Local Scattering Operators for $P(\varphi)_2$ Models and the Time-dependent Schrödinger Equation

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Abstract

We establish the existence of Bogoliubov's local scattering operators for $P(\varphi)_2$ models of constructive quantum field theory in a non-perturbative way. To this end, we use the technique of evolution semigroups to prove a new result on wellposedness of the Cauchy problem for the time-dependent Schrödinger equation under very general assumptions.

Zusammenfassung

Wir beweisen die Existenz von lokalen Streuoperatoren im Sinn von Bogoliubov für $P(\varphi)_2$-Modelle der konstruktiven Quantenfeldtheorie ohne Anwendung störungstheoretischer Methoden. Zu diesem Zweck beweisen wir einen neuen Satz über Existenz und Eindeutigkeit der Lösung des Cauchyproblems der zeitabhängigen Schrödingergleichung unter sehr allgemeinen Voraussetzungen und verwenden hierfür die Technik der Evolutionshalbgruppen.

Contents

Introduction

Perturbative quantum field theory is a successful framework for the description of high energy physics. In quantum electrodynamics, perturbative calculations lead to amazing agreement with the experiments. For weak interactions the agreement is still impressive. Physical quantities are expanded in formal power series in 'physical' parameters like the coupling constant. The terms of these expansions are in general ill-defined, but by renormalization it is possible to obtain results which are verified by the experiments to great accuracy.

However, on a conceptional level, it is not clear what an interacting quantum field theory really is. The program of *constructive quantum field theory* was started to clarify this point. Beginning in the late 1960s, physicists were looking for models which satisfy a set of mathematically clear-cut axioms describing what a quantum field theory could be [94]. Despite huge efforts and a lot of interesting results for lower spacetime dimensions and simplified interactions, the question whether a physically relevant quantum field theory in four dimensions exists is still open. In the attempts to construct a $SU(2)$ Yang-Mills theory, the high energy behavior can be controlled by advanced methods of nonperturbative renormalization [58], but it seems that the low energy behavior is much harder to handle. The *adiabatic limit* is out of reach and thus also crucial quantities like Schwinger functions which rely on global concepts.

Another irritating problem in quantum field theory is the failure to include gravitational fields. Classical general relativity does not lead to a renormalizable theory if 'quantized' analogously to the other fundamental forces. This might give the impression that quantum field theory with its nonlocal concepts inherited from quantum theory is not the right kind of theory to describe all fundamental forces. In fact, one could conjecture that quantum field theory is an effective theory and other approaches as for instance string theory and its generalizations are better suited as basic theories.

But it might well be that a formulation of quantum field theory which incorporates locality in a fundamental way can overcome some of the aforementioned problems. Quantum field theory on curved spacetimes turned out to be fruitful as a testing ground. Because of the weakness of gravitation compared to the electroweak and the strong interaction, it is a reasonable approximation to incorporate relativistic effects as a fixed background and to neglect back reactions.

R. Brunetti, K. Fredenhagen and R. Verch have given a definition of quantum fields in an intrinsically local and covariant way [9]. A rigorous formulation of perturbation theory fits into this framework. It is independent of global concepts and allows for an extension of perturbative renormalization to curved spacetimes [7]. A central tool are the so-called local scattering operators. These operators arise in the Stückelberg-Bogoliubov-Epstein-Glaser renormalization theory [6, 75]. In the context of the generally covariant approach, they can be considered as generalized quantum fields and connect different quantum field theories. They allow for a construction of the algebras of local observables for an interacting theory from the free fields in the sense of formal power series. Here, only local concepts come into play. Therefore, it would be interesting to investigate also models of constructive quantum field theory in the generally covariant context. The strategy is to construct the local scattering operators in a nonperturbative way. This is the aim of the present work. The local scattering operators are defined by the time evolution in the interaction picture for interactions localized in compact spacetime regions and for a fixed frame of reference. Because of the localization in spacetime, the Hamiltonian becomes time-dependent and the time evolution is the solution of the associated time-dependent Schrödinger equation. Even for the simplest constructive models, where the localized Hamiltonian can be defined in the Fock space of the free fields, the existent theory on the well-posedness of the Cauchy problem of the time-dependent Schrödinger equation is not sufficient for our aims [95]. Therefore we develop a new result on well-posedness of the time-dependent Cauchy problem for evolution equations of the Schrödinger type. We obtain solutions which are sufficiently regular to show the existence of the local scattering operators for $P(\varphi)_2$ models. To our knowledge, this is the first complete proof of existence of these objects in a quantum field theory with nonlinear field equation. A different approach to a special case of our result is due to W. Wreszinski and collaborators [96]. By our nonperturbative existence result for local scattering operators, we relate a model of constructive quantum field theory to the generally covariant formulation of quantum field theory. From the rigorous local scattering operators one gets the algebras of local observables as C^*-algebras and the interacting theory in a local way. The infrared and the ultraviolet problem are disentangled. Hence, as an example, it is possible to obtain the local net for a theory of free massless bosons in two dimensions, an infrared divergent theory without vacuum state.

The methods of constructive quantum field theory have found numerous applications outside of quantum field theory [73]. We think that our new solvability theorem for the time-dependent Schrödinger equation will also be applicable to and useful in other branches of quantum theory.

The plan of this thesis is as follows:

In Chapter 1 we give a survey of the axiomatic and constructive approaches to quantum field theory. We indicate the rôle of local scattering operators $S(g)$ in perturbation theory. They are used to obtain the algebras of local observables.

Because this construction turns out to be independent of perturbation theory, it opens up the possibility to construct the local net via local scattering operators in a constructive setting, that is without using perturbation theory. For this reason, we propose a nonperturbative definition of the local scattering operators. They are defined as being identical to the time evolution operator in the interaction picture for a localized interaction, where the time evolution is evaluated over an interval containing the time support of the interaction. To this end we need a nonperturbative solution of the time-dependent Schrödinger equation.

The topic of Chapter 2 is the Cauchy problem for hyperbolic nonautonomous evolution equations. The time-dependent Schrödinger equation is a special case of this type of equations. We investigate the properties of the autonomous and nonautonomous Cauchy problem and define various notions of solvability. Weak solutions exist under very general assumptions, but are not sufficiently regular to be interpreted in terms of an evolution operator or *propagator*. We present the theory of wellposedness established by T. Kato and some of its implications. Due to technical difficulties, Kato's theory is not applicable to the localized Schrödinger equation arising in simple models of constructive quantum field theory.

In Chapter 3 we relate the nonautonomous Cauchy problem to an autonomous one by a technique due to J. Howland. We give a survey and clarify the notions of solvability which arise naturally in this approach: mild solutions are associated with the closure of a special operator G_0 generating an evolution semigroup. A new uniqueness and continuity result for weak solutions follows. However, these techniques are not sufficient for the application we have in mind. We generalize the setting by considering the situation where the closure of G_0 is not necessarily a generator. Choosing a certain extension of G_0 with the generation property, we arrive at the main result in this chapter (Theorem 3.21). This is new result on the wellposedness of the time-dependent Schrödinger equation, where we use approximative solutions. The main results of this Chapter are obtained in collaboration with R. Schnaubelt and can be found in [77].

Chapter 4 is devoted to the existence of the local scattering operators for $P(\varphi)_2$ models. First, we demonstrate that the techniques of Chapter 2 are not sufficient for our purpose. But with the existence theorem for approximative solutions, we formulate sufficient conditions for the existence of local scattering operators in Theorem 4.3. In particular, we derive an existence result for the local scattering operators for $P(\varphi)_2$ models, including cases of non-semibounded polynomials $P(\lambda)$. This shows the disentanglement of the ultraviolet and the infrared problem which is the main advantage of this approach. As a special case, we get a simple construction of the algebra of local observables for the massless scalar field in two dimensions, a theory without ground state. Finally, we indicate shortly what in our opinion could be interesting topics for further investigations.

The Appendix provides some basic facts about $P(\varphi)_2$ models, that is scalar, massive, polynomial self-interacting quantum field theories in two dimensions. Moreover, we present an auxiliary result about the sum of maximally accretive operators which we will find useful.

For our approach, we need a considerable amount of semigroup theory. For an introduction to this topic we refer to Pazy's monograph [67]. Another recommendable reference is the book of Engel and Nagel [22]. The latter book contains also an introduction to the theory of evolution semigroups. For the theory of vector-valued integration see [100] or [17].

Notation

By X and Y we denote Banach spaces, $\mathcal{B}(X,Y)$ is the space of bounded linear operators from X to Y, and $\mathcal{B}(X) := \mathcal{B}(X,X)$. In the chapters dealing mainly with quantum field theory, we work in the setting of a Hilbert space $\mathcal{H}$.

For intervals $I \subset \mathbb{R}$, we set $D_I = \{(t,s) \in I^2 : t \geq s\}$ and $I_s = [s,\infty) \cap I$ for $s \in I$. We consider various Banach spaces of X-valued functions: $L^p(I,X), 1 \leq p \leq \infty$, denotes the space of (equivalence classes of) strongly measurable functions $f : I \to X$ such that $\|f\|_p^p := \int_I \|f(t)\|^p \, dt < \infty$. The spaces $C(I,X)$ resp. $C^n(I,X)$, are the sets of continuous resp. n-times continuously differentiable functions endowed with the appropriate sup–norms. Frequently we use the abbreviation $E_p := L^p(I,X)$, sometimes we even omit the subscript p if the meaning is clear from the context. By $W^{n,p}(I,X)$ we denote the Sobolev space of vector-valued functions whose nth derivative is a function in $L^p(I,X)$. A subscript '0' (e.g. $C_0(I,X)$) indicates that the functions of the corresponding class vanish at infinity (if I is unbounded) and at finite end points of I which are not contained in I. A subscript 'c' (e.g. $C_c(I,X)$) denotes a set of functions with compact support in I . The set $C_b(\mathbb{R},X)$ denotes the continuous functions which are bounded in norm. By $\mathcal{S}$ we denote the Schwartz space of rapidly decreasing, smooth functions. The translation operators are denoted by $\tau_\sigma : E_p \to E_p$,

$$\tau_\sigma f(t) = \begin{cases} f(t-\sigma) & \text{if } t-\sigma \in I \\ 0 & \text{if } t-\sigma \notin I. \end{cases}$$

The domain $D(A)$ of a closed operator A on X is always endowed with the graph norm of A. If the meaning is clear from the context, we omit the identity operator $\mathbb{1}$. For example, we write the sum of an operator A with a multiple $\lambda\mathbb{1}$ of the identity operator as $A + \lambda$. For a normal operator A we denote by $\rho(A) \subset \mathbb{C}$ the resolvent set and by $\sigma(A) \subset \mathbb{C}$ the spectrum. For $\lambda \in \rho(A)$, the resolvent of A is $R(\lambda, A) := (\lambda - A)^{-1}$. By $C^\infty(A)$ we denote the intersection $\bigcap_{n\in\mathbb{N}} D(A^n)$.

The Poincaré group is denoted by $\mathcal{P}$ and the open forward and backward lightcone in Minkowski space by V_+ resp. V_-.

Frequently, we designate a generic constant by the letter c.

Chapter 1

Local Scattering Operators

Since the early days of quantum field theory, formal perturbation theory has proved to be a reliable guide to high energy physics. In the last years, it was possible to formulate perturbation theory in a way which is suitable to an understanding in the context of axiomatic quantum field theory. The effort leading to this considerable improvement was motivated by an investigation of quantum field theory on curved spacetimes [7, 46]. In this context, methods relying on global symmetries are not applicable. Hence it is crucial to emphasize locality. The appropriate framework of this approach is algebraic quantum field theory. In particular, it is possible to define the local net of an interacting theory without having to address the adiabatic limit first. Thus the ultraviolet and infrared behavior of a theory can be studied independently of each other. A crucial observation for our work is that this mechanism, leading to the disentanglement of high and low energy properties, is in fact independent of perturbation theory.

The central objects in these developments are the local scattering operators $S(g)$ which arise in the Stückelberg-Bogoliubov-Epstein-Glaser formulation of perturbation theory. In this context, they are the generating functionals of the time-ordered products. But it is possible to characterize the local scattering operators directly as solutions of a time-dependent Schrödinger equation with an interaction localized in a compact region of spacetime.

One may take a more abstract point of view. In the formulation of locally covariant quantum field theories as covariant functors [9, 8], the local scattering operators can be interpreted as generalized quantum fields and arise as natural transformations in the sense of category theory.

In this chapter, we will shortly summarize the axiomatic approach to quantum field theory. We will define the local scattering operators and describe their relation to the algebras of local observables.

1.1 Quantum field theory

As mentioned in the introduction, formal perturbative quantum field theory was successfully applied to high energy physics. But there remains a logical puzzle: Is quantum field theory the appropriate language for the description of nature and a mathematically consistent theory at the same time?

To put the discussion of this question on solid grounds, a set of axioms is a suitable starting point to clarify the assumptions and to test their consistency. In the approach of Gårding and Wightman [30], fields are distributions which take values in the set of operators on a Hilbert space. An alternative formulation emphasizes the algebraic structure of bounded operators representing observables which are measurable in fixed spacetime regions. This formulation is due to Haag and Kastler [43]. The two sets of axioms are not equivalent, there are theories which fulfill the Haag-Kastler axioms but not the Wightman axioms. A generally covariant formulation of algebraic quantum theory was recently proposed by R. Brunetti, K. Fredenhagen and R. Verch [9].

Having an axiomatic definition of quantum field theory at hand, one might search for examples fulfilling the axioms and leading to a nontrivial scattering matrix. Because of huge technical and conceptional difficulties, it was not possible to investigate theories which are expected to correctly describe physics with interaction. The idea of constructive quantum field theory is to start with simplified models which are suitable for the development of skills necessary to address the more difficult ones. Despite the enormous effort which was put into this program, up to the time of writing it did not achieve its aim, the rigorous construction of a Yang-Mills theory with the correct gauge group in four-dimensional spacetime. Nevertheless, it was possible to construct interacting quantum field theories fulfilling the Haag-Kastler or the Wightman axioms and to gain considerable physical insights, not only concerning the existence question but also for scattering theory, particle interpretation, phase space analysis. The methods developed in constructive quantum field theory and nonperturbative renormalization found further applications in other fields, for example in statistical or solid state physics or the analysis of partial differential equations [73].

1.1.1 Axiomatic quantum field theory

The Wightman axioms state conditions for a quantum field theory which are close in spirit to the traditional Hilbert space formulation of quantum physics. They incorporate the requirements of special relativity by a unitary representation of the Poincaré group. On the other hand, the Haag-Kastler axioms start with algebras of local observables. The Poincaré group acts via automorphisms of these local algebras, hence in this approach it is possible to discuss the implementability of a Hilbert space representation afterwards.

1.1.1.1 Wightman axioms

The Wightman axioms set the following framework for a quantum theory of fields:

Hilbert space The pure states are rays in a Hilbert space $\mathcal{H}$ with scalar product $(\cdot\,,\,\cdot)$ which carries a unitary representation of the covering group $\overline{\mathcal{P}}$ of the Poincaré group $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^4$, where $\mathcal{L}$ denotes the proper, orthochronous Lorentz group. There is exactly one vacuum state, that is a Poincaré-invariant ray with $U(a, \Lambda(\alpha))\Omega = \Omega$ where $a \in \mathbb{R}^4, \alpha \in SL(2,\mathbb{C}) = \overline{\mathcal{L}}$. The translations $U(a, \mathrm{id}) = e^{iP_\mu a^\mu}$ are generated by the self-adjoint energy-momentum operators P^μ. Their spectrum is a subset of the closed forward lightcone $\overline{V}_+ = \{p \in \mathbb{R}^4 : p^2 \geq 0,\ p^0 \geq 0\}$. This is the *spectrum condition*.

Fields For every Schwartz function $f \in \mathcal{S}(\mathbb{R}^4)$ the field $\varphi(f) = \int \varphi(x)f(x)d^4x$ is an unbounded operator, defined on a dense set $\mathcal{D} \subset \mathcal{H}$ common to all $\varphi(f)$ and invariant under their application. We say that the fields are operator-valued distributions. The domain $\mathcal{D}$ contains the vacuum Ω and is invariant under application of $U(a, \Lambda(\alpha))$ for all $a \in \mathbb{R}^4, \alpha \in SL(2,\mathbb{C})$. In general, there are several fields (type i) which may have several spinor or tensor components (index λ). Hence the general expression for the fields as operator-valued distribution are

$$\varphi(f) = \sum_{i,\lambda} \int \varphi^i_\lambda(x) f^{i,\lambda}(x)\, d^4x.$$

The set of fields contains with φ also its hermitean conjugate φ^*, defined as a sesquilinear form via $(\psi_1, \varphi(x)^*\psi_2) = \overline{(\psi_2, \varphi(x)\psi_1)}$, $\psi_1, \psi_2 \in \mathcal{H}$.

Transformation properties Let $\alpha \in SL(2,\mathbb{C})$ and $M^{(i)}(\alpha)$ be a finite-dimensional representation matrix of α. The fields transform under $\overline{\mathcal{P}}$ as

$$U(a,\alpha)\varphi^i_\lambda(x)U(a,\alpha)^{-1} = \sum_\rho M^{(i)\rho}_\lambda(\alpha^{-1})\varphi^i_\rho(\Lambda(\alpha)x + a)$$

in the sense of distributions.

Causality If the supports of f and g are spacelike separated, then the fields obey causal commutation relations: $[\varphi^i(f), \varphi^j(g)] = 0$ for bosonic fields or $[\varphi^i(f), \varphi^j(g)]_+ = 0$ with the anticommutator $[\cdot,\cdot]_+$ in the fermionic case.

Completeness Every operator on $\mathcal{H}$ can be approximated by linear combinations of products of the $\varphi(f)$.

Time-slice axiom There exists a dynamical law which allows for the computation of the fields at arbitrary times in terms of the fields in a small time slice $\mathcal{O}_{t,\epsilon} := \{x \in \mathbb{R}^4 : |x^0 - t| < \epsilon\}$.

One can formulate the axioms equally well in terms of the vacuum expectation values $w^{(n)}(x_1, \ldots, x_n) = (\Omega, \varphi(x_1) \ldots \varphi(x_n)\Omega)$, the so-called *Wightman functions.* Given a set of tempered distributions $\{w^{(n)}\}$, $n \in \mathbb{N}$, fulfilling these axioms, one can reconstruct the quantum fields and the Hilbert space $\mathcal{H}$. The *Schwinger functions* $S^{(n)}$ are the continuation of the Wightman functions to purely imaginary times. The spectrum condition ensures the analyticity of the $w^{(n)}$. It is also possible to reverse the argument: Starting from the Euclidean Schwinger functions, a Wightman quantum field theory on Minkowski space can be recovered if the Schwinger functions $\{S^{(n)}\}$, $n \in \mathbb{N}$, satisfy the *Osterwalder-Schrader* axioms, see for example [39].

1.1.1.2 Haag-Kastler axioms

To every finite, contractible open subset $\mathcal{O}$ of the Minkowski space one assigns the set $\mathcal{A}(\mathcal{O})$ of bounded observables which can be measured inside of $\mathcal{O}$. The algebras of local observables $\mathcal{A}(\mathcal{O})$ are often defined in such a way that they are C^*-algebras. The following axioms are imposed:

Isotony If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

Covariance There is a representation β of the Poincaré group $\mathcal{P}$ by automorphisms: $\{a, \Lambda\} \mapsto \beta_{\{a,\Lambda\}}$ such that if $A \in \mathcal{A}(\mathcal{O})$ then $\beta_{\{a,\Lambda\}}(A) \in \mathcal{A}(\Lambda\mathcal{O}+a)$.

Causality If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $\mathcal{A}(\mathcal{O}_1) \subset (\mathcal{A}(\mathcal{O}_2))'$, that is $[A_1, A_2] = 0$ for all $A_1 \in \mathcal{A}(\mathcal{O}_1),\ A_2 \in \mathcal{A}(\mathcal{O}_2)$.

Time-slice axiom The algebra belonging to a neighborhood of a Cauchy surface* of a region equals the algebra of the full region (existence of a hyperbolic equation of motion).

The *quasilocal algebra* $\mathcal{A}$ is the inductive limit $\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$, which can be defined if the regions $\{\mathcal{O}\}$ form a directed set. This is the case for open, relatively compact subsets of Minkowski space. If the algebras $\mathcal{A}(\mathcal{O})$ are C^*-algebras, we define the quasilocal algebra by closure of the inductive limit in norm. Without loss of generality we assume that $\mathcal{A}$ contains an identity $\mathbb{1}$.

A *state* ρ is a complex-linear functional on $\mathcal{A}$ which is positive and normalized, that is it fulfills $\rho(A^*A) \geq 0$ for all $A \in \mathcal{A}$ and $\rho(\mathbb{1}) = 1$.

A state is invariant under a group G, represented by automorphisms β on $\mathcal{A}$, if it satisfies

$$\rho(\beta_g(A)) = \rho(A).$$

for all $g \in G$.

For a given state one can get a representation of the quasilocal algebra on a Hilbert space. This is the *GNS construction*, see [42].

*A *Cauchy surface* is a subset of a region in spacetime, which is intersected exactly once by every inextendible curve, which has no spacelike tangent vectors.

Theorem 1.1. *Let $\mathcal{A}$ be a C^*-algebra and ρ a state on $\mathcal{A}$. Then there exist a Hilbert space $\mathcal{H}_\rho$, a vector Ω_ρ and a representation π_ρ of $\mathcal{A}$ by bounded operators on $\mathcal{H}_\rho$ such that*

$$\rho(A) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho) \quad \text{for all } A \in \mathcal{A}.$$

The vector Ω_ρ is cyclic for $\pi_\rho(\mathcal{A})$. If ρ is invariant under a group G then there exists a representation $U_\rho(g)$ of elements $g \in G$ by unitary operators on $\mathcal{H}_\rho$ such that

$$\pi_\rho(\beta_g(A)) = U_\rho(g)\pi_\rho(A)U_\rho(g)^{-1}.$$

The vector Ω_ρ is invariant under G: $U_\rho(g)\Omega_\rho = \Omega_\rho,\ g \in G$.

We see that the transition from the algebraic level to a Hilbert space representation depends on the choice of a state. This choice is in general not unique and one can get different inequivalent representations of the quasilocal algebra.

Often one also wants to impose a stability condition. Then one assumes that the local algebra is a concrete algebra of operators on a Hilbert space and the automorphisms belonging to the Poincaré group are implemented by unitary operators. The joint spectrum of the generators of the unitary representatives of the translations should then be a subset of the forward lightcone. This assumption corresponds to the spectrum condition of the Wightman axioms.

1.1.2 A generally covariant approach

Algebraic quantum theory emphasizes locality. But it is not suitable to incorporate the covariance property of general relativity. A recent approach of R. Brunetti, K. Fredenhagen and R. Verch [9] generalizes the setting of algebraic quantum theory in a generally covariant way, allowing for the definition of a quantum field theory on all spacetimes of a certain class. We follow the presentation in [8]. A general covariant quantum field theory is considered as a functor between two categories. The first one describes the local relations. Its objects are certain topological spaces and its morphisms are structure preserving embeddings. The second category provides the information about the algebraic structure of observables. The standard choice for quantum physics is the category of C^*-algebras where the morphisms are unital embeddings. In classical physics, one considers Poisson algebras instead of C^*-algebras. Recently, also perturbative quantum field theory was incorporated into this concept. Here one deals with algebras which possess nontrivial representations as formal power series of Hilbert space operators. The principle of algebraic quantum field theory states that the functor $\mathcal{A}$ contains all physical information. Now we will put these ideas in more exact terms. We consider the categories $\mathfrak{L}$ and $\mathfrak{O}$.

The category $\mathfrak{L}$ is defined in the following way: The class of objects, $\mathrm{obj}(\mathfrak{L})$, consists of all ($d \geq 2$)-dimensional, smooth, globally hyperbolic[†] Lorentzian[‡] spacetimes M which are oriented and time-oriented. For two members M_1, M_2 of $\mathrm{obj}(\mathfrak{L})$ the morphisms $\psi \in \mathrm{hom}_{\mathfrak{L}}(M_1, M_2)$ are chosen to be isometric embeddings $\psi : M_1 \to M_2$ which satisfy the following conditions:

(i) If $\gamma : [a, b] \to M_2$ is an arbitrary causal curve[§] and $\gamma(a), \gamma(b) \in \psi(M_1)$ then the whole curve lies in $\psi(M_1)$, i.e. $\gamma(t) \in \psi(M_1)\ \forall\ t \in (a, b)$.

(ii) Every morphism preserves orientation and time-orientation of the embedded spacetime.

Composition is defined to be the composition of maps and the unit element in $\mathrm{hom}_{\mathfrak{L}}(M, M)$ is the identical embedding id_M.

Now we define $\mathfrak{O}$: The class of objects, $\mathrm{obj}(\mathfrak{O})$, is given by the unital C^*-algebras $\mathcal{A}$. The morphisms in $\mathrm{hom}_{\mathfrak{O}}(\mathcal{A}, \mathcal{B})$ are the faithful, injective, unit-preserving $*$-homomorphisms with the composition of maps. The unit element in $\mathrm{hom}_{\mathfrak{O}}(\mathcal{A}, \mathcal{A})$ for every $\mathcal{A} \in \mathrm{obj}(\mathfrak{O})$ is the identical map $\mathrm{id}_{\mathcal{A}} : A \mapsto A,\ A \in \mathcal{A}$.

This choice of the categories $\mathfrak{L}$ and $\mathfrak{O}$ may be changed to fit to the physical situation. In particular, for perturbation theory one would substitute the C^*-algebras by general topological $*$-algebras.

Definition 1.2. A *locally covariant quantum field theory* is a covariant functor A from $\mathfrak{L}$ to $\mathfrak{O}$ which has the covariance properties (denoting $\mathsf{A}(\psi)$ by α_ψ)

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}, \quad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathsf{A}(M)}$$

for all morphisms $\psi \in \mathrm{hom}_{\mathfrak{L}}(M_1, M_2)$, all morphisms $\psi' \in \mathrm{hom}_{\mathfrak{L}}(M_2, M_3)$ and all $M \in \mathrm{obj}(\mathfrak{L})$.

Moreover, a locally covariant quantum field theory described by a covariant functor A is called *causal* if the following holds: Consider two morphisms $\psi_j \in \mathrm{hom}_{\mathfrak{L}}(M_j, M),\ j = 1, 2$, such that the sets $\psi_1(M_1)$ and $\psi_2(M_2)$ are not connected by a causal curve in M. Then

$$[\alpha_{\psi_1}(\mathsf{A}(M_1)), \alpha_{\psi_2}(\mathsf{A}(M_2))] = \{0\},$$

where the element-wise commutation makes sense in $\mathsf{A}(M)$.

We will see that perturbative quantum field theory fits into this context and allows for a formulation on curved spacetimes [7]. A crucial object are the local scattering operators which fit into the generally covariant context as natural transformations, as we will see in Section 1.2.

[†] A spacetime is *globally hyperbolic* if it has a smooth foliation in Cauchy surfaces.
[‡] A *Lorentzian* spacetime of dimension n has a Pseudo-Riemannian metric of signature (1,n-1)
[§] A *causal curve* has no spacelike tangent vectors.

1.1.3 Constructive quantum field theory

Axiomatic quantum field theory was developed to be a rigorous foundation for the understanding of the dynamics of elementary particles. But in the early 1960s only the free fields were known to fulfill the axioms, thus showing their consistency. But the main question, whether the idealizations involved in the axioms result in a language suitable for practical purposes of elementary particle physics, remained unanswered. Therefore, as a first step, simplified models were examined. In the following, we will shortly review the development of constructive quantum field theory, see [66, 87].

The rigorous construction of examples for interacting quantum field theories is fundamentally affected by a famous result known as *Haag's Theorem* [41]. Whereas in quantum mechanics every representation of the canonical commutation relations is unitarily equivalent to the Schrödinger representation, this is no longer the case in a quantum field theory dealing with a system of infinitely many degrees of freedom. The appearance of *strange representations* can be traced back to the work of K. O. Friedrichs [28] and L. van Hove [92]. This turned out to be a generic situation and has consequences for the proposal to construct interacting quantum field theories starting from free fields.

Theorem 1.3. *Let φ be a free field on a Hilbert space $\mathcal{H}$ with Hamiltonian H_0. Let the space translations be implemented by unitary operators $U(\vec{x}) = U((0,\vec{x}),\mathrm{id})$. Assume that there is an operator-valued distribution $\tilde{\varphi}$ which satisfies:*

(i) coincidence with the free field at $t = 0$: $\tilde{\varphi}(x)\restriction_{x^0=0} = \varphi(x)\restriction_{x^0=0}$ and $\partial_0\tilde{\varphi}(x)\restriction_{x^0=0} = \partial_0\varphi(x)\restriction_{x^0=0}$;

(ii) translation covariance: $U(\vec{y})\tilde{\varphi}(x^0,\vec{x})U(\vec{y})^{-1} = \tilde{\varphi}(x^0,\vec{x}-\vec{y})$;

(iii) existence of the Hamiltonian: There is a self-adjoint operator H on $\mathcal{H}$ such that $\tilde{\varphi}(t,\vec{x}) = e^{itH}\tilde{\varphi}(0,\vec{x})e^{-itH}$.

Then H and H_0 differ only by an additive constant and $\tilde{\varphi} = \varphi$.

Thus, if one wants to work in the usual Fock space and to avoid dealing with strange representations, it is convenient to break the translation symmetry. This is done by placing the system under consideration in a finitely extended box V or by replacing the coupling constant by a compactly supported smooth function g on spacetime.

But another cut-off turns out to be necessary. The models are inspired by simple interaction Lagrangians built from the free field. To obtain the Hamiltonian as a well defined operator one has to introduce a high-momentum cut-off κ by keeping only the frequencies $\leq \kappa$ in the Fourier transform of the free field.

In this way the cut-off Yukawa theory Y_4 with the Hamiltonian

$$H_{\kappa,V} = H_{0,B,V} + H_{0,F,V} + \lambda \int_V :\psi_\kappa^+(\vec{x})\psi_\kappa(\vec{x}): \varphi_\kappa(\vec{x})\, d^3x$$

was investigated by O. Lanford in [53]. Here ψ is a fermion field and φ is a boson field. By $H_{0,B,V}$ and $H_{0,F,V}$ we denote the free fermionic resp. bosonic Hamiltonians in a box V with periodic boundary conditions. The colons denote Wick ordering, a prescription for the proper multiplication of the operator-valued distributions. An example where the coupling is of higher degree in φ_κ than the free Hamiltonian is the cut-off $(\varphi^4)_4$ model studied by A. Jaffe [48]. The Hamiltonian is given by

$$H_\kappa = H_0 + \lambda \int g(\vec{x}) :\varphi_\kappa^4(\vec{x}):\ d^3x.$$

In both models self-adjointness and semiboundedness of the Hamiltonians have been established. Moreover, uniqueness of the vacuum was proved.

The next step in the construction of the quantum field theories would be the removal of the cut-offs $V \to \mathbb{R}^3$ resp. $g \to$ const. and $\kappa \to \infty$. The limiting theories should satisfy the Wightman or Haag-Kastler axioms. But passing to this limit was impossible without a further significant simplification: The number of spacetime dimensions d had to be reduced to $d = 2$ and later $d = 3$. This is mainly related to the high-energy behavior of the theories which affects the $\kappa \to \infty$ limit. One has to add κ-depending terms to the Hamiltonian which diverge in the limit, renormalization is necessary.

Denote by H_{ren} the renormalized Hamiltonian and indicate the number of spacetime dimensions by a subscript. From formal perturbation theory the following behavior was predicted and confirmed by rigorous calculation [35]. In the $(\varphi^{2n})_2$ model on two-dimensional spacetime, one finds $D(H_{\mathrm{ren}}) \subset D(H_0)$. In this case, Wick ordering is sufficient to renormalize the Hamiltonian. Apart from that, only a finite constant has to be added which corresponds to a finite shift of the vacuum energy. For the mass shift model $(\varphi^2)_3$ the form domain of the renormalized Hamiltonian is still contained in the form domain of the free Hamiltonian $D(H_{\mathrm{ren}}^{1/2}) \subset D(H_0^{1/2})$, but for H_0 and H_{ren} themselves the inclusion of the domains is no longer true. For the models $(\varphi^2)_4$ and Y_2 even the form domain of H_{ren} is not contained in the form domain of the free Hamiltonian, only $D(H_{\mathrm{ren}}) \subset \mathcal{H}$ remains valid. The Yukawa model Y_2 needs infinite vacuum-energy and boson-mass renormalizations in the Hamiltonian. Even more singular are Y_3 and $(\varphi^4)_3$. These models on three-dimensional spacetime need an infinite wave-function renormalization: The domain of H_{ren} is no longer a subset of the Hilbert space $\mathcal{H}$ which is the Fock space of the free fields.

The models we mentioned up to now are superrenormalizable, that is, the counterterms are polynomials in the coupling constant and the degree of the divergences gets less severe in higher orders of perturbation theory. In this context, the *Hamiltonian strategy* led to some considerable insights. The idea is to describe an interacting theory by a construction of its dynamics in a Hilbert space. The easiest model where the Hamiltonian strategy is applicable is $(\varphi^4)_2$. Up to the middle of the 1970s it was known that the model exists without any cut-offs. It fulfills the Haag-Kastler axioms and most of the Wightman axioms [37]. These results were extended to the technically more difficult $P(\varphi)_2$ models, where $P(\lambda)$ is a semibounded polynomial of degree ≥ 4. Moreover, some features of Y_2 and $(\varphi^4)_3$ were accessible via the Hamiltonian strategy [34, 36].

Already in this work it turned out to be very useful to investigate the Hamiltonian H via its associated semigroup $(e^{-tH})_{t\geq 0}$. This can be regarded as a Euclidean method since it follows formally from the substitution $t \to -it$. But this was only the beginning of a powerful *Euclidean approach* [89] to constructive quantum field theory. This method is based on the fundamental correlations between boson quantum field theory and probability theory, the analyticity properties of the Wightman functions and the Schwinger functions as their Euclidean counterpart and, last but not least, the connection between Euclidean quantum field theory and classical statistical mechanics. Soon the study of Hamiltonians was abandoned in favor of the direct examination of the Schwinger functions via a Euclidean Gell-Mann-Low formula. The Schwinger functions are defined by functional integrals as moments of a certain probability measure on a function space. For their rigorous construction, one starts again from a regularized theory with cut-offs. For the removal of the cut-offs powerful renormalization methods were developed: *correlation inequalities* and the *cluster expansion*. By these methods, up to the beginning of the 1980s superrenormalizable models where under control. Examples are $P(\varphi)_2$, Y_2, the Sine-Gordon model $(\sin \epsilon\varphi)_2$ and the Hoegh-Krohn model $(e^{\alpha\varphi})_2$ in two dimensions. Further examples in three dimensions are $(\varphi^4)_3$ and Y_3. For these examples on two- and three-dimensional spacetime the existence of the Schwinger functions was proved, they define a quantum field theory with nontrivial scattering operator. One can analyze the particle spectrum and the equations of motion. Moreover, one can investigate phase transitions and symmetry breaking, and one finds Borel summability of a formal power series expansion. Thus the relation to perturbation theory is well understood.

In four dimensions there are new challenges. The counterterms are only known as formal power series in the coupling constant. An example for a renormalizable model which is no more superrenormalizable is $\lambda(\varphi^4)_4$. In this situation one needs a new technique, the *renormalization group*, which goes back to ideas of Wilson and Kadanoff. The integration over the function space is performed by a sequence of integrals with a fixed momentum scale. One can relate the counterterms to different momentum scales via the *flow equation*. This method works for models which are *asymptotically free*, the coupling decreases for high momenta. However, $\lambda(\varphi^4)_4$ is not asymptotically free for positive coupling constant λ. But for negative coupling it is, and a rigorous construction was given in [32]. But it is no physical model as it seems to be impossible to recover the quantum field theory on Minkowski space via the Osterwalder-Schrader reconstruction.

Renormalization group methods were successfully applied to the Gross-Neveu model in two dimensions, a model with quartic fermion interaction of several flavors [31]. Moreover, it was possible to investigate gauge theories in three and four dimensions, see [3, 4] and related papers by T. Balaban. In this program a Yang-Mills theory is investigated in a finite volume. With lattice regularization and block spin transformations, the high-energy limit of gauge-invariant observables as smoothed Wilson loops is tackled. But it seems that Balaban's ideas are not directly applicable to the Schwinger functions. These are the subject of the work of J. Magnen, V. Rivasseau and R. Sénéor [58] for an $SU(2)$ Yang-Mills theory in finite Euclidean volume. Here the ultraviolet problem seems to be under control. But for large volumes there are problems with the appearance of large fields, which at the moment seem insurmountable. Without control of the adiabatic limit, there is no possibility to define the interacting theory via its Schwinger functions.

Thus a local approach is expected to lead to an improvement of the understanding of interacting quantum field theory in the constructive context. Despite the fact that there are a lot of technical and even conceptional questions open, it would be interesting to develop a strategy to disentangle the infrared and the ultraviolet problem. This decoupling was achieved in the context of perturbation theory, but it is possible to carry over the main idea to constructive quantum field theory. The interacting theory is obtained via the local net [7], thus the theory is fixed without the need for a vacuum state or related global concepts. A crucial tool are the local scattering operators which we introduce in the next section.

1.2 Local scattering operators

As in constructive quantum field theory, in the Bogoliubov-Stückelberg-Epstein-Glaser formulation of perturbation theory problems with Haag's theorem are circumvented by replacing the coupling constant by a compactly supported, smooth function g. The time evolution for the localized Hamiltonian in the interaction picture leads to the *local scattering operators* $S(g)$. They are examples for a class of generalized quantum fields in the functorial sense and allow for a local formulation of perturbative quantum field theory. The proposal of the present work is to perform a similar strategy to obtain local scattering operators in a nonperturbative way for models of constructive quantum field theory. We will see that this is possible if we are able to find nonperturbative solutions of a time-dependent Schrödinger equation with localized interaction.

1.2.1 Definition of the local scattering operators

1.2.1.1 Local scattering operators and generalized quantum fields

For the interpretation of a physical theory it is crucial to compare measurements associated with different spacetime regions or actually with different spacetimes. This comparison can be done in terms of locally covariant quantum fields. To cover this kind of general situations, the locally covariant quantum fields are defined as natural transformations from the functor of quantum field theory to another functor on the category of spacetimes $\mathfrak{L}$. Here a standard choice is the functor D which associates to every spacetime M its space of compactly supported, smooth test functions $\mathcal{D}(M)$. The morphisms are the pushforwards $\mathcal{D}(\psi) = \psi_*$.

Definition 1.4. A *locally covariant quantum field* φ is a natural transformation between the functors D and A. That is for any object $M \in \mathfrak{L}$ there exists a morphism $\varphi_M : \mathcal{D}(M) \to \mathsf{A}(M)$ such that for any pair of objects M_1, M_2 and any morphism ψ between them the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(M_1) \arrow[r, "\varphi_{M_1}"] \arrow[d, "\psi_*"'] & \mathsf{A}(M_1) \arrow[d, "\alpha_\psi"] \\ \mathcal{D}(M_2) \arrow[r, "\varphi_{M_2}"'] & \mathsf{A}(M_2) \end{tikzcd} \tag{1.1}$$

A standard example for a quantum field according to this definition is the free Klein-Gordon field on all globally hyperbolic spacetimes and its Wick polynomials.

A more general locally covariant quantum field is the local scattering operator of Stückelberg-Bogoliubov-Epstein-Glaser. In contrast to the free Klein-Gordon field it is not linear. For $M \in \mathfrak{L}$ and $g \in \mathcal{D}(M)$, the local scattering operators are unitary operators $S_M(g)$ which fulfill the conditions

$$S_M(0) = \mathbb{1}, \tag{1.2}$$
$$S_M(f + h + g) = S_M(f + h)S_M^{-1}(h)S_M(h + g), \tag{1.3}$$

where in the latter *causality condition* the supports of g and f are separated by a Cauchy surface of M and the support of f lies in the future of this Cauchy surface. There is no restriction concerning the support of h.

Using the local scattering operators, it is possible to define a new quantum field theory. This approach leads to the axiomatic perturbation theory [7] where the local scattering operators are defined as formal power series. Hence the objects of $\mathfrak{O}$ are in this context $*$-algebras of operators defined as formal power series.

1.2.1.2 Local scattering operators in perturbation theory

Let $\mathcal{A}$ be the algebra of observables of a free quantum field theory. To be specific we could choose $\mathcal{A}$ to be the unital $*$-algebra generated by the smeared fields $\varphi(f)$, $f \in \mathcal{D}(\mathbb{R}^4)$, which obey the Klein-Gordon equation $(\Box + m^2)\varphi = 0$ in a distributional sense together with the appropriate commutation relation, $[\varphi(f), \varphi(g)] = i(f, \Delta * g)$. Here the propagator function $\Delta = \Delta_{\text{av}} - \Delta_{\text{ret}}$ is the difference of the retarded and advanced Green's functions of $(\Box + m^2)$. The free fields satisfy Wightman's axioms, hence the fields have an invariant domain $\mathcal{D}$. There are other fields A which are relatively local to φ, that is $[A(g), \varphi(f)] = 0$ if the support of f is spacelike to the support of g. These fields form the *Borchers' class.* If the fields from the Borchers' class can be evaluated at fixed times (that is, restricted to spacelike surfaces), they serve as building blocks for local interactions. We define the interaction Lagrangian as $\mathcal{L}_I(\vec{x}) = A(t, \vec{x})$ with $x^0 = t$.

For a given test function $g \in C_c^\infty(\mathbb{R}^4)$, the localized Hamiltonian in the interaction picture is

$$V(t; g) = -\int g(t, \vec{x}) A(t, \vec{x}) d^3x.$$

The corresponding time evolution operator $U(t, s)$ is formally obtained by a Dyson expansion [20]. We evaluate it over a time interval $(\sigma, \tau) \subset \mathbb{R}$ which is chosen in such a way that $\operatorname{supp} g \subset (\sigma, \tau) \times \mathbb{R}^3$. As $V(\tau; g) = V(\sigma; g) = 0$, we get the scattering operator depending on the localization function g,

$$S(g) = \mathbb{1} + \sum_{i=1}^{\infty} \frac{i^n}{n!} \int T(A(x_1) \dots A(x_n)) g(x_1) \dots g(x_n) d^4x_1 \dots d^4x_n, \tag{1.4}$$

where the operator-valued functionals $T(\dots)$ are the *time-ordered products.*

Unfortunately, the restriction of fields from the Borchers' class to spacelike surfaces is in general not possible. In the above example, only the free fields themselves together with their derivatives have this property. Hence a direct application of this strategy does not lead to interesting examples for interacting theories.

Nevertheless, the Stückelberg-Bogoliubov-Epstein-Glaser formulation of perturbation theory is based upon the definition of the local scattering operator as in (1.4). The emphasis it put on the time-ordered products. They are defined directly as multilinear mappings from the nth power of the Borchers' class to operator-valued distributions $T(A_1(x_1) \dots A_n(x_n))$ with domain $\mathcal{D}$ such that certain conditions are fulfilled [23, 75]. These conditions allow for the recursive construction of the time-ordered products. Attention has to be payed to the total diagonal in $\mathbb{R}^{4n}$, as in general distributions can not be multiplied at coinciding points. The extension to this hypersurface is a renormalization procedure [7]. We will not go into further detail.

One crucial property of the time-ordered products is the causal factorization,

$$T(A(x_1) \dots A(x_n)) = T(A(x_1) \dots A(x_k)) T(A(x_{k+1}) \dots A(x_n)), \tag{1.5}$$

if $(\{x_1,\dots,x_k\}+\overline{V}_+)\cap\{x_{k+1},\dots,x_n\}=\emptyset$. This property as well as the others have their counterpart on the level of the local scattering operators. In particular, the causal factorization in the form of equation (1.3) remains valid.

1.2.1.3 The abstract definition of local scattering operators

Following [6] we define local scattering operators as a family of operators, depending on test functions, fulfilling a set of conditions which are consistent with the requirements of the Epstein-Glaser approach of perturbation theory as well as the generally covariant approach. Note that the following definition does not refer to the Dyson expansion. Moreover, we allow spacetimes of arbitrary dimension $d\geq 2$.

Definition 1.5. Let $\mathcal{H}$ be a Hilbert space, carrying a unitary representation $\overline{\mathcal{P}}\to\mathcal{B}(\mathcal{H}),(a,\alpha)\mapsto U(a,\alpha)$ of the universal covering group $\overline{\mathcal{P}}=\mathbb{R}^d\rtimes G$ of the Poincaré group, where G is the universal covering group of the proper, orthochonous Lorentz group $SO^+(1,d-1)$, which is the identity component of the homogeneous symmetry group $O(1,d-1)$ of the d dimensional spacetime, $2\leq d\in\mathbb{N}$.

A family $\{S(g):g\in\mathcal{D}(\mathbb{R}^n,\mathbb{R})\}$ of linear operators on $\mathcal{H}$ is a family of local scattering operators if

(i) $S(0)=\mathbb{1}$,

(ii) $S(g)^*=S(g)^{-1}$,

(iii) $S(g)$ transforms covariantly under $\overline{\mathcal{P}}$: $U(a,\alpha)S(g)U(a,\alpha)^{-1}=S(g_{\langle a,\Lambda(\alpha)\rangle})$ with $g_{\langle a,\Lambda(\alpha)\rangle}(x)=g(\Lambda^{-1}(\alpha)(x-a))$, where $\Lambda(\alpha)\in SO^+(1,d-1)$,

(iv) causal factorization holds true: Let $f,g,h\in\mathcal{D}(\mathbb{R}^n)$ such that $(\operatorname{supp}f+\overline{V}_+)\cap\operatorname{supp}g=\emptyset$, then

$$S(f+h+g)=S(f+h)S(h)^{-1}S(h+g). \tag{1.6}$$

Notice that equation equation (1.6) is independent of the support properties of h.

Remark. In this definition, we restrict ourselves to scalar theories. This is sufficient for our application in Section 4.3. If one would consider local scattering operators associated with localized interactions constructed from free spinorial or tensorial fields, test functions with several components g^λ would come into play and the covariance condition would be changed, involving a finite-dimensional matrix representation of G in spinor space, analogously to the transformation properties of the fields in Section 1.1.1.1.

In the definition of the local scattering operators, we could restrict the causality condition associated to the causal factorization of the T-products (1.5): $S(f+g) = S(f)S(g)$ if $(\operatorname{supp} f + \overline{V}_+) \cap \operatorname{supp} g = \emptyset$. But we are aiming at a definition of the local net and therefore, the stronger relation (1.3) is necessary.

1.2.2 Interacting fields and the adiabatic limit

The definition of the interacting fields with the local scattering operator goes back to Bogoliubov and Shirkov [6]. It has regained considerable interest for the rigorous treatment of perturbation theory [7] and opens up the possibility to disentangle the infrared and the ultraviolet problem. One finds that the arguments are indeed independent of perturbation theory, if the local scattering operators are defined without recourse to the time-ordered products as in Definition 1.5. We follow the presentation of [7], see also [19].

Let $\mathcal{A}$ be a unital $*$-algebra and assume $\mathcal{V}$ to be the space of possible interaction Lagrangians. It is considered as an abstract, finite-dimensional, real vector space.

Given an assignment of test functions $f \in \mathcal{D}(\mathbb{R}^n, \mathcal{V})$ to unitary operators $S(f) \in \mathcal{A}$ which fulfill the conditions of Definition 1.5 and hence the causality condition (1.6), we can define a new family of unitary operators which satisfies the same functional equation by

$$S_g(f) := S(g)^{-1} S(g+f). \tag{1.7}$$

Here the localized interaction $g \in \mathcal{D}(\mathbb{R}^n, \mathcal{V})$ is fixed. These *relative scattering operators* are local objects: as a consequence of the causality condition one can show that

$$[S_g(f), S_g(h)] = 0$$

if $(x-y)^2 < 0$ for all $(x,y) \in \operatorname{supp} f \times \operatorname{supp} h$. Hence, if the functional derivatives of the relative local scattering operators exist, they are local fields,

$$A_g(x) := \frac{\delta}{\delta h(x)} S_g(hA)|_{h=0},$$

with respect to the interaction $g \in \mathcal{D}(\mathbb{R}^n, \mathcal{V})$. In this formula we have $h \in \mathcal{D}(\mathbb{R}^n)$ and $A \in \mathcal{V}$. For a constant interaction extended over the whole spacetime this is Bogoliubov's definition of the interacting field [6].

For every finite, contractive subset $\mathcal{O}$ of the spacetime the families $\{S_g(h) : h \in \mathcal{D}(\mathcal{O}, \mathcal{V})\}$ generate a $*$-algebra $\mathcal{A}_g(\mathcal{O})$. This algebra is the algebra of local observables. Notice that in perturbation theory this $*$-algebra consists of formal power series as the local scattering operators are obtained in this sense. If the local scattering operators are unitary operators on a Hilbert space and as such elements of a C^*-algebra, also the algebra of local observables is a C^*-algebra.

A crucial observation is that the algebra $\mathcal{A}_g(\mathcal{O})$ depends only locally on g.

Theorem 1.6. *Let $g, g' \in \mathcal{D}(\mathbb{R}^n, \mathcal{V})$ such that $g\restriction_{\mathcal{O}'} = g'\restriction_{\mathcal{O}'}$ for a causally closed region $\mathcal{O}' \supset \mathcal{O}$. Then there exists a unitary operator $V \in \mathcal{A}$ such that*

$$VS_g(h)V^{-1} = S_{g'}(h)$$

for all $h \in \mathcal{D}(\mathcal{O}, \mathcal{V})$.

For the proof see [7]. Again, the causal factorization in the form (1.3) enters crucially. As the structure of the algebras of local observables is independent of the behavior of the interaction outside of a neighborhood of an open region $\mathcal{O}$ of spacetime, the local net in the sense of the Haag-Kastler axioms in Section 1.1.1.2 is determined if one knows the relative scattering operators $f \mapsto S_g(f)$ for all test functions $g \in \mathcal{D}(\mathbb{R}^n, \mathcal{V})$.

Moreover, it is possible to obtain the quasilocal algebra $\mathcal{A}_{\mathcal{L}}$ for an interaction Lagrangian $\mathcal{L}$ which is no longer localized. This purely algebraic construction corresponds to the adiabatic limit, but in contrast to other formulations it is not necessary to extend the support of the interaction g explicitly to the whole spacetime.

The construction is based upon the following ideas (see [7]). Let $\Theta(\mathcal{O})$ be the set of all $g \in \mathcal{D}(\mathbb{R}^n)$ which are identically equal to 1 on a causally closed open neighborhood of $\mathcal{O}$. This set is the base of the bundle

$$\bigcup_{g \in \Theta(\mathcal{O})} \{g\} \times \mathcal{A}_{g\mathcal{L}}(\mathcal{O}). \tag{1.8}$$

Define $\mathcal{U}(g, g')$ to be the set of all unitary operators $V \in \mathcal{A}$ intertwining the relative scattering operators

$$VS_{g\mathcal{L}}(h) = S_{g'\mathcal{L}}(h)V$$

for all $h \in \mathcal{D}(\mathcal{O}, \mathcal{V})$. We define $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ to be the algebra of covariantly constant sections in the bundle (1.8). This means that if $A \in \mathcal{A}_{\mathcal{L}}(\mathcal{O})$, then $A = (A_g)_{g \in \Theta(\mathcal{O})}$, where $A_g \in \mathcal{A}_{g\mathcal{L}}(\mathcal{O})$ and $VA_g = A_{g'}V$ for all $V \in \mathcal{U}(g, g')$. In particular, the algebra $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ contains the elements $S_{\mathcal{L}}(h)$, given by the sections $(S_{\mathcal{L}}(h))_g = S_{g\mathcal{L}}(h)$.

To complete the construction of the net of algebras of local observables, we have to specify the embeddings which lead to the condition of isotony in the axioms 1.1.1.2. But the embedding $i_{21} : \mathcal{A}_{\mathcal{L}}(\mathcal{O}_1) \hookrightarrow \mathcal{A}_{\mathcal{L}}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$ is inherited from the inclusion $\mathcal{A}_{g\mathcal{L}}(\mathcal{O}_1) \subset \mathcal{A}_{g\mathcal{L}}(\mathcal{O}_2)$ for $g \in \mathcal{D}(\mathcal{O}_2)$ by restricting the sections from $\Theta(\mathcal{O}_1)$ to $\Theta(\mathcal{O}_2)$. These embeddings satisfy $i_{12} \circ i_{23} = i_{13}$ for $\mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1$. Hence they define an inductive system and we define the quasilocal algebra $\mathcal{A}_{\mathcal{L}}$ as the norm closure of the inductive limit of the algebras of local observables,

$$\mathcal{A}_{\mathcal{L}} := \overline{\bigcup_{\mathcal{O}} \mathcal{A}_{\mathcal{L}}(\mathcal{O})}.$$

The Poincaré covariance of the local scattering operators implies this property also for the relative scattering operators: Let $(a, \alpha) \in \overline{\mathcal{P}}$, then

$$U(a,\alpha)S_{g\mathcal{L}}(h)U(a,\alpha)^{-1} = S_{g_{\langle a,\Lambda(\alpha)\rangle}\mathcal{L}}(h_{\langle a,\Lambda(\alpha)\rangle}),$$

where $h_{\langle a,\Lambda(\alpha)\rangle} = h(\Lambda^{-1}(x-a))$ and we consider again Lorentz scalars for simplicity.

We define the automorphisms which implement Poincaré covariance of the local algebras,

$$(\beta_{\{a,\Lambda(\alpha)\}}(A))_g := U(a,\alpha)A_{g_{\langle a,\Lambda(\alpha)\rangle}}U(a,\alpha)^{-1},$$

for $A \in \mathcal{A}_{\mathcal{L}}(\mathcal{O})$ and $g \in \Theta(\Lambda(\alpha)\mathcal{O} + a)$. One has to check that $\beta_{\{a,\Lambda(\alpha)\}}(A)$ is again a covariantly constant section as defined above. Hence $\beta_{\{a,\Lambda(\alpha)\}}$ is an automorphism of the net of local algebras which implements the action of the Poincaré group,

$$\beta_{\{a,\Lambda(\alpha)\}}(\mathcal{A}_{\mathcal{L}}(\mathcal{O})) = \mathcal{A}_{\mathcal{L}}(\Lambda(\alpha)\mathcal{O} + a).$$

Furthermore, in perturbation theory it turns out that it is sufficient to localize the interaction in ‘small’ regions.

1.2.3 Local scattering operators and the time-dependent Schrödinger equation

There is a straightforward way to obtain the family of local scattering operators in a nonperturbative way. Instead of using a Dyson expansion to describe the time evolution in the interaction picture formally, we investigate the wellposedness of the Cauchy problem of the time-dependent Schrödinger equation rigorously.

Assume the time evolution $U(t,s)$ of a quantum theory is generated by a Hamiltonian of the form $H(t) = H_0 + V(t)$, hence it solves the Schrödinger equation,

$$i\frac{d}{dt}U(t,s) = H(t)U(t,s), \quad U(s,s) = \mathbb{1}. \tag{1.9}$$

The scattering operator is defined as the strong limit

$$S = \lim_{t\to\infty}\lim_{s\to-\infty} e^{iH_0t}U(t,s)e^{-iH_0s} \tag{1.10}$$

if it exists. This formula is simplified by transformation in the *Dirac* (or *interaction*) *picture*: Setting $V^D(t) = e^{iH_0t}V(t)e^{-iH_0t}$ and denoting by $U^D(t,s)$ the solution of the Schrödinger equation with respect to $V^D(t)$, one finds

$$S = \lim_{t\to\infty}\lim_{s\to-\infty} U^D(t,s). \tag{1.11}$$

Similar to the approach in perturbation theory we define

$$V(t;g) = -\int A(0,\vec{x})g(t,\vec{x})\, d^{d-1}x$$

for a localized coupling $g \in C_c^\infty(\mathbb{R}^d)$. The Hamiltonian in the interaction picture is

$$V^D(t;g) = e^{iH_0t}V(t;g)e^{-iH_0t} = -\int A(t,\vec{x})g(t,\vec{x})\,d^{d-1}x.$$

If the Cauchy problem of the time-dependent Schrödinger equation with respect to $V^D(t;g)$ is wellposed with propagator $U^D(t,s)$, the limit (1.11) exists trivially because of the localization of the interaction. We define the local scattering operator by

$$S(g) := U(\tau,\sigma), \tag{1.12}$$

where the time interval $(\sigma,\tau) \subset \mathbb{R}$ is chosen such that $\operatorname{supp} g \subset (\sigma,\tau) \times \mathbb{R}^n$. As the propagator is trivial outside of the time support of g, the definition of $S(g)$ does not depend on the choice of σ and τ as long as the support condition is fulfilled. Moreover, the properties of the time evolution as discussed in the next chapter lead to the conditions of Definition 1.5.

The field $A(x)$ describing the interaction comes from the Borchers' class of the free fields. Although the assumption of the restriction of A to fixed times remains problematic in four spacetime dimensions, at least for models on lower dimensional spacetimes this definition makes sense. As the approach is manifestly Hamiltonian, the strategy has a similar appearance as the constructive quantum field theory before the 'Euclidean revolution'. To test the approach we will concentrate on models which are accessible to the Hamiltonian strategy. But even for these models the interaction is an unbounded operator with complicated properties. We have to develop advanced methods to solve the corresponding time-dependent Schrödinger equation. This is our task for the next chapter.

Chapter 2

Evolution Equations

As we have seen, the existence question for local scattering operators outside of perturbation theory can be traced back to the question of solvability of a time-dependent Schrödinger equation. In the present Chapter we will address the wellposedness of the Cauchy problem for evolution equations of the type of the time-dependent Schrödinger equation. There are two main sources of difficulties: the time dependence of the Hamiltonian and the hyperbolic type of the Schrödinger equation.

For the time-independent case, that is for the Cauchy problem for autonomous evolution equations in Banach spaces, there is a well-developed theory. In fact, the wellposedness theory of the autonomous Cauchy problem is equivalent to the theory of strongly continuous operator semigroups on Banach spaces. Every strongly continuous semigroup is the solution of a Cauchy problem, and every solution gives rise to a strongly continuous semigroup. The semigroup property together with strong continuity implies features like exponential boundedness, differentiability, closedness of the generator. Hence wellposedness in the autonomous case can be formulated completely using generation theorems of Hille-Yosida type.

For the nonautonomous (time-dependent) Cauchy problem, the situation is quite different. If there exists a strongly continuous solution which depends continuously on the initial value, it can be interpreted in terms of a *propagator*. This is a family of operators satisfying a *causal factorization* equation, $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s$. But this property is considerably weaker than the semigroup property of the solutions in the autonomous context. It does *not* imply exponential boundedness or differentiability. In fact, not every propagator is related to a Cauchy problem for a nonautonomous evolution equation. This makes it very difficult to develop a general wellposedness theory in the time-dependent situation. Nevertheless, if the generator of the nonautonomous evolution equation is the generator of an analytic semigroup for every moment in time, there are quite sophisticated existence theorems due to P. Acquistapace, B. Terreni and others, see [79]. This is denoted as the *parabolic* case.

Unfortunately, the time-dependent Schrödinger equation is not of this type.

It is a *hyperbolic* nonautonomous evolution equation, in general the Hamiltonian generates a unitary group which is not analytic.

After discussing the general properties of the Cauchy problem for autonomous and nonautonomous evolution equations, we will give a review of existence theorems for various types of solutions in the hyperbolic case. Although weak solutions exist under very general assumptions, they have not enough regularity for our purposes. Moreover, weak solutions are in general not unique. Therefore we will give better suited concepts of solvability. The standard theorem for strong solutions is due to T. Kato. But there the assumptions are too restrictive to discuss the existence of local scattering operators in quantum field theory.

The key to a well-suited existence theory will be a technique due to J. Howland, which relates the time-dependent Cauchy problem to a time-independent one. But this is the subject of the next Chapter.

The theory of evolution equations has a wide range of applications, for example to partial differential equations with variable boundary conditions. Therefore, we formulate the results of the present and the following chapter not only for Hilbert spaces, but for general Banach spaces X if possible.

2.1 The autonomous Cauchy problem

For $t \geq s$ we consider the *autonomous Cauchy problem*, that is the initial value problem

$$\frac{d}{dt}u(t) = Au(t), \quad u(s) = x, \tag{2.1}$$

on a Banach space X, where A is a closed linear operator with domain of definition $D(A)$.

Definition 2.1. A *classical solution* of the Cauchy problem is a continuous X-valued function $t \mapsto u_x(t) \in D(A),\ t \geq s$, which is continuously differentiable and satisfies (2.1).

Existence and uniqueness of solutions in general depend on the choice of the initial value.

Definition 2.2. The autonomous Cauchy problem is called *wellposed* if

(i) $D(A)$ is dense in X and there exists a classical solution u_x of (2.1) for every $x \in D(A)$,

(ii) the solution is unique,

(iii) the solution depends continuously on the initial value: For every sequence $\{x_n\} \subset D(A), x_n \to 0$, we have $u_{x_n}(t) \to 0$ uniformly on compact time intervals.

Wellposedness of the autonomous Cauchy problem (2.1) can be completely characterized by properties of the operator A.

Theorem 2.3. *Let A be a densely defined linear operator with nonempty resolvent set. Then the Cauchy problem* (2.1) *with $s = 0$ is wellposed if and only if A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. For every $x \in D(A)$, $u_x(t) = T(t)x$ is a classical solution.*

For the proof see [67].

Every strongly continuous semigroup $(T(t))_{t\geq 0}$ is exponentially bounded. This means that there exist constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$ [67, Theorem 1.2.2]. If $\omega = 0$ and $M = 1$, we call $(T(t))_{t\geq 0}$ a strongly continuous semigroup of *contractions*. The theorem of Hille and Yosida gives a criterion for an operator A being the generator of a strongly continuous semigroup of contractions.

Theorem 2.4 (Hille-Yosida). *A linear operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$, if and only if*

(i) A is closed and densely defined,

(ii) $\mathbb{R}^+ \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for every $\lambda > 0$.

For the proof see [67]. In practice, the following notions turn out to be handy: For $x \in X$ denote by $F(x) \subset X^*$ the *duality set* $F(x) := \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}$. A linear operator A is *dissipative* if for every $x \in D(A)$ there is an $x^* \in F(x)$ such that $\operatorname{Re} x^*(Ax) \leq 0$. It is *accretive* if $-A$ is dissipative. A maximally dissipative operator A has no proper dissipative extension. Maximally accretive operators are defined analogously. The notion of dissipativity is related to the boundedness assumption on the resolvent in the Theorem of Hille-Yosida.

Theorem 2.5. *A linear operator A is dissipative if and only if $\|(\lambda - A)x\| \geq \lambda\|x\|$ for all $x \in D(A)$ and $\lambda > 0$.*

Thus we arrive at the following theorem.

Theorem 2.6 (Lumer-Phillips). *Let A be a densely defined, linear operator. If A is dissipative and there is a $\lambda_0 > 0$ such that* $\operatorname{Ran}(\lambda_0 - A) = X$, *then A generates a strongly continuous semigroup of contractions. Conversely, if A is the generator of a strongly continuous, contractive semigroup, then* $\operatorname{Ran}(\lambda - A) = X$ *for all $\lambda > 0$ and A is dissipative.*

For the proof of this and the preceding theorem, see again Pazy's monograph [67].

If the initial value $x \notin D(A)$, the Cauchy problem has in general no solution. However, the orbit $u : [0, \infty) \to X$, $t \mapsto T(t)x$ of the strongly continuous semigroup $(T(t))_{t\geq 0}$ solves (2.1) in a generalized way.

Definition 2.7. A continuous function $u : [0, \infty) \to X$ is called a *mild solution* of (2.1) if there is a sequence $\{x_n\} \subset D(A)$ such that $x_n \to u(0)$ and $T(t)x_n \to u(t)$ uniformly on bounded intervals as $n \to \infty$.

One can show that the generalized solution is independent of the choice of the sequence $\{x_n\}$ and coincides with the solution of (2.1) if $u(0) \in D(A)$. Hence, with the assumptions of Theorem 2.3, the Cauchy problem has a generalized solution for every initial value $x \in X$. All solutions are exponentially bounded.

This generalized notion of solvability is appropriate for the application to quantum mechanics: Usually one does not restrict the possible initial states of a quantum system to lie in the domain of the Hamiltonian.

For the initial value problem of the corresponding nonautonomous evolution equation there is no such easy characterization of solvability in terms of the generator. Therefore, it will be useful to relate the time-dependent situation to the time-independent one, where the question of solvability can be discussed with help from theorems like Theorem 2.4. In particular, we will develop also in the nonautonomous context a notion of solvability which is similar to the mild solutions above.

2.2 The nonautonomous Cauchy problem

For $t \geq s$ let $A(t)$ be linear operators on a Banach space X with domains of definition $D(A(t))$. The nonautonomous Cauchy problem is given by

$$\frac{d}{dt}u(t) = A(t)u(t), \quad u(s) = x, \tag{2.2}$$

where $x \in X$ is the initial value.

Definition 2.8. A classical solution of the nonautonomous Cauchy problem is a $C^1([s, \infty), X)$-function $u_{s,x}$ such that $u_{s,x}(t) \in D(A(t))$ and (2.2) holds.

An appropriate definition of wellposedness is given by R. Nagel and G. Nickel in [59].

Definition 2.9. The nonautonomous Cauchy problem (2.2) is called classically wellposed with regularity subspaces $(Y_s)_{s\in\mathbb{R}}$ if

(i) the subspace $Y_s = \{y \in X : \exists \text{ a classical solution } u_{s,y} \text{ of } (2.2)\} \subset D(A(s))$ and Y_s is dense in X for all $s \in \mathbb{R}$,

(ii) the solution $u_{s,y}$ is unique for every $y \in Y_s$,

(iii) the solution $u_{s,y}$ depends continuously on the initial data: Let $s_n \to s \in \mathbb{R}$, $Y_{s_n} \ni y_n \to y \in Y_s$ and for $z = y_n$ or $z = y$ set $u_{r,z}(t) = z$ if $t < r$. Then $\|u_{s_n,y_n}(t) - u_{s,y}(t)\| \to 0$ uniformly for $t \in I$, $I \subset \mathbb{R}$ compact.

If moreover $\|u_{s,y}(t)\| \leq Me^{\omega(t-s)}\|y\|$, the solution is called exponentially bounded.

Given a wellposed nonautonomous Cauchy problem with exponentially bounded solutions, these solutions give rise to *strongly continuous propagators*. Let $I \subseteq \mathbb{R}$ be an interval and $D_I = \{(t,s) \in I \times I : t \geq s\}$.

Definition 2.10. A strongly continuous, exponentially bounded *propagator* or *evolution family* is a family of bounded operators $\{U(t,s)\}_{(t,s)\in D_I}$ such that

(i) $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s$ and $U(t,t) = \mathbb{1}$,

(ii) $D_I \ni (t,s) \mapsto U(t,s)$ is strongly continuous,

(iii) $\|U(t,s)\| \leq Me^{\omega(t-s)}$.

Propagators arise naturally if one considers the properties of dynamical systems. Their properties reflect the conditions one would expect for a causal time evolution, hence the strong continuity of $(t,s) \mapsto U(t,s)$ seems to be a reasonable assumption. However, dealing with weaker notions of solvability and in the context of evolution semigroups in Chapter 3, it will turn out to be convenient to introduce also *weakly measurable* propagators. These are defined in the sense of Definition 2.10, but with condition (ii) replaced by

(ii') $D_I \ni (t,s) \mapsto U(t,s)$ is weakly measurable.

Propagators generalize the concept of strongly continuous semigroups in the context of nonautonomous evolution equations. Indeed, given a strongly continuous semigroup $(T(s))_{s\geq 0}$, the definition $U(t,s) := T(t-s)$ yields a strongly continuous, exponentially bounded propagator.

However, the theory of the nonautonomous Cauchy problem is very different from its autonomous counterpart. The semigroup property

$$T(t)T(s) = T(t+s), \quad t \geq s \geq 0,$$

together with strong continuity already implies strong differentiability of $t \mapsto T(t)$ on a dense set, exponential boundedness, and it fixes the properties of the generator A (for example closedness) together with its relation to the Cauchy problem [67]. In contrast, the causal condition

$$U(t,r)U(r,s) = U(t,s), \quad t \geq r \geq s,$$

does not allow for similar statements for the propagators $U(t,s)$, the family of generators $A(t)$ and the nonautonomous Cauchy problem. We give some examples which illustrate the difficulties in the time-dependent situation:

Example. Consider a continuous, nowhere differentiable function $t \mapsto f(t) > 0$ on $\mathbb{R}$. Then $U(t,s) := f(t)f(s)^{-1}$ is a propagator according to Definition 2.10, but nowhere differentiable. Hence it is not related to a classical solution of a nonautonomous Cauchy problem (2.2). The propagator $U(t,s)$ has no generator $A(t)$.

Example. The propagator $U(t,s) := e^{t^2-s^2}$ on $X = \mathbb{C}$ satisfies (i) and (ii), but it is not exponentially bounded.

Even if the nonautonomous Cauchy problem (2.2) is wellposed, the generators $A(t)$ may behave in a surprising way, compared to the autonomous situation. The following examples demonstrate some pathologies:

Example. The nonautonomous Cauchy problem (2.2) is wellposed with smooth solutions, but the intersection of the domains of the generators is trivial, $\bigcap_t D(A(t)) = \{0\}$, see Section 2.2.1 and [40].

Example. The nonautonomous Cauchy problem (2.2) is wellposed, but the regularity subspaces Y_t are strictly contained in $D(A(t))$ [62].

Example. The nonautonomous Cauchy problem (2.2) is wellposed, but $A(t)$ is not even *closable* [59].

Example. The nonautonomous Cauchy problem (2.2) is wellposed, but this property is not stable under perturbations of the $A(t)$ by *bounded* operators or scaling $A(t) \to \alpha A(t)$, $\alpha > 0$ [59].

Hence, to deal with this situation, the notion of classical solutions turns out to be too restrictive. Thus we will use concepts of solvability which are strictly weaker (if the generators $A(t)$ are unbounded). In analogy to the mild solutions in the autonomous case, we consider the orbits $u = U(\cdot, s)x$ for *every* $x \in X$.

Definition 2.11. Let $A(t)$, $t \in I$, be linear operators on X, $s < b$, $s \in I$, $x \in X$, and $1 \leq p \leq \infty$.

(a) Let $I = (a, b]$. A function $u \in C([s,b], X)$ is an *(E_p)-mild solution of* the nonautonomous Cauchy problem (2.2) if

  (i) $u(s) = x$,

  (ii) there are $u_n \in W^{1,p}((s,b), X)$ (respectively $C^1((s,b), X)$ if $p = \infty$), $n \in \mathbb{N}$, such that $u_n(t) \in D(A(t))$ for almost every $t \in [s,b]$, $A(\cdot)u_n \in L^p([s,b], X)$ (respectively $C([s,b], X)$ if $p = \infty$),

  (iii) $u_n \to u$ uniformly on $[s,b]$,

  (iv) $-\dot{u}_n + A(\cdot)u_n \to 0$ in $L^p([s,b], X)$ as $n \to \infty$.

(b) Let $I = \mathbb{R}$. A function $u \in C(\mathbb{R}, X)$ is an *(E_p)-mild solution on* $\mathbb{R}$ of the nonautonomous Cauchy problem (2.2) if

(i) $u(s) = x$,

(ii) for each $d > |s|$ there are $u_n \in W^{1,p}((-d,d),X)$ (respectively $C^1([-d,d],X)$ if $p=\infty$), $n \in \mathbb{N}$, such that $u_n(t) \in D(A(t))$ for almost every $t \in [-d,d]$, $A(\cdot)u_n \in L^p([-d,d],X)$ (respectively $C([-d,d],X)$ if $p=\infty$)),

(iii) $u_n \to u$ uniformly on $[-d,d]$,

(iv) $-\dot{u}_n + A(\cdot)u_n \to 0$ in $L^p([-d,d],X)$ as $n \to \infty$ for every $d > |s|$.

In Theorem 3.13 we will see that, with some special assumptions, mild solutions are closely related to *weak solutions*. For a reflexive Banach space X and positive p,q with $1 = \frac{1}{p} + \frac{1}{q}$ consider the function space $E_p = L^p(I,X)$ on the interval $I = [a,b], -\infty \le a < b \le \infty$. We identify $L^q(I,X^*)$ and E_p^*. By $\mathcal{T}$ we denote a space of test functions.

Definition 2.12. Given $p,q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $I \subset \mathbb{R}$, and define the test function space $\mathcal{T}$ by

$$\mathcal{T} := \{f \in L^q(I,X^*) : f(t) \in D(A^*(t))\ , f \text{ continuously differentiable}, \\ A^*(\cdot)f(\cdot) \text{ continuous}, f(b) = 0\}. \quad (2.3)$$

For $x \in X$ we call $u \in L^p(I,X)$ a *weak solution* of the nonautonomous Cauchy problem (2.2) with initial value x if

$$\int_I \Big[\Big(\dot{f}(t), u(t)\Big) + \big(A^*(t)f(t), u(t)\big) \Big]\, dt + \big(f(a), x\big) = 0 \quad (2.4)$$

for all $f \in \mathcal{T}$.

As we will see in Section 2.2.2, weak solutions of the nonautonomous Cauchy problem exist under very general assumptions. But, because we do not know whether weak solutions are continuous or unique, they are not well-suited for the definition of local scattering operators.

To get to a more appropriate notion of solvability, it is useful to distinguish those solutions which can be approximated by certain sequences of solutions of an evolution equation with bounded generator. We define the approximation of the generator in the following way:

Definition 2.13. Let $A(t)$, $t \in I$, $I \subset \mathbb{R}$, be operators on a Banach space X. We call the bounded operators $A_n(t)$, $n \in \mathbb{N}$, an *admissible bounded approximation* of $A(t)$ if for $(t,s) \in D_I$ and $n \in \mathbb{N}$,

(i) the function $t \mapsto A_n(t)$ is strongly continuous,

(ii) $\lim_{n\to\infty} A_n(t)y = A(t)y$ for all $y \in D(A(t))$,

(iii) $\|A_n(t)y\| \leq c(\|A(t)y\| + \|y\|)$ for all $y \in D(A(t))$ and a constant $c > 0$,

(iv) $\|U_n(t,s)\| = Me^{\omega(t-s)}$ where $\omega \in \mathbb{R}$, $M \geq 1$ and $U_n(t,s)$ is the propagator generated by $A_n(t)$.

The properties of $A_n(t)$ assure the existence of the propagators $U_n(t,s)$ associated with $A_n(t)$, see Theorem 2.18.

Remark. For operators $A(t)$, $t \in I$, $I \subset \mathbb{R}$, such that $Y \subset D(A(t))$ $\forall t \in \mathbb{R}$ with a Banach space $Y \subset X$, densely embedded in X, we could define admissible bounded approximations *on* Y by replacing (ii) and (iii) by

(ii') $\lim_{n\to\infty} A_n(t)y = A(t)y$ for all $y \in Y$,

(iii') $\|A_n(t)y\| \leq c\|y\|_Y$ for all $y \in Y$ and a constant $c > 0$.

This definition would be sufficient for the use in Theorem 3.20, but we will work with the conditions of Definition 2.13, which we find more natural.

A class of operators $A(t)$ which feature admissible bounded approximations are the so-called Kato-stable operators, see Definition 2.25. To obtain an approximation of an unbounded operator, in general a Kato approximation is suitable. For Hilbert spaces we obtain admissible bounded approximations via the functional calculus.

Lemma 2.14. *(i) Let $A(t), t \in I$, be Kato-stable operators with constants M, ω. Assume that $t \mapsto R(\omega', A(t)) = (\omega' - A(t))^{-1}$ is strongly continuous for $t \in I$ and some $\omega' > \omega$. Then $A_n(t) := nA(t)R(n, A(t)), n > \omega$, are admissible bounded approximations of $A(t)$.*

(ii) Let $A(t), t \in I$, be skew-adjoint operators on a Hilbert space X. Assume that $t \mapsto R(1, A(t))$ is strongly continuous for $t \in I$. Set $\varphi_n(i\tau) = i\tau$ for $|\tau| \leq n$ and $\varphi_n(i\tau) = \pm in$ for $\pm\tau \geq n$. Then $A_n(t) = \varphi_n(A(t))$ are skew-adjoint admissible bounded approximations of $A(t)$. The corresponding propagators $U_n(t,s)$ are unitary.

Proof. (i) For arbitrary $n > \omega$ the formula

$$R(n, A(t)) = [1 + (n - \omega')R(\omega', A(t))]^{-1}R(\omega', A(t))$$

shows the strong continuity of $t \mapsto R(n, A(t))$, and strong continuity of $t \mapsto A_n(t)$ follows. The remaining assertions are standard properties of the Kato approximation, see for example [22, Lemma II.3.4].
(ii) As above we see that $t \mapsto R(\omega, A(t))$ is strongly continuous for all $\omega \in \rho(A(t))$. Let $\{t_m\}$ be a sequence with $t_m \to t$ as $m \to \infty$. Then $R(\omega, A(t_m)) \to R(\omega, A(t))$ strongly, hence for every bounded, continuous function u on $i\mathbb{R}$ we have $u(A(t_m)) \to u(A(t))$ strongly. With $u = \varphi_n$ this is strong continuity of $t \mapsto A_n(t)$. The remaining assertions are shown using the functional calculus for normal operators [93]. $\square$

We use these generators with admissible bounded approximation to formulate the following notion of solvability.

Definition 2.15. Let $A(t)$, $t \in I$, $I \subset \mathbb{R}$, be linear operators on X with admissible bounded approximation $A_n(t)$. Let $s \in I$, $s < b$, $x \in X$, and $1 \leq p \leq \infty$.

(a) In the case $I = (a, b]$ a weakly continuous function $u : [s, b] \to X$ is an *(E_p)-approximative solution* of the nonautonomous Cauchy problem (2.2) if

  (i) $u(s) = x$,

  (ii) there are $u_n \in C^1([s, b], X)$, $n \in \mathbb{N}$, such that

    (1) $u_n(s) = x$,

    (2) $u_n(t) \to u(t)$ weakly as $n \to \infty$ all for $t \in [s, b]$,

    (3) $-\dot{u}_n + A_n(\cdot)u_n \to 0$ in $L^p([s, b], X)$ as $n \to \infty$.

(b) In the case $I = \mathbb{R}$ a weakly continuous function $u : \mathbb{R} \to X$ is an *(E_p)-approximative solution on* $\mathbb{R}$ of the nonautonomous Cauchy problem (2.2) if

  (i) $u(s) = x$,

  (ii) for each $d > |s|$ there are $u_n \in C^1(\mathbb{R}, X)$, $n \in \mathbb{N}$, such that

    (1) $u_n(s) = x$,

    (2) $u_n(t) \to u(t)$ weakly as $n \to \infty$ for $t \in \mathbb{R}$,

    (3) $-\dot{u}_n + A_n(\cdot)u_n \to 0$ in $L^p([-d, d], X)$ as $n \to \infty$ for every $d > |s|$.

We will see in Section 3.3 that the unique approximative solution of the time-dependent Schrödinger equations can be obtained with very general requirements. At the same time, this notion of solvability is considerably more regular than weak solutions and suitable for the definition of local scattering operators from the time evolution in the interaction picture, see Chapter 4.

2.2.1 Goldstein's example

In the course of this work, we will review some theorems about the wellposedness of the Cauchy problem for nonautonomous evolution equations. As we will see in the following section, weak solutions exist under very general assumptions.

However, if one aims at strong solutions one has to impose quite restrictive assumptions on the generators $A(t)$. In particular, one could ask whether it is necessary that the domains of the generators have a dense intersection. In other words: If the nonautonomous Cauchy problem (2.2) has a strong solution for all initial values from a dense set, and if the generators $A(t)$ are maximally dissipative for all times t, is the intersection $\bigcap_{t \geq 0} D(A(t))$ necessarily dense in X? Goldstein's example [40] shows that this is indeed not the case.

Goldstein shows that there are self-adjoint operators $H(t) \geq 0$ for $t \geq 0$ on a Hilbert space X such that the following conditions are fulfilled:

(i) The resolvent of $H(t)$ depends smoothly on t, that is for all $\lambda \in \mathbb{C}\backslash[0,\infty)$ and $x \in X$ the mapping $t \mapsto R(\lambda, H(t))^{-1}x$ is an element from $C^\infty([0,\infty), X)$.

(ii) There is a dense set $Y \subset X$ such that the nonautonomous Cauchy problem (2.2) with $A(t) = -iH(t)$ has a unique solution $u \in C^\infty([0,\infty), X)$ with $u(0) = y$ for all $y \in Y$. The solution depends continuously on the initial value.

(iii) $\bigcap_{t\geq 0} D(H(t)) = \{0\}$.

Let $S \geq 0$ be a self-adjoint unbounded operator with dense domain $D(S) \subset X$. According to a theorem of J. von Neumann, there is a unitary operator U_1 such that the nonnegative self-adjoint operator $T = U_1 S U_1^*$ satisfies $D(T) \cap D(S) = \{0\}$. Now we use the spectral theorem to represent U_1 as $U_1 = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda$ and define the bounded, nonnegative and self-adjoint operator L by $L := \int_0^{2\pi} \lambda\, dE_\lambda$. Choose nonnegative, nontrivial smooth functions φ, ψ and η with $\operatorname{supp}\varphi \subset [0,1)$, $\operatorname{supp}\eta \subset (1,2)$, $\operatorname{supp}\psi \subset (2,3)$ and $\int_0^2 \eta(t)\, dt = 1$. Define

$$H(t) := \varphi(t)S + \eta(t)L + \psi(t)T. \tag{2.5}$$

Then we have following result.

Theorem 2.16. *Let $Y := C^\infty(S)$. For every $y \in Y$ the nonautonomous Cauchy problem* (2.2) *has a unique solution $t \mapsto u(t)$ which fulfills the conditions (i), (ii) and (iii). It is given by*

$$u(t) := \begin{cases} e^{i\int_0^t \varphi(s)ds\, S} y & \text{for } 0 \leq t \leq 1 \\ e^{i\int_0^t \eta(s)ds\, L} u(1) & \text{for } 1 \leq t \leq 2 \\ e^{i\int_0^t \psi(s)ds\, T} u(2) & \text{for } 2 \leq t \leq \infty. \end{cases} \tag{2.6}$$

For the proof see [40].

This simple example describes a situation which we expect to find also in the context of quantum field theory. As we have seen in Section 1.1.3, even if the localized interaction Hamiltonian can be defined in the Fock space of the free fields, it will have a domain of definition which has trivial intersection with the domain of the free Hamiltonian if a nontrivial renormalization is necessary. Moreover, it may happen that the interaction term itself has a wildly varying domain for different moments in time. Goldstein's example gives a hint in the direction that the time evolution and hence the local scattering operators may exist nevertheless. We will return to this example in Section 3.3.3.

2.2.2 Existence of weak solutions

Discussing the time-dependent Schrödinger equation and the interaction picture in quantum theory, M. Reed and B. Simon state in [71] that weak solutions always exist if the Hamiltonian on a Hilbert space X can be written as $H(t) = H_0 + V(t)$ with a self-adjoint, possibly unbounded operator H_0 and $V(\cdot) : \mathbb{R} \to \mathcal{B}(X)$ strongly continuous.

However, weak solutions of the nonautonomous Cauchy problem (2.2) exist under much more general conditions. We cite a result due to H. Sohr [85] together with its proof which demonstrates the application of abstract methods known from partial differential equations to the Cauchy problem (2.2). We will return to the topic of weak solvability in Theorem 3.13.

Theorem 2.17. *Let X be a reflexive Banach space and let $A(t) : D(A(t)) \to X$ be a generator of a strongly continuous semigroup of contractions for every $t \in I = [a, b]$. For every given initial value $x \in X$ there exists a weak solution $u \in E_p$ of the nonautonomous Cauchy problem* (2.2) *in the sense of Definition 2.12 over the test function space $\mathcal{T}$ of equation* (2.3).

Proof. By the theorem of Hille and Yosida (Theorem 2.4), the resolvent $\big(1 - sA(t)\big)^{-1}$ is bounded and bijective from $\mathcal{H}$ to $D(A(t))$ with norm less or equal to 1 for every $s \geq 0$. With an equidistant partition of the interval I we can define an approximate solution of the nonautonomous Cauchy problem (2.2). Let $\delta_n := \frac{1}{n}(b-a)$, $t_\nu^{(n)} := a + \frac{\nu}{n}(b-a)$ and

$$C_{n,\nu} := \Big(1 - \delta_n A(t_\nu^{(n)})\Big)^{-1},\ C_{n,0} := \mathbb{1} \text{ for } n \geq 1,\ 0 \leq \nu \leq n.$$

With $W_{n,\nu} := C_{n,\nu} \cdot \ldots \cdot C_{n,0}$ define $u_n \in L^p(I, X)$ by $u_n(t) := W_{n,\nu-1}x$ for $t \in [t_{\nu-1}^{(n)}, t_\nu^{(n)}], 1 \leq \nu \leq n$.

Due to the boundedness of $u_n(t)$ by $\|x\|$, we have

$$\|u_n\|_p \leq \|u_0\| \, (b-a)^{1/p}.$$

With this sequence of approximations u_n we can establish the existence of the weak solution u.

Let $\{\mathbb{C}\}$ be the space of bounded, complex-valued sequences $\{c_n\}_{n\in\mathbb{N}}$. Define a semi-norm on $\{\mathbb{C}\}$ by $|||\{c\}||| := \limsup|c_n|$. By dividing out the zero-space $N = \{\{c\} \in \{\mathbb{C}\} : |||\{c\}||| = 0\}$ one gets a Banach space $\big[\mathbb{C}\big] = \{\mathbb{C}\}/N$.

The constant sequences form a closed subspace of $\{\mathbb{C}\}$, which is the image of an isometric embedding i:

$$\mathbb{C} \ni c \overset{i}{\hookrightarrow} [c] \in \big[\mathbb{C}\big]$$

By the theorem of Hahn-Banach, the bounded linear form $\tilde{F} : i(\mathbb{C}) \to \mathbb{C},\ \tilde{F}([c]) = c$ is extendable to $F : \big[\mathbb{C}\big] \to \mathbb{C},\ F\restriction_{i(\mathbb{C})} = \tilde{F},\ \|F\| = \|\tilde{F}\|$. Note that this extension is not unique.

For every $f \in L^q(I, X^*)$ the sequence $\{(f, u_n)\}_{n\in\mathbb{N}}$ is an element of $\{\mathbb{C}\}$, because the absolute values of its elements are bounded by $\|f\|_q\|x\|(b-a)^{1/p}$. Hence $|F([\{(f, u_n)\}_{n\in\mathbb{N}}])| \leq \|F\|\|f\|_q\|x\|(b-a)^{1/p}$, and $f \mapsto F([\{(f, u_n)\}_{n\in\mathbb{N}}])$ defines a bounded antilinear form on $E_p^* = L^q(I, X^*)$. Therefore, by the reflexiveness of the spaces involved and application of Riesz' Lemma, there exists a $u \in E_p$ such that $F([\{(f, u_n)\}_{n\in\mathbb{N}}]) = (f, u)$ for all $f \in L^q(I, X^*)$.

Now one shows that the function u solves the nonautonomous Cauchy problem (2.2) in the weak sense. Let $f \in \mathcal{T}$. Define $\Delta_n f$ and $T_n f \in L^q(I, X^*)$ by

$$\Delta_n f(t) = \delta_n^{-1}\left(f(t_\nu^{(n)}) - f(t_{\nu-1}^{(n)})\right) \quad \text{and} \quad T_n f(t) = f(t_\nu^{(n)})$$

for $t \in [f(t_{\nu-1}^{(n)}), f(t_\nu^{(n)})]$. By the assumptions on f, one has $\lim_{n\to\infty} T_n f = f$, $\lim_{n\to\infty} T_n A(\cdot) f = A(\cdot) f$ and $\lim_{n\to\infty} \Delta_n f = \dot{f}$ in the strong sense.

Because the norm on the space of sequences depends on a limes superior, one can use the approximate expressions instead of the original ones in the argument involving the linear form F:

$$(\dot{f}, u) = F([\{(\dot{f}, u_n)\}_{n\in\mathbb{N}}]) = F([\{(\Delta_n f, u_n)\}_{n\in\mathbb{N}}]).$$

Moreover, we calculate

$$(\Delta_n f, u_n) = -(f(a), x) - \sum_{\nu=1}^{n} \delta_n \left(A^*(t_\nu^{(n)}) f(t_\nu^{(n)}), W_{n,\nu} x\right).$$

By assumption, $t \mapsto A^*(t) f(t)$ is continuous and one gets for the equivalence classes,

$$\left[\left\{\sum_{\nu=1}^{n} \left(A^*(t_\nu^{(n)}) f(t_\nu^{(n)}), W_{n,\nu} x\right) \delta_n\right\}_{n\in\mathbb{N}}\right] = \left[\left\{\sum_{\nu=1}^{n} \left(A^*(t_\nu^{(n)}) f(t_\nu^{(n)}), W_{n,\nu-1} x\right) \delta_n\right\}_{n\in\mathbb{N}}\right],$$

because the difference of both sequences converges to zero. With this equality we find

$$\left[\left\{\sum_{\nu=1}^{n} \left(A^*(t_\nu^{(n)}) f(t_\nu^{(n)}), u_n\right) \delta_n\right\}_{n\in\mathbb{N}}\right] = \left[\left\{\left(T_n A^*(\cdot) f, u_n\right)\right\}_{n\in\mathbb{N}}\right] = \left[\left\{\left(A^*(\cdot) f, u_n\right)\right\}_{n\in\mathbb{N}}\right],$$

and one arrives at the assertion:

$$
\begin{aligned}
(\dot{f}, u) &= F([\{(\Delta_n f, u_n)\}_{n\in\mathbb{N}}]) \\
&= -F([\{(f(a), x)\}_{n\in\mathbb{N}}]) - F([\{\sum (A^*(t_\nu^{(n)}) f(t_\nu^{(n)}), u_n)\delta_n\}_{n\in\mathbb{N}}]) \\
&= -(f(a), x) - F([\{(A^*(\cdot) f, u_n)\}_{n\in\mathbb{N}}]) \\
&= -(f(a), x) - (A^*(\cdot) f, u) \\
\iff \int_a^b (\dot{f}(t), u(t))\, dt &+ \int_a^b (A^*(t) f(t), u(t))\, dt + (f(a), x) = 0.
\end{aligned}
$$

□

Evidently, the crucial point is to ensure that the test function space $\mathcal{T}$ is sufficiently large, at least not trivial. This gives the important conditions on the generators $A(t)$.

The theorem ensures the existence of weak solutions for a wide class of nonautonomous evolution equations. However, we have no information on uniqueness and regularity properties of the solution. In particular, it is not clear if the solution can be interpreted in terms of a propagator. Hence, in the present form this approach to nonautonomous evolution equations is not appropriate for our purpose.

A more recent result on weak solvability is for instance [5]. Although there are some statements about uniqueness and regularity, the context of this work is different from ours. For example, the domains of the generators are assumed to be closed subspaces. It does not seem to be possible to use similar methods to the setting we have in mind. A good starting point for a systematic discussion about results on weak methods mainly in the parabolic case is Carroll's book [12].

In the following, we return to stronger notions of solvability which come with uniqueness and regularity statements for the solutions.

2.2.3 Bounded generators

If the generators $A(t)$ are bounded operators, it is easy to solve the nonautonomous Cauchy problem (2.2). The method is closely related to the Dyson-Phillips expansion [20, 68].

Theorem 2.18. *Let X be a Banach space and let $A(t)$ be bounded operators for $t \in I$, where $I \subset \mathbb{R}$ is a compact interval with $s \in I$. Assume that $t \mapsto A(t)$ is a strongly continuous function. Then for every $x \in X$ there is a solution $t \mapsto u(t)$ of the nonautonomous Cauchy problem* (2.2).

Proof. We cite the proof of [67, Theorem 5.1]. Note that in this theorem continuity of $t \mapsto A(t)$ in the operator norm is assumed. However, this assumption is not necessary for the argument.

Set $I = [0,T]$ and $\alpha = \max_{t\in I} \|A(t)\|$. Define a mapping $S : C(I,X) \to C(I,X)$ by

$$(Su)(t) = x + \int_s^t A(\tau)u(\tau)\, d\tau. \tag{2.7}$$

Clearly,

$$\|Su(t) - Sv(t))\| \le \alpha(t-s)\|u-v\|_\infty,$$

and, by induction,

$$\|S^n u(t) - S^n v(t)\| \le \frac{1}{n!}\alpha^n (t-s)^n \|u-v\|_\infty.$$

Hence, for n large enough, $b = \frac{1}{n!}\alpha^n(T-s)^n < 1$ and

$$\|S^n u - S^n v\|_\infty \le b\|u-v\|_\infty,$$

so a generalization of Banach's contraction principle for the vector-valued context implies the existence of a unique fixed point $u \in C(I,X)$ which obeys

$$u(t) = x + \int_s^t A(\tau)u(\tau)\, d\tau. \tag{2.8}$$

The continuity assumptions on $A(\cdot)$ and $u(\cdot)$ together with the estimate $\|A(t)u(t) - A(s)u(s)\| \le \|(A(t) - A(s))u(t)\| + \|A(s)(u(t) - u(s))\|$ result in the differentiability of the right-hand side of (2.8). Therefore u is the unique solution of the nonautonomous Cauchy problem (2.2). □

2.2.4 The Theorem of Kato

The first article of T. Kato concerning the nonautonomous Cauchy problem goes back to 1953 [50]. In 1970, Kato published a more rigorous and detailed version of his ideas [51]. Kato's work is still the reference method for the treatment of the wellposedness problem for nonautonomous evolution equations of hyperbolic type, hence we cite the main statements in detail. Although there are lots of extensions and improvements of the original result, the main assumptions remain essentially unchanged, see [79].

The Theorem of Kato allows nonconstant domains of definition $D(A(t))$, but the intersection of these domains for different times has to contain a joint dense subspace with special features leading to the invariance of the subspace with respect to the propagator.

For the proofs of all statements in this section which we do not give explicitly we refer to Pazy's monograph [67, Chapter 5]. Another reference is Tanabe's book [90]. In the following, we will define *admissible subspaces* and *Kato-stable* families of operators before we come to the existence theorem in Section 2.2.4.3.

2.2.4.1 Admissible subspaces

Let A be a linear operator on a Banach space X and let Y be a subspace of X.

Definition 2.19. Y is an *invariant subspace* of A if A maps $D(A) \cap Y \to Y$.

Definition 2.20. The *part of A in Y* is the linear operator $\tilde{A}$ given by $\tilde{A}y = Ay$ on the domain $D(\tilde{A}) = \{y \in D(A) \cap Y : Ay \in Y\}$.

The restriction $A\restriction_Y$ is an extension of $\tilde{A}$. We have $A\restriction_Y = \tilde{A}$ if Y is an invariant subspace.

In the following we assume that Y is a Banach space with norm $\|\cdot\|_Y$ and Y is continuously embedded in X, that is the norm $\|\cdot\|_Y$ is stronger than the norm of X: There is a constant c such that $\|y\| \le c\|y\|_Y$ for $y \in Y$.

Definition 2.21. Let $(T(t))_{t\ge0}$ be a strongly continuous semigroup with generator A. The subspace Y is *A-admissible* if it is an invariant subspace of $T(t),\ t \ge 0$, and the restriction of $(T(t))_{t\ge0}$ to Y is again a semigroup in Y, strongly continuous with respect to $\|\cdot\|_Y$.

Criteria for admissible subspaces are given in the following theorems.

Theorem 2.22. *Let $(T(t))_{t\ge0}$ be a strongly continuous semigroup with generator A and $(\omega,\infty) \subset \rho(A)$, $\omega > 0$. The subspace Y is A-admissible if and only if Y is an invariant subspace of $R(\lambda, A)$, $\lambda > \omega$, and $\tilde{A}$, the part of A in Y, generates a strongly continuous semigroup in Y. Moreover, if Y is A-admissible, then $\tilde{A}$ is the infinitesimal generator of $(T(t))_{t\ge0}\restriction_Y$.*

Theorem 2.23. *Let Y be dense in X and let $S : Y \to X$ be an isomorphism. The subspace Y is A-admissible if and only if $A_1 = SAS^{-1}$ is the generator of a strongly continuous semigroup $(T_1(t))_{t\ge0}$. The semigroup is given by $T_1(t) = ST(t)S^{-1}$.*

For Hilbert spaces there is a related result due to Okazawa [65].

Theorem 2.24. *Let A be a linear operator in a Hilbert space X and $c \ge b \in \mathbb{R}$. Assume, that the closure of $A + b$ is maximally accretive. Let S be a self-adjoint, strictly positive operator with $D(S) \subset D(A)$ such that $D(S)$ is a core for the closure of $A + b$ and*

$$\mathrm{Re}(Au, Su) \ge -c(u, Su)$$

for all $u \in D(S)$. Then $D(S^{1/2})$ is A-admissible.

For the proof see [65]. Under the assumptions of the last theorem one can think of $S^{1/2}AS^{-1/2} + c$ as being maximally accretive.

2.2.4.2 Stable families of generators

Now let $\{A(t)\}_{t\in[0,T]}$ be a family of generators of strongly continuous semigroups on a Banach space X.

Definition 2.25. The family $\{A(t)\}_{t\in[0,T]}$ of generators of strongly continuous semigroups is said to be *stable* or *Kato-stable* if for any finite family $\{t_j\}$ with $0 \leq t_1 \leq \cdots \leq t_k \leq T$, $k \in \mathbb{N}$, there are positive constants M, ω such that $(\omega,\infty) \subset \rho(A(t))$ for all $t \in [0,T]$ and

$$\left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda-\omega)^{-k}, \quad \lambda > \omega. \tag{2.9}$$

An example for Kato-stable operators is easily obtained: Every family $\{A(t)\}$ of maximally dissipative operators fulfills the Definition 2.25 because of the Hille-Yosida theorem. An alternative formulation of Kato-stability due to H. Neidhardt is given in the following Lemma.

Lemma 2.26. *The family $\{A(t)\}_{t\in[0,T]}$ is Kato-stable with constants M,ω if and only if there are norms $\|\cdot\|_t$ on X and a constant $M \geq 1$ such that $\|x\| \leq \|x\|_t \leq \|x\|_s \leq M\|x\|$ and $\|R(\lambda, A(t))x\|_t \leq (\lambda-\omega)^{-1}\|x\|_t$ for $(t,s) \in D_I, \lambda > \omega$ and $x \in X$.*

Proof. Apply a rescaling $A(t) \to A(t)-\omega$ and observe that Kato-stability can be formulated in terms of the semigroups instead of resolvents ([67, Theorem 5.2.2]). Then the assertion follows from [64, Proposition 1.3]. □

Concerning stability in subspaces we have the following theorem:

Theorem 2.27. *Let $Y \subset X$ be a Banach space, continuously embedded in X. Assume there is a family of isomorphisms $Q(t) : Y \to X$, $t \in [0,T]$, such that $Q(t) \in \mathcal{B}(Y,X)$ and $Q^{-1}(t) \in \mathcal{B}(X,Y)$ are uniformly bounded by a constant c and the map $t \mapsto Q(t)$ is of bounded variation in the norm of $\mathcal{B}(Y,X)$. Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of generators and set $A_1(t) = Q(t)A(t)Q^{-1}(t)$. If $\{A_1(t)\}_{t\in[0,T]}$ is a stable family of generators in X, then Y is $A(t)$-admissible for $t \in [0,T]$, and $\{\tilde{A}(t)\}_{t\in[0,T]}$ is a stable family of generators in Y.*

The next theorem gives a perturbation result.

Theorem 2.28. *Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of generators of strongly continuous semigroups with stability constants M,ω, and let $\{B(t)\}_{t\in[0,T]}$ be a family of bounded linear operators such that $\|B(t)\| \leq c$ uniformly in t. Then $\{A(t)+B(t)\}_{t\in[0,T]}$ is a stable family of generators of strongly continuous semigroups with stability constants $M,\omega+cM$.*

2.2.4.3 The existence theorem

Now we can state Kato's existence theorem.

Theorem 2.29. *Let X and Y be Banach spaces, $Y \subset X$ dense and continuously embedded. For every $t \in [0,T]$ let $A(t)$ be the generator of a strongly continuous semigroup. Assume that:*

(i) $\{A(t)\}_{t\in[0,T]}$ *is a stable family with constants* M, ω.

(ii) Y *is* $A(t)$*-admissible for every* $t \in [0,T]$. *Moreover,* $\{\tilde{A}(t)\}_{t\in[0,T]}$, *the family given by the parts of* $A(t)$ *in* Y, *is a stable family in* Y *with constants* $\tilde{M}, \tilde{\omega}$.

(iii) $Y \subset D(A(t))$ *for every* $t \in [0,T]$. $A(t)$ *as an operator from* $Y \longrightarrow X$ *is bounded. The function* $t \mapsto A(t)$ *is norm-continuous with respect to* $\mathcal{B}(Y,X)$.

Then there exists a unique propagator $U(t,s), (t,s) \in D_{[0,T]}$, *such that*

(a) $\|U(t,s)\| \leq Me^{\omega(t-s)}$,

(b) the right derivative $\frac{\partial}{\partial t}^{+} U(t,s)y|_{t=s} = A(s)y$ *for every* $y \in Y, s \in [0,T]$,

(c) $\frac{\partial}{\partial s} U(t,s)y = -U(t,s)A(s)y$ *for every* $y \in Y, 0 \leq s \leq t \leq T$.

The derivatives are taken in the strong sense in X.

This theorem enables us to define a unique propagator associated with the family of generators $\{A(t)\}_{t\in[0,T]}$, but it is not strong enough to establish the classical wellposedness of the nonautonomous Cauchy problem (2.2). This is the point of the next theorem, where the solutions even take values in Y if the initial value stems from Y.

Theorem 2.30. *Given the assumptions of Theorem 2.29, where* (ii) *is replaced by*

(ii') There is a family of isomorphisms $\{Q(t)\}$ *where* $Q(t) : Y \to X$ *such that for every* $y \in Y, t \mapsto Q(t)y$ *is continuously differentiable in* X *and* $Q(t)A(t)Q^{-1}(t) = A(t) + B(t)$ *for a family* $\{B(t)\}$ *of bounded operators which is continuous in* t.

Then the conclusion of Theorem 2.29 holds true. Moreover,

(d) $U(t,s)Y \subset Y$,

(e) $(t,s) \mapsto U(t,s)y$ *is continuous in* Y *for* $y \in Y$.

This implies that for $y \in Y$, $u(\cdot) = U(\cdot,s)y \in C^1((s,T],X)$ *solves the nonautonomous Cauchy problem* (2.2) *with* $u(s) = y$.

Kato proves these results by approximation of the generators $A(t)$ and solving the corresponding Cauchy problem (2.2). The strategy is similar to the proof of Theorem 2.17, but, instead of an abstract existence argument based on Riesz' Theorem, a limit calculation with Duhamel's integral formula is used. It is possible to reproduce this proof using evolution semigroups and an approach similar to Theorem 3.10. This is done in G. Nickel's thesis [62].

With regard to the application in quantum field theory, we remark that Kato's Theorem is not very well-suited. In the simple $P(\varphi)_2$ models there are indeed common dense cores Y for the Hamiltonians $H(t)$, but invariance of these subspaces is in general problematic, due to the lacking smoothing properties of the time evolution. Even if one could establish invariance, Kato-stability in Y seems unlikely.

2.2.5 Time-independent domain

A simple reformulation of Theorem 2.30 is possible if the generators $A(t)$ have a common time-independent domain. This theorem can also be found in the books of Yosida [99] and Reed and Simon [71] with independent proofs. For Hilbert spaces it is possible to apply the result using a scale of spaces and thus to extend it to the case where the domain of a *quadratic form* is independent of time. This idea is due to Kisyński [52]. It allows for an application to a special situation of the $(\varphi^4)_2$ model, but it is not sufficient for the proof of the existence of local scattering operators.

Theorem 2.31. *Let X be a Banach space and $I \subset \mathbb{R}$ an open interval. Let $A(t)$ be maximally dissipative and assume that $0 \in \rho(A(t))$ for every $t \in I$. Moreover, assume that*

(i) the operators $A(t)$, $t \in I$, have a common domain of definition, $D(A(t)) = D$;

(ii) for each $x \in X$, $(t,s) \mapsto (t-s)^{-1}\left(A(t)A(s)^{-1} - 1\right)x$ is uniformly strongly continuous for $t \neq s$ in any fixed subinterval of I;

(iii) $\lim_{s\to t}(t-s)^{-1}\left(A(t)A(s)^{-1} - 1\right)x$ exists uniformly for t in every fixed subinterval of I. Moreover, the limit is bounded and continuous in t.

Then there exists a strongly continuous, bounded propagator $U(t,s)$ such that $t \mapsto U(t,s)x$ solves the nonautonomous Cauchy problem (2.2) *for all initial values $x \in D$.*

Proof. The operators $A(t)$ are maximally dissipative, hence Kato-stable. By assumption they are isomorphisms from $D \to X$. The conditions (ii) and (iii) imply the continuous differentiability of $t \mapsto A(t)$ as a mapping with values in $\mathcal{B}(D,X)$. So we can set $Q(t) = A(t)$ and apply Theorem 2.30 with $B(t) = 0$. □

In the case of a Hilbert space X the assumptions are fulfilled if for instance $A(t) = -iH(t) = -i(H_0 + V(t))$, H_0 is self-adjoint, $V(\cdot) : \mathbb{R} \to \mathcal{B}(X)$ and $V(t)$ maps $D(H_0)$ into $D(H_0)$, $[H_0, V(t)]$ is a bounded operator and $t \mapsto \|[H_0, V(t)]\|$ is locally bounded, see [71].

2.2.5.1 Application: The Theorem of Kisyński

As an application of the existence theorem for time-independent domains, we present a theorem due to Kisyński [52] which gives conditions of solvability for the nonautonomous Cauchy problem (2.2) in a Hilbert space if the form domain of $H(t)$ is constant. Note that the domain of definition of $H(t)$ as an operator in $\mathcal{H}$ may well depend on time. The price we have to pay for the improvement over Theorem 2.31 is that the solution of the evolution equation takes its value in a much larger space than the original Hilbert space. We present a simplified version of Kisyński's Theorem, which nevertheless shows the underlying idea.

Theorem 2.32. *Let H be a positive self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$ the scale of spaces with respect to H. For $t \in [0,T]$, $T > 0$, let $H(t)$ be a symmetric bilinear form on $\mathcal{H}_{+1} \times \mathcal{H}_{+1}$ such that*

$$c^{-1}(H+1) \leq H(t) + 1 \leq c(H+1) \tag{2.10}$$

for a constant $c > 0$. Suppose further that $H(t) \in \mathcal{B}(\mathcal{H}_{+1}, \mathcal{H}_{-1})$ is continuously differentiable in norm and $\pm\dot{H}(t) \leq c(H+1)$. Then there is a unitary propagator $U(t,s)$ on $\mathcal{H}$ such that for $x \in \mathcal{H}_{+1}$, $t \mapsto U(t,s)x \in \mathcal{H}_{+1}$ is continuous in t and such that for $x \in \mathcal{H}_{+1}$ one has $\frac{d}{dt}U(t,s)x = -iH(t)U(t,s)x$.

Proof. The main idea is contained in [52, Chapter 7]. The space $\mathcal{H}_{+1}$ is a Hilbert space; it equals $D(H^{1/2})$ together with the scalar product $(\cdot,\cdot)_{+1} = (\cdot,(H+1)\cdot)$ which induces a norm $\|\cdot\|_{+1}$. By equation (2.10), the bilinear form $(\cdot,\cdot)_t = (\cdot,(H(t)+1)\cdot)$ induces a norm $\|\cdot\|_t$ on $\mathcal{H}_{+1}$ which is equivalent to $\|\cdot\|_{+1}$. Hence the bilinear form $H(t)$ is closed as it gives an equivalent scalar product on $\mathcal{H}_{+1}$. The associated operator $H(t) : \mathcal{H}_{+1} \to \mathcal{H}_{-1}$ has the domain of definition $D(H(t)) = \{x \in \mathcal{H}_{+1} : y \mapsto (y,x)_t \text{ is continuous in } \mathcal{H}_{-1}\}$. Again by (2.10) we conclude that $D(H(t)) = \mathcal{H}_{+1}$, because by definition $\mathcal{H}_{-1}$ consists of the conjugate linear forms on $\mathcal{H}_{+1}$. By the form representation theorem, $H(t)$ is a self-adjoint, positive operator which maps $\mathcal{H}_{+1} \to \mathcal{H}_{-1}$.

Now set $X = \mathcal{H}_{-1}, D = \mathcal{H}_{+1}, A(t) = -iH(t)$, then the assumptions on the differentiability of $H(t)$ as a bounded operator from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$ imply that the assumptions of Theorem 2.31 are fulfilled and the assertions follow. □

In [18], J. Dimock investigates the $(\varphi^4)_2$ model with a coupling of the form $g = g_0 + g_1(t,x)$, where g_0 is a positive coupling constant and $g_1 \in C_c^\infty(\mathbb{R}^2)$ is small and localized: It is assumed that $(\operatorname{diam}(\operatorname{supp} g_1) + 1)\|g_1\|_\infty$ is sufficiently small. With this assumption it is possible to use Kisyński's Theorem in the aforementioned formulation to prove the existence of the time evolution. Dimock uses this fact to establish the time evolution for the $(\varphi^4)_2$ model on a two-dimensional, curved spacetime. We return to this topic in Section 3.3.2.

The crucial point about the splitting of the coupling into a constant and a small, localized term is that the scale of spaces $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$ is built with respect to the Hamiltonian $H = H_0 + g_0 \int : \varphi^4(x) : \, dx$ which describes a fixed φ^4 background. The background Hamiltonian is understood either to be contained in a sufficiently large box or it may result from a Euclidean construction, hence the underlying Hilbert space is not the Fock space of the free fields. In any case, the approach is not well-suited to tackle the existence question of local scattering operators. It is not possible to choose $g_0 = 0$ and to use Kisyński's Theorem with the space triple corresponding to the free Hamiltonian. The problem is that the bilinear form $H(t) = H_0 + \int g_1(t,x) : \varphi^4(x) : \, dx$ is not closed on $\mathcal{H}_{+1} = D(H_0)$. Furthermore, we did not succeed in performing the limit $g_0 \to 0$ in a way which would allow to patch together the time evolution inside of $\operatorname{supp} g_0$ and the free time evolution outside. A similar problem occurs in Section 4.1, where we are in fact able to make the interacting background arbitrarily small, but it remains impossible to connect to the case of a vanishing background in a continuous way.

We overcome these difficulties in Section 3.3 and Section 4.3 by applying a new wellposedness result. It is this result which enables us to show the existence of the time evolution for $P(\varphi)_2$ for curved spacetimes without the restrictions on g_1 as well as to prove the existence of the local scattering operators for $P(\varphi)_2$ models.

Chapter 3

Evolution Semigroups

Kato's approach to the Cauchy problem for hyperbolic nonautonomous evolution equations is inspired by the theory of ordinary differential equations. The strategy is to approximate the evolution equation by a sequence of equations which are easy to solve. Then the convergence of the solution operators is investigated.

There is a different approach to the present problem. Similar to the abstract methods for the study of the existence of weak solutions which are developed by Lions (see e.g. [12]), the evolution equation (2.2) is considered as a functional equation involving an operator sum.

In [13], DaPrato and Grisvard investigate the sum $-\frac{d}{dt} + A(\cdot)$ as an operator on the function spaces $E_p = L^p([0,T],X)$ or $C([0,T],X)$ with a suitable domain of definition. They derive conditions for the operator sum being densely defined and closable and apply their theory to parabolic and hyperbolic evolution equations. These ideas were improved in a paper by DaPrato and Iannelli [14]. Here the inhomogeneous Cauchy problem, $\dot{u}(t) - A(t)u(t) = f(t)$, $u(0) = x$, is mapped on the equation $Gu := \{\dot{u} - A(\cdot)u, u(0)\} = \{f, x\}$ and conditions implying invertibility of G are studied. The results extend Kato's theory in some respects, but the main restrictions which limit the usefulness of Kato's results in the context of quantum field theory do persist.

In this chapter we consider the so-called *evolution semigroups*. By functional methods similar to the aforementioned, it is possible to relate the nonautonomous Cauchy problem to an autonomous one. This idea goes back to J. Howland [47] for the study of scattering theory. It was generalized by D. Evans [24] and others, see [79]. The attractive feature of evolution semigroups is that the powerful theory of strongly continuous operator semigroups applies. From this one is enabled to derive results for the time-dependent situation. In recent years evolution semigroups have attracted new interest: They turned out to be useful for the study of spectral and asymptotic properties of propagators. The reader is again referred to [79] for a survey.

However, most of the results deal with the parabolic situation, and the question of existence of solutions has not been the subject of the main efforts. This is the starting point for the present work.

In the following, we define evolution semigroups and describe their properties. We show that the existence theory for solutions of the nonautonomous Cauchy problem (2.2) can be related to properties of generators of evolution semigroups. In Section 3.2 we see that the generator property of the closure of the operator $G_0 = -\frac{d}{dt} + A(\cdot)$ implies the existence of unique mild solutions of the nonautonomous Cauchy problem (2.2), and we investigate their relation to weak solutions. In the Hilbert space case, Section 3.2.2 contains sufficient requirements for this theory of solvability. In Section 3.3 we generalize these considerations to the situation where the closure of G_0 is not necessarily a generator. Under quite general assumptions, G_0 has an extension which is an evolution generator and which corresponds to a certain limit of bounded operators. For Hilbert spaces, the evolution groups associated to this extension lead to unique approximative solutions of the nonautonomous Cauchy problem (2.2). This is a new existence and uniqueness result for approximative solutions. It is this approach which we will find applicable to the quantum field theoretical problem we have in mind.

The main results of this chapter are obtained in collaboration with R. Schnaubelt and can be found in [77].

3.1 Definition and properties

For the definition of evolution semigroups as well as for their basic properties we follow the presentation in [78, 77].

Given an arbitrary interval $I \subset \mathbb{R}$ and a Banach space X, we define the function spaces $E_p = L^p(I, X)$, $1 \leq p < \infty$, with the usual L^p-norm as well as $E_\infty = C(I, X)$ with the sup-norm. For details on this kind of spaces of Banach-space-valued functions see [17, 100].

Now let $\{U(t,s)\}_{(t,s)\in D_I}$ be a strongly continuous, exponentially bounded propagator according to Definition 2.10. The *evolution semigroup* $(T(\sigma))_{\sigma\geq 0}$ on E_p is the strongly continuous semigroup formally defined by

$$(T(\sigma)f)(t) = \begin{cases} U(t, t-\sigma)f(t-\sigma) & \text{if } t, t-\sigma \in I, \\ 0 & \text{if } t \in I, t-\sigma \notin I. \end{cases} \tag{3.1}$$

We denote its generator by G.

The semigroup property of $(T(\sigma))_{\sigma\geq 0}$ is obvious. In the following we show strong continuity. To this end we consider $p < \infty$ and $p = \infty$ separately. It is sufficient to consider left half-open intervals: Assume that I is left closed and denote its left endpoint by a. Set $I' = I \setminus \{a\}$ if I is bounded from below and $I' = \mathbb{R}$ if $I = \mathbb{R}$. Given an exponentially bounded propagator on D_I, its restriction on $D_{I'}$ induces the same evolution semigroup. But in general a propagator on $D_{I'}$ has no continuous extension to $D_{\overline{I'}} = D_I$, as seen in the example $X = \mathbb{C}, I = (0, \infty)$

and $U(t,s) = p(t)/p(s)$ for $p(t) = 2 + \sin(1/t)$ [70]. We denote by $C_{c,I'}(I,X)$ the set of continuous functions compactly supported in I'.

Theorem 3.1. *Equation* (3.1) *defines a strongly continuous semigroup on E_p, the* evolution semigroup $(T(\sigma))_{\sigma\geq 0} = (e^{\sigma G})_{\sigma \geq 0}$.

Proof. First we consider the case $p = \infty$. Let $f \in C_{c,I'}(I,X)$. Boundedness of the propagator $U(t,s)$ in norm (by $Me^{\omega(t-s)}$) implies $\|T(\sigma)f\|_\sigma \leq Me^{\omega\sigma}\|f\|_\infty$, so $(T(\sigma))_{\sigma\geq 0}$ is a family of bounded operators. Furthermore, we see that $\|T(\sigma)f - f\| \to 0$ for $\sigma \to 0$ because of the uniform continuity of the function $(t,s) \mapsto U(t,s)x$ on compact sets. We conclude that $(T(\sigma))_{\sigma\geq 0}$ is a strongly continuous semigroup. This extends to $C_{0,I'}(I,X)$.

Second, let $1 \leq p < \infty$ and $f \in E_p$. We observe that $C_{c,I'}(I,X)$ is dense in E_p. Denote by τ_σ the semigroup of right translations on E_p: $(\tau_\sigma f)(t) = \chi_I(t-\sigma)f(t-\sigma)$. The function $T(\sigma)f$ is measurable and $\|(T(\sigma)f)(t)\|_X \leq Me^{\omega\sigma}\|(\tau_\sigma f)(t)\|_X$. So $T(\sigma)f \in E_p$ and $\|T(\sigma)\| \leq Me^{\omega\sigma}$. Furthermore we find that $T(\sigma)$ converges to $\mathbb{1}$ on $C_{c,I'}(I,X)$ for $\sigma \to 0$. Hence, $T(\sigma) \to \mathbb{1}$ as an operator on E_p and strong continuity follows. □

Note that the translations τ_σ constitute the evolution semigroup on E_p which is associated with the trivial propagator $U(t,s) = \mathbb{1}$.

We remark that the spaces $E_p = L^p(I,X)$ are not the most general function spaces which are admissible for the definition of evolution semigroups. For a given Banach function space F over a measure space (Ω, Σ, μ), we define the space $F(X)$ as the set of strongly measurable functions $f : \Omega \to X$ with the property that $\|f(\cdot)\|_X \in F$ with the Banach space norm $\|f\|_{F(X)} = \|\|f(\cdot)\|_X\|_F$. The conditions we have to impose on F are translation invariance, strong continuity of the semigroup of translations τ_σ and order continuity of the norm on E_p. For details see [69]. Examples for admissible spaces F are $L^p(\mathbb{R}) \cap L^q(\mathbb{R}), 1 \leq p,q < \infty$ or the Lorentz spaces $L_{p,q}(\mathbb{R}), 1 < p < \infty, 1 \leq q < \infty$.

We state several simple properties of evolution semigroups which follow easily from the definition (3.1). The operator norm of the evolution semigroup can be obtained via the propagators

$$\|T(t)\|_{\mathcal{B}(E)} = \sup_{s,s-t\in I} \|U(s,s-t)\|_{\mathcal{B}(X)} \tag{3.2}$$

for $t \geq 0$. Hence, by the exponential boundedness of the propagators, we conclude $\lambda \in \rho(G)$ if $\operatorname{Re}\lambda > \omega$ for the exponent $\omega \in \mathbb{R}$ in (2.10). If I is a bounded interval, the choice $\omega = 0$ is possible and it follows $\rho(G) = \mathbb{C}$. This fact will be of importance later.

The Laplace transform of $T(\cdot)$ is the resolvent of G. Thus, by a change of variables, we get the formula

$$(R(\lambda,G)f)(t) = \int_a^t e^{-\lambda(t-s)}U(t,s)f(s)\,ds, \quad t \in I, \operatorname{Re}\lambda > w,\ f \in E. \tag{3.3}$$

We recall that a strongly continuous semigroup $S(\cdot)$ with generator B can be embedded in a strongly continuous group if some operator $S(t)$, $t > 0$, is invertible, see e.g. [67, §1.6]. Moreover, $(S(t)^{-1})_{t\geq 0}$ is then generated by $-B$. An evolution semigroup on a function space on a bounded interval I is never invertible because of the boundary condition. For the case of an unbounded time interval, we get the following relation between the propagator and the evolution semigroup.

Lemma 3.2. *Let $U(\cdot,\cdot)$ be a propagator with time interval $I = \mathbb{R}$ and let $T(\cdot)$ be the associated evolution semigroup on E_p, $1 \leq p \leq \infty$. Then the following assertions hold.*

(i) $T(t)$ is an isometry for some $t > 0$ if and only if $U(s, s-t)$ is an isometry for $s \in \mathbb{R}$ and any $t > 0$.

(ii) $T(t)$ is invertible for some (and hence for all) $t > 0$ if and only if $U(\cdot,\cdot)$ is invertible. Then $(T(t)^{-1}f)(s) = U(s, s+t)f(s+t)$, $s \in \mathbb{R}$.

(iii) Let $p = 2$ and X be a Hilbert space. Then $T(t)$ is unitary if and only if $U(s, s-t)$ is unitary for $s \in \mathbb{R}$.

Proof. We observe that assertion (iii) follows from (i) and (ii). In (i) and (ii) the implications '$\Leftarrow$' are easy to check. The converse implications are shown for $p \in [1,\infty)$; the case $p = \infty$ can be established in a similar way.

Assume that $T(\tau)$ is an isometry for some $\tau > 0$. Take $x \in X$, $s \in \mathbb{R}$, and $\varepsilon > 0$. Set $f = \chi_{[s-\tau, s-\tau+\varepsilon]}x$ with the characteristic function χ_J of an interval $J \subset \mathbb{R}$. Then we obtain

$$\|x\|^p = \frac{1}{\varepsilon}\,\|f\|_p^p = \frac{1}{\varepsilon}\int_{\mathbb{R}} \|(T(\tau)f)(\sigma)\|^p\,d\sigma = \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \|U(\sigma,\sigma-\tau)x\|^p\,d\sigma. \tag{3.4}$$

In the limit $\varepsilon \to 0$, we arrive at $\|x\| = \|U(s, s-\tau)x\|$. Thus (i) holds.

Assume that $T(\tau)$ is invertible for some $\tau > 0$. There are constants $M' \geq 0$ and $w' \in \mathbb{R}$ such that $\|f\|_p \leq M'e^{w'\tau}\|T(\tau)f\|_p$ for $f \in E_p$. As in (3.4), one verifies that $\|x\| \leq M'e^{w'\tau}\|U(s, s-\tau)x\|$ for $s \in \mathbb{R}$ and $x \in X$. Observe that $T(\tau)D(G) = D(G)$. As stated below in Proposition 3.5, $D(G)$ is dense in $C_0(\mathbb{R}, X)$. Hence, $T(\tau)D(G)$ is dense in $C_0(\mathbb{R}, X)$ so that $U(s, s-\tau)$ has dense range for $s \in \mathbb{R}$. Therefore we have established assertion (ii). □

To address the question of wellposedness of evolution equations with evolution semigroups as our main tool, we need an abstract characterization of these special semigroups without referring to a propagator. The idea for this abstract characterization goes back to the original paper of Howland [47]. Let X be a Hilbert space and $\varphi \in C^1(\mathbb{R})$ be a continuously differentiable function. Assume that there is a skew-adjoint operator G on $L^2(\mathbb{R}, X)$ such that $\varphi D(G) \subset D(G)$

and $G(\varphi f) - \varphi G f = -\dot{\varphi}$ for all $f \in D(G)$. Moreover, let Q be the multiplication operator which is induced by the variable t, that is $(Qf)(t) = tf(t)$ for $f \in D(Q)$. Clearly, Q is self-adjoint. Howland observes that G has the same commutation relation with Q as $-\frac{d}{dt}$. Hence iG and Q constitute a representation of the canonical commutation relations. In the Weyl form this means

$$e^{\sigma G} e^{-i\sigma' Q} = e^{i\sigma\sigma'} e^{-i\sigma' Q} e^{\sigma G}$$

for $\sigma, \sigma' \in \mathbb{R}$, see e.g. [21]. By uniqueness of the Schrödinger representation of the CCR, there is a unitary multiplication operator U on $L^2(\mathbb{R}, X)$ such that $UQU^* = Q$ and $e^{\sigma G} = U\tau_\sigma U^*$. Setting $U(t,s) := U(t)U(s)^*$, we find

$$(e^{\sigma G} f)(t) = U(t, t-\sigma) f(t-\sigma),$$

thus, according to Theorem 3.1, $(e^{\sigma G})_{\sigma \geq 0}$ is an evolution semigroup. Further generalizations of this idea can be found in [24, 60, 69].

Crucial for this characterization is the notion of a multiplication operator. For $p = \infty$, a bounded operator $M \in \mathcal{B}(E_p)$ is a multiplication operator if and only if $M\varphi f = \varphi M f$ for all $\varphi \in C_b(\mathbb{R})$ and $f \in E_p$, see [24]. For $p < \infty$ and without assuming separability of X, we need special subspaces [60, 69].

Definition 3.3. Let $\mathcal{M} \in \mathcal{B}(E_p), 1 \leq p < \infty$. A subspace F of E_p is called $\mathcal{M}$*-determining* if

(i) F and $\mathcal{M}F$ consist of continuous functions,

(ii) $F^2 = \{f(s) : f \in F\}$ is dense in X,

(iii) F is dense in E_p.

Now we can characterize multiplication operators in a similar way as in the situation of spaces of continuous functions.

Theorem 3.4. *Let $I \subset \mathbb{R}$ be an interval. Consider $\mathcal{M} \in \mathcal{B}(E_p), 1 \leq p < \infty$. Then $\mathcal{M} = M(\cdot) \in C_b(I, \mathcal{B}(X))$ is an operator-valued function, continuous in the strong topology if and only if there is an $\mathcal{M}$-determining subspace $F \subset E_p$ such that $\mathcal{M}(\varphi f) = \varphi \mathcal{M}(f)$ for all $f \in F$ and $\varphi \in L^\infty(I)$.*

For the proof see [69]. Now, Howland's idea for the characterization of evolution semigroups should be compatible with the Definition 2.10 of propagators. Therefore, one has to include a condition which leads to strong continuity of $(t,s) \mapsto U(t,s)$. A suitable requirement for this purpose is that G generates a strongly continuous semigroup not only on E_p, but also on E_∞.

Theorem 3.5. *Let G generate a C_0–semigroup $T(\cdot)$ on E_p for some $1 \leq p \leq \infty$. Then the following assertions are equivalent.*

(i) $T(\cdot)$ *is an evolution semigroup given by an evolution family* $U(\cdot,\cdot)$.

(ii) $T(\sigma)\,(\varphi f) = (\tau_\sigma\varphi)\,T(\sigma)f$ *for* $f \in E$, $\varphi \in C_b(I)$, $t \geq 0$. $D(G)$ *is densely and continuously embedded in* $C_0(I, X)$.

(iii) For all f *contained in a core of* G *and* $\varphi \in C_c^1(I)$ *we have* $\varphi f \in D(G)$ *and* $G(\varphi f) = \varphi\, Gf - \dot{\varphi} f$. $D(G)$ *is densely and continuously embedded in* $C_0(I, X)$.

For the proof see [70]. This theorem relates the characterisation of evolution semigroups to properties of multiplication operators: The evolution semigroup behaves like a semigroup of multiplication operators, up to a translation. In the last theorem, $D(G)$ is endowed with the graph norm of G. Notice that the second condition in (ii) and (iii) is trivially satisfied if $p = \infty$. However, if $p < \infty$, (ii) does not imply (i). Consider for example a function p such that p and $1/p$ are discontinuous elements of $L^\infty(\mathbb{R})$, but $p(s) \to p(t)$ as $s \nearrow t$ for a.e. $t \in \mathbb{R}$. Then set $X = \mathbb{C}$, $E = L^1(\mathbb{R})$ and use $U(t, s) = p(t)/p(s)$ to define a strongly continuous semigroup which satisfies (ii) without the second condition, but which does not lead to an evolution semigroup associated with a strongly continuous propagator [70]. However, dropping the inclusion $D(G) \subset C_0(I, X)$ in (ii) and (iii) still allows for the characterization of an evolution semigroup associated with a strongly *measurable* propagator, see [47, 24]. In the context of the next section this gives no additional freedom, but in Section 3.3 the use of evolution semigroups which do not fulfill the second condition in (ii) and (iii) will be crucial.

For the application of our results to models of quantum field theory, the following result will be useful:

Theorem 3.6. *Let* $R = R(\cdot) \in C_b(I, \mathcal{B}(x))$ *be a bounded, invertible multiplication operator with bounded inverse. If* $(T(\sigma))_{\sigma\geq 0}$ *is an evolution semigroup with generator* $(G, D(G))$, *then the similar semigroup* $(\tilde{T}(\sigma))_{\sigma\geq 0}$, *defined by* $\tilde{T}(\sigma) = RT(\sigma)R^{-1}$, *is an evolution semigroup with generator* $(\tilde{G},\ D(\tilde{G}))$ *given by* $D(\tilde{G}) = RD(G), \tilde{G} = RGR^{-1}$.

Proof. Clearly, $(\tilde{T}(\sigma))_{\sigma\geq 0}$ is an evolution semigroup. Observe that R^{-1} is also a multiplication operator, hence it commutes with scalar multiplications. For $\varphi \in C_b(I)$ we have $\tilde{T}(\sigma)\,(\varphi f) = RT(\sigma)R^{-1}(\varphi f) = (\tau_\sigma\varphi)\,\tilde{T}(\sigma)f$ for $f \in E$. The continuous embedding of $D(\tilde{G})$ in $C_0(I, X)$ follows by boundedness of R and continuity of $t \mapsto R(t)$. □

In the course of this chapter, we will need a technical result concerning cores of the generator G of an evolution semigroup.

Lemma 3.7. *Let* $T(\cdot)$ *be an evolution semigroup with generator* G *on* E_p, $1 \leq p \leq \infty$, *with* $I = \mathbb{R}$. *Let* D *be a core for* G *such that* $\varphi f \in D$ *for* $f \in D$ *and*

$\varphi \in C_c^1(\mathbb{R})$. *Then* $D_c = D \cap C_c(\mathbb{R}, X)$ *is also a core for* G. *If* $f \in D(G) \cap C_c(\mathbb{R}, X)$, *then we can take approximating functions* $f_n \in D_c$ *whose supports are contained in a bounded interval* $J \supset \operatorname{supp} f$.

Proof. Choose $\varphi_n \in C_c^1(\mathbb{R})$ with $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $[-n, n]$, and $\|\dot{\varphi}_n\|_\infty \leq 2$ for $n \in \mathbb{N}$. Let $f \in D$ and set $f_n = \varphi_n f \in D_c$. Then $f_n \to f$ and $Gf_n = \varphi_n Gf - \dot{\varphi}_n f \to Gf$ in E_p as $n \to \infty$. Thereby we have used that Gf is the limit of $(T(\tau)f - f)/\tau$ as $\tau \to 0$ for $f \in D(G)$ by definition, and that we obtain $\varphi u \in D(G)$ and $G(\varphi u) = -\dot{\varphi} u + \varphi G u$ if $u \in D(G)$ and $\varphi \in C_c^1(I)$. So we have shown the first assertion. To verify the second one, take $f \in D(G) \cap C_c(\mathbb{R}, X)$. Then there are $g_n \in \mathcal{D}$, $n \in \mathbb{N}$, converging in $D(G)$ to f. Let $\varphi \in C_c^1(\mathbb{R})$ be equal to 1 on the support of f. Then the supports of the functions $f_n = \varphi g_n \in \mathcal{D}_c$, $n \in \mathbb{N}$, are contained in $J := \operatorname{supp} \varphi$. Moreover, $f_n \to f$ and $Gf_n \to Gf$ as $n \to \infty$, analogously to above. □

3.2 Wellposedness and mild solutions

For $I = \mathbb{R}$ and $p = \infty$, G. Nickel has investigated the relation between wellposedness of the nonautonomous Cauchy problem (2.2) and properties of evolution semigroups in [62, 63]. Let $C^1 = \{f \in C(\mathbb{R}, X) : f,\ d/dt f \in C_0(\mathbb{R}, X)\}$.

Theorem 3.8. *Let* X *be a Banach space and let* $\{A(t)\}_{t\in\mathbb{R}}$ *be a family of linear operators. The nonautonomous Cauchy problem* (2.2) *is wellposed if and only if there exists a unique evolution semigroup* $T(\cdot)$ *on* $E_\infty(\mathbb{R}) = C_0(\mathbb{R}, X)$ *with generator* $(G, D(G))$ *and an invariant core* $D \subset C^1 \cap D(G)$ *such that* $Gf = -\frac{d}{dt} f + A(\cdot) f$ *for* $f \in D$.

For the proof see [62].

Assuming the hypotheses of the Theorem of Kato (Theorem 2.29), Nickel uses the preceding theorem to prove wellposedness in the hyperbolic case, thereby giving a simplified proof of Kato's theorem using evolution semigroups. In doing so, he utilizes an approximating evolution semigroup. A similar strategy leads to Theorem 3.10.

The last Theorem 3.8 is remarkable as it allows us to characterize wellposedness of the nonautonomous Cauchy problem (2.2) by properties of the generator G of an evolution semigroup. It establishes a close analogy to the time-independent situation. Unfortunately, for an application to concrete examples we again need assumptions of Kato type.

For the general case $p < \infty$ and $I \subset \mathbb{R}$ an analogous equivalence result is not known. But there is a theorem giving the implication in one direction: If the nonautonomous Cauchy problem (2.2) is wellposed, the generator of the evolution semigroup is the closure of $-\frac{d}{dt} + A(\cdot)$.

Theorem 3.9. *Assume the nonautonomous Cauchy problem* (2.2) *is wellposed in the sense of Definition 2.9 with regularity subspaces Y_t and bounded propagator $U(t,s)$. Moreover, let $(G, D(G))$ be the generator of the evolution semigroup $T(\sigma)$ on $E_p(I) = L^p(I,\mathbb{R})$ associated with $U(t,s)$. Then*

$$F_p = \{f \in W_0^{1,p}(I,X) : f(t) \in D(A(t)) \text{ for a.e. } t \in I,\ A(\cdot)f(\cdot) \in L^p(I,X)\} \tag{3.5}$$

is a core for G and $Gf = -\frac{d}{dt}f + A(\cdot)f$ for all $f \in F_p$.

For the proof see [79, 78].

Our strategy proceeds in the opposite way. We define an operator G_0 and investigate under which assumptions it has extensions that are the generators of evolution semigroups. With regard to Howland's characterization of evolution semigroups in Hilbert spaces, we could paraphrase our strategy in the following way: We start with an incomplete Weyl pair $\{iG_0, Q\}$, and the construction of the evolution semigroup amounts to complete $\{iG_0, Q\}$ to a Weyl pair $\{iG, Q\}$ with $G_0 \subset G$, see for example [49]. This point of view is advocated in Neidhardt's article [61], where he uses von Neumann's theory of defect indices to classify the self-adjoint extensions of iG_0. This interesting analysis unfortunately does not lead to conditions which meet the concreteness requirements needed to obtain wellposedness results for nonautonomous evolution equations.

Given the nonautonomous Cauchy problem (2.2), we endow the multiplication operator $f \mapsto A(\cdot)f(\cdot)$ on E_p with the maximal domain

$$D(A(\cdot)) = \{f \in E_p : f(t) \in D(A(t)) \text{ for a.e. } t \in I,\ A(\cdot)f(\cdot) \in E_p\}. \tag{3.6}$$

We define the sum

$$G_0 = -\tfrac{d}{dt} + A(\cdot), \qquad D(G_0) = F_p,$$

in E_p on the maximal domain from equation (3.5).

As remarked above, wellposedness of the nonautonomous Cauchy problem (2.2) would imply that the closure of G_0 generates an evolution semigroup (analogous to [79, Prop. 4.1]).

We want to prove the opposite implication, saying that G_0 possesses a closure G which generates a semigroup $T(\cdot)$. The following theorem is due to R. Schnaubelt. It is formulated in a rather general way in order to make clear under which circumstances a Kato-stable family of generators $A(t)$ on X 'generates' an evolution semigroup on E_p. The idea of this theorem can already be found in [79].

Theorem 3.10. *Let $A(t)$, $t \in I$, be Kato-stable generators on X with constants (M, w) such that $t \mapsto R(w', A(t))$ is strongly continuous for some $w' > w$. Let $1 \le p \le \infty$. Assume that the space $(w' - G_0)F_p$ is dense in E_p and that F_p is dense in E_p and E_∞. Then G_0 with domain F_p possesses a closure G in E_p which generates an evolution semigroup $T(\cdot)$, given by an evolution family $U(\cdot,\cdot)$ on X.*

Proof. The proof is closely related to Nickel's treatment of the hyperbolic evolution equations in the $p = \infty$–situation [62]. Consider the Yosida approximation $A_n(t) = nA(t)R(n, A(t)) = n^2R(n, A(t)) - n$ for $n > w$ and $t \in I$. Notice that $t \mapsto A_n(t)$ is strongly continuous and that $A_n(\cdot)$ is the Yosida approximation of the generator $A(\cdot)$ on E_p, see for instance Theorem III.4.8 and Paragraph III.4.13 in [22]. Because $A_n(\cdot)$ is a bounded perturbation of $-\frac{d}{dt}$, it is then clear that $G_n = -\frac{d}{dt} + A_n(\cdot)$ with domain $W_0^{1,p}(I, X)$ generates an evolution semigroup $T_n(\cdot)$ on E_p which is given by the evolution family $U_n(\cdot, \cdot)$ generated by $A_n(\cdot)$. For $u \in F_p$ we have $G_n u \to Gu$ in E_p. Due to the Kato stability of $A(\cdot)$, there are norms $\|\cdot\|_t$ on X satisfying the conditions of Lemma 2.26. In particular, $\|R(\lambda, A(t))\|_t \leq (\lambda - w)^{-1}$ for $\lambda > w$. This fact yields

$$\|e^{\tau A_n(t)}\|_t = e^{-n\tau}\|\exp(\tau n^2 R(n, A(t)))\|_t \leq e^{-n\tau}\exp(\tau n^2(n-w)^{-1}) \leq e^{w_1\tau}$$

for $w_1 := (w + w')/2$, all $n \geq n_0$, and some $n_0 \geq w$. Hence the operators $A_n(t)$, $t \in I$, satisfy Lemma 2.26 with the same norms and the exponent w_1. Thus they are Kato-stable with uniform constants M and w_1. Kato's existence theorem (Theorem 2.29) then shows that

$$\|U_n(t,s)\| \leq Me^{w_1(t-s)}, \quad (t,s) \in D_I\,; \text{ hence } \; \|T_n(r)\| \leq Me^{w_1 r}, \quad r \geq 0. \tag{3.7}$$

The Trotter–Kato theorem now implies that the closure G of G_0 exists and generates a semigroup $T(\cdot)$ on E_p, see [22, Thm. III.4.9] or [67, Thm. 3.4.5]. Observe that the first condition of Theorem 3.5(iii) holds on the core F_p of G on E_p. To check the second condition, it suffices to consider u contained in the core $F_p \subset C_0(I, X)$ and to show that $\|u\|_\infty = \|R(w', G)f\|_\infty \leq c\,\|f\|_p$ where $f := R(w', G_0)u$. Recall that $D(G)$ is dense in $C_0(I, X)$ by assumption. We use the approximation G_n once more. Due to (3.3) and (3.7), we have $\|R(w', G_n)g\|_\infty \leq c\,\|g\|_p$ for $g \in E_p$ and a constant $c > 0$. This estimate implies that

$$R(w', G_n)f - R(w', G)f = R(w', G_n)(G_n - G_0)u = R(w', G_n)(A_n(\cdot) - A(\cdot))u,$$

and

$$\|R(w', G_n)f - R(w', G)f\|_\infty \leq c\,\|(A_n(\cdot) - A(\cdot))u\|_p \longrightarrow 0$$

as $n \to \infty$. As a result, $\|R(w', G)f\|_\infty \leq c\,\|f\|_p$. Theorem 3.5 thus shows that $T(\cdot)$ is an evolution semigroup. □

If one wants to apply the above result, the Kato stability can possibly be checked using dissipativity of $A(t)$, hence it is not problematic in the Hilbert space situation with a self-adjoint Hamiltonian. The density of F_p in $L^p(I, X)$ and $C_0(I, X)$ can be established in two situations. First, if the resolvents $R(\lambda, A(t))$, $\lambda > w$, are strongly continuously differentiable in t. Second, if there is a dense subset Y of X contained in all $D(A(t))$ and $A(\cdot)y$ is continuous for $y \in Y$ (then

$C_c^1(I,Y) \subset F_p$). Later on we will work in the latter setting. In this case, $R(\lambda, A(\cdot))$ is strongly continuous if in addition Y is a core for all $A(t)$. The most difficult problem is the verification of the range condition:

$$(w' - G_0)F_p \text{ is dense in } L^p(I,X). \tag{3.8}$$

Before giving sufficient conditions leading to the range condition to be fulfilled in Theorem 3.15, in the next result we establish differentiability properties on spaces Y as above.

Theorem 3.11. *Suppose that the assumptions of Theorem 3.10 hold. Let $Y \subset D(A(t))$ for all $t \in I$ and let $A(\cdot)y$ be continuous in X for $y \in Y$. Then the derivatives*

$$\frac{\partial}{\partial s} U(t,s)y = -U(t,s)A(s)y, \tag{3.9}$$

$$\frac{\partial^+}{\partial t} U(t,s)y|_{t=s} = A(t)y, \tag{3.10}$$

exist for $(t,s) \in D_I$ and $y \in Y$. (In (3.9) *one has to take the one–sided derivative if $t = s$.) If $U(\cdot,\cdot)$ is invertible and $I = \mathbb{R}$, then one may take $t,s \in \mathbb{R}$ in* (3.9) *and two-sided derivatives at $t = s$.*

Proof. Take $t, s, s' \in I$ with $t \geq s$, $t \geq s'$, $s \neq s'$, $y \in Y$, and $\varphi \in C_c^1(I)$ which is equal to 1 on an interval containing s and s'. Set $f = \varphi y$. Then $f \in F_p$ and $Gf = -\dot{\varphi} y + \varphi A(\cdot)y$. Thus standard semigroup theory yields

$$\begin{aligned} U(t,s)y - U(t,s')y &= (T(t-s)f)(t) - (T(t-s')f)(t) = \int_{t-s'}^{t-s} (T(\tau)Gf)(t)\,d\tau \\ &= \int_{t-s'}^{t-s} U(t,t-\tau)(-\dot{\varphi}(t-\tau)y + \varphi(t-\tau)A(t-\tau)y)\,d\tau \\ &= -\int_{s'}^{s} U(t,r)A(r)y\,dr. \end{aligned}$$

Multiplying by $(s-s')^{-1}$ and letting $s - s' \to 0$ we deduce (3.9). Using this result, we conclude

$$U(t,s)y - y = -\int_s^t \frac{\partial}{\partial \tau} U(t,\tau)y\,d\tau = \int_s^t U(t,\tau)A(\tau)y\,d\tau,$$

which implies (3.10). The final assertions are verified in the same way. $\square$

If we knew that $U(t,s)Y \subset Y$, then wellposedness of the nonautonomous Cauchy problem (2.2) would follow from the above proposition and the equality $U(t+h,s)y - U(t,s)y = (U(t+h,t)-1)U(t,s)y$. Unfortunately, the invariance of Y is hard to verify. Again one has to impose the restrictive conditions necessary in Kato's theory. It seems that this problem is not tackled more easily in the framework of evolution semigroups, see [62].

It turns out to be fruitful to attenuate the notion of solvability and to ask for *mild solutions* in the sense of Definition 2.11. The next theorem due to R. Schnaubelt shows that mild solutions arise naturally in the context of Theorem 3.10. They coincide with the orbits $u = U(\cdot, s)x$ for *every* $x \in X$. In the following we will use cut-off functions of the following form: For $s \in I = (a, b]$ and $\varepsilon \in (0, b-s]$ we take a function $\varphi_\varepsilon \in C_c^1((a, b])$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon = 0$ on $(a, s]$ and $\varphi_\varepsilon = 1$ on $[s+\varepsilon, b]$. For $I = \mathbb{R}$ and $d > |s|$, we take $\psi_d \in C_c^1(\mathbb{R})$ such that $0 \le \psi_d \le 1$ and $\psi_d = 1$ on $[-d, d]$. Set $I_s = I \cap [s, \infty)$.

Theorem 3.12. *Suppose that the assumptions of Theorem 3.10 hold. Let $s \in I$, $x \in X$, and define φ_ε and ψ_d as above.*

(i) Let $I = (a, b]$. Then $u = U(\cdot, s)x \in C([s, b], X)$ is the unique (E_p)-mild solution of (2.2), where one may take $u_n \in F_p$ in Definition 2.11. Moreover, u is the only function in $C(I_s, X)$ such that $u(s) = x$, $\varphi_\varepsilon u \in D(G)$, and $G(\varphi_\varepsilon u) = 0$ on $[s+\varepsilon, b]$ for all $\varepsilon \in (0, b-s]$.

(ii) Let $I = \mathbb{R}$ and $U(\cdot, \cdot)$ be invertible. Then $u = U(\cdot, s)x \in C(\mathbb{R}, X)$ is the unique (E_p)-mild solution on $\mathbb{R}$ of (2.2). Moreover, u is the only function in $C(\mathbb{R}, X)$ such that $u(s) = x$, $\psi_d u \in D(G)$, and $G(\psi_d u) = 0$ on $[-d, d]$ for all $d > |s|$.

Proof. Set

$$\tilde{U}(t, s) = \begin{cases} U(t, s) & (t, s) \in D_I, \\ 0 & t < s,\ t, s \in I. \end{cases}$$

Since Gf is the limit of $(T(t)f - f)/t$ as $t \to 0$ for $f \in D(G)$, we obtain $\varphi u \in D(G)$ and

$$G(\varphi u) = -\dot{\varphi} u + \varphi G u, \text{ if } u \in D(G) \text{ and } \varphi \in C_c^1(I), \tag{3.11}$$

$$G(\varphi u) = -\dot{\varphi} u, \text{ if } \varphi \in C_c^1(I) \text{ with } \varphi(t) = 0,\ a < t \le s, \text{ and } u = \tilde{U}(\cdot, s)x, \tag{3.12}$$

where $x \in X$ and $s \in I$ in the second line. In the invertible case, (3.12) also holds for $v(t) = \varphi(t)U(t, s)x$, $t \in I$, with $\varphi \in C_c^1(I)$. Now consider case (i). Let $u = U(\cdot, s)x$ and set $v_n = \tilde{\varphi}_n \tilde{U}(\cdot, s - \frac{1}{n})x$ for a function $\tilde{\varphi}_n \in C^1(I)$ with $0 \le \tilde{\varphi}_n \le 1$, $\tilde{\varphi}_n = 1$ on $[s, b]$ and $\tilde{\varphi}_n = 0$ on $(a, s - \frac{1}{n}]$. Then

$$\sup_{s \le t \le b} \|u(t) - v_n(t)\| \le c\, \|x - U(s, s - \tfrac{1}{n})x\| \longrightarrow 0$$

as $n \to \infty$. Moreover, $v_n \in D(G)$ and $Gv_n(t) = 0$ for $t \ge s$, due to (3.12). There are $w_n \in F_p$ such that $\|v_n - w_n\|_p + \|Gv_n - Gw_n\|_p \le 1/n$. Since $D(G)$ is continuously embedded in $C_0(I, X)$, we obtain

$$\|v_n - w_n\|_{L^\infty([s,b],X)} \le \tfrac{c}{n} \quad \text{and} \quad \|Gw_n\|_{L^p([s,b],X)} = \|-\dot{w}_n + A(\cdot)w_n\|_{L^p([s,b],X)} \le \tfrac{1}{n}\,.$$

Hence u is a mild solution of the nonautonomous Cauchy problem (2.2).

Let v be another mild solution with approximating functions v_n as in Definition 2.11. Take $s < t - r < t \le b$ and a function $\varphi \in C_c^1(I)$ which is equal to 1 on $[t-r, t]$ and equal to 0 on $(a, s]$. Then $\varphi v_n \in F_p$ and

$$G(\varphi v_n) = -\dot{\varphi} v_n + \varphi(-\dot{v}_n + A(\cdot)v_n).$$

This identity implies that

$$\begin{aligned} U(t,t-r)v_n(t-r) - v_n(t) = (T(r)\varphi v_n)(t) - (\varphi v_n)(t) &= \int_0^r [T(\tau)G(\varphi v_n)](t)\,d\tau \\ &= \int_{t-r}^t U(t,\sigma)(-\dot{v}_n(\sigma) + A(\sigma)v_n(\sigma))\,d\sigma. \end{aligned} \tag{3.13}$$

Since $v_n \to v$ uniformly and the integrand converges to 0 in L^p as $n \to \infty$, we arrive at $v(t) = U(t, t-r)v(t-r)$ for all $t > t - r > s$. We thus obtain $u = v$ taking the limit $r \to t - s$.

The function $u = U(\cdot, s)x$ satisfies $G(\varphi_\varepsilon u) = 0$ on $[s + \varepsilon, b]$, due to (3.12). Conversely, let $v \in C(I_s, X)$ be given with $v(s) = x$ and $G(\varphi_\varepsilon v) = 0$ on $[s+\varepsilon, b]$. We can approximate $\varphi_\varepsilon v$ in the graph norm of G by $v_n \in F_p$. As in (3.13), this fact implies that $v(t) = U(t, s+\varepsilon)v(s+\varepsilon)$ so that again $u = v$.

The assertions in the invertible case (ii) can be shown in a similar way. Here one starts with $v_n(t) = \psi_n(t)U(t, s)x$ for $n \ge |s|$ and $t \in \mathbb{R}$. □

Under some additional assumptions we can also prove that $u = U(\cdot, s)x$ is the *unique, strongly continuous* weak solution of the nonautonomous Cauchy problem (2.2). As we have seen in Section 2.12, weak solutions exist in a rather general setting. We point out that uniqueness and continuity are the crucial results which are not guaranteed in the general situation. We now assume that X is reflexive, that $1 < p < \infty$, and that the operators $A(t)$ and their adjoints $A(t)^*$, $t \in I$, satisfy the assumptions of Theorem 3.10. Let $q = p/(p-1)$. Replacing $-d/dt$ and the right shift by $+d/dt$ and the left shift, one can repeat the above proofs for $G_0' = d/dt + A(\cdot)^*$ defined on

$$\begin{aligned} F_q' = \{f \in W^{1,q}(I,X) : f(t) \in D(A(t)) \text{ for a.e. } t \in I, \\ A(\cdot)f(\cdot) \in L^q(I,X), f(b) = 0\}. \end{aligned}$$

If $I = \mathbb{R}$ the condition $f(b) = 0$ has to be dropped. In particular, G_0' has a closure G' in $L^q(I, X^*) = E_p^*$ which generates a strongly continuous semigroup. Since G is the closure of G_0, it is straightforward to check that $G_0' \subset G^*$. Consequently, $G' \subset G^*$, and thus $G^* = G'$.

Theorem 3.13. *Under the above assumptions, let $s \in I$ and $x \in X$.*

(i) Let $I = (a, b]$. Then $u = U(\cdot, s)x$ is the only function in $C(I_s, X)$ with $u(s) = x$ such that

$$\int_s^b \big(u(\tau), \dot{v}(\tau) + A(\tau)^* v(\tau)\big)\, d\tau = \big(x, v(s)\big) \qquad \text{for all } \ v \in F_q'. \tag{3.14}$$

(ii) Let $I = \mathbb{R}$ and $U(\cdot, \cdot)$ be invertible. Then $u = U(\cdot, s)x$ is the only function in $C(\mathbb{R}, X)$ with $u(s) = x$ such that

$$\int_s^b \big(u(\tau), \dot{v}(\tau) + A(\tau)^* v(\tau)\big)\, d\tau = 0 \qquad \text{for all } \ v \in F_q' \cap C_c(\mathbb{R}, X^*).$$

Proof. (i) Let $v \in F_q'$, $u = U(\cdot, s)x$, and u_n be the approximating functions from Definition 2.11. Integrating by parts we then obtain

$$\begin{aligned}
\int_s^b \big(u(\tau), \dot{v}(\tau) + A(\tau)^* v(\tau)\big)\, d\tau &= \lim_{n\to\infty} \int_s^b \big(u_n(\tau), \dot{v}(\tau) + A(\tau)^* v(\tau)\big)\, d\tau \\
&= \lim_{n\to\infty} \int_s^b \big(-\dot{u}_n(\tau) + A(\tau)u_n(\tau), v(\tau)\big)\, d\tau + \big(u_n(s), v(s)\big) \\
&= \big(x, v(s)\big)\,.
\end{aligned}$$

Conversely, assume that $u \in C(I_s, X)$ with $u(s) = x$ satisfies (3.14) for all $v \in F_q'$. Take $\varphi_\varepsilon \in C^1(I)$ as above. Using (3.14) for $\varphi_\varepsilon v \in F_q'$, we deduce

$$\int_a^b \big(\varphi_\varepsilon(\tau)u(\tau), \dot{v}(\tau) + A(\tau)^* v(\tau)\big)\, d\tau = -\int_a^b \big(\dot{\varphi}_\varepsilon(\tau)u(\tau), v(\tau)\big)\, d\tau,$$

because $\varphi_\varepsilon(t) = 0$ for $a < t \le s$. Since F_q' is a core for G^*, this equality yields

$$\big(\varphi_\varepsilon u, G^* v\big)_{E_p} = -\big(\dot{\varphi}_\varepsilon u, v\big)_{E_p} \qquad \text{for all } \ v \in D(G^*).$$

This implies $\varphi_\varepsilon u \in D(G)$ and $G(\varphi_\varepsilon u) = -\dot{\varphi}_\varepsilon u$. Theorem 3.12 then shows that $u = U(\cdot, s)x$.

(ii) This assertion can be established in the same way, now using the functions ψ_d. One only has to verify that $F_q' \cap C_c(\mathbb{R}, X^*)$ is a core of G^*, proceeding as in the proof of Lemma 3.7. □

3.2.1 Perturbations of evolution semigroups

The simplest examples for evolution semigroups are obtained if the generator $A(t) = A$ does not depend on time. Provided that the corresponding autonomous Cauchy problem (2.1) is wellposed, the generator G of the associated evolution semigroup is the closure of $-\frac{d}{dt} + A$. One might ask wether it is possible to obtain new evolution semigroups as perturbations of these or other evolution semigroups belonging to solvable nonautonomous evolution equations.

In [78, 69, 70] bounded perturbations and Myadera perturbations (see [22]) are investigated. The latter formulation of perturbation theory applies in the parabolic situation.

For Hilbert spaces Howland gives a perturbation result suitable for the application to the time-dependent Schrödinger equation in [47]. In the following we discuss the Theorem resulting from his approach.

Recall that, if S is a closed, densely defined operator with $\sigma(S) \subset \mathbb{R}$, then a closed, densely defined operator A is S-smooth if and only if $D(S) \subset D(A)$ and

$$\sup_{\epsilon>0} \int_{-\infty}^{\infty} \|A(Sr \pm i\epsilon)^{-1} f\|^2 \, dr < \infty$$

for all $f \in E$. For properties of such operators see for instance [72].

Theorem 3.14. *Let X be a Hilbert space and $E = L^2(\mathbb{R}, X)$. Let G be the skew-adjoint generator of an evolution group on E. Assume that A and B are closed, densely defined operators on E which are iG-smooth. Moreover, assume that $\varphi A \subset A\varphi$ and $\varphi B^* \subset B^*\varphi$ for every bounded scalar function $t \mapsto \varphi(t)$ and that $(Af, Bg) = (Bf, Ag)$ for all $f, g \in D(A) \cap D(B)$. Let $R_\lambda := -R(G, \lambda)$ for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$ and let there be a constant c such that*

$$\|AR_\lambda B^* f\| \leq c\|f\|$$

for all $f \in D(B^*)$. Then the unique bounded extension $Q(\lambda)$ of $AR_\lambda B^*$ has strong non-tangential boundary values $Q(r \pm i0)$ for almost every $r \in \mathbb{R}$. Assume that $Q(r \pm i0)$ is quasi-nilpotent almost everywhere and that there is a finite increasing function $t \mapsto \rho(t)$ such that

$$\|(I + \kappa Q(\lambda))^{-1}\| \leq \rho(|\kappa|)$$

for real κ and uniformly in λ. Under these assumptions, the bounded operator

$$\tilde{R}(\lambda, \kappa) := R_\lambda - \kappa(BR_{\overline{\lambda}})^*(I + \kappa Q(\lambda))^{-1} AR_\lambda$$

is the resolvent of a self-adjoint extension $i\tilde{G}(\kappa)$ of $iG + \kappa B^* A$ which is the generator of a unitary evolution group.

Howland applies this theorem to the time-dependent Schrödinger equation (2.2) with $A(t) = -i(-\nabla + \kappa q(x,t))$ on $X = L^2(\mathbb{R}^n)$, where $q(t,x)$ is a potential such that for some $\infty > p > n/2$ and $\epsilon > 0$,

$$v_p(t) := (\int_{\mathbb{R}^n} |q(t,x)|^p \, d^n x)^{1/p}$$

is an element of $L^{r+\epsilon}(\mathbb{R}) \cap L^{r-\epsilon}(\mathbb{R})$, $r := \frac{2p}{2p-n}$. However, this perturbation result is clearly not applicable in a quantum field theoretical context. The iG-smoothness of the perturbation implies relative iG-boundedness with arbitrarily small bound [72][Theorem XIII.22]. Even for the $(\varphi^4)_2$ model, the interaction Hamiltonian is not bounded relative to H_0, and on the level of the evolution semigroups the situation does not improve. This can be seen in the following way: Assume $H_0 > 0$ is self-adjoint, $V(t)$ is symmetric, $D(V(t)) \supset D(H_0)$ and it is relatively H_0-bounded with bound smaller than 1, that is $\|V(t)x\| \leq a(t)\|H_0x\| + b(t)\|x\|$, where $0 < a(t) < 1$ uniformly in t, $0 \leq b(t)$. Then $H(t) = H_0 + V(t)$ is self-adjoint on $D(H(t)) = D(H_0)$. Let $I = (a, b]$ be a bounded interval and $E = L^2(I, X)$. With additional smoothness conditions for $t \mapsto a(t)$ and $t \mapsto b(t)$, one might conjecture that on E, the dissipative multiplication operator $iV(\cdot)$ is relatively G-bounded with bound smaller than 1, where G is the closure of $G_0 := -\frac{d}{dt} - iH_0$. Thus $G - iV(\cdot)$ with domain $D(G)$ would be the generator of an evolution semigroup [67, Corollary 3.3.3]. Now the autonomous Cauchy problem with respect to $A = -iH_0$ is wellposed, hence the generator G of the evolution semigroup associated with the propagator $U(t,s) = e^{-i(t-s)H_0}$ is indeed the closure of G_0. Hence we would like to deduce boundedness of the operator $iV(\cdot)R(\lambda, G)$ from the boundedness of $V(\cdot)(H_0 + c)^{-1}$. This would be possible if $iH_0R(\lambda, G)$ is a bounded operator on E. But, in general, this is wrong: In the special case under consideration, formula (3.3) yields

$$(R(\lambda, G)f)(t) = \int_a^t e^{-\lambda(t-s)} e^{-i(t-s)H_0} f(s)\, ds,$$

and we see that $R(\lambda, G)$ in general does not map arbitrary $f \in E$ into the domain of H_0 considered as a multiplication operator on E. The reason is the lack of an appropriate smoothing property of the Schrödinger group $(e^{-itH_0})_{t\in\mathbb{R}}$. This expresses the fact that G_0 is not closed on $F_{p=2}$. These difficulties are typical for the hyperbolic context. In contrast, in the parabolic case one can find the situation where G_0 is already closed on F_2, hence boundedness of $iH_0R(\lambda, G)$ follows. The Schrödinger semigroup $(e^{-tH_0})_{t\geq 0}$ is smoothing [84].

However, also in the hyperbolic context the domain of a Hamiltonian obtained from a small, relatively H_0-bounded perturbation equals $D(H_0)$ and this is independent of time. Hence, with suitable smoothness assumptions, we can apply Theorem 2.31 to show wellposedness of the time-dependent Schrödinger equation and we obtain the evolution semigroup from the associated propagator. Thus we find perturbation techniques not to be promising for the topic of our work. In the next Section, we return to a more direct approach.

3.2.2 The Theorem of Sohr

In [86] H. Sohr develops a link between parabolic and hyperbolic evolution equations for Hilbert spaces X. In the parabolic context it is possible to define G_0

as an operator sum on the intersection of the domains of the time derivative and the generators. The sum is closed and accretive on this domain. Then it is possible to recover the hyperbolic evolution equation by a perturbation argument and a strategy similar to 'Konrady's trick' [71]. We present a simplified version of Sohr's theorem, but for this setting we extend the result showing unitarity for the propagators.

Theorem 3.15. *Let $H(t)$, $t \in \mathbb{R}$, be self-adjoint operators on a Hilbert space X. Set $A(t) = -iH(t)$. We assume that for every $r > 0$ there are positive constants $\beta = \beta(r)$ and $k = k(r)$ such that $H(t)$ is bounded from below by $1 - \beta$, $t \mapsto (\beta + H(t))^{-1}$ is weakly continuously differentiable, and*

$$\frac{1}{2}\frac{d}{dt}(x, (\beta + H(t))^{-1}x) + k(x, (\beta + H(t))^{-1}x) \geq 0 \tag{3.15}$$

for all $x \in X$ and $|t| \leq r$. Then the conditions of Theorem 3.10 and Theorem 3.13 with $p = 2$ are fulfilled. In particular, there exists a unitary evolution family $U(t, s)$, $t, s \in \mathbb{R}$, such that $U(\cdot, s)x$ is the unique mild, thus unique, continuous weak solution of the nonautonomous Cauchy problem (2.2) on $I = \mathbb{R}$.

Proof. Since $A(t)$ is skew-adjoint, the family $\{A(t)\}_{t\in\mathbb{R}}$ is Kato-stable. Let $r > 0$, $t \in I := (-r, r]$, and $\lambda > 0$. We first observe that $t \mapsto (\beta(r)+H(t))^{-1}$ is Lipschitz on $[-r, r]$ since

$$\begin{aligned}&\left|\left(\left[(\beta(r) + H(t))^{-1}x - (\beta(r) + H(s))^{-1}x\right], y\right)\right| \\ &\qquad = \left|\int_s^t \frac{\partial}{\partial\tau}((\beta(r) + H(\tau))^{-1}x, y)\, d\tau\right| \leq c\,|t - s|\, \|x\|\, \|y\|\end{aligned}$$

for $-r \leq t, s \leq r$ and $x, y \in X$, where $c = c(r)$ does not depend on x and y by the principle of uniform boundedness. We further compute

$$\begin{aligned}R(\lambda, A(t)) &= i(i\lambda - \beta(r) + \beta(r) + H(t))^{-1} \\ &= i[1 + (i\lambda - \beta(r))(\beta(r) + H(t))^{-1}]^{-1}(\beta(r) + H(t))^{-1}\end{aligned}$$

using [22, Thm.IV.1.13]. After multiplication with $\beta(r) + H(t)$, the above equation implies that the operators $[\cdots]^{-1}$ are uniformly bounded for $t \in I$. Hence $t \mapsto R(\lambda, A(t))$ is Lipschitz on $[-r, r]$. We take $p = 2$. For $f \in C_c^1(I, X)$ and $n \in \mathbb{N}$, we define $f_n = nR(n, A(\cdot))f$. Then f_n is Lipschitz and thus $f_n \in F(I) = W_0^{1,2}(I, X) \cap D(A(\cdot))$. Moreover, $f_n \to f$ in $E(I)$ and $C_0(I, X)$ as $n \to \infty$, see the proof of Theorem 3.10. Hence $F(I)$ is dense in $E(I)$ and in $C_0(I, X)$.

The derivative $\frac{d}{dt}$ is a maximally accretive operator on $W_0^{1,2}(I, X)$. Its adjoint is $\frac{d}{dt}^*$ on $D(\frac{d}{dt}^*) = \{f \in W^{1,2}(I, X) : f(r) = 0\}$. Using the assumptions, we calculate

$$
\begin{aligned}
&\mathrm{Re}\left((\frac{d}{dt} + k)^* f, (\beta + H(\cdot))^{-1} f \right) = \\
&\mathrm{Re} \int_{-r}^{r} \left[\left(-\frac{d}{dt} f(t), (\beta + H(t))^{-1} f(t) \right) + k \left(f(t), (\beta + H(t))^{-1} f(t) \right) \right] \, dt \geq \\
&\int_{-r}^{r} \left[\frac{1}{2} \left(f(t), \frac{d}{dt} (\beta + H(t))^{-1} f(t) \right) + k \left(f(t), (\beta + H(t))^{-1} f(t) \right) \right] \, dt \geq 0
\end{aligned}
\tag{3.16}
$$

for $f \in D((\frac{d}{dt} + k)^*)$. By Lemma B.1 we conclude maximal accretivity of $\frac{d}{dt} + H(\cdot) + \beta + k$ on $F = W_0^{1,2}(I, X) \cap D(H(\cdot))$. Using the transformation $\tilde{f}(t) = e^{kt} f(t)$, we see that the constant k can be arbitrarily adjusted and so also $\frac{d}{dt} + H(\cdot) + \beta$ on F is maximally accretive.

Positivity and self-adjointness of $\beta + H(t)$ imply $\mathrm{Re}\left(\frac{d}{dt} f, (\beta + H(\cdot)) f\right) \geq \frac{1}{2}\left((\beta + H(\cdot)) f, \frac{d}{dt} (\beta + H(\cdot))^{-1} (\beta + H(\cdot)) f\right)$. Assumption (3.15) now implies

$$
\mathrm{Re}\left((\frac{d}{dt} + iH(\cdot) + k) f, (\beta + H(\cdot)) f \right) \geq 0,
$$

and we conclude

$$
\left\| \left(\frac{d}{dt} + H(\cdot) + iH(\cdot) + \beta + k \right) \right\|^2 \geq \|(\beta + H(\cdot)) f\|^2
$$

for all $f \in F$. Consequently, $-(\beta + H(\cdot))$ is bounded relative to $\frac{d}{dt} + H(\cdot) + iH(\cdot) + \beta + k$ with bound 1. By a semigroup version of Wüst's Theorem ([67, Corollary 3.3.5]), we conclude that $-\left(\frac{d}{dt} + iH(\cdot) + k\right)$ defined on F has a closure which is maximally dissipative and generates a strongly continuous semigroup, hence it is surjective. This ensures the range condition for $G_0 = -\frac{d}{dt} + A(\cdot)$ defined on $F(I)$, and Theorem 3.10 shows that its closure G_I exists and generates an evolution semigroup $T_I(\cdot)$ on $L^2(I, X)$ corresponding to a propagator $U_I(t, s)$, $(t, s) \in D_I$. Since G_0 is dissipative, $T_I(\tau)$ and $U_I(t, s)$ are contractive.

Using the uniqueness result from Theorem 3.12, we can define the propagator $U(t, s) = U_I(t, s)$ for $(t, s) \in D_{\mathbb{R}}$, where $(t, s) \in D_I$. The corresponding evolution semigroup $T(\cdot)$ with generator G satisfies $T(\tau) f(s) = T_I(\tau) f(s)$ for $0 \leq \tau \leq 1$ and $s \in I = [-r, r]$, if f has compact support in $[-r + 1, r]$ for some $r > 1$. This shows that (the restriction of) $f \in D(G) \cap C_c(\mathbb{R}, X)$ belongs to $D(G_I)$ for some

I whose interior contains the support of f, and $Gf = G_I f$ on I. Thus there are $f_n \in F(I)$ such that $G_0 f_n \to Gf$ in $L^2(I,X)$ and $f_n \to f$ in $C_0(I,X)$ as $n \to \infty$. Take $\varphi \in C_c^1((-r,r))$ with $\varphi = 1$ on the support of f, and extend the functions φf_n by 0 on the complement of $\operatorname{supp} f$. Then $\varphi f_n \in F_c(\mathbb{R}) = F(\mathbb{R}) \cap C_c(\mathbb{R},X)$. Since $G_0(\varphi f_n) = -\varphi' f_n + \varphi G_0 f_n$ on I, the functions $G_0(\varphi f_n)$ converge to Gf in $L^2(\mathbb{R},X)$. On the other hand, $D(G) \cap C_c(\mathbb{R},X)$ is a core of $D(G)$ by Lemma 3.7. As a result, $F_c(\mathbb{R})$ is a core for G and $Gf = G_0 f$ for $f \in F_c(\mathbb{R})$. Moreover, G_0 is skew-symmetric on $F_c(\mathbb{R})$, so that G is skew-adjoint. Therefore the assertions follow from Lemma 3.2, Theorem 3.12, and Theorem 3.13. □

3.2.2.1 Application: Quantum mechanics

Define the *Rollnik class* by

$$\begin{aligned} R = \{V : \mathbb{R}^3 \to \mathbb{C} : V \text{ is measurable},\\ \|V\|_R^2 := \iint |V(x)V(y)|\,|x-y|^{-2}\,dx\,dy < \infty\}. \end{aligned}$$

It is a Banach space with the norm $\|V\|_R$, and it contains physically reasonable potentials which are not covered by the Kato class. The main significance of the Rollnik condition is that it assures the Kato-smallness of $|V|^{1/2}$ with respect to $H_0^{1/2}$ and the Hilbert-Schmidt property for operators as $(H_0+\beta)^{-1/2}|V|(H_0+\beta)^{-1/2}$ or $|V|^{1/2}(H_0+\beta)^{-1}|V|^{1/2}$ for suitable constants β. We refer to [83] for further properties of R. In quantum mechanics one would allow potentials $V \in R + L^\infty(\mathbb{R}^3)$. For simplicity we restrict ourselves to the Rollnik class, but the example can easily be modified to include an L^∞-part.

Theorem 3.16. *Let $V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be a measurable function such that $t \mapsto V(t,x)$ is continuously differentiable for a.e. $x \in \mathbb{R}^3$ with partial derivative $\frac{\partial}{\partial t}V(t,x) = \dot V(t,x)$. Suppose that $|V(t,x)|, |\dot V(t,x)| \le W(x)$ for $t \in (a,b)$ and a function $W \in R$ possibly depending on the interval (a,b). Then the assertions of Theorem 3.15 hold.*

Proof. Let $H_0 = -\Delta$ with $D(H_0) = W^{2,2}(\mathbb{R}^3)$. Since $|V(t)| \le W$, $|V(t)|^{1/2}$ is a small perturbation of $H_0^{1/2}$ locally uniformly in t, that is, we find a constant $b_1 > 0$, uniformly on compact intervals $[-r,r]$, such that

$$\|\,|V(t)|^{1/2}x\|^2 \le \tfrac14 \|H_0^{1/2}\|^2 + b_1\|x\|^2$$

for all $x \in D(H_0^{1/2})$ and $|t| \le r$, see the proof of [83, Thm. I.21]. So we conclude from the KLMN theorem [71, Thm. X.17] that there is a self-adjoint operator $H(t)$ such that $D(|H(t)|^{1/2}) = D(H_0^{1/2})$ and $H(t) = H_0 + V(t)$ as a quadratic form on $D(H_0^{1/2})$. Moreover, $H(t) \ge -b_1$ is semibounded from below. Since also $\dot V(t) \in R$, we can repeat this argument to define a self-adjoint operator $\tilde H(t)$

with $D(\tilde{H}(t)^{1/2}) = D(H_0^{1/2})$ and $\tilde{H}(t) = H_0 + V(t) - \dot{V}(t) + b_1$ as a quadratic form on $D(H_0^{1/2})$. There is a $b_2 = b_2(r) > 0$ such that $\tilde{H}(t) \geq -b_2$ for $|t| \leq r$. Choosing a sufficiently large $c \geq 1$, we set $\beta = \beta(r) = b_1 + b_2 + c$. Then we find $H(t) \geq 1 - \beta$ and $H(t) + \beta \geq \dot{V}(t) + c$ for $|t| \leq r$. Moreover, Tiktopoulos' formula [83, Thm. II.12(a)] shows that

$$(H(t) + \beta)^{-1} = (H_0 + \beta)^{-1/2}(1 + B(t))^{-1}(H_0 + \beta)^{-1/2}, \tag{3.17}$$

where $B(t) = (H_0 + \beta)^{-1/2}V(t)(H_0 + \beta)^{-1/2}$ is a bounded operator with $0 \leq \|B(t)\| \leq q < 1$ for $|t| \leq r$ and sufficiently large $\beta = \beta(r)$. We observe that then $(1 + B(t))^{-1}$ is uniformly bounded for $|t| \leq r$. Our assumptions and Lebesgue's Theorem imply the differentiability of $t \mapsto (f, B(t)g)$ for $f, g \in L^2(\mathbb{R}^3)$. Therefore, $t \mapsto B(t)$ is Lipschitz continuous on $[-r, r]$ by the principle of uniform boundedness. It follows from

$$(1 + B(t))^{-1} - (1 + B(s))^{-1} = (1 + B(t))^{-1}(B(s) - B(t))(1 + B(s))^{-1}$$

that also $(1 + B(\cdot))^{-1}$ is Lipschitz-continuous. Furthermore, the equation

$$\begin{aligned}
&(t-s)^{-1}([(1 + B(t))^{-1} - (1 + B(s))^{-1}]f, g) \\
&\quad = ([B(s) - B(t)](1 + B(s))^{-1}f, (t-s)^{-1}[(1 + B(t))^{-1} - (1 + B(s))^{-1}]g) \\
&\qquad\qquad + ((t-s)^{-1}[B(t) - B(s)](1 + B(s))^{-1}f, (1 + B(s))^{-1}g)
\end{aligned}$$

implies the weak differentiability of $t \mapsto (1+B(t))^{-1}$, as the first term on the right-hand side vanishes and the second one converges in the limit $t \to s$. By (3.17) we conclude that $(H(t) + \beta)^{-1}$ is weakly continuously differentiable. Finally, we calculate

$$\begin{aligned}
-\frac{d}{dt}(x, (H(t) + \beta)^{-1}x) &\leq ((H(t) + \beta)^{-1}x, (\dot{V}(t) + c)(H(t) + \beta)^{-1}x) \\
&\leq (x, (H(t) + \beta)^{-1}x),
\end{aligned}$$

so that the assumptions of Theorem 3.15 with $k = \frac{1}{2}$ are fulfilled. □

3.2.2.2 Application: $(\varphi^4)_2$ with spatial localization

We apply Theorem 3.15 to the $(\varphi^4)_2$ model. For the notation and some basic facts see Appendix A. In the context of this chapter we consider the case $P(\lambda) = \lambda^4$, that is the interaction Hamiltonian is formally given as

$$V(t; g) = \int g(t, x) : \varphi(x)^4 : \, dx$$

Under some additional assumptions on the localization function g in the interaction, we can show the existence of the time evolution with Theorem 3.15.

Set $Y = D(H_0) \cap D(N^2)$. We allow localization functions which are not compactly supported in t and choose them from the set $\tilde{\mathcal{S}}(\mathbb{R}^2)$, where

$$\tilde{\mathcal{S}}_I = \{g \in \mathcal{S} : x \mapsto g(t,x) \in C^\infty_{c,I}(\mathbb{R})\},$$

and $C^\infty_{c,I}(\mathbb{R})$ are the smooth functions with support in a fixed compact interval I.

Theorem 3.17. *Let $g \in \tilde{\mathcal{S}}_I$ be given in such a way that for every $r > 0$ there exists a $k > 0$ such that*

$$kg(t,x) - \frac{1}{2}\dot{g}(t,x) \geq 0 \tag{3.18}$$

for all $x \in \mathbb{R}, |t| \leq r$. Then the time-dependent Schrödinger equation with $H(t) = H_0 + V(t;g)$ has a unique mild solution with corresponding evolution family $U(t,s)$.

We remark that compactly supported localization functions do not fulfill (3.18) in general. An example for an admissible localization $g_\alpha \in \tilde{\mathcal{S}}_I$ is obtained by starting with $g \in C^\infty_c(\mathbb{R}^2)$, $\operatorname{supp} g \subset I \times I$, and convoluting it with a smooth, fast decreasing approximation of the δ distribution in time, for example $g_\alpha(t,x) = (\delta_\alpha *_t g)(t,x)$ where $\delta_\alpha(t) = \frac{1}{\alpha\sqrt{\pi}} e^{-\frac{t^2}{\alpha^2}}$ for $\alpha > 0$.

For the proof of the Theorem, we need two Lemmas.

Lemma 3.18. *Let $g, h \in \tilde{\mathcal{S}}_\mathbb{R}$. For $\beta > 0$ sufficiently large,*

$$(\beta + H(t))^{-1} V(t;h) (\beta + H(t))^{-1}$$

extends to a bounded operator for every t.

Proof. Fix $t \in \mathbb{R}$ and set $H := H(t) = H_0 + V(t;g), V(g) = V(t;g)$ and $V(h) = V(t;h)$. H is self-adjoint on $D(H_0) \cap D(V(g))$, so in particular it is closed. Notice that this fact is not yet proved for interactions with higher powers of φ than 4. In this case only essential self-adjointness on the aforementioned domain is proved in the literature.

Choose $\beta > E > 0$ with a lower bound E of H, which exists according to Lemma A.9. From closedness of H it follows that there is a constant c_1 such that

$$H_0^2 + V(g)^2 \leq c_1(\beta + H^2) \tag{3.19}$$

as a quadratic form on $Y \times Y$. So $(\beta + H_0)^2 \leq c_1((\beta + H)^2 + \beta)$. Moreover, by Theorem A.1, there is another constant c_2 such that $V(h) \leq c_2(H_0 + \beta)^2$. It follows that

$$V(h) \leq 2c_1c_2(\beta + H)^2.$$

Because Y is a core for H, the set $Z = (\beta + H)Y$ is dense. This implies that the operator $(\beta + H)^{-1} V(h) (\beta + H)^{-1}$ is bounded on a dense set and thus extends to a bounded operator. $\square$

Lemma 3.19. *Let $I \subset \mathbb{R}$ be a compact interval. The mapping $t \mapsto H(t)y$ on I is differentiable for all $y \in Y$. The derivative is given by the Wick polynomial $\dot{H}(\cdot) = V(\cdot; \dot{g})$ with $Y \subset D(\dot{H}(\cdot))$. Moreover, $(\beta + H(t))^{-1}$ is weakly differentiable, and*

$$\frac{d}{dt}\Big(x, (\beta + H(t))^{-1}x\Big) = -\Big(x, (\beta + H(t))^{-1}\dot{H}(t)(\beta + H(t))^{-1}x\Big)$$

for all $x \in X$ and $\beta > 0$ sufficiently large. The function $t \mapsto \frac{d}{dt}\Big(x, (\beta + H(t))^{-1}x\Big)$ is bounded.

Proof. We show that the first assertion is implied by the smoothness of g in the standard way, using Theorem A.1: Let $w(t, k_1, \dots, k_4)$ be the scalar kernel as defined in equation (3.3.1). We estimate $\|V(t; g)y\| \le c\|w(t, \cdot)\|_{L^2(\mathbb{R}^4)}\|(N+1)^2 x\|$ for $y \in Y$, and thus $H(t)\restriction_Y$ is uniformly bounded as an operator from $Y \to X$.

Choose β to be larger than the locally uniform lower bound E of $H(t)$, according to Lemma A.9. The difference $H(t+h) - H(t)$ is defined on Y, as well as $V(t; \dot{g})$. The latter operator has the same structure as $V(t; g)$, but g is replaced by its time derivative. For $y \in Y$ and $s > 0$ we find

$$\begin{aligned}
&\| \left(\frac{1}{s}(H(t+s) - H(t)) - V(t; \dot{g})\right) y\| \\
&\le \| \left(\frac{1}{s}(V(t+s; g) - V(t; g)) - V(t; \dot{g})\right)(N+1)^{-2}\| \|(N+1)^2 y\| \\
&\le c\|\frac{1}{s}(w(t+s, \cdot) - w(t, \cdot) - \frac{\partial}{\partial t}w(t, \cdot)\|_{L^2(\mathbb{R}^4)}\|(N+1)^2 y\|
\end{aligned}$$

Now we turn to the second assertion. Using Lemma 3.18 with $h = \dot{g}$, one sees that it is sufficient to show weak differentiability of $(\beta + H(t))^{-1}$ on a dense set.

Let $y \in Y$. By the differentiability of $H(t)$ on Y and the continuity of its resolvent in y, one has

$$\lim_{s\to 0} \frac{-1}{s}(\beta H(t+s))^{-1}\big(H(t+s) - H(t)\big)\, y = -(\beta + H(t))^{-1}\dot{H}(t)y.$$

For all $x \in Z = (\beta + H(t))Y$ it follows

$$\begin{aligned}
\lim_{s\to 0} \frac{-1}{s}&\Big(x, (\beta + H(t+s))^{-1}(H(t+s) - H(t))(\beta + H(t))^{-1}x\Big) \\
&\qquad - \Big(x, (\beta + H(t))^{-1}\dot{H}(t)(\beta + H(t))^{-1}x\Big) = 0,
\end{aligned}$$

and this implies differentiability on X. Local boundedness of

$$t \mapsto \frac{d}{dt}\Big(x, (\beta + H(t))^{-1}x\Big)$$

follows from its continuity, which is clear for $x \in Z$ and extends to $x \in X$ by boundedness. □

Proof of Theorem 3.17. Fix $r > 0$ and $g \in \tilde{\mathcal{S}}_I$ which satisfies (3.18). By Lemma A.9 we can find an $E = E(r)$ locally uniformly on intervals $[-r, r]$, such that $H(t) \geq -E$ for $|t| \leq r$. Thus we set $\beta = 1 + E + c$ with a constant $c > 0$ which will be fixed in a moment. The weak continuous differentiability of $t \mapsto (\beta + H(t))^{-1}$ is fulfilled according to Lemma 3.19. Because of (3.18) and by eventually increasing k, it is possible to plug $g - \frac{1}{2k}\dot{g}$ instead of g into the interaction term $V(t; g)$. Lemma A.9 assures the existence of a $c > 0$ such that

$$H_0 + \int : \varphi^4(x) : \left(g(t,x) - \frac{1}{2k}\dot{g}(t,x) \right) \, dx = H(t) + V(t; -\frac{1}{2k}\dot{g}) \geq c \qquad (3.20)$$

as a quadratic form on a core of $H(t)$ and $t \in [-r, r]$. Representing this core as the image of a dense set under $(\beta + H(t))^{-1}$ and using the formula for the weak derivative of $t \mapsto (\beta + H(t))^{-1}$, we can reformulate this equation as

$$k(\beta + H(t))^{-1} + \frac{1}{2}\frac{d}{dt}(\beta + H(t))^{-1} \geq 0$$

on a dense set and this extends to all of X by boundedness. Hence the assumptions of Theorem 3.15 are satisfied. □

Unfortunately, we are not able to localize the interaction also in time. We will see in Section 4.1 that we can already get scattering operators in this situation, but this is not sufficient for a proof of the existence of local scattering operators. The problem is due to the condition (3.18), which was necessary to ensure the range condition (3.8) in the context of Theorem 3.15. This motivates us to go further on to avoid the range condition.

3.3 Wellposedness and approximative solutions

Up to this point, we have considered a notion of solvability which leads to unique, continuous solutions of the nonautonomous Cauchy problem. If quantum field theory comes into play, one would not expect to have complete control over the time evolution for every moment, due to particle creation and annihilation. Only scattering situations, where the time evolution asymptotically equals the free time evolution, are considered as being meaningful. Hence we can abandon some regularity properties of the solutions and we attenuate the conditions on a solution of the nonautonomous Cauchy problem (2.2) further. In the language of Section 3.2 this means that G_0 is allowed to have several extensions which are generators of evolution semigroups. We choose the one which can be approximated in a certain way. For Kato-stable generators on a separable Banach space we obtain operator families $U(t, s)$ fulfilling the properties (i) and (iii) of Definition 2.10. In the special case of the time-dependent Schrödinger equation in a separable Hilbert space we obtain also weak continuity and strong continuity up to a null set, hence approximative solutions in the sense of Definition 2.15.

Theorem 3.20. *Let $I = (a,b], -\infty < a < b < \infty$, be a bounded interval. Assume that there is a family of Kato-stable generators with constants (M,ω) on a separable, reflexive Banach space X such that*

(i) $t \mapsto R(\lambda, A(t))$ *and* $t \mapsto R(\lambda, A(t)^*)$ *are strongly continuous for some* $\lambda \geq 0$*;*

(ii) there are Banach spaces Y and Y' which are densely embedded in X resp. X^* such that $Y \hookrightarrow D(A(t))$ and $Y' \hookrightarrow D(A(t)^*)$, where the embeddings are uniformly bounded in t;

(iii) the mappings $t \mapsto A(t)y$ and $t \mapsto A(t)^*y^*$ are strongly continuous on I for all $y \in Y$ and $y^* \in Y'$.

Then there exists an operator family $U(t,s), (t,s) \in D_I$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad U(s,s) = \mathbb{1}.$$

Moreover,

$$U(t,r)U(r,s) = U(t,s) \tag{3.21}$$

for $(t,s) \in D_I$ and $r \in [t,s] \setminus N$ for a set $N = N(I)$ of measure 0. The mapping $D_I \ni (t,s) \mapsto U(t,s)$ is weakly continuous and strongly measurable, $s \mapsto U(t,s)$ is strongly continuous uniformly in t and $U(t,s) \to \mathbb{1}$ as $(t,s) \to (r,r)$ in I.

Furthermore, we find

$$\frac{\partial}{\partial s} U(t,s)y = -U(t,s)A(s)y, \tag{3.22}$$

$$\frac{\partial^+}{\partial t} U(t,s)y|_{t=s} = A(t)y \tag{3.23}$$

for $(t,s) \in D_I$ and $y \in Y$. The operators $U(t,s)$ are weak limits of the propagators $U_{n_l}(t,s)$ which are generated by an admissible bounded approximation $A_{n_l}(t)$ of $A(t)$.

The operator $G_0 = -\frac{d}{dt} + A(\cdot)$ defined on $\tilde{F} = W_0^1(I,X) \cap L^2(I,Y)$ has an extension G which generates a strongly continuous semigroup $(T(\sigma))_{\sigma \geq 0}$ defined by

$$(T(\sigma)f)(t) = \begin{cases} U(t,t-\sigma)f(t-\sigma) & t, t-\sigma \in I, \\ 0 & t \in I, t-\sigma \notin I, \end{cases} \tag{3.24}$$

in analogy to an evolution semigroup.

Proof. Choose an admissible bounded approximation $A_n(t)$ of $A(t)$ such that, at the same time, $-A_n(t)^*$ is an admissible bounded approximation of $-A(t)^*$ on X^*, for example as in Lemma 2.14. We organize the proof in several steps. Starting from weak limit points, we use evolution semigroup techniques to establish (3.21) and the other assertions.

Step 1. For $i, j, k \in \mathbb{N}$ we start with a numbering of the pairs of rational times $(t_i, s_j) \in D_I \cap \mathbb{Q}$ and a dense sequence $\{x_k\}$ in X. Let $q \in \mathbb{N}$ be a numbering of $\mathbb{N}^3$. We fix (i, j, k) and consider the sequence $\{z_n(q)\}_{n\in\mathbb{N}} = \{U_n(t_i, s_j)x_k\}_{n\in\mathbb{N}}$. This sequence is bounded, hence we can choose a weakly convergent subsequence $\{z_{\nu_m(q)}(q)\}_{m\in\mathbb{N}}$. Now we define the diagonal sequence $\{n_l\}_{l\in\mathbb{N}} = \{\nu_l(l)\}_{l\in\mathbb{N}}$ and conclude that $U_{n_l}(t_i, s_j)x_k$ converges weakly for $l \to \infty$ and every (i, j, k). We denote the limit by $U(t_i, s_j)x_k$. By boundedness, we extend $U(t_i, s_j)$ to all $x \in X$ and conclude $\text{w-lim}_{l\to\infty} U_{n_l}(t_i, s_j)x = U(t_i, s_j)x$ as well as $\|U(t_i, s_j)\| \leq Me^{\omega(t_i - s_j)}$ for all $x \in X, i, j \in \mathbb{N}$.

Step 2. For the propagators associated with the admissible bounded approximation $A_n(t)$ we have

$$U_{n_l}(t, s)y - U_{n_l}(t, r)y = -\int_r^s U_{n_l}(t, \tau)A_{n_l}(\tau)y \, d\tau. \tag{3.25}$$

It follows that

$$\|U_{n_l}(t, s)y - U_{n_l}(t, r)y\| \leq c|s - r|\|y\|_Y \tag{3.26}$$

with a constant c which can be chosen uniformly in t and independent of l, r, s and y. To obtain an estimate involving the weak limit of $U_{n_l}(t, s)$, we take $t_i, s_j, s_k \in \mathbb{Q}$, $s_j \leq t_i$ and $s_k \leq t_i$ and find

$$\|U(t_i, s_j)y - U(t_i, s_k)y\| \leq \limsup_{l\to\infty} \|U_{n_l}(t_i, s_j)y - U_{n_l}(t_i, s_k)y\| \leq c|s_j - s_k|\|y\|_Y.$$

For fixed t_i this estimate enables us to extend $s \mapsto U(t_i, s)y$ from $s \in (a, t_i] \cap \mathbb{Q}$ to a continuous function on $(a, t_i]$, which is bounded by $Me^{\omega(t_i - s)}\|y\|$. Hence we can extend $U(t_i, s)$ by linearity to an operator on X, bounded by $Me^{\omega(t_i - s)}$. Next we show the strong continuity of $s \mapsto U(t_i, s)y$ on $(a, t_i]$, uniformly in t_i. To this end, we estimate

$$\begin{aligned}(y, U_{n_l}(t_i, s)x - U(t_i, s)x) &\leq (y, U_{n_l}(t_i, s_j)x - U(t_i, s_j)x) \\ &\quad + (\|U_{n_l}(t_i, s)x - U_{n_l}(t_i, s_j)x\| + \|U(t_i, s)x - U(t_i, s_j)x\|)\|y\|\end{aligned}$$

for rational s_j such that $|s - s_j| < \epsilon$. Thus we see that $U_{n_l}(t_i, s)x \to U(t_i, s)x$ weakly in the limit $l \to \infty$. Since, by assumption, $A_n(\tau)y \to A(\tau)y$ for $y \in Y$ and $n \to \infty$, we find $U_{n_l}(t_i, \tau)A_{n_l}(\tau)y \to U(t_i, \tau)A(\tau)y$ weakly for $l \to \infty$ and $\tau \in (a, t_i]$. Hence (3.25) leads to

$$U(t_i, s)y - U(t_i, r)y = -\int_r^s U(t_i, \tau)A(\tau)y \, d\tau \tag{3.27}$$

for $y \in Y$ and $s, r \in (a, t_i]$. This implies

$$\|U(t_i, s)y - U(t_i, r)y\| \leq c|s - r|\|y\|_Y. \tag{3.28}$$

Step 3. Up to this point, we have considered rational t. To define $U(t,s)$ for $t \notin \mathbb{Q}$, we use a solution of the backward Cauchy problem associated with the admissible bounded approximation of $A(t)^*$ on X^*:

$$\dot{v}(s) = -A_n(s)^* v(s), \quad s \in (a,t],\ v(t) = x^*.$$

The solution of this Cauchy problem is given by $v(s) = U_n(t,s)^* x^* =: V_n(t,s)x^*$ with $(t,s) \in D_I$. The equality

$$U_{n_l}(t,s)^* y^* - U_{n_l}(r,s)^* y^* = \int_r^t U_{n_l}(\tau,s)^* A_{n_l}(\tau)^* y^* \, d\tau$$

for $y^* \in Y'$ together with the observation that $U_{n_l}(t_i,s)^* \to U(t_i,s)^*$ weakly allows us to apply the argument of Step 2 with t and s interchanged. Using the assumptions on $A(t)^*$, we extend the propagator $U(t,s)^*$ to an exponentially bounded propagator $V(t,s)$ on X^* defined on D_I. Furthermore, the mapping $t \mapsto V(t,s)$ is strongly continuous and $V_{n_l}(t,s) \to V(t,s)$ weakly for $l \to \infty$ and $(t,s) \in D_I$.

This enables us to define $U(t,s) := V(t,s)^*$ for $(t,s) \in D_I$ with $t \in \mathbb{R}\backslash\mathbb{Q}$. So we have $U(t,s) = V(t,s)^*$, $\|U(t,s)\| \leq Me^{\omega(t-s)}$, $t \mapsto U(t,s)$ is weakly continuous, $U_{n_l}(t,s) \to U(t,s)$ weakly as $l \to \infty$ on D_I. For coinciding times $U(t,t) = \mathbb{1}$ for $t \in I$. Our last norm estimate extends to real t,

$$\|U(t,s)y - U(t,r)y\| \leq c|s-r|\|y\|_Y, \tag{3.29}$$

for $(t,s), (t,r) \in D_I$ and $y \in Y$. The constant c remains independent of t. Using the density of Y in X and again an $\epsilon/3$ argument, we show that $s \mapsto U(t,s)$ is strongly continuous uniformly in t. From our previous considerations for rational t_i we deduce

$$U(t,s)y - U(t,r)y = -\int_r^s U(t,\tau)A(\tau)y \, d\tau \tag{3.30}$$

for $y \in Y$ and $a < s, r \leq t \leq b$. This formula enables us to conclude that $U(t,s) \to \mathbb{1}$ strongly for $(t,s) \to (r,r)$ and (3.22), (3.23) hold. The latter fact follows analogous to the proof of Theorem 3.11.

Step 4. As we have seen in Step 1 to Step 3, it is possible to deduce some properties of $U(t,s)$ from its weak approximants $U_{n_l}(t,s)$. But one crucial property of the propagators does not survive the weak limit: The causal factorization $U_{n_l}(t,s) = U_{n_l}(t,r)U_{n_l}(r,s)$, $t \geq r \geq s$, does not imply a similar property for $U(t,s)$, as the multiplication of operators is not continuous in the weak topology.

However, we are able to prove the causal factorization using an extension of G_0 which generates a strongly continuous semigroup $(T(\sigma))_{\sigma \geq 0}$. This semigroup has a similar relation to $U(t,s)$ as an evolution semigroup has to a strongly continuous propagator: The difference to the definition of an evolution semigroup is that $U(t,s)$ lacks not only the causal factorization, but also strong continuity in t.

Let $\lambda \in \mathbb{C}$ and $f \in E = L^2(I,X)$. For $t \in I$ we define the operator R_λ by

$$(R_\lambda f)(t) = \int_a^t e^{-\lambda(t-s)} U(t,s) f(s)\, ds. \tag{3.31}$$

Fubini's theorem assures strong measurability of $R_\lambda f$, so $R_\lambda : E \to E$ and it is bounded.

By assumption, the operators $A(t)$ are Kato-stable. By Lemma 2.26, there is a monotone family of norms $\|\cdot\|_t$ such that

$$\|x\| \le \|x\|_t \le \|x\|_s \le M\|x\| \text{ and } \|R(\lambda, A(t))x\|_t \le (\lambda - w)^{-1}\|x\|_t \tag{3.32}$$

for $(t,s) \in D_I, \lambda > \omega$ and $x \in X$.

We now introduce an equivalent norm on E which is constructed in such a way that one has dissipativity of $A(\cdot)$ and $-\frac{d}{dt}$ and hence of G_0 with respect to this norm. Let f be a simple function, then $\alpha(t) = \|f(t)\|_t$ is measurable as a sum of decreasing functions with disjoint supports. By approximation, α is measurable for each $f \in E$ and we are able to introduce the norm

$$|||f||| = \left(\int_I \|f(t)\|_t^2 \, dt \right)$$

which fulfills $\|f\| \le |||f||| \le M\|f\|$. As in the proof of Kato's Theorem in [67, Theorem 5.3.1], we have

$$U_n(t,s)x = \lim_{k\to\infty} e^{d_k A_n(t_k)} e^{d_k A_n(t_{k-1})} \cdot \ldots \cdot e^{d_k A_n(t_0)} x,$$

where $d_k = (t-s)/k$ and $t_j = s + j(t-s)/k$ for $j = 0, 1, \cdots, k$. With (3.32) we estimate

$$\|e^{d_l A_n(t_l)}\|_{t_l} \le e^{-nd_l} \|\exp(d_l n^2 R(n, A_n(t_l)))\|_{t_l} \le e^{d_l n\omega(n-\omega)^{-1}}.$$

This implies that $\|U_n(t,s)x\|_t \le e^{w_n(t-s)}\|x\|_s$ with $w_n = n\omega(n-\omega)^{-1} \to w$ as $n \to \infty$, and thus $\|U(t,s)x\|_t \le e^{w(t-s)}\|x\|_s$ for $(t,s) \in D_I$. As a result we obtain

$$|||R_\lambda f||| \le (\lambda - w)^{-1}\, |||f|||, \qquad \lambda > w,\ f \in E, \tag{3.33}$$

by use of Young's inequality.

Step 5. On E and $E^* = L^2(I, X^*)$ we define

$$G_0 = -\frac{d}{dt} + A(\cdot) \quad \text{and} \quad G_0' = \frac{d}{dt} + A(\cdot)^* \tag{3.34}$$

with $D(G_0) = \tilde{F} = W_0^1(I,X) \cap L^2(I,Y)$ and $D(G_0') = \tilde{F}' := \{f \in W^1((a,b),X^*) \cap L^2(I,Y') : f(b) = 0\}$, respectively. Since $\tilde{F}'$ is dense in E^* by our assumptions, we can further set

$$G_1 = (G_0')^*. \tag{3.35}$$

Clearly, $G_0 \subset G_1$. By Step 3 of this proof, equation (3.22) holds which implies

$$R_\lambda(\lambda - G_0)u = u \qquad \text{for } u \in \tilde{F}. \tag{3.36}$$

Define $G_n = -\frac{d}{dt} + A_n(\cdot)$ on $D(G_n) = W_0^1(I,X)$. Its resolvent $R(\lambda, G_n),\ \lambda \in \mathbb{C}$, can be described as the Laplace transform of the corresponding evolution semigroup, leading to a formula analogous to (3.3). Using the Theorem of Dominated Convergence we verify that $R(\lambda, G_{n_l}) \to R_\lambda$ weakly as $l \to \infty$. Moreover, $A_n(t)^*y^* \to A(t)^*y^*$ for $y^* \in D(A(t)^*)$ and $t \in I$ as $n \to \infty$. For $v \in \tilde{F}'$ and $f \in E$ we thus obtain

$$\begin{aligned} &(G_{n_l}R(\lambda, G_{n_l})f, v) = (R(\lambda, G_{n_l})f, G_{n_l}^* v) \longrightarrow (R_\lambda f, G_0' v) \qquad \text{as } \ l \to \infty, \\ &\quad |(R_\lambda f, G_0' v)| \le c\,\|f\|_E\,\|v\|_{E^*}. \end{aligned}$$

These facts imply that $\operatorname{Ran} R_\lambda \subset D(G_1)$ and

$$(\lambda - G_1)R_\lambda f = f \qquad \text{for } f \in E. \tag{3.37}$$

From these equations we can easily check that the kernel of R_λ is trivial, hence R_λ is injective. We define $D_\lambda = \operatorname{Ran} R_\lambda$ and the operators $B_\lambda = R_\lambda^{-1}$ with domain $D(B_\lambda) = D_\lambda$. The operators B_λ are closed as the inverses of bounded operators. Then we see that $(\lambda - G_0) \subset B_\lambda \subset (\lambda - G_1)$ because of (3.36) and (3.37).

Step 6. Our strategy is to define G by inverting B_λ for $\lambda = 0$. This is possible if D_λ does not depend on λ. Aiming at a proof of this assertion, we show that multiplication with $\alpha \in C^1([a,b])$ maps D_λ into itself.

The *extrapolation space* $E_{\lambda,-1}$ is the closure of E with respect to the norm $\|f\|_{\lambda,-1} = \|R_\lambda f\|$. The operator $B_\lambda : D_\lambda \to E$ can be extended to an isometric operator $B_{\lambda,-1} : E \to E_{\lambda,-1}$ which is isomorphic to B_λ. The crucial fact about extrapolation spaces which will be used in the following is that $u \in D_\lambda$ if and only if $B_{\lambda,-1}u \in E$. Details can be found in [22, §II.5].

The density of $\tilde{F}$ in E implies the density of $B_\lambda \tilde{F}$ in $E_{\lambda,-1}$. For $u \in \tilde{F}$, $f = B_\lambda u$, and $\alpha \in C^1([a,b])$, we find the following equation

$$B_\lambda(\alpha u) = (\lambda - G_0)(\alpha u) = \dot{\alpha}u + \alpha B_\lambda u,$$

which enables us to estimate

$$\|\alpha f\|_{\lambda,-1} = \|\alpha u - R_\lambda(\dot{\alpha}u)\| \le c\,\|u\| = c\,\|R_\lambda f\| = c\,\|f\|_{\lambda,-1}.$$

We see that multiplication with α, regarded as an operator on $B_\lambda \tilde{F}$, is bounded. So the operator $f \mapsto \alpha f$ can be extended to a bounded operator $M_\alpha : E_{\lambda,-1} \to E_{\lambda,-1}$. By approximation, we deduce

$$B_{\lambda,-1}(\alpha u) = \dot{\alpha} u + M_\alpha \, B_{\lambda,-1} u$$

for all $u \in E$. In particular, if $u \in D_\lambda$, then the right-hand side belongs to E and hence $\alpha u \in D_\lambda$. This completes the intermediate step.

Step 7. Set $e_\lambda(t) = e^{-\lambda t}$. For given $f \in E$ and $\lambda \neq \mu \in \mathbb{C}$ we have

$$R_\lambda f = e_{\lambda-\mu} R_\mu (e_{\mu-\lambda} f).$$

Due to this equation and Step 6, every $g = R_\lambda f \in D_\lambda$ is also an element of D_μ and vice versa. Therefore $D_\lambda =: D$ is independent of $\lambda \in \mathbb{C}$. We define $G = -B_{\lambda=0}$ with $D(G) = D$. Hence $G_0 \subset G \subset G_1$ so that $\lambda - G = B_\lambda$ and $R_\lambda = (\lambda - G)^{-1}$ by (3.37). With estimate (3.33) we deduce $|||(\lambda - G)^{-1} f||| = (\lambda - w)^{-1} |||f\|$ for $\lambda > w$. Application of the Theorem of Hille and Yosida (Theorem 2.4) shows that G generates a semigroup $S(\cdot)$ on $(E, ||| \cdot |||)$ and hence on E, as the norms are equivalent.

On the other hand, we define

$$(T(\tau)f)(t) = \begin{cases} U(t, t-\tau) f(t-\tau) & t, t-\tau \in I, \\ 0 & t \in I, t-\tau \notin I, \end{cases} \tag{3.38}$$

for $f \in E$ and $\tau \geq 0$. The continuity properties of $U(t,s)$ enable us to check that $\tau \mapsto T(\tau) f \in E$ is continuous and that the Laplace transform of $T(\cdot) f$ is equal to $R_\lambda f$. But the Laplace transformation is unique, so $T(\tau) = S(\tau)$ for all $\tau \geq 0$. Hence $T(\cdot)$ is a semigroup with generator G.

The semigroup property of $(T(\sigma))_{\sigma \geq 0}$ has its counterpart on the level of the weak limits $U(t,s)$ of the sequence of propagators $U_{n_l}(t,s)$—it corresponds to the causal factorization which we want to prove. However, $(T(\sigma))_{\sigma \geq 0}$ acts on an L^2 space and we obtain the causal factorization only up to a set of measure zero.

Notice that $(T(\sigma))_{\sigma \geq 0}$, as defined in (3.38), is already an evolution semigroup in the sense of Howland's definition by construction, but the additional condition about the continuous embedding of $D(G)$ in $C_0(I, X)$, necessary in Theorem 3.5, is not clear. However, we are able to derive more properties of $T(\cdot)$ and the propagators $U(t,s)$ associated with it. Thereby we establish regularity of the solutions of the nonautonomous Cauchy problem (2.2), which is not accessible in Howland's original approach.

Let $\{x_k\}$ be a dense sequence in X and set $t \mapsto f_k(t) := x_k \in E$. The weak limit $U(t,s)$ is weakly continuous in (t,s), hence weakly measurable and strongly measurable by Pettis' Theorem. For $0 \leq \tau, \sigma \leq n$ the semigroup property of $T(\cdot)$

states $T(\tau)T(\sigma)f_k(t) = T(\tau+\sigma)f_k(t)$ in $L^2([0,n]^2 \times I, X)$. We evaluate this as an equation in X. Thus for the propagator

$$U(t,t-\tau)U(t-\tau,t-\tau-\sigma)x_k = U(t,t-\tau-\sigma)x_k \tag{3.39}$$

for all $k \in \mathbb{N}$ and all (t,τ,σ) belonging to a set $\Omega \subset \{(t,\tau,\sigma) \in I \times \mathbb{R}_+ \times \mathbb{R}_+ : t-\tau-\sigma > a\} =: \Delta$ so that $N_k'' := \Delta \setminus \Omega$ has measure 0. This is the set, where (3.39) fails to hold. Equation (3.39) can be extended for fixed $(t,\tau,\sigma) \in \Omega$ to all $x \in X$ by approximation. We obtain an exceptional set $\bigcup_{k\in\mathbb{N}} N_k'' = N''$ of measure 0 as a countable union of null sets. We transform the set of variables in a linear way by setting $r = t-\tau$. As the image of a null set under a Lipschitz continuous function is again a null set [1, Theorem IX.5.9], the image of N'' under this transformation is again a set of measure 0. In the following, we need another fact about null sets: Let $M \subset \mathbb{R}^n$ be a set of Lebesgue measure zero. Then there exists a set $M' \subset \mathbb{R}^{n-1}$ such that for all $x \in \mathbb{R}^{n-1} \setminus M'$ the sets $\{y \in \mathbb{R} : (x,y) \in M\}$ have measure zero. For the proof, see [54, Lemma VI.8.3]. Applied to N'', we conclude that there is a set $N' \subset D_I$ of measure 0 such that

$$U(t,r)U(r,r-\sigma)x = U(t,r-\sigma)x \tag{3.40}$$

for all $x \in X$, $(t,r) \in D_I \setminus N'$, and a.e. $\sigma \in [0, r-a]$. Varying σ for fixed (t,r), we then obtain

$$U(t,r)U(r,s) = U(t,s) \tag{3.41}$$

for all $a \le s \le r$. As above, we conclude that there is a set $N \subset I$ of measure 0 such that (3.41) holds for all $r \notin N$ and for a.e. $t \ge r$. Using the weak continuity of $t \mapsto U(t,s)$, we extend (3.41) to all $(t,s) \in D_I$ and $r \in [s,t] \setminus N$.

Finally, if $(t',s') \in D_I$ converges to (t,s) with $t > s$, we fix $r \notin N$ between $\min\{s,s'\}$ and $\max\{t,t'\}$. The continuity results from Step 3 then show that the difference

$$U(t',s') - U(t,s) = U(t',r)(U(r,s') - U(r,s)) + (U(t',r) - U(t,r))U(r,s)$$

tends weakly to 0. □

Notice that in all arguments involving the admissible bounded approximations it would be possible to use the alternative definition of an admissible bounded approximation *on* Y, see the remark after Definition 2.13.

For the special case of X being a Hilbert space with skew-adjoint $A(t)$, we obtain a better result. We can show that in this situation $U(t,s)$ is indeed independent of the choice of the approximating sequence $U_{n_l}(t,s)$. Moreover, we derive continuity properties and unitarity of $U(t,s)$ up to a set of measure 0.

We do need the assumption of a bounded time interval no longer and thus assume $I = \mathbb{R}$ in the following. This is no loss of generality: Given operators $H(t)$, $t \in [a,b]$, fulfilling the assumptions of the next theorem, we can extend this operator family by $H(t) = H(a)$ if $t < a$ and $H(t) = H(b)$ if $t > b$.

Theorem 3.21. *Let $H(t)$, $t \in \mathbb{R}$, be self-adjoint operators on a separable Hilbert space X and let Y be a Banach space densely embedded in X. Assume that*

(i) $Y \hookrightarrow D(H(t))$ *locally uniformly in* t,

(ii) $t \mapsto (i + H(t))^{-1}$ *is strongly continuous,*

(iii) $t \mapsto H(t)y$ *is continuous for* $y \in Y$.

Then there are contractive propagators $U(t,s)$ with $U(t,s)^* = U(s,t)$, $(t,s) \in \mathbb{R}^2$, on X which fulfill the causal factorization property,

$$U(t,r)U(r,s) = U(t,s), \tag{3.42}$$

for all $t, s \in \mathbb{R}$ and $r \in \mathbb{R} \setminus N$ for a set N of measure 0. Moreover,

(a) these operators are surjective for all $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus N$, isometric for all $t \in \mathbb{R} \setminus N$ and $s \in \mathbb{R}$, hence unitary with $U(t,s) = U(s,t)^{-1}$ for all $t, s \in \mathbb{R} \setminus N$;

(b) $(t,s) \mapsto U(t,s)$ *is weakly continuous on* $\mathbb{R}^2$;

(c) $(t,s) \mapsto U(t,s)$ *is strongly continuous at* (r,r) *and on* $(\mathbb{R} \setminus N) \times \mathbb{R}$;

(d) these operators have the differentiability properties

$$\frac{\partial}{\partial s} U(t,s)y = -U(t,s)A(s)y, \tag{3.43}$$

$$\frac{\partial^+}{\partial t} U(t,s)y|_{t=s} = A(t)y \tag{3.44}$$

for $t, s \in \mathbb{R}$ and $y \in Y$.

The generator of the strongly continuous semigroup $(T(\sigma))_{\sigma \geq 0}$, given by

$$(T(\sigma)f)(t) = U(t, t-\sigma)f(t-\sigma) \tag{3.45}$$

on $E := L^2(\mathbb{R}, X)$, is an extension of $G_0 = -d/dt + A(\cdot)$ defined on $\tilde{F} := W_0^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, Y)$. Moreover, for all self-adjoint admissible bounded approximations $H_n(t)$ of $H(t)$ with generated evolution family $U_n(t,s)$ one has $U_n(t,s) \to U(t,s)$ weakly for $(t,s) \in \mathbb{R}^2$ and strongly for $(t,s) \in (\mathbb{R} \setminus N) \times \mathbb{R}$ as $n \to \infty$.

The function $u = U(\cdot, s)x$ is the unique approximative solution of the nonautonomous Cauchy problem (2.2) with $s \in \mathbb{R}$ and $x \in X$.

Proof. Set $A(t) = -iH(t)$. Again we organize the proof in several steps.

Step 1. Let $A_n(t)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, be skew-adjoint admissible bounded approximations of $A(t)$, e.g. as obtained in Lemma 2.14(ii). Let $U_n(t,s)$, $(t,s) \in \mathbb{R}^2$, be the unitary evolution family generated by $A_n(\cdot)$. Step 1–Step 3 of the proof of Theorem 3.20 work also for our present operators $A_n(t)$ and $U_n(t,s)$ with $(t,s) \in \mathbb{R}^2$, $M = 1$ and $w = 0$. The constants c are uniform on compact time intervals. Thus we obtain a subsequence n_l and contractive operators $U(t,s)$, $(t,s) \in \mathbb{R}^2$, such that $U_{n_l}(t,s) \to U(t,s)$ weakly as $l \to \infty$ for $(t,s) \in \mathbb{R}^2$. This fact implies that $U(t,s)^* = U(s,t)$ for $t,s \in \mathbb{R}$. We further have the differentiability properties (3.43) and (3.44) for $t,s \in \mathbb{R}$ and $y \in Y$. By Step 4–Step 7 of the proof of Theorem 3.20, $U(t,s)$ satisfies the continuity properties stated in this theorem for all $-n < s \le t \le n$ and $n \in \mathbb{N}$, hence for all $(t,s) \in D_{\mathbb{R}}$. In particular, the causal factorization (3.21) holds for all $t,s \in \mathbb{R}$ and $r \in \mathbb{R} \setminus N$ for a set N of measure 0.

Now we want to examine the case $t \le s$. For this reason, we introduce the skew-adjoint operators $B(t) = -A(-t)$ for $t \in \mathbb{R}$ and take the skew-adjoint admissible bounded approximation $B_n(t) = -A_n(-t)$ of $B(t)$. Consequently, $B_n(\cdot)$ generates the evolution family $V_n(\tau,\sigma) = U_n(-\tau,-\sigma)$, $(\tau,\sigma) \in D_{\mathbb{R}}$. Repeating our argument, we see that the operators $U(t,s)$, $-\infty < t \le s < \infty$, also satisfy the causal factorization (3.21) up to an exceptional null set N and the continuity properties stated in Theorem 3.20. These facts imply that $\mathbb{R}^2 \ni (t,s) \mapsto U(t,s)$ is weakly continuous and strongly measurable, and it is strongly continuous at $(r,r), r \in \mathbb{R}$.

Step 2. To establish the invertibility properties of $U(t,s)$, we define $T(\tau)f(s) = U(s,s-\tau)f(s-\tau)$ for $\tau \ge 0$, a.e. $s \in \mathbb{R}$ and $f \in E$. Due to the properties obtained in Step 1, $T(\cdot)$ is a contractive, strongly continuous semigroup. Its generator is denoted by G. By reflexivity, its adjoint G^* is also the generator of a strongly continuous semigroup which coincides with $(T(\sigma)^*)_{\sigma\ge 0}$, see [67, §1.10]. Moreover, by $S(\tau)g = U(\cdot,\cdot-\tau)g(\cdot-\tau)$, $\tau \le 0$, we define another contractive strongly continuous semigroup on $L^2(\mathbb{R},X)$ with time interval $\mathbb{R}_-$ generated by an operator $\hat{G}$. This is seen in a similar way as in the case of $T(\sigma)$. Note that we do not yet know that $S(-\tau) = T(\tau)^{-1}$, $\tau \ge 0$.

We calculate

$$(T(-\tau)^*g)(s) = U(s-\tau,s)^*g(s-\tau) = U(s,s-\tau)g(s-\tau) = (S(\tau)g)(s)$$

for $\tau \le 0$ and a.e. $s \in \mathbb{R}$. This means that $-G^* = \hat{G}$. Hence G^* generates a contractive C_0–group, see [22, §II.3.11] or [67, §1.6]. Therefore, this group is isometric and thus unitary. As a consequence, G is skew-adjoint and

$$T(-\tau) = T(\tau)^{-1} = T(\tau)^* = S(-\tau), \qquad \tau \ge 0.$$

The map $\mathbb{R} \ni s \mapsto U(s,s-\tau)x$ is weakly continuous, hence strongly measurable, for all $\tau \in \mathbb{R}$ and $x \in X$. We take a dense sequence $\{x_k\}$ in X. For $\varepsilon > 0$

set $f = \chi_{[s-\tau, s-\tau+\varepsilon]} x_k$. Then we obtain

$$\|x\|^2 = \frac{1}{\varepsilon}\,\|f\|_2^2 = \frac{1}{\varepsilon}\int_{\mathbb{R}} \|(T(\tau)f)(\sigma)\|^2\, d\sigma = \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \|U(\sigma, \sigma-\tau)x\|^2\, d\sigma. \tag{3.46}$$

Now, sending $\varepsilon \to 0$, we obtain

$$\|U(s, s-\tau)x_k\| = \|x_k\|$$

for $\tau \in \mathbb{Q}$, $k \in \mathbb{N}$ and $s \in \mathbb{R} \setminus N_1'$ with a null set N_1'. Forming the countable union of these null sets, we get a null set N_1 such that this equality holds for all $\tau \in \mathbb{Q}$, $x \in X$ and $s \in \mathbb{R} \setminus N_1$. The strong continuity of $r \mapsto U(s, r)$ further implies that $U(s, r)$ is an isometry for $s \in \mathbb{R} \setminus N_1$ and all $r \in \mathbb{R}$. Therefore, $t \mapsto U(t, s)$ is strongly continuous on $\mathbb{R} \setminus N_1$ for all $s \in \mathbb{R}$ and $U_{n_l}(t, s)$ converges strongly to $U(t, s)$ as $l \to \infty$ for $t \in \mathbb{R} \setminus N_1$ and $s \in \mathbb{R}$.

Take functions $f_{k,n} \in C_c(\mathbb{R}, X)$ equal to x_k on $[-n, n]$. Then the equality $[T(\tau)T(-\tau)f_{k,n}](\cdot + \tau) = f_{k,n}(\cdot + \tau)$ for $\tau \in \mathbb{R}$ yields

$$x_k = U(s+\tau, s)U(s, s+\tau)x_k$$

for all $\tau \in \mathbb{Q}$, $k \in \mathbb{N}$ and $s \in \mathbb{R} \setminus N_2$ with a null set N_2. Varying τ and using the density of $\{x_k\}$, we thus obtain

$$x = U(r, s)U(s, r)x$$

for all $x \in X$, $r \in \mathbb{R}$ and $s \notin N_2$. Hence, $U(r, s)$ is surjective for $r \in \mathbb{R}$ and $s \in \mathbb{R} \setminus N_2$. Consequently, $U(t, s)$ is unitary and $U(t, s)^{-1} = U(s, t) = U(t, s)^*$ for $t, s \notin N$, where we may assume without loss of generality that we have the same null set as in Step 1. It is then easy to check that $U(t, r)U(r, s) = U(t, s)$ for $t, r, s \in \mathbb{R} \setminus N$. Using the continuity properties of U, this equation holds for all $t, s \in \mathbb{R}$ and $r \in \mathbb{R} \setminus N$.

Let $(t', s') \to (t, s)$ in $\mathbb{R}^2$ with $t \notin N$. We fix $r \notin N$ and write

$$U(t', s') - U(t, s) = U(t', r)(U(r, s') - U(r, s)) + U(t', s) - U(t, s).$$

Thus $(t, s) \mapsto U(t, s)$ is strongly continuous on $(\mathbb{R} \setminus N) \times \mathbb{R}$.

Step 3. We now check the uniqueness of $U(t, s)$. Let $A_l^{(1)}(t)$ and $A_l^{(2)}(t)$ be skew-adjoint admissible bounded approximations of $A(t)$ generating unitary evolution families $U_l^{(i)}(t, s)$ which converge weakly to $U^{(i)}(t, s)$ ($i = 1, 2$, $l \in \mathbb{N}$, $t, s \in \mathbb{R}$). These approximations may result from different subsequences in Step 1 of the proof of Theorem 3.20 or from different approximations of $A(t)$. The operators $U^{(i)}(t, s)$ have the properties established so far, in particular, they are unitary up to a set of measure 0. Let $G^{(i)}$ be the generators of the corresponding 'evolution semigroups' on $E(I) = L^2(I, X)$ given as in (3.38) for some $I = (a, b]$.

We now define $U(t,s) = \frac{1}{2}U^{(1)}(t,s) + \frac{1}{2}U^{(2)}(t,s)$ for $t,s \in \mathbb{R}$. Observe that $U(t,s)^* = U(s,t)$ and that $U(\cdot,\cdot)$ satisfies the continuity properties stated in Theorem 3.20.

We show that $U(\cdot,\cdot)$ is a unitary evolution family almost everywhere, proceeding as above and as in Theorem 3.20. This fact will lead to the desired equality $U(t,s) = U^{(1)}(t,s) = U^{(2)}(t,s)$. Equation (3.30) implies that

$$\begin{aligned}
U(t,s)y - U(t,r)y &= \frac{1}{2}\Big(U^{(1)}(t,s)y - U^{(1)}(t,r)y + U^{(2)}(t,s)y - U^{(2)}(t,r)y\Big) \\
&= -\frac{1}{2}\int_r^s U^{(1)}(t,\tau)A(\tau)y\,d\tau - \frac{1}{2}\int_r^s U^{(2)}(t,\tau)A(\tau)y\,d\tau \\
&= -\int_r^s U(t,\tau)A(\tau)y\,d\tau
\end{aligned}$$

for $t \geq r,s$ and $y \in Y$. Thus (3.43) holds for $U(t,s)$ and $A(t)$. Now we can deduce (3.36) as in Step 5 of the proof of Theorem 3.20, defining R_λ, G_0, G_0' and G_1 on $E = L^2(I,X)$ as before. We define analogously $R_\lambda^{(i)}$, and we recall that $R_\lambda^{(i)} = R(\lambda, G^{(i)})$ for $\lambda \in \mathbb{C}$ and $G^{(i)} \subset G_1$ by the proof of Theorem 3.20. We thus obtain

$$\begin{aligned}
(\lambda R_\lambda f, v) &= \frac{1}{2}\,(\lambda R_\lambda^{(1)} f, v) + \frac{1}{2}\,(\lambda R_\lambda^{(2)} f, v) \\
&= (f,v) + \frac{1}{2}\,(G_1 R_\lambda^{(1)} f, v) + \frac{1}{2}\,(G_1 R_\lambda^{(2)} f, v) \\
&= (f,v) + (R_\lambda, G_0' v)
\end{aligned}$$

for $f \in L^2(I,X)$ and $v \in \tilde{F}$. Therefore also (3.37) holds. Now we can repeat Step 6 and Step 7 in order to construct $G = G_I$ on $E(I)$ such that $R_\lambda = R(\lambda, G)$ and G_I generates a semigroup $T_I(\tau)$ given as in (3.38). Moreover, $U(t,s)$ satisfies $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = \mathbb{1}$ for $(t,s) \in D_I$, $r \in [s,t] \setminus N(I)$, and some null set $N(I)$. This property then holds for $I = \mathbb{R}$. By duality, it also valid for $(s,t) \in D_\mathbb{R}$ and $r \in [t,s] \setminus N$.

Reasoning as in Step 2, we derive that the 'evolution semigroup' $T(\tau)$, $\tau \geq 0$, associated to $U(t,s)$ on $L^2(\mathbb{R},X)$ can be embedded into a unitary group given by operators $U(t,s)$, $(t,s) \in \mathbb{R}^2$, which are unitary for $t,s \in \mathbb{R} \setminus N$. But then we have,

$$\begin{aligned}
4\|x\|^2 &= 4(U(t,s)x, U(t,s)x) \\
&= 2\,\|x\|^2 + (U^{(1)}(t,s)x, U^{(2)}(t,s)x) + (U^{(2)}(t,s)x, U^{(1)}(t,s)x),
\end{aligned}$$

for $x \in X$ and $t,s \in \mathbb{R} \setminus N$. This means that the numerical range, and hence the spectrum, of the unitary operator $U^{(2)}(t,s)^* U^{(1)}(t,s) = U^{(2)}(t,s)^{-1} U^{(1)}(t,s)$ is contained in the line $\operatorname{Re}\lambda = 1$, so that $U^{(2)}(t,s)^{-1}U^{(1)}(t,s) = \mathbb{1}$. As a result,

$U(t,s) = U^{(1)}(t,s) = U^{(2)}(t,s)$ for $t,s \in \mathbb{R} \setminus N$, and this equality extends to $t,s \in \mathbb{R}$ by the continuity properties of $U(t,s)$ and $U^{(i)}(t,s)$. Thus we have shown that $U(t,s)$ does not depend on the approximation, as asserted. Consequently, the sequence $U_n(t,s)$ from Step 1 of this proof converges strongly to $U(t,s)$ as $n \to \infty$ for all $t,s \in \mathbb{R} \setminus N$ and weakly for $t,s \in \mathbb{R}$.

Step 4. We set $u(t) = U(t,s)x$ for $t \in \mathbb{R}$ and some $s \in \mathbb{R}$ and $x \in X$. Take the sequence $U_n(t,s)$ from Step 1 of this proof. It is clear from the properties of $(t,s) \mapsto U_n(t,s)$ that u is an approximative solution on $\mathbb{R}$ of the nonautonomous Cauchy problem (2.2). If v is another approximative solution corresponding to some skew-adjoint admissible bounded approximations $A_n(t)$, then the calculation

$$\begin{aligned} U(t,s)v(s) - v(t) &= \lim_{n\to\infty} U_n(t,s)v_n(s) - v_n(t) \\ &= \lim_{n\to\infty} \int_s^t U_n(t,\sigma)\left(-\dot{v}_n(\sigma) - A_n(\sigma)v_n(\sigma)\right)\, d\sigma = 0 \end{aligned} \tag{3.47}$$

shows that $u = v$. □

Also in this theorem, the use of admissible bounded approximations on Y according to the remark after Definition 2.13 would be possible.

We want to point out that there might be other extensions of G_0 which generate evolution groups. But the corresponding evolution families give no approximative solutions. If G_0 is essentially skew-adjoint on F, there exists exactly one extension of G_0 that is an evolution generator with the range condition (3.8) fulfilled. In this case the approximative solution is actually a mild solution.

For the proof of the independence of the construction of the admissible bounded approximation we use the fact that only the trivial convex combination of unitary operators is again unitary. One might have the impression, that a similar argument works yet on the level of the semigroups. This would allow for a generalization of the theorem to the context of dissipative generators. However, the conjecture that only the trivial convex combination of strongly continuous semigroups is again a strongly continuous semigroup is wrong. There is a counterexample due to R. Schnaubelt. Consider the Banach space $X = \mathbb{C}^2$ and the strongly continuous semigroups $(S(\tau))_{\tau\geq 0}$ and $(T(\tau))_{\tau\geq 0}$, given by

$$S(\tau) := \mathbb{1} \quad \text{and} \quad T(\tau) := \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

for $\tau \geq 0$. Then $pT(\tau) + (1-p)S(\tau)$, $p \in (0,1)$, is again a strongly continuous semigroup in τ. Hence, our argument does apply only to the self-adjoint context.

3.3.1 Application: Time evolution of the $P(\varphi)_2$ model

We fix a real, semibounded polynomial of degree $2n$,

$$P(\lambda) = \sum_{j=0}^{2n} a_j \lambda^j,$$

and choose a real test function $g \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq g(t,x) \leq 1$. Define the interaction Hamiltonian localized in a compact spacetime region by

$$V(t) = V(t;g) = \int g(t,x) : P(\varphi(x)) : \, dx.$$

The Hamiltonian of the $P(\varphi)_2$ model, given by $H(t) = H_0 + V(t;g)$, is densely defined (according to the perturbation theory predictions mentioned in Section 1.1.3). It is essentially self-adjoint on $D(H_0) \cap D(V(t;g))$. For details and references to the literature see Appendix A. For the $P(\varphi)_2$ model, we can prove the existence of the time evolution by the existence theorem for approximative solutions.

Theorem 3.22. *Let $H(t)$, $t \in \mathbb{R}$, be the Hamiltonian of the massive $P(\varphi)_2$ model with localized interaction, defined as above with $V(t;g)$ for a test function $g \in C_c^\infty(\mathbb{R}^2)$, $0 \leq g \leq 1$ and a polynomial $P(\lambda)$ of order $2n$, $n \in \mathbb{N}$ with $a_{2n} > 0$. Then the time-dependent Schrödinger equation (2.2) with $A(t) = -iH(t)$ has a unique approximative solution given by operators $U(t,s)$ with the properties stated in Theorem 3.21.*

Proof. Denote by Y the intersection $D(H_0) \cap D(N^n)$ endowed with the sum of the graph norms of H_0 and N^n. This space is a core for $H(t), t \in \mathbb{R}$, see for example the proof of [36, Thm. 3.2.1].

Consider the term of highest order in the interaction $V(t)$:

$$V_{2n}(t) = a_{2n} \int g(t,x) : \varphi(x)^{2n} : \, dx.$$

For $y \in Y$ the the interaction term is dominated by a power of the number operator, see Theorem A.1,

$$\begin{aligned} \|V_{2n}(t)y\| &\leq \|V_{2n}(t)(N+1)^{-n}\| \|(N+1)^n y\| \\ &\leq c \|w(t,\cdot)\|_{L^2(\mathbb{R}^{2n})} \|(N+1)^n y\|, \end{aligned}$$

where c is a t-independent constant and $w(t,k_1,\ldots,k_{2n})$ denotes the numeric kernel of the expansion of $V_{2n}(t)$ into Wick monomials. It is given by

$$w(t,k_1,\ldots,k_{2n}) = \hat{g}(t, k_1 + \cdots + k_{2n}) \omega(k_1)^{-1/2} \cdot \cdots \cdot \omega(k_{2n})^{-1/2},$$

where $\hat{g}$ denotes the Fourier transform of g with respect to x, and the L^2-norm is evaluated with respect to $(k_1, \ldots, k_{2n}) \in \mathbb{R}^{2n}$, see [16, §6.1] By repeated use of Young's inequality, we estimate

$$\|w(t,\cdot)\|_{L^2(\mathbb{R}^{2n})} \leq c\|g(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

(cf. [16, Lem. 6.1]). The other summands of $V(t)$ can be treated in the same way. These inequalities imply the uniform boundedness of the embedding $Y \hookrightarrow D(H(t))$. Similarly one checks the continuity of $t \mapsto H(t)y$ for all $y \in Y$.

Because Y is a core for $H(t)$, the strong continuity of $t \mapsto -R(-i, H(t)) = (i + H(t))^{-1}$ follows. Now we can apply Theorem 3.21 to this problem. □

3.3.2 Application: Time evolution in curved spacetime

We generalize the former result about the existence of the time evolution for the $P(\varphi)_2$ model on a two-dimensional Lorentzian manifold. This extends a result of J. Dimock [18], who considers $P(\lambda) = \lambda^4$ and has to restrict himself to interactions with a localization which is 'small' compared to a given background, see Section 2.2.5.1.

Assume (M, g) is a d-dimensional Lorentzian manifold. Consider the covariant field equation in local coordinates,

$$(|g|^{-1/2}\partial_\mu g^{\mu\nu}|g|^{1/2}\partial_\nu + m^2)\varphi + 2n\lambda\varphi^{2n-1} = 0,$$

where $|g| = |\det\{g^{\mu\nu}\}|$ and $\lambda > 0$. The coordinates are chosen such that the hypersurfaces $x^0 = t = \text{const}$ are spacelike. Define the canonical conjugate field π to be the normal derivative density

$$\pi = |g|^{1/2} g^{\nu\mu}\partial_\mu\varphi.$$

With the notation $\alpha = (g^{00})^{-1/2}$, $\beta^j = g^{j0}(g^{00})^{-1}$, $\gamma_{ij} = -g_{ij}$ and $\gamma = \det\{\gamma_{ij}\}$, the field equation can be written as a Hamiltonian system

$$\frac{\partial\varphi}{\partial t} = \alpha\gamma^{-1/2}\pi + \beta^i\partial_i\varphi, \tag{3.48}$$

$$\frac{\partial\pi}{\partial t} = \partial_i\gamma^{ij}\alpha\gamma^{1/2}\partial_j\varphi - \alpha\gamma^{1/2}(m^2 + 2n\lambda\varphi^{2n-1}) + \partial_j\beta^j\pi, \tag{3.49}$$

where $1 \leq i, j \leq d-1$. The corresponding Hamiltonian is given by

$$\begin{aligned} H(t,\varphi,\pi) = \frac{1}{2}\int_{x_0=t} \Big(&\alpha\gamma^{-1/2}\pi^2 + \alpha\gamma^{1/2}(\gamma^{ij}\partial_i\varphi\partial_j\varphi + m^2\varphi) \\ &+ \alpha\gamma^{1/2}\lambda\varphi^{2n} + \pi\beta^j\partial_j\varphi\Big). \end{aligned} \tag{3.50}$$

Now we specialize to $d = 2$. For a two-dimensional manifold one can always choose local coordinates in such a way that $g_{\mu\nu} = \Lambda\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowskian metric and $\Lambda = \Lambda(t,x)$ is a smooth function on $\mathbb{R}^2$. Hence, $|g|^{1/2} = \Lambda$ and $g^{\mu\nu} = \Lambda^{-1}\eta^{\mu\nu}$. The field equation becomes

$$(\Box + \Lambda m^2)\varphi + 2n\lambda\Lambda\varphi^{2n-1} = 0,$$

the normal derivative is $\pi = \frac{\partial\varphi}{\partial t}$, and the Hamiltonian is

$$H(t,\varphi,\pi) = \frac{1}{2}\int_{x_0=t} \left(\pi^2 + (\nabla\varphi)^2 + m^2\Lambda\varphi^2 + \lambda\Lambda\varphi^{2n}\right).$$

Proceeding in the same way as Dimock in [18], we assume that $\Lambda = \Lambda_0 + \Lambda_1 > 0$ with $\Lambda_0 > 0$ a constant and $\Lambda_1 \in C_0^\infty(\mathbb{R}^2)$. Defining H_0 and the Wick product with respect to $\Lambda_0^{1/2}$, we obtain the formal expression for the Hamiltonian which contains a non localized term. To define the Hamiltonian as an operator on the Fock space of the free field, we introduce an ultraviolet cut-off by restricting the space integration in the corresponding term to a compact interval. Denote by χ_l the characteristic function of the interval $[-l,l]$, $l \in \mathbb{N}$. The Hamiltonian is given by

$$\begin{aligned} H(t) = H_0 + \int \lambda(\Lambda_0\chi_l(x) + \Lambda_1(t,x)) : \varphi^{2n}(x) : \, dx \\ + \int \frac{m^2}{2}\Lambda_1(t,x) : \varphi^2(x) : \, dx. \end{aligned} \tag{3.51}$$

Theorem 3.23. *The time-dependent Schrödinger equation with the Hamiltonian* (3.51) *has an approximative solution given by an almost everywhere unitary evolution family $U_l(t,s)$, depending on the localization l.*

Proof. We verify the prerequisites of Theorem 3.21 analogous to the preceding section. As before, the Hamiltonian is an essentially self-adjoint operator on $D(H_0) \cap D(V(t))$, where

$$V(t) = \int \left(\lambda(\Lambda_0\chi_l(x) + \Lambda_1(t,x)) : \varphi^{2n}(x) : + \frac{m^2}{2}\Lambda_1(t,x) : \varphi^2(x) :\right) dx.$$

Again, we set $Y = D(H_0) \cap D(N^n) \subset D(H(t)) \ \forall\ t$. The continuity of $t \mapsto H(t)y$ for $y \in Y$ is verified as in the proof of Theorem 3.22, as well as the other requirements of Theorem 3.21. Thus there exists a propagator $U_l(t,s)$ associated with an approximative solution of the time-dependent Schrödinger equation describing the time evolution associated with $H(t)$. $\Box$

The propagator depends on the ultraviolet cut-off l. But we can use [18, Theorem 2.3] to show that the time evolution induced on local observables is independent of l if l is chosen large enough. Clearly, all the critical comments by Dimock concerning this construction apply to our setting as well, in particular covariance with regard to coordinate dependence is unclear. Moreover, the results of Torre and Varadarajan [91] show that it is not possible to generalize a similar result to curved spacetimes of dimension larger than two, even for free fields. Nevertheless, it might be interesting to reexamine the free field setting with regard to the existence result of Theorem 3.21.

3.3.3 Application: Goldstein's example

For a direct application of Theorem 3.21 we need a sufficiently large intersection of the domains of the time-dependent generators. As we have seen, this is the case for $P(\varphi)_2$ models. But for models which require an infinite renormalization, we do not expect this to be fulfilled. As a testing ground for a situation, where the intersection of the domains of the generators is trivial, we examine again Goldstein's example.

As in Section 2.2.1, we define the operators S, T, L and U_1 as well as the test functions φ, η and ψ and the Hamiltonian $H(t)$ of equation (2.5). Moreover, we choose a smooth function $\kappa \in C^\infty(\mathbb{R})$ such that $\kappa(t) = 0$ for $0 \leq t < 1$ and $\kappa(t) = 1$ for $2 \leq t < \infty$.

Consider the Hamiltonian

$$\tilde{H}(t) := (\varphi(t) + \psi(t))S + (\eta(t) + \dot{\kappa}(t))L$$

with $\tilde{H}(t) = \tilde{H}(0)$ for $t \leq 0$. It is self-adjoint for all $t \in \mathbb{R}$ and $Y := D(S) \hookrightarrow D(\tilde{H}(t))$. Continuity of $t \mapsto \tilde{H}(t)y$ for $y \in Y$ and strong continuity of $t \mapsto (i + H(t))^{-1}$ follow easily from the continuity of the test functions and their support properties. Hence, by Theorem 3.21, there is a unique approximative solution $\tilde{u}(t) = \tilde{U}(t,s)x$ for all $x \in X$ with a unitary propagator (up to a set of measure zero). Associated with it there is an evolution semigroup $(\tilde{T}(\sigma))_{\sigma \geq 0}$ with generator $\tilde{G}$ which is an extension of $\tilde{G}_0 = -\frac{d}{dt} - i\tilde{H}(\cdot)$ with domain $W_0^{1,2}(\mathbb{R}, X) \cap L^2(\mathbb{R}, Y)$. Now set $R(t) = e^{i\kappa(t)L}$ and notice that $R(t) = \mathbb{1}$ for $t \in (0,1)$ and $R(t) = U_1$ for $t \geq 2$. It corresponds to a unitary operator $R = R(\cdot)$ on $E = L^2(\mathbb{R}, X)$.

By Theorem 3.6, $T(\sigma) = R\tilde{T}(\sigma)R^{-1}$ is an evolution semigroup with generator $G = R\tilde{G}R^{-1}$. As a result $(t,s) \mapsto U(t,s) = R(t)\tilde{U}(t,s)R(s)^{-1}$ is again a propagator with the same properties as $\tilde{U}(t,s)$. We transform $\tilde{G}_0$ correspondingly and find formally

$$G_0 = R\tilde{G}_0R^{-1} = -\frac{d}{dt} - i(\varphi(\cdot)S + \eta(\cdot)L + \psi(\cdot)T) = -\frac{d}{dt} - iH(\cdot).$$

In this sense $U(t,s)$ is the propagator associated with the nonautonomous Cauchy problem of Goldstein's example. For $x \in C^\infty(S)$ $u(\cdot)$ coincides with the solution of Theorem 2.16 by uniqueness.

Chapter 4

Existence of Local Scattering Operators

In Section 3.3 we have developed a new wellposedness theorem for the time-dependent Schrödinger equation. Furthermore, we have seen that this wellposedness result is suitable to show the existence of the time evolution for $P(\varphi)_2$ models with localized interaction.

The main result of the present chapter is the proof of existence of local scattering operators for the $P(\varphi)_2$ model. However, before we address this topic, we indicate to which extent scattering theory is covered by the approach of Section 3.2.2.2 for $(\varphi^4)_2$ with spatially localized interaction. After this, we show the existence of the local scattering operators for general $P(\varphi)_2$ models of which the $(\varphi^4)_2$ interaction is a special case. This result enables us to give an easy construction of the algebra of local observables for massless bosons in two dimensions, a theory without a ground state. We hereby demonstrate the advantage of the local construction: the disentanglement of the infrared and the ultraviolet behavior of a theory.

In this chapter, we switch back to the notation of Chapter 1 and work in the setting of a Hilbert space $\mathcal{H}$.

4.1 Scattering operator for the spatially localized $(\varphi^4)_2$ model

To establish the existence of local scattering operators, we would like to exhibit the existence of a solution of the time-dependent Schrödinger equation with an interaction which is compactly supported in space and time. In a first step, we investigate the example of Section 3.2.2.2. According to Theorem 3.17, there exists a unique propagator $U(t,s)$ associated with a mild solution, but the interaction is not localized in a compact spacetime region. Therefore, in the context of Theorem 3.17 and using the notion of mild solutions, we are not able to define the local scattering operators: in general, the condition (3.18) in Theorem 3.17 is not satisfied for compactly supported localization functions.

We investigate what amount of information about the scattering operator can be gathered from the approach in Theorem 3.17. In a first step, we can overcome the difficulty with (3.18) by convoluting g in time with an approximated δ distribution.

Example. Given a rapidly decaying approximation of the δ-distribution,

$$\delta_\alpha(t) = \frac{1}{\alpha\sqrt{\pi}} e^{-\frac{t^2}{\alpha^2}}, \tag{4.1}$$

converging weakly in $\mathcal{D}'(\mathbb{R})$ to $\delta(t)$ as $\alpha \to 0$. Let $g \in C_c^\infty(\mathbb{R}^2)$, and set $g_\alpha(t,x) = \big(\delta_\alpha *_t g\big)(t,x)$. Then we can choose $k > 0$ such that the positivity condition (3.18) is satisfied, uniformly in $t \in [-r, r]$, $r > 0$. We calculate

$$kg_\alpha(t,x) - \frac{1}{2}\dot{g}_\alpha(t,x) = \int g(s,x)\delta_\alpha(t-s)\left(k - \frac{1}{\alpha^2}(t-s)\right) ds$$

and find that the choice $k := \frac{1}{\alpha^2}(r - s')$ is sufficient, where $s' := \inf\{s \in \mathbb{R} : (s,x) \in \operatorname{supp} g\}$.

In the following, let g_α be chosen according to the preceding example for a $g \in C_c^\infty(\mathbb{R}^2)$. Denote by $U_{g_\alpha}(t,s)$ the propagator solving the Schrödinger equation to $H(t) = H_0 + V(t; g_\alpha)$ depending on g_α. The Hamiltonian is an operator on the Hilbert space $\mathcal{H}$, the Fock space of the free, massive, scalar field.

Our first result states that outside of the time support of g, the evolution family $U_{g_\alpha}(t,s)$ is close to the evolution family of the free Hamiltonian $U_0(t,s) = e^{-iH_0(t-s)}$.

Theorem 4.1. *Let $\rho > 0$ and suppose* $\operatorname{supp} g \subset [-\rho,\rho] \times [-\rho,\rho]$. *Let $\psi \in \mathcal{H}$ and $t \geq \rho$. Then*

$$\|(U_{g_\alpha}(t+s,t) - U_0(t+s,t))\psi\| \to 0 \quad \textit{for} \quad \alpha \to 0 \tag{4.2}$$

and arbitrary $s > 0$.

Proof. Choose a function $s \mapsto f(s) = u(s)\psi \in D = C_c^\infty(\mathbb{R}) \otimes Y$ with $u(t) = 1$. On these functions we can write the difference of the propagators by evaluating the image of f under the difference of the evolution groups at the point $t+s$:

$$\begin{aligned}(U_{g_\alpha}(t+s,t) - U_0(t+s,t))\psi &= \left(\left(T_{g_\alpha}(s) - T_0(s)\right) f\right)(t+s) \\ &= i\left(\int_0^s T_{g_\alpha}(\sigma)(G_0 - G_{g_\alpha})T_0(s-\sigma) f \, d\sigma\right)(t+s),\end{aligned}$$

where G_0, G_{g_α} are the generators of T_0, T_{g_α} respectively. Observe that T_0 maps D to D, thus the difference of the evolution generators $G_0 - G_{g_\alpha}$ is given by the interaction Hamiltonian. For the norm we get the estimate

$$\|(U_{g_\alpha}(t+s,t) - U_0(t+s,t))\psi\| \leq \int_0^s \| \Big(T_{g_\alpha}(\sigma)V(\cdot;g_\alpha)T_0(s-\sigma)f\Big)(t+s)\| \, d\sigma.$$

The integrand can be calculated,

$$\begin{aligned}\| &\Big(T_{g_\alpha}(\sigma)V(\cdot;g_\alpha)T_0(s-\sigma)f\Big)(t+s)\| \\ &\leq |u(t+s-2\sigma)| \|V(t+s-\sigma;g_\alpha))(N+1)^{-2}\| \|(N+1)^2\psi\|. \end{aligned} \tag{4.3}$$

Thus, we may estimate

$$\begin{aligned}&\|(U_{g_\alpha}(t+s,t) - U_0(t+s,t))\psi\| \\ &\qquad \leq \sup_{t'\in\mathbb{R}} |u(t')| \sup_{t'\in[t,t+s]} \|V(t';g_\alpha))(N+1)^{-2}\| \|(N+1)^2\psi\|.\end{aligned}$$

Application of the estimate (A.2) shows,

$$\sup_{t'\in[t,t+s]} \|V(t';g_\alpha)(N+1)^{-2}\| \leq C \sup_{t'\in[t,t+s]} \|\hat{g}_\alpha(t', \sum_{l=1}^4 k_l)\|_{L^2(\mathbb{R}^4)} \tag{4.4}$$

$$\longrightarrow 0 \quad \text{for } \alpha \to 0, \tag{4.5}$$

because with $g_\alpha \to g$ also $\hat{g}_\alpha \to \hat{g}$ and t' is outside of the support of g. $\square$

The next result shows that the scattering operators exist.

Theorem 4.2. *For fixed α, the scattering operator*

$$S(g_\alpha) = \lim_{\substack{t\to\infty \\ s\to-\infty}} e^{itH_0} U_{g_\alpha}(t,s) e^{isH_0} \tag{4.6}$$

exists as a strong limit.

Proof. According to [47], the existence of the scattering operator for the time-dependent Schrödinger equation is equivalent to the scattering problem for the evolution group. To show the existence of the wave operators for the evolution generator, it is convenient to use Cook's method [72, Theorem XI.4].

As above, let $f \in D = C_c^\infty(\mathbb{R}) \otimes Y$. We have $e^{\pm i\sigma G_0} Y \subset Y$, thus it is sufficient to show that

$$\int_{\sigma_0}^\infty \|V(\cdot;g_\alpha)e^{\pm\sigma G_0} f\| \, d\sigma < \infty. \tag{4.7}$$

To this end, observe that $t \mapsto \tilde{g}(t) = \|\hat{g}_\alpha(t, \sum k_i)\|_{L^2(\mathbb{R}^4)}$ is fast decreasing and u is compactly supported:

$$\begin{aligned}\int \|V(t; g_\alpha)(e^{\pm\sigma G_0} f)(t)\|^2 \, dt &= \int \|V(t; g_\alpha) e^{\pm i\sigma H_0} \psi\|^2 |u(t \pm \sigma)|^2 \, dt \\ &\leq c \int \tilde{g}^2(t) |u(t \pm \sigma)|^2 \, dt \\ &= c(g^2 * |\check{u}|^2)(\pm\sigma) \end{aligned} \tag{4.8}$$

is fast decreasing, and thus its square root is integrable. Hence $\|V(\cdot; g_\alpha) e^{\pm\sigma G_0} f\| = (\int \|V(t; g_\alpha)(e^{\pm\sigma G_0} f)(t)\|^2 \, dt)^{1/2}$ is integrable as a function of σ and (4.7) holds. □

Thus the dynamics $U_{g_\alpha}(t, s)$ is asymptotically free in the sense of scattering theory, for $t \to \infty, s \to -\infty$. With respect to (4.2), one sees that the dynamics is also asymptotically free in the limit $\alpha \to 0$ on arbitrary time intervals not intersecting the time support of g.

Although we have a fair amount of information on the propagators $U_{g_\alpha}(t, s)$, it is not possible to define the local scattering operators by a limit $g_\alpha \to g$. For smooth, compactly supported functions the condition (3.18) is not satisfied, as such a function can not dominate its time derivative on the boundary of its support. To perform the limit $\lim_{\alpha \to 0} U_{g_\alpha}(t, s)$ in the strong sense, one would need Kato-type assumptions. Weak limit points exist, but then the causal factorization is lacking. Here we could envisage a similar strategy as in Section 3.3, but then we arrive at approximative solutions. In view of this, it is much simpler and more general to use admissible bounded approximations. Thus we start directly from the wellposedness theory leading to approximative solutions. We obtain the $(\varphi^4)_2$ model as a special case.

4.2 The existence theorem

We give an existence theorem for local scattering operators. The underlying spacetime is the d-dimensional Minkowski space, $2 \leq d \leq 4$, with its symmetry group $O(1, d-1)$. The proper, orthochronous Lorentz group is the connection component of the identity $SO^+(1, d-1)$. We denote its universal covering group by G. As in Definition 1.5, we consider scalar theories.

Theorem 4.3. *Let $\mathcal{H}$ be a separable Hilbert space which carries a unitary, irreducible representation $\overline{\mathcal{P}} \to \mathcal{B}(\mathcal{H}), (a, \alpha) \mapsto U(a, \alpha)$ of the universal covering group $\overline{\mathcal{P}} = \mathbb{R}^d \rtimes G$ of the Poincaré group on d-dimensional spacetime, $2 \leq d \in \mathbb{N}$. Let $V(t; g)$ be a self-adjoint operator with domain $D(V(t; g))$, depending on $t = x_0 \in \mathbb{R}$, and $g \in C_c^\infty(\mathbb{R}^d)$ such that*

(i) $V(t, g) = 0$ if $g(t, \vec{x}) = 0$ for all $\vec{x} \in \mathbb{R}^{d-1}$,

(ii) $t \mapsto R(i, V(t;g))$ *is strongly continuous,*

(iii) *there is a Banach space* Y*, dense in* $\mathcal{H}$ *and continuously embedded in* $D(V(t;g))$*, locally uniformly in* t*, such that* $t \mapsto V(t,g)\psi$ *is continuous for all* $\psi \in Y$*,*

(iv) *the time translations* $U((t,\vec{0}),\mathrm{id}) = e^{-iH_0t}$ *fulfill* $e^{-iH_0t}Y \subset Y$ *and* $(e^{-iH_0t})_{t\in\mathbb{R}}$ *is strongly continuous in* Y*,*

(v) $e^{iH_0t}V(t;g_{\langle a,\Lambda(\alpha)\rangle})e^{-iH_0t} = U(a,\alpha)^*e^{iH_0t}V(t;g)e^{-iH_0t}U(a,\alpha)$*, where* $\Lambda(\alpha) \in SO^+(1,d-1)$*,* $\alpha \in G$*,* $a \in \mathbb{R}^d$ *and* $g_{\langle a,\Lambda(\alpha)\rangle}(x) = g(\Lambda(\alpha)^{-1}(x-a))$*.*

Then the local scattering operators $S(g)$ *exist.*

Proof. We organize the proof in two steps. In the first one we consider the solution of the time-dependent Schrödinger equation in the interaction picture. The second step deals with the local scattering operators.

Step 1. We transform the interaction to the *Dirac* (or *interaction*) *picture*,

$$V^D(t;g) := e^{itH_0}V(t;g)e^{-itH_0} \quad \text{with domain} \quad D(V^D(t;g)) := e^{itH_0}D(V(t;g)),$$

where H_0 is the generator of the time translations, $e^{-itH_0} = U((t,\vec{0}),\mathrm{id})$. It is easy to see that $V^D(t;g)$ is self-adjoint on $D(V^D(t;g))$: Let $\theta \in \mathcal{H}$ be arbitrary and set $\tilde{\theta} := e^{-itH_0}\theta$. Because of the self-adjointness of $V(t;g)$ there is a $\tilde{\psi} \in D(V(t;g))$ such that $\tilde{\theta} = (V(t;g)+i)\tilde{\psi}$. Thus $\psi := e^{itH_0}\tilde{\psi} \in D(V^D(t;g))$ is the pre-image of θ under $V^D(t;g)+i$. A similar argument shows $\mathrm{Ran}(V^D(t,g)-i) = \mathcal{H}$. Because of condition (iv), $Y \subset D(V^D(t;g))$. Furthermore, condition (iii), together with the unitarity of $e^{\pm itH_0}$, implies that the embedding of Y in $D(V^D(t;g))$ is also continuous, locally uniformly in t. The mapping $t \mapsto R(i, V^D(t;g)) = e^{itH_0}R(i,V(t;g))e^{-itH_0}$ is strongly continuous as a product of strongly continuous functions of bounded operators. Moreover, we find continuity of $t \mapsto V^D(t;g)\psi$ for $\psi \in Y$. We calculate

$$\begin{aligned}\|(V^D(s;g) - V^D(t;g))\psi\| &\le \|e^{isH_0}V(s;g)(e^{-isH_0} - e^{-itH_0})\psi\| \\ &\quad + \|e^{isH_0}(V(s;g)-V(t;g))e^{-itH_0}\psi\| + \|(e^{isH_0}-e^{itH_0})V(t;g)e^{-itH_0}\psi\|.\end{aligned}$$

In the limit $s \to t$, the first term on the right-hand side vanishes because Y is continuously embedded in $D(V(t;g))$ locally uniformly in t and $(e^{-itH_0})_{t\in\mathbb{R}}$ is a group in Y, strongly continuous with respect to the norm $\|\cdot\|_Y$. The second term converges to 0 because of the invariance of Y under e^{-itH_0} and the continuity properties of $V(t;g)$. The third term obviously vanishes too.

By Theorem 3.21, there exists an approximative solution of the Cauchy problem of the time-dependent Schrödinger equation with respect to $A(t) = -iV^D(t;g)$, given by a propagator $U_g^D(t,s)$, which is unique as well as strongly continuous and unitary almost everywhere in $t, s \in \mathbb{R}$. Furthermore, $U_g^D(t,s)$ is the weak limit of a sequence of propagators $\{U_{g,n}^D(t,s)\}_{n\in\mathbb{N}}$, where each element solves the time-dependent Schrödinger equation with respect to a bounded admissible approximation $V_n^D(t;g)$ of $V^D(t;g)$.

Step 2. Now we define the local scattering operators. Choose arbitrary, but fixed test functions $f, g, h \in C_c^\infty(\mathbb{R}^d)$ and an arbitrary Poincaré transformation $(a, \Lambda(\alpha))$. Moreover, we choose $\sigma, \tau \in \mathbb{R}$ such that $\sigma < \min I_g \cup I_{g_{\langle a,\Lambda(\alpha)\rangle}} \cup I_f \cup I_h$ and $\tau > \max I_g \cup I_{g_{\langle a,\Lambda(\alpha)\rangle}} \cup I_f \cup I_h$ where we set $I_g := \{t \in \mathbb{R} : (t, \vec{x}) \in \operatorname{supp} g\}$ for arbitrary test functions $g \in C_0^\infty(\mathbb{R}^d)$. We define

$$S(g) := U_g^D(\tau, \sigma),$$

and we verify the conditions of Definition 1.5. Note, that $U_g^D(t,s) = \mathbb{1}$ if $[t,s] \cap I_g = \emptyset$. Thus, by uniqueness of the propagator, the definition of $S(g)$ does not depend on τ and σ as long as $I_g \subset [\sigma, \tau]$. Moreover, it is clear that the null set N is a subset of $I_g \cup I_{g_{\langle a,\Lambda(\alpha)\rangle}} \cup I_f$. Hence $S(g)^{-1} = S(g)^*$ and $S(g) = \mathbb{1}$ if $g = 0$.

With a similar argument, involving uniqueness, we verify the causality condition. Clearly, if two Hamiltonians $H(t)$ and $H'(t)$ coincide for $t \in I \subset \mathbb{R}$, then the admissible bounded approximation $H_n'(t)$ is an admissible bounded approximation of $H(t)$ at the same time. By uniqueness of the approximative solution and the calculation (3.47) we find $U(t,s) = U'(t,s)$ for $t, s \in I$. Now suppose that there exists a $t_0 \in \mathbb{R}$ such that $\max I_g < t_0 < \min I_f$. Then $t_0 \notin N$ and, using $V^D(t; g+f) = V^D(t;g)$ for $t < t_0$ as well as $V^D(t; g+f) = V^D(t;f)$ for $t > t_0$, we see that $U_{g+f}^D(t,s) = U_g^D(t,s)$ if $t, s \le t_0$ and $U_{g+f}^D(t,s) = U_f^D(t,s)$ if $t, s \ge t_0$. Hence

$$S(g+f) = U_{g+f}^D(\tau, t_0) U_{g+f}^D(t_0, \sigma) = U_f^D(\tau, t_0) U_g^D(t_0, \sigma) = S(f)S(g),$$

where $U_g^D(t,s)$ and $U_f^D(t,s)$ are extended by $\mathbb{1}$ outside the time support of g respective f. Moreover we establish the causality condition (1.6) using the same argument. The test function $h \in C_c^\infty(\mathbb{R}^d)$ has arbitrary time support I_h. By possibly varying t_0 in an open neighborhood between $\max I_g$ and $\min I_f$, we achieve $t \notin N$, where N is the exceptional null set of the propagator $U_h^D(t,s)$ where the causal factorization fails to hold. With this assumtion, again the uniqueness of the approximative solution results in

$$U_{f+h+g}^D(\tau, t_0) U_h^D(t_0, \sigma) = U_{f+h}^D(\tau, \sigma).$$

Using this relation and an analogous one, we see that

$$\begin{aligned}
S(f+h+g) &= U_{f+h+g}^D(\tau, \sigma) \\
&= U_{f+h+g}^D(\tau, t_0) U_h^D(t_0, \sigma) (U_h^D(t_0, \sigma))^* (U_h^D(\tau, t_0))^* U_h^D(\tau, t_0) U_{f+h+g}^D(t_0, \sigma) \\
&= U_{f+h}^D(\tau, \sigma) (U_h^D(t_0, \sigma))^* (U_h^D(\tau, t_0))^* U_{h+g}^D(\tau, \sigma) \\
&= S(f+h) S(h)^* S(h+g).
\end{aligned}$$

Now, we consider covariance. By assumption (v),

$$V^D(t; g_{\langle a,\Lambda(\alpha)\rangle}) = U(a,\alpha)^* V^D(t; g) U(a,\alpha).$$

We can repeat the argument of Step 1 for $V^D(t; g_{\langle a,\Lambda(\alpha)\rangle})$ and end up with an approximative solution of the time-dependent Schrödinger equation, given by a propagator $U^D_{g_{\langle a,\Lambda(\alpha)\rangle}}(t,s)$, which is unique as well as strongly continuous and unitary almost everywhere in $t, s \in \mathbb{R}$. As above, $U^D_{g_{\langle a,\Lambda(\alpha)\rangle}}(t,s)$ is the weak limit of a sequence of propagators $\{U^D_{g_{\langle a,\Lambda(\alpha)\rangle},n}(t,s)\}_{n\in\mathbb{N}}$, where each element is strongly differentiable in t and solves the time-dependent Schrödinger equation with respect to an admissible bounded approximation $V^D_n(t; g_{\langle a,\Lambda(\alpha)\rangle})$ of $V^D(t; g_{\langle a,\Lambda(\alpha)\rangle})$. But, considering $U(a,\alpha)^* U^D_{g,n}(t,s) U(a,\alpha)$, we find

$$i\frac{\partial}{\partial t} U(a,\alpha)^* U^D_{g,n}(t,s) U(a,\alpha)\theta = \\ U(a,\alpha)^* V^D_n(t; g) U(a,\alpha) U(a,\alpha)^* U^D_{g,n}(t,s) U(a,\alpha)\theta$$

for every $\theta \in \mathcal{H}$. If we show that $U(a,\alpha)^* V^D_n(t; g) U(a,\alpha)$ is an admissible bounded approximation of $V^D(t; g_{\langle a,\Lambda(\alpha)\rangle})$, then uniqueness of the approximative solution implies

$$U^D_{g_{\langle a,\Lambda(\alpha)\rangle}}(t,s) = U(a,\alpha)^* U^D_g(t,s) U(a,\alpha)$$

and hence covariance of the local scattering operators,

$$S(g_{\langle a,\Lambda(\alpha)\rangle}) = U(a,\alpha)^* S(g) U(a,\alpha).$$

The strong continuity of $t \mapsto U(a,\alpha)^* V^D_n(t; g) U(a,\alpha)$ is clear, as well as the boundedness of the corresponding propagators. We establish the conditions on $U(a,\alpha)^* V^D_n(t; g) U(a,\alpha)$ in the following calculations: Let $\psi \in D(V^D(t; g_{\langle a,\Lambda(\alpha)\rangle}))$, then

$$\|(U(a,\alpha)^* V^D_n(t; g) U(a,\alpha) - V^D(t; g_{\langle a,\Lambda(\alpha)\rangle}))\psi\| \leq \|(V^D_n(t; g) - V^D(t; g))\tilde{\psi}\| \to 0$$

for $n \to \infty$, where $\tilde{\psi} = U(a,\alpha)\psi \in D(V^D(t; g))$, because $V^D_n(t; g)$ is an admissible bounded approximation of $V^D(t; g)$. With the same argument,

$$\|U(a,\alpha)^* V^D_n(t; g) U(a,\alpha)\psi\| \leq c(\|V^D(t; g)\tilde{\psi}\| + \|\tilde{\psi}\|) \\ \leq c(\|V^D(t; g_{\langle a,\Lambda(\alpha)\rangle})\psi\| + \|\psi\|).$$

So the family of unitary operators $\{S(g) : g \in C^\infty_c(\mathbb{R}^d)\}$ satisfies the requirements of Definition 1.5 and the statement follows. ☐

For the definition of the local scattering operators $S(g)$ we use the time evolution $U^D_g(t,s)$, generated by the Hamiltonian $V^D(t; g)$ in the interaction picture. However, if also the Hamiltonian $H(t) = H_0 + V(t; g)$ satisfies the requirements of Theorem 3.21, we could start with the propagator $U_g(t,s)$ in the Schrödinger picture and define the local scattering operators with its transformation

$$\tilde{U}_g^D(t,s) := e^{itH_0} U_g(t,s) e^{-isH_0}.$$

But this is not a good choice: Let $U_{g,n}(t,s)$ be the propagator associated with an admissible bounded approximation $H_n(t)$ of $H(t)$. Then $\tilde{U}_{g,n}^D(t,s)\theta$ is not differentiable for arbitrary $\theta \in \mathcal{H}$. On a suitable subspace we find

$$i\frac{\partial}{\partial t}\tilde{U}_{g,n}^D(t,s)\psi = e^{itH_0}(H_n(t) - H_0)e^{-isH_0}\tilde{U}_{g,n}^D(t,s)\psi,$$

but in general $H_n(t) - H_0$ is *not* an admissible bounded approximation of $V(t;g)$. Nevertheless, it is possible to establish the equivalence of both strategies if we require $D(H_0^p) \subset D(V(t;g))$ for a $p \in \mathbb{N}$ and specialize to $Y := D(H_0^p)$. Then we could choose the admissible bounded approximation *on* Y by $H_n(t) = H_{0,n} + V_n(t;g)$ with $H_{0,n}$, and $V_n(t;g)$ being admissible bounded approximations on Y of H_0 and $V(t;g)$ respectively, see the remark after Definition 2.13. Then it is easy to see that the propagator in the interaction picture coincides with the transformed time evolution operator in the Schrödinger picture. But we do not go into further detail here, because we prefer the definition of the local scattering operators starting with the propagator $U_g^D(t,s)$ generated by $V^D(t;g)$. In the following, we will see that it is advantageous to work directly in the interaction picture.

4.3 Local scattering operators for the $P(\varphi)_2$ model

We establish the existence of the local scattering operators $S(g)$ for the $P(\varphi)_2$ model. As far as we know, this is the first proof of the existence of local scattering operators for a quantum field theory with nonlinear field equations. The $(\phi^4)_2$ model of Section 4.1 is a special case.

According to Theorem 3.22, there exists a unique approximative solution of the time-dependent Schrödinger equation for the $P(\varphi)_2$ model with interaction localized in the support of a test function $g \in C_c^\infty(\mathbb{R}^2)$, $0 \leq g \leq 1$, and P a semibounded polynomial. Associated with this solution, there is an almost everywhere unitary and strongly continuous propagator $U_g(t,s)$. For the definition of the local scattering operator we could transform the propagator to the interaction picture, as indicated in the last section: Choose an interval $[\sigma,\tau]$ large enough such that $I_g \subset [\sigma,\tau]$, where $I_g := \{t \in \mathbb{R} \ : (t,\vec{x}) \in \operatorname{supp} g\}$. In this case we would define

$$S(g) := e^{i\tau H_0} U_g(\tau,\sigma) e^{-i\sigma H_0}. \tag{4.9}$$

However, if we aim at the local algebras, as defined in Section 1.2.2, the positivity requirement on the test function g is problematic. The condition arises from the proof of essential self-adjointness of the total Hamiltonian $H(t) = H_0 + V(t;g)$. We can avoid this kind of restriction on the choice of the test function by working directly in the interaction picture, see Theorem 4.3. Moreover, we do not need the restriction to semibounded polynomials P and positive coupling constants any longer. These conditions, ensuring stability, are not necessary in a local construction.

Let $\mathcal{H}$ be the Fock space of a free, massive boson field as in Appendix A. We consider the interaction of the $P(\varphi)_2$ model,

$$V(t;g) = \int g(t,x) : P(\varphi(x)) : \, dx,$$

where $P(\lambda)$ is a real polynomial. The test function $g \in C_c^\infty(\mathbb{R}^2)$ is assumed to be real, but there are no restrictions in other respects. With Theorem 4.3 we demonstrate the existence of the local scattering operators.

The existence question for these operators in the special case $(\varphi^4)_2$ was formerly addressed in the work of W. Wreszinski. In [95], the Cauchy problem of the time-dependent Schrödinger equation for the localized $(\varphi^4)_2$ model with *factorizing test function* is solved, using Kato's theory for the case of time-independent domains (Theorem 2.31). The restriction to factorizing test functions $g(t,x) = u(t)v(x),\ u, v \in C_c^\infty(\mathbb{R})$ is necessary to achieve time-independence of the domains of the Hamiltonians. This is a considerable loss of generality: For the approximation of an arbitrary compactly supported test function, one needs linear combinations of such factorizing functions. But for these linear combinations, the Hamiltonians do not have time-independent domains anymore. Recently, the question of existence of local scattering operators for the $(\varphi^4)_2$ model is addressed [96], using the Theorem of Kisyński (see Theorem 2.32). In this article, a scale of Hilbert spaces $\mathcal{F}_{+2} \subset \mathcal{H} \subset \mathcal{F}_{-2}$ with respect to the free Hamiltonian H_0 is considered. The space $\mathcal{F}_{+2}$ equals $D(H_0)$ with the norm $\|\psi\|_{+2} = \|(H_0+1)\psi\|$. In the course of the argument, closedness of the sesquilinear form

$$S(\theta,\psi) = \Big((H(t)+M)^{1/2}\theta, (H(t)+M)^{1/2}\psi\Big)$$

on $\mathcal{F}_{+2}$ is assumed, where $M > 0$ is a constant such that $H(t) + M \geq 0$, see [96, Equation 2.21]. But if $S(\theta,\psi)$ would be closed with respect to $\mathcal{F}_{+2}$, this would imply that $S(\psi,\psi)$ induces an equivalent norm on $\mathcal{F}_{+2}$. Or in other words, there would exist a constant $c > 0$ such that

$$c^{-1}(H_0+1)^2 \leq (H(t)+M) \leq c(H_0+1)^2.$$

While the inequality on the right-hand side can be fulfilled using Theorem A.1, the left-hand inequality is *not* satisfied for arbitrary t in the *localized* $(\varphi^4)_2$ model.

Hence, S is not closed and Kisyński's Theorem is not applicable. Notice that this problem does not arise in Dimock's application [18], as in this case the scale of Hilbert spaces is constructed with respect to the *interacting* Hamiltonian.

As we will see in the following, the notion of approximative solutions is appropriate to discuss the existence question for local scattering operators. This setting results in a considerable simplification compared to previous attempts.

Theorem 4.4. *Let $\mathcal{H}$ be the Fock space of a free, massive boson field φ in two spacetime dimensions. Consider the interaction Hamiltonain of the $P(\varphi)_2$ model,*

$$V(t;g) = \int g(t,x) : P(\varphi(x)) : \, dx,$$

where $P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ is a real polynomial, not necessarily semibounded, and $g \in C_c^\infty(\mathbb{R}^2)$ is a real test function. Then the local scattering operator $S(g)$ exists.

Proof. The spacetime under consideration is the hyperbolical plane, that is $\mathbb{R}^2$ with the Lorentzian metric $g_{\mu\nu}$. Its homogeneous isometry group is $O(1,1)$ and the proper Lorentz group is $SO^+(1,1)$. The latter group is simply connected and has the one-parameter form

$$\mathbb{R} \to SO^+(1,1), \quad \eta \mapsto \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix},$$

as a group isomorphism, see [44]. Hence the universal covering group G equals $SO^+(1,1)$ itself, and the Poincaré group is $\overline{\mathcal{P}} = \mathbb{R}^2 \rtimes SO^+(1,1)$. The Fock space $\mathcal{H}$ over the one-particle space $L^2(\mathbb{R})$ for the theory of a scalar, massive quantum field carries a unitary representation $\overline{\mathcal{P}} \to \mathcal{B}(\mathcal{H})$, $\langle a, \Lambda\rangle \mapsto U(a,\Lambda)$ of the Poincaré group, see [11]. The interaction Hamiltonian is a sum of Wick monomials of the free time-zero fields $\varphi(x) = \varphi(0,x)$, which we denote by the same symbol. Transformed to the time t by the free time evolution, the Wick monomials transform covariantly,

$$\begin{aligned} U(a,\alpha)^* e^{iH_0 t} : \varphi^n(x) : e^{-iH_0 t} U(a,\alpha) = U(a,\alpha)^* : \varphi^n(t,x) : U(a,\alpha) = \\ : \varphi^n(\Lambda(t,x) + a) : . \end{aligned}$$

This implies the transformation property (v) in Theorem 4.3.

The interaction Hamiltonian $V(t;g)$, $g = \overline{g} \in C_c^\infty(\mathbb{R}^2)$, is essentially self-adjoint on $\mathcal{D}$, the set of Fock space vectors with finite particle number and Schwartz wavefunctions (see [37] and Theorem A.3). Notice that neither positivity of g nor semiboundedness of P are necessary. Hence, using Theorem A.1, we see that $V(t;g)$ is essentially self-adjoint on $Y := D(N^m)$, where $m \in \mathbb{N}$ is larger or equal to $n/2$. Clearly, $V(t;g)$ vanishes outside the time support of g.

The free Hamiltonian H_0 and the particle number operator N commute, thus $e^{-itH_0}Y = Y$ and $(e^{-itH_0})_{t\in\mathbb{R}}$ is strongly continuous in the Banach space norm of Y.

Consider the term of highest order in the interaction $V(t;g)$:

$$V_n(t;g) = a_n \int g(t,x) : \varphi(x)^n : \, dx.$$

For $\psi \in Y$ the interaction Hamiltonian is dominated by a power of the number operator, see Theorem A.1,

$$\begin{aligned}\|V_n(t;g)\psi\| &\le \|V_n(t;g)(N+1)^{-m}\| \|(N+1)^m\psi\| \\ &\le c\|w(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|(N+1)^m\psi\|,\end{aligned}$$

where c is a t-independent constant and $w(t,k_1,\dots,k_n)$ denotes the numeric kernel of the expansion of $V_n(t;g)$ into Wick monomials. It is given by

$$w(t,k_1,\dots,k_n) = \hat{g}(t,k_1+\dots+k_n)\omega(k_1)^{-1/2}\cdot\dots\cdot\omega(k_n)^{-1/2},$$

where $\hat{g}$ denotes the Fourier transform of g with respect to x, and the L^2-norm is evaluated with respect to $(k_1,\dots,k_n) \in \mathbb{R}^n$, see [16, §6.1]. By repeated use of Young's inequality, we estimate

$$\|w(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le c\|g(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

(cf. [16, Lem. 6.1]). The other summands of $V(t;g)$ can be treated in the same way. These inequalities imply the uniform boundedness of the embedding $Y \hookrightarrow D(V(t;g))$. Similarly we verify the continuity of $t \mapsto V(t;g)\psi$ for all $y \in Y$. Strong continuity of $t \mapsto R(i, V(t;g))$ follows, because Y is a core for $V(t;g)$. Hence, by Theorem 4.3, the local scattering operators exist. □

4.3.1 The massless boson field in two dimensions

For the massless boson field in two dimensions, the two-point function is divergent. The theory has no vacuum state. Thus, the massless boson field in $d = 2$ is not a field theory in the sense of Wightman's axioms in Section 1.1.1.1.

However, it is possible to define the theory in the algebraic sense, hence to construct the algebras of local observables. Problems do not arise until the state space is analyzed.

In Wightman's Cargèse lectures [94] he addresses the massless boson in two dimensions by using spaces with indefinite metric, similar to Schroer's treatment of the derivative coupling model [81]. The algebra of observables is constructed by Streater and Wilde in [88].

By our results on the existence of the local scattering operator for $P(\varphi)_2$ models, it is possible to get the algebra of local observables for the massless boson field in two dimensions in an easier way, using the formalism of Section 1.2.2.

The classical Hamiltonian in the theory of a free massive boson field φ with mass $m > 0$ is given by

$$H_0 = \frac{1}{2} \int \pi(x)^2 + (\nabla\varphi(x))^2 + m^2\varphi(x)^2 \, dx. \tag{4.10}$$

The field $\pi(x) = \frac{d}{dt}\varphi(t,x)\restriction_{t=0}$ is the canonically conjugate object to the time-zero field $\varphi(x)$. We consider the formal interaction term

$$V_\lambda(t; g) = \frac{1}{2}\lambda \int g(t,x)\varphi(x)^2 \, dx$$

with $g \in C_c^\infty(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}$. If we choose $\lambda := -m^2$ and $g(t,x) = 1$ for (t,x) in an open, contractible region $\mathcal{O}$ of spacetime, the density of the Hamiltonian $H_0 + V_\lambda(t; g)$ coincides with the Hamiltonian density of the massless theory inside of $\mathcal{O}$. On $\mathcal{H}$, the Fock space over $L^2(\mathbb{R})$, belonging to a free, massive, scalar quantum field φ, we apply the unitary transformation R_g, which transforms the interacting Hamiltonian $H_0 + V_\lambda(t; g)$ of the theory into the free one with modified dispersion relation. That such a transformation exists is shown by Rosen in [74] for $\lambda = -m^2 + \epsilon$, $\epsilon > 0$. After addition of a vacuum energy renormalization constant, the Hamiltonian is transformed into $H(g) = \int a^*(k_1)\mu_{g,t}(k_1,k_2)a(k_2)\, dk_1\, dk_2$ with the dispersion relation $\mu_{g,t}^2 = -\Delta + m^2 + \lambda g(t,x) = \mu^2 + \lambda g(t,x)$. Hence, for $g \to 1$ and $\epsilon \to 0$, the dispersion relation approaches the $(m = 0)$-case $\mu^2 = -k^2$.

Since the existence result for the local scattering operator of the $P(\varphi)_2$ model (Theorem 4.4) gives no restriction on the sign of the coupling constant respective the localization function, we can apply it to the special case $P(\varphi) = -m^2\varphi^2$. The local scattering operators generate the algebra of observables of a massless boson field in two dimensions according to the formalism of Section 1.2.2.

Indeed we might even choose $\lambda < -m^2$. The local scattering operators exist by Theorem 4.4 and the associated relative scattering operators generate the local algebras of a bosonic field theory with negative squared mass. As in the massless case, for these theories there are no vacuum states. But they are well defined theories in the algebraic sense. By coupling of another field or addition of a suitable selfinteraction, these *tachyonic fields* with imaginary mass may acquire a stable vacuum state. This process is often called *tachyon condensation*.

4.3.2 Going further

Up to this point, we have established the existence of the local scattering operators for a quantum field theory with nonlinear field equation, but without infinite renormalization (apart from Wick ordering). The next step would be to investigate models which require this technique.

The simplest model which shows a UV divergence is the exactly solvable *van Hove* model [15]. The question of the existence of propagators for time-dependent van Hove models was addressed in [76]. The strategy is similar to our approach to Goldstein's example in Section 3.3.3 and uses dressing transformations.

Consider a 3-dimensional spacetime and the theory of a free, massive, scalar field $\varphi(\vec{x})$ on the Fock space $\mathcal{H}$. The Hamiltonian of the time-localized van Hove model is formally given as

$$H(t) = H_0 + g(t)\varphi(\vec{0}),$$

with $g \in C_c^\infty(\mathbb{R})$ and the time-zero field φ. To get a meaningful expression, we choose test functions $\rho_l \in C_c^\infty(\mathbb{R}^2)$, $l \in \mathbb{N}$, such that $\hat{\rho}_l \to 1$ for $l \to \infty$. The function $\vec{k} \mapsto \hat{\rho}_l(\vec{k})$ serves as a UV cut-off. We define $H_l(t) = H_0 + V_l(t)$, where

$$V_l(t) := g(t) \int \rho_l(\vec{x})\varphi(\vec{x})\, d^2x.$$

In the limit $l \to \infty$, it turns out that $H_l(t)$ is not defined as an operator on $\mathcal{H}$. It is necessary to renormalize the Hamiltonian by addition of a c-number $E_l(t)$,

$$H_{\mathrm{ren},l}(t) := H_l(t) + E_l(t),$$

where $E_l(t) \to \infty$ but $H_{\mathrm{ren},l}(t)$ is a well defined operator for $l \to \infty$. Perturbation theory suggests $E_l(t) := g(t)^2 \int (2\mu(\vec{k}))^{-2} |\rho_l(\vec{k})|\, d^2k$.

The renormalized Hamiltonian is unitarily equivalent to the free Hamiltonian. There exists a unitary dressing transformation $R_l(t)$, such that

$$R_l^{-1}(t) H_{\mathrm{ren},l}(t) R_l(t) = H_0.$$

The dressing transformation is exact. It is defined by $R_l(t) = e^{-\Gamma V_l(t)}$, involving the inverse adjoint action of H_0, that is $\Gamma V_l(t)$ is the operator which satisfies $[H_0, \Gamma V_l(t)] = V_l(t)$. The Γ operation produces an additional factor $\mu(\vec{k})^{-1}$ in the numerical kernel of $V_l(t)$, hence it leads to a faster decrease of the kernel for large $|\vec{k}|$. Thus, $R(t) := \lim_{l\to\infty} R_l(t)$ is well defined as a unitary operator and we define $H_{\mathrm{ren}}(t) := R(t) H_0 R^{-1}(t)$ with domain $D(H_{\mathrm{ren}}(t)) = R(t)D(H_0)$. The intersection of the domains of H_0 and $H_{\mathrm{ren}}(t)$ is trivial.

To discuss the time evolution, we formally define the candidate for the generator of an evolution group by $iG_0 := H_{\mathrm{ren}}(t) - i\frac{d}{dt}$ on $E = L^2(\mathbb{R}, \mathcal{H})$ and use the dressing transformation as an operator on E to calculate

$$iG_0 = R \left(H_0 - i\Gamma\dot{V} - i\frac{d}{dt} - \tilde{E} \right) R^{-1},$$

where $\tilde{E}$ is a bounded, c-number-valued function. In [76] we discussed essential self-adjointness of $i\tilde{G}_0 = H_0 - i\Gamma\dot{V} - i\frac{d}{dt} - \tilde{E}$. This is possible due to the simple structure of the model. But our results of Section 3.3 show that it is sufficient to deal with approximative solutions which belong to a self-adjoint extension of $i\tilde{G}_0$, which exists under more general assumptions. The existence theorem for approximative solutions (Theorem 3.21) and hence Theorem 4.3 make it possible to envisage a similar strategy for the proof of existence of local scattering operators for more complicated models. Locality of the interaction ensures $R(t) = \mathbb{1}$ for sufficiently large t, thus it is sufficient to investigate existence of local scattering operators for the regularized interaction $H_0 - iR(t)^{-1}(\frac{d}{dt}R(t))$. We obtain a regularization procedure similar to the one developed by Mickelsson and Langmann in [55]. They work on the level of the one-particle space and then discuss the implementability of the scattering operators in Fock space. This is possible for models with linear field equations, but the quantization of the scattering operators leaves a phase factor undefined. Our Theorem 4.3 allows for a direct discussion of the existence question of the local scattering operators in Fock space, thereby avoiding the mentioned ambiguity and extending the scope of application of the regularization procedure to models with non-linear field equations.

Possible candidates for a further investigation of models with linear field equations are fermions in external fields with a dressing transformation as in [27] or massive bosons with a localized φ^2 interaction in four dimensions [33, 74]. Let $\mathcal{H}$ be the Fock space over $L^2(\mathbb{R}^3)$ and let $\varphi(\vec{x})$ be a scalar, massive time-zero field. For a real test function $g \in C_c^\infty(\mathbb{R}^4)$ we consider formally $H(t) := H_0 + V(t)$, where

$$V(t) := \int g(t,\vec{x}) :\varphi^2(\vec{x}): \, d^3x.$$

The interaction term is the sum of terms with zero, one and two creation operators. We introduce an ultraviolet cut-off $\sigma > 0$ in the numerical kernel by the characteristic function χ_σ of the set $[-\sigma,\sigma]^3 \subset \mathbb{R}^3$ and define

$$\begin{aligned} V_{0,\sigma}(t) = V_{2,\sigma}(t)^* &= \int \tfrac{\hat{g}(t,\vec{k}+\vec{k}')\chi_\sigma(\vec{k})\chi_\sigma(\vec{k}')}{\sqrt{\mu(\vec{k})\mu(\vec{k}')}} a(\vec{k})a(\vec{k}')\, d^3k\, d^3k', \\ V_{1,\sigma}(t) &= 2\int \tfrac{\hat{g}(t,\vec{k}-\vec{k}')\chi_\sigma(\vec{k})\chi_\sigma(\vec{k}')}{\sqrt{\mu(\vec{k})\mu(\vec{k}')}} a^*(\vec{k})a(\vec{k}')\, d^3k\, d^3k', \end{aligned}$$

where $\hat{g}(t,\cdot)$ is the Fourier transform of g with respect to $\vec{x}$. To define the Hamiltonian, a counter term is necessary. By $\underbrace{AB}_{n}$ we denote the term with n contractions in the product of two Wick monomials A and B. Perturbation theory predicts that addition of

$$E_\sigma(t) := V_{0,\sigma}\underbrace{(t)\Gamma V_{2,\sigma}}_{2}(t)$$

to $H(t)$ leads to a renormalized Hamiltonian,

$$H_{\mathrm{ren},\sigma}(t) := H_0 + V_{0,\sigma}(t) + V_{1,\sigma}(t) + V_{2,\sigma}(t) + E_\sigma(t) = H_0 + V_\sigma(t) + E_\sigma(t),$$

which is well-defined as an operator in the limit $\sigma \to \infty$. Notice that in this limit the counter term $E_\sigma(t)$ diverges. As above, $\Gamma = \mathrm{ad}_{H_0}^{-1}$.

We define the dressing transformation in the following way: By Nelson's theorem of analytical vectors, we can establish essential skew-adjointness of $W_\sigma(t) := \Gamma(V_{0,\sigma}(t) + V_{2,\sigma}(t))$ and set $R_\sigma(t) := e^{-W_\sigma}$. This dressing transformation is unitary. We investigate the candidate for an evolution generator, which we obtain by the transformation of $iG_{0,\sigma} := H_{\mathrm{ren},\sigma}(\cdot) - i\frac{d}{dt}$ on E, and calculate

$$\begin{aligned} R^{-1}\left(H_{\mathrm{ren},\sigma} - i\frac{d}{dt}\right) R &= H_0 - i\frac{d}{dt} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\mathrm{ad}_{W_\sigma}\right)^{n-1}\left[\mathrm{ad}_{W_\sigma}(V_\sigma) - V_{0,\sigma} - V_{2,\sigma} - i\,\mathrm{ad}_{W_\sigma}\left(\frac{d}{dt}\right)\right], \end{aligned}$$

again using analytical vectors. The counter term as well as the interaction term containing two creators but no Γ-factor cancel exactly. Expanding the terms up to second order explicitly, we find

$$\begin{aligned} R^{-1}\left(H_{\mathrm{ren},\sigma} - i\frac{d}{dt}\right) R &= H_0 - i\frac{d}{dt} + V_{1,\sigma} \\ &+ \Gamma \underbrace{V_{0,\sigma}V_{2,\sigma}}_{1} - \underbrace{V_{0,\sigma}\Gamma V_{2,\sigma}}_{1} + \frac{1}{2}\underbrace{V_{0,\sigma}\Gamma V_{2,\sigma}}_{1} - \frac{1}{2}\Gamma\underbrace{V_{0,\sigma}V_{2,\sigma}}_{1} \\ &+ \underbrace{V_{0,\sigma}V_{1,\sigma}}_{1} - \underbrace{V_{1,\sigma}\Gamma V_{2,\sigma}}_{1} + i\Gamma\dot{V}_{0,\sigma} + i\Gamma\dot{V}_{2,\sigma} \\ &+ \frac{1}{2}\left[(\Gamma V_{0,\sigma} + \Gamma V_{2,\sigma}), \left(\Gamma\underbrace{V_{0,\sigma}V_{2,\sigma}}_{1} - \underbrace{V_{0,\sigma}\Gamma V_{2,\sigma}}_{1} - \underbrace{V_{1,\sigma}\Gamma V_{2,\sigma}}_{1}\right)\right] \\ &+ \frac{1}{2}[\Gamma V_{0,\sigma}, \Gamma\dot{V}_{2,\sigma}] + \frac{1}{2}[\Gamma V_{2,\sigma}, \Gamma\dot{V}_{0,\sigma}] \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!}\left(\mathrm{ad}_{W_\sigma}\right)^{n-1}\left[\mathrm{ad}_{W_\sigma}(V_\sigma) - V_{0,\sigma} - V_{2,\sigma} - i\,\mathrm{ad}_{W_\sigma}\left(\frac{d}{dt}\right)\right]. \end{aligned} \tag{4.11}$$

There are no dangerous terms left and one could envisage the discussion of the existence of local scattering operators for the regularized interaction of the right-hand side. To this end, we first notice that the right-hand side is symmetric. All terms have at most two external creators or annihilators which are not contracted. The higher orders can be discussed in terms of connected Friedrichs' graphs [29]. For superrenormalizable models, the results of Hepp [45, Theorem 4.5] and Glimm [34, Theorem 2.2.1] indicate that, as the number of vertices in a graph increases, the kernel decreases more rapidly. Therefore, because there are no dangerous terms left in lower orders, the situation becomes even better in higher orders and we could envisage an analysis of summability of (4.11) by similar arguments as in [80]. Then applicability of Theorem 4.3 for the existence of local scattering operators could be investigated and we conjecture an affirmative outcome. Furthermore, we expect that this strategy can be extended to cover other superrenormalizable models like Y_2 or even $(\varphi^4)_3$. However, it might be possible to choose the dressing transformation in a more convenient way. As it is important to retain unitarity of the dressing transformation, constructions similar to Federbush's in [25, 26] or a Padé approximation of the formal wave operator may be interesting starting points. Unfortunately, we are not able to present results in this direction in the present work.

Conclusions and Outlook

In this thesis, we have established the existence of local scattering operators for $P(\varphi)_2$ models. To this end, we investigated the Cauchy problem for the time-dependent Schrödinger equation in a general context. We used the theory of evolution semigroups to attack the wellposedness problem. The strategy is as follows: We investigated the extensions of a certain operator and found sufficient conditions that an extension exists which is the generator of an evolution semigroup. For Hilbert spaces and self-adjoint Hamiltonians we saw that this extension corresponds to a unique approximative solution of the Cauchy problem of the time-dependent Schrödinger equation. This is a new wellposedness result requiring considerably less assumptions than Kato's classical wellposedness theory. We sacrifice some regularity of the solution.

In future investigations, it may be possible to eliminate the exceptional set N of measure 0, where strong continuity of $t \mapsto U(t,s)$ fails to hold. One may have the impression that this set is a technical artefact. If it can be ruled out, the notion of approximative solutions may be modified to involve strong instead of weak convergence of the approximating solutions to an admissible bounded approximation of the generators.

However, at the moment we do not see if it is possible to develop an even more general wellposedness theory without requiring fundamentally new concepts. In special situations it is possible to use similarity transformations to extend the scope of application of our wellposedness result, as we demonstrated with Goldstein's example.

In the course of our discussion, we investigated the time evolution of a φ^4 theory on a two-dimensional, curved spacetime and extended a result of Dimock. We do not expect that it is possible to extend similar results to higher dimensions because of the results of Torre and Varadarajan [91]: It is problematic to formulate the concept of time evolution from Cauchy surface to Cauchy surface even for free theories. But there are various other fields of physics where our solvability result for the time-dependent Schrödinger equation may be useful: In applications to quantum optics [82], a formal method due to Lewis and Riesenfeld [57] is used. In the context of general relativity, the influence of an expanding universe on a quantum system is investigated in [56]. Yajima uses techniques close to Howland's ideas to investigate the Stark effekt [97], compare also [98].

The approach of Asch et.al. [2] for the examination of the adiabatic properties of a Landau Hamiltonian correspond to the mild solutions in the present work, an extension to approximative solutions might give a considerable simplification.

Our main application aims at quantum field theory. We establish the existence of local scattering operators for $P(\varphi)_2$ models. To our knowledge, this is the first proof of existence of these in a quantum field theoretical model with nonlinear field equation. The local scattering operators enable us to access the algebras of local observables by the general construction of R. Brunetti and K. Fredenhagen [7], even for models having no ground state. This demonstrates the disentanglement of the ultraviolet and the infrared problem, the main advantage of the approach. Examples are models with a non-semibounded polynomial $P(\lambda)$ or the case of a massless boson in two dimensions. For future investigations, we find it interesting to examine the isomorphy of the algebra of local observables in this case as constructed using local scattering operators or by the indefinite-metric approach of Streater and Wilde explicitly. Furthermore, the direct and easily accessible construction of the local algebras via local scattering operators may have some benefit for the investigation of other properties of interacting fields as for example nuclearity in the sense of Buchholz and Wichmann [10].

Evidently, future investigations should focus on local scattering operators for models with infinite renormalizations. We conjecture that our approach together with the technique of dressing transformations is sufficiently general to include superrenormalizable models which are accessible via a Hamiltonian construction, for example Y_2. However, we also see the limitations of the approach: On the side of the existence result, a further generalization seems to be out of reach. From a quantum field theoretical point of view, it is the manifestly Hamiltonian formulation which makes it difficult to use advanced methods from constructive quantum field theory. We think that an attempt to analyze the existence of local scattering operators for renormalizable models, as the Gross-Neveu model in two dimensions, may show whether the approach to the local net via local scattering operators is powerful enough to revitalize the interest in constructive quantum field theory.

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Appendix A

Scalar quantum field theory in two dimensions

We give a short summary of some basic facts about polynomial self-interacting, scalar massive quantum field theory in two dimensional spacetime, mainly to fix our notation. The proofs of all results which are not mentioned explicitly can be found in the standard reference [37]. We follow the exposition given there.

Let $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{H}_n = \{\psi \in L^2(\mathbb{R}^n) : \psi(k_1, \dots, k_n) \text{ is symmetric}\}$. The Fock space is defined as $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. The subspace $\mathcal{H}_n$ is the "n-particle subspace" of $\mathcal{H}$, the vector $\Omega = 1 \subset \mathcal{H}_0$ is called the "vacuum". For $\psi = \{\psi_0, \psi_1, \dots, \psi_n, \dots\} \in \mathcal{H}$, the number operator N is defined by

$$\psi \mapsto N\psi = \{0, \psi_1, 2\psi_2, \dots, n\psi_n, \dots\}$$

on the domain $D(N) = \{\psi \in \mathcal{H} : \sum_{n=0}^{\infty} \|n\psi_n\|_2^2 < \infty\}$.

Let $\mu(k) = (k^2 + m^2)^{1/2}$ for $k \in \mathbb{R}$ and $m > 0$. The free Hamiltonian H_0 maps ψ on $H_0\psi$ given by

$$\psi \mapsto H_0\psi = \{0, \mu(k_1)\psi_1, (\mu(k_1) + \mu(k_2))\psi_2, \dots, \textstyle\sum_{i=1}^{n} \mu(k_i)\psi_n, \dots\} \tag{A.1}$$

for ψ in $D(H_0) = \{\psi \in \mathcal{H} : \sum_{n=1}^{\infty} \| \sum_{i=0}^{n} \mu(k_i)\psi_n\|_2^2 < \infty\}$.

Denote by S_n the projection of $L^2(\mathbb{R}^n)$ on its symmetric part $\mathcal{H}_n$. The finite linear combinations of Ω and of vectors of the form $S_n f_1(k_1) \dots f_j(k_j)$ for $f_i \in \mathcal{S}(\mathbb{R})$ span the dense subset $\mathcal{D}$ of $\mathcal{H}$. It is an invariant domain for the annihilation operator $a(k), k \in \mathbb{R}$, defined by

$$(a(k)\psi)_n(k_1, \dots, k_n) = (n+1)^{1/2}\psi_{n+1}(k, k_1, \dots, k_n).$$

The operator $a(k)$ is not closable, its formal adjoint $a^*(k)$ is defined as a quadratic form on $\mathcal{D} \times \mathcal{D}$. The commutation relations of a and a^* are $[a(k), a^*(k')] = \delta(k - k')$. For $\psi_1, \psi_2 \in \mathcal{D}$, $(\psi_1, a^*(k_1) \dots a^*(k_m)a(k'_1) \dots a'(k_n)\psi_2)$ is a function in

$\mathcal{S}(\mathbb{R}^{m+n})$. Therefore, every distribution $w \in \mathcal{S}'(\mathbb{R}^{m+n})$ defines a bilinear form W on $\mathcal{D} \times \mathcal{D}$ by

$$W = \int a^*(k_1) \dots a^*(k_m) w(k_1, \dots, k_m; k_1', \dots, k_n') a(k_1') \dots a(k_n') \, d^m k \, d^n k'.$$

It is called a Wick monomial of order (m, n).

An important property of Wick monomials is that they can be dominated by powers of N if their kernels are sufficiently regular.

Theorem A.1. *Let W be a Wick monomial with kernel w which is a bounded operator from $S_n L^2(\mathbb{R}^n) \to S_m L^2(\mathbb{R}^m)$ with norm $\|w\|$. Then $(N+1)^{-m/2} W (N+1)^{-n/2}$ is a bounded operator and*

$$\|(N+1)^{-m/2} W (N+1)^{-n/2}\| \leq \|w\|. \tag{A.2}$$

Moreover, let $a + b \geq m + n$. Then

$$\|(N+1)^{-a/2} W (N+1)^{-b/2}\| \leq (1 + |m-n|)^{|a-m|/2} j^{-(a+b)/2} \|w\|.$$

Corollary A.2. *With the same assumptions on the kernel of W, the bilinear form defines an operator in the domain $D(N^{(m+n)/2})$.*

Denote by $\mathcal{V}(\mathcal{S}')$ the set of bilinear forms on $\mathcal{D} \times \mathcal{D}$ which are given as

$$V = \sum_{j=0}^{p} \binom{p}{j} \int v(k_1, \dots, k_p) a^*(k_1) \dots a^*(k_j) a(-k_{j+1}) \dots a(-k_p) \, d^p k$$

with $v \in \mathcal{S}'$ symmetric with real Fourier transform. These are special Wick monomials. For $\mathcal{X} \subset \mathcal{S}'$ let $\mathcal{V}(\mathcal{X})$ be the subset of $\mathcal{V}(\mathcal{S}')$ with kernels restricted to $\mathcal{X}$.

There is a different realization of $\mathcal{H}$ as square-integrable functions on a finite measure space (Q, dq), the so-called Schrödinger or Q-space. From $L^2(Q, dq)$, Fock space is recovered as a Hermite expansion. The advantage of the Q-space realization is that every $V \in \mathcal{V}(L^2)$ is represented by a multiplication operator:

Theorem A.3. *Let $V \in \mathcal{V}(L^2)$. Then V is essentially selfadjoint on $\mathcal{D}$. Moreover, $V \in L^r(Q, dq)$ for all $r < \infty$. If $v = \ker V$ is the integral kernel of V, then $\|V\|_r \leq \|v\|_2$.*

The free Hamiltonian H_0 from (A.1) can be written as a Wick monomial $H_0 = \int \mu(k) a^*(k) a(k) \; dk$ with a kernel from $\mathcal{S}'(\mathbb{R}^2)$. Nevertheless, it is an operator in contrast to the free field, which is defined by $\varphi(x) = (4\pi)^{-1/2} \int e^{-ikx} \mu(k)^{-1/2} (a^*(k) + a(-k)) \, dk$ for $x \in \mathbb{R}$ as a bilinear form.

The free Hamiltonian and the free field are the building blocks for the $P(\varphi)_2$ model. Let $P(\lambda) := \sum_{i=0}^{n} a_i \lambda^i$ be a real polynomial. The Hamiltonian is formally given as $H(t) := H_0 + V(t;g)$ with $V(t;g) := \int g(t,x) : P(\varphi(x)) : dx$. The function $g : \mathbb{R}^2 \to \mathbb{R}$ is chosen such that the Wick monomials in $V(t;g)$ are elements in $\mathcal{V}(L^2)$. In particular, the choice $g \in C_c^\infty(\mathbb{R}^2)$ is possible. In the following, we concentrate on the special case $P(\lambda) = \lambda^p$, $p \in \mathbb{N}$, for simplicity. Then the interaction Hamiltonian is given by

$$\begin{aligned} V(t;g) &= \int g(t,x) : \varphi^p(x) : dx \\ &= (4\pi)^{-p/2} \sum_{j=0}^{p} \binom{p}{j} \int \hat{g}(t, \textstyle\sum_{i=1}^p k_i) \prod_{i=1}^p \mu(k_i)^{-1/2} \times \\ &\qquad\qquad \times a^*(k_1) \dots a^*(k_j) a(-k_{j+1}) \dots a(-k_p)\, d^p k, \end{aligned} \tag{A.3}$$

where $\hat{g}(t,k) = \int e^{-ikx} g(t,x)\, dx$ is the Fourier transform of g with respect to x. For $x \mapsto g(t,x) \in L^2(\mathbb{R})$ the Hamiltonian $V(t;g)$ is an element from $\mathcal{V}(L^2(\mathbb{R}^p))$. This is seen by writing the kernel as a convolution and using Young's inequality [71]. Hence, by Corollary A.2, $V(t;g)$ is a densely defined operator on $\mathcal{H}$ and Theorem A.3 implies essential self-adjointness on $\mathcal{D}$.

To deal with the operator sum $H(t) = H_0 + V(t;g)$, further assumptions are necessary. In the following, let p be even and $g \geq 0$. Introducing the free field with an ultraviolet cut-off κ by

$$\varphi_\kappa = (4\pi)^{-1/2} \int_{|k| \leq \kappa} e^{-ikx} \mu(k)^{-1/2} (a^*(k) + a(-k))\, dk,$$

we define the cut-off interaction Hamiltonian: $V_\kappa(t;g) = \int : \varphi_\kappa^p(x) : g(t,x)\, dx$. It has an expansion in Wick monomials similar to (A.3) with the numerical kernels replaced by

$$v_\kappa(k_1, \dots, k_p) = \textstyle\prod_{i=1}^p \chi_\kappa(k_i) \mu(k_i)^{-1/2} \hat{g}(t, \sum_{i=1}^p k_i),$$

where χ_κ is the characteristic function of the interval $[-\kappa, \kappa]$. For fixed $t \in \mathbb{R}$ and $\kappa > 0$, $V_\kappa(t;g)$ is essentially self-adjoint on $\mathcal{D}$. This follows directly from the above result on Wick monomials with L^2-kernels.

Whereas $V(t;g)$ is an unbounded operator, $V_\kappa(t;g)$ is semibounded from below.

Lemma A.4. *Given a function $g \geq 0$ on $\mathbb{R}^2$ such that $g(t,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sup_{t \in \mathbb{R}} \|g(t,\cdot)\|_1 < \infty$, the operator $V_\kappa(t;g)$ is semibounded from below uniformly in t. The lower bound is $\mathcal{O}((\log \kappa)^{p/2})$.*

Proof. Let $c_\kappa := (4\pi)^{-1} \int_{|k| \leq \kappa} \mu^{-1}(k)\, dk = \mathcal{O}(\log \kappa)$. Wick's theorem states

$$: \varphi_\kappa^p(x) := \sum_{j=0}^{[p/2]} (-1)^j \frac{p!}{(p-2j)!\, j!\, 2^j} c_\kappa^j \varphi^{p-2j}(x)$$

as an operator identity on $\mathcal{D}$, see [37]. Since p is even, the right-hand side is a semibounded polynomial in φ. Therefore $:\varphi_\kappa^p(x):\geq -Mc_\kappa^{p/2}$ and $H_I^g(t;\kappa)\geq -Mc_\kappa^{p/2}\sup_{t\in\mathbb{R}}\|g(t,\cdot)\|_1$ $\square$

The semiboundedness of $V_\kappa(t;g)$ enables one to prove the next theorem.

Theorem A.5. *$H_\kappa(t) := \overline{H_0+V_\kappa(t;g)}$ is essentially self-adjoint on $D(H_0)\cap D(V_\kappa(t;g))$. Moreover,* $\operatorname{Ran} e^{-rH(t;\kappa)}\subset D(H_0)\cap D(\tilde V)$ *for every $r>0$ and $\tilde V\in\mathcal{V}(L^2(\mathbb{R}^p))$.*

By this theorem $V_\kappa(t;g)$ fullfills the prerequisites for the application of the following theorem in Q-space.

Theorem A.6. *Let $V\in L^r(Q,dq)$ for some $r\geq 1$ such that H_0+V is essentially self-adjoint on $D(H_0)\cap D(V)$. Suppose that $-M\leq V$ for some $M\geq 0$. Then the integral kernels fulfill*

$$0\leq \ker e^{-r(H_0+V)}\leq \ker e^{rM}e^{-rH_0}. \tag{A.4}$$

The proof uses the Trotter product formula. The Duhamel formula,

$$e^{-rH_\kappa(t)}=e^{-rH_{\kappa'}(t)}-\int_0^r e^{-sH_{\kappa'}(t)}(H_\kappa(t)-H_{\kappa'}(t))e^{-(r-s)H_\kappa(t)}\,ds, \tag{A.5}$$

can be used to generate an expansion of $e^{-rH_\kappa(t)}$ in κ. This expansion gives essential self-adjointness and a lower bound for $H(t)$.

Theorem A.7. *$H(t)=\overline{H_0+V(t;g)}$ is essentially self-adjoint on $D(H_0)\cap D(V(t;g))$ and semibounded from below. Moreover,* $\operatorname{Ran} e^{-rH(t)}\subset D(H_0)\cap D(\tilde V)$ *for $r>0$ and $\tilde V\in\mathcal{V}(L^2(\mathbb{R}^p))$.*

Remark. Other cores for $H(t)$ are $D(H_0)\cap D(N^{p/2})$ and $C^\infty(H_0)=\bigcap_{n=0}^\infty D(H_0^n)$, see [36].

Theorem A.8. *Let $0\leq g\leq 1$ and let $(t,x)\mapsto h(t,x)$ be a real function with $\|h\|_\infty\leq 1$. Set $D=\operatorname{diam}\operatorname{supp} h<\infty$. Then there exists a constant M, independent of g and h, such that*

$$|\inf\sigma(H_0+V(t;g+h))|\leq MD.$$

For the proof, see [38].

The full Hamiltonian is defined by $H(t)=H_0+V(t;g)+E$ for every $t\in\mathbb{R}$, where $E>0$ is a finite renormalization constant. As we see in the following, it can be chosen in such a way that $H(t)$ is positive on compact time intervals.

Lemma A.9. *Let $I=[a,b]\subset\mathbb{R}$ be an arbitrary compact interval. Then there is a constant $E>0$ such that $H(t)>0$ for all $t\in I$.*

Proof. By [16, Theorem 5.8], there are $c > 0$ and $p \in \mathbb{N}$ independent of the interaction Hamiltonian such that $H(t)$ is semibounded. Set $\tilde{H}(t) := H_0 + V(t; g)$. Explicitly one finds

$$\tilde{H}(t) \geq -c - \ln \|e^{-V(t;g)}\|_{L^p(Q,d\mu)},$$

where the measure space Q refers to the Q-space representation of the Fock space, see above or [16, Section 5]. For $\lambda > 0$ sufficiently large, the resolvent $R(\lambda, \tilde{H}(t))$ of $\tilde{H}(t)$ is defined and, as Y is a core, we conclude norm continuity of $t \mapsto R(\lambda, \tilde{H}(t))$. In turn we have norm continuity of the semigroup $e^{i\lambda' \tilde{H}(t)}$ in the parameter t (which is not the semigroup parameter). The formula

$$\inf \sigma(\tilde{H}(t)) = \lambda'^{-1} \ln \|e^{i\lambda' \tilde{H}(t)}\|$$

shows that the infimum of the spectrum of $\tilde{H}(t)$ is continuous in t. As we have t in a compact interval, we can define $E = 1 + |\min_t \inf \sigma(\tilde{H}(t))|$ and get $H(t) > 0$. $\square$

Appendix B

Maximal accretivity of an operator sum

For the proof of the Theorem of Sohr we need a lemma which goes back to [86]. This lemma shows in particular the closedness of the sum $B+C$ on $D(B) \cap D(C)$ provided that the maximally accretive operators B and C satisfy the 'angle condition' (B.1).

Lemma B.1. *Let B and C be maximally accretive operators on a Hilbert space X such that $C^{-1} \in \mathcal{B}(X)$. Let $\mathcal{D}$ be a core of B^*. If there exists $0 \leq a < 1$ such that*

$$\operatorname{Re}(B^*x, C^{-1}x) + a\|x\|^2 \geq 0 \tag{B.1}$$

for every $x \in \mathcal{D}$, then $B+C$ is maximally accretive on $D(B+C) = D(B) \cap D(C)$.

With respect to our application we remark that in general the angle condition is violated for B being the time derivative and C being the generator of a *hyperbolic* evolution equation.

Proof. On Hilbert spaces maximal accretivity of an operator A is equivalent to accretivity and $\operatorname{Ran} A + \beta = \mathcal{H}$ for $\beta > 0$. Clearly, the sum $B + C$ is accretive, so it remains to prove $\operatorname{Ran} B + C + \beta = \mathcal{H}$.

Given $y \in \mathcal{H}$, we are looking for $x \in D(B) \cap D(C)$ with $(B + C + \beta)x = y$. The Yosida approximants $B_n = B(1+\frac{1}{n}B)^{-1}$ and $C_n = C(1+\frac{1}{n}C)^{-1}$ are bounded and maximally accretive operators for every $n \in \mathbb{N}$, which converge strongly on $D(B)$ respectively $D(C)$.

Assume that

$$\|B_n x + C_n x\| \geq (1 - a)\|C_n x\| \tag{B.2}$$

for all $x \in \mathcal{H}$ and $n \in \mathbb{N}$. By the boundedness and accretivity of $B_n + C_n$, there exists for every n an x_n such that $(B_n+C_n+\beta)x_n = y$. The sequences $\{B_n x_n\}_{n\in\mathbb{N}}$,

$\{C_n x_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ are bounded. This follows from

$$\begin{aligned}\|y\|^2 &= \|B_n x_n + C_n x_n\|^2 + \beta^2\|x_n\|^2 + 2\beta\,\mathrm{Re}(B_n x_n + C_n x_n, x_n)\\ &\geq (1-a)\|C_n x_n\|.\end{aligned}$$

Without loss of generality we can assume the sequence itself to be weakly convergent. Let x be the weak limit of $\{x_n\}_{n\in\mathbb{N}}$. One shows $x \in D(B^{**}) = D(B)$. For $z \in D(B^*)$ one finds

$$|\lim_{n\to\infty}(x_n, (B_n^* - B^*)z| \leq \sup_{n\in\mathbb{N}}\|x_n\| \lim_{n\to\infty}\|(B_n^* - B^*)z\| = 0$$

because of the strong convergence of $B_n^* z$ to $B^* z$. Therefore, one has

$$\lim_{n\to\infty}(B_n x_n, z) = \lim_{n\to\infty}(x_n, B^* z) = (x, B^* z)$$

and it follows $|(x, B^*z)| \leq \sup_{n\in\mathbb{N}}\|B_n x_n\|\|z\|$. This is the continuity of the linear functional $z \mapsto (x, B^*z)$, so $x \in D(B^{**}) = D(B)$ and the equation $\lim_{n\to\infty}(B_n x_n, z) = (Bx, z)$ extends to all $z \in \mathcal{H}$. With the same reasoning one shows $x \in D(C)$ and $\lim_{n\to\infty}(C_n x_n, z) = (Cx, z)$ for all $z \in \mathcal{H}$. One concludes from $y = (B_n + C_n + \beta)x_n$ for every n that $y = (B + C + \beta)x$ and $x \in D(B) \cap D(C) = D(B + C)$.

It remains to prove the estimate (B.2): $\|B_n x + C_n x\| \geq (1-a)\|C_n x\|$.

Let $y_n := C_n x$ and $z_n := (1 + \frac{1}{n}B^*)^{-1} y_n$. By the accretivity of B_n, B^* and C^{-1}, we find

$$\begin{aligned}\mathrm{Re}(B_n x, C_n x) &= \mathrm{Re}(B_n C_n^{-1} y_n, y_n) = \mathrm{Re}(B_n(C^{-1} + \tfrac{1}{n})y_n, y_n)\\ &\geq \mathrm{Re}(B_n C^{-1} y_n, y_n) = \mathrm{Re}(C^{-1}(1 + \tfrac{1}{n}B^*)z_n, B^* z_n)\\ &\geq \mathrm{Re}(C^{-1} z_n, B^* z_n).\end{aligned}$$

Besides we have $\|y_n\|^2 \geq \|z_n\|^2$ and conclude

$$\begin{aligned}\|(B_n + C_n)x\|\|C_n x\| &\geq \mathrm{Re}((B_n + C_n)x, C_n x) = \mathrm{Re}(B_n x, C_n x) + \|C_n x\|^2\\ &= \mathrm{Re}(B_n x, C_n x) + a\|y_n\|^2 + (1-a)\|C_n x\|^2\\ &\geq \underbrace{\mathrm{Re}(C^{-1} z_n, B^* z_n) + a\|z_n\|^2}_{\geq 0 \text{ by assumption}} + (1-a)\|C_n x\|^2\\ &\geq (1-a)\|C_n x\|^2.\end{aligned}$$

□

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