Proton Decay in Supersymmetric Grand Unified Theories

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Sören Wiesenfeldt

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Gutachter der Dissertation:	Prof. Dr. W. Buchmüller Prof. Dr. G. Mack
Gutachter der Disputation:	Prof. Dr. W. Buchmüller Prof. Dr. B. Kniehl
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Vorsitzender des Promotionsausschusses:	Prof. Dr. R. Wiesendanger
Dekan des Fachbereichs Physik:	Prof. Dr. G. Huber

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Abstract

The instability of the proton is a crucial prediction of supersymmetric GUTs, and the long-lasting search for proton decay makes it possible to constrain such models. We consider the decay in minimal supersymmetric $\mathsf{SU}(5)$, which is dominated by dimension-five operators, and analyze the implications of the failure of Yukawa unification for the decay rate. In a consistent $\mathsf{SU}(5)$ model with higher dimensional operators, where $\mathsf{SU}(5)$ relations among Yukawa couplings hold, the proton decay rate can be several orders of magnitude smaller than the present experimental bound. We extend the operator analysis to $\mathsf{SO}(10)$ as well. Finally, we discuss a 6D $\mathsf{SO}(10)$ orbifold GUT model, where proton decay is mediated by dimension-six operators. The branching ratios differ significantly from those in four dimensions.

Protonenzerfall in supersymmetrischen vereinheitlichten Theorien

Zusammenfassung

Die Instabilität des Protons ist eine zentrale Vorhersage von supersymmetrischen GUTs, und die jahrelange Suche nach Protonenzerfällen ermöglicht es, verschiedene Modelle zu testen. Wir untersuchen den Zerfall im minimalen supersymmetrischen SU(5)-Modell, der durch Dimension-fünf-Operatoren bestimmt ist, und diskutieren die Auswirkungen, die sich aus der Verletzung der Vereinigung der Yukawakopplungen ergeben. Dieses Problem kann durch höherdimensionale Operatoren gelöst werden, und in einem solchen konsistenten SU(5)-Modell kann die Protonenlebensdauer mehrere Größenordnungen oberhalb der experimentellen Grenze liegen. Wir betrachten solche Operatoren auch in SO(10)-Modellen. Schließlich betrachten wir ein 6D SO(10) Orbifold-GUT-Modell, in dem Protonen mittels Dimension-sechs-Operatoren zerfallen. Die Zerfallsbreiten unterscheiden sich zum Teil deutlich von denen in vier Dimensionen.

Contents

Introduction

Is the proton stable? Although the decay of a proton could not be detected, physicists have been dealing with this question for decades. Due to the non-observation of such a process, Stückelberg invented a "conservation law of heavy charge" in 1938 [1]:

> *Apart from ... the conservation law of electricity there exists evidently a further conservation law: No transmutations of heavy particles (neutron and proton) into light particles (electron and neutrino) have yet been observed in any transformation of matter. We shall therefore demand a conservation law of heavy charge.*

This approach was taken up by Wigner in the late 40's [2,3][1] and motivated experimentalists to test this conservation law [4]. Exactly 50 years ago, Reines, Cowan and Goldhaber reported $\tau_p > 1 \times 10^{22}$ years as the result of the first systematic experiment [5]; the current lower limit is $\mathcal{O}\left(10^{33}\,\text{years}\right)$. Hence there was no reason not to believe in baryon number conservation, but on the other hand, no satisfying framework could be found explaining the conservation law.

The situation changed because of two reasons: First, in 1976, 't Hooft found that due to Bell-Jackiw anomalies in gauge theories, non-perturbative effects can give rise to interactions that violate baryon and lepton number [6]. In the electroweak theory, however, such processes are suppressed by the factor $\exp\left\{-16\pi^2/g^2\right\} \simeq 10^{-37}$. Second, exactly thirty years ago, Georgi and Glashow proposed the idea that the standard model of particle physics can be embedded in a grand unified theory (GUT) based on the group $\mathsf{SU(5)}$ [7]. As a result, the additional gauge and charged Higgs bosons lead to baryon and lepton number violating interactions. Even one year earlier, Pati and Salam had considered the possibility of baryons and leptons being members of the same fermionic multiplet yielding such interactions as well [8,9]. Thus both mechanisms state that baryon number does not correspond to an absolute symmetry and, moreover, explain why the lifetime of protons is tremendously long.

The standard model of particle physics (SM) describes the particles and their interactions at the electroweak scale M_{EW} extremely well and there is no experiment in

[1] "It is conceivable ... that a conservation law for the number of heavy particles (protons and neutrons) is responsible for the stability of the protons in the same way as the conservation law for charges is responsible for the stability. Without the conservation law in question, the proton could disintegrate, under emission of a light quantum, into a positron." [2]

contradiction to it. But there are several evidences, both theoretical and experimental, that require the extension of the standard model. Several approaches have been pursued to extend the standard model to a more fundamental theory and in these extensions, baryon and lepton number violating operators arise naturally. We can therefore expect those baryon and lepton number violating operators to be generated at a higher scale, where new physics takes place.

A very attractive framework is provided by supersymmetric GUTs, combining the ideas of grand unification and supersymmetry, the latter broken at a scale $\mathcal{O}\,(1\,\text{TeV})$. This low-scale supersymmetry solves the naturalness problem of gauge hierarchies. Remarkably, with the renormalization group equations (RGEs) of the minimal supersymmetric extension of the standard model (MSSM) above the TeV scale, the three gauge couplings meet almost exactly at $M_{\text{GUT}} = 2 \times 10^{16}\,\text{GeV}$ (Figure 1.1), where strong, electromagnetic and weak interactions are then unified in a single gauge group with one gauge coupling. It is striking that the experimental evidence for small but non-vanishing neutrino masses fits nicely in this framework as M_{GUT} has the right order for Majorana masses of neutrinos.

A key assumption in this picture is that no new phenomena occur between M_{EW} and M_{GUT}, which covers over 14 orders of magnitude; this is often called the big desert. Moreover, supersymmetric GUTs are intermediate theories as well and have to be extended at least at the Planck scale, where gravity becomes as strong as the other interactions.

Supersymmetric GUTs have been studied for more than twenty years. Recently the simplest version, minimal supersymmetric $\mathsf{SU(5)}$ [10, 11], was claimed to be excluded due to the current bound on proton decay [12, 13]. This minimal model is the "prototype" GUT model and its exclusion is an important result. In this work, we reanalyze proton decay in this model and discuss the underlying assumptions, in particular the dependence on flavor mixing. The decay is dominated by dimension-five operators that are mediated by color-charged Higgs particles and involve two fermions and two scalar partners of fermions, where the latter are integrated out at the supersymmetry breaking scale. The flavor dependence occurs due to the failure of down quark and charged lepton Yukawa couplings to unify. Thus the theory requires additional interactions which account for the difference between down quark and charged lepton masses. Such interactions are conveniently parameterized by higher dimensional operators, which are naturally expected as a result of interactions at a higher scale, where the GUT model is extended to a more fundamental theory. In particular, the GUT scale is only about two orders below the Planck scale. Interestingly, the operators induced by Planck-scale effects have the right order to explain the differences between down quarks and charged leptons; we therefore call this model a consistent $\mathsf{SU(5)}$ model. Moreover, we will show that these operators can reduce the proton decay rate by several orders of magnitude and make it consistent with the experimental upper bound.

We will extend our operator analysis to $\mathsf{SO(10)}$, where all particles of the standard model are unified in only one 16-dimensional representation, together with one additional particle being a singlet with respect to the standard model. This particle can be identified with the right-handed neutrino. The breaking path of $\mathsf{SO(10)}$, however, and the Higgs sector is not unique which makes the theory less predictive. Therefore a large number of $\mathsf{SO(10)}$ models exists that try to match the full set of experimental data available today.

Starting in a six-dimensional space-time simplifies the breaking pattern if the $\mathsf{SO(10)}$ GUT symmetry is broken to the standard model by utilizing GUT-symmetry violating boundary conditions on a singular orbifold compactification. These orbifold GUTs have the additional advantage that they separate the GUT and electroweak scale in an elegant way and avoid the dimension-five operators. Proton decay is now dominated by exchange of the additional gauge bosons. The branching ratios significantly differ from those in four dimensions which should make it possible to distinguish orbifold and four-dimensional GUTs if proton decay is observed in the future [15].

This work is organized as follows: At the beginning, we review the standard model and its problems. After that we discuss approaches which have been done to solve these problems, in particular supersymmetry and grand unification. In Chapter 2, we consider proton decay via dimension-five operators. We study the minimal $\mathsf{SU(5)}$ model in detail and show that a consistent model with minimal particle content can fulfill the $\mathsf{SU(5)}$ predictions and is still in agreement with the experimental bound on proton decay. We then turn to $\mathsf{SO(10)}$ models and, finally, discuss proton decay induced at the Planck scale.

Chapter 3 is devoted to proton decay via dimension-six operators. We start with a general analysis of those operators and then turn to a six-dimensional $\mathsf{SO(10)}$ model, which is compactified on a torus at the four-dimensional GUT scale. The results are summarized and discussed in the last chapter. Details are given in the Appendices.

Chapter 1

Supersymmetric Grand Unification

The standard model is a very successful but only effective theory which has to be extended at higher energies. We briefly review the basics of the standard model and discuss its limitations and problems. Then we explore different approaches for extensions, in particular supersymmetry and grand unification.

1.1 Standard Model and its Limitations

The standard model of particle physics is based on the gauge interactions of the strong and electroweak interactions with gauge group $\mathsf{G}_{\text{SM}} = \mathsf{SU(3)_C} \times \mathsf{SU(2)_L} \times \mathsf{U(1)_Y}$ [16]. It is described by Quantum Chromodynamics (QCD) and the Glashow-Salam-Weinberg (GSW) theory and contains twelve gauge bosons with spin 1: eight gluons of $\mathsf{SU(3)_C}$, three $\mathsf{SU(2)_L}$ weak bosons and the hypercharge boson of $\mathsf{U(1)_Y}$ (Table 1.1).

The fundamental fermionic entities are leptons, which do not feel the strong interaction, and quarks, the constituents of hadrons. They appear in three distinct generations and are described by left-handed Weyl fields ξ_α, which are representations of the Lorentz group $\mathsf{SO(1,3)}$ (cf. Appendix A). The first generation is shown in Table 1.2. There is no left-handed antineutrino in the standard model which would be neutral with respect to all interactions, i.e. (1,1,0).

The multiplets can also be identified by their quantum numbers. Each fermion family is given by the sum

$$(3, 2, \tfrac{1}{6}) \oplus (3^*, 1, -\tfrac{2}{3}) \oplus (3^*, 1, \tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2}) \oplus (1, 1, 1)\ , \tag{1.1}$$

gauge boson	gauge group	quantum numbers	coupling
gluons	$\mathsf{SU(3)_C}$	(8,1,0)	g_3
W bosons	$\mathsf{SU(2)_L}$	(1,3,0)	g_2
B boson	$\mathsf{U(1)_Y}$	(1,1,0)	g'

Table 1.1: *The SM gauge bosons. The quantum numbers are due to* $\mathsf{SU(3)_C}, \mathsf{SU(2)_L}$ *and* $U(1)_Y$.

quark	quantum numbers	charge	lepton	quantum numbers	charge
$Q = \binom{u_\mathrm{L}}{d_\mathrm{L}}$	$(3, 2, \frac{1}{6})$	$+2/3$ $-1/3$	$L = \binom{\nu_{e\mathrm{L}}}{e_\mathrm{L}^-}$	$(1, 2, -\frac{1}{2})$	0 -1
u_L^C	$(3^*, 1, -\frac{2}{3})$	$-2/3$			
d_L^C	$(3^*, 1, \frac{1}{3})$	$+1/3$	e_L^+	$(1, 1, 1)$	$+1$

Table 1.2: *The fermions of the standard model.*

the gauge bosons by

$$(8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0) \ . \tag{1.2}$$

The standard model is based on renormalizable gauge theories. This makes it possible to describe the particles and their interactions in some energy range up to a scale Λ, where perturbation theory breaks down, with good approximation. Here, the divergencies of the bare parameters of the theory are absorbed in physical, running parameters that depend on the energy scale. Furthermore, the Noether current associated with the symmetries of the theory is conserved. This property can be destroyed by anomalies which appear whenever a classical symmetry is neccessarily violated at quantum level. The anomaly coefficients are proportional to the trace over the group matrices,

$$\mathrm{tr}\left(\left\{T^a, T^b\right\} T^c\right) . \tag{1.3}$$

An $\mathsf{SU}(n)$ gauge theory is anomaly free if the coefficients of the various irreducible components of the fermion multiplet sum to zero; this is exactly what happens in the standard model. Remarkably, the theory requires equal numbers of quark and lepton doublets.

Due to renormalizability, the Lagrangean of a gauge theory must be exactly invariant under gauge transformations. Since vector boson mass terms are not gauge invariant, gauge bosons must be exactly massless (as it is the photon in QED). For fermions, however, there are two possible mass terms. The Dirac mass term reads

$$m\,\xi^\alpha \xi^\mathrm{C}_\alpha + m\,\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\mathrm{C}\dot{\alpha}} \ . \tag{1.4}$$

It is not invariant because ξ and ξ^C transform according to different irreducible representations. The Majorana mass term is given by

$$m\,\xi^\alpha \xi_\alpha + m\,\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \tag{1.5}$$

where $\bar{\xi}$ denote the right-handed particles. It is only possible for singlets. In general, gauge-invariant mass terms exist if the representation containing the fermions is strictly real. This is not the case in the standard model, so there are no direct mass terms.

Nevertheless mass terms can be generated, namely via the Higgs-Kibble mechanism which implies a spontaneous breakdown of the electroweak symmetry $\mathsf{SU}(2)_\mathsf{L} \times \mathsf{U}(1)_\mathsf{Y}$ to the electromagnetic $\mathsf{U}(1)_\mathsf{em}$ so that Dirac mass terms are possible. A color-neutral doublet

$\Phi = (1, 2, \frac{1}{2})$ of scalar fields ϕ^+ and ϕ^0, the Higgs field, is introduced. For $\mu^2 < 0$, its potential

$$V(\Phi) = +\frac{1}{2}\mu^2\,\Phi^\dagger\Phi + \frac{1}{4}\lambda\,(\Phi^\dagger\Phi)^2 \tag{1.6}$$

has the nonvanishing vacuum expectation value (vev)

$$\langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \quad v = \sqrt{-\frac{\mu^2}{\lambda}} = \sqrt{\frac{1}{\sqrt{2}\,G_F}} \simeq 246\,\mathrm{GeV} \tag{1.7}$$

where G_F is the Fermi constant. Three gauge bosons become massive, the charged bosons $W^\pm$ and the neutral Z boson, whereas one boson remains massless, which can be identified with the photon,

$$\begin{aligned} W^\pm &= \frac{1}{\sqrt{2}}\,(W_1 \mp i\,W_2)\;, & M_W &= \frac{1}{2}\,g_2\,v \\ Z &= B\sin\theta_\mathrm{w} + W_3\cos\theta_\mathrm{w}\;, & M_Z &= \frac{1}{2}\sqrt{g'^2 + g_2^2}\;v \\ A &= B\cos\theta_\mathrm{w} - W_3\sin\theta_\mathrm{w}\;, & M_A &= 0\;. \end{aligned} \tag{1.8}$$

The weak mixing angle,

$$\cos\theta_\mathrm{w} = \frac{g_2}{\sqrt{g'^2 + g_2^2}}\;, \tag{1.9}$$

determines the ratio of $W^\pm$ and Z boson masses as well as the ratio of the couplings g' and g_2 (cf. Table. 1.1). Furthermore, it gives a relation between the electric charge e and g_2: $e = g_2\sin\theta_\mathrm{w}$. The experimental value is $\sin^2\theta_\mathrm{w}(M_Z) = 0.23$ [17].

Three components of Φ are eaten to give masses to the gauge bosons, hence only one is left, the scalar Higgs particle which is the only particle in the standard model that could not be detected so far. The experimental lower bound for its mass, $M_H = \sqrt{2\lambda}\,v$, is $M_H > 114.4\,\mathrm{GeV}$ [18]. For the SM to be self-consistent up to the Planck scale, the self-coupling is restricted such that $M_H \lesssim 190\,\mathrm{GeV}$ [17]. Furthermore, the scattering amplitude for the longitudinal components of W and Z violates unitary at $\mathcal{O}(1\,\mathrm{TeV})$.

Yukawa couplings between fermion and Higgs fields,[1]

$$Y_u\,Q\,u^\mathrm{c}\,\Phi + Y_d\,Q\,d^\mathrm{c}\,\overline{\Phi} + Y_e\,e^\mathrm{c}L\,\overline{\Phi}\;, \tag{1.10}$$

generate fermion masses which are of Dirac type and free parameters of the standard model. The connection between mass and flavor eigenstates is given by the CKM matrix, which can be parametrized by three angles and one $\mathcal{CP}$ violating phase.

Counting the free parameters of the standard model, there is one missing. The Lagrangean contains another invariant,

$$\frac{\theta}{32\pi^2}\,F^a_{\mu\nu}\,\widetilde{F}^{\mu\nu\,a}\;, \tag{1.11}$$

[1] In view of $\mathsf{SU(5)}$ GUTs, the structure of the lepton couplings differ from those of the quarks.

which violates $\mathcal{P}$ and $\mathcal{CP}$. This term is actually a 4-divergence which only contributes surface terms to the action. In electroweak processes, θ can be set to zero due to the non-existence of explicit mass terms. In QCD, there are such mass terms which break the chiral symmetry. There is a combination that is invariant under $\mathsf{U(1)}_A$ rotations,

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \mathrm{Arg}\,(\det M) \ , \tag{1.12}$$

with the quark mass matrix $M = (M_u, M_d)$. But in pure QCD, $\mathcal{P}$ and $\mathcal{CP}$ seem to be exact symmtries, and the non-observation of a neutron dipole moment constrains $\bar{\theta} \leq 10^{-9}$. This small number gives rise to the θ problem of QCD.[2]

Incidentally, the additional term (1.11) is obtained by non-perturbative phenomena, namely instantons. Those are Yang-Mills field configurations in Euclidean space for which the surface term is nonzero, even though their action is finite [19]. In Minkowski spacetime, they describe tunneling processes between the various vacua, separated by a finite energy barrier E_{sph}. Instantons are of interest in non-perturbative QCD [20], whereas they are strongly suppressed in electroweak processes due to the much smaller coupling. But at energies above E_{sph}, i.e. in the early universe and in high energy collisions, electroweak sphaleron processes which violate $(B + L)$ can play an important role [21].

Problems and Open Questions

The standard model successfully describes or is at least consistent with all known facts of elementary particle physics. It is a consistent field theory in which electromagnetic, strong and weak interactions are basically gauge interactions. For several reasons, however, it is no fundamental but only an effective theory which has to be extended at a higher scale.

Open questions arise due to the structure of the model. The pattern of groups and representations is complicated and arbitrary. The gauge group is a direct product of three different factors, and it requires chiral fermions in the GSW theory but not in QCD. The values of the three gauge couplings are much different, and the electric charge is not quantized. Furthermore, there are three generations, where the second and third are merely more massive repetitions of the first family, and the Yukawa couplings as well as the Higgs parameters are free parameters of the theory. Altogether, there are nineteen free parameters: three gauge couplings, two Higgs parameters, nine fermion masses, three mixing angles and one phase of the CKM matrix and the θ parameter of $\mathsf{SU(3)}$.

One hint for new physics comes from neutrino physics. In the standard model, neutrinos are massless but several experiments consistently require small but nonvanishing neutrino masses. As will be discussed in the following section, these masses, being several orders smaller than those of all other leptons and quarks, point to a new scale above the electroweak breaking scale M_{EW}.

[2] θ_{QCD} could be put to zero if the up-quark was massless or via a $\mathsf{U(1)}_A$ symmetry giving rise to axions.

Another scale for new physics is indicated by the Higgs sector. The Higgs field is introduced as a scalar because only scalars can have nonvanishing vacuum expectation values without breaking Lorentz invariance. On the other hand, scalar masses are subject to quadratic divergences in perturbation theory. In order for the Higgs mass to be naturally in the $\mathcal{O}\,(100 - 1000\,\mathrm{GeV})$ range, either new physics, which couples to the Higgs sector, should appear in the TeV region or below to cut off the quadratically divergent contributions, or large bare contributions with the opposite sign must appear. For Planck scale corrections, however, the cancellation must be accurate to $\mathcal{O}\left(\frac{M_{\mathrm{Pl}}^2}{M_{\mathrm{EW}}^2}\right) \sim 10^{32}$, which makes it rather unnatural.

Finally, the most fundamental aspect concerns gravity, the fourth interaction observed in nature. Gravitational interactions do not appear in the standard model so that there are two distinct theories, the standard model dealing with light particles on small scales and general relativity, valid on large scales. We know that around the Planck scale, $M_{\mathrm{Pl}} = \sqrt{\frac{\hbar c}{8\pi G_N}} = 2.4 \times 10^{18}\,\mathrm{GeV}$, both theories have to be combined to a theory of quantum gravity, and the standard model is no longer valid.

Several approaches have been taken extending the standard model to a more — or even the ultimate — fundamental theory. The basic ideas arose already in the 70's (or even before) and have been studied in great detail in the last decades. At low energies, new physics can be integrated out and its effects can be parametrized in terms of higher dimensional operators involving only standard model fields [22]. Precision measurements constrain the sizes of various higher dimensional operators and consequently the scale of the corresponding new physics [23]. The most stringent bounds are on operators which break the (approximate) symmetries of the standard model, such as baryon number, flavor and $\mathcal{CP}$ symmetries.

1.2 Neutrino Physics

The standard model neutrinos are strictly massless, due to the absence of left-handed antineutrinos and renormalizable couplings to the Higgs boson. There is, however, compelling evidence in favor of massive neutrinos. It arises from experiments which require neutrinos to oscillate and, at the end, quote mass square differences between the mass eigenstates. These values turn out to be small, which can explain the problem of measuring the absolute values of neutrino masses.

The idea of neutrino oscillations traces back to Pontecorvo in 1957 [24]. If neutrinos are massive, weak and mass eigenstates no longer coincide; they are connected by a unitary matrix, which can pe parametrized by three angles θ_{12}, θ_{13}, θ_{23} and one $\mathcal{CP}$ violating phase δ (analogously to the CKM matrix in the quark sector). If the masses are not degenerate, the neutrinos oscillate between the different eigenstates; a detailed discussion of neutrino oscillations is given in Ref. [25].

The current status of the experiments is as follows (see e. g. Ref. [26]): The solar experiments together with the KamLAND reactor experiment are explained in terms of two-neutrino $\nu_e \leftrightarrow \nu_\mu$ oscillations with the best-fit point

$$\Delta m^2_{\text{sun}} \simeq 7 \times 10^{-5}\ \text{eV}^2\ , \quad \tan^2 \vartheta_{\text{sun}} \simeq 0.4\ ; \tag{1.13}$$

thus $\vartheta_{\text{sun}} \simeq \theta_{12}$. The atmospheric neutrino experiments favor $\nu_\mu \leftrightarrow \nu_\tau$ oscillations with

$$\Delta m^2_{\text{atm}} \simeq 2 \times 10^{-3}\ \text{eV}^2\ , \quad \sin^2 2\vartheta_{\text{atm}} \simeq 1 \tag{1.14}$$

so that $\vartheta_{\text{atm}} \simeq \theta_{23}$. The two-neutrino scenarios are viable due to a small value for the third angle, $\sin\theta_{13} \lesssim 0.1$; finally, the phase δ cannot be measured so far.[3]

The understanding of the origin of neutrino masses and mixing requires knowledge of the absolute values of neutrino masses. The small values for Δm^2, however, suggest that this is rather challenging unless the neutrinos are quasi-degenerate; the current experimental limits are $\mathcal{O}\,(1\,\text{eV})$ [27].

In the standard model as an effective theory, neutrino masses can be described by the non-renormalizable operator

$$L_i\, L_j\, \Phi\, \Phi \tag{1.15}$$

of dimension five, hence it is suppressed by some mass M, which marks a scale of new physics. This operator can simply be generated by adding left-handed antineutrinos to the fermions of the standard model which can get masses both via the Higgs mechanism and via Majorana mass terms because they are singlets with repect to the standard model. Diagonalizing the mass matrix by assuming $M \gg v$ leads to light and heavy neutrinos with masses

$$m_\nu \simeq \frac{Y_\nu^2\, v^2}{M}\ , \qquad\qquad m_R \simeq M\ , \tag{1.16}$$

where Y_ν denote the neutrino Yukawa couplings and M the Majorana mass matrix. This result is known as the seesaw mechanism [28]. With $m_\nu = \mathcal{O}\,(0.1\ \text{eV})$ and $Y_\nu = \mathcal{O}\,(1)$, we obtain $M = \mathcal{O}\,(10^{14}\ \text{GeV})$. Grand unified theories (see Section 1.4) in general require left-handed antineutrinos. Since the unification scale M_{GUT} is $\mathcal{O}\,(10^{15-16}\ \text{GeV})$, Majorana masses can be generated when the GUT symmetry is broken.

1.3 Supersymmetry

In 1967, Coleman and Mandula [29] showed that it is impossible to unify space-time symmetry and internal symmetries in a master group with only bosonic generators, which

[3] The only experiment whose result does not fit to the others is LSND. The best-fit values for this accelerator experiment are $\sin^2 \vartheta_{\text{LSND}} \simeq 3 \times 10^{-3}$, $\Delta m^2_{\text{LSND}} \simeq 1.2\ \text{eV}^2$. This result is usually neglected until it will be confirmed by other experiments within the next years [26, 27].

fulfill commutation relations. The way to avoid this no-go theorem was proposed in 1971 by Gol'fand and Likhtman [30], followed up by Volkov and Akulov [31]. They realized that the Poincaré algebra can be extended if one includes symmetry operations whose generators obey anticommutation relations. Hence fermionic symmetry operators that carry spin $\frac{1}{2}$ are introduced. Haag, Łopuszański and Sohnius proved that supersymmetry is the only possible extension of the Poincaré algebra [32].

Supersymmetric theories are very attractive for several reasons: First of all, they relate bosons and fermions, which are strictly separated in the standard model. Moreover, they introduce a supersymmetric partner for every known particle whose contributions in perturbation theory appear with opposite sign and hence exactly cancel the quadratic divergences in the standard model, leaving only logarithmic divergences. Furthermore, mass and Yukawa coupling constants receive no quantum corrections and the corrections in the kinetic term are only logarithmically divergent. Thus the superpotential (the analogue to the potential in non-supersymmetric theories) is not renormalized by higher-loop corrections and any fine-tuning of the potential will not receive any contributions from renormalization. This feature is known as the non-renormalization theorem.

Next, supersymmetry suggests a new symmetry which implies that the newly introduced supersymmetric partners can only be produced in pairs. This, in particular, means that the lightest supersymmetric particle is stable. This LSP is one of the most promising candidates for dark matter in the universe.

Finally, the gauging of supersymmetry leads to supersymmetric gravity, supergravity, since local supersymmetric transformations include general coordinate transformations. This still gives a non-renormalizable quantum theory like gravity, but it might be an appropriate effective theory below $M_{\rm Pl}$ (see reviews [33]).

These features have made supersymmetry very popular, though there has not been any experimental evidence so far. In the following, we will discuss basics of ($N = 1$) supersymmetry and formulate the supersymmetric extension of the standard model. For detailed discussions, we refer to the reviews and textbooks Refs. [34].

Supersymmetry operations transform bosons to fermions and vice versa. The generator Q_α ($\alpha = 1, 2$) is a left-handed Weyl spinor with spin $\frac{1}{2}$ and is invariant under translations but not under rotations and Lorentz-boosts. These generators fulfill anticommutation relations among themselves; the superalgebra is given in Appendix A. It follows immediately that $H = P^0 \geq 0$, so the non-degenerate ground state has zero energy whereas all other states are degenerate with positive energy.

Since supersymmetry transformations relate bosons and fermions, the representation space of the superalgebra (superspace) can be divided into a bosonic (Minkowski) and a fermionic subspace. The bosonic generators map the subspaces into themselves, whereas the supersymmetry generators map the bosonic subspace into the fermionic and vice versa. Thus both subspaces have the same dimension. The superspace can be parameterized by eight coordinates, four bosonic coordinates of spacetime, x^μ, and four fermionic, two-

Superfield		$S=\frac{1}{2}$	$S=0$
(s)quark	$Q=(U,D)$	$q=(u,d)$	$\tilde{q}=(\tilde{u},\tilde{d})$
	U^{C}	u^{C}	$\tilde{u}^{\mathrm{C}}$
	D^{C}	d^{C}	$\tilde{d}^{\mathrm{C}}$
(s)lepton	$L=(N,E)$	$l=(\nu,e)$	$\tilde{l}=(\tilde{\nu},\tilde{e})$
	E^{C}	e^{C}	$\tilde{e}^{\mathrm{C}}$
Higgs(ino)	$H_u=(H_u^+,H_u^0)$	$\tilde{h}_u=(\tilde{h}_u^+,\tilde{h}_u^0)$	$h_u=(h_u^+,h_u^0)$
	$H_d=(H_d^0,H_d^-)$	$\tilde{h}_d=(\tilde{h}_d^0,\tilde{h}_d^-)$	$h_d=(h_d^0,h_d^-)$

Table 1.3: *The chiral superfields of the standard model. Here and in the following, Higgs fields are denoted by H.*

component Grassmann numbers θ^α and $\overline{\theta}^{\dot{\alpha}}$.

The superfields $\phi(x,\theta,\overline{\theta})$ can be expanded as a finite Taylor series in θ and $\overline{\theta}$ with coefficients which are themselves local fields over Minkowski space. In general, they have 16 bosonic and 16 fermionic field components with equal mass, charge, weak isospin etc. and are not irreducible. We can, however, impose constraints on superfields to obtain smaller multiplets.

The covariant constraint, $\overline{\mathcal{D}}_{\dot{\alpha}}\phi=0$, defines chiral superfields. Each contains a complex scalar field φ, a Weyl fermion ψ_α and an auxiliary complex scalar field F, which is needed for the off-shell closure of the algebra and does not propagate. In general, the highest component of the supermultiplets only transforms into derivatives of the other fields. This can be used to construct Lagrangeans which transform into a total derivate under supersymmetry transformations leaving the corresponding action invariant.

Changing the parameterization of the superspace, $x^\mu \to y^\mu = x^\mu + i\,\theta\sigma^\mu\overline{\theta}$, the chiral multiplet can be expressed independently of $\overline{\theta}$,

$$\phi(y,\theta)=\varphi(y)+\theta\,\psi_\alpha(y)+\theta^2\,F(y)\ . \tag{1.17}$$

For a supersymmetric extension of the standard model, the fermions are put into chiral multiplets together with scalar partners, the sfermions (Table 1.3). The same happens to the Higgs field which gets a fermionic partner. Due to its $U(1)_Y$ hypercharge, however, a second Higgs field $(1,2,\frac{1}{2})$ has to be introduced not to upset the anomaly cancellation condition. This leads to five Higgs bosons, two $\mathcal{CP}$-even (h^0, H^0), one $\mathcal{CP}$-odd (A^0) and a charged pair ($H^\pm$). The mass of the lightest Higgs boson is predicted to be $m_{h^0} \lesssim 150$ GeV.

The vector superfield is defined by the reality condition, $V=V^\dagger$. It contains a massless boson A_μ, a Weyl fermion λ and its adjoint $\overline{\lambda}$ and a real scalar field D, which is again an auxiliary field. Vector superfields are clearly needed for the gauge bosons, that get spin-$\frac{1}{2}$ partners, gauginos.

Altogether, we see that the particle spectrum of this minimal supersymmetric standard model (MSSM) is more than doubled. Since no supersymmetric partner has been detected in experiments so far, supersymmetry cannot be realized unbroken in nature. The breaking of supersymmetry, however, is an unsolved problem. The first ansatz, spontaneous breaking, does not work because of phenomenological difficulties. Supersymmetry has to be broken explicitly but softly, i. e. by insertion of weak scale mass terms in the Lagrangean. Those terms can arise from the spontaneous breaking of supergravity. In the MSSM, all possible terms are just added to the Lagrangean, hence the origin of these terms, the mechanism that leads to supersymmetry breaking is postponed to a more fundamental theory. Due to the quadratic divergence of the Higgs boson, the masses of the superpartners must not be bigger than $\mathcal{O}\,(1 \text{ TeV})$.

To define a Lagrangean, we use the Berezin integral (A.11). The integral of any superfield over the whole superspace will be an invariant,

$$\delta \int d^4x \; d^2\theta \; d^2\overline{\theta} \; \phi(x, \theta, \overline{\theta}) = 0 \; . \tag{1.18}$$

For chiral superfields, $\int d^4x \, d^2\theta$ alone is already an invariant integral, thus the Lagrangean of a supersymmetric gauge theory reads

$$\mathscr{L} = \int d^2\theta \; d^2\overline{\theta} \; \mathscr{L}_K \left(\overline{\phi}, \phi, V\right) + \left(\int d^2\theta \; W(\phi) + \text{h.c.} \right) \; , \tag{1.19}$$

where the superpotential

$$W = \frac{1}{2} \, m_{ij} \, \phi_i \, \phi_j + \frac{1}{3} \, y_{ijk} \, \phi_i \, \phi_j \, \phi_k \; , \tag{1.20}$$

is a holomorphic function of the scalar fields and the Kähler potential K is real. Here, m_{ij} are mass parameters, while the y_{ijk} are Yukawa couplings. The chiral Lagrangean is given by

$$\mathscr{L}_{\text{chiral}} = \partial_\mu \varphi_i^* \, \partial^\mu \varphi_i - i \, \overline{\psi}_i \, \overline{\sigma}^\mu \, \partial_\mu \psi_i + F_i^* F_i - \left(\frac{1}{2} \, m_{ij} \, \psi_i \, \psi_j + y_{ijk} \, \psi_i \, \psi_j \, \varphi_k + \text{h.c.} \right) . \tag{1.21}$$

In the MSSM, the Yukawa interactions read

$$W_{\text{Y}} = h_u \, Q \, U^{\text{c}} H_u + h_d \, Q \, D^{\text{c}} H_d + h_e \, E^{\text{c}} L \, H_d + \mu \, H_u \, H_d \; . \tag{1.22}$$

The Higgs fields acquire the vacuum expectation values $\langle H_d \rangle = v_d$ and $\langle H_u \rangle = v_u$, where $v_d^2 + v_u^2 = v^2 = (246 \text{ GeV})^2$. Their ratio, $\tan\beta = \frac{v_u}{v_d}$, can be restricted by requiring the Yukawa couplings not to be non-perturbatively large. This gives the rough constraint $2.5 \lesssim \tan\beta \lesssim 65$. Because of electroweak symmetry breaking, the Higgsinos and electroweak gauginos mix with each other, forming four neutral and two charged mass eigenstates called neutralinos and charginos, respectively.

There are, unfortunately, more gauge-invariant terms of mass dimension 4,

$$L_i\, L_j\, E_k^{\mathrm{C}} \;, \quad Q_i\, D_j^{\mathrm{C}}\, L_k \;, \quad U_i^{\mathrm{C}}\, D_j^{\mathrm{C}}\, D_k^{\mathrm{C}} \;, \tag{1.23}$$

which violate baryon and lepton number. Allowing those operators, the search for proton decay requires the coefficients being $\mathcal{O}\,(10^{-7})$. Therefore, to avoid the dimension-four operators, an additional symmetry is introduced [35].

One convenient choice is R-parity, a $\mathbb{Z}_2$ symmetry which is defined by the multiplicative quantum number

$$R_p = (-1)^{3B+L+2S} \;, \tag{1.24}$$

so the particles of the standard model and their supersymmetric partners have positive and negative parity, respectively. As a consequence, the latter can only be produced in pairs and the lightest supersymmetric particle (LSP) is absolutely stable. R-parity forbids the dimension-four operators (1.23) and, moreover, all dimension-five operators except

$$Q_i\, Q_j\, Q_k\, L_m \;, \quad U_i^{\mathrm{C}}\, U_j^{\mathrm{C}}\, D_k^{\mathrm{C}}\, E_m^{\mathrm{C}} \;, \quad L_i\, L_j\, H_u\, H_u \;. \tag{1.25}$$

As discussed in the previous section, the latter induces neutrino masses, the others violate baryon and lepton number. To be consistent with the negative proton decay searches, they are suppressed by a mass $\mathcal{O}\,(10^{16}\,\mathrm{GeV})$. We will discuss these operators in detail in Chapter 2.

The lightest neutralino is mostly assumed to be the LSP. Since it is electrically neutral, it interacts only weakly with ordinary matter, and so can make an attractive candidate for non-baryonic dark matter which seems to be required by cosmology (for a recent review see e. g. Ref. [36]).

An equivalent description is matter parity which is defined by

$$R_m = (-1)^{3(B-L)} \;. \tag{1.26}$$

Supersymmetric extensions of the standard model are usually defined to conserve R-parity, which is somewhat ad hoc from a theoretical point of view. There are no internal inconsistencies if it is not conserved, furthermore, the known discrete symmetries in the standard model ($\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$) are inexact symmetries. On the other hand, exactly conserved discrete symmetries can exist (they only have to satisfy certain anomaly cancellation conditions), and one particular way this could occur is if a continuous $\mathsf{U(1)}_{B-L}$ gauge symmetry is spontaneously broken at some high energy scale. Such a symmetry appears in GUTs based on gauge groups like $\mathsf{SO(10)}$ and R_m is a discrete subgroup of the continuous $\mathsf{U(1)}_{B-L}$ group. Therefore, if gauged $\mathsf{U(1)}_{B-L}$ is broken by scalar vevs which carry only integer values of $3\,(B-L)$, then R_m will automatically survive as an exactly conserved remnant. A systematic study of all discrete symmetries (even R-symmetries) that can be embedded in some $\mathsf{U(1)}$ gauge symmetry was done by Ibanez and Ross [37].

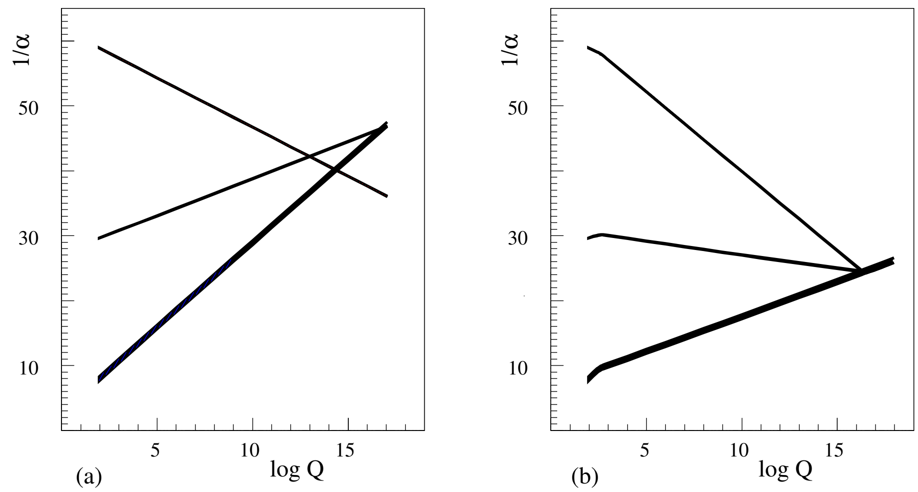


Figure 1.1: *Extrapolation in energy of the gauge couplings of the standard model, g_3, g_2 and $g_1 = \sqrt{5/3}\, g'$, where $g = \sqrt{4\pi\,\alpha}$, in (a) the standard model, (b) the MSSM [38].*

1.4 Grand Unification

The standard model contains three gauge groups with different gauge couplings. The basic idea of grand unification is that above a high scale, G_{SM} is embedded in a larger underlying group G_{GUT} which is simple, i. e. it has only one gauge coupling. The additional symmetries of G_{GUT} restrict some of the features that are arbitrary in the standard model. At M_{GUT}, G_{GUT} is spontaneously broken resulting in the observed pattern of couplings at low energies: the values of $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ increase at smaller momentum scales due to their asymptotically free renormalization group equations (RGEs), while the value of the $\mathsf{U(1)}$ decreases.

If we take the gauge couplings at M_{EW}, extrapolate them to high energies using the RGEs of the standard model and — in view of $\mathsf{SU(5)}$ — redefine the $\mathsf{U(1)}$ coupling, $g_1 = \sqrt{\frac{5}{3}}\, g'$, we get the picture shown in Figure 1.1a: The coupling constants do come close together at 10^{14-15} GeV, though they do not meet. But with the RGEs of the MSSM above a scale M_s where the supersymmetric particles are integrated out, we end up with Figure 1.1b, where the three gauge couplings meet accurately within their current uncertainties at $M_{\text{GUT}} = 2\times 10^{16}$ GeV.[4] Therefore assuming the MSSM to describe particle physics above M_s, one can indeed set a high energy scale M_{GUT} at which the MSSM is extended to a supersymmetric GUT. M_s is determined to be between 100 GeV and 1 TeV, which fits perfectly with the masses of the superpartners predicted to be below the TeV scale.

Thus the simple picture of GUTs looks like

$$\mathsf{G}_{\text{GUT}} \xrightarrow[M_{\text{GUT}}]{} \mathsf{G}_{\text{SM}} \xrightarrow[M_{\text{EW}}]{} \mathsf{SU(3)_C} \times \mathsf{U(1)}_{\text{em}}\ , \tag{1.27}$$

[4]This picture changes slightly at 2-loop but here, the threshold effects at M_{GUT} are important as well. Furthermore, gauge-coupling unification is not only successful in the MSSM, see e. g. Ref. [39].

where the GUT symmetry is spontaneously broken, if necessary in more than one step. Forcing the electric charge operator Q to be a generator of the GUT gauge group, this embedding already explains the quantization of electric charge because of $\operatorname{tr} Q = 0$.

There are only four classes of simple Lie groups: $\mathsf{SU}(n)$, $\mathsf{SO}(2n)$, $\mathsf{SO}(2n+1)$ and $\mathsf{Sp}(2n)$, where n is an integer number. In addition, there are five so-called exceptional groups, namely G_2, F_4, E_6, E_7 and E_8. The groups are discussed in detail in Ref. [40].

The search for the GUT group is guided by two general features: Firstly, to embed the standard model, it has to be at least of rank four and, in particular, contain a $\mathsf{SU}(3)$ subgroup. The rank is the maximal number of commuting generators. The standard model has four, namely the color generators T_3 and T_8 of $\mathsf{SU}(3)$, T_3 of weak isospin, and the hypercharge Y. On the other hand, in order not to add too many new particles and interactions, the rank should not increase too much.

Secondly, the representations must allow for the correct reproduction of the particle content of the observed fermion spectrum, at least for one generation of fermions. This requirement implies that G_{GUT} must possess complex representations as well as it must be free from anomalies in order not to spoil the renormalizability of the grand unified theory by an incompatibility of regularization and gauge invariance. The requirement of complex fermion representations is based on the fact that embedding the known fermions in real representations leads to difficulties: Mirror fermions must be added which must be very heavy. But then the conventional fermions would in general get masses of order M_{GUT}. Hence all light fermions should be components of a complex representation of G_{GUT}.

The requirements restrict the possible grand unified models to the gauge groups $\mathsf{SU}(n)$ starting with $n = 5$, $\mathsf{SO}(4n+2)$, $n \geq 2$, and E_6. In the following, we briefly discuss the (non-supersymmetric) Georgi-Glashow model [7] which is based on $\mathsf{SU}(5)$; for more details, see Refs. [40, 41].

1.4.1 Georgi-Glashow Model

The group $\mathsf{SU}(5)$ is defined by its fundamental representation, the group of 5×5 unitary matrices with determinant one. A general transformation can be written as

$$U = \exp\left\{ -i \sum_{j=1}^{24} \beta^j L^j \right\} , \tag{1.28}$$

where the generators $L^j = \frac{1}{2}\lambda^j$ are Hermitean and traceless and normalized so that $\operatorname{tr}(L^i L^j) = \frac{1}{2}\delta^{ij}$. The matrices λ^j are given in Appendix B; in case of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$, they correspond to the Pauli and Gell-Mann matrices, respectively. Since we embed a group of the form $\mathsf{SU(N)} \times \mathsf{SU(M)} \times \mathsf{U}(1)$ into $\mathsf{SU(N+M)}$, we choose $\mathsf{SU(N)}$ to act on the first N indices and $\mathsf{SU(M)}$ on the last M indices. Both of these subgroups commute with the $\mathsf{U}(1)$ which we can take to be M on the first N indices and $-\mathsf{N}$ on the last M.

There are 24 Hermitean gauge fields A^j. It is convenient to define the 5×5 matrix A,

$$A \equiv \sqrt{2} \sum_{j=1}^{24} L^j A^j \; ; \tag{1.29}$$

explicitly, it reads

$$A = \left(\begin{array}{ccc|cc} G_1^1 - \frac{2B}{\sqrt{30}} & G_2^1 & G_3^1 & \bar{X}^1 & \bar{Y}^1 \\ G_1^2 & G_2^2 - \frac{2B}{\sqrt{30}} & G_3^2 & \bar{X}^2 & \bar{Y}^2 \\ G_1^3 & G_2^3 & G_3^3 - \frac{2B}{\sqrt{30}} & \bar{X}^3 & \bar{Y}^3 \\ \hline X_1 & X_2 & X_3 & \frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & \frac{W_1 + iW_2}{\sqrt{2}} \\ Y_1 & Y_2 & Y_3 & \frac{W_1 - iW_2}{\sqrt{2}} & -\frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{array} \right) . \tag{1.30}$$

The entries represent the gauge bosons transforming according to the adjoint representation which decomposes into

$$\mathsf{24} \to (8,1,0) \oplus (1,3,0) \oplus (1,1,0) \oplus (3,2^*,-\tfrac{5}{6}) \oplus (3^*,2,\tfrac{5}{6}) \; . \tag{1.31}$$

We identify the SM gauge bosons (1.2) and find twelve new ones, the X and Y bosons. They carry color and electric charge and can mediate baryon and lepton number violating interactions.

Next we turn to the fermions. $\mathsf{SU(5)}$ has two five-dimensional representations, $\mathsf{5}$ and $\mathsf{5}^*$, which have the decompositions

$$\mathsf{5} \to (3,1,-\tfrac{1}{3}) \oplus (1,2,+\tfrac{1}{2}) \; , \quad \mathsf{5}^* \to (3^*,1,\tfrac{1}{3}) \oplus (1,2^*,-\tfrac{1}{2}) \; . \tag{1.32}$$

Thus we can group the anti-down quarks and lepton doublet into $\mathsf{5}^*$ because the $\mathsf{2}^*$ of $\mathsf{SU(2)}$ is equivalent to $\mathsf{2}$. Hence we are left with $(3^*,1,-\frac{2}{3}) \oplus (3,2,\frac{1}{6}) \oplus (1,1,1)$ which is ten-dimensional and fits perfectly into the ten dimensional representations, $\mathsf{10} = [\mathsf{5} \times \mathsf{5}]_A$, so we can group every fermion generation (1.1) into $\mathsf{5}^* \oplus \mathsf{10}$. Ignoring mixings, we get

$$\mathsf{5}^* = \left(\begin{array}{c} d_1^{\mathrm{C}} \\ d_2^{\mathrm{C}} \\ d_3^{\mathrm{C}} \\ \hline e^- \\ -\nu_e \end{array} \right)_{\mathrm{L}} , \quad \mathsf{10} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} 0 & u_3^{\mathrm{C}} & -u_2^{\mathrm{C}} & -u^1 & -d^1 \\ -u_3^{\mathrm{C}} & 0 & u_1^{\mathrm{C}} & -u^2 & -d^2 \\ u_2^{\mathrm{C}} & -u_1^{\mathrm{C}} & 0 & -u^3 & -d^3 \\ \hline u^1 & u^2 & u^3 & 0 & -e^{\mathrm{C}} \\ d^1 & d^2 & d^3 & e^{\mathrm{C}} & 0 \end{array} \right)_{\mathrm{L}} . \tag{1.33}$$

Remarkably, $\mathsf{5}^* \oplus \mathsf{10}$ is anomaly free.

The kinetic energy terms for fermions read

$$\begin{aligned} \mathscr{L}_{\mathrm{kin},f} &= i\, \overline{\mathsf{5}}^*_a (\not{D}\, \mathsf{5}^*)^a + i\, \overline{\mathsf{10}}_{ac} (\not{D}\, \mathsf{10}\,)^{ac} \\ &= \overline{\mathsf{5}}^*_a \left[i\not{\partial}\, \delta^a_b + \frac{g_5}{\sqrt{2}} \not{A}^a_b \right] \mathsf{5}^{*\,b} + \overline{\mathsf{10}}_{ac} \left[i\not{\partial}\, \delta^a_b + \sqrt{2} g_5 \not{A}^a_b \right] \mathsf{10}^{bc} \; , \end{aligned} \tag{1.34}$$

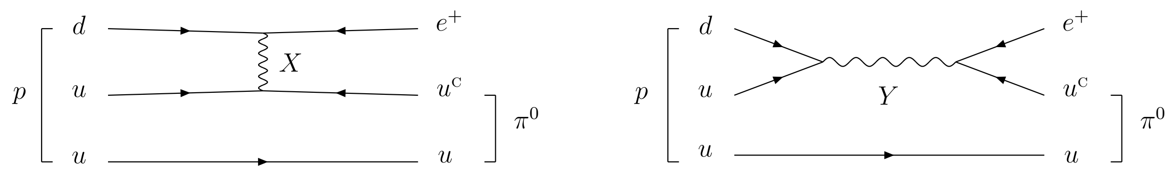


Figure 1.2: *Proton decay via X and Y bosons*

where the antisymmetry of $\mathsf{10}^{bc}$ is used, and g_5 is the $\mathsf{SU(5)}$ gauge coupling. Since quarks and leptons appear together in representations, baryon and lepton number violating interactions are possible which lead to proton decay (Fig. 1.2). Finally, by decomposing Eqn. (1.34) in SM fields, we must take $g_1 = \sqrt{\frac{5}{3}}\, g'$ due to the fact that $\sqrt{\frac{3}{5}}\, Y$ is a properly normalized $\mathsf{SU(5)}$ generator.

Spontaneous Symmetry Breaking

We first break $\mathsf{SU(5)}$ by an adjoint representation Σ to G_{SM} where

$$\Sigma(\mathsf{24}) = \begin{pmatrix} \Sigma_8 & \Sigma_{(3,2)} \\ \Sigma_{(3^*,2)} & \Sigma_3 \end{pmatrix} + \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \Sigma_{24} \ . \tag{1.35}$$

It acquires the vacuum expectation value

$$\langle \Sigma \rangle = \sigma \,\mathrm{diag}(2,2,2,-3,-3) \tag{1.36}$$

so that the X and Y bosons become massive,

$$M_V \equiv M_X = M_Y = 5\sqrt{2} g_5 \sigma \ , \tag{1.37}$$

whereas the SM particles remain massless. The components Σ_8 and Σ_3 of Σ both acquire the mass

$$M_\Sigma \equiv M_8 = M_3 = \tfrac{5}{2} m \ , \tag{1.38}$$

where $m = \mathcal{O}\left(M_{\text{GUT}}\right)$, while $\Sigma_{(3,2)}$ and $\Sigma_{(3^*,2)}$ form vector multiplets of mass M_V together with the gauge multiplets. The mass of the singlet component Σ_{24} is $\frac{1}{2} m$. As discussed in Appendix B, m can be constrained by the RGEs.

Next we break G_{SM} to $\mathsf{SU(3)_C} \times \mathsf{U(1)}_{\text{em}}$ by a five-dimensional Higgs representation $H(\mathsf{5})$. The fermions become massive via the Yukawa couplings

$$\mathsf{5}^*_a\, \mathcal{C}\, \mathsf{10}^{ab}\, H^\dagger_b + \tfrac{1}{4}\, \epsilon_{abcde}\, \mathsf{10}^{ab}\, \mathcal{C}\, \mathsf{10}^{cd}\, H^e + \text{h.c.} \ , \tag{1.39}$$

which predicts $Y_d = Y_e$. $H(\mathsf{5})$ contains the SM Higgs doublet H_f, which acquires the VEV, as well as a color triplet H_{C} which gets a mass

$$M_{H_{\text{C}}} = 5\lambda\sigma \ , \tag{1.40}$$

by a mixing term $\lambda H^\dagger \left(\Sigma + 3\sigma\right) H$. But this means that the mass parameters of H have to be fine-tuned $\mathcal{O}\left(\frac{v}{\sigma}\right) \sim 10^{-13}$ in all orders, which is the doublet-triplet-splitting problem [42].

Features of the Model

As the minimal GUT model, the Georgi-Glashow model contains the standard model group as a maximal subgroup. Grouping every fermion generation into the reducible and anomaly free $\mathsf{5}^* \oplus \mathsf{10}$, the structure becomes much simpler without adding new particles but is still not one irreducible representation. Electric charge is quantized because the electric charge operator is an $\mathsf{SU(5)}$ generator. Therefore $\operatorname{tr} L_Q = 0$, and the sum of the charges of the particles in each multiplet must be zero which gives $q_d = \frac{1}{3}\, q_e = -\frac{1}{3}$. Baryon and lepton number violating interactions naturally appear, whereby the large unification mass can give an explanation why the decay of a proton could not be measured so far.

On the other hand, the model is only valid below the Planck scale because gravity is still not taken into account. Furthermore, there are other aspects which need the model to be extended. One issue is the doublet-triplet splitting problem; next the number of free parameters increases. There are altogether 23 free parameters: one gauge coupling, one θ and nine Higgs parameters, six fermion masses, and six mixing angles and $\mathcal{CP}$ violating phases. Two additional phases originate from the phase matrix P in the $\mathsf{10} - \mathsf{10} - H(\mathsf{5})$ coupling (Eqn. (B.2)).

Moreover, the neutrinos remain massless unless we add a singlet, as in the case of the standard model. This problem is solved by choosing a larger GUT group, which implies massive neutrinos. As already discussed, the GUT scale is of the right order for neutrino masses. Larger groups can also offer a solution for the question of why there are three families.

As theories being valid in the range $M_{\text{GUT}} - M_{\text{Pl}}$, supersymmetric GUTs provide a beautiful framework for theories beyond the standard model. It is remarkable that the neutrino data fit well in this concept, hence can give – as well as the search for proton decay – constraints on different models. We will discuss several models in detail in the next chapter.

1.5 Other Approaches

We conclude this chapter with a brief discussion of two further approaches beyond the standard model we will use later on, namely extra dimension and family symmetries.

Extra Dimensions

Inspired by the close ties between Minkowski's four-dimensional spacetime and Maxwell's unification of electricity and magnetism, Nordström tried already in 1914 [43] (even before Einstein's theory) and (independently) Kaluza in 1919 [44] to unify gravity and electromagnetism in a theory of five dimensions (for reviews see Refs. [45]). Kaluza used Einstein's tensor potential and could demonstrate that general relativity, when interpreted as a five-dimensional theory in vacuum, contained four-dimensional general relativity in the presence of an electromagnetic field, together with Maxwell's laws of electromagnetism. In 1926, Klein showed that Kaluza's cylinder condition – that physics takes place on a four-dimensional hypersurface – would arise naturally, if the fifth coordinate was compactified and had a circular topology [46]. From the gauge-invariant point of view, a $\mathsf{U(1)}$ gauge-invariance with respect to coordinate transformations along the fifth dimension is added, giving rise to the electromagnetic field as a vector gauge field in four dimensions. The theory can then be Fourier-expanded with all Fourier modes ("KK modes") above the ground state being unobservable and be effectively independent of the fifth dimension.

Theories with extra dimensions have been of great interest, in particular supergravity in eleven dimensions and ten-dimensional string theories. Recently, new attention was drawn in the context of orbifold GUTs. Here, the GUT gauge symmetry is realized in more than four space-time dimensions and broken to the standard model by compactification on an orbifold, utilizing boundary conditions that violate the GUT symmetry [47].

Consider as an example the five-dimensional factorized spacetime $M^4 \times S^1/\left(\mathbb{Z}_2 \times \mathbb{Z}_2'\right)$, where the circle S^1 has the radius $R \sim 1/M_{\mathrm{GUT}}$. The $\mathbb{Z}_2$ transformation imposes on the fifth coordinate $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the equivalence relation $y \sim -y$, so $S^1/\mathbb{Z}_2$ is not smooth but has a singularity at $y = 0$; such a space is known as an orbifold. The second transformation $\mathbb{Z}_2'$ imposes the relation $y' \sim -y'$ with $y' \equiv y + \frac{\pi}{2}R$, thus the physical space reduces to the interval $y \in \left[0, \frac{\pi}{2}\right]$ with two fixed points at $y = 0$ and $y = \frac{\pi}{2}R$.

The action of the equivalences P, P' on the fields $\phi\left(x, y\right)$ are given by $P^{(\prime)} : \phi\left(x, y^{(\prime)}\right) \sim P_\phi^{(\prime)} \phi\left(x, -y^{(\prime)}\right)$. The fields of ϕ can be classified by their (P, P') eigenvalues $(\pm, \pm)$ and have KK expansions, which involve $\cos\frac{ky}{R}$ and $\sin\frac{ky}{R}$. From the 4D perspective, the KK modes acquire a mass $\frac{k}{R}$, so only the ϕ_{++} possess a massless zero mode. Moreover, ϕ_{-+} and ϕ_{--} vanish at $y = 0$ while ϕ_{+-} and again ϕ_{--} vanish at $y = \frac{\pi}{2}R$. The action of the identifications of P, P' can utilize all symmetries of the bulk theory such as gauge transformation, discrete parity transformations and R-symmetry transformations. The fixed points respect only the $P^{(\prime)}$ invariant subgroups.

In Chapter 3, we study a six-dimensional N=1 supersymmetric $\mathsf{SO(10)}$ model compactified on a torus with three $\mathbb{Z}_2$ parities. The four fixed points respect 4d N=1 SUSY as well as $\mathsf{SO(10)}$ or one of its three subgroups, hence at the end, only the standard model symmetry remains. In this model, the doublet-triplet splitting problem is solved because the Higgs color triplets do not have massless zero modes.

Family Symmetries

The standard model contains three generations, where the mass pattern is hierarchical. This fact can be explained by family (flavor) symmetries, where some fields X couple differently to the distinct generations. The symmetry is then broken by the expectation values of X. The symmetry breaking is assumed to be transmitted to quarks and leptons by means of particles with mass M so that the low energy Yukawa couplings are constructed out of powers of $\epsilon \equiv \langle X\rangle / M$ with a texture dictated by the family symmetry [48]. Such models have been used to explain the large mixing angles in the neutrino sector [49]. We will consider one of those models in the context of proton decay induced at the Planck scale in Chapter 2.4 and in comparison to the six-dimensional model in Chapter 3.2.

In the following, different supersymmetric GUT models are studied in view of proton decay. First, in Chapter 2, we consider four-dimensional models, where proton decay is dominated by dimension five operators. Then we turn to a six-dimensional $\mathsf{SO(10)}$ model, where dimension-five operators are forbidden, so proton decay is mediated by the new gauge bosons (Chapter 3).

Chapter 2

Proton Decay in Conventional Supersymmetric GUT Models

Linking supersymmetry and GUTs, it was quickly realized that supersymmetry can solve the naturalness problem of gauge hierarchies [50]. Dimopoulos and Georgi [10] and Sakai [11] formulated the supersymmetric Georgi-Glashow model being broken at a TeV scale. With the precision data from LEP it became clear that the gauge couplings do not unify in the standard model but in its supersymmetric extension (Fig. 1.1). The higher value of the unification scale and the smaller gauge coupling at M_{GUT} reduce the decay width such that the lifetime via dimension-six operator is $\mathcal{O}\left(10^{36}\,\text{years}\right)$.

On the other hand, supersymmetric theories involve dimension-five operators, which lead to much faster proton decay, as realized by Sakai and Yanagida [51] and Weinberg [52].

In this chapter, we analyze four-dimensional models where proton decay is dominated by dimension-five operators. We start with the supersymmetric extension of the Georgi-Glashow model which will then be extended to a consistent model, where $\mathsf{SU}(5)$ relations among Yukawa couplings hold. After that we turn to $\mathsf{SO}(10)$ models. Finally, we consider proton decay induced at the Planck scale.

2.1 Analysis of Dimension-five Operators

Sakai and Yanagida as well as Weinberg required dimension-five operators to be forbidden. More careful analyses showed that they were not in conflict with the experimental bounds [53, 54]. In this section, we review the evolution of the proton decay rate. We focus on the leading process $p \to K^+\bar{\nu}$ as it is usually done in the analyses, though the discussion is valid for the other decay channels as well.

Let us start with the Yukawa couplings in minimal $\mathsf{SU}(5)$,

$$W_Y = \tfrac{1}{4}\, Y_1^{ij}\, \mathsf{10}_i\; \mathsf{10}_j\; H(\mathsf{5}) + \sqrt{2}\, Y_2^{ij}\, \mathsf{10}_i\; \mathsf{5}_j^*\; \overline{H}(\mathsf{5}^*)\ , \tag{2.1}$$

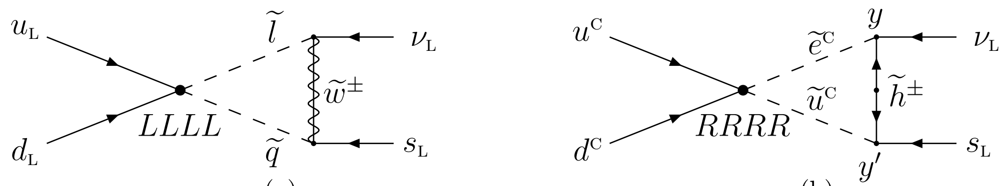


Figure 2.1: Proton Decay via dimension five operators: They result from exchange of the leptoquarks followed by gaugino or higgsino dressing.

which involves the couplings

$$\tfrac{1}{2}Y_{qq}^{ij}\, Q_i\, Q_j\, H_C + Y_{ql}^{ij}\, Q_i\, L_j\, \overline{H}_C + Y_{ue}^{ij}\, u_i^C\, e_j^C\, H_C + Y_{ud}^{ij}\, u_i^C\, d_j^C\, \overline{H}_C\ . \tag{2.2}$$

Analogously to the MSSM, we introduced a second Higgs field, $\mathbf{5}^*$. Integrating out the leptoquarks, two dimension five operators remain which lead to proton decay (Fig. 2.1),

$$W_5 = \frac{1}{M_{H_C}}\left[\frac{1}{2}\, Y_{qq}^{ij}\, Y_{ql}^{km}\ (Q_i\, Q_j)\, (Q_k\, L_m) + Y_{ue}^{ij}\, Y_{ud}^{km}\ \left(u_i^C\, e_j^C\right)(u_k^C\, d_m^C)\right]\ , \tag{2.3}$$

called the $LLLL$ and $RRRR$ operator, respectively. The scalars are transformed to their fermionic partners by exchange of a gauge or Higgs fermion. Neglecting external momenta, the triangle diagram factor reads, up to a coefficient κ depending on the exchange particle,

$$\int \frac{d^4k}{i(2\pi)^4}\, \frac{1}{m_1^2 - k^2}\, \frac{1}{m_2^2 - k^2}\, \frac{1}{M - \not{k}} = \frac{1}{(4\pi)^2} f(M; m_1, m_2)\ , \tag{2.4}$$

with

$$f(M; m_1, m_2) = \frac{M}{m_1^2 - m_2^2}\left(\frac{m_1^2}{m_1^2 - M^2}\ \ln\frac{m_1^2}{M^2} - \frac{m_2^2}{m_2^2 - M^2}\ \ln\frac{m_2^2}{M^2}\right)\ , \tag{2.5}$$

where M and m_j denote the gaugino and sfermion masses, respectively.

As a result of Bose statistics for superfields, the total anti-symmetry in the colour index requires that these operators are flavor non-diagonal [53]. The dominant decay mode is therefore $p \to K\bar{\nu}$. Since the dressing with gluinos and neutralinos is flavor diagonal, the chargino exchange diagrams are dominant [55, 56]. The wino exchange is related to the $LLLL$ operator and the charged higgsino exchange to the $RRRR$ operator, so that the coefficients of the triangle diagram factor are

$$\kappa_L = 2g^2\ , \quad \kappa_R = y\, y'\ . \tag{2.6}$$

Here y and y' denote the corresponding Yukawa couplings (cf. Fig. 2.1(b)) and g is the gauge coupling.

The Wilson coefficients $C_{5L} = Y_{qq}Y_{ql}$ and $C_{5R} = Y_{ue}Y_{ud}$ are evaluated at the GUT scale. Then they have to be evolved down to the SUSY breaking scale, leading to a short-distance renormalization factor A_s. In practice, the Wilson coefficients are evaluated at

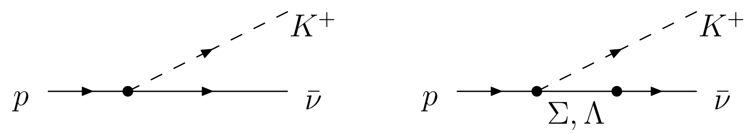


Figure 2.2: *Feynman diagrams on the hadronic level contributing to $p \to K^+\bar{\nu}$.*

$M_{\rm SUSY}$ using the renormalization group equations (B.10) and (B.11). Now the sparticles are integrated out, as described above, and the operators give rise to the effective four-fermion operators of dimension 6. The renormalization group procedure goes on to the scale of the proton mass, $m_p \sim 1\,\mathrm{GeV}$, leading to a second, long-distance renormalization factor A_l,

$$A_l = \left[\frac{\alpha_3(\mu_{\rm had})}{\alpha_3(M_Z)}\right]^{\frac{6}{33-2n_f}} \to \left[\frac{\alpha_3(\mu_{\rm had})}{\alpha_3(m_c)}\right]^{\frac{2}{9}} \left[\frac{\alpha_3(m_c)}{\alpha_3(m_b)}\right]^{\frac{6}{25}} \left[\frac{\alpha_3(m_b)}{\alpha_3(M_Z)}\right]^{\frac{6}{23}} . \tag{2.7}$$

At 1 GeV, the link to the hadronic level is made using the chiral Lagrangean method (see Appendix C) [57, 58].[1] Thus the decay width can be written as

$$\Gamma = \sum \left| \mathcal{K}_{\rm had}\, A_l\, \frac{\kappa}{(4\pi)^2} f(M; m_1, m_2)\, A_s\, \frac{1}{M_{H_{\rm c}}}\, C_5 \right|^2 , \tag{2.8}$$

where we sum over all neutrino flavors as well as over all possible diagrams, as will discussed below.

The hadronic factors $\mathcal{K}_{\rm had}$ are calculated via lattice simulations nowadays and agree well with the predictions of chiral Lagrangean technique. According to the different diagrams (see Figures C.1, C.2), the decay width then reads

$$\begin{aligned}\Gamma(p \to K^+\bar{\nu}) = \frac{(m_p^2 - m_K^2)^2}{32\pi m_p^3 f_\pi^2} \sum_\nu \Bigg| & C_5^{usd\nu}\, \frac{2m_p}{3m_B} D + C_5^{uds\nu} \left(1 + \frac{m_p}{3m_B}(3F + D)\right) \\ & + C_5^{dsu\nu} \left(1 - \frac{m_p}{3m_B}(3F - D)\right) \Bigg|^2 .\end{aligned} \tag{2.9}$$

Here, m_p and m_K denote the masses of the proton and kaon, respectively, and f_π the pion decay constant. m_B is an average baryon mass according to contributions from diagrams with virtual Σ and Λ (Fig. 2.2) [57]. D and F are the symmetric and antisymmetric $\mathsf{SU(3)}$ reduced matrix elements for the axial-vector current.

According to the two Wilson coefficients, the coefficients C_5 split into two parts,

$$C_5 = \beta\, C_{\rm LL} + \alpha\, C_{\rm RL} , \tag{2.10}$$

with

$$C_{\rm LL/RL} = \frac{1}{M_{H_{\rm c}}}\, C_{5{\rm L}/5{\rm R}}\, A_s\, A_l\, \frac{\kappa_{\rm L/R}}{(4\pi)^2} f(M; m_1, m_2) \tag{2.11}$$

[1] Other approaches such as the nonrelativistic quark model, the bag model and QCD sum rules have also been used to calculate the proton decay rates; the results coincide [59].

proton mass	m_p	938.3 MeV	hadron matrix elements	α, β	0.003 GeV^3
kaon mass	m_K	493.7 MeV	renormalization factor	A_l	1.43
baryon mass	m_B	1150 MeV		D	0.81
π decay constant	f_π	131 MeV		F	0.44

Table 2.1: *Parameter values for the analysis of the dimension-five operators.*

and the hadron matrix elements α and β [60],

$$\begin{aligned} \alpha\, P_{\mathrm{L}}\, u_p &= \epsilon_{\alpha\beta\gamma} \left\langle 0 \middle| \left(d_{\mathrm{R}}^{\alpha}\, u_{\mathrm{R}}^{\beta}\right) u_{\mathrm{L}}^{\gamma} \middle| p \right\rangle \ , \\ \beta\, P_{\mathrm{L}}\, u_p &= \epsilon_{\alpha\beta\gamma} \left\langle 0 \middle| \left(d_{\mathrm{L}}^{\alpha}\, u_{\mathrm{L}}^{\beta}\right) u_{\mathrm{L}}^{\gamma} \middle| p \right\rangle \ , \end{aligned} \tag{2.12}$$

from which all other elements can be calculated; u_p denotes the proton spinor.

The renormalization group effects in SUSY GUTs have first been discussed in Ref. [54]. At that time, not only the high top mass was unknown ($m_t = 20\,\mathrm{GeV}$ was assumed), but since there were no data at M_Z, the values at 1 GeV were taken to calculate the decay rate. Hence the renormalization factors A_S and A_L were defined, which include the running factor of the Yukawa couplings from low to high scale. In this work, we use the Yukawa couplings at M_Z and M_{SUSY} and evaluate their values at M_{GUT}. These are taken as input parameters for the calculation, so our factors A_s and A_l differ from A_S and A_L in Refs. [54, 56]. For the long-distance part, this discrepancy was stressed in Ref. [61].

Parameter Values

Apart from the Wilson coefficients and the leptoquark mass, the parameter values are independent of the GUT model. First there are the parameters that appear in the chiral Lagrangean method and are determined by QCD. The others are the masses and mixings of the supersymmetric particles.

First of all, the masses of the baryons and mesons as well as the pion decay constant are well known. As discussed in Appendix C, the other factors, α, β, D and F, are less but fairly known. Their values are summarized in Table 2.1.

The so far unsuccessful search for supersymmetric particles bounds their masses to be heavier than $\mathcal{O}\,(100\,\mathrm{GeV})$, on the other hand, they are expected not to be much heavier than $\mathcal{O}\,(1\,\mathrm{TeV})$. Looking at the dressing diagram we notice that when taking the sfermions to be degenerate at a TeV, the triangle diagram factor (2.5) is given by

$$f(M;m) = \frac{M}{(M^2-m^2)^2}\left(m^2 - M^2 - M^2\,\ln\frac{m^2}{M^2}\right) \xrightarrow{M \ll m} \frac{M}{m^2}\ . \tag{2.13}$$

To get a small decay width, one therefore assumes the sfermions to have masses of 1 TeV. An exception is often made for top squarks. Since the off-diagonal entries of the mass matrix are proportional to m_t, the mixing is expected to be large, with at least one eigenvalue much below 1 TeV. In analyses, one typically uses 400 GeV, 800 GeV, or 1 TeV for $m_{\tilde{t}}$. For the other sfermions, the mixings are neglected. The proton decay rate is further suppressed by light gauginos and higgsinos. Note that the experimental limit for charginos is $m_{\tilde{\chi}^{\pm}} > 67.7\,\mathrm{GeV}$ [17].

Since proton decay is dangerously large, also the decoupling scenario [62] has been studied, where the scalars of the first and second generation can be as heavy as $10\,\mathrm{TeV}$ [13]. Such an adjustment has been motivated by the supersymmetric flavor problem. The numerous parameters of the soft SUSY-breaking sector are a priori arbitrary and generically will give rise to phenomenologically dangerous flavor-changing neutral current effects. One proposal for avoiding these difficulties is to decouple the first two generations of superpartners. In this scenario, proton decay via dimension-five operators is clearly dominated by the third generation.

2.2 Minimal and Consistent $\mathsf{SU(5)}$

With the technique derived in the last section, we will now calculate the proton decay rate in supersymmetric $\mathsf{SU(5)}$.

2.2.1 Supersymmetric $\mathsf{SU(5)}$ GUTs

The superpotential of minimal $\mathsf{SU(5)}$ is given by

$$\begin{aligned} W &= \tfrac{1}{2} m \,\mathrm{tr}\,\Sigma^2 + \tfrac{1}{3} a \,\mathrm{tr}\,\Sigma^3 + \lambda\, \overline{H}(\mathsf{5}^*)\,(\Sigma + 3\sigma)\, H(\mathsf{5}) \\ &\quad + \tfrac{1}{4}\, Y_1^{ij}\, \mathsf{10}_i\; \mathsf{10}_j\; H(\mathsf{5}) + \sqrt{2}\, Y_2^{ij}\, \mathsf{10}_i\; \mathsf{5}_j^*\; \overline{H}(\mathsf{5}^*)\ . \end{aligned} \tag{2.14}$$

The term $\mathsf{10}\;\mathsf{5}^*\;\mathsf{5}^*$ which contains the dimension-four operators of Eqn. (1.23) is forbidden by R-parity.

Expressed in terms of SM superfields, the Yukawa interactions are

$$\begin{aligned} W_Y &= Y_u^{ij}\, Q_i\, u_j^{\mathrm{C}}\, H_f + Y_d^{ij}\, Q_i\, d_j^{\mathrm{C}}\, \overline{H}_f + Y_e^{ij}\, e_i^{\mathrm{C}}\, L_j\, \overline{H}_f \\ &\quad + \tfrac{1}{2} Y_{qq}^{ij}\, Q_i\, Q_j\, H_{\mathrm{C}} + Y_{ql}^{ij}\, Q_i\, L_j\, \overline{H}_{\mathrm{C}} + Y_{ue}^{ij}\, u_i^{\mathrm{C}}\, e_j^{\mathrm{C}}\, H_{\mathrm{C}} + Y_{ud}^{ij}\, u_i^{\mathrm{C}}\, d_j^{\mathrm{C}}\, \overline{H}_{\mathrm{C}}\ , \end{aligned} \tag{2.15}$$

where

$$Y_u = Y_{qq} = Y_{ue} = Y_1\ , \tag{2.16}$$

$$Y_d = Y_e = Y_{ql} = Y_{ud} = Y_2\ . \tag{2.17}$$

In particular the Yukawa couplings of down quarks and charged leptons are unified. While $m_b = m_\tau$ can be fulfilled at the GUT scale, it fails for the first and second generation. This problem can be solved by adding higher-dimensional operators due to physics at the Planck scale so that [63, 64]

$$W_\Sigma = \tfrac{1}{2} m \,\mathrm{tr}\,\Sigma^2 + \tfrac{1}{3} a \,\mathrm{tr}\,\Sigma^3 + b\frac{(\mathrm{tr}\,\Sigma^2)^2}{M_{\mathrm{Pl}}} + c\frac{\mathrm{tr}\,\Sigma^4}{M_{\mathrm{Pl}}}\ . \tag{2.18}$$

Now the masses of Σ_3 and Σ_8 are no longer identical, which will affect the constraints on the leptoquark mass. Including possible couplings up to order $1/M_{\rm Pl}$, the Yukawa interactions read

$$\begin{aligned} W_Y = \frac{1}{4}\,\epsilon_{abcde}\left(Y_1^{ij}\,\mathsf{10}_i^{ab}\,\mathsf{10}_j^{cd}\,H^e + f_1^{ij}\,\mathsf{10}_i^{ab}\,\mathsf{10}_j^{cd}\,\frac{\Sigma_f^e}{M_{\rm Pl}}\,H^f + f_2^{ij}\,\mathsf{10}_i^{ab}\,\mathsf{10}_j^{cf}\,H^d\,\frac{\Sigma_f^e}{M_{\rm Pl}}\right) \\ + \sqrt{2}\left(Y_2^{ij}\,\overline{H}_a\,\mathsf{10}_i^{ab}\,\mathsf{5}_{jb}^* + h_1^{ij}\,\overline{H}_a\,\frac{\Sigma_b^a}{M_{\rm Pl}}\,\mathsf{10}_i^{bc}\,\mathsf{5}_{jc}^* + h_2^{ij}\,\overline{H}_a\,\mathsf{10}_i^{ab}\,\frac{\Sigma_b^c}{M_{\rm Pl}}\,\mathsf{5}_{jc}^*\right)\ . \end{aligned} \tag{2.19}$$

Then the Yukawa couplings are given by

$$Y_u = Y_1 + 3\frac{\sigma}{M_{\rm Pl}}f_1^S + \frac{1}{4}\frac{\sigma}{M_{\rm Pl}}\left(3f_2^S + 5f_2^A\right)\ , \tag{2.20a}$$

$$Y_d = Y_2 - 3\frac{\sigma}{M_{\rm Pl}}h_1 + 2\frac{\sigma}{M_{\rm Pl}}h_2\ , \tag{2.20b}$$

$$Y_e = Y_2 - 3\frac{\sigma}{M_{\rm Pl}}h_1 - 3\frac{\sigma}{M_{\rm Pl}}h_2\ . \tag{2.20c}$$

Here $\sigma/M_{\rm Pl} \sim \mathcal{O}\left(10^{-2}\right)$, and S and A denote the symmetric and antisymmetric parts of the matrices, respectively. Thus the three Yukawa matrices, which are related to masses and mixing angles at M_Z by the RGEs, are determined by six matrices.

From Eqs. (2.20) one reads off,

$$Y_d - Y_e = 5\frac{\sigma}{M_{\rm Pl}}h_2\ . \tag{2.21}$$

Hence the failure of Yukawa unification is naturally accounted for by the presence of h_2. Note that we do not need to introduce any additional field at $M_{\rm GUT}$ to obtain this relation; it just arises from corrections $\mathcal{O}\left(\sigma/M_{\rm Pl}\right)$. Therefore this model is a *consistent* supersymmetric SU(5) GUT model.

In the minimal model, $Y_{qq} = Y_{ue} = Y_u$; furthermore, one usually chooses $Y_{ql} = Y_{ud} = Y_d$. Note, however, that the choices $Y_{ql} = Y_{ud} = Y_e$ or $Y_{ql} = Y_d$, $Y_{ud} = Y_e$ would be equally justified. As we shall see, this ambiguity strongly affects the proton decay rate.

2.2.2 Minimal Model

As discussed in Appendix B, two physical bases are used to calculate the decay amplitudes, with either a diagonal up quark matrix [56] or a diagonal down quark matrix [12]. Assuming

$$Y_{qq} = Y_{ue} = Y_u\ , \qquad Y_{ql} = Y_{ud} = Y_d\ , \tag{2.22}$$

the Wilson coefficients at the GUT scale can be written as

$$\begin{aligned} C_{5L}^u &= Y_{qq}^u Y_{ql}^u = (\mathcal{D}_u\,P)(V_{\rm CKM}\,\mathcal{D}_d)\ , \\ C_{5R}^u &= Y_{ue}^u Y_{ud}^u = (\mathcal{D}_u\,V_{\rm CKM}^*)(P^*\,V_{\rm CKM}\mathcal{D}_d) \end{aligned} \tag{2.23}$$

in the former and

$$\begin{aligned} C^d_{5L} &= Y^d_{qq} Y^d_{ql} = (V^T_{\text{CKM}} P\, \mathcal{D}_u V_{\text{CKM}})(\mathcal{D}_d)\ , \\ C^d_{5R} &= Y^d_{ue} Y^d_{ud} = (V^T_{\text{CKM}} \mathcal{D}_u)(P^* V^*_{\text{CKM}} \mathcal{D}_d) \end{aligned} \tag{2.24}$$

in the latter case. Here $\mathcal{D}_u$ and $\mathcal{D}_d$ are the diagonalized Yukawa coupling matrices evaluated from Y_u and Y_d, respectively, V_{CKM} is the CKM matrix and P is the additional phase matrix as given in Eqn. (B.2).

As already mentioned, it is possible to constrain the leptoquark mass using the renormalization group equations (see Appendix B). These constraints depend strongly on the Higgs representations. We will choose the most conservative value $M_{H_{\text{c}}} = 2 \times 10^{16}$ GeV $=$ M_{GUT} in order to study, if proton decay via dimension-five operators is already ruled out by the experimental limit.

Now $\tan\beta$ remains as a free parameter. Since the decay width is proportional to $\tan\beta$, as discussed below, low values are preferred to obtain a small decay rate. On the other hand, the top Yukawa coupling becomes non-perturbative for low $\tan\beta$ as $y_t \simeq \frac{1}{\sin\beta}$. We will therefore vary $\tan\beta$ starting with $\tan\beta \simeq 2.5$.

$\boldsymbol{LLLL}$ versus $\boldsymbol{RRRR}$ contribution

The $RRRR$ contribution was neglected for a long time. For large $\tan\beta$, however, it becomes important because it is proportional to $(\tan\beta + \frac{1}{\tan\beta})^2$, whereas the $LLLL$ contribution is proportional to $\frac{1}{\sin 2\beta} = \frac{1}{2}\,(\tan\beta + \frac{1}{\tan\beta})$. Additionally, due to the large top quark Yukawa coupling, the triangle diagram factor becomes large for third generation sparticles so that the $RRRR$ contribution dominates the decay channel $p \to K^+\bar{\nu}_\tau$ [65]. As long as the top mass was believed to be less than 100 GeV, the decay width was almost given by the $LLLL$ contribution only and could be suppressed sufficiently by adjusting the phase matrix P, given in Eqn. (B.2).

In Ref. [12], the $RRRR$ contribution was studied in the minimal $\mathsf{SU(5)}$ model. It was found that the total width is even affected for low $\tan\beta$ because the phase dependence of $p \to K^+\bar{\nu}_\mu$ and $p \to K^+\bar{\nu}_\tau$ now differs, so the two channels cannot be reduced simultaneously.

Flavor Dependence of the Decay Rate

Fig. 2.3 shows the results of the following three cases: (i) all sfermions have masses of 1 TeV; (ii) $m_{\tilde{t}}$ is changed to 400 GeV; (iii) decoupling scenario, where the scalars of the first and second generation have masses of 10 TeV. The values for the phases in P (Eqn. (B.2)) are chosen such that the amplitude is minimal. The dash-dotted line represents the experimental limit $\tau = 6.7 \times 10^{32}$ years as given by the SuperKamiokande experiment [17, 66], the dotted line is the newer limit $\tau = 1.9 \times 10^{33}$ years [67].[2] The

[2] Recently, the experimental limit was raised to $\tau = 2.2 \times 10^{33}$ years [68]. This slight improvement, however, does not change our results [14].

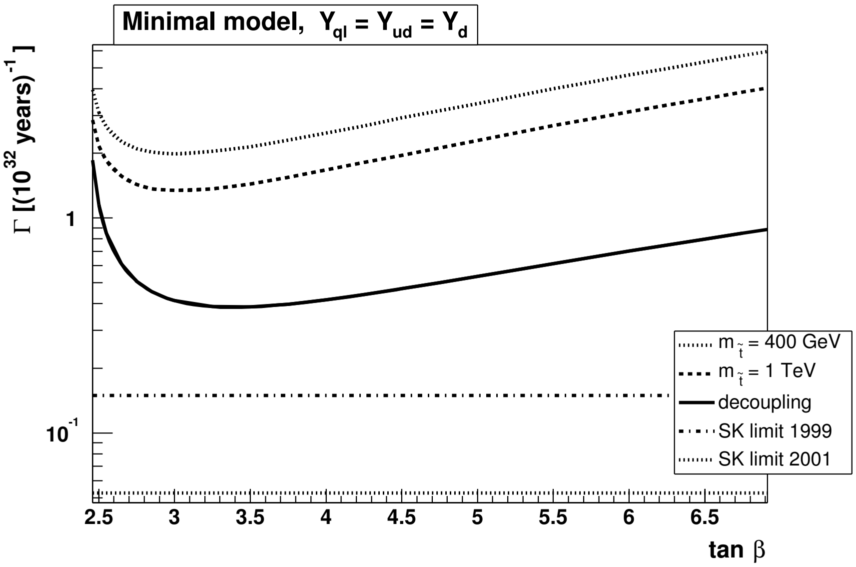


Figure 2.3: Decay rate $\Gamma(p \to K^+\bar{\nu})$ as function of $\tan\beta$ in the minimal model with $Y_{ql} = Y_{ud} = Y_d$. The experimental limits are given by SuperKamiokande experiment [66, 67].

amplitude is always above the experimental limit, which led to the claim that minimal SU(5) is excluded [12, 13].

But as already discussed, there is no compelling reason for the assumption $Y_{ql} = Y_{ud} = Y_d$ (2.22)! In order to illustrate the strong dependence of the decay rate on flavor mixing and therefore on Yukawa unification, let us study the case

$$Y_{qq} = Y_{ue} = Y_u \ , \qquad Y_{ql} = Y_{ud} = Y_e \ . \tag{2.25}$$

The Wilson coefficients now read

$$\begin{aligned} C^u_{5L} &= (\mathcal{D}_u\, P)(\mathcal{M}\, \mathcal{D}_e) \ , \\ C^u_{5R} &= (\mathcal{D}_u\, \mathcal{M}^*)(P^*\, \mathcal{M}\mathcal{D}_e) \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} C^d_{5L} &= (\mathcal{M}^T P\, \mathcal{D}_u \mathcal{M})(\mathcal{D}_e) \ , \\ C^d_{5R} &= (P^* \mathcal{M}^* \mathcal{D}_e)(\mathcal{M}^T \mathcal{D}_u) \ , \end{aligned} \tag{2.27}$$

where $\mathcal{M} = U_u^\dagger\, U_e$ replaces the CKM matrix V_{CKM}. Note that the mixing matrix in Y_u or Y_d (cf. Eqs. (B.4) and (B.5)) is still given by V_{CKM}. Since $Y_d \neq Y_e$, the masses and mixing of quarks and leptons are different and $\mathcal{M}$ is undetermined.

We first ignore mixing, i. e. $\mathcal{M} = \mathbb{1}$, and calculate the decay rate – P is still chosen such that the values are minimal. The results are shown in Fig. 2.4. Without mixing, only scalars of the first and second generation take part so that the decay rate can be reduced significantly in the decoupling scenario where the triangle diagram factor (2.5) changes by almost two orders of magnitude.

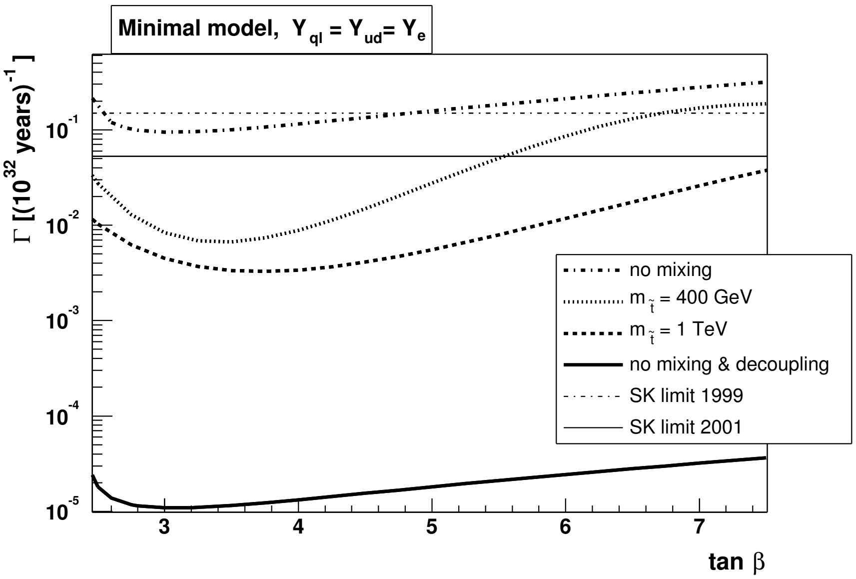


Figure 2.4: Decay rate $\Gamma(p \to K^+\bar{\nu})$ as a function of $\tan\beta$ with $Y_{ql} = Y_{ud} = Y_e$. The mixing matrix $\mathcal{M}$ is taken arbitrary or $\mathcal{M} = \mathbb{1}$.

Now we take $\mathcal{M}$ totally arbitrarily and minimize the decay rate. As can be seen in Fig. 2.4, it is possible to push the amplitude below the experimental limit even for smaller sfermion masses. In the case $m_{\tilde{t}} = 400\,\mathrm{GeV}$, this is only possible for small values of $\tan\beta$.

The fact that a sufficiently low decay rate can be found illustrates the dependence on flavor mixing and therefore the uncertainty due to the failure of Yukawa unification. Minimal supersymmtric $\mathsf{SU}(5)$ can only be excluded by the mismatch between the Yukawa couplings of down quarks and charged leptons, analogously to the exclusion of non-supersymmetric $\mathsf{SU}(5)$ by the failure of gauge unification.

2.2.3 Consistent Model

The coefficients of the operators can be derived from the superpotential (2.19),

$$\begin{aligned} Y_{qq} &= Y_1 - 2\frac{\sigma}{M_{\mathrm{Pl}}} f_1^S - \frac{1}{2}\frac{\sigma}{M_{\mathrm{Pl}}} f_2^S \ , \\ Y_{ue} &= Y_1 - 2\frac{\sigma}{M_{\mathrm{Pl}}} f_1^S - \frac{1}{2}\frac{\sigma}{M_{\mathrm{Pl}}} \left(f_2^S + 5 f_2^A\right) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} Y_{ql} &= Y_2 + 2\frac{\sigma}{M_{\mathrm{Pl}}} h_1 - 3\frac{\sigma}{M_{\mathrm{Pl}}} h_2 \ , \\ Y_{ud} &= Y_2 + 2\frac{\sigma}{M_{\mathrm{Pl}}} h_1 + 2\frac{\sigma}{M_{\mathrm{Pl}}} h_2 \ . \end{aligned} \tag{2.29}$$

Note that $Y_{ql} - Y_{ud} = Y_e - Y_d$, which means that Y_{ql} and Y_{ud} cannot be zero at the same time.

It is instructive to express these Yukawa matrices in terms of the quark and charged lepton Yukawa couplings and the additional matrices f and h (cf. relations Eqs. (B.7)),

$$\begin{aligned} Y_{qq} = Y_{qq}^S = Y_{ue}^S = Y_u^S - 5\frac{\sigma}{M_{\text{Pl}}}\left(f_1^S + \frac{1}{4}f_2^S\right) \ , \\ Y_{ue}^A = Y_u^A - \frac{5}{2}\frac{\sigma}{M_{\text{Pl}}} f_2^A \ , \end{aligned} \tag{2.30}$$

$$\begin{aligned} Y_{ql} &= Y_e + 5\frac{\sigma}{M_{\text{Pl}}} h_1 \ , \\ Y_{ud} &= Y_d + 5\frac{\sigma}{M_{\text{Pl}}} h_1 \ . \end{aligned} \tag{2.31}$$

To avoid proton decay via dimension-five operators, both $C_{5L} = Y_{qq}Y_{ql}$ and $C_{5R} = Y_{ue}Y_{ud}$ must vanish. For this purpose the couplings have to fulfill the relations

$$\begin{aligned} f_1^S + \frac{1}{4}f_2^S &= \frac{M_{\text{Pl}}}{5\,\sigma} Y_u^S \ , \\ f_2^A &= \frac{2}{5}\frac{M_{\text{Pl}}}{\sigma} Y_u^A \ , \end{aligned} \tag{2.32}$$

which can easily be read off from Eqs. (2.30). This is only possible if we allow the (3,3)-component of f_1 and f_2 to be $\mathcal{O}\left(\frac{M_{\text{Pl}}}{\sigma}\right) \gg 1$. But even if we restrict ourselves to 'natural matrices', i. e. couplings up to $\mathcal{O}\,(1)$, we can considerably reduce the decay amplitudes. We will illustrate this with two simple examples where either the $RRRR$ or the $LLLL$ contribution vanishes at the GUT scale.

Let us assume that Y_{qq}, Y_{ql}, Y_{ue} and Y_{ud} are all diagonal by a suitable choice of matrices. The simplest form of Y_{qq} and Y_{ue} is then

$$Y_{qq} = Y_{ue} = \text{diag}(0, 0, y_t) \ , \tag{2.33}$$

where y_t are the top Yukawa coupling at M_{GUT}.

In the first model, we spread $Y_e - Y_d$ such that

$$\begin{aligned} Y_{ud} &= \text{diag}(0, y_s - y_\mu, y_b - y_\tau) \ , \\ Y_{ql} &= \text{diag}(y_e - y_d, 0, 0) \ . \end{aligned} \tag{2.34}$$

Clearly $C_{5R}^{ijkm} = Y_{ue}^{ij}\, Y_{ud}^{km}$ is zero whenever a particle of the first generation takes part. But according to Figs. C.1(d) and C.2(b), at least one particle of the first generation is needed, thus the $RRRR$ contribution vanishes completely. Furthermore, only the decay channel $p \to K^+\bar{\nu}_e$ remains.

After RGE evolution by means of Eqs. (B.10) and (B.11), the simple structure of Wilson coefficients changes slightly, but the $RRRR$ contribution and the decay channel $p \to K\bar{\nu}_\mu$ are still negligible whereas $p \to K\bar{\nu}_\tau$ becomes dominant. Fig. 2.5 shows the

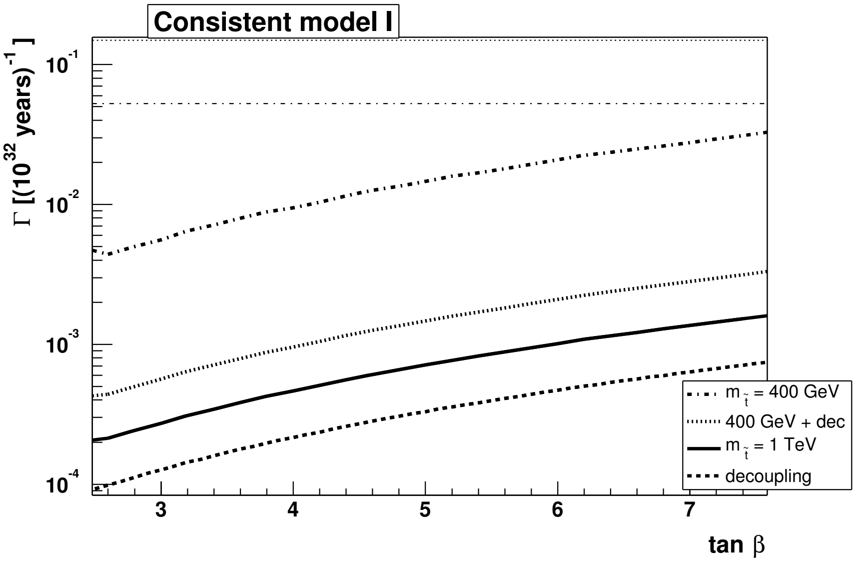


Figure 2.5: Decay rate $\Gamma(p \to K^+\bar{\nu})$ as function of $\tan\beta$ in the consistent model.

results for different sfermion masses. The decay amplitude is always well below the experimental limit, in the case $m_{\tilde{t}} = 1\,\mathrm{TeV}$ even more than two orders of magnitude.

If we choose the matrices Y_{ql} and Y_{ud} as

$$\begin{aligned} Y_{ud} &= \mathrm{diag}(y_d - y_e, y_s - y_\mu, y_b)\ , \\ Y_{ql} &= \mathrm{diag}(0, 0, y_\tau)\ , \end{aligned} \tag{2.35}$$

the $LLLL$ contribution vanishes at M_{GUT} because now $C_{5L}^{ijkm} = Y_{qq}^{ij}\,Y_{ql}^{km}$ is only different from zero for $i = j = k = m = 3$, but the decay has to be non-diagonal. Only the $RRRR$ contribution with a low absolute value remains. After renormalization, the $RRRR$ contribution is still dominated by third generation scalars so that decoupling of the first and second generation does not change the result. The $LLLL$ operator contributes only via $p \to K\bar{\nu}_\tau$.

As shown in Fig. 2.6, the proton decay rate is even smaller in this model. Furthermore, due to the smaller (3,3)-component of h_1 compared to the first model, it can easily be used for higher values of $\tan\beta$.

We have shown that the higher-dimensional operators can reduce the proton decay rate by several orders of magnitude and make it consistent with the experimental upper bound. This impressing fact leads to the question, if there is any mechanism which would naturally lead to the required relations among Yukawa couplings. We can think of two possibilities, the first of which is to start with some ad-hoc textures as a result of an unknown additional symmetry as hoped in Refs. [63, 69]. We will follow the second approach, namely to extend the analysis to another group, in order to obtain additional symmetry restrictions, and study $\mathsf{SO(10)}$ GUT models.

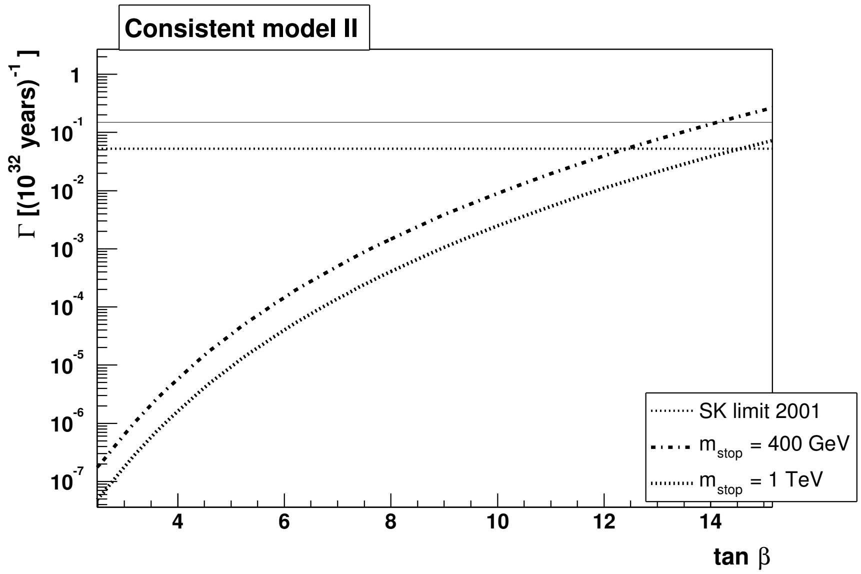


Figure 2.6: Decay rate $\Gamma(p \to K^+\bar{\nu})$ in the second consistent model.

2.3 Higher-dimensional Operators in $\mathsf{SO}(10)$

$\mathsf{SO}(10)$ is probably the most natural GUT group, since it unifies the matter fields in one representation by only requiring one additional field [70]. This field is a singlet with respect to the standard model and can be identified with the left-handed antineutrino. Thus $\mathsf{SO}(10)$ in general involves massive neutrinos. Since it is rank-5, it contains an additional $\mathsf{U}(1)$ symmetry which can be referred to as $(B-L)$. If this symmetry is broken at a scale M_{B-L}, heavy Majorana masses are generated which then explain the smallness of neutrino masses.

There are two possible breaking scenarios,[3]

$$\mathsf{SO}(10) \to \begin{cases} \mathsf{SU}(5) \times \mathsf{U}(1) \to \mathsf{G}_{\mathrm{SM}} \\ \mathsf{G}_{\mathrm{PS}} \to \mathsf{SU}(3)_{\mathsf{C}} \times \mathsf{SU}(2)_{\mathsf{L}} \times \mathsf{SU}(2)_{\mathsf{R}} \times \mathsf{U}(1) \to \mathsf{G}_{\mathrm{SM}} \end{cases} , \tag{2.36}$$

where $\mathsf{G}_{\mathrm{PS}} = \mathsf{SU}(4) \times \mathsf{SU}(2)_{\mathsf{L}} \times \mathsf{SU}(2)_{\mathsf{R}}$ is the Pati-Salam group [9]. $\mathsf{SO}(10)$ therefore contains a left-right symmetric subgroup; in particular, the left-handed antifermions transform nontrivially under $\mathsf{SU}(2)_{\mathsf{R}}$. The group $\mathsf{SU}(3)_{\mathsf{C}} \times \mathsf{SU}(2)_{\mathsf{L}} \times \mathsf{SU}(2)_{\mathsf{R}}$ can then be broken to the standard model at an intermediate scale M_{LR}. Finally, as will be seen below, $\mathsf{SO}(10)$ is anomaly free.

[3] The $\mathsf{SU}(5)$ is not necessarily the Georgi-Glashow group we discussed before. There is another possibility, flipped $\mathsf{SU}(5)$ (cf. Section 3.2 and Appendix B).

SO(10) spinor state	SU(5) dim.	SO(10) spinor state	SU(5) dim.
$\lvert 0\rangle$	1	$b_i^\dagger b_j^\dagger b_k^\dagger \lvert 0\rangle$	10
$b_i^\dagger \lvert 0\rangle$	5	$b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger \lvert 0\rangle$	5
$b_i^\dagger b_j^\dagger \lvert 0\rangle$	10	$b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger \lvert 0\rangle$	1

Table 2.2: *The states of the spinor representation of* SO(10) *and their* SU(5) *dimension [71].*

The Algebra of SO(10)

Consider a set of operators b_j $(j = 1, \dots 5)$ plus their Hermitean conjugates, $b_j^\dagger$, satisfying

$$\left\{b_i, b_j\right\} = \left\{b_i^\dagger, b_j^\dagger\right\} = 0 \; , \qquad \left\{b_i, b_j^\dagger\right\} = \delta_{ij} \; . \tag{2.37}$$

Then the operators $T^i_j \equiv b_i^\dagger b_j$ satisfy the U(5) algebra,

$$\left[T^i_j, T^k_l\right] = \delta^k_j T^i_l - \delta^i_l T^k_j \; . \tag{2.38}$$

To express the algebra of SO(10) in the U(5) basis, we define the Γ matrices [71]

$$\begin{aligned} \Gamma_{2j-1} &= -i\left(b_j - b_j^\dagger\right) \\ \Gamma_{2j} &= \quad\left(b_j + b_j^\dagger\right) \qquad (j = 1, \dots 5) \; , \end{aligned} \tag{2.39}$$

which form the Clifford algebra of rank 5 $(\mu, \nu = 1, \dots 10)$,

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\,\delta_{\mu\nu} \; . \tag{2.40}$$

With the Γ matrices we can construct the generators of SO(10), $\Sigma_{\mu\nu}$, as

$$\Sigma_{\mu\nu} = \frac{1}{2i}\left[\Gamma_\mu, \Gamma_\nu\right] \; . \tag{2.41}$$

The dimensionality of the spinor representation is $2^5 = 32$. To write it in terms of the SU(5) basis, we define an SU(5) invariant vacuum state $\lvert 0\rangle$. The SO(10) spinor states and their SU(5) dimensionality are then given by Table 2.2. This representation can be split into two 16-dimensional representations $\Psi_\pm$ by chiral projection,

$$\tfrac{1}{2}\left(1 \pm \Gamma_0\right) \; , \tag{2.42}$$

where

$$\Gamma_0 = i\,\Gamma_1\Gamma_2\cdots\Gamma_{10} = \prod_{j=1}^{5}\left[b_j, b_j^\dagger\right] = \prod_{j=1}^{5}\left(1 - 2\,b_j^\dagger b_j\right) = (-1)^{\sum_j n_j} \; ; \tag{2.43}$$

$n_j \equiv b_j^\dagger b_j$ is the number operator. It follows $[\Sigma_{\mu\nu}, \Gamma_0] = 0$. Each irreducible chiral subspace is characterized by an even or odd number of the b operators and we can write the two 16-dimensional representations as [71]

$$\mathsf{16} = |\Psi_+\rangle = |0\rangle\, \psi_0 + \frac{1}{2!} b_i^\dagger b_j^\dagger\, |0\rangle\, \psi^{ij} + \frac{1}{4!} \epsilon^{ijklm}\, b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_i\ , \tag{2.44}$$

$$\mathsf{16}^* = |\Psi_-\rangle = b_i^\dagger\, |0\rangle\, \psi^i + \frac{1}{2 \cdot 3!} \epsilon^{ijklm}\, b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_{ij} + b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger\, |0\rangle\, \overline{\psi}_0\ . \tag{2.45}$$

The SM fermions are assigned to $\mathsf{16}$ where we identify $\overline{\psi}_i$ and ψ^{ij} with the 5^* and 10-dimensional representations of $\mathsf{SU(5)}$, respectively. The singlet ψ_0 denotes the left-handed anti-neutrino.

As already discussed, the theory must be anomaly free. In $\mathsf{SO(10)}$, the anomaly reads

$$\mathrm{tr}\left[\left\{\Sigma^{\lambda\mu}, \Sigma^{\nu\rho}\right\} \Sigma^{\sigma\tau}\right] \tag{2.46}$$

which cannot be written as a constant tensor (or proportional to such a tensor) with the indices $\mu \ldots \tau$, hence $\mathsf{SO(10)}$ is anomaly free.[4] Thus the miraculous cancellation of anomalies in the standard model as well as in the Georgi-Glashow model can be explained by the property of $\mathsf{SO(10)}$ to be anomaly free, where all fermions are grouped in one representation.

Yukawa Couplings

The simplest possibility is to introduce a 10-dimensional Higgs field $\mathsf{10}_H$ so that

$$\mathsf{16}\ \mathsf{16}\ \mathsf{10}_H = \widetilde{\Psi} B\, \Gamma_\mu \Psi\, \phi_\mu = \langle \Psi^* | B\, \Gamma_\mu | \Psi \rangle\, \phi_\mu \tag{2.47}$$

where we write Ψ instead of Ψ_+ for simplicity. The matrix B is the equivalent of the charge conjugation matrix $\mathcal{C}$ (which is dropped here) for $\mathsf{SO(10)}$,

$$B = \prod_{\mu = \mathrm{odd}} \Gamma_\mu\ . \tag{2.48}$$

Similarly, we can introduce 120 and 252-dimensional Higgs representations and write down the couplings (cf. Eqn. (B.20))

$$\widetilde{\Psi} B\, \Gamma_\mu \Gamma_\nu \Gamma_\rho \Psi\, \phi_{\mu\nu\rho}\ , \tag{2.49}$$

$$\widetilde{\Psi} B\, \Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma \Gamma_\tau \Psi\, \phi_{\mu\nu\rho\sigma\tau}\ . \tag{2.50}$$

The latter can again be split into $\mathsf{126} + \mathsf{126}^*$ where only the $\mathsf{126}^*$ couples to matter.

In the following, we express the Yukawa couplings of Eqn. (2.47) in $\mathsf{SU(5)}$ fields. Note that these couplings already include all $\mathsf{SU(5)}$ couplings (2.15) but are symmetric. This

[4] $\mathsf{SO(N)}$ groups are anomaly free in general except $\mathsf{SO(6)}$, where such a constant tensor ϵ^{ijklmn} exists.

can be changed by considering the couplings with the $\mathsf{120}_H$ (2.49) which are antisymmetric. Finally, Majorana masses can be generated via the couplings $\mathsf{16}\ \mathsf{16}\ \mathsf{126}^*_H$ (2.50) both for the anti-neutrino via $\mathsf{1}_{\mathsf{126}}$ and for the neutrino via $\mathsf{15}^*_{\mathsf{126}}$ (cf. Eqn. (B.19)). The latter leads to the type II seesaw mechanism.

The $\mathsf{SO(10)}$ fields $\mathsf{10}$ and $\mathsf{16}$ have the $\mathsf{SU(5)}$ decompositions [72]

$$\mathsf{10} = \mathsf{5} + \mathsf{5}^* \ , \tag{2.51}$$

$$\mathsf{16} = \mathsf{1} + \mathsf{5}^* + \mathsf{10} \ , \tag{2.52}$$

where we identify the $\mathsf{1}$, $\mathsf{5}^*$ and $\mathsf{10}$ of Eqn. (2.52) with ψ_0, $\overline{\psi}_i$ and ψ^{ij}, respectively. Now we write the $\mathsf{10}$ in $\mathsf{SU(5)}$ fields as [73]

$$\mathsf{10}_H : \phi_\mu = \begin{cases} \phi_{2j} &= \frac{1}{2}\left(\phi_{c_j} + \phi_{\bar{c}_j}\right) \\ \phi_{2j-1} &= \frac{1}{2i}\left(\phi_{c_j} - \phi_{\bar{c}_j}\right) \end{cases} \quad , \tag{2.53}$$

where ϕ_{c_j} and $\phi_{\bar{c}_j}$ transform like $\mathsf{SU(5)}$ representations. Thus we are able to compute the $\mathsf{SO(10)}$ in $\mathsf{SU(5)}$ fields which then only have to be deduced to irreducible representations. Therefore we obtain

$$\Gamma_\mu\, \phi_\mu = b_j\, \phi_{c_j} + b_j^\dagger\, \phi_{\bar{c}_j} \ . \tag{2.54}$$

To have a canonical kinetic term for the $\mathsf{SU(5)}$ Higgs fields H, $\overline{H}$,

$$-\partial_\alpha H_j\, \partial^\alpha H_j^\dagger - \partial_\alpha\, \overline{H}_j \partial^\alpha\, \overline{H}_j^\dagger \ , \tag{2.55}$$

we normalize the fields

$$\begin{aligned} \phi_{\bar{c}_j} &= \sqrt{2} H_j \ , \\ \phi_{c_j} &= \sqrt{2}\ \overline{H}_j \end{aligned} \tag{2.56}$$

and find ($\mathsf{5}_H \equiv H\,,\ \mathsf{5}^*_H \equiv \overline{H}$)

$$W_Y^{(10)} = \sqrt{2} i f_{ab} \left[- \left(\mathsf{1}_a\ \mathsf{5}^*_b\ + \mathsf{5}^*_a\ \mathsf{1}_b\right) \mathsf{5}_H + \left(\mathsf{10}_a\ \mathsf{5}^*_b + \mathsf{5}^*_a\ \mathsf{10}_b\right) \mathsf{5}^*_H + \frac{1}{4}\ \mathsf{10}_a\ \mathsf{10}_b\ \mathsf{5}_H \right] . \tag{2.57}$$

We identify the couplings in the second line with the $\mathsf{SU(5)}$ couplings (2.14), which are symmetric now, whereas those in the first line are the additional couplings for the neutrinos. Since all fermions are grouped in one multiplet, we obtain the relation

$$Y_u = Y_d = Y_e = Y_\nu^D \tag{2.58}$$

at GUT scale, where Y_ν^D denotes the neutrino Dirac Yukawa matrix, hence $\tan\beta \simeq 50$.

To keep $\tan\beta$ as a free parameter, different proposals have been put forward. One is to introduce a 16-dimensional Higgs representation $\mathsf{16}_H$ and its conjugate $\mathsf{16}^*_H$ where the $\mathsf{16}_H$ acquires vevs both $\mathcal{O}\left(M_{\text{GUT}}\right)$ and $\mathcal{O}\left(M_{\text{EW}}\right)$: the former for the G_{SM} and $\mathsf{SU(5)}$ singlet component to give Majorana masses to the anti-neutrino, the latter for the $\mathsf{SU(2)} \times \mathsf{U(1)}$ breaking component, i. e. for the doublet in the $\mathsf{5}^*$ of $\mathsf{16}_H$ which couples only to the down quarks and charged leptons [74, 75]. These additional interactions are described by the non-renormalizable operator $\mathsf{16\ 16\ 16}_H\,\mathsf{16}_H$. Another is to allow the $H(\mathsf{5})$ and $\overline{H}(\mathsf{5}^*)$ of the $\phi(\mathsf{10})$ to have different vevs such that

$$Y_u = Y_\nu^D \ , \qquad Y_d = Y_e \ , \tag{2.59}$$

and $\tan\beta$ remains as a free parameter. Alternatively, two 10-dimensional Higgs representations ϕ^1 and ϕ^2 are introduced: The $\mathsf{SU(5)}$-fields H and $\overline{H}$ are contained in ϕ^1 and ϕ^2, respectively, such that the Yukawa relations of Eqs. (2.59) hold [76]. This idea is realized in $\mathsf{SO(10)}$ orbifold GUTs, where only one SM doublet of $\phi(\mathsf{10})$ remains as a zero mode and, moreover, only one field leads to an anomalous low-energy theory [77, 78]. We will study such a model in detail in Chapter 3.

Following the consistent $\mathsf{SU(5)}$ model, we now consider the higher-dimensional operator $\mathsf{16\ 16\ 10}_H\,\mathsf{45}_H$, which includes the $\mathsf{SU(5)}$-operators (2.19).

Higher-dimensional Operators

With the tensor products (B.20), the operator $\mathsf{16\ 16\ 10}_H\,\mathsf{45}_H$ appears in four different invariants,

$$\begin{array}{ll} (\mathsf{16\ 16})_{10}\ (\mathsf{10}_H\ \mathsf{45}_H)_{10} & \qquad (\mathsf{16\ 10}_H)_{16^*}\ (\mathsf{16\ 45}_H)_{16} \\ (\mathsf{16\ 16})_{120}\ (\mathsf{10}_H\ \mathsf{45}_H)_{120} & \qquad (\mathsf{16\ 10}_H)_{144^*}\ (\mathsf{16\ 45}_H)_{144} \end{array} \tag{2.60}$$

To calculate the different couplings, we use the generalizations of Eqn. (2.53) [73, 79],

$$\phi_{\cdots\,\mu\,\cdots} = \begin{cases} \phi_{\cdots\,2j\,\cdots} & = \frac{1}{2}\left(\phi_{\cdots\,c_j\,\cdots} + \phi_{\cdots\,\bar{c}_j\,\cdots}\right) \\ \phi_{\cdots\,2j-1\,\cdots} & = \frac{1}{2i}\left(\phi_{\cdots\,c_j\,\cdots} - \phi_{\cdots\,\bar{c}_j\,\cdots}\right) \end{cases} , \tag{2.61}$$

$$\phi^\dagger_{\cdots\,\mu\,\cdots} = \begin{cases} \phi^\dagger_{\cdots\,2j\,\cdots} & = \frac{1}{2}\left(\phi^\dagger_{\cdots\,c_j\,\cdots} + \phi^\dagger_{\cdots\,\bar{c}_j\,\cdots}\right) \\ \phi^\dagger_{\cdots\,2j-1\,\cdots} & = \frac{i}{2}\left(\phi^\dagger_{\cdots\,c_j\,\cdots} - \phi^\dagger_{\cdots\,\bar{c}_j\,\cdots}\right) \end{cases} , \tag{2.62}$$

so that

$$\Sigma_{\mu\nu}\,\phi_{\mu\nu} = -i\,\left(b_i^\dagger b_j^\dagger\,\phi_{c_i c_j} + b_i b_j\,\phi_{\bar{c}_i\bar{c}_j} + 2\,b_i^\dagger b_j\,\phi_{c_i\bar{c}_j} - \phi_{c_n\bar{c}_n}\right) , \tag{2.63}$$

$$\begin{aligned} \Gamma_\mu\Gamma_\nu\Gamma_\lambda\phi_{\mu\nu\lambda} = {} & b_i^\dagger b_j^\dagger b_k^\dagger\,\phi_{c_i c_j c_k} + b_i b_j b_k\,\phi_{\bar{c}_i\bar{c}_j\bar{c}_k} \\ & + 3\,b_i^\dagger b_j b_k\,\phi_{c_i\bar{c}_j\bar{c}_k} + 3\,b_i^\dagger b_j^\dagger b_k\,\phi_{c_i c_j\bar{c}_k} + 3\,b_i\,\phi_{\bar{c}_n c_n\bar{c}_i} + 3\,b_i^\dagger\,\phi_{\bar{c}_n c_n c_i} \ . \end{aligned} \tag{2.64}$$

The reducible tensors $\phi_{\cdots}$ can be decomposed into irreducible $\mathsf{SU(5)}$ fields which are given in Appendix B.

For the first term we need the coupling $\mathsf{10}-\mathsf{10}-\mathsf{45}$, which can be decomposed as

$$\sqrt{2}\;[(\mathsf{5}_{10}\,\mathsf{5}^*_{10}+\mathsf{5}^*_{10}\,\mathsf{5}_{10})\,\mathsf{1}_{45}+\mathsf{5}_{10}\,\mathsf{5}_{10}\,\mathsf{10}^*_{45}+\mathsf{5}^*_{10}\,\mathsf{5}^*_{10}\,\mathsf{10}_{45}+(\mathsf{5}_{10}\,\mathsf{5}^*_{10}+\mathsf{5}^*_{10}\,\mathsf{5}_{10})\,\mathsf{24}_{45}]\,. \tag{2.65}$$

Since the vev of the $\mathsf{45}_H$ is taken in the $\mathsf{24}$-direction of $\mathsf{SU(5)}$, only the last two terms are relevant. Now we integrate out the heavy field $\mathsf{10}$ in Eqs.(2.57,2.65) by means of

$$W^{10}_M=2\;M_{10}\;\mathsf{5}\;\mathsf{5}^*\;, \tag{2.66}$$

and obtain the coupling given in Eqn. (2.73a).

The calculation for the second term is straightforward. We compute

$$\begin{aligned} W^{(120)}_Y = \frac{i}{\sqrt{3}}\,f_{ab}\Big[&(-\mathsf{1}_a\;\mathsf{10}_b+\mathsf{10}_a\;\mathsf{1}_b)\,\mathsf{10}^*_H+2\cdot\mathsf{5}^*_a\;\mathsf{5}^*_b\;\mathsf{10}_H+2\,(\mathsf{1}_a\;\mathsf{5}^*_b-\mathsf{5}^*_a\;\mathsf{1}_b)\,\mathsf{5}_H \\ &+(\mathsf{5}^*_a\;\mathsf{10}_b-\mathsf{10}^*_a\;\mathsf{5}^*_b)\,\mathsf{5}^*_H-\frac{1}{2}\,\mathsf{10}_a\;\mathsf{10}_b\;\mathsf{45}_H+(\mathsf{5}^*_a\;\mathsf{10}_b-\mathsf{10}_a\;\mathsf{5}^*_b)\,\mathsf{45}^*_H\Big] \end{aligned} \tag{2.67}$$

and calculate the relevant terms of the coupling $\mathsf{10}-\mathsf{45}-\mathsf{120}$,

$$\sqrt{3}\,[2\,(\mathsf{5}_{10}\;\mathsf{24}_{45}\;\mathsf{45}^*_{120}+\mathsf{5}^*_{10}\;\mathsf{24}_{45}\;\mathsf{45}_{120})-\mathsf{5}_{10}\;\mathsf{24}_{45}\;\mathsf{5}^*_{120}-\mathsf{5}^*_{10}\;\mathsf{24}_{45}\;\mathsf{5}_{120}]+\ldots \tag{2.68}$$

With the mass term

$$W^{120}_M=M_{120}\left(\frac{1}{2}\,\mathsf{10}\;\mathsf{10}^*+\mathsf{45}\;\mathsf{45}^*-2\cdot\mathsf{5}\;\mathsf{5}^*\right)\,, \tag{2.69}$$

we then get the result of Eqn.(2.73b).

The remaining two operators read

$$(\mathsf{16}\;\mathsf{10}_H)_{\mathsf{16}^*}\;\;(\mathsf{16}\;\mathsf{45}_H)_{\mathsf{16}}=\left(\widetilde{\Psi}B\,\Gamma_\mu\phi_\mu\right)\left(\Sigma_{\nu\rho}\Psi\,\phi_{\nu\rho}\right)\;, \tag{2.70}$$

$$(\mathsf{16}\;\mathsf{10}_H)_{\mathsf{144}^*}\;\;(\mathsf{16}\;\mathsf{45}_H)_{\mathsf{144}}=\left(\widetilde{\Psi}B\,\phi_\mu\right)\left(\Gamma_\nu\Psi\,\phi_{\mu\nu}\right)-\Big((2.70)\Big)\;. \tag{2.71}$$

The first expression in Eqn. (2.71), $(\widetilde{\Psi}B\,\phi_\mu)(\Gamma_\nu\Psi\,\phi_{\mu\nu})$, describes the reducible **160** representation. Since the **144** requires

$$\Gamma_\mu\tilde{\phi}_\mu=0\;, \tag{2.72}$$

we add $\Gamma^2_\mu=\mathbb{1}$ to project out the $\mathsf{16}$ contribution which is already calculated in Eqn. (2.70). Then we get the $\mathsf{144}$ contribution just by the difference of the two terms. We calculate both terms directly by means of the decompositions (2.54,2.63).

Altogether, the couplings of the four operators read

$$\widehat{Y}_{10}=\frac{h^{10}_{ij}}{M_{10}}\left\{\frac{1}{2}\,\epsilon_{abcde}\,\mathsf{10}^{ab}_i\,\mathsf{10}^{cd}_j\,\Sigma^e_f\,H^f-2\,\overline{H}_a\,\Sigma^a_b\left(\mathsf{10}^{bc}_i\,\mathsf{5}^*_{jc}+\mathsf{10}^{bc}_j\,\mathsf{5}^*_{ic}\right)\right\}+\ldots \tag{2.73a}$$

$$\begin{aligned}\widehat{Y}_{120}=\frac{h^{120}_{ij}}{M_{120}}\Big\{&-2\,\epsilon_{abcde}\,\mathsf{10}^{ab}_i\,\mathsf{10}^{cf}_j\,H^d\,\Sigma^e_f-\overline{H}_a\,\Sigma^a_b\left(\mathsf{10}^{bc}_i\,\mathsf{5}^*_{jc}-\mathsf{10}^{bc}_j\,\mathsf{5}^*_{ic}\right)\\ &-4\,\overline{H}_a\,\Sigma^c_b\left(\mathsf{10}^{ab}_i\,\mathsf{5}^*_{jc}-\mathsf{10}^{ab}_j\,\mathsf{5}^*_{ic}\right)\Big\}+\ldots\end{aligned} \tag{2.73b}$$

$$\widehat{Y}_{16} = \frac{h^{16}_{ij}}{M_{16}} \left\{ \frac{1}{2}\,\epsilon_{abcde}\, \mathsf{10}^{ab}_i\, \mathsf{10}^{cf}_j\, H^d\, \Sigma^e_f + 2\,\overline{H}_a\, \Sigma^a_b\, \mathsf{10}^{bc}_i\, \mathsf{5}^*_{jc} - \overline{H}_a\, \mathsf{10}^{ab}_i\, \Sigma^c_b\, \mathsf{5}^*_{jc} \right\} + \dots \tag{2.73c}$$

$$\widehat{Y}_{144} = \frac{h^{144}_{ij}}{M_{144}} \left\{ \epsilon_{abcde}\, \mathsf{10}^{ab}_i\, \mathsf{10}^{cd}_j\, \Sigma^e_f\, H^f - \frac{1}{2}\,\epsilon_{abcde}\, \mathsf{10}^{ab}_i\, \mathsf{10}^{cf}_j\, H^d\, \Sigma^e_f \right.$$
$$\left. + 2\,\overline{H}_a\, \Sigma^a_b\, \mathsf{10}^{bc}_i\, \mathsf{5}^*_{jc} + \overline{H}_a\, \mathsf{10}^{ab}_i\, \Sigma^c_b\, \mathsf{5}^*_{jc} \right\} + \dots\ , \tag{2.73d}$$

where we only list the $\mathsf{SU(5)}$ relevant terms. Note that there is no connection between the matrices $\widehat{Y}_k$ and the Yukawa matrices Y_j of the previous section.

Without loss of generality, we can assume that the heavy particles all have the same mass and can compare the couplings with those of $\mathsf{SU(5)}$ (2.19),

$$f_1 = \tfrac{1}{2}h^{10} + h^{144}\ , \tag{2.74a}$$
$$f_2 = -2\,h^{120} + \tfrac{1}{2}h^{16} - \tfrac{1}{2}h^{144}\ , \tag{2.74b}$$
$$h_1 = -2\,h^{10} - h^{120} + 2\,h^{16} + 2\,h^{144}\ , \tag{2.74c}$$
$$h_2 = -4\,h^{120} - h^{16} + h^{144}\ . \tag{2.74d}$$

Here, h^{10} is symmetric whereas h^{120} is antisymmetric. The other matrices, h^{16} and h^{144}, are not restricted by symmetry requirements. We see that $\mathsf{SO(10)}$ does not restrict the contributions from the higher-dimensional operators. Thus we are left with the fact that these solve the problem of Yukawa unification in $\mathsf{SU(5)}$ and can further reduce the proton decay rate by several orders of magnitude by a suitable choice of matrices — but without a mechanism that explains the pattern of matrices.

Higher-dimensional operators have also been studied in $\mathsf{SO(10)}$ models which are broken via the Pati-Salam group [75]. Here the vev of the $\mathsf{45}_H$ is chosen along the $(1, 1, 15)$-direction (under G_{PS}) that is proportional to $B - L$. This choice constrains the contributions (2.73). The dimension-five operators arise at the breaking scale of $\mathsf{SO(10)}$ which is not as restricted as the GUT scale in the Georgi-Glashow scenario. Moreover, the model in Ref. [75] needs several additional Higgs fields. At the end, an effective mass as the relevant scale for dimension-five proton decay is defined which turns out to be $\mathcal{O}\,(10^{18}\,\text{GeV})$. Thus we are already at the scale, where gravitational effects should become important as well.

2.4 Proton Decay Induced at Planck Scale

Independent of Grand Unification, it is natural to expect baryon and lepton number violating operators to appear at the Planck scale [80]. The analysis of the dimension-five operators with a coefficient $\lambda \simeq 1$, however, requires the suppressing mass to be $\mathcal{O}\,(10^{24}\,\text{GeV})$. Conversely with the reduced Planck scale, λ has to be smaller than $5 \cdot 10^{-8}$. Hence, there is a second problem with dimension-five operators, now at the Planck scale.

	$\mathbf{10}_3$	$\mathbf{10}_2$	$\mathbf{10}_1$	$\mathbf{5}^*_3$	$\mathbf{5}^*_2$	$\mathbf{5}^*_1$	$\mathbf{1}_3$	$\mathbf{1}_2$	$\mathbf{1}_1$	Φ
Q_F	0	1	2	a	a	$a+1$	0	$1-a$	$2-a$	-1

Table 2.3: $\mathsf{U}(1)_F$ *charges of the* $\mathsf{SU}(5)$ *fields and* Φ*;* $a = 0, 1$ [82].

One way to understand why λ is so small is to introduce a flavor symmetry according to the Froggatt-Nielsen mechanism based on a spontaneously broken global $\mathsf{U}(1)_F$ symmetry (cf. Chapter 1.5) [48]. The Yukawa couplings arise from non-renormalizable interactions after a gauge singlet field Φ acquires a vev,

$$Y_{ij} = g_{ij} \left(\frac{\langle \Phi \rangle}{\Lambda} \right)^{Q_i + Q_j} . \tag{2.75}$$

Here, g_{ij} are couplings $\mathcal{O}(1)$ and Q_i are the charges of the various fermions. Motivated by the atmospheric neutrino anomaly, different realizations of the idea were studied with a large $\nu_\mu - \nu_\tau$ mixing angle. We will focus on the symmetry $\mathsf{SU}(5) \times \mathsf{U}(1)_F$ as discussed by Sato and Yanagida [81]; the $\mathsf{U}(1)_F$ charges are given in Table 2.3. They originate from the observed mass ratios, the CKM matrix and a large $\nu_\mu - \nu_\tau$ mixing angle.

The value of a is restricted by $\tan\beta \sim \epsilon^{a-1}$ and by applying the model to leptogenesis [82]. The mass ratios, however, are independent of the value of a,

$$m_u : m_c : m_t \sim \epsilon^4 : \epsilon^2 : 1 , \tag{2.76}$$

$$m_e : m_\mu : m_\tau \sim m_d : m_s : m_b \sim \epsilon^a \left(\epsilon^3 : \epsilon : 1 \right) , \tag{2.77}$$

$$m_{\nu_1} : m_{\nu_2} : m_{\nu_3} \sim \epsilon^2 : 1 : 1 , \tag{2.78}$$

with

$$\epsilon = \frac{\langle \Phi \rangle}{\Lambda} \sim \frac{1}{17} . \tag{2.79}$$

The phenomenology of neutrino oscillations depends on the unspecified coefficients g_{ij} which are $\mathcal{O}(1)$. The LMA solution (1.13) requires the determinant of the 2-3 matrix of m_ν,

$$m_\nu \sim \epsilon^{2a} \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix} , \tag{2.80}$$

to be $\mathcal{O}(\epsilon)$ [83]. Finally, the mass ratio of the heavy neutrinos reads

$$M_1 : M_2 : M_3 \sim \epsilon^{2a} \left(\epsilon^{4-2a} : \epsilon^{2-2a} : 1 \right) . \tag{2.81}$$

We apply this model to proton decay induced at the Planck scale. The dimension-five operator, $\mathsf{10}_i\, \mathsf{10}_j\, \mathsf{10}_k\, \mathsf{5}^*_m$, gets its main contribution for

$$Q_1\, Q_1\, Q_2\, L_2 \;:\quad W = \frac{1}{M_{\rm Pl}} \left(\frac{\langle \Phi \rangle}{\Lambda} \right)^{5+a} . \tag{2.82}$$

$a = 0$ corresponding to high $\tan\beta$ gives only $7 \cdot 10^{-7}$ and is excluded. For $a = 1$ we obtain

$$\epsilon^6 \simeq 4 \times 10^{-8} \qquad \Longrightarrow \qquad t \simeq 1.5 \times 10^{33}\,\text{years}\;, \tag{2.83}$$

slightly below the experimental limit. Due to the unknown factors $\mathcal{O}(1)$, this is the correct order of magnitude, hence such a flavor symmetry is appropriate to reduce the operator at the Planck scale.

Although the models are in agreement with the experimental limits on proton decay, we would like to have either a mechanism which would naturally lead to the required relations among Yukawa couplings or avoid dimension-five operators. We can forbid those either with other parities or symmetries than R-parity that forbid these operators or by avoiding the mass term of the Higgs triplets.

Ibanez and Ross studied systematically disrete $\mathbb{Z}_N$ symmetries and $\mathbb{Z}_N$ R-symmetries (see Appendix A), in particular for $N = 2, 3$ [37]. They focussed on discrete gauge symmetries because gauge symmetries are stable under gravitational or other high energy physics corrections. As in usual gauge theories, anomalies appear in those discrete gauge symmetries [84], and the cancellation of these anomalies constrains the massless fermion content of models. They found that only two possible generalized parities with the particle content of the MSSM are discrete anomaly free, one of which is R-parity. The other symmetry, which they called Baryon parity $B_3 \equiv R_3 L_3$, requires additional fermions to cancel anomalies which makes possible models complicated. Of course, allowing additional singlets (anti-neutrinos) and Higgs multiplets, more symmetries can be made anomaly free but there is no compelling model so far.

Thus it is more promising to consider theories, where the common mass term of the Higgs triplets is generically absent. This is the case in orbifold GUT models to which we will turn in the next chapter.

Chapter 3

Proton Decay in Orbifold GUTs

The study of physics beyond four-dimensional spacetime opens a new window for physics beyond the standard model. In particular, the existence of extra dimensions is essential in superstring theory which is, at the moment, the most promising candidate for quantum gravity. Here, the fundamental objects are one-dimensional objects, either open or closed, and a consistent descriptions of these strings requires ten space-time dimensions. The spectrum of oscillations of closed strings includes a particle with spin 2 and zero mass, with the right type of interactions to be the graviton. Thus the spectrum contains chiral fermions coupled to gravity, and the effective low-energy theory contains supergravity.

Orbifold constructions have been used to compactify the extra dimensions, in particular because they allow chiral fermions [85]. The compactification procedure is mostly considered to go from ten to four dimensions in one step but there is no need to restrict the radii of all extra dimensions to the same size. Hence it is reasonable to study field theories in $4 + d$ dimensions, where $d < 6$. In view of the phenomenological success of gauge coupling unification in the MSSM, the energy range between M_{GUT} and M_{Pl} is the natural domain for higher-dimensional field theories. Thus orbifold GUTs are very attractive theories, in particular since they enable us to deal with the doublet-triplet splitting problem and dimension-five proton decay.

In this chapter, we consider a supersymmetric $\mathsf{SO(10)}$ model in six dimensions compactified on a torus with three $\mathbb{Z}_2$ parities. Since dimension-five operators are forbidden by a $\mathsf{U(1)}_R$ symmetry, only proton decay via dimension-six operators appears. Those operators can be generated by gauge boson exchange and by additional "derivative" operators. As we will see below, the former ones are given by the $\mathsf{SU(5)}$ gauge bosons, hence we start the chapter with these interactions in $\mathsf{SU(5)}$ [86, 87].

3.1 Analysis of Dimension-six Operators

The exchange of the $\mathsf{SU(5)}$ gauge bosons leads to dimension-six operators because they arise from D-terms of the kinetic part of the Lagrangean and there are no F terms of both chiral and anti-chiral functions. Contrary to the dimension-five operators, they do not involve any dressing through supersymmetric partners and therefore are not sensitive to the SUSY breaking scale (apart for the weak dependence of the GUT scale from the supersymmetric mass spectrum). The effective vertex can be obtained simply by integrating out the heavy gauge bosons, similarly as in the Fermi theory.

The coupling of the representations $\mathsf{5}^*$ and $\mathsf{10}$ of $\mathsf{SU(5)}$ to the X and Y gauge bosons are given by their kinetic terms,

$$\int d^2\theta \, d^2\bar{\theta} \, \sum_{\text{reps}} \overline{\Phi}_i \, e^{2V} \, \Phi_i \ , \tag{3.1}$$

so that

$$\mathscr{L}_{\text{kin}} = i \frac{g_5}{\sqrt{2}} \, V_\mu^a \left[2 \, \text{tr} \left(\overline{\mathsf{10}}_i \gamma^\mu T^a \mathsf{10}_i \right) + \bar{\mathsf{5}}_k^* \gamma^\mu (T^a)^t \mathsf{5}_k^* \right] + \text{h.c.} \ . \tag{3.2}$$

We express the $\mathsf{SU(5)}$ representations in terms of SM fields and obtain the baryon and lepton number violating operators

$$\mathscr{L}_{\not{B}} = -i \frac{g_5}{\sqrt{2}} V_\mu^\alpha \left[\epsilon_{\alpha\beta\gamma} \bar{q}_\beta \gamma^\mu u_\gamma^{\mathrm{C}} + \bar{e}^{\mathrm{C}} \gamma^\mu \left(i\sigma_2 \right) q_\alpha - \bar{d}_\alpha^{\mathrm{C}} \gamma^\mu \left(i\sigma_2 \right) l \right] + \text{h.c.} \tag{3.3}$$

with $V = (X, Y)$. Integrating out the heavy gauge bosons with masses M_V (1.37), we have the effective operators relevant for proton decay,

$$\mathscr{L}_{\text{eff}} = -\frac{g_5^2}{2M_V^2} \, \epsilon_{\alpha\beta\gamma} \, \overline{u^{\mathrm{C}}_{\alpha i}} \, \gamma^\mu q_{\beta i} \ \left[\overline{e^{\mathrm{C}}_j} \, \gamma_\mu \left(i\sigma_2 \right) q_{\gamma j} - \overline{d^{\mathrm{C}}_{\gamma k}} \, \gamma_\mu \left(i\sigma_2 \right) l_k \right] + \text{h.c.} \ , \tag{3.4}$$

where i, j count the generations of the $\mathsf{10}$ and k the generations of the $\mathsf{5}^*$. With Fierz reordering, one can write the operators in supersymmetric Weyl notation in the form,

$$\mathscr{L}_{\text{eff}} = -\frac{g_5^2}{M_V^2} \, \epsilon_{\alpha\beta\gamma} \ \left[\overline{e^c_j} \, \overline{u^c_{\alpha,i}} \, Q_{\beta,i} \, Q_{\gamma,j} - \overline{d^c_{\alpha,k}} \, \overline{u^c_{\beta,i}} \, Q_{\gamma,i} \, L_k \right] + \text{h.c.} \, . \tag{3.5}$$

The calculation of the decay width is straightforward, only the coefficients C change. The formulae for the remaining decay channels are given in Table 3.1, where the coefficients are already split into the gauge coupling and the flavor dependent part $\widehat{C}$. In 4D $\mathsf{SU(5)}$, the gauge coupling is simply given by

$$G_G^{[\text{4D } \mathsf{SU(5)}]} = \frac{g_5^2 A_l A^{\text{SD}}}{M_V^2} \ , \tag{3.6}$$

where the short-distance renormalization is now given by [14]

$$A^{\text{SD}} = \left[\frac{\alpha_1(M_Z)}{\alpha_5} \right]^{\frac{23}{30\, b_1}} \left[\frac{\alpha_2(M_Z)}{\alpha_5} \right]^{\frac{3}{2\, b_2}} \left[\frac{\alpha_3(M_Z)}{\alpha_5} \right]^{\frac{4}{3\, b_3}} = 2.37 \ , \tag{3.7}$$

according to the three gauge couplings; the first part is an approximate calculation [89].

$$\Gamma(p \to e_j^+ \pi^0) = \frac{(m_p^2 - m_{\pi^0}^2)^2}{32\pi m_p^3 f_\pi^2}\, \alpha^2 \left(\frac{1+D+F}{\sqrt{2}} \right)^2 \left(G_G\, \widehat{C}_{udue_j} \right)^2$$

$$\Gamma(p \to \bar{\nu}_j \pi^+) = \frac{(m_p^2 - m_{\pi^\pm}^2)^2}{32\pi m_p^3 f_\pi^2}\, \alpha^2 \,(1+D+F)^2 \left(G_G\, \widehat{C}_{udd\nu_j} \right)^2$$

$$\Gamma(p \to e_j^+ K^0) = \frac{(m_p^2 - m_{K^0}^2)^2}{32\pi m_p^3 f_\pi^2}\, \alpha^2 \left(1 + (D+F)\frac{m_p}{m_B} \right)^2 \left(G_G\, \widehat{C}_{usue_j} \right)^2$$

$$\Gamma(p \to e_j^+ \eta) = \frac{(m_p^2 - m_\eta^2)^2}{32\pi m_p^3 f_\pi^2}\, \alpha^2 \left(\frac{1+D+F}{\sqrt{6}} \right)^2 \left(G_G\, \widehat{C}_{udue_j} \right)^2$$

Table 3.1: Decay widths of the remaining channels [88].

3.2 A 6D $\mathsf{SO}(10)$ GUT Model

As discussed in the last chapter, $\mathsf{SO}(10)$ is one of the most attractive GUT groups. If we take the step forward to orbifold GUTs, the simplest possibility is to add just one extra dimension. The breaking of five-dimensional $\mathsf{SO}(10)$, however, gives only one of its maximal symmetric subgroups, $\mathsf{SU}(5) \times \mathsf{U}(1)$ or G_{PS} because there is only one reflection and one translation [78]. This problem is avoided in six dimensions, where the two translations can be used to break $\mathsf{SO}(10)$ to the extended standard model, $\mathsf{G}_{\text{SM}} \times \mathsf{U}(1)'$.[1]

6D $\mathsf{SO}(10)$ on $T^2/\left(\mathbb{Z}_2 \times \mathbb{Z}_2' \times \mathbb{Z}_2''\right)$

The spinors in six dimensions have eight components, twice as much as supersymmetry in four-dimensional $N = 1$, thus six-dimensional $N = 1$ supersymmetry corresponds to $N = 2$ in four dimensions. The multiplets are given in Table 3.2.

The Lagrangean for the vector multiplet reads [91]

$$\mathscr{L}_6^{\text{YM}} = \operatorname{tr}\left(-\frac{1}{2} V_{MN} V^{MN} + i\,\overline{\Lambda}\,\Gamma^M D_M \Lambda \right) , \tag{3.8}$$

where $V_M = V_M^A\, T^A$, $\Lambda = \Lambda^A\, T^A$, $D_M\Lambda = \partial_M \Lambda - ig\,[V_M, \Lambda]$ and $V_{MN} = \frac{1}{ig}[D_M, D_N]$. The Γ-matrices are

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} , \quad \Gamma^5 = \begin{pmatrix} 0 & i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix} , \quad \Gamma^6 = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix} . \tag{3.9}$$

The gaugino has negative 6D chirality, $\Gamma_7 \Lambda = -\Lambda$, with $\Gamma_7 = \operatorname{diag}(\gamma_5, -\gamma_5)$.

Now we perform the compactification on $T^2/\left(\mathbb{Z}_2 \times \mathbb{Z}_2' \times \mathbb{Z}_2''\right)$, as sketched in Figure 3.1. After compactifying on the torus, the fields $\Phi = (V_M, \Lambda)$ get the mode expansion [77]

$$\Phi(x, y, z) = \frac{1}{2\pi\sqrt{R_5 R_6}} \sum_{m,n} \Phi^{(m,n)}(x)\, \exp\left\{ i \left(\frac{my}{R_5} + \frac{nz}{R_6} \right) \right\} , \tag{3.10}$$

[1] In general, the rank does not have to be preserved as discussed in Ref. [90].

6D $N=1$ SUSY	corresponding fields in 4D $N=1$ SUSY	
Vector multiplet	Vector multiplet	$V=(V_\mu,\lambda_1,D^3)$
$\left(V_M,\Lambda=\binom{\lambda_1}{-i\lambda_2},D^a\right)$	$\oplus$ chiral multiplet	$\Sigma=\left(V_6+i\,V_5,-i\,\lambda_2,D^1+i\,D^2\right)$
hypermultiplet	chiral multiplet	(H,ψ_L,F)
$\left(\binom{H}{\bar{H}},\Psi_\mathrm{Dirac},\binom{F}{\bar{F}}\right)$	$\oplus$ anti-chiral multiplet	$(\bar{H},\psi_\mathrm{R},\bar{F})$

Table 3.2: Multiplets of a six-dimensional $N=1$ supersymmetric theory.

where R_j are the radii of the torus and $y=x^5,\ z=x^6$. Since the vector field is Hermitean, the corresponding coefficients satisfy the relation $V_M^{(-m,-n)}=V_M^{(m,n)\dagger}$.

To work out the 4D Lagrangean, we integrate over the extra dimensions. For the 4D scalars a convenient reparameterization is

$$\Pi_1^{(m,n)}(x)=\frac{i}{M(m,n)}\left(\frac{m}{R_5}V_5^{(m,n)}(x)+\frac{n}{R_6}V_6^{(m,n)}(x)\right)\ , \tag{3.11}$$

$$\Pi_2^{(m,n)}(x)=\frac{i}{M(m,n)}\left(-\frac{n}{R_6}V_5^{(m,n)}(x)+\frac{m}{R_5}V_6^{(m,n)}(x)\right)\ , \tag{3.12}$$

where $M(m,n)=\sqrt{\left(\frac{m}{R_5}\right)^2+\left(\frac{n}{R_6}\right)^2}$. The kinetic term for gauge and scalar fields is then given by

$$\begin{aligned}\mathscr{L}_4^{(1)}=\sum_{m,n}\mathrm{tr}\Bigg(&-\frac{1}{2}\widetilde{V}_{\mu\nu}^{(m,n)\dagger}\widetilde{V}^{(m,n)\mu\nu}+M(m,n)^2\,V_\mu^{(m,n)\dagger}V^{(m,n)\mu}+\partial_\mu\Pi_2^{(m,n)\dagger}\partial^\mu\Pi_2^{(m,n)}\\&+M(m,n)^2\,\Pi_2^{(m,n)\dagger}\Pi_2^{(m,n)}+\partial_\mu\Pi_1^{(m,n)\dagger}\partial^\mu\Pi_1^{(m,n)}\\&-M(m,n)\left(V_\mu^{(m,n)\dagger}\partial^\mu\Pi_1^{(m,n)}+\partial^\mu\Pi_1^{(m,n)\dagger}V_\mu^{(m,n)}\right)\Bigg)\ ,\end{aligned} \tag{3.13}$$

where $\widetilde{V}_{\mu\nu}^{(m,n)}=\partial_\mu V_\nu^{(m,n)}-\partial_\nu V_\mu^{(m,n)}$. Massless states are obtained for $m=n=0$ (zero modes). The mass generation for the massive Kaluza-Klein (KK) states is analogous to the Higgs mechanism. Here $\Pi_1^{(m,n)}$ play the role of the Nambu-Goldstone bosons and $M(m,n)$ correspond to the Higgs vacuum expectation values.

Similarly, one obtains for the gauginos,

$$\begin{aligned}\mathscr{L}_4^{(2)}=\sum_{m,n}\mathrm{tr}\Bigg(&i\,\overline{\lambda}_1^{(m,n)}\gamma^\mu\partial_\mu\lambda_1^{(m,n)}+i\,\overline{\lambda}_2^{(m,n)}\gamma^\mu\partial_\mu\lambda_2^{(m,n)}\\&-\left(\frac{m}{R_5}-i\frac{n}{R_6}\right)\overline{\lambda}_1^{(m,n)}\lambda_2^{(m,n)}+\mathrm{h.c.}\Bigg)\ .\end{aligned} \tag{3.14}$$

This is the kinetic term for the Dirac fermion $\lambda_D=(\lambda_1,\lambda_2)$ with mass $M(m,n)$.

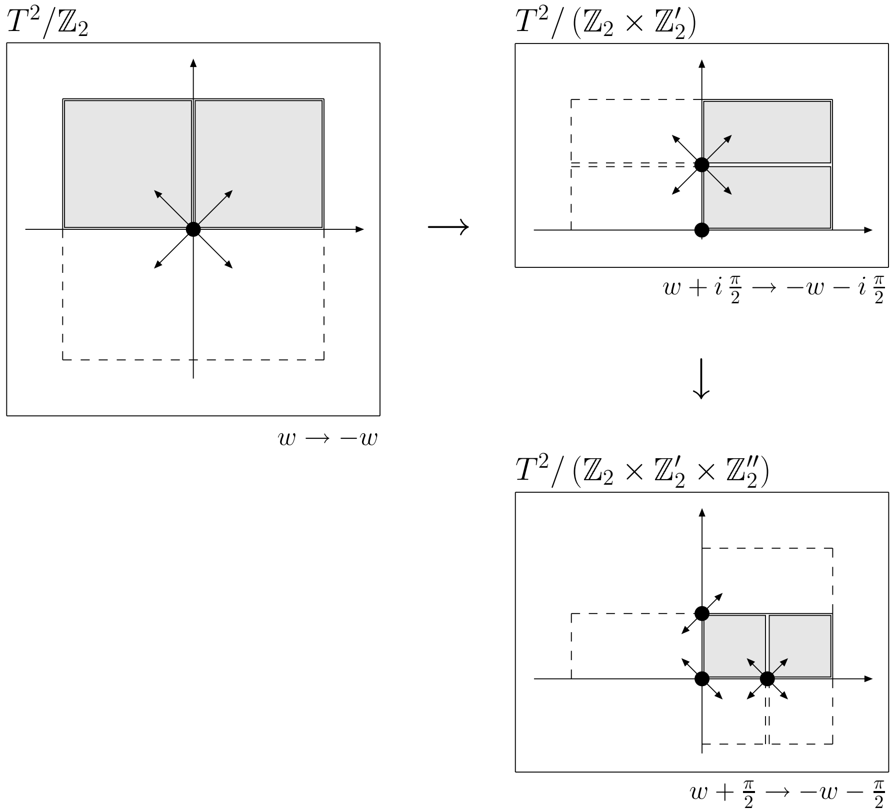


Figure 3.1: The $T^2/\left(\mathbb{Z}_2 \times \mathbb{Z}_2' \times \mathbb{Z}_2''\right)$ orbifold in the $w = x + iy$ plane.

The unwanted $N = 2$ supersymmetry is broken by the first $\mathbb{Z}_2$-parity. Under the corresponding reflection $(y, z) \to (-y, -z)$, vectors and scalars are even and odd, respectively,

$$P\,V_\mu\,(x, -y, -z)\,P^{-1} = +V_\mu\,(x, y, z)\,, \quad P\,V_{5,6}\,(x, -y, -z)\,P^{-1} = -V_{5,6}\,(x, y, z)\,, \tag{3.15}$$

where $P = \mathbb{1}$. This implies for the Kaluza-Klein modes,

$$V_\mu^{(-m,-n)} = +V_\mu^{(m,n)}\,, \qquad V_{5,6}^{(-m,-n)} = -V_{5,6}^{(m,n)}\,, \tag{3.16}$$

so that scalar zero modes are eliminated. Further, the number of massive KK modes is halved. Since the derivatives $\partial_{5,6}$ are odd under reflection, the two Weyl fermions λ_1 and λ_2 must have opposite parities,

$$P\,\lambda_1(x, -y, -z)P^{-1} = +\lambda_1(x, y, z)\,, \quad P\,\lambda_2(x, -y, -z)P^{-1} = -\lambda_2(x, y, z)\,. \tag{3.17}$$

Comparison of Eqs. (3.15) and (3.17) shows that $V = (V_\mu, \lambda_1)$ and $\Sigma = (V_{5,6}, \lambda_2)$ form vector and chiral multiplets, respectively (Table 3.2), and only vector multiplets have zero modes. The orbifold compactification breaks the extended supersymmetry which one obtains from the six-dimensional theory by dimensional reduction.

Now the zero modes obtained by compactification on the orbifold $T^2/\mathbb{Z}_2$ form a $N=1$ supersymmetric $\mathsf{SO(10)}$ theory in four dimensions. A breaking of the full $\mathsf{SO(10)}$ gauge group can be achieved by using the two parities P_{GG} and P_{PS} which define the symmetric subgroups $\mathsf{G}_{\text{GG}} = \mathsf{SU(5)} \times \mathsf{U(1)}$ and $\mathsf{G}_{\text{PS}} = \mathsf{SU(4)} \times \mathsf{SU(2)} \times \mathsf{SU(2)}$, respectively. In the vector representation, the parities can be taken as

$$P_{\text{GG}} = \text{diag}\left(\sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2\right) \ , \quad P_{\text{PS}} = \text{diag}\left(-\sigma_0, -\sigma_0, -\sigma_0, \sigma_0, \sigma_0\right) \ . \tag{3.18}$$

For the vector fields $V = (V_\mu, \lambda_1)$, one demands

$$P_{\text{GG}}\, V\left(x, -y, -z - \tfrac{\pi}{2}R_6\right) P_{\text{GG}}^{-1} = V\left(x, y, z + \tfrac{\pi}{2}R_6\right) \ , \tag{3.19}$$

$$P_{\text{PS}}\, V\left(x, -y - i\,\tfrac{\pi}{2}R_5, -z\right) P_{\text{PS}}^{-1} = V\left(x, y + i\,\tfrac{\pi}{2}R_6, z\right) \ . \tag{3.20}$$

Component fields belonging to the symmetric subgroup $\mathsf{G_s}$ then have positive parity, those of the coset space $\mathsf{SO(10)}/\mathsf{G_s}$ have negative parity. The restrictions of the discrete symmetry $\mathbb{Z}_2$ require the opposite parities for the chiral fields $\Sigma = (V_{5,6}, \lambda_2)$,

$$P_{\text{GG}}\, \Sigma\left(x, -y, -z - \tfrac{\pi}{2}R_6\right) P_{\text{GG}}^{-1} = -\Sigma\left(x, y, z + \tfrac{\pi}{2}R_6\right) \tag{3.21}$$

$$P_{\text{PS}}\, \Sigma\left(x, -y - i\,\tfrac{\pi}{2}R_5, -z\right) P_{\text{PS}}^{-1} = -\Sigma\left(x, y + i\,\tfrac{\pi}{2}R_6, z\right) \ , \tag{3.22}$$

thus the component fields are split again. The parities for the different G_{SM} representations contained in the $\mathsf{45}$-plet of $\mathsf{SO(10)}$ are summarized in Table 3.3. The explicit mode expansion is given in Ref. [77]; here we only need

$$\Phi_{+++}(x,y,z) = \frac{1}{\pi\sqrt{R_5 R_6}} \sum_{m,n} \frac{1}{2^{\delta_{m,0}\delta_{n,0}}} \phi_{+++}^{(2m,2n)}(x) \cos\left(\frac{2my}{R_5} + \frac{2nz}{R_6}\right), \tag{3.23}$$

$$\Phi_{++-}(x,y,z) = \frac{1}{\pi\sqrt{R_5 R_6}} \sum_{m,n} \phi_{++-}^{(2m,2n+1)}(x) \cos\left(\frac{2my}{R_5} + \frac{(2n+1)z}{R_6}\right), \tag{3.24}$$

$$\Phi_{+-+}(x,y,z) = \frac{1}{\pi\sqrt{R_5 R_6}} \sum_{m,n} \phi_{+-+}^{(2m+1,2n)}(x) \cos\left(\frac{(2m+1)y}{R_5} + \frac{2nz}{R_6}\right). \tag{3.25}$$

Only fields for which all parities are positive have zero modes; they form an $N=1$ massless vector multiplet in the adjoint representation of the unbroken group G'_{SM}. All other fields with one or more negative parities combine to massive vector multiplets. In the limiting cases $R_5 \to 0$ with R_6 fixed, and R_5 fixed with $R_6 \to 0$, we get the Pati-Salam group G_{PS} and $\mathsf{G}_{\text{GG}} \times \mathsf{U(1)}$, respectively. Thus we have three fixed points in the extra dimensions (branes), $O = (0,0)$, $O_{\text{PS}} = (\frac{\pi}{2}R_5, 0)$ and $O_{\text{GG}} = (0, \frac{\pi}{2}R_6)$, where the unbroken subgroups are $\mathsf{SO(10)}$, G_{PS} and G_{GG}, respectively. In addition, there is a fourth fixed point at $O_{\text{fl}} = (\frac{\pi}{2}R_5, \frac{\pi}{2}R_6)$ [78], which is obtained by combining the three discrete symmetries $\mathbb{Z}_2$, $\mathbb{Z}_2^{\text{PS}}$ and $\mathbb{Z}_2^{\text{GG}}$,

$$P_{\text{fl}} V\left(x, -y + \tfrac{\pi}{2}R_5, -z + \tfrac{\pi}{2}R_6\right) P_{\text{fl}}^{-1} = +V\left(x, y + \tfrac{\pi}{2}R_5, z + \tfrac{\pi}{2}R_6\right). \tag{3.26}$$

			$(V_\mu,\, \lambda_1)$			$(V_{5,6},\, \lambda_2)$		
G'_{SM}	G_{GG}	G_{PS}	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\text{GG}}$	$\mathbb{Z}_2^{\text{PS}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\text{GG}}$	$\mathbb{Z}_2^{\text{PS}}$
(8, 1, 0, 0)	(24, 0)	(15,1,1)	+	+	+	−	−	−
(3, 2,−5, 0)	(24, 0)	(6,2,2)	+	+	−	−	−	+
$(\bar{3}$, 2, 5, 0)	(24,0)	(6,2,2)	+	+	−	−	−	+
(1, 3, 0, 0)	(24,0)	(1,3,1)	+	+	+	−	−	−
(1, 1, 0, 0)	(24,0)	(1,1,3)	+	+	+	−	−	−
(3, 2, 1, 4)	(10,4)	(6,2,2)	+	−	−	−	+	+
$(\bar{3}$, 1, −4, 4)	(10,4)	(15,1,1)	+	−	+	−	+	−
(1, 1, 6, 4)	(10,4)	(1,1,3)	+	−	+	−	+	−
$(\bar{3}$, 2, −1, −4)	$(\overline{10}$,−4)	(6,2,2)	+	−	−	−	+	+
(3, 1, 4, −4)	$(\overline{10}$,−4)	(15,1,1)	+	−	+	−	+	−
(1, 1, −6, −4)	$(\overline{10}$,−4)	(1,1,3)	+	−	+	−	+	−
(1, 1, 0, 0)	(1,0)	(15, 1, 1)	+	+	+	−	−	−

Table 3.3: Parity assignment for the components $V_M^A = \frac{1}{2}\,\mathrm{tr}(T^A V_M)$ of the $\mathbf{45}$-plet of SO(10).

The unbroken subgroup at O_{fl} is flipped $\mathsf{SU(5)}$ (see Appendix B). Only for finite R_5 and R_6 one obtains the extended standard model group G'_{SM}.

The physical region is obtained by folding the shaded regions in Fig. 3.1 and gluing the edges. The result is a 'pillow' with the four fixed points as corners (Fig. 3.2).

Higgs and Matter Fields

Quarks, leptons and Higgs fields are incorporated by adding $\mathbf{10}$ and $\mathbf{16}$-plets in the bulk and on the fixed points. We first consider Higgs fields $\mathbf{10}$. They contain two complex scalars, H and H', and a fermion (h, h') with opposite 4D chiralities, $\gamma_5 h = h$, $\gamma_5 h' = -h'$, and positive 6D chirality $\Gamma_7 h = h$; for details see Ref. [77]. The N=1 supermultiplets $H = (H, h)$ and $H' = (H', h')$ get opposite parities with respect to every $\mathbb{Z}_2$. For the $\mathbb{Z}_2$, the parity is chosen as

$$PH\,(x, -y, -z) = +H\,(x, y, z)\,, \qquad PH'\,(x, -y, -z) = -H'\,(x, y, z)\,, \tag{3.27}$$

so that the extended supersymmetry is broken. For $\mathbb{Z}_2^{\text{GG}}$, we have

$$P_{\text{GG}}H\left(x, -y, -z + \tfrac{\pi}{2}R_6\right) = +H\left(x, y, z + \tfrac{\pi}{2}R_6\right), \tag{3.28}$$

$$P_{\text{GG}}H'\left(x, -y, -z + \tfrac{\pi}{2}R_6\right) = -H'\left(x, y, z + \tfrac{\pi}{2}R_6\right), \tag{3.29}$$

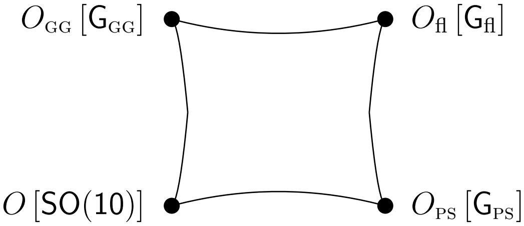


Figure 3.2: The three SO(10) subgroups at the corresponding fixed points of the orbifold $T^2/\mathbb{Z}_2 \times \mathbb{Z}_2' \times \mathbb{Z}_2''$.

where the $\mathsf{5}$ and $\mathsf{5}^*$ of SU(5) have opposite parities with respect to $\mathbb{Z}_2^{\mathrm{GG}}$. The parity P_{PS} yields the desired doublet-triplet splitting. With

$$P_{\mathrm{PS}} H\left(x, -y + \tfrac{\pi}{2} R_5, -z\right) = +H\left(x, y + \tfrac{\pi}{2} R_5, z\right), \tag{3.30}$$

$$P_{\mathrm{PS}} H'\left(x, -y + \tfrac{\pi}{2} R_5, -z\right) = -H'\left(x, y + \tfrac{\pi}{2} R_5, z\right) \tag{3.31}$$

one obtains one SU(2) doublet $N = 1$ supermultiplet as zero modes, as given in Table 3.4; the related SU(3) triplet is heavy. Note that there is an ambiguity in the global sign of each parity, as long as we do not consider a superpotential for the matter fields. The opposite choice of sign in Eqn. (3.30) leads to a massless colour triplet and a heavy weak doublet.

In order to obtain the wanted two Higgs doublets as zero modes one has to introduce two 10-plets H_1 and H_2. Their parities must be different with respect to Z_2^{GG}. Their irreducible 6D gauge anomalies cancel the one of the 45-plet [90].

Introducing matter fields, the guiding principles in this model are the cancellation of the anomalies and an embedding into the gauge group E_8 [92]. The anomalies of the vector and hypermultiplets are related by

$$a\,(\mathsf{45}) = -2\,a\,(\mathsf{10}) \qquad\qquad a\,(\mathsf{16}) = a\,(\mathsf{16}^*) = -\,a\,(\mathsf{10}) \ . \tag{3.32}$$

Hence the matter fields should not be bulk fields because those require too many additional fields in order to cancel the anomalies. The breaking of $\mathsf{U(1)}_{B-L}$ can be achieved by a vev of the singlet component of an additional $\Phi = \mathsf{16}$ and $\Phi^{\mathrm{C}} = \mathsf{16}^*$. If we then add two more 10 hypermultiplets, H_3 and H_4, the irreducible and reducible bulk anomalies as well as all brane anomalies cancel, together with the gauge-gravity mixed anomaly [93].

Fermion masses and mixings are determined by the brane superpotential. The allowed terms are restricted by R-invariance and an additional $\mathsf{U(1)}_{\tilde{\mathsf{X}}}$ symmetry [92]. The fields H_1, H_2, Φ and Φ^{C}, which acquire a vacuum expectation value, have vanishing R-charge. All matter fields have R-charge one.

The main idea to generate fermion mass matrices is as follows. The three sequential 16-plets ψ_j are located on the three branes where SO(10) is broken to its three GUT subgroups, namely ψ_1 at O_{GG}, ψ_2 at O_{fl} and ψ_3 at O_{PS}. The three 'families' are then

SO(10)	10							
$\mathsf{G}_{\rm PS}$	(1, 2, 2)		(1, 2, 2)		(6, 1, 1)		(6, 1, 1)	
$\mathsf{G}_{\rm GG}$	5^*_{-2}		5_2		5^*_{-2}		5_2	
	$H^{\rm C}$		H		$G^{\rm C}$		G	
	$\mathbb{Z}_2^{\rm PS}$	$\mathbb{Z}_2^{\rm GG}$	$\mathbb{Z}_2^{\rm PS}$	$\mathbb{Z}_2^{\rm GG}$	$\mathbb{Z}_2^{\rm PS}$	$\mathbb{Z}_2^{\rm GG}$	$\mathbb{Z}_2^{\rm PS}$	$\mathbb{Z}_2^{\rm GG}$
H_1	$+$	$+$	$+$	$-$	$-$	$+$	$-$	$-$
H_2	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
H_3	$-$	$+$	$-$	$+$	$+$	$+$	$+$	$-$
H_4	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
H_5	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$-$
H_6	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$

Table 3.4: Parity assignment for the bulk Higgs 10-plets of SO(10).

separated by distances large compared to the cutoff scale M_*. Hence, they can only have diagonal Yukawa couplings with the bulk Higgs fields. Direct mixings are exponentially suppressed. To have mixing as observed, we add an additional pair of bulk 16-plets, ϕ and $\phi^{\rm c}$, together with two 10-plets, H_5 and H_6, to cancel bulk anomalies [94]. The brane fields can mix with the bulk field zero modes, (cf. Tab. 3.4 and 3.5),

$$L_\phi = \begin{pmatrix} \nu_4 \\ e_4 \end{pmatrix}, \qquad L^{\rm C}_{\phi^{\rm c}} = \begin{pmatrix} \nu_4^{\rm C} \\ e_4^{\rm C} \end{pmatrix}, \qquad G_5^{\rm C} = d_4^c, \qquad G_6 = d_4 \; . \tag{3.33}$$

These mixings take place only among left-handed leptons and right-handed down-quarks, which leads to a characteristic pattern of mass matrices. Since ψ_j and ϕ have the same charges, we combine them to the quartet $\psi_\alpha = (\psi_j, \phi),\ \alpha = 1, \ldots, 4$.

The most general brane superpotential up to quartic terms is given by [94]

$$\begin{aligned} W_Y = {} & M^d\, H_5 H_6 + M^l_\alpha\, \psi_\alpha\, \phi^c + \tfrac{1}{2} h^{(1)}_{\alpha\beta}\, \psi_\alpha \psi_\beta H_1 + \tfrac{1}{2} h^{(2)}_{\alpha\beta}\, \psi_\alpha \psi_\beta H_2 + f_\alpha \Phi\, \psi_\alpha H_6 \\ & + \frac{h^N_{\alpha\beta}}{2\, M_*}\, \psi_\alpha \psi_\beta\, \Phi^c\, \Phi^c + \frac{g^d_\alpha}{M_*} \Phi^c \psi_\alpha H_5 H_1 + f^D \Phi^c\, \Phi^c H_3 + f^G \Phi\, \Phi H_4 \; , \end{aligned} \tag{3.34}$$

$$\begin{aligned} W_r = {} & f_5\, \Phi^c \phi^c H_5 + M_{13}\, H_1 H_3 + M_{23} H_2 H_3 + \frac{k_1}{M_*} H_1^2 H_5^2 + \frac{k_2}{M_*} H_1 H_2 H_5^2 \\ & + \frac{k_3}{M_*} H_2^2 H_5^2 + \frac{k_4}{M_*} \Phi\, \Phi^c H_1 H_3 + \frac{k_5}{M_*} \Phi\, \Phi^c H_2 H_3 \\ & + \frac{g^u_\alpha}{M_*} \Phi^c \psi_\alpha H_5 H_2 + \frac{g^d}{M_*} \Phi\, \phi^c H_5 H_1 + \frac{g^u}{M_*} \Phi\, \phi^c H_5 H_2 + \frac{k^d_\alpha}{M_*} \Phi\, \Phi^c \psi_\alpha\, \phi^c \\ & + \frac{k^l_\alpha}{M_*} \Phi\, \Phi^c \psi_\alpha\, \phi^c + \frac{k^l}{M_*} \Phi\, \Phi \phi^c \phi^c + \frac{\lambda_1}{M_*} S H_1 \Phi\, \Phi^c + \frac{\lambda_2}{M_*} S H_2 \Phi\, \Phi^c \; . \end{aligned} \tag{3.35}$$

Only the terms in W_Y give rise to couplings for the zero modes and the matter fields.

SO(10)	16							
G_{PS}	$(4,2,1)$		$(4,2,1)$		$(4^*,1,2)$		$(4^*,1,2)$	
G_{GG}	10_1		5^*_{-3}		10_1		$5^*_{-3}, 1_5$	
	Q		L		$U^{\text{C}}, E^{\text{C}}$		$D^{\text{C}}, N^{\text{C}}$	
	$\mathbb{Z}_2^{\text{PS}}$	$\mathbb{Z}_2^{\text{GG}}$	$\mathbb{Z}_2^{\text{PS}}$	$\mathbb{Z}_2^{\text{GG}}$	$\mathbb{Z}_2^{\text{PS}}$	$\mathbb{Z}_2^{\text{GG}}$	$\mathbb{Z}_2^{\text{PS}}$	$\mathbb{Z}_2^{\text{GG}}$
Φ	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
ϕ	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$+$

Table 3.5: Parity assignment for the bulk 16-plets of SO(10).

Fermion Masses and Mixing

The vacuum expectation values $\langle H_1^c\rangle = v_d$, $\langle H_2\rangle = v_u$ and $\langle \Phi^{\text{C}}\rangle = \langle\Phi\rangle = v_N$ yield the mass terms [94]

$$W = d_\alpha m^d_{\alpha\beta} d^{\text{C}}_\beta + e^{\text{C}}_\alpha m^e_{\alpha\beta} e_\beta + n^{\text{C}}_\alpha m^D_{\alpha\beta}\nu_\beta + u^{\text{C}}_i m^u_{ij} u_j + \tfrac{1}{2} n^{\text{C}}_i M_{ij} n^{\text{C}}_j \ . \tag{3.36}$$

Here m^d, m^e and m^D are 4×4 matrices,

$$m^d = \begin{pmatrix} h^d_{11}v_d & 0 & 0 & g^d_1\frac{v_N}{M_*}v_d \\ 0 & h^d_{22}v_d & 0 & g^d_2\frac{v_N}{M_*}v_d \\ 0 & 0 & h^d_{33}v_d & g^d_3\frac{v_N}{M_*}v_d \\ f_1 v_N & f_2 v_N & f_3 v_N & M^d \end{pmatrix} , \tag{3.37}$$

$$m^e = \begin{pmatrix} h^d_{11}v_d & 0 & 0 & h^e_{14}v_d \\ 0 & h^e_{22}v_d & 0 & h^e_{24}v_d \\ 0 & 0 & h^d_{33}v_d & h^e_{34}v_d \\ M^l_1 & M^l_2 & M^l_3 & M^l_4 \end{pmatrix} , \tag{3.38}$$

$$m^D = \begin{pmatrix} h^D_{11}v_u & 0 & 0 & h^D_{14}v_u \\ 0 & h^u_{22}v_u & 0 & h^D_{24}v_u \\ 0 & 0 & h^u_{33}v_u & h^D_{34}v_u \\ M^l_1 & M^l_2 & M^l_3 & M^l_4 \end{pmatrix} , \tag{3.39}$$

whereas m^u and m^N are diagonal 3×3 matrices,

$$m^u = \begin{pmatrix} h^u_{11}v_u & 0 & 0 \\ 0 & h^u_{22}v_u & 0 \\ 0 & 0 & h^u_{33}v_u \end{pmatrix} , \qquad m^N = \begin{pmatrix} h^N_{11}\frac{v_N^2}{M_*} & 0 & 0 \\ 0 & h^N_{22}\frac{v_N^2}{M_*} & 0 \\ 0 & 0 & h^N_{33}\frac{v_N^2}{M_*} \end{pmatrix} . \tag{3.40}$$

In the matrices m^d, m^e and m^D, corrections $\mathcal{O}\left(v_N/M_*\right)$ are neglected. The diagonal elements satisfy four GUT relations which correspond to the unbroken SU(5), flipped SU(5) and Pati-Salam subgroups of SO(10).

The crucial feature of the matrices m^d, m^e and m^D,

$$m = \begin{pmatrix} \mu_1 & 0 & 0 & \tilde{\mu}_1 \\ 0 & \mu_2 & 0 & \tilde{\mu}_2 \\ 0 & 0 & \mu_3 & \tilde{\mu}_3 \\ \widetilde{M}_1 & \widetilde{M}_2 & \widetilde{M}_3 & \widetilde{M}_4 \end{pmatrix} , \tag{3.41}$$

are the mixings between the six brane states and the two bulk states. The first three rows of the matrices are proportional to the electroweak scale. The corresponding Yukawa couplings have to be hierarchical in order to obtain a realistic spectrum of quark and lepton masses. This corresponds to different strengths of the Yukawa couplings at the different fixed points of the orbifold. The fourth row, proportional to M^d, M^l and v_N, is of order of the unification scale and, in general, not hierarchical.

The matrix (3.41) can be diagonalized using the unitary matrices

$$m = U_4 U_3 \,\mathcal{D}\, V_3^\dagger V_4^\dagger \tag{3.42}$$

where the matrices U_4, V_4 single out the heavy mass eigenstate, that can then be integrated out, while the matrices U_3, V_3 act only on the SM flavour indices and perform the final diagonalization in the 3×3 subspace.

We define a set of four-dimensional unit vectors,

$$\left(\widetilde{M}_1, \ldots, \widetilde{M}_4\right) = \widetilde{M} e_4^\top , \qquad\qquad e_\alpha^\top e_\beta = e_{\alpha\gamma}^\top e_{\beta\gamma} = \delta_{\alpha\beta}, \tag{3.43}$$

with $\widetilde{M} = \sqrt{\sum_j \widetilde{M}_j^2}$. Then U_4, V_4 can be given as (neglecting phases) [15]

$$U_4 \simeq \begin{pmatrix} 1 & 0 & 0 & \frac{\mu_1 \widetilde{M}_1 + \tilde{\mu}_1 \widetilde{M}_4}{\widetilde{M}^2} \\ 0 & 1 & 0 & \frac{\mu_2 \widetilde{M}_2 + \tilde{\mu}_2 \widetilde{M}_4}{\widetilde{M}^2} \\ 0 & 0 & 1 & \frac{\mu_3 \widetilde{M}_3 + \tilde{\mu}_3 \widetilde{M}_4}{\widetilde{M}^2} \\ -\frac{\mu_1 \widetilde{M}_1 + \tilde{\mu}_1 \widetilde{M}_4}{\widetilde{M}^2} & -\frac{\mu_2 \widetilde{M}_2 + \tilde{\mu}_2 \widetilde{M}_4}{\widetilde{M}^2} & -\frac{\mu_3 \widetilde{M}_3 + \tilde{\mu}_3 \widetilde{M}_4}{\widetilde{M}^2} & 1 \end{pmatrix}, \tag{3.44}$$

$$V_4 = \begin{pmatrix} \frac{\widetilde{M}_4}{\widetilde{M}_{14}} & 0 & -\frac{\widetilde{M}_1 \widetilde{M}_{23}}{\widetilde{M}\widetilde{M}_{14}} & \frac{\widetilde{M}_1}{\widetilde{M}} \\ 0 & \frac{\widetilde{M}_3}{\widetilde{M}_{23}} & \frac{\widetilde{M}_2 \widetilde{M}_{14}}{\widetilde{M}\widetilde{M}_{23}} & \frac{\widetilde{M}_2}{\widetilde{M}} \\ 0 & -\frac{\widetilde{M}_2}{\widetilde{M}_{23}} & \frac{\widetilde{M}_3 \widetilde{M}_{14}}{\widetilde{M}\widetilde{M}_{23}} & \frac{\widetilde{M}_3}{\widetilde{M}} \\ -\frac{\widetilde{M}_1}{\widetilde{M}_{14}} & 0 & -\frac{\widetilde{M}_4 \widetilde{M}_{23}}{\widetilde{M}\widetilde{M}_{14}} & \frac{\widetilde{M}_4}{\widetilde{M}} \end{pmatrix} , \tag{3.45}$$

where $\widetilde{M}_{ij} = \sqrt{\widetilde{M}_i^2 + \widetilde{M}_j^2}$; so in general V_4 contains large mixings, while U_4 is approximately the unit matrix up to terms $\mathrm{O}(v/\widetilde{M})$. U_3 and V_3 diagonalize

$$m' = U_4^\dagger \, m \, V_4 = \begin{pmatrix} \widehat{m} & 0 \\ 0 & \widetilde{M} \end{pmatrix} + \mathcal{O}\left(\frac{v^2}{\widetilde{M}}\right), \tag{3.46}$$

where the 3×3 matrix $\widehat{m}$ is given by

$$\widehat{m} = \begin{pmatrix} \mu_1 \left(V_4\right)_{1j} + \tilde{\mu}_1 \left(V_4\right)_{4j} \\ \mu_2 \left(V_4\right)_{2j} + \tilde{\mu}_2 \left(V_4\right)_{4j} \\ \mu_3 \left(V_4\right)_{3j} + \tilde{\mu}_3 \left(V_4\right)_{4j} \end{pmatrix} . \tag{3.47}$$

Note that the rows of $\widehat{m}$ scale each like $\mu_j, \tilde{\mu}_j$. U_3 and V_3 both have only a nontrivial 3×3 part

$$U_3 = \begin{pmatrix} V_{\text{CKM}}^{\dagger} & 0 \\ 0 & 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} \widehat{V} & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.48}$$

The hierarchy of the row vectors suggests to perform a further change of basis such that all remaining mixings are small. With respect to this new basis the matrix m has triangular form [94],

$$\overline{m} = \begin{pmatrix} \bar{\mu}_1 \gamma & \bar{\mu}_1 & \bar{\mu}_1 \beta \\ 0 & \bar{\mu}_2 & \bar{\mu}_2 \alpha \\ 0 & 0 & \bar{\mu}_3 \end{pmatrix} . \tag{3.49}$$

Now we can derive the mass eigenvalues and mixing angles. The parameters can be chosen in such a way to give a consistent quark mass pattern and the CKM matrix, in particular

$$\mu_1^u : \mu_2^u : \mu_3^u \sim m_u : m_c : m_t \ , \qquad \tilde{\mu}_2^d : \tilde{\mu}_3^d, \mu_3 \sim m_s : m_b \ . \tag{3.50}$$

Then we can easily maintain a kind of top-bottom unification, corresponding to large $\tan\beta$, for $\mu_3^d, \tilde{\mu}_3 \simeq \frac{1}{\tan\beta}\, m_t$. The CKM matrix determines the remaining parameter $\tilde{\mu}_1$ to give

$$V_{us} = \theta_{\text{C}} \simeq \frac{\tilde{\mu}_1^d}{\tilde{\mu}_2^d} \ , \qquad V_{cb} \sim \frac{\tilde{\mu}_2^d}{\tilde{\mu}_3^d} = \frac{m_s}{m_b} \simeq 2 \times 10^{-2} \ , \tag{3.51}$$

$$V_{ub} \sim \frac{\tilde{\mu}_1^d}{\tilde{\mu}_3^d} = \theta_{\text{C}} \frac{m_s}{m_b} \simeq 4 \times 10^{-3} \ . \tag{3.52}$$

The matrix $\hat{V}$ takes a simple form in the limit $\mu_{1,2} \to 0$, where $\alpha = \beta$ and $\gamma = 0$, so that the mass matrix (3.47) has one zero eigenvalue,

$$\widehat{V} = \begin{pmatrix} \frac{-\widetilde{M}_2 \widetilde{M}_4}{\widetilde{M}_{12} \widetilde{M}_{14}} & \frac{\widetilde{M}_1 \left(\tilde{\mu}_3 \widetilde{M}_3 \widetilde{M}_4 - \mu_3 \left(\widetilde{M}_1^2 + \widetilde{M}_2^2 + \widetilde{M}_4^2\right)\right)}{\bar{\mu} \widetilde{M} \widetilde{M}_{12} \widetilde{M}_{14}} & \frac{-\tilde{\mu}_3}{\bar{\mu}} \frac{\widetilde{M}_1}{\widetilde{M}_{14}} \\ \frac{\widetilde{M}_1 \widetilde{M}_3}{\widetilde{M}_{12} \widetilde{M}_{23}} & \frac{\widetilde{M}_2 \left(\tilde{\mu}_3 \left(\widetilde{M}_1^2 + \widetilde{M}_2^2 + \widetilde{M}_3^2\right) - \mu_3 \widetilde{M}_3 \widetilde{M}_4\right)}{\bar{\mu} \widetilde{M} \widetilde{M}_{12} \widetilde{M}_{23}} & \frac{-\mu_3}{\bar{\mu}} \frac{\widetilde{M}_2}{\widetilde{M}_{23}} \\ \frac{\widetilde{M}_1 \widetilde{M}_2 \tilde{M}}{\widetilde{M}_{12} \widetilde{M}_{14} \widetilde{M}_{23}} & \frac{-\left(\tilde{\mu}_3 \widetilde{M}_1^2 \widetilde{M}_3 + \mu_3 \widetilde{M}_2^2 \widetilde{M}_4\right)}{\bar{\mu} \widetilde{M}_{12} \widetilde{M}_{14} \widetilde{M}_{23}} & -\frac{\tilde{\mu}_3}{\bar{\mu}} \frac{\widetilde{M}_4 \widetilde{M}_{23}}{\widetilde{M} \widetilde{M}_{14}} + \frac{\mu_3}{\bar{\mu}} \frac{\widetilde{M}_3 \widetilde{M}_{14}}{\widetilde{M} \widetilde{M}_{23}} \end{pmatrix} , \tag{3.53}$$

with

$$\bar{\mu}^2 = \tilde{\mu}_3^2 \left(1 - \frac{\widetilde{M}_4^2}{\widetilde{M}^2}\right) + \mu_3^2 \left(1 - \frac{\widetilde{M}_3^2}{\widetilde{M}^2}\right) - 2\,\mu_3\,\tilde{\mu}_3\,\frac{\widetilde{M}_3\widetilde{M}_4}{\widetilde{M}^2}\,. \tag{3.54}$$

This limit is actually of physical interest, since it automatically gives a lightest down-type state with

$$\frac{m_d}{m_s} \sim \frac{\mu_2^d\,\tilde{\mu}_1^d}{\tilde{\mu}_2^d\,\tilde{\mu}_2^d} \sim \;\theta_{\mathrm{C}}\,\frac{m_c m_b}{m_t m_s} \simeq \;0.03, \tag{3.55}$$

in agreement with analysis of weak decays. In this case, the remaining diagonalization simplifies to a 2×2 part for the second and third generation, parameterized by an angle θ_R. Thus the diagonalization matrix for right-handed down quarks reads ($\tilde{\mu}_3 \equiv \tilde{\mu}_3^d$)

$$U_R^d = \begin{pmatrix} -\frac{\widetilde{M}_2}{\widetilde{M}_{12}} & \frac{\widetilde{M}_1\left(\tilde{\mu}_3\widetilde{M}_3-\mu_3\widetilde{M}_4\right)}{\bar{\mu}\,\widetilde{M}\,\widetilde{M}_{12}} & -\frac{\widetilde{M}_1\left(\tilde{\mu}_3\widetilde{M}_4+\mu_3\widetilde{M}_3\right)}{\bar{\mu}\,\widetilde{M}^2} & \frac{\widetilde{M}_1}{\widetilde{M}} \\ \frac{\widetilde{M}_1}{\widetilde{M}_{12}} & \frac{\widetilde{M}_2\left(\tilde{\mu}_3\widetilde{M}_3-\mu_3\widetilde{M}_4\right)}{\bar{\mu}\,\widetilde{M}\,\widetilde{M}_{12}} & -\frac{\widetilde{M}_2\left(\tilde{\mu}_3\widetilde{M}_4+\mu_3\widetilde{M}_3\right)}{\bar{\mu}\,\widetilde{M}^2} & \frac{\widetilde{M}_2}{\widetilde{M}} \\ 0 & -\frac{\tilde{\mu}_3}{\bar{\mu}}\,\frac{\widetilde{M}_{12}}{\widetilde{M}} & -\frac{\tilde{\mu}_3\widetilde{M}_3\widetilde{M}_4-\mu_3\left(\widetilde{M}_1^2+\widetilde{M}_2^2+\widetilde{M}_4^2\right)}{\bar{\mu}\,\widetilde{M}^2} & \frac{\widetilde{M}_3}{\widetilde{M}} \\ 0 & \frac{\mu_3}{\bar{\mu}}\,\frac{\widetilde{M}_{12}}{\widetilde{M}} & \frac{\tilde{\mu}_3\left(\widetilde{M}_1^2+\widetilde{M}_2^2+\widetilde{M}_3^2\right)-\mu_3\widetilde{M}_3\widetilde{M}_4}{\bar{\mu}\,\widetilde{M}^2} & \frac{\tilde{M}_4}{\widetilde{M}} \end{pmatrix} \tag{3.56}$$

up to the two-dimensional mixing matrix for the second and third generation.

The charged lepton mass matrix m^e has the same structure as the down-quark mass matrix, but the large mixings are now between the left-handed states e_j. We will assume that the mixing matrix U_R^e coincides with U_L^d. Note that while for the large eigenvalue of m^e one can keep $m_\tau \simeq m_b$ by requiring $\tilde{\mu}_j^d \simeq \tilde{\mu}_j^e$, the muon and electron masses can be easily accommodated since the usual $\mathsf{SU(5)}$ relations do not hold for the second row of the mass matrices, $\mu_2^e \neq \mu_2^d$.

3.2.1 Proton Decay

The up-quarks are confined to one fixed point each, in particular the up quark is located on the Georgi-Glashow one. It is therefore clear that dimension-six proton decay can arise via the exchange of the $\mathsf{SU(5)}$ X and Y bosons as in the traditional four-dimensional picture. For this, we obtain from Eqn. (3.5)

$$\begin{aligned} \mathcal{L}_{\mathrm{eff}} = \frac{g_5^2}{M_V^2}\,\epsilon_{\alpha\beta\gamma}\Big[\,\overline{e^{\mathrm{C}}}{}'_k\left(U_R^{e\top}\right)_{kj}\overline{u^{\mathrm{C}}}_{\alpha\,i}\left(d'_{\beta\,m}\left(U_L^d\right)_{im}u_{\gamma\,j} - u_{\beta\,i}\left(U_L^d\right)_{jl}d'_{\gamma\,l}\right) \\ + \,\overline{d^{\mathrm{C}}}{}'_{\alpha\,l}\left(U_R^{d\top}\right)_{lk}\overline{u^{\mathrm{C}}}_{\beta\,i}\left(u_{\gamma\,i}\left(U_L^e\right)_{kj}e'_j - d'_{\gamma\,m}\left(U_L^d\right)_{im}\left(U_L^\nu\right)_{kj}\nu'_j\right)\Big]\;. \end{aligned} \tag{3.57}$$

There are though two very important differences in the six-dimensional case, as we will discuss in the following.

Effective Operator from Gauge Boson Exchange

First we have to take into account the presence of the Kaluza-Klein tower with masses

$$M_V^2(n,m) = \frac{(2n+1)^2}{R_5^2} + \frac{(2m)^2}{R_6^2} \ ; \tag{3.58}$$

the lowest possible mass is $M_V(0,0) = \frac{1}{R_5}$, as given by the $\mathsf{SU(5)}$ breaking parity. Moreover, if we define the 4D gauge coupling as the effective coupling of the zero modes, the KK modes interact more strongly by a factor of $\sqrt{2}$ due to their bulk normalization.

To obtain the low energy effective operator, we have to sum over the Kaluza-Klein modes. This sum is logarithmically divergent; since the theory is valid only below the scale M_*, where the theory becomes strongly coupled and also 6D gravity corrections are not negligible any more, we can hope that at that scale some mechanism arises to cancel the divergence. With this particular assumption, we can regulate the sum with the cut-off M_*, and obtain

$$\begin{aligned}
\sum_{n,m=0}^{\infty} \frac{1}{M_V^2(n,m)} &= \sum_{n,m=0}^{\infty} \frac{R_5^2}{(2n+1)^2 + \frac{R_5^2}{R_6^2}(2m)^2} \\
&= \sum_{n=0}^{\infty} \left\{ \frac{R_5^2}{2\,(2n+1)^2} + \frac{\pi R_5 R_6}{4\,(2n+1)} \left[1 + \frac{2\exp\left\{-(2n+1)\,\pi\frac{R_6}{R_5}\right\}}{1-\exp\left\{-(2n+1)\,\pi\frac{R_6}{R_5}\right\}} \right] \right\} \\
&= \frac{\pi}{8} R_5^2 \left[\frac{\pi}{2} + g\left(\frac{R_5}{R_6}\right) \right] + \frac{\pi}{8} R_5 R_6 \,\ln\left(M_* R_5\right),
\end{aligned} \tag{3.59}$$

where the function $g(x)$ can be computed numerically,

$$g(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\,x} \frac{\exp\left\{-(2n+1)\,\frac{\pi}{x}\right\}}{1-\exp\left\{-(2n+1)\,\frac{\pi}{x}\right\}} = \begin{cases} \frac{\pi}{2} & \text{for } x = \infty \\ 0.18 & \text{for } x = 1 \\ 0 & \text{for } x = 0 \end{cases} \tag{3.60}$$

In the limit $R_6 \to 0$, we regain the 5D result up to the factor of 2 contained in the coupling [95],

$$\sum_{n=0}^{\infty} \frac{R_5^2}{(2n+1)^2} = \frac{\pi^2 R_5^2}{8} \ ; \tag{3.61}$$

for $R_5 = R_6 = 1/M_c$, the expression simplifies to

$$\sum_{n,m=0}^{\infty} \frac{1}{M_V^2(n,m)} \simeq \frac{\pi}{8\,M_c^2} \left[\frac{\pi}{2} + g(1) + \ln\left(\frac{M_*}{M_c}\right) \right] \ . \tag{3.62}$$

Then the coupling in the 6D model is given by

$$G_G^{[\text{6D }\mathsf{SO(10)}]} = A_l\, A^{\text{SD}} \frac{\pi\, g_5^2}{4\,M_c^2} \left[\frac{\pi}{2} + g(1) + \ln\left(\frac{M_*}{M_c}\right) \right] \ . \tag{3.63}$$

$\frac{M_*}{M_c}$	Eqn. (3.62)	Eqn. (3.64)	Explicit sum
10	1.59	1.86	1.79
20	1.86	2.14	2.08
30	2.02	2.30	2.24
50	2.22	2.50	2.44

Table 3.6: Result of the KK summation in units $1/M_c^2$.

The result is not independent of the specific summation procedure. If we first sum over n, we obtain

$$\sum_{n,m=0}^{\infty} \frac{1}{M_V^2(n,m)} \simeq \frac{\pi}{8\,M_c^2}\left[\pi - \ln 2 + \ln\left(\frac{M_*}{M_c}\right)\right]\,, \tag{3.64}$$

where we neglect a small addend with a minus sign, similar to $g(1)$. Therefore we use the explicit sum up to the cutoff-scale M_* in the calculation. The results agree well, as shown in Table 3.6.

Another important difference is the non-universal coupling of the gauge bosons. In fact, due to the parities and the $\mathsf{SO(10)}$ breaking pattern, their wavefunctions must vanish on the fixed points with broken symmetry, O_{PS} and O_{fl}, and therefore no coupling arises via the kinetic term with the charm and top quark or to the brane states $d^c_{2,3}, l_{2,3}$. We have, in principle, couplings to the bulk states d^c_4, d_4 and l_4, l^c_4 but due to the embedding of the zero modes in full $\mathsf{SU(5)}$ multiplets together with massive KK modes, like $(d^c_4, L_4)\,, (d_4, L^c_4)\,, (D^c_4, l_4)\,, (D_4, l^c_4)$, the charged current interaction always mixes the light states with the heavy ones, hence it is irrelevant for the low energy proton decay [95].

Additional Operators

Apart from the kinetic term couplings, additional couplings can arise at any brane which contain the derivative along the extra dimensions of the locally vanishing gauge bosons. Such operators have first been discussed in 5D and are of the type [95]

$$\mathscr{L}_d = \frac{c}{M}\,\delta(y')\int d^2\theta\; d^2\bar\theta\;\overline{\Phi}_1\,\mathcal{D}\,e^{2V}\,\Phi_2 + \text{h.c.}\;. \tag{3.65}$$

Even though the X and Y components vanish on the brane, their ∂_5-derivatives appearing in $\left(\mathcal{D}\,e^{2V}\right)$ are non-zero. The prefactors include a dimensionless $\mathcal{O}\,(1)$ coefficient c and the fundamental scale $M \simeq M_*$.

The 6D generalization of these operators are

$$\mathscr{L}_d = \sum_{\text{fixed points}} \delta_j(z)\,\frac{c^j_{5/6}}{M_*}\int d^2\theta\; d^2\bar\theta\;\overline{\Phi}_1\,\mathcal{D}_{5/6}e^{2V}\,\Phi_2 + \text{h.c.} \tag{3.66}$$

where $c^j_{5/6}$ are unknown brane coefficients, $\mathcal{D}_{5/6} = \partial_{5/6} + iA_{5/6}$ is the covariant derivative in the extra dimensions and Φ_j are any two different local group representations, such to give a singlet combination with the generators of the broken direction. Note that if we extended these brane states to bulk fields, they would have opposite parity so that the operator in Eqn. (3.66) has positive parity.

These supersymmetric terms produce couplings with the flipped $\mathsf{SU(5)}$ leptoquark gauge bosons on the GG brane, whose derivatives do not vanish on the brane, and on the flipped $\mathsf{SU(5)}$ brane couplings containing the derivative of the gauge bosons X, Y. Due to these additional vertices, three different classes of operators can arise:

- operators coming from V'-exchange on the GG brane: these can produce additional contributions to the effective operator

$$\overline{d^{\mathrm{c}}_k}\, \overline{u^{\mathrm{c}}_1}\, Q_1 L_j \tag{3.67}$$

  with $j, k = 1, 4\,;$

- operators coming from V-exchange on the flipped $\mathsf{SU(5)}$ brane: these usually involve the charm quark instead of the up quark and are irrelevant for proton decay;

- interbrane operators from both the exchanges of V and V' gauge bosons: they generate usually operators of the type

$$\begin{aligned}
\overline{e^{\mathrm{c}}_j}\, \overline{u^{\mathrm{c}}_1}\, u_1\, d_k - \overline{d^{\mathrm{c}}_k}\, \overline{u^{\mathrm{c}}_1}\, Q_1 L_j \quad & j,k = 2,3,4 && V \text{ exchange}, && (3.68)\\
- \overline{d^{\mathrm{c}}_2}\, \overline{u^{\mathrm{c}}_1}\, d_2\, \nu_k \quad & k = 1,4 && V' \text{ exchange GG}-\text{fl}, && (3.69)\\
- \overline{d^{\mathrm{c}}_j}\, \overline{u^{\mathrm{c}}_1}\, d_l\, \nu_k \quad & j,l = 3,4\,;\; k = 1,4 && V' \text{ exchange GG}-\text{PS}. && (3.70)
\end{aligned}$$

  but they are usually suppressed compared to the kinetic term operators by a factor M_c/M_*.

Let us consider the KK summation contained in these additional operators with one or two derivative vertices, where we restrict ourselves to the case $R_5 = R_6 = 1/M_c$. Even if suppressed by M_*, these operators can be more dangerous than the usual ones, since the derivative enhances the divergence of the KK summation.

From the exchange of the V' bosons on the GG brane, we obtain the sum

$$\frac{2}{M_*^2} \sum_{n,m} \frac{\| c_5\,(2n+1) + c_6\,(2m+1)\,\|^2}{(2n+1)^2 + (2m+1)^2}\,, \tag{3.71}$$

which is quadratically divergent. We write the sum as an integration with cutoff M_* and obtain

$$\frac{\pi}{16\, M_c^2} \left(|c_5|^2 + |c_6|^2 + \frac{4}{\pi}\, \mathrm{Re}\,[c_5 c_6^*] \right) \left(1 - 2\,\frac{M_c^2}{M_*^2} \right)\,. \tag{3.72}$$

The exchange of gauge bosons between different branes is less dangerous because the propagator acquires a factor $(-1)^n$ accounting for the different parities of the derivative and kinetic term vertices. Therefore the summations are finite, and in general suppressed by a factor M_c/M_*. The summation for exchange of V bosons between the GG and the flipped SU(5) brane gives

$$\frac{2}{M_c M_*}\sum_{n,m}(-1)^n \frac{c_5\,(2n+1)+c_6\,(2m)}{(2n+1)^2+(2m)^2}\ . \tag{3.73}$$

This summation is only logarithmically divergent; the result oscillates depending on the cut-off (i. e. even or odd n_{max}). To estimate it, we split the sum into parts with $n=2k$ and $n=2k+1$; the first part of the sum then reads, up to the coefficient $\frac{2\,c_5}{M_c M_*}$,

$$\sum_{m,n}(-1)^n \frac{(2n+1)}{(2n+1)^2+(2m)^2} = \sum_{m,k}\frac{2\left[(4k+1)\,(4k+3)-(2m)^2\right]}{\left[(4k+1)^2+(2m)^2\right]\left[(4k+3)^2+(2m)^2\right]}. \tag{3.74}$$

Now we can go to the continuum. While the first part of the sum, Eqn. (3.74), can only be calculated numerically, we can perform the second part analytically and finally obtain

$$\frac{1}{2M_c M_*}\left[0.8\,c_5 + c_6\left(\frac{1}{8}-\frac{3}{8}\ln 3+\frac{1}{4}\ln\frac{M_*}{M_c}+\frac{1}{3}\frac{M_c}{M_*}+\frac{2}{3}\frac{M_c^3}{M_*^3}+\mathcal{O}\left(\frac{M_c^4}{M_*^4}\right)\right)\right], \tag{3.75}$$

which is suppressed compared with the previous pieces. The V' interbrane exchange gives a similar result,

$$\frac{1}{2M_c M_*}\left[0.4\,c_5 + c_6\left(\ln\left(\frac{M_*}{M_c}\right)+\frac{2-\sqrt{2}}{4}-\frac{3}{8}\ln(6)\right.\right.$$
$$\left.\left.-\frac{1}{4}\ln(2+\sqrt{2})+\frac{M_c}{6M_*}+\mathcal{O}\left(\frac{M_c^3}{M_*^3}\right)\right)\right]\ . \tag{3.76}$$

Regarding the $N=2$ scalar superpartners of the SU(5) gauge bosons, $V_{5,6}$, the effective brane terms in Eqn. (3.66) do not generate any coupling between them and the fermion fields. Their derivatives appear in the D-terms of the gauge bosons $N=1$ supersymmetry, as well, but those do not give rise to coupling with the brane fermions either.

Results

We start with the simplest case of $U_R^d = U_L^e$ and degenerate masses $\widetilde{M}$ in the limit $\tilde{\mu}_3^{d,e}=\mu_3$, which we denote case I. The mixing matrix for the right-handed down quarks, Eqn. (3.56), is simply given by

$$U_R^d = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = U_L^e\ , \tag{3.77}$$

decay channel	Branching Ratios [%]		
	6D SO(10)		$\mathsf{SU}(5) \times \mathsf{U}(1)_F$
	case I	case II	models A & B
$e^+\pi^0$	75	71	54
$\mu^+\pi^0$	4	5	<1
$\bar{\nu}\,\pi^+$	19	23	27
e^+K^0	1	1	<1
μ^+K^0	<1	<1	18
$\bar{\nu}\,K^+$	<1	<1	<1
$e^+\eta$	<1	<1	<1
$\mu^+\eta$	<1	<1	<1

Table 3.7: Resulting branching ratios and comparision with $\mathsf{SU}(5) \times \mathsf{U}(1)_F$.

thus the state d_{R1} has no strange-component. Let us first consider only the standard operators, where we obtain from Eqs. (3.5) and (3.57)

$$\begin{aligned}\mathscr{L}_{\text{eff}} = \frac{g_5^2}{(M_V^{\text{eff}})^2}\epsilon_{\alpha\beta\gamma}\Big[&2\,V_{ud}^2\,\overline{e^{\mathrm{c}}}\,\overline{u_\alpha^{\mathrm{c}}}\,d_\beta\,u_\gamma + \frac{1}{2}\,\overline{d_\alpha^{\mathrm{c}}}\,\overline{u_\beta^{\mathrm{c}}}\,u_\gamma\,e + 2\,V_{ud}V_{us}\,\overline{\mu^{\mathrm{c}}}\,\overline{u_\alpha^{\mathrm{c}}}\,d_\beta\,u_\gamma \\ &+ 2\,V_{ud}V_{us}\,\overline{e^{\mathrm{c}}}\,\overline{u_\alpha^{\mathrm{c}}}\,s_\beta\,u_\gamma + 2\,V_{us}^2\,\overline{\mu^{\mathrm{c}}}\,\overline{u_\alpha^{\mathrm{c}}}\,s_\beta\,u_\gamma \\ &- \sum_{j=1}^{3}\frac{1}{\sqrt{2}}\,(U_L^\nu)_{1j}\;\overline{u_\alpha^{\mathrm{c}}}\,\overline{d_\beta^{\mathrm{c}}}\left\{V_{ud}\,d_\gamma + V_{us}\,s_\gamma\right\}\nu_j\Big]\,, \end{aligned} \tag{3.78}$$

where the fermions are in their in mass eigenstates. From this equation, we can read off the coefficients of the various operators so that those in the decay width, $\widehat{C}_{ijkm}$, are given by

$$\begin{aligned} &\widehat{C}^2_{udue} = 4\,V_{ud}^4 + \frac{1}{4}\,, \quad \widehat{C}^2_{udu\mu} = 4\,V_{us}^2V_{ud}^2\,, \quad \widehat{C}^2_{udd\nu} = \frac{1}{2}\,V_{ud}^2\,, \\ &\widehat{C}^2_{usue} = 4\,V_{ud}^2V_{us}^2\,, \quad \widehat{C}^2_{usu\mu} = 4\,V_{us}^4\,, \quad \widehat{C}^2_{uds\nu} = \frac{1}{2}\,V_{us}^2\,, \quad \widehat{C}^2_{usd\nu} = 0\,, \end{aligned} \tag{3.79}$$

where we used $\sum_{j=1}^3 (U_L^\nu)_{1j}\,(U_L^\nu)^*_{1j} = 1$. The derivative operators are suppressed with respect to the standard operators; we take them inot account with $c_5 = c_6 = 1$. The branching ratios are listed in Table 3.7.

In general, the strange component in d_{R1} does not vanish but is smaller than the bottom component (see Eqn. (3.56)). Let us consider a second case, where $\tilde{\mu}_3^d = 2\mu_3$ and $\tilde{\mu}_3^e = 3\mu_3$, furthermore the heavy masses are chosen to be non-degenerate, $\frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1$ for the down quarks and $\frac{1}{2} : \frac{1}{\sqrt{2}} : 1 : \frac{1}{2}$ for the charged leptons. The branching ratios are listed in Table 3.7; the differences to the simplest case are small.

To compare the branching ratios with those in four dimensions, we consider the $\mathsf{SU}(5)$-model with a spontaneously broken global $\mathsf{U}(1)_F$ symmetry again [48]. We consider two

Q_F	$\mathbf{10}_3$	$\mathbf{10}_2$	$\mathbf{10}_1$	$\mathbf{5}^*_3$	$\mathbf{5}^*_2$	$\mathbf{5}^*_1$
model A	0	1	2	a	a	$a+1$
model B	0	3	5	0	0	2

Table 3.8: $\mathsf{U}(1)_F$ *charges of the* $\mathsf{SU}(5)$ *fields;* $a = 0, 1$.

realizations; the charges are given in Table 3.8. While the first one is already familiar from Section 2.4 (in the following denoted by model A), we additionally consider a second one, where the order parameter is [96]

$$\lambda \sim 0.35 \ . \tag{3.80}$$

In model A, the coefficients are given by

$$\widehat{C}^2_{udue} \sim \widehat{C}^2_{usu\mu} \sim 4 \ , \quad \widehat{C}^2_{udu\mu} \sim \widehat{C}^2_{usue} \sim \widehat{C}^2_{usd\nu} \sim 4\,\epsilon^2 \ , \quad \widehat{C}^2_{udd\nu} \sim \widehat{C}^2_{uds\nu} \sim \frac{1}{2} \ ; \tag{3.81}$$

In model B, ϵ is replaced by λ^2. Since $\lambda^2 > \epsilon$, the branching ratios of case A and B slightly differ; the differences, however, are smaller than the predictivity of these models. The correspondent branching ratios are listed in Table 3.7.

The most striking difference is the decay channel $p \to \mu^+ K^0$. It is suppressed by two orders of magnitude in the 6D model with respect to 4D due to the absence of the second and third generation in the six-dimensional model and the small (12)-component in U_R^d. Therefore it is interesting to determine an upper limit for this channel in the 6D model. We vary the heavy masses $\widetilde{M}_j/\widetilde{M} = 0.1, \ldots, 1$ and $\tilde{\mu}_3^{d,e}/\mu_3 = 1, \ldots, 5$ and find

$$\left. \frac{\Gamma(p \to \mu^+ K^0)}{\Gamma(p \to e^+ \pi^0)} \right|_{\max} = 5\,\% \ , \tag{3.82}$$

one order of magnitude smaller than in the four-dimensional case. This significant difference can make it possible to distinguish between conventional and orbifold GUTs.

The position of the lightest quark generation is crucial for the result. If the up-quark was located on the Pati-Salam brane, the dimension-six operators coming from the kinetic terms would be completely absent since both the V gauge bosons of $\mathsf{SU}(5)$ and V' of flipped $\mathsf{SU}(5)$ vanish there. This gives us a mean to avoid the dominant dimension-six operator completely, leaving unfortunately the undetermined derivative couplings.

Finally, a limit on the compactification scale can be derived from the decay width of the dominant channel $p \to e^+ \pi^0$. Since the corrections from the derivative operators are negligible, we can give the analytic expression

$$\Gamma \simeq \left(\mathcal{K}^{\pi^0}_{\mathrm{had}}\right)^2 \frac{\pi^2}{16\,M_c^4} \left(\frac{\pi}{2} + \ln\left(\frac{M_*}{M_c}\right)\right)^2 \left[4V_{ud}^4 + \frac{\widetilde{M}_2^{d\,2}}{\widetilde{M}_1^{d\,2} + \widetilde{M}_2^{d\,2}} \frac{\widetilde{M}_2^{e\,2}}{\widetilde{M}_1^{e\,2} + \widetilde{M}_2^{e\,2}}\right] \tag{3.83}$$

With $M_* = 10^{17}\,\mathrm{GeV}$ and $\widetilde{M}_{1,2}^{d,e} = \mathcal{O}\,(1)$, the limit $\tau \geq 5.4 \times 10^{33}$ yields $M_c \geq 9 \times 10^{15}\,\mathrm{GeV}$, roughly half of the 4D GUT scale.

Chapter 4

Discussion

The instability of protons is a crucial prediction of Grand Unified Theories, and the long-lasting search for proton decay has put a strong constraint on such models. In this work, we have studied proton decay in various supersymmetric GUT models.

First, we considered minimal $\mathsf{SU(5)}$ in four dimensions which was claimed to be excluded due to the SuperKamiokande bound on proton decay. We pointed out that the minimal model is inconsistent due to failure of Yukawa unification and emphasized the strong dependence of the decay amplitude on flavor mixing. A consistent $\mathsf{SU(5)}$ model requires additional interactions which account for the difference of down quark and charged lepton masses. Such interactions are conveniently parameterized by higher dimensional operators. We have shown that these operators can reduce the proton decay rate by several orders of magnitude and make it consistent with the experimental upper bound.

We extended the operator analysis to the next possible GUT group, $\mathsf{SO(10)}$, which involves right-handed neutrinos. We found that the additional symmetries do not restrict the higher-dimensional operators, hence do not offer a mechanism which would naturally lead to the required relations among Yukawa couplings.

We should stress here that we could introduce additional fields that couple differently to down quarks and charged leptons as well, as it is done in the missing-partner mechanism (see Appendix B). Then the leptoquark mass may have a higher value. Higher dimensional operators, however, are naturally expected as a result of interactions at the Planck scale.

Thus on the one hand, proton decay does not rule out consistent supersymmetric $\mathsf{SU(5)}$ models, where the decay is dominated by dimension-five operators. On the other hand, the Yukawa couplings have to fulfill special relations among themselves to be consistent with the experimental limit. There are even more uncertainties related to sfermion mixing [97], which is neglected in most analyses, although there are restrictions from flavor changing neutral currents [98].

In this situation, orbifold GUTs provide an elegant solution, as the dimension-five operators are absent, furthermore they solve the doublet-triplet splitting problem and, in general, do not require Yukawa unification of the first two generations. We have analyzed proton decay in a six-dimensional $\mathsf{SO(10)}$ model, which is compactified on a torus with three $\mathbb{Z}_2$ symmetries. In four dimensions, we obtain an effective $N = 1$ supersymmetric theory with the extended standard model gauge group as the intersection of the Georgi-Glashow and Pati-Salam subgroups of $\mathsf{SO(10)}$. The compactification scale is of the same order as the four-dimensional GUT scale, hence the successful gauge unification in four dimensions remains.

Since the up-quarks of the first generation are confined on the $\mathsf{SU(5)}$ brane, the $\mathsf{SU(5)}$ analysis gives the main contribution to proton decay. Nevertheless, we have demonstrated that due to the different flavor mixing, the branching ratios can differ significantly from those in four dimensions, which could make it possible to distinguish orbifold and four dimensional GUTs, if proton decay is observed in the future.

As already mentioned, orbifold GUTs are interesting in view of a fundamental theory which combines all interactions we observe in nature. At the moment, it seems to be most promising to add extra dimensions and in particular, orbifold constructions are well suited to work out the effective 4D theory. Since there is no compelling reason for the extra dimensions to be of the same size, it is tempting to study orbifold models in more than four dimensions. Higher-dimensional theories, however, are non-renormalizable and require an explicit cut-off in order to regularize all the divergences. It is essential to obtain an UV completion which introduces new physics at the cut-off scale. In our specific model, the cut-off scale coincides with the 6D Planck scale, so we expect gravitational effects to be important.

Recently a five-dimensional orbifold model based on G_{PS} was constructed by compactifying the heterotic string on a particular orbifold, which can easily be extended to a six-dimensional model based on $\mathsf{SO(10)}$ [99]. The resulting model is similar to the one considered in this work, which motivates to study these models in more detail.

There are, of course, many open questions, both in the general setup and the specific model. In particular, it would be desirable to understand the additional operators. They could even be dependent on the generation because of some flavor symmetry. Moreover, we have to allow complex couplings and study $\mathcal{CP}$ violation.

Finally, the most important point regards the observation of proton decay. The experimental setup is already impressive and there are several proposals for future experiments, which aim to reach at least $\mathcal{O}\,(10^{35}\,\text{years})$ [100]. Hence there is a well-justified hope that such a decay will be observed in the future, though there might be only a few events. The dominance of the channel $p \to \bar{\nu} K^+$ would strongly point at dimension-five operators, whereas $p \to e^+ \pi^0$ refers to dimension-six decay. In the latter case, the absence of the process $p \to \mu^+ K^0$ can probably only disfavor the naïve GUT model in four dimensions but would nevertheless support the idea that the different generations are spatially separated.

The verification of proton decay, the proof that protons are unstable would be a strong push for physics beyond the standard model and attest us to be potentially on the right path to a fundamental theory.

Appendix A

Spinors and supersymmetry

Weyl, Dirac and Majorana spinors

The γ matrices are defined by the Clifford algebra $[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}$, the chiral (Weyl) representation reads

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \,, \quad \sigma^\mu = (1, \sigma^i)\,,\ \bar{\sigma}^\mu = (1, -\sigma^i)\,, \tag{A.1}$$

where σ^i are the Pauli matrices.

Chiral fermions are described by two-component, complex, anti-commuting objects, Weyl spinors; ξ_α ($\alpha = 1, 2$) denote a left-handed, $\bar{\chi}^{\dot{\beta}}$ right-handed particles. Under rotations $\vec{\theta}$ and Lorentz boosts $\vec{\beta}$, they transform as

$$\xi \to (1 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\beta}\cdot\vec{\sigma})\xi\,, \quad \bar{\chi} \to (1 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\beta}\cdot\vec{\sigma})\bar{\chi}\,. \tag{A.2}$$

For a given spinor ξ in representation R, one can construct the right-handed adjoint $\bar{\xi}$ in R^*,

$$\bar{\xi}_{\dot{\alpha}} \equiv (\xi_\alpha)^* \,. \tag{A.3}$$

In the standard model, the fermions are chiral, hence $R \neq R^*$. One can show that the scalar products

$$\xi^\alpha\,\xi_\alpha = \epsilon_{\alpha\beta}\,\xi_\beta\,\xi_\alpha\,, \quad \xi^\alpha\,\chi_\alpha\,, \quad \bar{\chi}_{\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}\,, \quad \bar{\chi}_{\dot{\beta}}\,\bar{\xi}^{\dot{\beta}} \tag{A.4}$$

are Lorentz scalar whereas

$$\xi^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\chi}^{\dot{\beta}} \tag{A.5}$$

is a Lorentz vector.

Now one can put two Weyl spinors together to get a Dirac spinor,

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}\,, \quad \overline{\Psi}_{\mathrm{D}} = (\chi^\alpha, \bar{\xi}_{\dot{\alpha}})\,, \quad \Psi_{\mathrm{D}}^C = \begin{pmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} = \mathcal{C}\overline{\Psi}_{\mathrm{D}}^T\,. \tag{A.6}$$

The Weyl spinors can be projected out by the chirality operators $\mathcal{P}_{\mathrm{L},R} = \frac{1}{2}(1 \mp \gamma_5)$,

$$\Psi_{\mathrm{L}} = \mathcal{P}_{\mathrm{L}}\Psi_{\mathrm{D}} = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} , \quad \Psi_{\mathrm{R}} = \mathcal{P}_{\mathrm{L}}\Psi_{\mathrm{D}} = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} . \tag{A.7}$$

A Majorana-Spinor is defined via $\Psi_{\mathrm{M}} = \Psi_{\mathrm{M}}^{\mathrm{C}}$. In four-component notation, it is given by one Weyl spinor ξ and its adjoint $\bar{\xi}$,

$$\Psi_{\mathrm{M}} = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} = \Psi_{\mathrm{M}}^{\mathrm{C}} . \tag{A.8}$$

Finally, a Dirac spinor Ψ_{D} can be written as a sum of two Majorana spinors,

$$\Psi_{\mathrm{D}} = \Psi_{\mathrm{M1}} + i\Psi_{\mathrm{M2}} . \tag{A.9}$$

Grassmann numbers

In a n-dimensional Grassmann algebra, the generators η_j obey

$$\{\eta_i \, , \eta_j\} = 0 \; , \quad i,j = 1,2\ldots n \; . \tag{A.10}$$

The Berezin integral is defined by

$$\int d\eta = 0 \; , \quad \int d\eta \, \eta = 1 \tag{A.11}$$

for each η. Since a function of any one anticommuting η is always of the form

$$f(\eta) = f_0 + \eta f_1 \; , \tag{A.12}$$

these definitions are sufficient to define a general $\int d\eta \; f(\eta)$. Assuming that η is not a multiple of ζ, the translational invariance of the integral follows,

$$\int d\eta \, f(\eta + \zeta) = \int d\eta \; (f_0 + \eta f_1 + \zeta f_1) = f_1 = \int d\eta \, f(\eta) \; . \tag{A.13}$$

Formally, differentiation and integration are the same,

$$\int d\eta \, f(\eta) = \frac{\partial}{\partial \eta} f(\eta) \equiv \partial_\eta \, f(\eta) \; . \tag{A.14}$$

Supersymmetry algebra

Supersymmetry transforms bosons into fermions and vice versa, hence the generators transform in a spin-$\frac{1}{2}$ representation of the Lorentz group. The simplest realization is the pair of a left-handed Weyl spinor Q_α and its Hermitean adjoint $\bar{Q}_{\dot{\beta}}$,

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} &= 0 \; , \\ \left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} \, P_\mu \; , \end{aligned} \tag{A.15}$$

where P_μ is the energy-momentum operator. Now, the graded Lie algebra (superalgebra) is given by

$$\begin{aligned} [Q_\alpha, P_\mu] = \left[\bar{Q}_{\dot\beta}, P_\mu\right] &= 0 \\ [Q_\alpha, M_{\mu\nu}] &= \frac{1}{2}\left(\sigma_{\mu\nu}\right)_\alpha^\beta Q_\beta \\ \left[\bar{Q}_{\dot\beta}, M_{\mu\nu}\right] &= \frac{1}{2}\left(\bar\sigma_{\mu\nu}\right)_{\dot\beta}^{\dot\alpha} \bar{Q}_{\dot\alpha} \ , \end{aligned} \tag{A.16}$$

where $\sigma_{\mu\nu} = \frac{1}{4}\left(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu\right)$, $\bar\sigma_{\mu\nu} = \frac{1}{4}\left(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu\right)$. and $M_{\mu\nu}$ denote the Lorentz generators. Using two-component Grassmann numbers θ, $\bar\theta$, we can rewrite Eqn. (A.15) in terms of commutators,

$$\left[\theta^\alpha Q_\alpha, \bar\theta^{\dot\beta}\bar{Q}_{\dot\beta}\right] = 2\,\theta^\alpha \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,P_\mu \ . \tag{A.17}$$

The covariant derivatives are defined by

$$\begin{aligned} &\mathcal{D}_\mu = \partial_\mu \ , \\ &\mathcal{D}_\alpha = \partial_\alpha - i\,\left(\sigma^\mu\,\overline{\theta}\right)_\alpha \partial_\mu \ , \\ &\overline{\mathcal{D}}_{\dot\alpha} = -\,\partial_{\dot\alpha} + i\,\left(\theta\,\sigma^\mu\right)_{\dot\alpha} \partial_\mu \ . \end{aligned} \tag{A.18}$$

R-symmetry

R-symmetries are those not commuting with supersymmetry. In case of $\mathbb{Z}_N$ R-symmetries, the Grassmann coordinate and chiral superfields transform like

$$\theta \to e^{2i\pi\alpha/N}\theta \ , \quad \phi \to e^{2i\pi R\alpha/N}\phi \ . \tag{A.19}$$

Choosing $N = 2$, we obtain

$$\theta \to e^{i\alpha}\theta \ , \quad \phi \to e^{i\pi R\alpha}\phi \ , \tag{A.20}$$

so that the fields possess the R-charges

$$R\,(\theta) = 1 \ , \quad R\,(\phi) = R \ . \tag{A.21}$$

The theory is invariant if

$$R\,(W) = 2 \ . \tag{A.22}$$

To allow the Yukawa interactions (1.22), we can choose

$$R\,(Q, L, U^{\mathrm{c}}, D^{\mathrm{c}}, E^{\mathrm{c}}) = \frac{1}{2} \ , \quad R\,(H_u, H_d) = 1 \ , \tag{A.23}$$

so only the operators $H\bar{H}$, $QQQL$ and $U^{\mathrm{c}}U^{\mathrm{c}}D^{\mathrm{c}}E^{\mathrm{c}}$ are allowed. But since this symmetry forbids gaugino mass terms, it has to be broken. This breaking generates axions which are experimentally excluded. Thus another choice seems to be more promising,

$$R\left(Q, L, U^{\mathrm{c}}, D^{\mathrm{c}}, E^{\mathrm{c}}\right) = 1 \ , \quad R\left(H_u, H_d\right) = 0 \ , \tag{A.24}$$

which forbids not only the dangerous dimension five operators but also the μ-term, $\mu H_u H_d$. Therefore even in this case, R-symmetry has to be broken which leads to R-parity.

Appendix B

Addendum to GUT groups

In this chapter, we present details of $\mathsf{SU}(5)$ and $\mathsf{SO}(10)$. We give the explicit form of the $\mathsf{SU}(5)$ generators and study the Yukawa sector in order to calculate the Wilson coefficients for proton decay. Then we present the decompositions of $\mathsf{SO}(10)$ fields in $\mathsf{SU}(5)$. For more details, see Ref. [72].

The generators of $\mathsf{SU}(5)$

$\mathsf{SU}(5)$ is a Lie group of rank 4, with 24 generators. Therefore we have 24 gauge fields, the Standard Model gauge bosons plus 12 additional gauge bosons,

$$A_\mu\left(\mathbf{24}\right) = \frac{1}{2}\lambda^a A_\mu^a\ , \quad a = 1, \ldots, 24\ , \tag{B.1}$$

where the λ^a are given by:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \lambda_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \lambda_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

$$\lambda_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \lambda_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

The $\mathsf{SU(5)}$ Yukawa sector and specific bases

Minimal model

In the minimal theory, the $\mathsf{SU(5)}$ Yukawa sector of the superpotential reads

$$W_Y = \tfrac{1}{4}\, Y_1^{ij}\, \mathsf{10}_i\, \mathsf{10}_j\, H(\mathsf{5}) + \sqrt{2}\, Y_2^{ij}\, \mathsf{10}_i\, \mathsf{5}_j^*\, \overline{H}(\mathsf{5}^*)\ .$$

From the superpotential one can immediately conclude that Y_1 is a symmetric complex matrix. With $Y_u = Y_1$ and $Y_d = Y_2$, the Yukawa matrices have the form

$$Y_u = U_u\, \mathcal{D}_u\, P\, U_u^\top\ , \qquad Y_d = U_d\, \mathcal{D}_d\, U_{d\,R}^\dagger\ .$$

Here, P is an additional phase matrix with $\det P = 1$ which is usually parameterized as

$$P = e^{i\varphi}\ \mathrm{diag}(e^{i\phi_{13}}, e^{i\phi_{23}}, 1)\ . \tag{B.2}$$

These phases cannot be absorbed by field redefinitions [87]. The CKM matrix is then defined as

$$V_{\mathrm{CKM}} = U_u^\dagger\, U_d\ . \tag{B.3}$$

The most general weak basis transformation which leaves the interactions invariant is:

$$\mathsf{10}_i \to \mathsf{10}_i' = \mathcal{U}_{ij}\mathsf{10}_j\ , \qquad \mathsf{5}_i^* \to \mathsf{5}_i^{*\prime} = \mathcal{V}_{ij}\mathsf{5}_j^*\ .$$

Then the Yukawa matrices transform like

$$Y_u \to \mathcal{U}^\top Y_u \mathcal{U} \,, \qquad\qquad Y_d \to \mathcal{U}^\top Y_d \mathcal{V} \,.$$

The superpotential of the $\mathsf{SU(5)}$ Yukawa interactions expressed in terms of SM superfields is given by Eqn. (2.15). Transforming the singlets fields by $\Phi \to \mathcal{W}_\Phi \Phi$, the superpotential transforms like

$$\begin{aligned} W_Y = {} & Q^\top (\mathcal{U}^\top Y_u \mathcal{U} \mathcal{W}_u)\, u^{\mathrm{c}} H_f + Q^\top (\mathcal{U}^\top Y_d \mathcal{V} \mathcal{W}_d)\, d^{\mathrm{c}} \overline{H}_f + e^{\mathrm{c}\top} (\mathcal{W}_e^\top \mathcal{U}^\top Y_e \mathcal{V})\, L \overline{H}_f \\ & + \tfrac{1}{2} Q^\top (\mathcal{U}^\top Y_{qq} \mathcal{U})\, Q H_{\mathrm{C}} + Q^\top (\mathcal{U}^\top Y_{ql} \mathcal{V})\, L \overline{H}_{\mathrm{C}} \\ & + u^{\mathrm{c}\top} (\mathcal{W}_u^\top \mathcal{U}^\top Y_{ue} \mathcal{U} \mathcal{W}_e)\, e^{\mathrm{c}} H_{\mathrm{C}} + u^{\mathrm{c}\top} (\mathcal{W}_u^\top \mathcal{U}^\top Y_{ud} \mathcal{V} \mathcal{W}_d)\, d^{\mathrm{c}} \overline{H}_{\mathrm{C}} \,. \end{aligned}$$

There are two possible physical bases now, namely diagonal up quark and diagonal down quark matrices, which can be realized by a suitable choice of all transformation matrices. With $Y_{qq} = Y_{ue} = Y_u$ and $Y_{ql} = Y_{ud} = Y_d$, the Yukawa interactions read

$$\begin{aligned} W_Y = {} & Q^\top \mathcal{D}_u\, u^{\mathrm{c}} H_f + Q^\top (V_{\text{CKM}} \mathcal{D}_d)\, d^{\mathrm{c}} \overline{H}_f + e^{\mathrm{c}\top} \mathcal{D}_e\, L \overline{H}_f \\ & + \tfrac{1}{2} Q^\top (\mathcal{D}_u P)\, Q H_{\mathrm{C}} + Q^\top (V_{\text{CKM}} \mathcal{D}_d)\, L \overline{H}_{\mathrm{C}} \\ & + u^{\mathrm{c}\top} (\mathcal{D}_u V^*_{\text{CKM}})\, e^{\mathrm{c}} H_{\mathrm{C}} + u^{\mathrm{c}\top} (P^* V_{\text{CKM}} \mathcal{D}_d)\, d^{\mathrm{c}} \overline{H}_{\mathrm{C}} \end{aligned} \tag{B.4}$$

in the first and

$$\begin{aligned} W_Y = {} & Q^\top (V^\dagger_{\text{CKM}} \mathcal{D}_u)\, u^{\mathrm{c}} H_f + Q^\top \mathcal{D}_d\, d^{\mathrm{c}} \overline{H}_f + e^{\mathrm{c}\top} \mathcal{D}_e\, L \overline{H}_f \\ & + \tfrac{1}{2} Q^\top (V^\dagger_{\text{CKM}} \mathcal{D}_u P V^*_{\text{CKM}})\, Q H_{\mathrm{C}} + Q^\top \mathcal{D}_d\, L \overline{H}_{\mathrm{C}} \\ & + u^{\mathrm{c}\top} (\mathcal{D}_u V^*_{\text{CKM}})\, e^{\mathrm{c}} H_{\mathrm{C}} + u^{\mathrm{c}\top} (P^* V_{\text{CKM}} \mathcal{D}_d)\, d^{\mathrm{c}} \overline{H}_{\mathrm{C}} \end{aligned} \tag{B.5}$$

in the second basis. The former is used in Ref. [56], the latter in Ref. [12].

In principle, these formulae are only valid for unbroken supersymmetry where one can use the same transformations for the fermions and their supersymmetric partners. Broken supersymmetry gives small corrections to these transformations [55].

Consistent model

Expanding the superpotential by higher dimensional operators, the identities (2.16) and (2.17), $Y_u = Y_{qq} = Y_{ue} = Y_1$ and $Y_d = Y_{ql} = Y_{ud} = Y_2$, at M_{GUT} no longer hold. Instead, one can derive the following relations between the matrices:

$$\begin{aligned} Y_{qq} - Y_{ue} &= \frac{5}{2} \frac{\sigma}{M_{\text{Pl}}} f_2^A \,, \\ Y_u - Y_{qq} &= 5 \frac{\sigma}{M_{\text{Pl}}} f_1^S + \frac{5}{4} \frac{\sigma}{M_{\text{Pl}}} \left(f_2^S + f_2^A \right), \\ Y_u - Y_{ue} &= 5 \frac{\sigma}{M_{\text{Pl}}} f_1^S + \frac{5}{4} \frac{\sigma}{M_{\text{Pl}}} \left(f_2^S + 3 f_2^A \right), \end{aligned} \tag{B.6}$$

$$
\begin{aligned}
Y_d - Y_e = Y_{ud} - Y_{ql} &= 5\frac{\sigma}{M_{\mathrm{Pl}}} h_2 \ , \\
\tfrac{3}{5} Y_d + \tfrac{2}{5} Y_e &= Y_2 - 3\frac{\sigma}{M_{\mathrm{Pl}}} h_1 \ , \\
Y_{ql} - Y_e = Y_{ud} - Y_d &= 5\frac{\sigma}{M_{\mathrm{Pl}}} h_1 \ .
\end{aligned}
\tag{B.7}
$$

The antisymmetric part of f_2 is determined by the difference between Y_{qq} and Y_{ue}, then only symmetric terms of f_1 and f_2 remain.

Renormalization group equations

Yukawa couplings

The one-loop renormalization group equations, in the $\overline{\mathrm{MS}}$ scheme, can be written for general Yukawa matrices [101]

$$
\begin{aligned}
16\pi^2 \frac{d\,Y_u}{dt} &= \left[T_u - G_u + \frac{3}{2}\left(b\,Y_u\,Y_u^\dagger + c\,Y_d\,Y_d^\dagger \right) \right] Y_u \ , \\
16\pi^2 \frac{d\,Y_d}{dt} &= \left[T_d - G_d + \frac{3}{2}\left(b\,Y_d\,Y_d^\dagger + c\,Y_u\,Y_u^\dagger \right) \right] Y_d \ , \\
16\pi^2 \frac{d\,Y_e}{dt} &= Y_e \left(T_e - G_e + \frac{3}{2} b\,Y_e^\dagger\,Y_e \right) \ ,
\end{aligned}
\tag{B.8}
$$

where $t = \log \mu / M_Z$ and the traces T_u, T_d, T_e are given by

$$
\begin{aligned}
T_u &= \mathrm{tr}\left(3\,Y_u\,Y_u^\dagger + 3\,a\,Y_d\,Y_d^\dagger + a\,Y_e^\dagger\,Y_e\right) \ , \\
T_d = T_e &= \mathrm{tr}\left(3\,a\,Y_u\,Y_u^\dagger + 3\,Y_d\,Y_d^\dagger + \,Y_e^\dagger\,Y_e\right) \ .
\end{aligned}
\tag{B.9}
$$

The constants a, b and c as well as the functions G_u, G_d and G_e, are summarized in the Table B.1.

The equations for the Wilson coefficients read [12]

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} C_{5L}^{ijkl} &= \left(-8\,g_3^2 - 6\,g_2^2 - \tfrac{2}{3} g_1^2\right) C_{5L}^{ijkl} + C_{5L}^{mjkl}\left(Y_d Y_d^\dagger + Y_u Y_u^\dagger\right)^i_{\ m} \\
&\quad + C_{5L}^{imkl}\left(Y_e^\dagger Y_e\right)^j_{\ m} + C_{5L}^{ijml}\left(Y_d Y_d^\dagger + Y_u Y_u^\dagger\right)^k_{\ m} + C_{5L}^{ijkm}\left(Y_d Y_d^\dagger + Y_u Y_u^\dagger\right)^l_{\ m} ,
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} C_{5R}^{ijkl} &= \left(-8\,g_3^2 - 4\,g_1^2\right) C_{5R}^{ijkl} + C_{5R}^{mjkl}\left(2\,Y_u^\dagger Y_u\right)^i_{\ m} \\
&\quad + C_{5R}^{imkl}\left(2\,Y_d^\dagger Y_d\right)^j_{\ m} + C_{5R}^{ijml}\left(2\,Y_e Y_e^\dagger\right)^k_{\ m} + C_{5R}^{ijkm}\left(2\,Y_u^\dagger Y_u\right)^l_{\ m} .
\end{aligned}
\tag{B.11}
$$

Gauge couplings, Constraint on M_{H_c}

In minimal supersymmetric $\mathsf{SU(5)}$, the RGE at one-loop level are given by [56]:

$$
\alpha_1^{-1}(M_Z) = \alpha_5^{-1}(\Lambda) + \frac{1}{2\pi}\left[\left(-\frac{2}{3}n - \frac{1}{2}\right)\log\frac{M_S}{M_Z} + \left(2n + \frac{3}{5}\right)\log\frac{\Lambda}{M_Z} - 10\log\frac{\Lambda}{M_V} + \frac{2}{5}\log\frac{\Lambda}{M_{H_c}}\right],
$$

	SM	MSSM
(a, b, c)	$(1,\, 1,\, -\frac{3}{2})$	$(0,\, 2,\, 1)$
G_u	$\frac{17}{20}\, g_1^2 + \frac{9}{4}\, g_2^2 + 8\, g_3^2$	$\frac{13}{15}\, g_1^2 + 3\, g_2^2 + \frac{16}{3}\, g_3^2$
G_d	$\frac{1}{4}\, g_1^2 + \frac{9}{4}\, g_2^2 + 8\, g_3^2$	$\frac{7}{15}\, g_1^2 + 3\, g_2^2 + \frac{16}{3}\, g_3^2$
G_e	$\frac{9}{4}\, g_1^2 + \frac{9}{4}\, g_2^2$	$\frac{9}{5}\, g_1^2 + 3\, g_2^2$
b_1	$\frac{4}{3}n + \frac{1}{10}m$	$2\,n + \frac{3}{10}m$
b_2	$\frac{4}{3}n + \frac{1}{6}m - \frac{22}{3}$	$2\,n + \frac{1}{2}m - 6$
b_3	$\frac{4}{3}n - 11$	$2\,n - 9$

Table B.1: Coefficients to (B.8) and (B.9). The running gauge coupling constant at 1-loop is given by $g_i^2(t) = g_i^2(0)/\left(1 - \frac{b_i}{8\pi^2}\, g_i^2(0)\, t\right)$. The integers n and m stand for number of generations and Higgs doublets, respectively.

$$\alpha_2^{-1}(M_Z) = \alpha_5^{-1}(\Lambda) + \frac{1}{2\pi}\left[\left(-\frac{2}{3}n - \frac{13}{6}\right)\log\frac{M_S}{M_Z} + (2n-5)\log\frac{\Lambda}{M_Z} - 6\log\frac{\Lambda}{M_V} + 2\log\frac{\Lambda}{M_3}\right],$$

$$\alpha_3^{-1}(M_Z) = \alpha_5^{-1}(\Lambda) + \frac{1}{2\pi}\left[\left(-\frac{2}{3}n - 2\right)\log\frac{M_S}{M_Z} + (2n-9)\log\frac{\Lambda}{M_Z} - 4\log\frac{\Lambda}{M_V} + 3\log\frac{\Lambda}{M_8} + \log\frac{\Lambda}{M_{H_c}}\right],$$

where α_5 is the $\mathsf{SU(5)}$ coupling, n the number of generations and M_S the SUSY breaking scale. The combinations

$$\left(-\alpha_1^{-1} + 3\alpha_2^{-1} - 2\alpha_3^{-1}\right)(M_Z) = \frac{1}{2\pi}\left[-2\,\log\frac{M_S}{M_Z} + \frac{12}{5}\log\left(\frac{M_{H_c}}{M_Z}\left(\frac{M_3}{M_8}\right)^{\frac{5}{2}}\right)\right]\,, \quad \text{(B.12)}$$

$$\left(5\alpha_1^{-1} - 3\alpha_2^{-1} - 2\alpha_3^{-1}\right)(M_Z) = \frac{1}{2\pi}\left[8\,\log\frac{M_S}{M_Z} + 12\,\log\frac{M_V^2\sqrt{M_3 M_8}}{M_Z^3}\right] \quad \text{(B.13)}$$

allow to derive constraints on the products $M_{H_c}(M_3/M_8)^{5/2}$ and $M_V^2\sqrt{M_3 M_8} \equiv M_{\rm GUT}^3$. At two-loop level, the relation cannot be expressed analytically any more.

In the minimal model, $M_8 = M_3 = M_\Sigma$ (1.38) and the expressions simplify to M_{H_c} and $M_{\rm GUT}^3 = M_V^2 M_\Sigma$. The analysis has been done for a long time already. The most recent calculation leads to the constraint [13],

$$3.5 \times 10^{14}\,\text{GeV} \le M_{H_c} \le 3.6 \times 10^{15}\,\text{GeV} \quad (90\%\ \text{C.L.})\,, \quad \text{(B.14)}$$

with M_{H_c} well below the GUT scale and

$$1.7 \times 10^{16}\,\text{GeV} \le M_{\rm GUT} \equiv (M_V^2 M_\Sigma)^{1/3} \le 2.0 \times 10^{16}\,\text{GeV} \quad (90\%\ \text{C.L.})\,. \quad \text{(B.15)}$$

Even with minimal matter content, we can simply avoid the constraint, if we allow $M_8 \neq M_3$ as is the case with additional higher-dimensional operators. We easily estimate that $M_3 = 2\, M_8$ is enough to raise the limit to $M_{\rm GUT}$.

Furthermore, the constraints are not valid in more realistic models. In the missing-partner model, one employs a $\mathsf{75}$-dimensional representation to break $\mathsf{SU(5)}$ instead of the adjoint $\mathsf{24}$ and further introduces $\mathsf{50} \oplus \mathsf{50}^*$ which contains $\mathsf{3} \oplus \mathsf{3}^*$ and mix with the color-triplet Higgs to make them massive [102]. The doublets remain massless because of missing partners. The large representations raise the size of the GUT-scale threshold corrections, which shift the value of $M_{H_{\rm C}}$ to larger values. Thus the dimension-five operators are much more suppressed. On the other hand, the model becomes non-perturbative well below the Planck scale due to these large representations and needs to be complicated. Furthermore, the other components of the $\mathsf{50} \oplus \mathsf{50}^*$ must be made heavy as well but the inclusion of a mass term brings back a possible mass term for the Higgs doublets. These problems can be solved by an additional Peccei-Quinn (as in Ref. [103]) or flavor symmetry (as in Ref. [104]).

Flipped $\mathsf{SU(5)}$

As mentioned in Section 3.2, the breaking of $\mathsf{SO(10)}$ does not lead automatically to the Georgi-Glashow $\mathsf{SU(5)}$ model (up to an additional $\mathsf{U(1)}$). There is another possible breaking scenario $\mathsf{SO(10)} \to \mathsf{SU(5)}' \times \mathsf{U(1)}_X$, where $\mathsf{G}_{\rm SM}$ is not contained in $\mathsf{SU(5)}'$, as first discussed by Barr in 1982 [105]. $\mathsf{SU(5)}'$ can be decomposed in $\mathsf{SU(3)_C} \times \mathsf{SU(2)_L} \times \mathsf{U(1)_Z}$, and the weak hypercharge Y is then a linear combination of X and Z,

$$\frac{1}{2} Y = \alpha Z + \beta X \; . \tag{B.16}$$

Beside the solution $(\alpha, \beta) = (1, 0)$ (Georgi-Glashow), a second solution exists, where

$$(\alpha, \beta) = \left(-\frac{1}{5}, \frac{1}{5} \right) \; . \tag{B.17}$$

Barr called this solution flipped $\mathsf{SU(5)}$, as the multiplets can be derived from the familiar Georgi-Glashow $\mathsf{SU(5)}$ by "flipping"

$$u \leftrightarrow d \qquad\qquad u^{\rm c} \leftrightarrow d^{\rm c} \qquad\qquad e \leftrightarrow \nu \qquad\qquad e^{\rm c} \leftrightarrow N \; , \tag{B.18}$$

where N is the additional singlet neutrino. The group is broken to $\mathsf{G}_{\rm SM}$ by $\mathsf{10} \oplus \mathsf{10}^*$, then to $\mathsf{SU(3)_C} \times \mathsf{U(1)_{em}}$ by doublets from the $\mathsf{5} \oplus \mathsf{5}^*$ as usual [106]. Hereby, only the triplets $H_{\rm C}$ and $\overline{H}_{\rm C}$ can be massive with partners in $\mathsf{10} \oplus \overline{10}$, whereas the doublets have no partner and remain massless. Therefore the missing-partner mechanism is naturally contained in flipped $\mathsf{SU(5)}$.

SO(10) field decompositions

The decompositions of the SO(10) fields with respect to SU(5) are

$$
\begin{aligned}
\mathsf{10} &= \mathsf{5} \oplus \mathsf{5}^* \ , \\
\mathsf{16} &= \mathsf{1} \oplus \mathsf{5}^* \oplus \mathsf{10} \ , \\
\mathsf{45} &= \mathsf{1} \oplus \mathsf{10} \oplus \mathsf{10}^* \oplus \mathsf{24} \ , \\
\mathsf{120} &= \mathsf{5} \oplus \mathsf{5}^* \oplus \mathsf{10} \oplus \mathsf{10}^* \oplus \mathsf{45} \oplus \mathsf{45}^* \ , \\
\mathsf{126} &= \mathsf{1} \oplus \mathsf{5}^* \oplus \mathsf{10} \oplus \mathsf{15}^* \oplus \mathsf{45} \oplus \mathsf{50}^* \ ;
\end{aligned}
\tag{B.19}
$$

their tensor products read

$$
\begin{aligned}
\mathsf{16} \otimes \mathsf{16} &= \mathsf{10}_\mathsf{s} \oplus \mathsf{120}_\mathsf{a} \oplus \mathsf{126}_\mathsf{s} \ , \\
\mathsf{16}^* \otimes \mathsf{10} &= \mathsf{16} \oplus \mathsf{144} \ , \\
\mathsf{45} \otimes \mathsf{10} &= \mathsf{10} \oplus \mathsf{120} \oplus \mathsf{320} \ , \\
\mathsf{45} \otimes \mathsf{16} &= \mathsf{16} \oplus \mathsf{144} \oplus \mathsf{560} \ ,
\end{aligned}
\tag{B.20}
$$

The reducible spinor representation,

$$
\begin{aligned}
|\Psi\rangle = |0\rangle\, \psi_0 + b_i^\dagger\, |0\rangle\, \psi^i + \frac{1}{2}\, b_i^\dagger b_j^\dagger\, |0\rangle\, \psi^{ij} + \frac{1}{12}\epsilon^{ijklm}\, b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_{ij} \\
+\ \frac{1}{24}\epsilon^{ijklm}\, b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_i + b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger\, |0\rangle\, \overline{\psi}_0 \ ,
\end{aligned}
\tag{B.21}
$$

can be split into two irreducible ones with the chiral projector,

$$
\Psi_\pm = \frac{1 \pm \Gamma_0}{2}\, \Psi : \tag{B.22}
$$

$$
|\Psi_+\rangle = |0\rangle\, \psi_0 \ + \ \frac{1}{2}\, b_i^\dagger b_j^\dagger\, |0\rangle\, \psi^{ij} \ + \ \frac{1}{24}\epsilon^{ijklm}\, b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_i \tag{B.23}
$$

$$
|\Psi_-\rangle = b_i^\dagger\, |0\rangle\, \psi^i \ + \ \frac{1}{12}\epsilon^{ijklm}\, b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\, \overline{\psi}_{ij} \ + \ b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger\, |0\rangle\, \overline{\psi}_0 \ ; \tag{B.24}
$$

the corresponding "kets" read

$$
\langle \Psi_+^* | = \psi_0\ \langle 0| \ - \ \frac{1}{2}\psi^{ij}\, \langle 0|\, b_i b_j \ + \ \frac{1}{24}\epsilon^{ijklm}\, \overline{\psi}_i\, \langle 0|\, b_j b_k b_l b_m \tag{B.25}
$$

$$
\langle \Psi_-^* | = \psi^i\, \langle 0|\, b_i \ - \ \frac{1}{12}\epsilon^{ijklm}\, \overline{\psi}_{ij}\, \langle 0|\ b_k b_l b_m \ + \ \overline{\psi}_0\, \langle 0|\, b_1 b_2 b_3 b_4 b_5 \tag{B.26}
$$

The tensors $\phi_{\dots}$ are in general reducible and not normalized. We rewrite the tensor as

$$
\begin{aligned}
\phi_{\mu\nu\cdots\lambda} = \frac{i^{\mu+\nu+\cdots+\lambda}}{2^n} \big(&\phi_{c_i\, c_j \cdots c_k} + (-1)^\mu\, \phi_{\bar{c}_i\, c_j \cdots c_k} + (-1)^\nu\, \phi_{c_i\, \bar{c}_j \cdots c_k} \\
&+ (-1)^\lambda \phi_{c_i\, c_j \cdots \bar{c}_k} + (-1)^{\mu+\nu}\, \phi_{\bar{c}_i\, \bar{c}_j \cdots c_k} + (-1)^{\mu+\lambda}\, \phi_{\bar{c}_i\, c_j \cdots \bar{c}_k} \\
&+ (-1)^{\nu+\lambda}\, \phi_{c_i\, \bar{c}_j \cdots \bar{c}_k} + \ldots + (-1)^{\mu+\nu+\cdots+\lambda}\, \phi_{\bar{c}_i\, \bar{c}_j \cdots \bar{c}_k} \big) \ ,
\end{aligned}
\tag{B.27}
$$

and obtain

$$\partial_\alpha \phi_{\mu\nu\cdots\lambda}\, \partial^\alpha \phi^\dagger_{\mu\nu\cdots\lambda} = \frac{1}{2^n} \bigg(\partial_\alpha \phi_{c_i\, c_j\, \cdots\, c_k}\, \partial^\alpha \phi^\dagger_{c_i\, c_j\, \cdots\, c_k} + \frac{n!}{(n-n_b)!\, n_b!}\, \partial_\alpha \phi_{c_i\, c_j\, \cdots\, \bar{c}_k}\, \partial^\alpha \phi^\dagger_{c_i\, c_j\, \cdots\, \bar{c}_k} + \ldots + \partial_\alpha \phi_{\bar{c}_i\, \bar{c}_j\, \cdots\, \bar{c}_k}\, \partial^\alpha \phi^\dagger_{\bar{c}_i\, \bar{c}_j\, \cdots\, \bar{c}_k} \bigg) \quad \text{(B.28)}$$

The tensors of $\phi_{\mu\nu\lambda}$ can be decomposed into their irreducible forms as [73]

$$\phi_{c_i\, c_j\, \bar{c}_k} = f^{ij}_k + \frac{1}{4}\left(\delta^i_k\, f^j - \delta^j_k\, f^i\right), \qquad \phi_{c_i\, \bar{c}_j\, \bar{c}_k} = f^i_{jk} - \frac{1}{4}\left(\delta^i_j\, f_k - \delta^i_k\, f_j\right), \quad \text{(B.29)}$$

$$\phi_{c_i\, c_j\, c_k} = \epsilon^{ijklm}\, f_{lm}\ , \qquad \phi_{\bar{c}_i\, \bar{c}_j\, \bar{c}_k} = \epsilon_{ijklm}\, f^{lm}, \quad \text{(B.30)}$$

$$\phi_{\bar{c}_n\, c_n\, c_i} = f^i\ , \qquad \phi_{\bar{c}_n\, c_n\, \bar{c}_i} = f_i\ . \quad \text{(B.31)}$$

We identify the $\mathsf{5},\ \mathsf{10},\ \mathsf{45},\ \mathsf{5}^*,\ \mathsf{10}^*$ and $\mathsf{45}^*$-plet of $\mathsf{120}$, which are normalized as

$$f^i = \frac{4}{\sqrt{3}}\, h^i\ , \qquad f^{ij} = \frac{1}{\sqrt{3}}\, h^{ij}\ , \qquad f^{ij}_k = \frac{2}{\sqrt{3}}\, h^{ij}_k\ , \quad \text{(B.32)}$$

$$f_i = \frac{4}{\sqrt{3}}\, h_i\ , \qquad f_{ij} = \frac{1}{\sqrt{3}}\, h_{ij}\ , \qquad f^i_{jk} = \frac{2}{\sqrt{3}}\, h^i_{jk}\ . \quad \text{(B.33)}$$

Analogously, we have for $\phi_{\mu\nu}$ [79]

$$\phi_{c_n\, \bar{c}_n} = h\ , \qquad \phi_{c_i\, \bar{c}_j} = h^i_j + \frac{1}{5}\, \delta^i_j\, h\ , \qquad \phi_{c_i\, c_j} = h^{ij}\ , \qquad \phi_{\bar{c}_i\, \bar{c}_j} = h_{ij} \quad \text{(B.34)}$$

with the $\mathsf{1},\ \mathsf{10}^*,\ \mathsf{10}$ and $\mathsf{24}$-plet, which are normalized as

$$h = \sqrt{10}\, H\ , \qquad h_{ij} = \sqrt{2}\, H_{ij}\ , \qquad h^{ij} = \sqrt{2}\, H^{ij}\ , \qquad h^i_j = \sqrt{2}\, H^i_j. \quad \text{(B.35)}$$

Appendix C

Chiral Langrangean Technique and Proton Decay Diagrams

Chiral Langrangian Technique

To calculate proton decay rates, we have to translate the operators at quark level to those at hadron level. We then have to evaluate the hadron matrix elements $\langle PS|\mathcal{O}|\,p\rangle$, which describes the transition of the proton via the three-quark operator $\mathcal{O}$ into the pseudoscalar meson. The baryon number violating operators are classified into four types [86, 107],

$$\mathcal{O}^{(1)}_{abcd} = \left(\bar{D}^c_{\alpha a\mathrm{R}} U_{\beta b\mathrm{R}}\right)\left(\bar{Q}^c_{i\gamma c\mathrm{L}} L_{jd\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\epsilon_{ij}\ , \tag{C.1a}$$

$$\mathcal{O}^{(2)}_{abcd} = \left(\bar{Q}^c_{i\alpha a\mathrm{L}} Q_{j\beta b\mathrm{L}}\right)\left(\bar{U}^c_{\gamma c\mathrm{R}} L_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\epsilon_{ij}\ , \tag{C.1b}$$

$$\mathcal{O}^{(3)}_{abcd} = \left(\bar{Q}^c_{i\alpha a\mathrm{L}} Q_{j\beta b\mathrm{L}}\right)\left(\bar{Q}^c_{k\gamma c\mathrm{L}} L_{md\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\epsilon_{im}\epsilon_{jk}\ , \tag{C.1c}$$

$$\mathcal{O}^{(4)}_{abcd} = \left(\bar{D}^c_{\alpha a\mathrm{R}} U_{\beta b\mathrm{R}}\right)\left(\bar{U}^c_{\gamma c\mathrm{R}} L_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\ . \tag{C.1d}$$

The operators relevant to non-strange final states are [57]

$$\mathcal{O}^{(1)}_{d} = \left(\bar{d}^c_{\alpha\mathrm{R}} u_{\beta\mathrm{R}}\right)\left(\bar{u}^c_{\gamma\mathrm{L}} e_{d\mathrm{L}} - \bar{d}^c_{\gamma\mathrm{L}}\nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.2a}$$

$$\mathcal{O}^{(2)}_{d} = \left(\bar{d}^c_{\alpha\mathrm{L}} u_{\beta\mathrm{L}}\right)\left(\bar{u}^c_{\gamma\mathrm{R}} e_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.2b}$$

$$\mathcal{O}^{(3)}_{d} = \left(\bar{d}^c_{\alpha\mathrm{L}} u_{\beta\mathrm{L}}\right)\left(\bar{u}^c_{\gamma\mathrm{L}} e_{d\mathrm{L}} - \bar{d}^c_{\gamma\mathrm{L}}\nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.2c}$$

$$\mathcal{O}^{(4)}_{d} = \left(\bar{d}^c_{\alpha\mathrm{R}} u_{\beta\mathrm{R}}\right)\left(\bar{u}^c_{\gamma\mathrm{R}} e_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.2d}$$

whereas those for the strange final states are [57]

$$\tilde{\mathcal{O}}^{(1)}_{d} = \left(\bar{s}^c_{\alpha\mathrm{R}} u_{\beta\mathrm{R}}\right)\left(\bar{u}^c_{\gamma\mathrm{L}} e_{d\mathrm{L}} - \bar{d}^c_{\gamma\mathrm{L}}\nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.3a}$$

$$\tilde{\mathcal{O}}^{(2)}_{d} = \left(\bar{s}^c_{\alpha\mathrm{L}} u_{\beta\mathrm{L}}\right)\left(\bar{u}^c_{\gamma\mathrm{R}} e_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.3b}$$

$$\tilde{\mathcal{O}}^{(3)}_{d} = \left(\bar{s}^c_{\alpha\mathrm{L}} u_{\beta\mathrm{L}}\right)\left(\bar{u}^c_{\gamma\mathrm{L}} e_{d\mathrm{L}} - \bar{d}^c_{\gamma\mathrm{L}}\nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.3c}$$

$$\tilde{\mathcal{O}}^{(4)}_{d} = \left(\bar{s}^c_{\alpha\mathrm{R}} u_{\beta\mathrm{R}}\right)\left(\bar{u}^c_{\gamma\mathrm{R}} e_{d\mathrm{R}}\right)\epsilon_{\alpha\beta\gamma}\ , \tag{C.3d}$$

$$\tilde{\mathcal{O}}_d^{(5)} = \left(\bar{d}^c_{\alpha\mathrm{R}} u_{\beta\mathrm{R}}\right)\left(\bar{s}^c_{\gamma\mathrm{L}} \nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma} \ , \tag{C.3e}$$

$$\tilde{\mathcal{O}}_d^{(6)} = \left(\bar{d}^c_{\alpha\mathrm{L}} u_{\beta\mathrm{L}}\right)\left(\bar{s}^c_{\gamma\mathrm{L}} \nu_{d\mathrm{L}}\right)\epsilon_{\alpha\beta\gamma} \ . \tag{C.3f}$$

Here d denotes the generation. These operators are translated to those written in terms of baryon and meson fields with the aid of chiral perturbation theory [57, 58].

The chiral Lagrangian is an effective field theory which describes the interactions of the pseudo-Goldstone bosons associated with the spontaneous breakdown of chiral $\mathsf{SU(3)}_L \times \mathsf{SU(3)}_R$ symmetry. These bosons are incorporated in a 3×3 unitary matrix,

$$\Sigma = \exp\left\{\frac{2\,i\,M}{f_\pi}\right\} \ , \tag{C.4}$$

where f_π is the pion decay constant and

$$M = \begin{pmatrix} \sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix} \ . \tag{C.5}$$

The baryon fields can be written as

$$B = \begin{pmatrix} \sqrt{\frac{1}{2}}\Sigma^0 + \sqrt{\frac{1}{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\sqrt{\frac{1}{2}}\Sigma^0 + \sqrt{\frac{1}{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\Lambda \end{pmatrix} \ . \tag{C.6}$$

The most general Lagrangean for the strong interactions of mesons and baryons reads

$$\begin{aligned} \mathscr{L}_0 = {} & \frac{1}{8} f_\pi^2 \,\mathrm{tr}\,(\partial_\mu\Sigma)(\partial_\mu\Sigma)^\dagger + \mathrm{tr}\,\bar{B}\,(\not{\partial} - M_B)\,B + \frac{i}{2}\,\mathrm{tr}\,\bar{B}\gamma^\mu\left[\zeta(\partial_\mu\zeta^\dagger) + \zeta^\dagger(\partial_\mu\zeta)\right]B \\ & + \frac{i}{2}\,\mathrm{tr}\,\bar{B}\gamma^\mu B\left[(\partial_\mu\zeta)\,\zeta^\dagger + \left(\partial_\mu\zeta^\dagger\right)\zeta\right] - \frac{i}{2}(D-F)\,\mathrm{tr}\,\bar{B}\gamma^\mu\gamma_5 B\left[(\partial_\mu\zeta)\,\zeta^\dagger - \left(\partial_\mu\zeta^\dagger\right)\zeta\right] \\ & + \frac{i}{2}(D+F)\,\mathrm{tr}\,\bar{B}\gamma^\mu\gamma_5\left[\zeta\left(\partial_\mu\zeta^\dagger\right) - \zeta^\dagger(\partial_\mu\zeta)\right]B \ , \end{aligned}$$

with $\zeta = \exp\{iM/f_\pi\}$. The parameters D and F are the symmetric and antisymmetric $\mathsf{SU(3)}$ reduced matrix elements for the axial-vector current [108]. Their values can be determined from hyperon decays measurements [109].

The operators (C.2) and (C.3) transform under $\mathsf{SU(3)}_L \times \mathsf{SU(3)}_R$ as

$$\begin{aligned} &(3,\bar{3}) : \mathcal{O}_d^{(1)}, \tilde{\mathcal{O}}_d^{(1)}, \tilde{\mathcal{O}}_d^{(5)}, \qquad && (8,1) : \mathcal{O}_d^{(3)}, \tilde{\mathcal{O}}_d^{(3)}, \tilde{\mathcal{O}}_d^{(6)}, \\ &(\bar{3},3) : \mathcal{O}_d^{(2)}, \tilde{\mathcal{O}}_d^{(2)}, && (1,8) : \mathcal{O}_d^{(4)}, \tilde{\mathcal{O}}_d^{(4)} \ . \end{aligned}$$

These transformation properties are realized by $\zeta B\zeta \in (3,\bar{3})$, $\zeta^\dagger B\zeta^\dagger \in (\bar{3},3)$, $\zeta B\zeta^\dagger \in (8,1)$ and $\zeta^\dagger B\zeta \in (1,8)$, with which we can express the operators as

$$\begin{aligned} \mathcal{O}_d^{(1)} &= \alpha\left(\bar{e}^c_{d\mathrm{L}}\,\mathrm{tr}\,\mathcal{F}\zeta B_L\zeta - \bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\mathcal{F}'\zeta B_L\zeta\right), \qquad & \mathcal{O}_d^{(2)} &= \alpha\,\bar{e}^c_{d\mathrm{R}}\,\mathrm{tr}\,\mathcal{F}\zeta^\dagger B_R\zeta^\dagger, \\ \mathcal{O}_d^{(3)} &= \beta\left(\bar{e}^c_{d\mathrm{L}}\,\mathrm{tr}\,\mathcal{F}\zeta B_L\zeta^\dagger - \bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\mathcal{F}'\zeta B_L\zeta^\dagger\right), & \mathcal{O}_d^{(4)} &= \beta\,\bar{e}^c_{d\mathrm{R}}\,\mathrm{tr}\,\mathcal{F}\zeta^\dagger B_R\zeta, \end{aligned}$$

$$\tilde{\mathcal{O}}_d^{(1)} = \alpha\left(\bar{e}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}\zeta B_L\zeta - \bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}'\zeta B_L\zeta\right), \qquad \tilde{\mathcal{O}}_d^{(2)} = \alpha\,\bar{e}^c_{d\mathrm{R}}\,\mathrm{tr}\,\tilde{\mathcal{F}}\zeta^\dagger B_R\zeta^\dagger,$$
$$\tilde{\mathcal{O}}_d^{(3)} = \beta\left(\bar{e}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}\zeta B_L\zeta^\dagger - \bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}'\zeta B_L\zeta^\dagger\right), \qquad \tilde{\mathcal{O}}_d^{(4)} = \beta\,\bar{e}^c_{d\mathrm{R}}\,\mathrm{tr}\,\tilde{\mathcal{F}}\zeta^\dagger B_R\zeta,$$
$$\tilde{\mathcal{O}}_d^{(5)} = \alpha\,\bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}''\zeta B_L\zeta, \qquad \tilde{\mathcal{O}}_d^{(6)} = \beta\,\bar{\nu}^c_{d\mathrm{L}}\,\mathrm{tr}\,\tilde{\mathcal{F}}''\zeta B_L\zeta^\dagger,$$

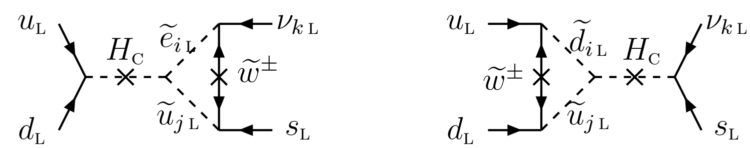
where α and β, which are already defined in Eqn. (2.12), are unknown coefficients associated with the $(3,\bar{3})$ and $(\bar{3},3)$ operators and the $(8,1)$ and $(1,8)$ operators respectively; $\mathcal{F}$, $\mathcal{F}'$, $\tilde{\mathcal{F}}$, $\tilde{\mathcal{F}}'$ and $\tilde{\mathcal{F}}''$ are projection matrices in the flavor space. We finally obtain the tree-level results for the independent matrix elements [88]

$$\langle\pi^0|(ud_{\mathrm{R}})u_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[\frac{1}{\sqrt{2}f_\pi} + \frac{D+F}{\sqrt{2}f_\pi}\right], \tag{C.7a}$$
$$\langle\pi^0|(ud_{\mathrm{L}})u_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[\frac{1}{\sqrt{2}f_\pi} + \frac{D+F}{\sqrt{2}f_\pi}\right], \tag{C.7b}$$
$$\langle\pi^+|(ud_{\mathrm{R}})d_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[\frac{1}{f_\pi} + \frac{D+F}{f_\pi}\right], \tag{C.7c}$$
$$\langle\pi^+|(ud_{\mathrm{L}})d_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[\frac{1}{f_\pi} + \frac{D+F}{f_\pi}\right], \tag{C.7d}$$
$$\langle K^0|(us_{\mathrm{R}})u_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[-\frac{1}{f_\pi} - \frac{D-F}{f_\pi}\frac{m_N}{m_B}\right], \tag{C.7e}$$
$$\langle K^0|(us_{\mathrm{L}})u_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[\frac{1}{f_\pi} - \frac{D-F}{f_\pi}\frac{m_N}{m_B}\right], \tag{C.7f}$$
$$\langle K^+|(us_{\mathrm{R}})d_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[+\frac{2D}{3f_\pi}\frac{m_N}{m_B}\right], \tag{C.7g}$$
$$\langle K^+|(us_{\mathrm{L}})d_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[+\frac{2D}{3f_\pi}\frac{m_N}{m_B}\right], \tag{C.7h}$$
$$\langle K^+|(ud_{\mathrm{R}})s_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[\frac{1}{f_\pi} + \frac{D+3F}{3f_\pi}\frac{m_N}{m_B}\right], \tag{C.7i}$$
$$\langle K^+|(ud_{\mathrm{L}})s_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[\frac{1}{f_\pi} + \frac{D+3F}{3f_\pi}\frac{m_N}{m_B}\right], \tag{C.7j}$$
$$\langle\eta|(ud_{\mathrm{R}})u_{\mathrm{L}}|p\rangle \simeq \alpha P_{\mathrm{L}}u_p\left[-\frac{1}{\sqrt{6}f_\pi} - \frac{D-3F}{\sqrt{6}f_\pi}\right], \tag{C.7k}$$
$$\langle\eta|(ud_{\mathrm{L}})u_{\mathrm{L}}|p\rangle \simeq \beta P_{\mathrm{L}}u_p\left[\frac{3}{\sqrt{6}f_\pi} - \frac{D-3F}{\sqrt{6}f_\pi}\right]. \tag{C.7l}$$

The matrix elements α and β are evaluated in lattice QCD calculations; their absolute values differ in the range $(0.003-0.03)\,\mathrm{GeV}^3$. The most recent results of the CP-PACS and JLQCD as well as the RBC collaboration seem to agree at $0.01\,\mathrm{GeV}^3$ [110]. In Section 2.2, however, we use the lowest possible value $0.003\,\mathrm{GeV}^3$ in order to study, if dimension-five proton decay is in agreement with the experimental limit at all.

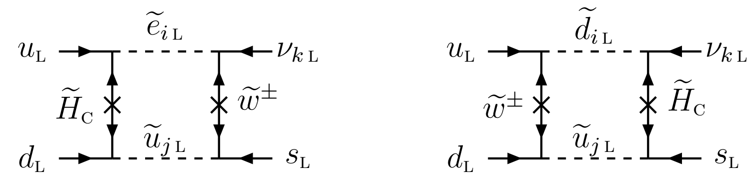
Diagrams

Fig. C.1 lists the diagrams for the decay $p \to K^+\bar{\nu}$ with chargino dressing. Those with right-handed fermions incoming can be divided into two groups depending on the dressing before (Fig. C.1(c)) or after the decay operator (Fig. C.1(d)); therefore they are called $RRLL$ and $RRRR$ diagrams, respectively. The latter case is the only one related to the $RRRR$ operator and $C_{5\mathrm{R}}$ because there are no right-handed neutrinos in the model. As discussed, the dimension five operators are flavour non-diagonal, hence several diagrams are suppressed.



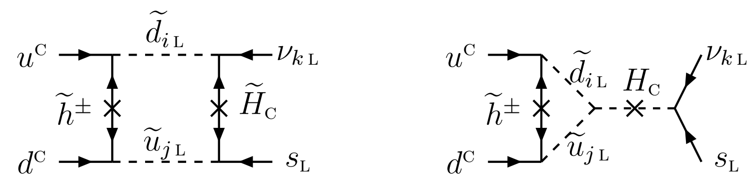
$Y_{qq}(1,1)\,Y_{ql}(j,i)$ $(j \neq 1)$ $Y_{qq}(i,j)\,Y_{ql}(2,k)$ $(i \neq 2)$

(a) *LLLL* triangle diagrams



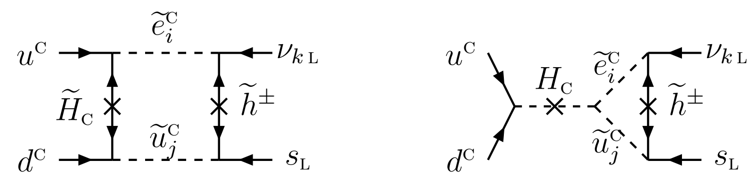
$Y_{qq}(1,j)\,Y_{ql}(1,i)$ $(j \neq 1)$ $Y_{qq}(j,2)\,Y_{ql}(i,k)$ $(i \neq 2)$

(b) *LLLL* box diagrams



$Y_{qq}(j,2)\,Y_{ql}(i,k)$ $(i \neq 2)$ $Y_{qq}(i,j)\,Y_{ql}(2,k)$ $(i \neq 2)$

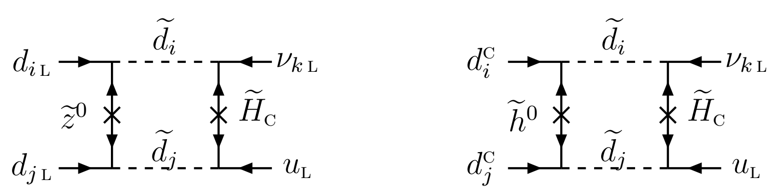
(c) *RRLL* diagrams



$Y_{ue}(1,i)\,Y_{ud}(j,1)$ $(j \neq 1)$ $Y_{ue}(j,i)\,Y_{ud}(1,1)$ $(j \neq 1)$

(d) *RRRR* diagrams

Figure C.1: Diagrams with chargino dressing



$Y_{qq}(j,1)\,Y_{ql}(i,k)$ $(i \neq j)$ $Y_{qq}(j,1)\,Y_{ql}(i,k)$ $(i \neq j)$

(a) *LLLL* diagrams (b) *RRRR* diagrams

Figure C.2: Diagrams with neutralino dressing

Bibliography

[1] E. C. G. Stückelberg, *Die Wechselwirkungskräfte in der Elektrodynamik und in der Feldtheorie der Kernkräfte (Teil II und III)*, Helv. Phys. Acta **11**, 299 (1938).

[2] E. P. Wigner, *Invariance In Physical Theory*, Proc. Am. Phil. Soc. **93**, 521 (1949).

[3] E. P. Wigner, Proc. Natl. Acad. Sci. **38**, 449 (1952).

[4] H. S. Gurr, W. R. Kropp, F. Reines and B. Meyer, *Experimental Test of Baryon Conservation*, Phys. Rev. **158**, 1321 (1967).

[5] F. Reines, C. L. Cowan and M. Goldhaber, *Conservation of the Number of Nucleons*, Phys. Rev. **96**, 1157 (1954).

[6] G. 't Hooft, *Symmetry breaking through Bell-Jackiw anomalies*, Phys. Rev. Lett. **37**, 8 (1976).

[7] H. Georgi and S. L. Glashow, *Unity of all elementary particle forces*, Phys. Rev. Lett. **32**, 438 (1974).

[8] J. C. Pati and A. Salam, *Unified lepton - hadron symmetry and a gauge theory of the basic interactions*, Phys. Rev. **D 8**, 240 (1973).

J. C. Pati and A. Salam, *Is baryon number conserved?*, Phys. Rev. Lett. **31**, 661 (1973).

[9] J. C. Pati and A. Salam, *Lepton number as the fourth color*, Phys. Rev. **D 10**, 275 (1974).

[10] S. Dimopoulos and H. Georgi, *Softly broken supersymmetry and SU(5)*, Nucl. Phys. **B 193**, 150 (1981).

[11] N. Sakai, *Naturalness in Supersymmetric GUTs*, Z. Phys. **C 11**, 153 (1981).

[12] T. Goto and T. Nihei, *Effect of an RRRR dimension 5 operator on proton decay in the minimal SU(5) SUGRA GUT model*, Phys. Rev. **D 59**, 115009 (1999).

[13] H. Murayama and A. Pierce, *Not even decoupling can save the minimal supersymmetric SU(5) model*, Phys. Rev. **D 65**, 055009 (2002).

[14] D. Emmanuel-Costa and S. Wiesenfeldt, *Proton decay in a consistent supersymmetric SU(5) GUT model*, Nucl. Phys. **B 661**, 62 (2003).

[15] W. Buchmüller, L. Covi, D. Emmanuel-Costa and S. Wiesenfeldt, *Flavour structure and proton decay in a 6D SO(10)-GUT*, DESY 03-202 (to appear).

[16] M. Kaku, *Quantum Field Theory*, Oxford University Press, 1993.

M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley, Reading, Massachusetts, 1995.

S. Weinberg, *The making of the standard model*, hep-ph/0401010.

[17] K. Hagiwara et al., *Review of particle physics*, Phys. Rev. **D 66**, 010001 (2002).

[18] LEP Working Group for Higgs boson searches, *Search for the standard model Higgs boson at LEP*, Phys. Lett. **B 565**, 61 (2003).

[19] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov and M. A. Shifman, *ABC of instantons*, Sov. Phys. Usp. **24**, 195 (1982).

[20] T. Schäfer and E. V. Shuryak, *Instantons in QCD*, Rev. Mod. Phys. **70**, 323 (1998).

[21] V. A. Rubakov and M. E. Shaposhnikov, *Electroweak baryon number non-conservation in the early universe and in high-energy collisions*, Usp. Fiz. Nauk **166**, 493 (1996).

[22] S. Weinberg, *Phenomenological Lagrangians*, Physica **A 96**, 327 (1979).

[23] W. Buchmüller and D. Wyler, *Effective Lagrangian Analysis Of New Interactions And Flavor Conservation*, Nucl. Phys. **B268**, 621 (1986).

[24] S. M. Bilenky, *Early years of neutrino oscillations*, hep-ph/9908335.

[25] S. M. Bilenky, C. Giunti and W. Grimus, *Phenomenology of neutrino oscillations*, Prog. Part. Nucl. Phys. **43**, 1 (1999).

[26] V. Barger, D. Marfatia and K. Whisnant, *Progress in the physics of massive neutrinos*, Int. J. Mod. Phys. **E 12**, 569 (2003).

[27] S. M. Bilenky, C. Giunti, J. A. Grifols and E. Masso, *Absolute values of neutrino masses: Status and prospects*, Phys. Rept. **379**, 69 (2003).

[28] M. Gell-Mann, P. Ramond and R. Slansky, *Complex spinors and unified theories*, in: *Supergravity* (ed. P. van Nieuvenheizen and D. Z. Freedman), North Holland Publ. Co., 1979.

T. Yanagida, *Horizontal Gauge Symmetry and Masses of Neutrinos*, in: *Proceedings of the Workshop on unified Theories and Baryon Number Violation in the universe*, Tsukauba 1979 (ed. O. Sawada and A. Sugamoto), KEK Report Nr. 79-18, 1979.

[29] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Phys. Rev. **159**, 1251 (1967).

[30] Y. A. Gol'fand and E. P. Likhtman, *Extension of the algebra of Poincare group generators and violaton of P invariance*, JETP Lett. **13**, 323 (1971).

[31] D. V. Volkov and V. P. Akulov, *Possible universal neutrino interaction*, JETP Lett. **16**, 438 (1972).

[32] R. Haag, J. T. Łopuszański and M. F. Sohnius, *All possible generators of supersymmetries of the S matrix*, Nucl. Phys. **B88**, 257 (1975).

[33] N. Dragon, U. Ellwanger and M. G. Schmidt, *Supersymmetry and Supergravity*, Prog. Part. Nucl. Phys. **18**, 1 (1987).

P. van Nieuwenhuizen, *Supergravity*, Phys. Rept. **68**, 189 (1981).

[34] M. F. Sohnius, *Introducing Supersymmetry*, Phys. Rept. **128**, 39 (1985).

H. P. Nilles, *Supersymmetry, supergravity and particle physics*, Phys. Rept. **110**, 1 (1984).

J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Series in Physics, Princeton University Press, 1983.

I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, IOP Publishing, 1998.

[35] F. Fayet and S. Ferrara, *Supersymmetry*, Phys. Rept. **32**, 249 (1977).

[36] M. Roncadelli, *Searching for dark matter*, astro-ph/0307115.

[37] L. E. Ibanez and G. G. Ross, *Discrete gauge symmetries and the origin of baryon and lepton number conservation in supersymmetric versions of the standard model*, Nucl. Phys. **B 368**, 3 (1992).

[38] U. Amaldi, W. de Boer and H. Fürstenau, *Comparision of Grand Unified Theories with elektroweak and strong coupling constants measured at LEP*, Phys. Lett. **B 260**, 240 (1991).

[39] S. Willenbrock, *Triplicated trinification*, Phys. Lett. **B 561**, 130 (2003).

[40] H. Georgi, *Lie Algebras in Particle Physics*, Perseus Books, 1999.

[41] P. Langacker, *Grand unified theories and proton decay*, Phys. Rept. **72**, 185 (1981).

G. Ross, *Grand unified theories*, Frontiers in Physics, vol. 60, Benjamin/Commings, 1985.

[42] A. J. Buras, J. Ellis, M. K. Gaillard and D. V. Nanopoulos, *Aspects of the Grand Unification of strong, weak and electromagnetic interactions*, Nucl. Phys. **B 135**, 66 (1978).

[43] G. Nordström, *Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen*, Phys. Zeitschr. **15**, 504 (1914).

[44] T. Kaluza, *Zum Unitaritätsproblem der Physik*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), 966 (1921).

[45] T. Appelquist, A. Chodos and P. G. O. Freund, *Modern Kaluza-Klein Theories*, Addison-Wesley, Reading, Massachusetts, 1987.

J. M. Overduin and P. S. Wesson, *Kaluza-Klein gravity*, Phys. Rept. **283**, 303 (1997).

[46] O. Klein, *Quantum theory and five-dimensional theory of relativity*, Z. Phys. **37**, 895 (1926).

[47] Y. Kawamura, *Gauge symmetry reduction from the extra space S(1)/Z(2)*, Prog. Theor. Phys. **103**, 613 (2000).

Y. Kawamura, *Triplet-doublet splitting, proton stability and extra dimension*, Prog. Theor. Phys. **105**, 999 (2001).

G. Altarelli and F. Feruglio, *SU(5) grand unification in extra dimensions and proton decay*, Phys. Lett. **B 511**, 257 (2001).

L. J. Hall and Y. Nomura, *Gauge unification in higher dimensions*, Phys. Rev. **D 64**, 055003 (2001).

A. Hebecker and J. March-Russell, *A minimal S(1)/(Z(2) × Z'(2)) orbifold GUT*, Nucl. Phys. **B 613**, 3 (2001).

[48] C. D. Froggatt and H. B. Nielsen, *Hierarchy of quark masses, Cabibbo angles and CP violation*, Nucl. Phys. **B 147**, 277 (1979).

[49] W. Buchmüller and M. Plümacher, *Neutrino masses and the baryon asymmetry*, Int. J. Mod. Phys. **A 15**, 5047 (2000).

[50] S. Dimopoulos, S. Raby and F. Wilczek, *Supersymmetry And The Scale Of Unification*, Phys. Rev. **D 24**, 1681 (1981).

[51] N. Sakai and T. Yanagida, *Proton decay in a class of supersymmetric grand unified models*, Nucl. Phys. **B 197**, 533 (1982).

[52] S. Weinberg, *Supersymmetry at ordinary energies. masses and conservation laws*, Phys. Rev. **D 26**, 287 (1982).

[53] S. Dimopoulos, S. Raby and F. Wilczek, *Proton decay in supersymmetric models*, Phys. Lett. **B 112**, 133 (1982).

[54] J. R. Ellis, D. V. Nanopoulos and S. Rudaz, *A phenomenological comparison of conventional and supersymmtric GUTs*, Nucl. Phys. **B 202**, 43 (1982).

[55] P. Nath, A. H. Chamseddine and R. Arnowitt, *Nucleon decay in supergravity unified theories*, Phys. Rev. **D 32**, 2348 (1985).

[56] J. Hisano, H. Murayama and T. Yanagida, *Nucleon decay in the minimal supersymmetric SU(5) grand unification*, Nucl. Phys. **B 402**, 46 (1993).

[57] M. Claudson, M. B. Wise and L. J. Hall, *Chiral Lagrangian for deep mine physics*, Nucl. Phys. **B 195**, 297 (1982).

[58] S. Chadha and M. Daniel, *Chiral Lagrangian calculation of nucleon decay modes induced by $D=5$ supersymmetric operators*, Nucl. Phys. **B 229**, 105 (1983).

[59] W. Lucha, *Proton Decay in Grand Unified Theories*, Fortschritte der Physik **33**, 547 (1985).

[60] S. J. Brodsky, J. Ellis, J. S. Hagelin and C. T. Sachrajda, *Baryon wave functions and nucleon decay*, Nucl. Phys. **B 238**, 561 (1984).

[61] R. Dermísek, A. Mafi and S. Raby, *SUSY GUTs under siege: proton decay*, Phys. Rev. **D 63**, 035001 (2000).

[62] A. Kagan M. Dine and S. Samuel, *Naturalness in supersymmetry, or raising the supersymmetric breaking scale*, Phys. Lett. **B 243**, 250 (1990).

S. Dimopoulos and G.F. Giudice, *Naturalness Constraints in Supersymmetric Theories with Non-Universal Soft Terms*, Phys. Lett. **B 357**, 573 (1995).

A. Pomarol and D. Tommasini, *Horizontal symmetries for the supersymmetric flavor problem*, Nucl. Phys. **B 4661**, 3 (1996).

A. G. Cohen, D.B. Kaplan and A.E. Nelson, *The more minimal supersymmetric standard model*, Phys. Lett. **B 388**, 588 (1996).

[63] P. Nath, *Hierarchies and Textures in Supergravity Unification*, Phys. Rev. Lett. **76**, 2218 (1996).

P. Nath, *Textured Minimal and Extended Supergravity Unification and Implications for Proton Stability*, Phys. Lett. **B 381**, 147 (1996).

[64] B. Bajc, P. F. Perez and G. Senjanović, *Minimal supersymmetric SU(5) theory and proton decay: where do we stand?*

[65] V. Lucas and S. Raby, *Nucleon Decay in a realistic SO(10) SUSY GUT*, Phys. Rev. **55**, 6986 (1997).

[66] Y. Hayato et al., *Search for Proton Decay through* $p \rightarrow \bar{\nu}K^+$ *in a Large Water Cherenkov Detector*, Phys. Rev. Lett. **83**, 1529 (1999).

[67] K. S. Ganezer, *The Search for Proton Decay at Super-Kamiokande*, Salt Lake City 1999, Cosmic ray **2**, 328 (1999).

[68] M. Shiozawa, *http://www-sk.icrr.u-tokyo.ac.jp/noon2003*, in: *Proceedings of the 4th International Workshop on Neutrino Oscillations and their Origin*, Kanazawa 2003 (ed. Y. Suzuki, M. Nakahata, Y. Itow, M. Shiozawa and Y. Obayashi), World Scientific Publishing, 2004 (to appear).

[69] Z. Berezhiani, Z. Tavartkiladze and M. Vysotsky, $d = 5$ *operators in SUSY GUT: Fermion masses versus proton decay*, hep-ph/9809301.

[70] H. Georgi, *The State of the Art—Gauge Theories*, in: *Particles and fields* (ed. C. Carlson), proceedings of the Williamsburg Meeting of the Division of Particles and Fields of the APS, 1975.

H. Fritzsch and P. Minkowski, *Unified interactions of leptons and hadrons*, Ann. Phys. **93**, 193–266 (1975).

[71] R. N. Mohapatra and B. Sakita, *SO(2N) grand unification in an SU(N) basis*, Phys. Rev. **D 21**, 1062 (1980).

[72] R. Slansky, *Group Theory For Unified Model Building*, Phys. Rept. **79**, 1 (1981).

[73] P. Nath and R. M. Syed, *Analysis of couplings with large tensor representations in SO(2N) and proton decay*, Phys. Lett. **B 506**, 68 (2001).

[74] K. S. Babu, J. C. Pati and F. Wilczek, *Suggested new modes in supersymmetric proton descay*, Phys. Lett. **B 423**, 337 (1998).

[75] K. S. Babu, J. C. Pati and F. Wilczek, *Fermion masses, neutrino oscillations, and proton decay in the light of SuperKamiokande*, Nucl. Phys. **B 566**, 33 (2000).

[76] W. Buchmüller and D. Wyler, *CP violation, neutrino mixing and the baryon asymmetry*, Phys. Lett. **B521**, 291 (2001).

[77] T. Asaka, W. Buchmüller and L. Covi, *Gauge unification in six dimensions*, Phys. Lett. **B 523**, 199 (2001).

[78] L. J. Hall, Y. Nomura, T. Okui and D. R. Smith, *SO(10) unified theories in six dimensions*, Phys. Rev. **D 65**, 035008 (2002).

[79] P. Nath and R. M. Syed, *Complete cubic and quartic couplings of 16 and 16-bar in SO(10) unification*, Nucl. Phys. **B 618**, 138 (2001).

[80] J. R. Ellis, J. S. Hagelin, D. V. Nanopoulos and K. Tamvakis, *Observable Gravitationally Induced Baryon Decay*, Phys. Lett. **B 124**, 484 (1983).

H. Murayama and D. B. Kaplan, *Family symmetries and proton decay*, Phys. Lett. **B 336**, 221 (1994).

[81] J. Sato and T. Yanagida, *Large Lepton Mixing in Seesaw Models — Coset-space Family Unification*, Nucl. Phys. B. Proc. Suppl. **77**, 293 (1999).

[82] W. Buchmüller and T. Yanagida, *Quark lepton mass hierarchies and the baryon asymmetry*, Phys. Lett. **B 445**, 399 (1999).

[83] F. Vissani, *Large mixing, family structure, and dominant block in the neutrino mass matrix*, JHEP **11**, 25 (1998).

[84] L. E. Ibanez and G. G. Ross, *Discrete gauge symmetry anomalies*, Phys. Lett. **B 260**, 291 (1991).

[85] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Strings On Orbifolds*, Nucl. Phys. **B 261**, 678 (1985).

[86] S. Weinberg, *Baryon- and lepton-nonconserving processes*, Phys. Rev. Lett. **43**, 1566 (1979).

F. Wilczek and A. Zee, *Operator analysis of nucleon decay*, Phys. Rev. Lett. **43**, 1571 (1979).

[87] J. R. Ellis, M. K. Gaillard and D. V. Nanopoulos, *On the effective Lagrangian for baryon decay*, Phys. Lett. **B 88**, 320 (1979).

[88] S. Aoki et al., *Nucleon decay matrix elements from lattice QCD*, Phys. Rev. **D 62**, 014506 (2000).

[89] L. E. Ibanez and C. Munoz, *Enhancement Factors For Supersymmetric Proton Decay In The Wess-Zumino Gauge*, Nucl. Phys. **B 245**, 425 (1984).

[90] A. Hebecker and J. March-Russell, *The structure of GUT breaking by orbifolding*, Nucl. Phys. **B 625**, 128 (2002).

[91] P. Fayet, *Supersymmetric Grand Unification In A Six-Dimensional Space-Time*, Phys. Lett. **B 159**, 121 (1985).

[92] T. Asaka, W. Buchmüller and L. Covi, *Exceptional coset spaces and unification in six dimensions*, Phys. Lett. **B 540**, (2002).

[93] T. Asaka, W. Buchmüller and L. Covi, *Bulk and brane anomalies in six dimensions*, Nucl. Phys. **B 648**, 231 (2003).

[94] T. Asaka, W. Buchmüller and L. Covi, *Quarks and leptons between branes and bulk*, Phys. Lett. **B 563**, 209–216 (2003).

[95] A. Hebecker and J. March-Russell, *Proton decay signatures of orbifold GUTs*, Phys. Lett. **B 539**, 119 (2002).

[96] G. Altarelli, F. Feruglio and I. Masina, *Models of neutrino masses: Anarchy versus hierarchy*, JHEP **01**, 35 (2003).

[97] B. Bajc, P. F. Perez and G. Senjanović, *Proton decay in minimal supersymmetric SU(5)*, Phys. Rev. **D 66**, 075005 (2002).

[98] J. R. Ellis and D. V. Nanopoulos, *Flavor Changing Neutral Interactions In Broken Supersymmetric Theories*, Phys. Lett. **B 110**, 44 (1982).

R. Barbieri and R. Gatto, *Conservation Laws For Neutral Currents In Spontaneously Broken Supersymmetric Theories*, Phys. Lett. **B 110**, 211 (1982).

[99] T. Kobayashi, S. Raby and R.-J. Zhang, *Constructing 5d orbifold grand unified theories from heterotic strings*, hep-ph/0403065.

[100] D. B. Cline, F. Sergiampietri, J. G. Learned and K. McDonald, *LANNDD: A massive liquid argon detector for proton decay, supernova and solar neutrino studies, and a neutrino factory detector*, Nucl. Instrum. Meth. **A 503**, 136 (2003).

C. K. Jung et al., *http://superk.physics.sunysb.edu/nngroup/uno/UNO_Narrative.pdf.*

K. Nakamura, *Hyper-Kamiokande: A next generation water Cherenkov detector*, Int. J. Mod. Phys. **A 18**, 4053 (2003).

Y. Y. Suzuki et al., *Multi-Megaton water Cherenkov detector for a proton decay search: TITAND (former name: TITANIC)*, hep-ex/0110005.

[101] V. D. Barger, M. S. Berger and P. Ohmann, *Supersymmetric grand unified theories: Two loop evolution of gauge and Yukawa couplings*, Phys. Rev. **D 47**, 1093 (1993).

V. D. Barger, M. S. Berger and P. Ohmann, *niversal evolution of CKM matrix elements*, Phys. Rev. **D 47**, 2038 (1993).

V. D. Barger, M. S. Berger, P. Ohmann and R. J. N. Phillips, *Phenomenological implications of the m(t) RGE fixed point for SUSY Higgs boson searches*, Phys. Lett. **B 314**, 351 (1993).

[102] A. Masiero, D. V. Nanopoulos, K. Tamvakis and T. Yanagida, *Naturally Massless Higgs Doublets In Supersymmetric SU(5)*, Phys. Lett. **B 115**, 380 (1982).

B. Grinstein, *A Supersymmetric SU(5) Gauge Theory With No Gauge Hierarchy Problem*, Nucl. Phys. **B 206**, 387 (1982).

[103] J. Hisano, T. Moroi, K. Tobe and T. Yanagida, *Suppression of proton decay in the missing partner model for supersymmetric SU(5) GUT*, Phys. Lett. **B 342**, 138 (1995).

[104] G. Altarelli, F. Feruglio and I. Masina, *From minimal to realistic supersymmetric SU(5) grand unification*, JHEP **11**, 040 (2000).

[105] S. M. Barr, *A New Symmetry Breaking Pattern For SO(10) And Proton Decay*, Phys. Lett. **B 112**, 219 (1982).

[106] J. P. Derendinger, J. E. Kim and D. V. Nanopoulos, *Anti - SU(5)*, Phys. Lett. **B 139**, 170 (1984).

I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, *Supersymmetric Flipped SU(5) Revitalized*, Phys. Lett. **B 194**, 231 (1987).

[107] L. F. Abbott and M. B. Wise, *The Effective Hamiltonian For Nucleon Decay*, Phys. Rev. **D 22**, 2208 (1980).

[108] R. E. Shrock and L.-L. Wang, *New, Generalized Cabibbo Fit and Application to Quark Mixing Angles in the Sequential Weinberg-Salam Model*, Phys. Rev. Lett. **41**, 1692 (1978).

M. Bourquin, *Measurements Of Hyperon Semileptonic Decays At The Cern Super Proton Synchrotron. 4. Tests Of The Cabibbo Model*, Z. Phys. **C21**, 27 (1983).

[109] N. Cabibbo, E. C. Swallow and R. Winston, *Semileptonic hyperon decays*, Ann. Rev. Nucl. Part. Sci. **53**, 39 (2003).

[110] Y. Aoki, *Nucleon decay matrix elements for domain-wall fermions*, Nucl. Phys. Proc. Suppl. **119**, 380 (2003).

N. Tsutsui et al., *Lattice QCD calculation of the proton decay matrix element in the continuum limit*, hep-lat/0402026.

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