R_{\mathfrak{g}}(\Lambda_{l,0})\big)\,. \tag{4.41}$$

Let us denote the this component of the product by $V_{2l} = V_{2l}(z, \bar{z})$. To summarize, we have now constructed a field multiplet V_{2l} in the WZNW model that transforms in the representation $\Lambda_{l,0}$ with respect to both the left and the right action of $\mathfrak{osp}(4|2)$ and in the representations $\Lambda_{2l,0}$ with respect to the diagonal action. In the WZNW model, i.e. the free Gross-Neveu model, this field possesses weights $\big(h_0(V_{2l}), \bar{h}_0(V_{2l})\big) = (l^2/2, l^2/2)$.

Since the representation $\Lambda_{2l,0}$ of $\mathfrak{osp}(4|2)$ in which the field V_{2l} transforms is $\frac{1}{2}$BPS, we can apply the results of the previous subsection to compute its dimension for any value of the coupling g and in particular at the point $g = -1$. With the help of the leading term in eq. (4.36) we obtain

$$h(V_{2l})_{g=-1} = h_0(V_{2l}) - \frac{1}{8} 4l^2 = 0. \tag{4.42}$$

Hence, we obtain precisely the spectrum provided by the spherical harmonics $\Lambda_{2l,0}$ in the sigma model, i.e. at least one half of the zero modes of the supersphere sigma model. [2] Remarkably, this identification is also consistent with what we know about the one-loop anomalous dimensions in the sigma model. In fact, if we keep the next to leading term in eq. (4.36) we find

$$h(V_{2l})_g = \frac{l^2}{2} + \frac{gl^2}{1-g} = \frac{2l^2}{R^2}\,. \tag{4.43}$$

This should be compared with the result (4.13) for the one-loop anomalous dimension of the sigma model vertex operators $V_{\Lambda_{2l,0},\lambda_0}$. We see that also the one-loop corrections to the scaling law agree. In the deformed WZNW model, the formula (4.36) is actually exact, i.e. there are no further corrections by terms involving higher powers of the sigma model coupling $1/R^2$. The duality therefore predicts that the anomalous dimensions of zero mode fields in the sigma model are one-loop exact. It should not be too difficult to check this prediction through a direct computation along the lines of [51, 52], where anomalous dimensions of tachyonic vertex operators in bosonic $\mathrm{O}(N)$-models were computed up to four-loop order. The general structure of Wegner's results suggest that higher order corrections indeed vanish for the conformal supersphere models, but we have not yet completed an honest derivation.

Since our fields $\mathcal{O}_l^{L/R}$ are the only ones satisfying the bound (4.37) and the bulk field V_{2l} the only fields we could build from them that solve the no-winding condition (4.35), the deformed WZNW model contains no further field of weight $(h_\infty, \bar{h}_\infty) = (0,0)$ at $g = -1$. Moreover, because of the bound (4.37), all other WZNW fields that solve the no-winding condition end up with $h_\infty + \bar{h}_\infty > 0$ for $g = -1$. In the free sigma model, the conformal

[2] One would expect to obtain the missing zero modes V_{2l+1} from other sectors of the Gross-Neveu model. Without the inclusion of additional states, the Gross-Neveu model is related to an orbifold theory $S^{3|2}/\mathbb{Z}_2$ rather than the supersphere sigma model.

weights are determined by the number of derivatives and hence they are certainly non-negative. So, our results are in beautiful agreement with the proposed duality.

Let stress that the match of zero modes only works for the WZNW model at $k = 1$, i.e. it does make crucial use of the exact position of singular vectors. In order to illustrate this point let us consider the space of states $\mathcal{H}_k^{(l)}$ of conformal weight $h = 2$ ($\bar{h} = 0$). For an $\mathfrak{osp}(4|2)$ WZNW model with $k > 1$, these transform in

$$\mathcal{H}^{(2)} \cong \Lambda_{0,1} + \Lambda_{0,1} \odot \Lambda_{0,1} = \Lambda_{0,0} + \Lambda_{0,1} + \Lambda_{2,-1} + 2\Lambda_{2,0} + \Lambda_{2,1} + [2,0,0]\,. \quad (4.44)$$

The term $\Lambda_{0,1}$ originates from the action of the modes J_{-2}^{AB} while the term $\Lambda_{0,1} \odot \Lambda_{0,1}$ contains the contributions of $J_{-1}^{AB} J_{-1}^{CD} |0\rangle$. A formula for the symmetric tensor product $\odot$ of the adjoint $\Lambda_{0,1}$ can be found at the end of appendix B. Note that there appear four different multiplets in which the Casimir element has the maximal value $\mathbf{Cas}_{\mathfrak{g}}(\Lambda) = 4$, namely the multiplets $\Lambda = \Lambda_{2,k}, k = 0, \pm 1$. At level $k = 1$, the first singular vectors appear at $h = 2$ and these reduce the spectrum to

$$\mathcal{H}_{k=1}^{(2)} \cong \Lambda_{0,0} + \Lambda_{0,1} + \Lambda_{2,0} + [2,0,0] \quad (4.45)$$

so that the representations with maximal Casimir are reduced to a single one, namely $\Lambda_{2,0}$. This is the unique multiplet in $\mathcal{H}_{k=1}^{(2)}$ that is used to build a zero mode at $g = -1$. WZNW models with level $k > 1$ contain many more zero modes and hence cannot be dual to the supersphere sigma model.

Spectrum of gradient operators

After our success in matching the zero modes of the sigma model with fields in the deformed WZNW theory, we want to move on to gradient fields in the sigma model. Some of them are very easy to find. This applies in particular to the operators of weight $(h_\infty, \bar{h}_\infty) = (1, 0)$. Their spectrum was described in eq. (4.14). Most of these fields emerge from the WZNW model derivative operators ∂V_{2l} with $l = 1, 2, \ldots$. The fields V_{2l} were constructed above. The bulk operators ∂V_{2l} have conformal weight $(h_0, \bar{h}_0) = (l^2/2 + 1, l^2/2)$ and they transform in the representation $\Lambda_{2l,0}$. By the same reasoning as above we obtain a family of fields with weight $(h_\infty, \bar{h}_\infty) = (1, 0)$ at the point $g = -1$ which transform in the $\Lambda_{2l,0}$ representations of $\mathfrak{osp}(4|2)$. Their one-loop anomalous dimension coincides with that of the corresponding zero modes. Of course, the match with the operators of weight $(h_\infty, \bar{h}_\infty) = (1, 0)$ is not surprising since they are obtained as derivatives in both the WZNW and the sigma model description.

There is one more set of operators at $(h_\infty, \bar{h}_\infty) = (1, 0)$, namely the Noether currents of the sigma model that sit in the representation $\Lambda_{0,1}$. It is obvious that these arise from the chiral currents J^{AB} in the WZNW model. In fact, the currents of the WZNW model transform in the representation $\Lambda^L = \Lambda_{0,1}$ and $\Lambda^R = \Lambda_{0,0}$ with respect to the left and right action of $\mathfrak{osp}(4|2)$, respectively. Under the diagonal action, the transformation law is described by the tensor product $\Lambda^D = \Lambda_{0,1} \otimes \Lambda_{0,0} = \Lambda_{0,1}$. Since all these representations possess vanishing Casimir, the no-winding condition (4.35) is satisfied and the anomalous contribution to the conformal weight vanishes. Hence, we can identify the deformation of the WZNW currents with the Noether currents of the sigma model.

Let us now turn to the operators of conformal weight $(h_\infty, \bar{h}_\infty) = (1, 1)$ in the sigma model. Their spectrum in the sigma model is given by eq. (4.17). Obviously, we can obtain some of these from the operators $\partial\bar{\partial}V_{2l}, l = 1, 2, \ldots$ in the WZNW model. But these fields are not even close to exhausting the content of eq. (4.17). In particular, the sigma model contains 145 marginal fields with vanishing one-loop anomalous dimension that we discussed extensively in Section 4.1 and so far we have not seen any of them.

These 145 fields belong to mutiplets $\Lambda_{0,0} + 2\Lambda_{0,1} + \Lambda_{0,2}$, all of which have vanishing Casimir. Hence, in the WZNW model they must appear with $(h_0, \bar{h}_0) = (1, 1)$. So, let us count the fields in the WZNW model that have weights $(h_0, \bar{h}_0) = (1, 1)$ and vanishing Casimir. All of these fields must arise among $J_A \bar{J}_B$, i.e. sit in the tensor product of the adjoint representation of $\mathfrak{osp}(4|2)$ with itself. This tensor product is given by

$$\Lambda_{0,1} \otimes \Lambda_{0,1} \cong \Lambda_{0,0} + 2\Lambda_{0,1} + \Lambda_{0,2} + \Lambda_{2,-1} + 2\Lambda_{2,0} + \Lambda_{2,1} + [2, 0, 0] \,. \quad (4.46)$$

Indeed, this contains exactly 145 fields in representations from the block of the trivial representations for which the anomalous dimension vanishes to all orders in the coupling and hence also around $g = -1$, in perfect agreement with the sigma model results. Since the space of marginal fields in the sigma model is truncated by the equations of motion, the deformed WZNW model has the sigma model equations of motion built in!

This is a remarkable agreement. On the other hand, looking back at the sigma model spectrum (4.17) we realize that the content of what looks like $\mathcal{P}_{\Lambda_{2l,0}}, l = 1, 2, \ldots$ is still missing. Additional fields in these representations that acquire weights $(h_\infty, \bar{h}_\infty) = (1, 1)$ at $g = -1$ do exist in the WZNW, but these turn out not to match the one-loop data near $g = -1$. This is the first discrepancy between the Gross-Neveu and the sigma model.

Let us point out that, once again, the singular vectors are absolutely crucial in order for the WZNW model to respect the sigma model equations of motion. As an example let us look at the operators of the form $\partial\bar{\partial}V_4$. These give rise to a single marginal sigma model field in the representation $\Lambda_{4,0}$. If it was not for the singular vectors of conformal weight $h = 2$, the WZNW model would give many more marginal fields in the same block. In fact, the tensor product

$$\begin{aligned}(2\Lambda_{2,0} + \Lambda_{2,1} + \Lambda_{2,-1}) \otimes (2\Lambda_{2,0} + \Lambda_{2,1} + \Lambda_{2,-1}) \cong \\ \cong \Lambda_{4,-2} + 4\Lambda_{4,-1} + 6\Lambda_{4,0} + 4\Lambda_{4,1} + \Lambda_{4,2} + \cdots\end{aligned} \tag{4.47}$$

where $+\cdots$ stand for multiplets Λ with $\mathbf{Cas}_{\mathfrak{g}}(\Lambda) < 16$, none of which satisfy the no-winding condition. But those that do clearly outnumber the spectrum of marginal sigma model fields.

4.3 Beyond one loop

As we have stressed above, eq. (4.36) is actually exact, that is in its derivation no terms of higher order in $1/R^2$ were dropped. Its only R-dependent term agrees with the one-loop result in eq. (3.38), provided that $\mathbf{Cas}_{\mathfrak{h}}(\mu) + \mathbf{Cas}_{\mathfrak{h}}(\overline{\mu}) = 0$. This implies that, if the proposed duality is to hold, at least for a subsector of the supersphere sigma model, the anomalous dimensions of this subsector need to be one-loop exact. The setup described in Chapter 3 can be used to compute the corrections to the scaling weights also to higher orders in $1/R^2$. For now we will restrict to the tachyonic vertex operators in conformal sigma models, i.e. the ground states of the sigma model at $R = \infty$, and leave more general computations for future research. The computation simplifies substantially for tachyonic vertex operators since the tail contribution is trivial. The anomalous dimensions of tachyonic vertex operators of non-linear sigma models on the bosonic symmetric spaces have been computed to four-loop order [51, 52].

From the expansion (3.32) of vertex operators, it is clear that the anomalous dimension of tachyonic vertex operators vanishes at two-loop order. The only corrections to the two-point function are of the form

$$\langle V^{(2)}(u, \bar{u}) \otimes V^{(2)}(v, \bar{v})\rangle \tag{4.48}$$

which diverges like $\ln^2(\varepsilon)$ while contributions to the anomalous dimension only arise from divergences of order $\ln(\varepsilon)$. The interaction vertices Ω are of at least quartic order for symmetric cosets and $V^{(i)}$ contains i copies of the coordinate fields. Therefore, there are no further contributions at two-loop order. This statement is consistent with known results for bosonic symmetric cosets, where the two-loop correction vanishes as well.

The simplicity of the tachyonic states allows us to proceed to third order. The correction due to $V^{(3)}$ vanishes for the same reasons as before. Two possible contributions remain. The integrand of the first one is

$$\int_{G/H} d\mu(g_0H)\langle V^{(1)}(u,\bar{u}) \otimes V^{(1)}(v,\bar{v})\Omega_4(z,\bar{z})\Omega_4(w,\bar{w})\rangle. \tag{4.49}$$

The group structure of (4.49) contains an insertion of a rank two tensor built from the structure constants that appear in Ω_4. For symmetric spaces, the group structure of Ω_4 has a simple geometric interpretation. It is proportional to the Riemann curvature tensor $R_{ijkl} = f_{ija}f^a{}_{kl}$ of the coset space. The condition on the vanishing of the β-function is that *any* contraction of any number of Riemann tensors which results in a rank two tensor must vanish. Hence, this first contribution vanishes, not only for the supersphere but for all conformal sigma models on symmetric superspaces.

The remaining possible contribution arises from the integrand

$$\int_{G/H} d\mu(g_0H)\langle V^{(2)}(u,\bar{u}) \otimes V^{(2)}(w,\bar{w})\Omega_4(z,\bar{z})\rangle \tag{4.50}$$

as well as two further "unbalanced" contributions with $V^{(3)} \otimes V^{(1)}$ which, however, have the same group structure. The resulting group structure is

$$\int_{G/H} d\mu(g_0H) L_\Lambda\big(\mathrm{Ad}_{g_0}(R_{ijkl}t^it^jt^kt^l + \text{perm.})\big)\mathcal{D}_{\Lambda\lambda}(g_0)\mathcal{D}_{\Xi\xi}(g_0), \tag{4.51}$$

where $\lambda = \xi = 0$. The order of the t^i can be changed using the commutation relations. For tachyonic vertex operators any resulting terms of $t^a \in \mathfrak{h}$ can be commuted through and, due to equation (3.33), eventually dropped. This process produces additional insertions of structure constants and it is easy to see that they combine again to contributions proportional to R_{ijkl}. Therefore, this last correction can be separated into two pieces. One is proportional to the completely symmetric part of the Riemann tensor which vanishes. The other one is again a rank two tensor built from two copies of the Riemann tensor and therefore also zero.

These arguments hold for *any* conformal sigma model defined on a symmetric superspace. They show that for the ground state spectrum of the supersphere $S^{3|2}$ the duality with the $\mathfrak{osp}(4|2)$ WZNW holds to at least three-loop order. Starting at fourth order, tensor structures begin to appear that cannot be shown to vanish due to symmetry alone. The contribution in question has the form

$$\int_{G/H} d\mu(g_0H) L_\Lambda\big(\mathrm{Ad}_{g_0}(f_{ija}f^a{}_{kl}f^j{}_{mb}f^{bk}{}_n t^it^lt^mt^n)\big)\mathcal{D}_{\Lambda\lambda}(g_0)\mathcal{D}_{\Xi\xi}(g_0)\,. \tag{4.52}$$

We stress that the indices $i, j, \ldots$ run over a basis of $\mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h}$, while a and b run over a basis of $\mathfrak{h}$. Due to the fact that

$$L_\Lambda(\mathrm{Ad}_{g_0} t^a)\mathcal{D}_{\Lambda\lambda}(g_0) = -R_\lambda(\mathrm{Ad}_{g_0} t^a)\mathcal{D}_{\Lambda\lambda}(g_0) \tag{4.53}$$

we can extend the range of the index n to run over all of $\mathfrak{g}$ without changing the result, as long as $\lambda = 0$. In the next step, we can also extend the range of the indices b and k to run over all of $\mathfrak{g}$. The $\mathbb{Z}_2$-structure ensures that the additional terms are identically zero. The result is that the term (4.52) results in an operator which commutes with the $\mathfrak{g}$-action and contains a contribution of the structure constants of $\mathfrak{g}$. Such operators must act trivially on $\mathfrak{g}$-representations which have non-zero superdimension. These are in particular the maximally atypical representations. The same argument was used in [39] to derive the all-loop result (2.25) for deformed WZNW models. At higher loop-orders, more complicated structures appear. However, we believe that the same argument can be applied.

To summarize, we have shown that the two- and three-loop contributions to the anomalous dimensions of tachyonic vertex operators in conformal sigma models on symmetric spaces vanish. We further argued that all higher-order contributions vanish as well provided that the vertex operator transforms in a maximally atypical representation of the numerator group. For this argument it was critical that the target-space was symmetric and that the vertex operators had no tail components. If we were to relax the latter condition, interactions between the tails would contribute to the anomalous dimensions and it is currently not clear under which conditions those additional contributions would cancel. The result suggests that the anomalous dimensions of all tachyonic vertex operators of the supersphere $S^{3|2}$ model is one-loop exact, as was predicted by the duality.

Chapter 5

High-Gradient Operators in the $\mathfrak{psl}(2|2)$ Gross-Neveu Model

In the introduction, we mentioned the existence of an instability in non-linear sigma models and deformed WZNW models. In chapter 3 we then saw the origin of this instability in sigma models from the one-loop anomalous dimension (3.40). In this chapter we revisit the instability of WZNW models. We exploit the result (2.25) on the perturbation theory of WZNW models and the representation theory of $\mathfrak{psl}(2|2)$ to prove that the deformed $\mathfrak{psl}(2|2)$ WZNW model at $k = 1$ is free of strongly RG-relevant $\mathfrak{psl}(2|2)$ invariant operators in any order of perturbation theory. In addition, we evaluate the spectrum of all fields that transform in maximally atypical ($\frac{1}{2}$BPS) representations of the target space symmetry $\mathfrak{psl}(2|2)$ up to scaling weight $\Delta \leq 5$. Very remarkably, the spectrum at infinite coupling turns out to assume half-integer values only, nurturing hopes it might be described by a dual free field theory. In fact, it has been argued that such a dual model is provided by the $\mathbb{C}\mathrm{P}^{1|2}$ NLSM [70]. Our results, however, do not provide evidence for a duality with this sigma model.

The plan of this chapter is as follows. In order to apply the results reviewed in section 2.5 to the $\mathfrak{psl}(2|2)$ Gross-Neveu model, we need some background from representation theory which is collected in section 5.2. All-loop stability is established in section 5.3 before we compute the low lying spectrum at infinite coupling in section 5.4.

5.1 The BPS spectrum

We now specialize the general formula (2.25) for the anomalous dimension of $\frac{1}{2}$BPS operators to the case $\mathfrak{g} = \mathfrak{psl}(2|2)$. The superalgebra $\mathfrak{psl}(2|2)$ has only one atypicality condition and the quadratic Casimir vanishes on all atypical representations. Thus, eq. (2.25) simplifies to

$$\delta_g = -\frac{g}{2(1+kg)}\big(\mathbf{Cas}^L_{\mathfrak{g}} + \mathbf{Cas}^R_{\mathfrak{g}}\big). \tag{5.1}$$

We are particularly interested in operators that are invariant under the diagonal action of the symmetry algebra since such operators could be used to generate a $\mathfrak{g}$ preserving perturbation. The assumption of $\mathfrak{g}$ invariance does not simplify our formula (5.1) any further but it restricts it to operators for which the tensor product of left and right action contains the trivial representation. The finite-dimensional representation theory of $\mathfrak{psl}(2|2)$ has been worked out in detail in [71]. The results imply that the only way to obtain an invariant with $\Lambda_R = \Lambda$ is to tensor with the same representation $\Lambda_L = \Lambda$.

5.2 Review of $\mathfrak{psl}(2|2)$ representation theory

In this section we give a brief review of the pertinent facts regarding the Lie superalgebra $\mathfrak{psl}(2|2)$ and its finite dimensional representation theory. The algebra $\mathfrak{psl}(2|2)$ has rank two and its even subalgebra is $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Consequently, all finite dimensional representations are uniquely characterized by a pair of $\mathfrak{sl}(2)$ weights $j, l \in \frac{1}{2}\mathbb{Z}$. Representations of $\mathfrak{psl}(2|2)$ can satisfy one shortening, or atypicality, condition which is simply given by $j = l$. We will denote typical representations of $\mathfrak{psl}(2|2)$ by $[j, l]$ and atypical irreducibles by $[j]$. Irreducible representations of the even subalgebra will be denoted by (j, l).

Upon restriction to the even subalgebra $\mathfrak{g}_{\bar{0}}$ the irreducible representations decompose as

$$[j]\big|_{\mathfrak{g}_{\bar{0}}} \simeq (j+\tfrac{1}{2}, j-\tfrac{1}{2}) \oplus 2(j,j) \oplus (j-\tfrac{1}{2}, j+\tfrac{1}{2}) \qquad (\text{for } j>0) \tag{5.2}$$

$$[j,l]\big|_{\mathfrak{g}_{\bar{0}}} \simeq (j,l) \otimes \big[2(0,0) \oplus 2(\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \oplus (1,0)\big] \tag{5.3}$$

and $[0]$ is the trivial representation. Let us also remark that $[\frac{1}{2}]$ corresponds to the adjoint representation. Atypical irreducibles of Lie superalgebras can form indecomposables. If one is not interested in the precise form in which such indecomposables are built from their constituents, all tensor products of finite dimensional $\mathfrak{g}$ representations may be determined by restricting the factors to the even subalgebra, tensoring the associated $\mathfrak{g}_{\bar{0}}$ representations and combining the resulting products back into representations of $\mathfrak{g}$. The first and last step require no more than our decomposition formulas (5.2) and (5.3). The tensor products of irreducibles, including the indecomposable structures, have been worked out in [71].

We will also need to know the eigenvalue of the quadratic Casimir invariant $\mathbf{Cas}_{\mathfrak{g}}$. It is given in terms of the highest weights by

$$\begin{aligned} \mathbf{Cas}_{\mathfrak{g}}\big([j,l]\big) &= -j(j+1) + l(l+1) \\ \mathbf{Cas}_{\mathfrak{g}}\big([j]\big) &= 0. \end{aligned} \tag{5.4}$$

Note that its value in atypical representations is given by evaluating the Casimir for typicals on weights which satisfy the shortening condition $j = l$.

Let us conclude with a few scattered comments on the notation we are about to use. As we mentioned above, atypical irreducibles can combine to form complicated indecomposables. We will not concern ourselves with this indecomposable structure of the spectrum and simply look at the constituent irreducible representations. For this reason, we shall not use the symbol $\oplus$ in our formulas but simply write $+$ instead. Many of the sums of representations we are about to see are in fact not direct. Since traces are blind to the indecomposable structures, our formulas for representations encode true identities among their characters χ_Λ in which $+$ and tensor products are ordinary sums and products of characters.

5.3 Absence of relevant high-gradient operators

The spectrum of WZNW models on type I supergroups is quite well understood, see [72]. Almost all of these models give rise to logarithmic conformal field theories, see also [73, 27], and hence their Hamiltonian (generator of dilations) is not diagonalizable. In our analysis of the spectrum we shall only be concerned with the generalized eigenvalues of the dilation operator. This information is encoded in the partition function of the WZNW model. The latter decomposes into a sum of products of characters for representations of the left- and right moving chiral algebra. This does not mean that these models experience holomorphic factorization – they do not. But the trace we take when we compute the partition function cannot see the intricate coupling between left and right movers.

The representations of the affine $\widehat{\mathfrak{psl}}(2|2)_k$ algebra along with their characters have been worked out for arbitrary level k in [71]. When $k = 1$, the theory contains a single sector which is based on the vacuum representation of the current algebra. Using the results of [71] one can obtain the branching functions for the decomposition of the affine modules into irreducible representations of the zero-mode subalgebra $\mathfrak{psl}(2|2)$. In case of the vacuum representation of the affine $\mathfrak{psl}(2|2)$ at level $k = 1$ the branching functions into representations (j, l) of the even subalgebra $\mathfrak{g}_{\bar{0}}$ read

$$\begin{aligned}\psi^{(0)}_{(j,l)} = \frac{q^{\frac{1}{12}}}{\phi(q)^4} \sum_{s\in\mathbb{Z}} \sum_{m,n=0}^{\infty} &(-1)^{m+n}\, q^{\frac{m(m+1)+n(n+1)}{2}+s(s+m-n)-j(m+n+1)} \\ &\times \left(1 - q^{-(m+n+1)}\right)\left(1 - q^{2l+1}\right) q^{l^2},\end{aligned} \tag{5.5}$$

where $j, l \in \mathbb{Z}$ and with $s \to s + 1/2$ for $j, l \in \mathbb{Z} + 1/2$. From these formulas one can determine the branching functions into representations of the superalgebra $\mathfrak{g}$ with the help of eqs. (5.2) and (5.3). For the first few levels, the resulting decomposition of the vacuum character $\hat{\chi}_0$ reads

$$\begin{aligned}\hat{\chi}_0(q, x, y) = q^{\frac{1}{12}} \big(&q^0 \chi_{[0]} + q^1 \chi_{[\frac{1}{2}]} + q^2 (\chi_{[1,0]} + \chi_{[\frac{1}{2}]} + \chi_{[0]}) \\ &+ q^3 (\chi_{[2,0]} + \chi_{[1,0]} + 2\chi_{[1]} + 3\chi_{[\frac{1}{2}]} + 4\chi_{[0]}) \\ &+ q^4 (\chi_{[3,0]} + \chi_{[2,0]} + 3\chi_{[1,0]} + \chi_{[0,1]} + 2\chi_{[\frac{3}{2},\frac{1}{2}]} \\ &+ 2\chi_{[1]} + 4\chi_{[\frac{1}{2}]} + 5\chi_{[0]})\big) + \mathcal{O}(q^5)\ .\end{aligned} \tag{5.6}$$

Here, we expanded the vacuum character of the affine $\mathfrak{psl}(2|2)$ at level $k = 1$ into characters $\chi_\Lambda = \chi_\Lambda(x, y)$ of the zero mode algebra $\mathfrak{psl}(2|2)$. The arguments x, y keep track of the $\mathfrak{psl}(2|2)$ weights while q is associated with the eigenvalues of L_0, i.e. with the conformal weight h, as usual. In the partition function, $\hat{\chi}_0$ gets multiplied with an identical contribution from the anti-holomorphic sector, only that we need to replace q by $\bar{q}$.

From (5.5) it follows that the smallest conformal weight at which a bosonic module $(j, l) \neq (0, 0)$ appears is given by

$$h^{\min}_{g=0}(j, l) = \begin{cases} j + l^2 & j, l \in \mathbb{Z} \\ j + l^2 + \dfrac{1}{4} & j, l \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{5.7}$$

In addition we note that modules with $j, l \in \mathbb{Z} + \frac{1}{2}$ always appear with multiplicity two at their lowest weight. From the decomposition (5.3) of typical irreducible modules we can now deduce that the minimal weight $h^{\min}_{g=0}([j, l])$

of a module $[j,l]$, $j \neq l$, is given by the minimal weight $h^{\min}_{g=0}(j,l+1)$ of the bosonic module $(j,l+1)$,

$$h^{\min}_{g=0}\big([j,l]\big) = \begin{cases} j+(l+1)^2 & j,l \in \mathbb{Z} \\ j+(l+1)^2+\dfrac{1}{4} & j,l \in \mathbb{Z}+\frac{1}{2}, \end{cases} \tag{5.8}$$

for typical $[j,l]$. With the help of the decomposition (5.2) one can find a similar result for atypical representations,

$$h^{\min}_{g=0}\big([j]\big) = \begin{cases} j^2+2j & j \in \mathbb{Z} \\ j^2+2j-\dfrac{1}{4} & j \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{5.9}$$

Given the values (5.4) of the quadratic Casimir, it is clear that if we take $g \leq 0$ high-gradient operators become relevant for arbitrarily small values of the coupling, since their engineering dimension grows linearly in j, while the anomalous dimension grows like $-j^2$. So this direction of the perturbation cannot lead to a stable theory.

Let us therefore turn to the case $g \geq 0$. From [71] we know that operators that are invariant under the diagonal action of $\mathfrak{psl}(2|2)$ must transform in the same representation $\Lambda_L = \Lambda_R$ with respect to the left and right action. Eq. (5.1) implies that the only invariant operators that become more relevant as we increase the coupling g must sit in multiplets $\Lambda_L = [j,l] = \Lambda_R$ with $l > j$. Among those, the lowest lying ones at $g=0$, namely those with $j=0$, are also those that receive the largest correction to their conformal weights. From eq. (5.1), the anomalous dimension $\delta_g\big([0,l]\big)$ of invariant operators with $\Lambda_L = [0,l] = \Lambda_R$ is given by

$$\delta_g\big([0,l]\big) = -\frac{g}{1+g}\, l(l+1). \tag{5.10}$$

Comparing with eq. (5.8), we infer that these operators remain irrelevant for all finite values of the coupling. In conclusion, the models with $g \geq 0$ actually contain no RG-relevant invariant operators. Thereby, we have extended the one-loop result of [41] to all loop orders and all invariant operators.

5.4 The spectrum at infinite coupling

The limiting point $g=\infty$ is obviously of special interest. Let us therefore describe its spectrum in some more detail. From the above discussion we can conclude that there are no relevant invariant operators in the spectrum for any positive value of the coupling. Moreover, we see that as $g \to \infty$ the spectrum of operators in atypical ($\frac{1}{2}$BPS) representations under the diagonal action of $\mathfrak{psl}(2|2)$ is half-integer valued, i.e.

$$h_g := h_{g=0} + \delta_g \quad \text{satisfies} \quad h_\infty = \lim_{g\to\infty} h_g \in \tfrac{1}{2}\mathbb{Z} \ . \tag{5.11}$$

The multiplicities of $\frac{1}{2}$BPS states at any total conformal weight $\Delta = h + \bar{h}$ remain finite as the coupling g tends to infinity, as can be seen with the help of eq. (5.8) together with eq. (5.1). For $j_1, j_2 \in \mathbb{Z}$ we find

$$\Delta_\infty^{\min} = h_\infty^{\min}\big([j_1,l_1]\big) + \bar{h}_\infty^{\min}\big([j_2,l_2]\big) = j_1^2 + 2j_1 + j_2^2 + 2j_2 + l_1 + l_2 + 2. \tag{5.12}$$

When either j_1 or j_2 are half-integer, $\frac{1}{4}$ gets added to the above formula. If they are both half-integer, we must add $\frac{1}{2}$. Since all the labels are non-negative, the total energy grows strictly monotonically in them. Therefore, multiplicities of $\frac{1}{2}$BPS states remain finite for any given value of Δ_∞. Moreover, Δ_∞ remains non-negative and the only state that goes to $\Delta_\infty = 0$ is the ground state of the WZNW model.

We will now describe the spectrum at $g = \infty$ up to $\Delta_\infty = 5$. The analysis is organized according to the right moving conformal weight $\bar{h}_\infty$, i.e. we shall start by listing all the $\frac{1}{2}$BPS states that possess $\bar{h}_\infty = 0$, i.e. the chiral states of the Gross-Neveu model at strong coupling $g = \infty$. Obviously, all chiral $\frac{1}{2}$BPS states of the WZNW model, that is those that are built with the right moving vacuum state and hence have weights $(h_0, 0)$, do not acquire an anomalous contribution to their conformal weights. Hence, chiral states of the WZNW model give states with $(h_\infty = h_0, \bar{h}_\infty = 0)$. That does not mean, however, that the chiral $\frac{1}{2}$BPS spectrum at $g = \infty$ is the same as it is at $g = 0$. Indeed, starting from $h_\infty = 3$ we see new chiral states appearing. The first ones originate from an operator multiplet at $\big(h_0, \bar{h}_0\big) = (4,1)$ that transforms in the representation $[0,1]^L \otimes [\frac{1}{2}]^R$ in the WZNW model. Under the diagonal action D, this product decomposes into

$$[0,1] \otimes [\tfrac{1}{2}] = 6[0] + 6[\tfrac{1}{2}] + 4[1] + [\tfrac{3}{2}] + \text{typicals}. \tag{5.13}$$

Hence, this multiplet of the WZNW model contributes plenty of chiral fields at strong coupling. At $h_\infty = 4$ we only need to account for the holomorphic derivative of this operator. For $h_\infty = 5$, finally, there exist three multiplets in the representation $[0,1]^L \otimes [\frac{1}{2}]^R$. Additionally, we obtain a contribution from a multiplet that transforms in $[0,2]^L \otimes [0,1]^R$. Its $\frac{1}{2}$BPS content in the decomposition with respect to the diagonal action is the same as for the previous operator. Summing everything up, the chiral spectrum to this level is given by

$$\begin{array}{ll} h_\infty = 0 & [0] \\ h_\infty = 1 & [\frac{1}{2}] \\ h_\infty = 2 & [0] + [\frac{1}{2}] \\ h_\infty = 3 & 10[0] + 9[\frac{1}{2}] + 6[1] + [\frac{3}{2}] \\ h_\infty = 4 & 11[0] + 10[\frac{1}{2}] + 6[1] + [\frac{3}{2}] \\ h_\infty = 5 & 38[0] + 37[\frac{1}{2}] + 24[1] + 5[\frac{3}{2}] \ . \end{array} \tag{5.14}$$

The analysis for the next cases with $\bar{h}_\infty > 0$ proceeds along the same lines. For $\bar{h}_\infty = 1$ one finds,

$$\begin{array}{ll} h_\infty = 1 & 4[0] + 2[\frac{1}{2}] + 2[1] \\ h_\infty = 2 & 4[0] + 3[\frac{1}{2}] + 2[1] \\ h_\infty = 3 & 18[0] + 20[\frac{1}{2}] + 14[1] + 5[\frac{3}{2}] \\ h_\infty = 4 & 22[0] + 23[\frac{1}{2}] + 16[1] + 5[\frac{3}{2}] \ . \end{array} \tag{5.15}$$

Similarly, the results for $\bar{h}_\infty = 2$ read

$$\begin{array}{ll} h_\infty = 2 & 19[0] + 16[\frac{1}{2}] + 8[1] + [\frac{3}{2}] \\ h_\infty = 3 & 58[0] + 61[\frac{1}{2}] + 46[1] + 17[\frac{3}{2}] + 2[2]. \end{array} \tag{5.16}$$

For higher values of $\bar{h}_\infty \leq 5$ the multiplicities of $\frac{1}{2}$BPS multiplicities in the $g = \infty$ Gross-Neveu model can be inferred from the list we provided, exploiting that the spectrum is certainly symmetric under the exchange of left- and right movers.

There exist actually a few more states at $\Delta_\infty = 5$ that we have not listed yet. In fact, $\Delta_\infty = 5$ marks the first level at which states with negative left moving weight $h_\infty < 0$ appear in the spectrum. At the same time, $\Delta_\infty = 5$ is also the lowest value of the scaling weight at which half-integer conformal weights $(h_\infty, \bar{h}_\infty)$ are actually observed. The additional states are generated by two WZNW operators that transform in the representation $[\frac{1}{2}]^L \otimes [\frac{1}{2}, \frac{3}{2}]^R$. In this case, the decomposition of the diagonal action can be worked out to give

$$2[\tfrac{1}{2}] \otimes [\tfrac{1}{2}, \tfrac{3}{2}] = 4[0] + 8[\tfrac{1}{2}] + 12[1] + 8[\tfrac{3}{2}] + 2[2] + \text{typicals}. \tag{5.17}$$

Note that we multiplied the left hand side by a factor 2 so that the left hand side accounts for all operators that possess weights $\big(h_\infty, \bar{h}_\infty\big) = \big(-\frac{1}{2}, \frac{11}{2}\big)$ at $g = \infty$. Of course, the spectrum is symmetric under the exchange of the holomorphic and anti-holomorphic sectors so that the same content appears with $\left(h_\infty, \bar{h}_\infty\right) = \left(\frac{11}{2}, -\frac{1}{2}\right)$.

The fact that the spectrum becomes half-integer valued at $g = \infty$ suggests that the theory might possess a free-field description. It has been proposed in the past that such a dual model should be given by the $\mathbb{C}\mathrm{P}^{1|2}$ non-linear sigma model [70]. Unfortunately, the spectrum we have presented in this section bears no resemblance to that if the $\mathbb{C}\mathrm{P}^{1|2}$ model. It remains to be seen what a free field description of the strongly coupled $\mathfrak{psl}(2|2)$ Gross-Neveu model could be, if it exists.

Chapter 6

Deformations of Kazama-Suzuki models

When discussing WZNW models as potential duals of non-linear sigma models it is natural to further widen the scope of the discussion and include their GKO cosets. Then one can ask whether the resulting models have a marginal deformation analogous to the current-current deformation of WZNW models on simple supergroups. In the absence of worldsheet supersymmetry this turns out to be the case precisely if the coset is taken from the list (2.16) without further restrictions on the level k [36]. Note, however, that the setup here is quite different. In the GKO construction, the coset is taken with respect to the adjoint action of the denominator, while it was taken with respect to the right action in the sigma model case. Nevertheless, the GKO cosets based on the list (2.16) are believed to approach the sigma model on the corresponding space in the limit where the level k is taken to infinity [74, 75, 36].

Worldsheet $\mathcal{N} = 1$ supersymmetric extensions of the GKO cosets (2.16) still possess a marginal deformation, but the conditions on the denominator subgroup should be more relaxed. In this chapter, we will focus on the case where the supersymmetry is further enhanced to $\mathcal{N} = 2$. These are models that are associated to hermitian symmetric spaces G/H where again the dual Coxeter number $g^\vee$ of G vanishes. As we will see, all these models have at least one marginal chiral primary field which are known to be exactly marginal [76].

We begin with a brief review of the Kazama-Suzuki construction [77] which yields the $\mathcal{N}=2$ supersymmetric models Then, in section 6.2, we construct a marginal chiral primary field that is present in all Kazama-Suzuki models with vanishing dual Coxeter number of the numerator.

6.1 Kazama-Suzuki supercoset models

The construction of Kazama-Suzuki models begins with the numerator group G. We denote its Lie algebra by $\mathfrak{g}$. For the moment we shall only assume $\mathfrak{g}$ to be simple. Associated with our choice of the numerator group comes a WZNW model $\hat{\mathfrak{g}}$ of level k. Through the usual affine Sugawara construction we then obtain the generators of a Virasoro algebra with central charge

$$c^G = \mathrm{sdim}\,\mathfrak{g}\,\frac{k}{k+g^\vee}$$

with $g^\vee$ the dual Coxeter number of $\mathfrak{g}$.

Let us now choose some subgroup H of G with Lie algebra $\mathfrak{h}$. Following Kazama and Suzuki, there are some conditions on the choice of H that we want to impose. To begin with, we shall assume the existence of some order two automorphism of G that leaves the subgroup $H \subset G$ invariant and H is required to have the same rank as G. Furthermore, we assume that roots of H are also roots of G. The quotient space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ is a representation of the denominator algebra $\mathfrak{h}$. We require that $\mathfrak{m}$ splits into a direct sum of two conjugate representations. The embedding of H into G determines the level $k'_G = \kappa(k - g^\vee)$ of the embedded $\mathfrak{h}$ current algebra. The factor κ is known as the embedding index for the embedding of $\mathfrak{h}$ into $\mathfrak{g}$.

Once we have made our choice of $\mathfrak{h}$, we add a "fermionic" sector to the numerator. The additional fields possess conformal weight $h = 1/2$ and transform in the representation $\mathfrak{m}$ of the denominator subalgebra $\mathfrak{h}$. From these fields we can construct an $\mathfrak{osp}(2p|2q)$ affine current algebra at level $k = 1$, see Appendix E, which we reproduce here from [35] for convenience. Here $2p = \dim \mathfrak{m}_{\bar{0}}$ and $2q = \dim \mathfrak{m}_{\bar{1}}$ are the dimensions of the even and odd parts of $\mathfrak{m}$, respectively. Since $\mathfrak{m}$ was assumed to split into a direct sum of two conjugate representations, $2p$ and $2q$ are even. The Lie superalgebra $\mathfrak{osp}(2p|2q)$ possesses dual Coxeter number $g^F = 2p - 2q - 2$. Therefore, the central charge of the Virasoro algebra obtained for the level $k = 1$ algebra is $c^F = p - q$. The current algebra $\mathfrak{h}$ can be embedded into the fermionic $\mathfrak{osp}(2p|2q)$ we described in the previous paragraph. The index of this embedding determines the level $k'_F = \kappa g^\vee - h^\vee$ of the $\mathfrak{h}$ currents within the $\mathfrak{osp}(2p|2q)$ algebra.

In the coset model, we combine the embedding of the $\mathfrak{h}$ currents into the $\hat{\mathfrak{g}}$ model with the one that maps into the fermionic sector. This leaves us with an affine current algebra of level $k'_G + k'_F = k' - h^\vee$ where $k' = \kappa k$. The coset model has central charge

$$c = c^G + p - q - \sum_\nu \frac{\operatorname{sdim} H_\nu (k'_\nu - h^\vee_\nu)}{k'_\nu} .$$

In writing this formula we have allowed the subgroup H to consist of several simple subgroups H_ν. According to the usual results of Kazama and Suzuki, the conditions imposed on choice of H guarantee that the Virasoro symmetry can be extended to an $N = 2$ superconformal algebra, see [78] for an extension to supercosets.

The sectors of the coset model are labeled by triples $(\Lambda, \tilde{\Lambda}, \lambda)$ of indices. Here, the first index Λ is an integral highest weight of the current algebra $\mathfrak{g}$ at level $k - g^\vee$, λ is an integral highest weight of the current algebra $\mathfrak{h}$ at level $k' - h^\vee$ and the index $\tilde{\Lambda}$ is an integral highest weight of $\mathfrak{osp}(2p|2q)$ at level one. For our purposes it is simpler to use the fact that this "fermionic" factor is constructed from free fields. Since the latter contain bosons, there are infinitely many sectors that are related by spectral flow. The ground state of these sectors possesses conformal weight

$$h^S = \frac{p-q}{8} S^2 .$$

For an even number of spectral flows, the fundamental fields of the "fermionic" sector possess half-integer mode indices. Hence, the weight of states in this sectors may be shifted by multiples of $1/2$. When we apply an odd number of spectral flows, on the other hand, all states have weight h^S up to an integer. From now on we shall think of $\tilde{\Lambda} = (S, \sigma)$ with $S \in \mathbb{Z}$ and $\sigma = 0, 1$. We also set

$$h(\tilde{\Lambda}) = h(S, \sigma) = \begin{cases} h^S + 1/2 & \text{for} \quad S \text{ even} \ , \ \sigma = 1 \\ h^S & \qquad\quad \text{else} \end{cases}$$

Similarly, we want introduce a function $q(\tilde{\Lambda}) = q(S, \sigma)$ that measures the U(1) charge up to an integer,

$$q(\tilde{\Lambda}) = q(S, \sigma) = \frac{p-q}{2} S .$$

Note that $q(S,\sigma)$ does not depend on the choice of σ. The conformal weight and U(1) charge of the sectors $(\Lambda, \tilde{\Lambda}, \lambda)$ are given by

$$h = \frac{1}{k} C_{\mathfrak{g}}^{(2)}(\Lambda) + h(S,\sigma) - \frac{1}{k'} C_{\mathfrak{h}}^{(2)}(\lambda), \tag{6.1}$$

$$q = q(S,\sigma) - \frac{1}{k'} q(\lambda), \tag{6.2}$$

with

$$q(\lambda) = \frac{1}{k'} \sum_{\bar{\alpha} \in \bar{\Delta}^+} \bar{\alpha} \cdot \lambda, \tag{6.3}$$

where $\bar{\Delta}^+$ denotes the positive roots of $\mathfrak{g}$ that are not roots of $\mathfrak{h}$. In particular we have $q(\lambda = 0) = 0$.

6.2 Marginal chiral primaries

We want to study a special perturbation of our Kazama-Suzuki models. In order to describe the perturbing field, we decompose the representation of the denominator algebra $\mathfrak{h}$ on the Lie superalgebra $\mathfrak{g}$ according to $\mathfrak{g} = \mathfrak{m}^- \oplus \mathfrak{h} \oplus \mathfrak{m}^+$, where $\mathfrak{m}^\pm$ form conjugate representations of $\mathfrak{h}$. The spaces $\mathfrak{m}^\pm_{\bar{a}}$ possess the dimensions

$$\dim\, \mathfrak{m}^\pm_{\bar{0}} = p \quad , \quad \dim\, \mathfrak{m}^\pm_{\bar{1}} = q .$$

A basis of $\mathfrak{m}^\pm$ is labeled by an index $m_\pm = 1, \ldots, p+q$. The coset field we are going to perturb by is obtained from the decomposition

$$|\mathfrak{g}\rangle^G_{m_\pm} \otimes \psi^{m_\pm}_{-1/2} |0\rangle^f_{\text{NS}} = |(\mathfrak{g}, v, 0)\rangle_\pm \otimes |0\rangle^H$$

of the state on the left hand side. The sum over $m_\pm = 1, \ldots, p+q$ is performed without summing over $\pm$. Here, $|\mathfrak{g}\rangle^G_{m_\pm}$ denotes ground states in the sector of the adjoint representation of $\mathfrak{g}$, while $|0\rangle^f_{\text{NS}}$ is the ground state in the Neveu-Schwarz sector of the fermions. The operator $\psi^{m_\pm}_{-1/2}$ is taken to transform in the conjugate $\mathfrak{h}$ representation. The OPE of the fields $\psi^{m_\pm}(z)$ is defined as

$$\psi^m(z) \psi^n(w) \sim \frac{h^{mn}}{z-w} , \tag{6.4}$$

where h^{mn} is the inverse of the metric on the adjoint of $\mathfrak{g}$. In particular, the fields $\psi^\pm(z)$ have trivial OPE amongst themselves. It is easy to see that the above state is primary with respect to the H currents. Since it also transforms trivially under the zero modes of the H current algebra, it can be written as a product of the vacuum state $|0\rangle^H$ of the denominator and the ground states $|(\mathfrak{g}, v, 0)\rangle_\pm$ of the coset model. The corresponding fields of the coset model will be denoted by $\chi_\pm$. The perturbation we want to look at is generated by a descendent of the operator

$$\mathcal{O}(z,\bar{z}) = \overline{\chi}_-(\bar{z})\chi_-(z) + \overline{\chi}_+(\bar{z})\chi_+(z)\ . \tag{6.5}$$

From eqs (6.1) and (6.2) together with

$$h(\psi^{m_\pm}_{-1/2}) = \frac{1}{2}, \qquad q(\psi^{m_\pm}_{-1/2}) = \pm 1, \tag{6.6}$$

we see that the conformal weight h and charge q of the coset state are given by

$$h(\chi_\pm) = \frac{k+g^\vee}{k} C^{(2)}_{\mathfrak{g}}(\mathfrak{g}) + \frac{1}{2} \quad , \quad q(\chi_\pm) = \pm 1\ . \tag{6.7}$$

These equations follow from the fact that our coset fields have trivial entry $\lambda = 0$ for the denominator algebra. We see that our perturbing fields can be (anti-)chiral primary only if the dual Coxeter number vanishes, i.e. $g^\vee = C^{(2)}_{\mathfrak{g}}(\mathfrak{g}) = 0$. This is the case if G is one of the supergroups $\mathrm{PSU}(n|n)$, $\mathrm{OSP}(2n+2|2n)$ or $\mathrm{D}(2,1;\alpha)$. Note that the central charge c^G is then independent of the level. For $G = \mathrm{PSU}(n|n)$ the central charge is $c^G = -2$ while for the other two series we have $c^G = 1$.

Since WZNW models and, by extension, also Kazama-Suzuki models defined on supergroups are non-unitary, the vanishing of the dual Coxeter number is only a necessary condition for the fields $\chi_\pm$ to be (anti-)chiral primary. If we can show that they are primary, then vanishing of the dual Coxeter number implies that they are chiral as well, due to the superconformal algebra. A state in the Neveu-Schwarz sector is primary if it is annihilated by all positive modes of the supersymmetry generators $G^\pm = \psi^{m_\pm} J_{m_\pm}$, where $J_{m_\pm}$ are the components of the $\mathfrak{g}$-currents. The only case where we have to do a little bit of work is

$$\begin{aligned} G^+_{+\frac{1}{2}}|(\mathfrak{g}, v, 0)\rangle_- &= J_{m_+,0}|g\rangle_{m_-} \otimes \psi^{m_+}_{+\frac{1}{2}}\psi^{m_-}_{-\frac{1}{2}}|0\rangle^f_{\mathrm{NS}} \\ &= h^{m_+m_-} J_{m_+,0}|\mathfrak{g}\rangle_{m_-} \otimes |0\rangle^f_{\mathrm{NS}} \\ &= h^{m_+m_-} f_{m_+m_-}{}^n |\mathfrak{g}\rangle_n \otimes |0\rangle^f_{\mathrm{NS}} \\ &= 0\,, \end{aligned} \tag{6.8}$$

where $f_{m_+m_-}{}^n$ are structure constants of $\mathfrak{g}$. The computation for the case with $+$ and $-$ exchanged is analogous. We have shown that the fields $\chi_\pm$ are indeed (anti-)chiral primary if the dual Coxeter number of the numerator vanishes.

For unitary theories, perturbations that are associated to (anti-)chiral primary fields of weight $h = 1/2$ are known to be exactly marginal [76]. We expect that the same is true in these non-unitary models. In other words, Kazama-Suzuki models obtained from numerator groups G with vanishing dual Coxeter number possess at least one exactly marginal deformation (modulus). A closer inspection shows that there exist three families of such models,

$$\begin{aligned} H &= \mathrm{SU}(p|q) \times \mathrm{SU}(n-p|n-q) \subset G = \mathrm{PSU}(n|n) \\ H &= \mathrm{SU}(n+1|n) \subset G = \mathrm{OSP}(2n+2|2n) \\ H &= \mathrm{OSP}(2n|2n) \times \mathrm{SO}(2) \subset G = \mathrm{OSP}(2n+2|2n) \end{aligned} \tag{6.9}$$

not including those that are associated with the supergroup $G = \mathrm{D}(2,1;\alpha)$.

Chapter 7

Conclusions and Outlook

In this thesis we have reviewed and extended recent results on the spectrum of conformal coset sigma models. We were able to extend results for the one-loop spectrum of sigma models on symmetric spaces to those on semi-symmetric spaces, at least in the conformal case. We were able to show that if the sigma model on a coset space defined by a $\mathbb{Z}_4$-automorphism is conformal the one-loop anomalous dimension is formally identical to that of the symmetric case, in spite of the more complicated structure of the action. In order for the result to be expressible in terms of simple operators it was crucial that the coupling of the sigma model were fine tuned to ensure vanishing of the beta-function at one-loop.

We then used the results for symmetric spaces and the general methods for the construction of vertex operators which were developed in [38] and applied them to the supersphere sigma model with target space $S^{3|2}$. By comparing these methods with more traditional constructions, we illustrated the power of these methods and showed how they allow easy access to the spectrum of sigma models, at least to leading order in the coupling. We then used these results to test a conjectured duality of the $S^{3|2}$ model with a deformed $\mathfrak{osp}(4|2)$ WZNW model at level $k = 1$ which may also be regarded as a Gross-Neveu model. The conjecture posits that the small radius limit of the sigma model is described by the Gross-Neveu model close to the Wess-Zumino point. Using an all-order result from [39], we showed that the Gross-Neveu model reproduces the ground state spectrum of the sigma model, up to a $\mathbb{Z}_2$ orbifold. Using the general setup of [38] we also computed the anomalous dimension for conformal sigma models on symmetric spaces to third order in the coupling and showed that it vanishes. We further argued for the vanishing of all higher order corrections in the maximally atypical sector, thereby further supporting the duality with the deformed WZNW model at the level of the ground states. We were also able to recover a number of gradient fields and in particular argued that the Gross-Neveu model correctly implements the constraints and equations of motion of the sigma model.

We again used the all order result [39] on the anomalous dimensions of deformed WZNW models to revisit a question about the stability of sigma models and related theories. Following up on the observation made in [41] that the $\mathfrak{psl}(N|N)$ Gross-Neveu model might be free of relevant high-gradient operators at level $k = 1$, we presented an analysis of the spectrum for $N = 2$. We were able to show analytically that the $\mathfrak{psl}(2|2)$ Gross-Neveu model does not contain RG-relevant high-gradient operators, thereby extending the results of [41] to all orders and all invariant operators. This shows that the $\mathfrak{psl}(N|N)$ WZNW models, at least in the case of $N = 2$, take a special role among models with target-space supersymmetry. The only other case where similar stability statements have been established is the boundary $\mathfrak{osp}(4|2)$ Gross-Neveu model [35].

We also observed that, as the coupling tends to infinity, the spectrum of the $\mathfrak{psl}(2|2)$ model becomes half-integer valued, albeit with some peculiar features. This indicates that the theory could possess a free field description. It has been argued several times before that such a dual description should exist in the shape of the $\mathbb{CP}^{1|2}$ sigma model [36, 37]. A similar study has been performed for the boundary $\mathfrak{osp}(4|2)$ Gross-Neveu model in [35] where the resulting spectrum was identified with that of the free $S^{3|2}$ sigma model. The spectrum of the $\mathbb{CP}^{1|2}$ sigma model was worked out for the boundary case at infinite radius (zero coupling) in [79]. The analysis can be easily extended to the bulk, but unfortunately the resulting spectrum does not resemble the results we presented at the end of chapter 5.

In the final chapter, we investigated the existence of marginal deformations in GKO cosets based on supergroups. Our results suggest that all GKO cosets with $\mathcal{N} = 2$ extended worldsheet supersymmetry possess at least one exactly marginal deformation.

Moving forward, a number of open questions and interesting problems remain. First of all, it would be very interesting to repeat the analysis we presented for the supersphere $S^{3|2}$ sigma model in the semi-symmetric case and construct the vertex operators and their one-loop spectrum for an AdS background. The main obstacle here arises from the fact that AdS spaces are non-compact, which has implications for the construction of normalizable sections, at least when the denominator is also non-compact. A first step would be to investigate AdS_2, where the denominator is compact so that the construction of vertex operators that we reviewed in chapter 3 remains unchanged. One could then construct a string embedding using the hybrid approach and study the construction of physical states.

The duality between the $\mathfrak{osp}(4|2)$ Gross-Neveu model and the $S^{3|2}$ sigma model that was studied in chapter 4 also leaves us with some interesting questions. While we did find some remarkable agreement between the spectra, there were also a number of gradient operators, already at $(h_\infty, \bar{h}_\infty) = (1, 1)$ that the dual WZNW model cannot account for. And these are not the only fields that cannot be matched. In fact, only those fields of the sigma model for which $\mathbf{Cas}_\mathfrak{h}(\mu) + \mathbf{Cas}_\mathfrak{h}(\overline{\mu}) = 0$ can possibly have a counterpart in the Gross-Neveu model, at least in the sense we discussed. It may be that these discrepancies simply disprove the duality. On the other hand, the matchings we observed are already rather non-trivial. Further support for the duality comes from an analysis of the boundary spectra. The authors of [35] observed a surprising character identity that completely matches the chiral states of the two theories, although the one-loop data spoils this match. In order to better understand the duality in the bulk, it would be very interesting to extend the higher-loop analysis we presented for the sigma model ground states to gradient states and identify a subsector for which the scaling weights are one-loop exact, as required by the duality. A better understanding of the instabilities could also furnish a better understanding of the observed discrepancies, which might be influenced by the relevant operators.

While we have found no evidence to support an analogous duality between the $\mathbb{C}P^{1|2}$ sigma model and the deformed $\mathfrak{psl}(2|2)$ WZNW model, it would still be very interesting to identify a free field theory that gives rise to the spectrum we found for the $\mathfrak{psl}(2|2)$ model at $g = \infty$ and possibly understand its precise relation to the $\mathbb{C}P^{1|2}$ model.

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Appendix A

Derivation of integral formulas

First of all we recall a useful formula that was derived in [38]

$$\int_{\mathbb{C}_\epsilon} \frac{d^2z}{\pi} \frac{1}{(z-x)(z-y)(\bar{z}-\bar{x})(\bar{z}-\bar{y})} = \frac{2\ln\left|\frac{x-y}{\epsilon}\right|^2}{|x-y|^2} + \mathcal{O}(\epsilon)\ . \tag{A.1}$$

By taking now the appropriate number of derivatives in x, y, $\bar{x}$ or $\bar{y}$ on both sides of the above equation, we recover eq. (3.57).

Another useful integral is

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2z}{\pi} \frac{1}{(z-x)(\bar{z}-\bar{x})(\bar{z}-\bar{y})} &= \frac{1}{\bar{x}-\bar{y}} \int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2z}{\pi} \bar{\partial}_z \frac{\ln\left|\frac{z-x}{z-y}\right|^2}{z-x} \\ &= \frac{-1}{\bar{x}-\bar{y}} \oint_{\partial\mathbb{C}_\varepsilon} \frac{\mathrm{d}z}{2\pi i} \frac{\ln\left|\frac{z-x}{z-y}\right|^2}{z-x} = -\frac{\ln\left|\frac{x-y}{\varepsilon}\right|^2}{(\bar{x}-\bar{y})} + \mathcal{O}(\varepsilon)\end{aligned} \tag{A.2}$$

Note the differences with the previous case with four factors in the denominator. In particular, only taking derivatives with respect to barred variables retains the logarithmic factor. This explains the delta factor in the above formula. The difference by a factor $\frac{1}{2}$ is due to the difference in the number of poles. Using (A.1) and (A.2) and their derivatives we can calculate a series of double integrals:

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{u-z} \frac{1}{v-w} \frac{1}{(\bar{u}-\bar{z})^2} \frac{1}{(\bar{v}-\bar{w})^2} \frac{1}{(z-w)^2} = \\ = -\frac{2\ln\left|\frac{u-v}{\varepsilon}\right|^2}{(u-v)^2(\bar{u}-\bar{v})^2}\end{aligned} \tag{A.3}$$

That can be derived by starting with the w-integral:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{v-w} \frac{1}{(\bar{v}-\bar{w})^2} \frac{1}{(z-w)^2} = \\
&= -\bar{\partial}_v \partial_z \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{w-v} \frac{1}{\bar{w}-\bar{v}} \frac{1}{w-z} = \\
&= \bar{\partial}_v \partial_z \frac{\ln|\frac{v-z}{\varepsilon}|^2}{v-z} + \mathcal{O}(\varepsilon) = \frac{1}{(v-z)^2} \frac{1}{\bar{v}-\bar{z}} + \mathcal{O}(\varepsilon)
\end{aligned} \tag{A.4}$$

Plugging this in the double integral:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \frac{1}{u-z} \frac{1}{(v-z)^2} \frac{1}{(\bar{u}-\bar{z})^2} \frac{1}{\bar{v}-\bar{z}} = \\
&= -\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar{u}-\bar{v})^2}
\end{aligned} \tag{A.5}$$

Similarly one can calculate:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{u-w} \frac{1}{v-z} \frac{1}{(\bar{u}-\bar{z})^2} \frac{1}{(\bar{v}-\bar{w})^2} \frac{1}{(z-w)^2} = \\
&= +\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar{u}-\bar{v})^2}
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(u-z)^2} \frac{1}{(v-w)^2} \frac{1}{(\bar{u}-\bar{z})^2} \frac{1}{\bar{v}-\bar{w}} \frac{1}{z-w} = \\
&= -\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar{u}-\bar{v})^2}
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(v-z)^2} \frac{1}{(u-w)^2} \frac{1}{(\bar{u}-\bar{z})^2} \frac{1}{\bar{v}-\bar{w}} \frac{1}{z-w} = \\
&= +\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar{u}-\bar{v})^2}
\end{aligned} \tag{A.8}$$

In the main text we need also some double integrals containing logarithms. For this reason we need to calculate:

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(u-w)^2}\frac{1}{(\bar u-\bar w)^2}\ln\left|\frac{z-w}{\varepsilon}\right|^2 &=\\ =\partial_u\partial_{\bar u}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\partial_{\bar w}\left(\frac{\ln\left(\frac{\bar u-\bar w}{\bar u-\bar z}\right)\ln\left|\frac{z-w}{\varepsilon}\right|^2+\mathrm{Li}_2\left(\frac{\bar w-\bar z}{\bar u-\bar z}\right)}{w-u}\right) &=\\ =\partial_u\partial_{\bar u}\left(\frac{\pi^2}{6}+\ln\left(\frac{\varepsilon}{\bar u-\bar z}\right)\ln\left|\frac{u-z}{\varepsilon}\right|^2\right) &=\\ =-\frac{1}{u-z}\frac{1}{\bar u-\bar z}\end{aligned} \tag{A.9}$$

Where we have used the fact that $\mathrm{Li}_2(1)=\frac{\pi^2}{6}$. This result can be used to calculate:

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(u-w)^2}\frac{1}{(v-z)^2}\frac{1}{(\bar u-\bar w)^2}\frac{1}{(\bar v-\bar z)^2}\ln\left|\frac{z-w}{\varepsilon}\right|^2 &=\\ =\frac{2\ln\left|\frac{u-v}{\varepsilon}\right|^2}{(u-v)^2(\bar u-\bar v)^2}\end{aligned} \tag{A.10}$$

Other useful logarithmic integrals are:

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(u-w)^2}\frac{1}{(\bar z-\bar w)^2}\ln\left|\frac{z-w}{\varepsilon}\right|^2 &=\\ =-\partial_u\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\partial_{\bar w}\left(\frac{1+\ln\left|\frac{z-w}{\varepsilon}\right|^2}{(w-u)(\bar w-\bar z)}\right) &=\\ =-\partial_u\left(\frac{1+\ln\left|\frac{z-u}{\varepsilon}\right|^2}{(\bar u-\bar z)}\right) &=\\ =-\frac{1}{u-z}\frac{1}{\bar u-\bar z}\end{aligned} \tag{A.11}$$

and

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(u-w)^2}\frac{1}{(\bar v-\bar w)^2}\ln\left|\frac{z-w}{\varepsilon}\right|^2 &=\\ =\partial_u\partial_{\bar v}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\partial_{\bar w}\left(\frac{\ln\left(\frac{\bar v-\bar w}{\bar v-\bar z}\right)\ln\left|\frac{z-w}{\varepsilon}\right|^2+\mathrm{Li}_2\left(\frac{\bar w-\bar z}{\bar v-\bar z}\right)}{w-u}\right) &=\\ =\partial_u\partial_{\bar u}\left(\mathrm{Li}_2\left(\frac{\bar u-\bar z}{\bar v-\bar z}\right)+\ln\left(\frac{\varepsilon}{\bar u-\bar z}\right)\ln\left|\frac{u-z}{\varepsilon}\right|^2\right) &=\\ =\frac{(\bar z-\bar u)}{(\bar u-\bar v)(u-z)(\bar v-\bar z)}\end{aligned} \tag{A.12}$$

Appendix B

Representation theory of $\mathfrak{osp}(4|2)$

In the following we give a very basic introduction to the Lie superalgebra $\mathfrak{osp}(4|2)$ and (some of) its finite dimensional representations. The complex superalgebra $\mathfrak{g} := \mathfrak{osp}(4|2)$ may be realized as the set of supermatrices,

$$\mathfrak{osp}(4|2) = \left\{ \begin{pmatrix} A & B \\ J_2 B^t & D \end{pmatrix} : A^t = -A \text{ and } D^t J_2 = -J_2 D \right\} . \tag{B.1}$$

Here A is a 4×4 matrix, D is a 2×2 matrix and B is rectangular of size 4×2. In addition, we introduced the 2×2 matrix $J_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. As usual, the Lie superalgebra $\mathfrak{g}$ decomposes into an even, or bosonic, subalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(4) \oplus \mathfrak{sp}(2) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and an odd, or fermionic, subspace $\mathfrak{g}_{\bar{1}}$.

Our review of representations focuses on finite dimensional representations. As usual for superalgebras, irreducible representations fall into two different categories. On the one hand, there are the generic long multiplets. These are also known as typical representations in the more mathematical literature. On the other hand, a superalgebra also possesses short or BPS multiplets which mathematicians refer to as atypical representations. BPS multiplets can be put together into indecomposable representations. We will only work with one class of such indecomposables, namely the projective covers of atypical representations.

In order to make all this more precise, we note that an integral dominant highest weight $\Lambda = (j_1, j_2, j_3)$ of $\mathfrak{g}_{\bar{0}}$ is also one for the full superalgebra $\mathfrak{g}$ if it obeys the consistency conditions

$$j_1 = 0 \Rightarrow j_2 = j_3 = 0 \quad , \qquad j_1 = \frac{1}{2} \Rightarrow j_2 = j_3 \ . \tag{B.2}$$

The ordering of our the spins $j_i \in \frac{1}{2}\mathbb{Z}$ is such that the first spin is related to the symplectic subalgebra $\mathfrak{sp}(2)$ while the two others are associated with the orthogonal one. This is a bit unfortunate but agrees with conventions in earlier literature. We shall use the label $[\Lambda] = [j_1, j_2, j_3]$ to denote finite dimensional irreducibles.

With these labels introduced we can now spell out the shortening conditions we have mentioned above. A representation $[j_1, j_2, j_3]$ is atypical provided the spins satisfy any one of the following conditions

$$\begin{aligned} 2j_1 &= -j_2 - j_3 \ , \\ 2j_1 &= j_2 + j_3 + 2 \ , \\ 2j_1 &= \pm(j_2 - j_3) + 1 \ . \end{aligned} \tag{B.3}$$

Otherwise the representation $[j_1, j_2, j_3]$ is typical. The eigenvalue of the quadratic Casimir element in the irreducible representation $[\Lambda]$ is given by

$$\mathbf{Cas}_{\mathfrak{g}}(\Lambda) \ = \ -4j_1(j_1 - 1) + 2j_2(j_2 + 1) + 2j_3(j_3 + 1) \ . \tag{B.4}$$

If the spins satisfy one of the shortening conditions (B.3) the value of the quadratic Casimir element is a square, i.e. $\mathbf{Cas}_{\mathfrak{g}}(\Lambda) = l^2$ with $l \in \mathbb{N}$. The atypical weights $\Lambda = (j_1, j_2, j_3)$, i.e. those weights that satisfy one of the shortening conditions, can be divided into blocks β_l that contain all those representations $\Lambda \in \beta_l$ for which $\mathbf{Cas}_{\mathfrak{g}}(\Lambda) = l^2$. The corresponding atypical labels can be listed explicitly [59],

$$\begin{aligned} \beta_0 &= \left\{ \Lambda_{0,0} = (0,0,0) \, , \, \Lambda_{0,k} = \frac{1}{2}(k+1, k-1, k-1) \, , \, k \geq 1 \right\} \\ \beta_l &= \{ \Lambda_{l,k} \, , \, k \in \mathbb{Z} \} \end{aligned} \tag{B.5}$$

where

$$\Lambda_{l,k} = \begin{cases} \frac{1}{2}(-k+2, -k-l, -k+l) & \text{if } k \leq -l \\ \frac{1}{2}(-k+1, k+l-1, -k+l-1) & \text{if } -l+1 \leq k \leq 0 \\ \frac{1}{2}(k+1, k+l-1, -k+l-1) & \text{if } 0 \leq k \leq l-1 \\ \frac{1}{2}(k+2, k+l, k-l) & \text{if } l \leq k \end{cases} \ . \tag{B.6}$$

One sees easily, that the weights $\Lambda_{l,-k}$ for $l \geq 1$ may be obtained from $\Lambda_{l,k}$ by simply exchanging the second and the third Dynkin label. Furthermore, it is possible to distinguish the weights $\Lambda_{l,k}$ according to the atypicality condition (B.3) they obey. The only weight to fulfill the first condition is $\Lambda_{0,0}$. The weights belonging to the second condition are $\Lambda_{0,k}$ for $k \geq 1$ and $\Lambda_{l,\pm k}$ for $k \geq l$. Finally, those the satisfy the last atypicality relation are the $\Lambda_{l,\pm k}$ for $k < l$. In any case, each of the weights fulfills at most one of the shortening conditions. This means that all atypical representations of $\mathfrak{osp}(4|2)$ possess the same degree of atypicality, i.e. they are all what mathematicians refer to as maximally atypical and physicists call $\frac{1}{2}$BPS.

We can decompose all irreducible representations $[j_1, j_2, j_3]$ in terms of irreducible subrepresentations of the bosonic subalgebra $\mathfrak{g}_{\bar{0}}$. For typical representation one finds

$$\begin{aligned}
[j_1, j_2, j_3]_{\mathfrak{g}_{\bar{0}}} \cong \; & (j_1, j_2, j_3) \bigoplus_{\alpha,\beta=\pm\frac{1}{2}} (j_1 - \frac{1}{2}, j_2 + \alpha, j_3 + \beta) \\
& \bigoplus_{\alpha=\pm 1} \big[(j_1 - 1, j_2 + \alpha, j_3) \oplus (j_1 - 1, j_2, j_3 + \alpha)\big] \\
& \oplus 2(j_1 - 1, j_2, j_3) \bigoplus_{\alpha,\beta=\pm\frac{1}{2}} (j_1 - \frac{3}{2}, j_2 + \alpha, j_3 + \beta) \\
& \oplus (j_1 - 2, j_2, j_3) \ .
\end{aligned} \tag{B.7}$$

There are a few special cases for which the decomposition is not generic. If $j_1 \leq 2, j_2 \leq 1$ or $j_3 \leq 1$ then the above decomposition formula must be truncated at the point where one or more of the labels become negative. Moreover, there are two cases for which the multiplicity of the $(j_1 - 1, j_2, j_3)$ submodule has to be changed. If $j_1 = 1, j_2 > 0, j_3 > 0$ or $j_1 > 1, j_2 = 0, j_3 > 0$ or $j_1 > 1, j_2 > 0, j_3 = 0$, then this block will appear only once and if both j_2 and j_3 are null or $j_1 = 1$ and at least one between j_2 and j_3 is null, then it will not be present at all. From the decomposition into representations of the bosonic algebra we can determine the dimension of typical representations

$$\dim[j_1, j_2, j_3] = 16(2j_1 - 1)(2j_2 + 1)(2j_3 + 1) \ . \tag{B.8}$$

The decomposition (B.7) for $j_1 \geq 1$, is valid for the indecomposable Kac modules that emerge when the spins j_i satisfy one of the shortening conditions (B.3). These Kac modules are composites of irreducibles. More precisely, one finds

$$\begin{aligned} K_{\Lambda_{0,2}} &: \; [\Lambda_{0,2}] \longrightarrow [\Lambda_{0,0}] \oplus [\Lambda_{0,1}] \\ K_{\Lambda_{0,k}} &: \; [\Lambda_{0,k}] \longrightarrow [\Lambda_{0,k-1}] \text{ for } k \geq 3 \\ K_{\Lambda_{l,k}} &: \; [\Lambda_{l,k}] \longrightarrow [\Lambda_{l,k-1}] \text{ for } k \geq 1 \\ K_{\Lambda_{l,k}} &: \; [\Lambda_{l,k}] \longrightarrow [\Lambda_{l,k+1}] \text{ for } k \leq -1 \quad . \end{aligned} \tag{B.9}$$

The arrows mean that fermionic generators can take us from the representation on the left to the one on the right but not vice versa. Put differently, the representation on the right hand side of the arrows is a subrepresentation of the Kac module. If we quotient the Kac module by this subrepresentation, the corresponding factor representation is the one on the left hand side. The representations with $j_1 = \frac{1}{2}$ are somewhat special. In fact, when $j_1 = \frac{1}{2}$, the Kac module is irreducible and we obtain

$$\Lambda_{l+1,2}\big|_{\mathfrak{g}_{\bar{0}}} \cong \left(\frac{1}{2}, \frac{l}{2}, \frac{l}{2}\right) \oplus \left(0, \frac{l+1}{2}, \frac{l+1}{2}\right) \oplus \left(0, \frac{l-1}{2}, \frac{l-1}{2}\right) \quad . \tag{B.10}$$

From our description of the Kac modules it is possible to determine the dimensions of irreducible atypicals,

$$\begin{aligned} \dim[\Lambda_{0,0}] &= 1 \,, \qquad \dim[\Lambda_{0,1}] \;=\; 17 \,, \qquad \dim[\Lambda_{l,0}] \;=\; 4l^2 + 2 \\ \dim[\Lambda_{0,k}] &= (2k+1)\left[(2k+1)^2 - 3\right] \text{ for } k \geq 2 \\ \dim[\Lambda_{l,k}] &= (2k+1)\left[4(l^2-1) - (2k+1)^2 + 7\right] \text{ for } k \leq l-1 \\ \dim[\Lambda_{l,k}] &= (2k+3)\left[(2k+3)^2 - 4(l^2-1) - 7\right] \text{ for } k \geq l \;. \end{aligned} \tag{B.11}$$

We are finally prepared to describe the projective covers that feature so prominently in the construction of homogeneous vector bundles. While typical irreducibles $[\Lambda]$ coincide with their projective cover $\mathcal{P}_\Lambda = [\Lambda]$, the projective cover of an atypical representations is an indecomposable composite of atypicals. Its precise structure can be read off from the following diagrams

$$\mathcal{P}_{\Lambda_{0,0}} : \Lambda_{0,0} \to \Lambda_{0,2} \to \Lambda_{0,0} \tag{B.12}$$

$$\mathcal{P}_{\Lambda_{0,1}} : \Lambda_{0,1} \to \Lambda_{0,2} \to \Lambda_{0,1} \tag{B.13}$$

$$\mathcal{P}_{\Lambda_{0,2}} : \Lambda_{0,2} \to \Lambda_{0,3} \oplus \Lambda_{0,1} \oplus \Lambda_{0,0} \to \Lambda_{0,2} \tag{B.14}$$

$$\mathcal{P}_{\Lambda_{l,k}} : \Lambda_{l,k} \to \Lambda_{l,k+1} \oplus \Lambda_{l,k-1} \to \Lambda_{l,k} \qquad \text{otherwise} \tag{B.15}$$

The meaning of the arrows was explained in our discussion of Kac modules above. Note that all the atypicals that appear in any given projective cover belong to the same block β. It is actually not possible to build indecomposables from representations within different blocks.

Before we conclude this brief overview over representations of the Lie superalgebra $\mathfrak{osp}(4|2)$ we want to spell out a few tensor product decompositions between irreducible atypicals. These are used in our discussion of the low lying spectrum in the $\mathfrak{osp}(4|2)$ Gross-Neveu model.

$$\begin{aligned}
\Lambda_{0,1} \otimes \Lambda_{0,1} =& \Lambda_{0,0} + 2\Lambda_{0,1} + \Lambda_{0,2} + \Lambda_{2,-1} + 2\Lambda_{2,0} + \Lambda_{2,1} + [2,0,0] \\
\Lambda_{0,1} \odot \Lambda_{0,1} =& \Lambda_{0,0} + \Lambda_{2,-1} + 2\Lambda_{2,0} + \Lambda_{2,1} + [2,0,0] \\
\Lambda_{0,1} \otimes \Lambda_{0,2} =& \Lambda_{0,0} + \Lambda_{0,1} + 3\Lambda_{0,2} + \Lambda_{0,3} + \\
& + [1,1,1] + [\tfrac{3}{2},\tfrac{1}{2},\tfrac{3}{2}] + [\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}] + \\
& + [2,0,1] + [2,1,0] + [\tfrac{5}{2},\tfrac{1}{2},\tfrac{1}{2}] \\
\Lambda_{0,2} \otimes \Lambda_{0,2} =& 2\Lambda_{0,0} + 4\Lambda_{0,1} + 4\Lambda_{0,2} + 4\Lambda_{0,3} + \Lambda_{0,4} + \\
& + \Lambda_{2,-2} + 3\Lambda_{2,-1} + 4\Lambda_{2,0} + 3\Lambda_{2,1} + \Lambda_{2,2} + \\
& + \Lambda_{4,-1} + 2\Lambda_{4,0} + \Lambda_{4,1} + \\
& + [1,0,2] + 2[1,1,1] + [1,2,0] + \\
& + 2[\tfrac{3}{2},\tfrac{1}{2},\tfrac{3}{2}] + 2[\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}] + 2[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}] + \\
& + 2[2,0,0] + 2[2,0,1] + \\
& 2[2,1,0] + [2,1,2] + [2,2,1] + \\
& + 2[\tfrac{5}{2},\tfrac{1}{2},\tfrac{1}{2}] + 2[\tfrac{5}{2},\tfrac{1}{2},\tfrac{3}{2}] + 2[\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2}] + \\
& + [3,0,0] + [3,0,1] + [3,1,0] + [3,1,1] \\
\Lambda_{1,0} \otimes \Lambda_{1,0} =& \Lambda_{0,0} + \Lambda_{0,1} + \Lambda_{2,0} \\
\Lambda_{2,0} \otimes \Lambda_{2,0} =& \Lambda_{0,0} + \Lambda_{0,1} + \Lambda_{2,-1} + 2\Lambda_{2,0} + \Lambda_{2,1} + \Lambda_{4,0} + [1,1,1]
\end{aligned} \tag{B.16}$$

The $+$ on the right hand side requires a short comment. As we have stated above, atypical irreducibles can be combined to form larger indecomposables. This happens for many of the atypical representations that appear in the above tensor product decompositions. Hence, many of the atypicals are not direct summands. This is why we did not use $\oplus$. On the other hand, the sum is direct for all projective modules, i.e. for typicals and projective covers of atypicals. The symbol $\odot$ is used to denote the symmetric part of the tensor product. The tensor products (B.16) were obtained by implementing the decompositions (B.16) along with the product rules for $\mathfrak{su}(2)$-representations in a Mathematica script.

Appendix C

Representation theory of $\mathfrak{osp}(3|2)$

In this appendix we provide some background material on the Lie superalgebra $\mathfrak{osp}(3|2)$ and its finite dimensional representations. The basic definition of $\mathfrak{osp}(3|2)$ resembles the definition (B.1) we gave for $\mathfrak{osp}(4|2)$ only that now A is a 3×3 matrix and B is rectangular of size 3×2. In the case of $\mathfrak{h} = \mathfrak{osp}(3|2)$, the bosonic subalgebra is $\mathfrak{h}_{\bar{0}} = \mathfrak{so}(3) \oplus \mathfrak{sp}(2)$. Since $\mathfrak{h}_{\bar{0}}$ has rank two, highest weights are labeled by two numbers $\lambda = (q, p)$. In our conventions, the $\mathfrak{so}(3)$ spin p runs over non-negative integers while q is a non-negative half-integer. Note that once again, the order of the two labels is a bit unfortunate. As in the case of $\mathfrak{osp}(4|2)$, there is an additional constraint on the weights (q, p) that must be satisfied in order for (q, p) to label a representation of $\mathfrak{osp}(3|2)$, namely

$$q = 0 \;\Rightarrow\; p = 0 \; .$$

Once more we shall use the bracket notation $[\lambda] = [q, p]$ to denote the associated irreducible representation of $\mathfrak{osp}(3|2)$. The representation $[q, p]$ is typical (long) unless the labels q, p satisfy one of the following two shortening conditions

$$p + 2q = 0 \quad , \quad p - 2q + 1 = 0 \; . \tag{C.1}$$

These conditions are mutually exclusive. While the first one is only satisfied for the trivial representation $q = p = 0$, the latter singles out a one parameter family of (maximally) atypical (or $\frac{1}{2}$BPS) representations.

The eigenvalue of the quadratic Casimir element in an irreducible representation $[\lambda] = [q,p]$ is given by

$$\mathbf{Cas}_{\mathfrak{h}}([q,p]) \ = (p+2q)(p-2q+1) \ . \tag{C.2}$$

In particular, we conclude that the quadratic Casimir element vanishes for all atypical representations of $\mathfrak{osp}(3|2)$. This suggests that all atypicals belong to one and the same block, which is indeed the case. Representations in this unique block are given by

$$\lambda_0 = [0,0] \quad , \quad \lambda_q = [q, 2q-1] \ . \tag{C.3}$$

Let us also mention in passing that the Lie superalgebra $\mathfrak{osp}(3|2)$ possesses a fourth order Casimir element whose eigenvalues are given by

$$\mathbf{Cas}^{(4)}_{\mathfrak{h}}(\lambda) \ = \frac{1}{4}\mathbf{Cas}_{\mathfrak{h}}(\lambda)[3p(3p+1)+2(q+1)(2q-3)] \tag{C.4}$$

The fourth order Casimir element does not show up in the 1-loop anomalous dimensions but could enter starting from two-loops.

As in the case of $\mathfrak{osp}(4|2)$ it is useful to know how the irreducible representations decompose with respect to the bosonic subalgebra. For typical representations, this decomposition is given by

$$[q,p]_{\mathfrak{h}_{\bar 0}} \ \cong (q,p) \oplus \bigoplus_{\alpha=0,\pm 1} \big[(q-\tfrac{1}{2}, p+\alpha) \oplus (q-1, p+\alpha)\big] \oplus (q-\tfrac{3}{2}, p) \ . \tag{C.5}$$

Truncations are present whenever one or both labels on the right hand side become negative. When $q = \frac{1}{2}$ or $p = 0$ the term $(q-\frac{1}{2}, p)$ does not appear. For the adjoint representation the decomposition reads

$$[1,0]_{\mathfrak{h}_{\bar 0}} \cong (1,0) \oplus (\tfrac{1}{2}, 1) \oplus (0,1) \ . \tag{C.6}$$

Note that in the case of $\mathfrak{osp}(3|2)$ the adjoint representation is typical. Atypical representations with $q \geq 1$ possess the following decomposition

$$[\lambda_q]_{\mathfrak{h}_{\bar 0}} \ \cong (q, 2q-1) \oplus (q-\tfrac{1}{2}, 2q-1) \oplus (q-\tfrac{1}{2}, 2q) \oplus (q-1, 2q) \ . \tag{C.7}$$

The atypical trivial representation λ_0 and the fundamental $\lambda_{\frac{1}{2}}$ are special. While the decomposition of λ_0 is trivial, the fundamental representation gives

$$[\lambda_{\frac{1}{2}}]_{\mathfrak{h}_{\bar 0}} \cong (\tfrac{1}{2}, 0) \oplus (0,1) \ . \tag{C.8}$$

For completeness we also state the dimension of the these representations. In the case of typical long multiplets we have

$$\dim\big([q,p]\big) = 4(2p+1)(4p-1) \tag{C.9}$$

while the dimension of atypicals is given by

$$\begin{aligned}\dim[\lambda_0] = 1 \quad \dim[\lambda_{\frac{1}{2}}] = 5 \\ \dim[\lambda_q] = -2 + 32q^2 \ .\end{aligned} \tag{C.10}$$

As for any Lie superalgebra, atypical representations can be combined into larger indecomposables. For our analysis, the projective covers of atypicals are of particular importance. Their structure is given by

$$\mathcal{P}_{\lambda_0} : \lambda_0 \to \lambda_1 \to \lambda_0 \tag{C.11}$$

$$\mathcal{P}_{\lambda_{\frac{1}{2}}} : \lambda_{\frac{1}{2}} \to \lambda_1 \to \lambda_{\frac{1}{2}} \tag{C.12}$$

$$\mathcal{P}_{\lambda_1} : \lambda_1 \to \lambda_{\frac{3}{2}} \oplus \lambda_{\frac{1}{2}} \oplus \lambda_0 \to \lambda_1 \tag{C.13}$$

$$\mathcal{P}_{\lambda_q} : \lambda_q \to \lambda_{q+\frac{1}{2}} \oplus \lambda_{q-\frac{1}{2}} \to \lambda_q \qquad \text{otherwise} \tag{C.14}$$

The meaning of the arrows was explained in appendix B. The structure we display here is consistent with the fact that all atypical irreducibles λ_q of $\mathfrak{osp}(3|2)$ belong to the same block.

In our construction of coset vertex operators (3.10), and in particular in the analysis of the tail factors, we need some input about tensor products of $\mathfrak{osp}(3|2)$ representations. The first few powers of the fundamental representation $\lambda_{\frac{1}{2}}$ are given by

$$\lambda_{\frac{1}{2}}^{\otimes 2} = [1,0] + [\tfrac{1}{2},1] + \lambda_0 \tag{C.15}$$

$$\lambda_{\frac{1}{2}}^{\odot 2} = [\tfrac{1}{2},1] + \lambda_0 \tag{C.16}$$

$$\lambda_{\frac{1}{2}}^{\odot 3} = [\tfrac{1}{2},2] + \lambda_{\frac{1}{2}} \tag{C.17}$$

Here, we use the symbol $\odot$ to denote the graded symmetric part of the tensor product. The formulas we displayed are relevant e.g. for products such as $j\partial j$, j^2 and j^3, respectively. Let us also list a few additional tensor products of low dimensional representations,

$$
\begin{aligned}
[1,0] \otimes \lambda_{\frac{1}{2}} &= [\tfrac{3}{2},0] + 2\lambda_{\frac{1}{2}} + \lambda_1 \\
[\tfrac{1}{2},1] \otimes \lambda_{\frac{1}{2}} &= [\tfrac{1}{2},2] + 2\lambda_{\frac{1}{2}} + \lambda_1 \\
[\tfrac{1}{2},1] \otimes [\tfrac{1}{2},1] &= [1,2] + [1,0] + [\tfrac{1}{2},3] + [\tfrac{1}{2},1] + 2\lambda_0 + \lambda_1 \\
[\tfrac{1}{2},2] \otimes \lambda_{\frac{1}{2}} &= [1,2] + [\tfrac{1}{2},3] + [\tfrac{1}{2},1] \\
[\tfrac{1}{2},2] \otimes [\tfrac{1}{2},1] &= [1,3] + [1,2] + [\tfrac{1}{2},4] + [\tfrac{1}{2},2] + 2\lambda_{\frac{1}{2}} + \lambda_1 \\
[\tfrac{1}{2},2] \otimes [\tfrac{1}{2},2] &= [1,4] + [1,3] + [1,2] + [1,0] + [\tfrac{1}{2},5] + [\tfrac{1}{2},3] \\
&\quad + [\tfrac{1}{2},2] + [\tfrac{1}{2},1] + 2\lambda_0 + \lambda_1 \\
[\tfrac{1}{2},1] \otimes [1,0] &= [\tfrac{3}{2},1] + [1,0] + [1,2] + [\tfrac{1}{2},1] \\
[\tfrac{1}{2},2] \otimes [1,0] &= [1,3] + [\tfrac{1}{2},2] + \lambda_0 + \lambda_{\frac{1}{2}} + 2\lambda_1 + \lambda_{\frac{3}{2}} \\
[1,0] \otimes [1,0] &= [2,0] + [\tfrac{3}{2},1] + [1,0] + [\tfrac{1}{2},1] + 2\lambda_0 + \lambda_1
\end{aligned}
\tag{C.18}
$$

These are useful in order to carry the construction of vertex operators to higher gradient operators. Note that while it is not relevant for our discussion, the atypical representations in (C.18) always combine into projectives, while all other sums are direct. These tensor products were obtained by implementing the decompositions (C.5)–(C.8) along with the product rules for $\mathfrak{su}(2)$ representations in a Mathematica script.

Appendix D

Restriction of $\mathfrak{osp}(4|2)$ representations to $\mathfrak{osp}(3|2)$

As we explained in section 3.1.3, a key ingredient in constructing vertex operators on coset superspaces is the decomposition (3.11) of sections in homogeneous vector bundles into multiplets of the symmetry. According to the central formula, the multiplicity $n_{\Lambda\lambda}$ of a $\mathfrak{g}$ multiplet Λ in a bundle Γ_λ is given by eq. (3.13). It implies that $n_{\Lambda\lambda}$ can be computed through the decomposition

$$\mathcal{P}_\Lambda|_{\mathfrak{h}} = \bigoplus_\lambda n_{\Lambda\lambda}\,\mathcal{P}_\lambda = \bigoplus_\lambda [\mathcal{P}_\Lambda|_{\mathfrak{h}} \,:\, \mathcal{P}_\lambda]\;\mathcal{P}_\lambda\;.$$

Given what we know about the projective covers of both $\mathfrak{osp}(4|2)$ and $\mathfrak{osp}(3|2)$ it is not too difficult to work out the multiplicities $n_{\Lambda\lambda}$. We only need the results for atypical labels $\Lambda = \Lambda_{l,k}$. For representations $\Lambda_{0,k}$ in the block of the trivial representation one finds

$$\begin{aligned}
&\mathcal{P}_{\Lambda_{0,0}}|_{\mathfrak{osp}(3|2)} = \mathcal{P}_{\lambda_0} \oplus [\tfrac{3}{2},0] \oplus [\tfrac{3}{2},1]\\
&\mathcal{P}_{\Lambda_{0,1}}|_{\mathfrak{osp}(3|2)} = \mathcal{P}_{\lambda_{\frac{1}{2}}} \oplus [\tfrac{3}{2},0] \oplus [\tfrac{3}{2},1]\\
&\mathcal{P}_{\Lambda_{0,k}}|_{\mathfrak{osp}(3|2)} = \mathcal{P}_{\lambda_{\frac{k}{2}}} \oplus 2\bigoplus_{n=0}^{k-1}[\tfrac{k+1}{2},n] \oplus \bigoplus_{n=0}^{k}[\tfrac{k+2}{2},n] \oplus \bigoplus_{n=0}^{k-2}[\tfrac{k}{2},n]\,, \quad \forall\, k \geq 2\,.
\end{aligned} \tag{D.1}$$

Similarly one can decompose the projective covers of the symmetric traceless tensor representations $\Lambda_{l,0}$,

$$\begin{aligned}
\mathcal{P}_{\Lambda_{1,0}}\big|_{\mathfrak{osp}(3|2)} &= \mathcal{P}_{\lambda_0} \oplus \mathcal{P}_{\lambda_{\frac{1}{2}}} \oplus 2[\tfrac{3}{2},1] \\
\mathcal{P}_{\Lambda_{2,0}}\big|_{\mathfrak{osp}(3|2)} &= \mathcal{P}_{\lambda_0} \oplus \mathcal{P}_{\lambda_{\frac{1}{2}}} \oplus 2[\tfrac{1}{2},1] \\
\mathcal{P}_{\Lambda_{l,0}}\big|_{\mathfrak{osp}(3|2)} &= \mathcal{P}_{\lambda_0} \oplus \mathcal{P}_{\lambda_{\frac{1}{2}}} \oplus 2\bigoplus_{n=1}^{l-1}[\tfrac{1}{2},n] \oplus 2\bigoplus_{n=2}^{l-1}[1,n]\,, \quad \text{when} \quad l \geq 2\,.
\end{aligned} \tag{D.2}$$

Finally, generic projective covers possess the following decomposition into projectives of $\mathfrak{osp}(3|2)$,

$$\begin{aligned}
\mathcal{P}_{\Lambda_{l,k}}|_{\mathfrak{osp}(3|2)} = \mathcal{P}_{\lambda_{\frac{|k|+1}{2}}} &\oplus \bigoplus_{n=|k|}^{l-1}[\tfrac{|k|}{2},n] \oplus 2 \bigoplus_{n=|k|+1}^{l-1}[\tfrac{|k|+1}{2},n] \oplus \bigoplus_{n=l}^{|k|-1}[\tfrac{|k|+1}{2},n] \\
&\oplus 2\bigoplus_{n=l}^{|k|}[\tfrac{|k|+2}{2},n] \oplus \bigoplus_{n=|k|+2}^{l-1}[\tfrac{|k|+2}{2},n] \oplus \bigoplus_{n=l}^{|k|+1}[\tfrac{|k|+3}{2},n]\,.
\end{aligned} \tag{D.3}$$

This last formula holds whenever $l \geq 1$ and $|k| \geq 1$. Formulas (D.1)-(D.3) provide the main input for the construction of vertex operators in section 4.1.2. Let us note that in these formulas all sums are direct since the restriction of projective modules is a direct sum of projectives and projectives cannot appear as pieces of larger indecomposables.

In order to derive these decomposition formulas one starts from the following decomposition formula for representations of the bosonic subalgebra $\mathfrak{g}_{\bar{0}}$ into representations of $\mathfrak{h}_{\bar{0}}$,

$$(j_1,j_2,j_3)_{\mathfrak{h}_{\bar{0}}} \cong \bigoplus_{p=|j_2-j_3|}^{j_2+j_3}(j_1,p) \tag{D.4}$$

In a second step these decomposition formulas are exploited to determine how atypical irreducibles of $\mathfrak{osp}(4|2)$ decompose upon restriction to $\mathfrak{osp}(3|2)$.

The results read,

$$
\begin{aligned}
\Lambda_{0,0}|_{\mathfrak{osp}(3|2)} &= \lambda_0 \\
\Lambda_{0,k}|_{\mathfrak{osp}(3|2)} &= \lambda_{\frac{l}{2}} \oplus \bigoplus_{n=0}^{k-1}[\tfrac{k+1}{2}, n], \quad l > 0 \\
\Lambda_{l,0}|_{\mathfrak{osp}(3|2)} &= \bigoplus_{n=0}^{l-1}[\tfrac{1}{2}, n] \oplus \lambda_0, \quad l > 0 \\
\Lambda_{l,k}|_{\mathfrak{osp}(3|2)} &= \bigoplus_{n=|k|}^{l-1}[\tfrac{|k|+1}{2}, n], \quad 0 < |k| \leq l-1 \\
\Lambda_{l,k}|_{\mathfrak{osp}(3|2)} &= \bigoplus_{n=l}^{|k|}[\tfrac{|k|}{2}+1, n] \oplus \lambda_{\frac{|k|+1}{2}}, \quad 0 < l \leq |k|
\end{aligned}
\tag{D.5}
$$

Since we know how projective covers are built from atypicals, it is now straightforward to verify the decomposition formulas (D.1)-(D.3). All this was computed using the Mathematica scripts used to compute the tensor products of Appendices B and C.

Appendix E

The fermionic sector

In this appendix we briefly review the construction of the affine osp(2p|2q) algebra at level $\tilde{k} = 1$ in terms of free fermions and several bosonic ghost systems. Let us decompose all supermatrices $X \in \text{osp}(2p|2q)$ into blocks according to

$$X \; = \; \left(\begin{array}{c|cc} \mathcal{E} & \bar{\mathcal{T}} & \mathcal{T} \\ \hline -\mathcal{T}^t & \mathcal{F} & \mathcal{G} \\ \bar{\mathcal{T}}^t & \bar{\mathcal{G}} & -\mathcal{F}^t \end{array} \right) \tag{E.1}$$

where $\mathcal{E}$ is antisymmetric and $\mathcal{G}, \bar{\mathcal{G}}$ are symmetric. A basis for the various blocks in the supermatrix X is provided by

$$\begin{aligned} \mathcal{E}_{ij} \; &= \; e_{ij} - e_{ji} \qquad 1 \, \leq \, i \, < \, j \, \leq \, 2p \\ \mathcal{F}_{ab} \; &= \; e_{ab} \qquad 1 \, \leq \, a \, , \, b \, \leq \, q \\ \mathcal{G}_{ab} \; &= \; \bar{\mathcal{G}}_{ab} \; = \; e_{ab} + e_{ba} \qquad 1 \, \leq \, a \, \leq \, b \, \leq \, q \\ \mathcal{T}_{ia} \; &= \; \bar{\mathcal{T}}_{ia} \; = \; e_{ia} \qquad 1 \, \leq \, i \, \leq \, 2p \, , \, 1 \, \leq \, a \, \leq \, q \end{aligned} \tag{E.2}$$

where e_{mn} are elementary matrices. The matrices we have just introduced describe the various blocks in the supermatrix X. We agree to denote by E_{ij} the supermatrix of the form (E.1) where $\mathcal{E}$ is given by $\mathcal{E}_{ij}$ and all other blocks vanish. The basis elements $F_{ab}, G_{ab}, \bar{G}_{ab}, T_{ia}, \bar{T}_{ia}$ are defined similarly.

Now let us introduce M free fermions ψ_i and $2N$ bosons β_a, γ_a with the following basic operator products,

$$\psi_i(z)\psi_j(w) \; \sim \; \frac{\delta_{ij}}{z-w} \; , \qquad \beta_a(z)\gamma_b(w) \; \sim \; -\gamma_a(z)\beta_b(w) \; \sim \; \frac{\delta_{ab}}{z-w} \; . \tag{E.3}$$

We can define the free field representation of the osp(M|2N) current algebra through

$$
\begin{aligned}
E_{ij}(z) &= (\psi_i\psi_j)(z)\,, & F_{ab}(z) &= -(\beta_a\gamma_b)(z) \\
G_{ab}(z) &= (\beta_a\beta_b)(z)\,, & \bar{G}_{ab}(z) &= -(\gamma_a\gamma_b)(z) \\
T_{ia}(z) &= i(\psi_i\beta_a)(z)\,, & \bar{T}_{ia}(z) &= -i(\psi_i\gamma_a)(z)\,.
\end{aligned}
\tag{E.4}
$$

The invariant bilinear form for osp(M|2N) is $(X,Y) = \frac{1}{2}\mathrm{str}(XY)$. On the basis elements it takes the following from

$$
\begin{aligned}
(E_{ij}, E_{kl}) &= -\delta_{ik}\delta_{jl} \quad i<j \text{ and } k<l \\
(F_{ab}, F_{cd}) &= -\delta_{ad}\delta_{bc} \\
(G_{ab}, \bar{G}_{cd}) &= -\delta_{ac}\delta_{bd} \quad \text{for } a\neq b \text{ and } c\neq d \quad (G_{aa}, \bar{G}_{bb}) = -2\delta_{ab} \\
(T_{ia}, \bar{T}_{jb}) &= \delta_{ij}\delta_{ab}\,.
\end{aligned}
\tag{E.5}
$$

With the help of this form and assuming that $M \neq 2N+1$, the holomorphic part of the energy momentum tensor is given by the Sugawara construction

$$
\begin{aligned}
T(z) = \frac{(J^\mu J_\mu)(z)}{2(k+g^\vee)} &= \frac{1}{2(k+g^\vee)}\Big[-\sum_{i<j=1}^{M}(E_{ij}^2) \\
&\quad -\sum_{a,b=1}^{N}(F_{ab}F_{ba}) - \sum_{a<b=1}^{N}\big(\{G_{ab},\bar{G}_{ab}\}\big) \\
&\quad -\frac{1}{2}\sum_{a=1}^{N}\big(\{G_{aa},\bar{G}_{aa}\}\big) - \sum_{i=1}^{M}\sum_{a=1}^{N}\big([T_{ia},\bar{T}_{ia}]\big)\Big] \\
&= -\frac{1}{2}\sum_{i=1}^{M}(\psi_i\partial\psi_i) + \frac{1}{2}\sum_{a=1}^{N}\big((\beta_a\partial\gamma_a) - (\gamma_a\partial\beta_a)\big)
\end{aligned}
\tag{E.6}
$$

Here, the dual Coxeter number is given by $g^\vee = M - 2N - 2$ and the value of the level is $k=1$. The central charge of the system is easily seen to take the value $c = \frac{M}{2} - N$.

Consider the $\beta\gamma$-system. The mode expansions are [80]

$$
\beta(z) = \sum_{n\in\mathbb{Z}} z^{-n-1-S/2}\beta_{n+(1+S)/2}, \quad \gamma(z) = \sum_{n\in\mathbb{Z}} z^{-n-1+S/2}\gamma_{n+(1-S)/2} \tag{E.7}
$$

and the associated commutators read

$$[\beta_r, \gamma_s] = \delta_{r+s,0}. \tag{E.8}$$

Following [80] we choose the ground state such that it obeys the highest-weight conditions

$$\beta_{n+(1+S)/2}|\varphi_S\rangle = \gamma_{n+(1-S)/2}|\varphi_S\rangle = 0, \qquad n \geq 0. \tag{E.9}$$

The $U(1)$-current of system is given by

$$J_{\beta\gamma}(z) = :\gamma\beta:. \tag{E.10}$$

With this we find that the $U(1)$-charge q of the system is given by

$$q(S) = -\frac{S}{2}. \tag{E.11}$$

For the complex fermion system we proceed analogously. The mode expansions are given by

$$\psi(z) = \sum_{n\in\mathbb{Z}} z^{-n-1+S/2}\psi_{n+(1-S)/2}, \qquad \psi^*(z) = \sum_{n\in\mathbb{Z}} z^{-n-1-S/2}\psi^*_{n+(1+S)/2}. \tag{E.12}$$

and the highest weight conditions are now

$$\psi^*_{n+(1+S)/2}|\varphi_S\rangle = \psi_{n+(1-S)/2}|\varphi_S\rangle = 0, \qquad n \geq 0. \tag{E.13}$$

These definitions are consistent with

$$h^S = \frac{S^2}{8} \tag{E.14}$$

The $U(1)$-current is now

$$J_\psi = :\psi\psi^*: \tag{E.15}$$

which implies that the $U(1)$-charge of the ground state is

$$q(S) = +\frac{S}{2}. \tag{E.16}$$

Explicitly, for $S = 2$ the $U(1)$-charge is given by

$$J_0^{\psi\psi^*} = \psi_{-1/2}\psi^*_{1/2} \tag{E.17}$$

and we find

$$J_0^{\psi\psi^*}|\varphi_2\rangle = |\varphi_2\rangle. \tag{E.18}$$

The difference in sign compared $U(1)$ of the bosonic system is due to the anti-commuting nature of the fermions.

Bibliography

[1] A. M. Polyakov, *"Interaction of Goldstone Particles in Two-Dimensions. Applications to Ferromagnets and Massive Yang-Mills Fields"*, Phys.Lett. B59, 79 (1975).

[2] E. Brezin and J. Zinn-Justin, *"Renormalization of the nonlinear σ-model in $2+\epsilon$ dimensions. Application to the Heisenberg ferromagnets"*, Phys.Rev.Lett. 36, 691 (1976).

[3] D. H. Friedan, *"Nonlinear Models in $2+\epsilon$ Dimensions"*, Annals Phys. 163, 318 (1985).

[4] C. Callan and L. Thorlacius, *"Sigma models and string theory"*, in: *"Particles, strings and supernovae"*.

[5] E. Fradkin, *"Field theories of condensed matter systems"*, Addison-Wesley Redwood City (1991).

[6] A. M. Tsvelik, *"Quantum field theory in condensed matter physics"*, Cambridge university press (2006).

[7] J. Evans, D. Kagan, N. MacKay and C. Young, *"Quantum, higher-spin, local charges in symmetric space sigma models"*, JHEP 0501, 020 (2005), hep-th/0408244.

[8] A. Babichenko, *"Conformal invariance and quantum integrability of sigma models on symmetric superspaces"*, Phys.Lett. B648, 254 (2007), hep-th/0611214.

[9] I. Bena, J. Polchinski and R. Roiban, *"Hidden symmetries of the AdS(5) x S**5 superstring"*, Phys.Rev. D69, 046002 (2004), hep-th/0305116.

[10] C. A. Young, *"Non-local charges, Z(m) gradings and coset space actions"*, Phys.Lett. B632, 559 (2006), hep-th/0503008.

[11] J. M. Maldacena, *"The Large N limit of superconformal field theories and supergravity"*, Int.J.Theor.Phys. 38, 1113 (1999), hep-th/9711200.

[12] R. Metsaev and A. A. Tseytlin, *"Type IIB superstring action in AdS(5) x S**5 background"*, Nucl.Phys. B533, 109 (1998), hep-th/9805028.

[13] R. Roiban and W. Siegel, *"Superstrings on AdS(5) x S**5 supertwistor space"*, JHEP 0011, 024 (2000), hep-th/0010104.

[14] G. Arutyunov and S. Frolov, *"Superstrings on AdS(4) x CP**3 as a Coset Sigma-model"*, JHEP 0809, 129 (2008), arxiv:0806.4940.

[15] j. Stefanski, B., *"Green-Schwarz action for Type IIA strings on AdS(4) x CP**3"*, Nucl.Phys. B808, 80 (2009), arxiv:0806.4948.

[16] N. Berkovits, *"Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background"*, Nucl.Phys. B431, 258 (1994), hep-th/9404162.

[17] N. Berkovits, *"A New description of the superstring"*, hep-th/9604123.

[18] N. Berkovits, C. Vafa and E. Witten, *"Conformal field theory of AdS background with Ramond-Ramond flux"*, JHEP 9903, 018 (1999), hep-th/9902098.

[19] N. Berkovits, *"Super Poincare covariant quantization of the superstring"*, JHEP 0004, 018 (2000), hep-th/0001035.

[20] O. A. Bedoya and N. Berkovits, *"GGI Lectures on the Pure Spinor Formalism of the Superstring"*, arxiv:0910.2254.

[21] E. Witten, *"Perturbative gauge theory as a string theory in twistor space"*, Commun.Math.Phys. 252, 189 (2004), hep-th/0312171.

[22] S. Sethi, *"Supermanifolds, rigid manifolds and mirror symmetry"*, Nucl.Phys. B430, 31 (1994), hep-th/9404186.

[23] A. S. Schwarz, *"Sigma models having supermanifolds as target spaces"*, Lett.Math.Phys. 38, 91 (1996), hep-th/9506070.

[24] M. Aganagic and C. Vafa, *"Mirror symmetry and supermanifolds"*, hep-th/0403192.

[25] D. Bernard, *"(Perturbed) conformal field theory applied to 2-D disordered systems: An Introduction"*, hep-th/9509137.

[26] K. Efetov, *"Supersymmetry and theory of disordered metals"*, Advances in Physics 32, 53 (1983).

[27] T. Quella and V. Schomerus, *"Superspace conformal field theory"*, J.Phys. A46, 494010 (2013), arxiv:1307.7724.

[28] L. Frappat, A. Sciarrino and P. Sorba, *"Dictionary on Lie algebras and superalgebras"*, Academic Press San Diego (2000).

[29] S. R. Coleman, *"The Quantum Sine-Gordon Equation as the Massive Thirring Model"*, Phys.Rev. D11, 2088 (1975).

[30] S. Mandelstam, *"Soliton Operators for the Quantized Sine-Gordon Equation"*, Phys.Rev. D11, 3026 (1975).

[31] E. Witten, *"Phases of N=2 theories in two-dimensions"*, Nucl.Phys. B403, 159 (1993), hep-th/9301042.

[32] A. Polyakov, *"Supermagnets and sigma models"*, hep-th/0512310.

[33] C. Candu and H. Saleur, *"A lattice approach to the conformal $OSp(2S+2|2S)$ supercoset sigma model. Part I: Algebraic structures in the spin chain. The Brauer algebra"*, Nucl.Phys. B808, 441 (2009), arxiv:0801.0430.

[34] C. Candu and H. Saleur, *"A Lattice approach to the conformal $OSp(2S+2|2S)$ supercoset sigma model. Part II: The Boundary spectrum"*, Nucl.Phys. B808, 487 (2009), arxiv:0801.0444.

[35] V. Mitev, T. Quella and V. Schomerus, *"Principal Chiral Model on Superspheres"*, JHEP 0811, 086 (2008), arxiv:0809.1046.

[36] C. Candu and V. Schomerus, *"Exactly marginal parafermions"*, Phys.Rev. D84, 051704 (2011), arxiv:1104.5028.

[37] S. Ryu, C. Mudry, A. Ludwig and A. Furusaki, *"Global phase diagram of two-dimensional Dirac fermions in random potentials"*, Phys.Rev. B85, 235115 (2012), arxiv:1111.3249.

[38] C. Candu, V. Mitev and V. Schomerus, *"Spectra of Coset Sigma Models"*, Nucl.Phys. B877, 900 (2013), arxiv:1308.5981.

[39] C. Candu, V. Mitev and V. Schomerus, *"Anomalous Dimensions in Deformed WZW Models on Supergroups"*, JHEP 1303, 003 (2013), arxiv:1211.2238.

[40] V. Kravtsov, I. Lerner and V. Yudson, *"Anomalous Dimensions of High Gradient Operators in the Extended Nonlinear σ Model and Distribution of Mesoscopic Fluctuations"*, Phys.Lett. A134, 245 (1989).

[41] S. Ryu, C. Mudry, A. W. Ludwig and A. Furusaki, *"High-gradient operators in perturbed Wess-Zumino-Witten field theories in two dimensions"*, Nucl.Phys. B839, 341 (2010), `arxiv:1002.0118`.

[42] G. E. Castilla and S. Chakravarty, *"Is the phase transition in the Heisenberg model described by the (2+epsilon) expansion of the nonlinear sigma model?"*, Nucl.Phys. B485, 613 (1997), `cond-mat/9605088`.

[43] B. L. Al'Tshuler, V. E. Kravtsov and I. V. Lerner, *"Statistical properties of mesoscopic fluctuations and similarity theory"*, Soviet Journal of Experimental and Theoretical Physics Letters 43, 441 (1986).

[44] B. Al'tshuler, V. Kravtsov and I. Lerner, *"Statistics of mesoscopic fluctuations and instability of one-parameter scaling"*, Zh. Eksp. Teor. Fiz 91, 2276 (1986).

[45] B. Altshuler, V. Kravtsov and I. Lerner, *"Applicability of scaling description to the distribution of mesoscopic fluctuations"*, Physics Letters A 134, 488 (1989), `http://www.sciencedirect.com/science/article/pii/0375960189906919`.

[46] B. ALTSHULER, V. KRAVTSOV and I. LERNER, *"Chapter 10 - Distribution of Mesoscopic Fluctuations and Relaxation Processes in Disordered Conductors"*, in: *"Mesoscopic Phenomena in Solids"*, ed.: P. L. B.L. ALTSHULER and R. WEBB, Elsevier (1991), 449 - 521p, `http://www.sciencedirect.com/science/article/pii/B9780444884541500164`.

[47] A. Cagnazzo, V. Schomerus and V. Tlapak, *"On the Spectrum of Superspheres"*, `arxiv:1408.6838`.

[48] A. Cagnazzo, V. Schomerus and V. Tlapák, *"High-Gradient Operators in the psl(2|2) Gross-Neveu Model"*, `arxiv:1410.4560`.

[49] V. Mitev, *"Sigma models on supercosets (PhD Thesis)"*.

[50] C. Candu, *"Discrétisation des modèles sigma invariants conformes sur des supersphères et superespaces projectifs"*, `http://tel.archives-ouvertes.fr/tel-00494973`.

[51] F. Wegner, *"Anomalous Dimensions for the Nonlinear σ Model in (2+ϵ)-dimensions. 2"*, Nucl.Phys. B280, 210 (1987).

[52] F. Wegner, *"Anomalous Dimensions for the Nonlinear σ Model in (2+ϵ)-dimensions. 1"*, Nucl.Phys. B280, 193 (1987).

[53] F. Wegner, *"Anomalous dimensions of high-gradient operators in the n-vector model in $2+\epsilon$ dimensions"*, Zeitschrift für Physik B Condensed Matter 78, 33 (1990), http://dx.doi.org/10.1007/BF01317354.

[54] F. Wegner, *"Anomalous dimensions of high gradient operators in the unitary matrix model"*, Nucl.Phys. B354, 441 (1991).

[55] C. Candu, T. Creutzig, V. Mitev and V. Schomerus, *"Cohomological Reduction of Sigma Models"*, JHEP 1005, 047 (2010), arxiv:1001.1344.

[56] M. Bershadsky, S. Zhukov and A. Vaintrob, *"PSL($n|n$) sigma model as a conformal field theory"*, Nucl.Phys. B559, 205 (1999), hep-th/9902180.

[57] D. Kagan and C. A. Young, *"Conformal sigma-models on supercoset targets"*, Nucl.Phys. B745, 109 (2006), hep-th/0512250.

[58] R. Bott, *"Homogeneous vector bundles"*, Annals of Mathematics 66, 203 (1957).

[59] J. Germoni, *"Indecomposable representations of $osp(3|2)$, $D(2,1;\alpha)$ and $G(3)$"*, Bol. Acad. Nac. Cienc.(Córd 65, 147 (2000).

[60] K. Pilch and A. Schellekens, *"Formulae for the Eigenvalues of the Laplacian on Tensor Harmonics on Symmetric Coset Spaces"*, J.Math.Phys. 25, 3455 (1984).

[61] H. Mall and F. Wegner, *"Anomalous dimensions of high gradient operators in the orthogonal matrix model"*, Nucl.Phys. B393, 495 (1993).

[62] A. Vasiliev and A. Stepanenko, *"A Method of calculating the critical dimensions of composite operators in the massless nonlinear sigma model"*, Theor.Math.Phys. 94, 471 (1993).

[63] K. Lang and W. Ruhl, *"Critical nonlinear O(N) sigma models at $2 < d < 4$: The Degeneracy of quasiprimary fields and it resolution"*, Z.Phys. C61, 495 (1994).

[64] S. E. Derkachov, S. Kehrein and A. Manashov, *"High-gradient operators in the N-vector model"*, Nucl.Phys. B493, 660 (1997).

[65] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *"Superstring theory on AdS(2) $\times$ S(2) as a coset supermanifold"*, Nucl. Phys. B567, 61 (2000), hep-th/9907200.

[66] E. Witten, *"Nonabelian Bosonization in Two-Dimensions"*, Commun.Math.Phys. 92, 455 (1984).

[67] N. Read and H. Saleur, *"Exact spectra of conformal supersymmetric nonlinear sigma models in two-dimensions"*, Nucl.Phys. B613, 409 (2001), hep-th/0106124.

[68] J. Van Der Jeugt, *"Irreducible Representations Of The Exceptional Lie Superalgebras D(2,1;α)"*, J.Math.Phys. 26, 913 (1985).

[69] J. Van der Jeugt, *"Finite and infinite-dimensional representations of the orthosymplectic superalgebra OSP(3,2)"*, J.Math.Phys. 25, 3334 (1984).

[70] *"The statement has been communicated in the past by H. Saleur and by A. Ludwig. One argument has been written down in [41]. A different reasoning can be found in [36]"*.

[71] G. Gotz, T. Quella and V. Schomerus, *"Tensor products of psl(2|2) representations"*, hep-th/0506072.

[72] T. Quella and V. Schomerus, *"Free fermion resolution of supergroup WZNW models"*, JHEP 0709, 085 (2007), arxiv:0706.0744.

[73] V. Schomerus and H. Saleur, *"The GL(1|1) WZW model: From supergeometry to logarithmic CFT"*, Nucl.Phys. B734, 221 (2006), hep-th/0510032.

[74] V. Fateev and A. Zamolodchikov, *"Integrable perturbations of Z(N) parafermion models and O(3) sigma model"*, Phys.Lett. B271, 91 (1991).

[75] P. Fendley, *"Integrable sigma models and perturbed coset models"*, JHEP 0105, 050 (2001), hep-th/0101034.

[76] L. J. Dixon, *"INTRODUCTION TO CONFORMAL FIELD THEORY AND STRING THEORY"*.

[77] Y. Kazama and H. Suzuki, *"New N=2 Superconformal Field Theories and Superstring Compactification"*, Nucl.Phys. B321, 232 (1989).

[78] T. Creutzig, P. B. Ronne and V. Schomerus, *"N=2 Superconformal Symmetry in Super Coset Models"*, Phys.Rev. D80, 066010 (2009), arxiv:0907.3902.

[79] C. Candu, V. Mitev, T. Quella, H. Saleur and V. Schomerus, *"The Sigma Model on Complex Projective Superspaces"*, JHEP 1002, 015 (2010), arxiv:0908.0878.

[80] F. Lesage, P. Mathieu, J. Rasmussen and H. Saleur, *"The su (hat)(2)(-1/2) WZW model and the beta gamma system"*, Nucl.Phys. B647, 363 (2002), hep-th/0207201.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit den Spektren von konformen Sigma Modellen, die auf (verallgemeinerten) symmetrischen Räumen definiert sind. Die Räume, auf denen Sigma Modelle ohne Wess-Zumino-Term konform sind, sind Supermannigfaltigkeiten, also Mannigfaltigkeiten, die fermionische Richtungen aufweisen. Wir stellen die Konstruktion von Vertex Operatoren vor, gefolgt von der Hintergrundfeld-Entwicklung. Für semisymmetrische Räume berechnen wir anschließend die diagonalen Terme der anomalen Dimensionen dieser Operatoren in führender Ordnung. Das Resultat stimmt mit dem für symmetrische Räume überein, jedoch treten nichtdiagonale Terme auf, die hier nicht weiter betrachtet werden.

Anschließend präsentieren wir eine detaillierte Analyse des Spectrums des Supersphären $S^{3|2}$ Sigma Modells. Dies ist eins der einfachsten Beispiele für konforme Sigma Modelle auf symmetrischen Räumen und dient als Illustration für die Mächtigkeit der vorgestellten Methoden. Wir verwenden die erhaltenen Daten, um eine Dualität mit dem OSP$(4|2)$ Gross-Neveu Modell zu untersuchen, die von Candu und Saleur vorgeschlagen wurde. Wir verwenden dazu ein Resultat, welches die anomalen Dimensionen von $\frac{1}{2}$BPS Operatoren zu allen Ordnungen berechnet. Wir finden das gesamte Grundzustandsspektrum des Sigma Modells. Darüber hinaus legen wir dar, dass sowohl die Zwangsbedingungen als auch die Bewegungsgleichungen des Sigma Modells korrekt vom Gross-Neveu Modell implementiert werden. Die Dualität wird weiterhin durch ein neues exaktes Resultat für die anomalen Dimensionen der Grundzustände unterstützt. Andererseits beobachten wir für Operatoren mit mehreren Ableitungen Diskrepanzen. Es ist möglich, dass diese Diskrepanzen im Zusammenang mit einer bekannten Instabilität von Sigma Modellen stehen.

Die Instabilität von Sigma Modellen wird von Operatoren mit vielen Ableitungen verursacht, die bei beliebig kleiner Kopplung relevant werden. Diese Eigenschaft wurde bereits vor langer Zeit, zuerst im O(N)-Vektor-Modell, beobachtet. Gross-Neveu Modelle besitzen generisch eine ähnliche Instabilität. Ryu et al. haben beobachtet, dass solche Operatoren in $\mathfrak{psl}(N|N)$ Gross-Neveu Modellen möglicherweise nicht vorhanden sind. Die Beobachtung wurde für eine bestimmte Klasse von Operatoren in führender Ordnung bestätigt. Wir zeigen analytisch, dass im $\mathfrak{psl}(2|2)$ Modell in der Tat alle invarianten Operatoren irrelevant bleiben. Darüber hinaus bestimmen wir das Spektrum des BPS-Sektors für unendliche Kopplung. Wir finden keinen Hinweis auf einen Dualität mit dem $\mathbb{C}\mathrm{P}^{1|2}$ Sigma Modell. Wir schließen mit einer Diskussion von marginalen Deformation von Kazama-Suzuki-Modellen.

Abstract

In this thesis the spectra of conformal sigma models defined on (generalized) symmetric spaces are analysed. The spaces where sigma models are conformal without the addition of a Wess-Zumino term are supermanifolds, in other words spaces that include fermionic directions. After a brief review of the general construction of vertex operators and the background field expansion, we compute the diagonal terms of the one-loop anomalous dimensions of sigma models on semi-symmetric spaces. We find that the results are formally identical to the symmetric case. However, unlike for sigma models on symmetric spaces, off diagonal terms that lead to operator mixing are also present. These are not computed here.

We then present a detailed analysis of the one-loop spectrum of the supersphere $S^{3|2}$ sigma model as one of the simplest examples. The analysis illustrates the power and simplicity of the construction. We use this data to revisit a duality with the OSP(4|2) Gross-Neveu model that was proposed by Candu and Saleur. With the help of a recent all-loop result for the anomalous dimension of $\frac{1}{2}$BPS operators of Gross-Neveu models, we are able to recover the entire zero-mode spectrum of the supersphere model. We also argue that the sigma model constraints and its equations of motion are implemented correctly in the Gross-Neveu model, including the one-loop data. The duality is further supported by a new all-loop result for the anomalous dimension of the ground states of the sigma model. However, higher-gradient operators cannot be completely recovered. It is possible that this discrepancy is related to a known instability of the sigma model.

The instability of sigma models is due to symmetry preserving high-gradient operators that become relevant at arbitrarily small values of the coupling. This feature has been observed long ago in one-loop calculations of the O(N)-vector model and soon been realized to be a generic property of sigma models that persists to higher loop orders. A similar instability has been observed for Gross-Neveu models which can be seen as a certain deformation of WZNW models at level $k = 1$. Recently, Ryu et al. suggested that the $\mathfrak{psl}(N|N)$ Gross-Neveu models might be free of relevant high-gradient operators. They tested this proposal at one-loop level for a certain class of invariant operators. We extend the result to all invariant operators and all loops for the $\mathfrak{psl}(2|2)$ Gross-Neveu model. Additionally, we determine the spectrum of the BPS sector at infinite coupling and observe that all scaling weights become half-integer. Evidence for a proposed duality with the $\mathbb{C}\mathrm{P}^{1|2}$ sigma model is not found. We conclude with a brief discussion of marginal deformations of Kazama-Suzuki models.

This thesis is based on the following publications:

- A. Cagnazzo, V. Schomerus and V. Tlapák, *On the Spectrum of Superspheres*, arXiv:1408.6838, submitted to JHEP
- A. Cagnazzo, V. Schomerus and V. Tlapák, *High-Gradient Operators in the psl(2|2) Gross-Neveu Model*, arXiv:1410.4560

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 09. Dezember 2014