Simple Space-Time Symmetries in Quantum Field Theory

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Abstract

In this thesis a program to study quantum field theories with simple space-time symmetries is introduced. These symmetries are given by simple Lie groups which are required to act on the Hilbert space of states by unitary positive energy representations (spectrum condition). It is assumed that the action of the symmetry group on space-time leaves a globally causal structure invariant. The classical examples are conformal field theories in any dimension. The properties of the conformal groups and their action on space-time are generalized, and the simple Lie groups with these properties are shown to be universal coverings of the non-compact Lie groups $SO(2,D)$, $SO^*(4n)$, $SU(m,m)$, $Sp(l,\mathbb{R})$, and the exceptional group $E_{7(-25)}$. Space-time is given by the quotient ${}^{G}\!/_{H}$, where G is one of the listed simple Lie groups and H the identity component of a maximal parabolic subgroup. It is also shown that the classical conformal groups $SO(2,D)$ are the only allowed groups possessing an automorphism which can be interpreted as a time reflection. The remaining generalized conformal space-times have an intrinsic chiral structure. Methods for the explicit analysis of Lie algebras are developed and have been implemented in the computer algebra package LambdaLie. An application of these methods in the study of non-compact gauge groups is presented.

Zusammenfassung

In dieser Arbeit wird ein Programm vorgestellt, in dem Quantenfeldtheorien mit einfachen Raumzeitsymmetrien studiert werden. Diese Symmetrien sind durch einfache Liegruppen gegeben, von welchen gefordert wird, auf den Hilbertraum der Zustände durch unitäre positive Energiedarstellungen zu wirken (Spektrumsbedingung). Es wird angenommen, dass die Wirkung der Symmetriegruppe auf die Raumzeit eine global kausale Struktur invariant lässt. Die klassischen Beispiele sind konforme Feldtheorien in beliebiger Dimension. Die Eigenschaften der konformen Gruppen und ihrer Wirkung auf die Raumzeit werden verallgemeinert und es wird gezeigt, dass die einfachen Liegruppen mit diesen Eigenschaften universelle Überlagerungen der nichtkompakten Liegruppen $SO(2,D)$, $SO^*(4n)$, $SU(m,m)$, $Sp(l,\mathbb{R})$ und der exzeptionellen Gruppe $E_{7(-25)}$ sind. Die Raumzeit ist durch den Quotientenraum ${}^{G}\!/_{H}$ gegeben, wobei G eine der oben angeführten einfachen Liegruppen und H die Identitätskomponente einer maximalen parabolischen Untergrupppe ist. Es wird außerdem gezeigt, dass die klassischen konformen Gruppen $SO(2,D)$ die einzigen erlaubten Gruppen sind, die einen Automorphismus besitzen, der als Zeitreflektion interpretiert werden kann. Die übrigen verallgemeinerten konformen Raumzeiten haben eine intrinsische chirale Struktur. Methoden zur expliziten Analyse der Liealgebren werden entwickelt und sind in dem Computeralgebrapaket LambdaLie implementiert worden. Eine Anwendung dieser Methoden im Studium nichtkompakter Liegruppen wird präsentiert.

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Chapter 1

Introduction

The symmetries of space-time play a central role in the foundations of any physical theory. The central subject of this work is the generalization of conformal space-time symmetry. These are continuous symmetries which are described by simple real Lie groups. The focus will be on space-time symmetry in quantum field theory satisfying the spectrum condition.

The original motivation for studying generalized conformal symmetry was given by the appearance of conformal realizations of the exceptional real simple Lie group $E_{7(7)}$ in supergravity [Koe01, GKN01a]. This group cannot be used to describe space-time symmetries because it does not admit unitary positive energy representations. However, the exceptional group $E_{7(-25)}$, another real form of the complex Lie group E_7, is known to have that property.

The rich symmetry structure of the conformal group $SO(2,D)$, or more suitably its universal covering, is exploited in quantum field theory with D-dimensional conformal space-time symmetry to derive non-perturbative results [Mac77b, Mac73, Mac74a, Mac74b], which greatly restrict the theory. Studying generalized space-time symmetries which admit analogous results in quantum field theory is therefore very attractive.

The main goal is to classify all simple real Lie groups G together with their action on a space-time manifold $\mathbb{M}$ which admit an interpretation as space-time symmetry in quantum field theory satisfying the spectrum condition. The main properties of the conformal symmetry given by $SO(2,D)$ which enable the derivation of the non-perturbative results will also be assumed in this classification.

For a Lie group G to be admissible as space-time symmetry in quantum field theory it has to possess unitary representations U. In order to satisfy the spectrum condition, there has to be a generator H_0 (iH_0 is an element of the

Lie algebra of G.) which is represented by a self-adjoint operator $\mathsf{U}(H_0)$ having non-negative spectrum. Such a representation is called a unitary positive energy representation because iH_0 will be interpreted as a generator of time translations.

The operation of the space-time symmetry group G should also leave a globally causal structure in the space-time manifold $\mathbb{M}$ invariant. Such a causal structure is given by an assignment of a cone in the tangent space to each point in $\mathbb{M}$. The vectors in these cones specify the future-directed time-like directions at each point. Note that no reference is made to a metric on the manifold. It will turn out that the possible causal structures are strongly restricted by their invariance under the group action. This also imposes strong restrictions on the global structure of the space-time manifold.

The distinguishing feature of the conformal group $SO(2,D)$ when compared to the Poincaré group is that it is a non-compact simple real Lie group. The simple Lie groups have the richest structure among all Lie groups. This is the source of the strong restrictions imposed on the quantum field theory by the conformal invariance. Therefore, it is natural to restrict our attention to simple Lie groups.

The action of the group G on the space-time $\mathbb{M}$ is assumed to be transitive. This makes it possible to identify $\mathbb{M}$ with the quotient space G/H, where H is the subgroup of G which leaves an arbitrary point in $\mathbb{M}$ invariant. In the conformal case $SO(2,D)$, this quotient space is the universal covering of the conformally compactified Minkowski space. The stability subgroup H is the identity component of a parabolic subgroup. It is generated by the Lorentz group, dilatation group, and the group of special conformal transformations. Suitable generalizations of these subgroups of $SO(2,D)$ will be found in the general structure theory of parabolic subgroups. The generalized special conformal transformations are required to act trivially on the tangent space of $\mathbb{M}$. This is equivalent to the requirement that the translations, which are in G but not in H, commute. The analysis will show that this also implies a 3-grading of the Lie algebra of G.

For the classification of the simple Lie groups and parabolic subgroups which allow an interpretation as space-time symmetries, it is necessary to employ two complementary approaches to the structure theory of real simple Lie algebras. These approaches are based on two non-conjugate Cartan subalgebras. Note that non-compact real Lie algebras in general have more than one conjugacy class of Cartan subalgebras, which differ in the dimension of their compact subspace. It is necessary to use the structure theory based on a maximally compact Cartan subalgebra in order to check for the existence of positive energy representations. Parabolic subgroups, on the other hand, are described by starting with a maximally non-compact Cartan subalgebra.

The main result of this work is the following classification: The simple Lie groups which satisfy the above requirements are

$$SO(2,D), \;\; SO^*(4n), \;\; SU(m,m), \;\; Sp(l,\mathbb{R}), \quad \text{and} \quad E_{7(-25)} \; .$$

It turns out that of these groups only the classical conformal group $SO(2,D)$ has an automorphism which can be interpreted as a time reflection. This will have implications on an Euclidean version of the theories.

The general theory of Lie algebras and Lie groups necessary for the classification of simple Lie groups which describe a generalized space-time symmetry is presented in chapter 2. Most of the mathematical tools used can be found in several places in the mathematical literature. Automorphisms of Lie algebras are our fundamental tool and consequently the presentation of this subject will be quite exhaustive. An attempt has been made to unify two alternative approaches to automorphisms which are usually treated separately.

Even though the structure theory of complex Lie algebras in the literature seems complete and explicit, it is a non-trivial task to construct Lie algebras starting from Dynkin diagrams all the way down to the structure constants including their relative signs. An algorithm for determining these constants is presented. The structure theory of real Lie algebras and parabolic subalgebras is presented in the same constructive spirit.

One of the major accomplishments during this PhD work is the development of the computer algebra package LambdaLie presented in section 2.7. It is capable of constructing all real or complex simple Lie algebras starting from the type and rank and, for the real case, some additional information. Two alternative possibilities for the construction of real Lie algebras are implemented, based respectively on a maximally compact Cartan subalgebra and a maximally non-compact Cartan subalgebra. It can be searched for automorphisms of the root system with specific properties which then are checked for possible extensions to automorphisms of the Lie algebra and all parabolic subalgebras can be constructed.

The central results described above are developed in chapter 3.

In chapter 4 an outlook on the future development of quantum field theory with generalized conformal space-time symmetry is given. It is indicated which steps in the non-perturbative study of such theories carry over from conformal field theory and which aspects need more insight. Research field from mathematics and physics which are related to these studies are briefly reviewed. Techniques from the study of causal symmetric spaces, Jordan algebras, and from supergravity might be useful in future developments in generalized conformal symmetry.

The techniques described in chapter 2 also find an application when considering gauge theories which admit parallel transporters in a non-compact holonomy group H as is studied in [Leh04]. The results of this are given in chapter 5. The maximally compact subgroup of H is identified with the gauge group G. (The quotient H/G is then a symmetric space.) If the theory is symmetric only under gauge transformations in G, a non-trivial potential term for the parallel transporters is allowed in the action functional which might have a minimum different from zero. If the vacuum expectation value of the parallel transporter assumes such a minimum, the gauge symmetry is broken. The main result presented in chapter 5 is that the unbroken symmetry always is the intersection of a parabolic subgroup of H with the gauge group G.

Chapter 2

The Structure of Lie Algebras and Lie Groups

All the methods and tools used in the remainder of this work will be introduced in this chapter. Most of the mathematical results presented here were developed over a long period of time and can be found in several textbooks. Here, the theory is presented in a way which emphasizes the constructive aspects of the structure theory of semi-simple Lie algebras and groups. This is most apparent in section 2.2, where it is shown, how complex simple Lie algebras are explicitly constructed starting from the Cartan matrix. The author does not know of any publication which provides a complete construction of the structure constants as is given here.

Automorphisms of semi-simple Lie algebras play an important role in the structure theory of real semi-simple Lie algebras as well as in the study of space-time symmetries in chapter 3 and are therefore treated thoroughly in section 2.3. Two complementary approaches to the structure theory of real semi-simple Lie algebras are presented in section 2.4. Both will be important in chapter 3. One of them is a prerequisite for the definition of parabolic subalgebras as described in section 2.5 and the corresponding generalized Bruhat decomposition of real semi-simple Lie algebras. Explicit constructions of bases of the real Lie algebras and the parabolic subalgebras will be presented. General graded structures are an important tool needed in the generalization of conformal symmetry in chapter 3 and will be defined in section 2.6.

The constructive approach has been implemented by the author in the computer algebra package LambdaLie by the author which will be introduced in section 2.7. The feature which distinguishes LambdaLie from similar computer packages is the possibility to construct Lie algebras, parabolic subalgebras, and other related subalgebras completely starting from the Cartan matrix.

This chapter concludes with a detailed discussion of global structure of Lie groups and their relationships with Lie algebras in section 2.8. Semi-simple Lie groups with infinite center are of special interest to the study of space-time symmetries. The most important global result is the generalized Bruhat decomposition of semi-simple Lie groups.

2.1 Introduction to Lie Algebras

This section mentions only the most essential notions of Lie algebra theory and is in no way intended as a substitute for a thorough introduction which can be found in the literature [Cor84, Hel78].

A real or complex *Lie algebra* $\mathfrak{g}$ is a real or complex vector space equipped with a bi-linear antisymmetric product $[\cdot,\cdot]$, called *commutator* or *Lie multiplication*, which satisfies the Jacobi identity. A Lie algebra is *Abelian* if the commutator is always zero.

A *representation* of $\mathfrak{g}$ is a linear map of $\mathfrak{g}$ into the endomorphisms $\operatorname{End} V$ of a vector space V preserving the Lie product, which in $\operatorname{End} V$ is given as $[A,B] = AB - BA$. The obvious non-trivial representation of $\mathfrak{g}$ is the *adjoint representation*

$$\operatorname{ad}(X)Y := [X,Y] \qquad \text{for } X, Y \in \mathfrak{g} \ .$$

It is a representation, because of the Jacobi identity.

A subalgebra of $\mathfrak{g}$ which is invariant under $\operatorname{ad}(X)$ for all $X \in \mathfrak{g}$ is called an *invariant* subalgebra or *ideal*. $\mathfrak{g}$ is *semi-simple* if it does not contain an Abelian ideal and *simple* if it contains no ideal other than $\mathfrak{g}$ and $\{0\}$ and $\dim \mathfrak{g} > 1$. Each semi-simple Lie algebra has a unique decomposition into simple ideals:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n \ . \tag{2.1}$$

With this, a classification of all isomorphy classes of simple Lie algebras automatically gives all semi-simple Lie algebras. This can easily be extended to *reductive* Lie algebras which are defined to be the direct sum of a semi-simple and an Abelian ideal, which obviously is completely specified by its dimension.

The *Killing form* on a Lie algebra $\mathfrak{g}$ is defined to be

$$B(X,Y) := \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(X)\operatorname{ad}(Y)) \ . \tag{2.2}$$

It is a symmetric bi-linear form on $\mathfrak{g}$, non-degenerate if and only if $\mathfrak{g}$ is semi-simple, and the summands in (2.1) are mutually orthogonal. The adjoint representation of a semi-simple Lie algebra is faithful, i.e. the map $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End} \mathfrak{g}$ is injective.

2.1.1 Complex Lie Algebras

The structure theory and classification of semi-simple complex Lie algebras is based on *Cartan subalgebras* $\mathfrak{h}$ defined to be maximal Abelian subalgebras which consist of ad-semi-simple elements, where $h \in \mathfrak{h}$ is ad-semi-simple, if the endomorphism $\mathrm{ad}(h)$ is semi-simple. A semi-simple endomorphism of a vector space over a field K is defined to be diagonalizable in an algebraically closed field containing K. For complex Lie algebras this simplifies to the condition that $\mathfrak{h}$ contains only ad-diagonalizable elements.

With this, a basis may be chosen consisting of a basis h_i of $\mathfrak{h}$ and mutual eigenvectors e_α of all $\mathrm{ad}(h)$ for $h \in \mathfrak{h}$:

$$[h,e_\alpha] = \alpha(h)e_\alpha \quad \text{for } h \in \mathfrak{h} \ .$$

The function α is an element of $\mathfrak{h}^\vee$, the dual of $\mathfrak{h}$. Each $\alpha \in \mathfrak{h}^\vee$, $\alpha \neq 0$ is called a *root*, if the common eigenspace

$$\mathfrak{g}^\alpha = \{e \in \mathfrak{g} \mid [h,e] = \alpha(h)e \text{ for all } h \in \mathfrak{h}\}$$

is non-zero. The space $\mathfrak{g}^\alpha$ is called *root space*. The set of all roots is denoted by Φ. Since a $h \in \mathfrak{h}$ may be found such that the numbers $\alpha(h)$ for all roots $\alpha \in \Phi$ are different and non-zero, the $\mathfrak{g}^\alpha$ and $\mathfrak{h}$ are eigenspaces of $\mathrm{ad}(h)$ to different eigenvalues and therefore linearly independent. Since $\mathrm{ad}(h)$ is diagonalizable in $\mathbb{C}$, it holds for a complex semi-simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha \ , \tag{2.3}$$

and the root spaces $\mathfrak{g}^\alpha$ turn out to be one-dimensional. Let e_α be a non-zero element of $\mathfrak{g}^\alpha$, then the h_i and e_α form a basis of $\mathfrak{g}$. Also, in contrast to the situation of real Lie algebras, all Cartan subalgebras of a complex Lie algebra are conjugate to each other. The precise notion of conjugacy will be defined in section 2.3.

All roots span a real subspace E of $\mathfrak{h}^\vee$ whose complexification is $\mathfrak{h}^\vee$ and they form a *root system* in this real subspace. A root system Φ is a finite set of elements of a real vector space E which spans E and has the properties that for $\alpha, \beta \in \Phi$

$$A_{\alpha,\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \tag{2.4}$$

is an integer, called the *Cartan integer*, and the *Weyl reflection* $\omega_\alpha(\beta)$ of β along α, which is defined by

$$\omega_\alpha(\beta) = \beta - A_{\alpha,\beta}\alpha = \beta - \tfrac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \ , \tag{2.5}$$

is also an element of Φ. It can be shown that with $\alpha \in \Phi$ the element $m\alpha \in E$ can only be in Φ if $m = \pm\frac{1}{2}, \pm 1, \pm 2$. Φ is a *reduced* root system if only $m = \pm 1$ appears. Root systems of complex Lie algebras are reduced and we will always assume root systems to be reduced unless otherwise stated.

The dimension of the Cartan subalgebra $\mathfrak{h}$ is the *rank* l of the Lie algebra. A basis of $\mathfrak{h}^\vee$ consisting of roots α_i, $i = 1, \ldots, l$ for which $\alpha_i - \alpha_j$ and $\frac{1}{2}\alpha_i$ are not roots is called a *fundamental system* Ψ of roots. The individual roots $\alpha_i \in \Psi$ are called *simple roots*. In the expansion of any root in the simple roots the non-zero coefficients are either all positive or all negative integers. Therefore, they are divided into the set of *positive roots* Φ^+, which contains the simple roots, and *negative roots* Φ^-.

The Killing form defines to each root $\alpha \in \Phi$ a dual vector $h_\alpha \in \mathfrak{h}$ and a bi-linear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^\vee$ via

$$B(h_\alpha, h) := \alpha(h) \quad \text{and} \quad \langle \alpha, \beta \rangle := B(h_\alpha, h_\beta)\ .$$

The bi-linear form $\langle \cdot, \cdot \rangle$ turns out to be positive definite on the real subspace E of $\mathfrak{h}^\vee$, which is therefore an Euclidean space. This also means that the Killing form $B(\cdot, \cdot)$ is positive definite on the span of $h_\alpha, h_\beta, \ldots$ and negative definite on $ih_\alpha, ih_\beta, \ldots$.

The dual basis to Ψ consists of h_{α_i} for $i = 1, \ldots, l$. The *Cartan matrix* A is defined by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}\ . \tag{2.6}$$

It fixes a complex semi-simple Lie algebra up to isomorphy, as we will see in section 2.2. The main tool of the classification of complex simple Lie algebras is the *Dynkin diagram* from which the Cartan matrix A can be read of directly. A Dynkin diagram is essentially a directed multi-graph. Each node $---\circ---$ in the graph represents a row and a column of the Cartan matrix. If there is no edge between the nodes i and j, it is $A_{ij} = A_{ji} = 0$. If there are n edges directed from node i to node j, it is $A_{ij} = -1$ and $A_{ji} = -n$.The diagonal entries are $A_{ii} = 2$. If a Dynkin diagram is connected the corresponding Lie algebra is simple. For non-reduced root systems the simple roots α_i for which $2\alpha_i$ is also a root are marked by a double node $---\odot---$.

Each complex simple Lie algebra corresponds to one of the reduced Dynkin diagrams listed in table 2.1, which also contains the Dynkin diagram of the only non-reduced root system BC_l ([Hel78] sections X.3.3 and X.3.4). If two complex simple Lie algebras have the same Dynkin diagram, they are isomorphic. In section 2.2, we will see how a representative of each isomorphism class is explicitly constructed.

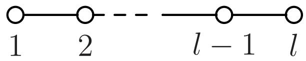
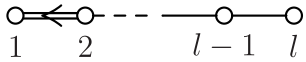
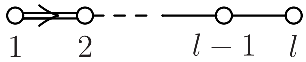
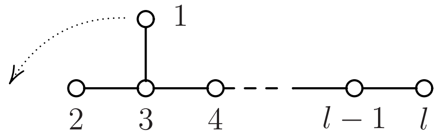
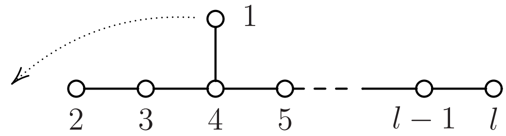
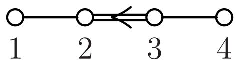
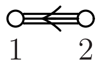
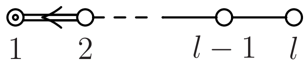
Type	Φ	l
A_l		$1, \ldots$
B_l		$2, \ldots$
C_l		$3, \ldots$
D_l		$4, \ldots$
E_l		$6, 7, 8$
F_4		4
G_2		2
BC_l		$2, \ldots$

Table 2.1: The reduced Dynkin diagrams of all simple complex Lie algebras and the only non-reduced Dynkin diagram BC_l, which does not describe a finite dimensional Lie algebra.

Note that all classes $A, \ldots, E$ may be defined starting with $l = 1$. However this results in several identical diagrams and therefore isomorphic Lie algebras: $A_1 \cong B_1 \cong C_1 \cong E_1$, $B_2 \cong C_2$, $D_2 \cong E_2 \cong A_1 \oplus A_1$ (semi-simple), $D_3 \cong A_3$, $E_3 \cong A_2 \oplus A_1$ (semi-simple), $E_4 \cong A_4$, $E_5 \cong D_5$. These isomorphisms are found again among the real Lie algebras. $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ is probably the best known example and corresponds to $A_1 \cong \mathfrak{b}_1$. Because of the isomorphy $\mathfrak{so}(2) \cong \mathfrak{u}(1)$ the notation D_1 denotes the one-dimensional complex Lie algebra, which has no Dynkin diagram.

If any connected directed multi-graph which does not appear in table 2.1 is interpreted as a Dynkin diagram, the resulting Lie algebra would be of infinite dimension (Kač-Moody algebras) or are even impossible to realize. The proof of this involves a detailed analysis which can be found e.g. in [Jac62].

2.1.2 Real Lie Algebras

Since, as mentioned, some of the facts vital for this way of classification are not valid for real Lie algebras, the classification of real Lie algebras is performed in two steps: Firstly, the complex simple Lie algebras ${}^{\mathbb{C}}\mathfrak{g}$ via Dynkin diagrams are classified, and then all *real forms*, i.e. all real Lie algebras $\mathfrak{g}$ with *complexification* $(\mathfrak{g})_{\mathbb{C}}={}^{\mathbb{C}}\mathfrak{g}$ are found. For the classification of simple real Lie algebras $\mathfrak{g}$ it is important to know that its complexification $(\mathfrak{g})_{\mathbb{C}}$ is either a complex simple Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$ or the direct sum of two complex simple Lie algebras which are isomorphic $(\mathfrak{g})_{\mathbb{C}} = {}^{\mathbb{C}}\mathfrak{g} \oplus {}^{\mathbb{C}}\mathfrak{g}$ [Gan39b]. In the latter case the real simple Lie algebra is just the complex simple Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$ considered as a real Lie algebra.

The complexification $(\mathfrak{g})_{\mathbb{C}}$ of a real Lie algebra $\mathfrak{g}$ comes with a *conjugation* σ with respect to $\mathfrak{g}$, which is defined to be an anti-linear involution (automorphism of order 2 – defined in section 2.3) of $(\mathfrak{g})_{\mathbb{C}}$ which is the identity on $\mathfrak{g}$ and $-\mathbb{1}$ on $i\mathfrak{g}$. Explicitly, $\sigma^2 = \mathbb{1}$, $\sigma(X + iY) = X - iY$ for $X, Y \in \mathfrak{g}$, and $\sigma([X,Y]) = [\sigma(X),\sigma(X)]$ for $X, Y \in (\mathfrak{g})_{\mathbb{C}}$. Given any simple complex Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$, its real forms may be found by finding all (classes of) anti-linear involutions σ of ${}^{\mathbb{C}}\mathfrak{g}$ and determining the real subalgebra of σ-stable vectors. This method will be employed in section 2.4.2 for the classification and structure theory of real Lie algebras.

A real semi-simple Lie algebra has a natural but non-unique decomposition into subspaces on which $B(\cdot, \cdot)$ is negative definite and positive definite, respectively:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \ . \tag{2.7}$$

This is the *Cartan decomposition*. It plays an important role in the structure theory of real semi-simple Lie algebras. To it corresponds the *Cartan involution* θ which has the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ to the eigenvalues 1 and -1, respectively. The classification of Cartan involutions is another method for finding real forms. It will be discussed in section 2.4.1.

A real Lie algebra is defined to be *compact* if its Killing form is negative definite. If this is the case, it is semi-simple. The name might be confusing, because the Lie algebra is clearly not compact as a topological space. It stems from the fact that any Lie group corresponding to it will be compact. Another complication with this notion is the fact that the so-called compact part of the Cartan decomposition $\mathfrak{k}$ might not be a compact Lie algebra: The Killing form of $\mathfrak{g}$ restricted to $\mathfrak{k}$ is in general not equal to the Killing form of $\mathfrak{k}$. (The trace "knows" the whole Lie algebra.) It turns out that $\mathfrak{k}$ is reductive. The semi-simple part is compact, while the Killing form is zero on the Abelian ideal, which is therefore not compact. This situation will arise at a prominent place in the discussion of space-time symmetries.

Given a basis h_{α_i}, e_α of ${}^{\mathbb{C}}\mathfrak{g}$ as described above there are two obvious real forms of ${}^{\mathbb{C}}\mathfrak{g}$ which are given by specifying a basis: The *compact form* $\mathfrak{g}_{\mathrm{cpt}}$ has the basis

$$(2.8)\qquad ih_{\alpha_i}, \quad e_\alpha + e_{-\alpha}, \quad ie_\alpha - ie_{-\alpha}\ ,$$

and the *normal form* $\mathfrak{g}_{\mathrm{nrm}}$ has the basis

$$(2.9)\qquad h_{\alpha_i}, \quad e_\alpha$$

for all simple roots α_i and all roots α. This is possible, because this basis of the complex Lie algebra may be normalized such that the structure constants are real and such that $B(\cdot,\cdot)$ is negative definite on (2.8). The details will be discussed in section 2.2.4.

The conjugation τ with respect to the compact form of a complex Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$ will be used frequently. It is given explicitly for (2.8) by

$$(2.10)\qquad \tau(h_{\alpha_i}) = -h_{\alpha_i}, \qquad \tau(e_\alpha) = e_{-\alpha}\ .$$

If the Cartan involution θ of any real form $\mathfrak{g}$ of ${}^{\mathbb{C}}\mathfrak{g}$ commutes with the conjugation τ of the compact form, it is related to the conjugation σ of $\mathfrak{g}$ via

$$(2.11)\qquad \sigma = \theta \circ \tau\ .$$

This shows the equivalence of the two approaches to the task of finding real forms described above.

From a Cartan decomposition of a real Lie algebra $\mathfrak{g}$ a compact form of its complexification $(\mathfrak{g})_{\mathbb{C}}$ can be constructed: $\mathfrak{g}_{\mathrm{cpt}} = \mathfrak{k} \oplus i\mathfrak{p}$. There is a basis of $(\mathfrak{g})_{\mathbb{C}}$ such that the conjugation in $(\mathfrak{g})_{\mathbb{C}} = (\mathfrak{g}_{\mathrm{cpt}})_{\mathbb{C}}$ with respect to $\mathfrak{g}_{\mathrm{cpt}}$ is given by τ in equation (2.10) because of conjugacy of compact forms and Cartan subalgebras.

2.2 Constructing Complex Lie Algebras

To the reader familiar with root systems the following constructive approach will appear rather unorthodox. The intention here is not to present yet another version of the arguments which can be found in many good text books, but to show how a Lie algebra may be explicitly constructed given only the Dynkin diagram. Normally, this is included in the theory in a rather implicit way, because, for mathematicians, it is more natural to assume the existence of a given object (Lie algebra) specified by a set of properties and then discover structure and prove facts based on these properties. The approach presented here tries to present a recipe which can be easily employed, e.g. for writing an algorithm on a computer. This has in fact been done by the author as will be described in section 2.7.

The first step, the construction of the Cartan matrix A, was already given in section 2.1.1. In the next sections the root system, the basis h_{α_i}, e_α, and the structure constants will be constructed. After that, different conventions will be discussed.

2.2.1 Constructing the Root System

Given a Cartan matrix A of dimension l the corresponding reduced root system Φ is constructed in an l dimensional real vector space E as follows. Choose a basis $\alpha_1, \ldots, \alpha_l$ of E – the elements of this basis will be the simple roots. The remaining roots are integer linear combinations of the simple roots with either all positive or all negative coefficients. Thus, they are divided in positive and negative roots. The root systems of semi-simple Lie algebras are finite. Some of the Cartan integers (2.4) can be obtained from the Cartan matrix by linear extension in the second index, but not in the first:

$$A_{\alpha_i,\beta} = \sum_{j=1}^{l} A_{ij}\beta^j$$

for the root $\beta = \sum \beta^i \alpha_i$. Given any two roots α and β the *α-string of roots through β* is defined to be the set of all roots of the form $\beta + k\alpha$. Theorem 13.5.IX in [Cor84] states that $\beta + k\alpha$ is a root for all $-p \leq k \leq q$ for some positive or zero integers p and q. For the α_i-string of roots through β these integers are constrained by

$$p - q = A_{\alpha_i,\beta}\ . \tag{2.12}$$

All positive roots can be constructed recursively from the simple roots by computing q with this equation: Let a root β be of *level* n, if $\sum \beta^i = n$ for $\beta = \sum \beta^i \alpha_i$. The simple roots are the roots of level 1. From all the roots β of level n, the roots of level $n+1$ are found by considering the α_i-string through β for all simple roots α_i. p is known because $\beta - p\alpha_i$ has to be a positive root of level not greater than n and p can be found by inspecting the roots of level n through 1 which have been constructed during the earlier steps of the induction. If q is greater zero, $\beta + \alpha_i$ is a root of level $n+1$. All positive roots in Φ can be found in that way. The negative roots can be obtained because $-\alpha$ is a negative root if α is a positive root and vice versa.

This procedure will be illustrated for $D_3 = A_3$, because it is the complex Lie algebra of the conformal group in four space-time dimensions $SO(4,2)$. The simple roots are

$$\begin{aligned}\alpha_1 &= (1,0,0)\\ \alpha_2 &= (0,1,0)\\ \alpha_3 &= (0,0,1)\ .\end{aligned}$$

The Cartan matrix derived from the Dynkin diagram is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the remaining positive roots are

$$\begin{aligned}\alpha_4 &= (1,1,0) && \text{because } A_{\alpha_1,\alpha_2} = -1 \text{ and } p = 0\ ,\\ \alpha_5 &= (0,1,1) && \text{because } A_{\alpha_2,\alpha_3} = -1 \text{ and } p = 0\ ,\\ \alpha_6 &= (1,1,1) && \text{because } A_{\alpha_4,\alpha_3} = -1 \text{ and } p = 0\ .\end{aligned}$$

$\beta = (1,0,1)$ is not a root, because $A_{\alpha_1,\alpha_3} = 0$ and $p = 0$, and $\beta = (2,1,1)$ is not a root, because $A_{\alpha_6,\alpha_1} = 1$ and $p = 1$. Finally, $\beta = (1,2,1)$ is not a root, because $A_{\alpha_6,\alpha_2} = 0$ and $p = 0$.

The scalar product $\langle\cdot,\cdot\rangle$ on E is needed, which is related to the Cartan matrix by (2.6). In order to determine the scalar product from the Cartan Matrix, the squares $\langle\alpha_i,\alpha_i\rangle$ of the simple roots are needed. However, the scalar product obeys

$$\langle\alpha,\beta\rangle = \sum_{\gamma\in\Phi}\langle\alpha,\gamma\rangle\langle\beta,\gamma\rangle\ ,$$

where Φ is the set of all roots (equation (13.24) of [Cor84]). This equation determines the squares of the simple roots as

$$\langle\alpha_i,\alpha_i\rangle = \Big(\tfrac{1}{2}\sum_{\gamma\in\Phi^+}\big(A_{\alpha_i,\gamma}\big)^2\Big)^{-1}\ , \tag{2.13}$$

where Φ^+ is the set of positive roots. In the example of A_3 this is e.g.

$$\langle\alpha_1,\alpha_1\rangle = 2\big((2)^2 + (-1)^2 + (0)^2 + (1)^2 + (-1)^2 + (1)^2 +\big)^{-1} = \tfrac{1}{4}\ .$$

If one square is calculated in this way, the others can be obtained with

$$\langle\alpha_i,\alpha_i\rangle = \frac{A_{ki}}{A_{ik}}\langle\alpha_k,\alpha_k\rangle\ . \tag{2.14}$$

The scalar product is thus

$$\langle\alpha,\beta\rangle = \tfrac{1}{2}\sum_{i,k=1}^{l}\alpha^i\beta^k\langle\alpha_i,\alpha_i\rangle A_{ik}\ . \tag{2.15}$$

2.2.2 Almost the Whole Lie Algebra

A root system with its scalar product describes a basis of a Lie algebra and most of the commutators of the basis elements immediately: The basis consists of h_{α_i} for each simple root α_i and of e_α for each root $\alpha \in \Phi$. The complex span of these is the complex (simple) Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$. The complex subspace spanned by the h_{α_i} is the defining Cartan subalgebra $\mathfrak{h}$. The real subspace $\mathfrak{h}^*$ spanned by the h_{α_i} with real coefficients is the dual of E. We will call it the *vector part* of the Cartan subalgebra. The scalar product $\langle\cdot,\cdot\rangle$ is lifted to $\mathfrak{h}$ via

$$B(h_{\alpha_i},h_{\alpha_j}) = \langle\alpha_i,\alpha_j\rangle$$

which will be the Killing form of the Lie algebra. The duality action of the simple roots on $h \in E$ is defined to be

$$\alpha_i(h) = B(h_{\alpha_i},h)\ .$$

The Killing form can be extended to the remaining Lie algebra by

$$\begin{aligned} B(e_\alpha,e_\beta) &= \begin{cases} B_\alpha & \text{for } \alpha = -\beta \\ 0 & \text{otherwise.}\end{cases} \\ B(h,e_\beta) &= 0\ . \end{aligned}$$

The constants $B_\alpha \in \mathbb{C}$ may be chosen freely.

The commutation relations are

$$\begin{aligned} \left[h_{\alpha_i},h_{\alpha_j}\right] &= 0 \\ [h,e_\alpha] &= \alpha(h)e_\alpha && \text{for } h \in \mathfrak{h} \\ [e_\alpha,e_{-\alpha}] &= B_\alpha h_\alpha && \text{with } h_\alpha = \alpha^i h_{\alpha_i} \\ [e_\alpha,e_\beta] &= n_{\alpha,\beta}e_{\alpha+\beta}\ . \end{aligned}$$

The only unspecified components are the structure constants $n_{\alpha,\beta}$ which will be dealt with in the next section.

2.2.3 The Structure Constants – $n_{\alpha,\beta}$

According to section 13.6. of [Cor84] the structure constants $n_{\alpha,\beta}$ for $\alpha, \beta \in \Phi$ and $\alpha \neq \beta$ satisfy

$$n_{\alpha,\beta} = 0 \qquad \text{for } \alpha,\beta \in \Phi \text{ and } \alpha+\beta \notin \Phi$$

$$n_{\alpha,\beta} = -\,n_{\beta,\alpha} \qquad \text{for } \alpha,\beta \in \Phi \tag{2.16}$$

$$n_{\alpha,\beta}B_\gamma = n_{\beta,\gamma}B_\alpha = n_{\gamma,\alpha}B_\beta \qquad \text{for } \alpha,\beta,\gamma \in \Phi \text{ and } \alpha+\beta+\gamma = 0 \tag{2.17}$$

$$n_{\alpha,\beta}\, n_{\gamma,\delta}B_{\alpha+\beta} + n_{\beta,\gamma}\, n_{\alpha,\delta}B_{\beta+\gamma} + n_{\gamma,\alpha}\, n_{\beta,\delta}B_{\alpha+\gamma} = 0 \tag{2.18}$$
$$\text{for } \alpha,\beta,\gamma,\delta \in \Phi \text{ and } \alpha+\beta+\gamma+\delta = 0$$
$$\text{and } x+y \neq 0 \text{ for } x,y = \alpha,\beta,\gamma,\delta$$

$$n_{\alpha,\beta}\, n_{-\alpha,-\beta} = -\frac{\langle\alpha,\alpha\rangle B_\alpha B_\beta}{2B_{\alpha+\beta}} q(p+1) \tag{2.19}$$
$$\text{for } \alpha,\beta \in \Phi \text{ and } q,\ p \text{ for the } \alpha\text{-string through } \beta.$$

The equations (2.17) and (2.18) follow from the Jacobi equation, and (2.19) can be deduced, when interpreting the span of all e_γ for γ in the α-string as a representation space of the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra spanned by h_α and $e_{\pm\alpha}$. There is also a relation

$$n_{\alpha,\beta} = \epsilon\, n_{-\alpha,-\beta}\ , \tag{2.20}$$

where ϵ can be chosen to be either $\epsilon = 1$ or $\epsilon = -1$. Other values of ϵ are only possible, if ϵ depends on α and β. We will only consider constant ϵ.

For the constructive determination of the $n_{\alpha,\beta}$ a complete ordering of the positive roots is needed which extends the partial ordering given by the level of each root. This can be chosen arbitrarily. The order in which the positive roots are generated in the construction described in section 2.2.1 may be used. In that case each root β of level $n+1$ is obtained via $\beta = \alpha + \sigma$, where α is of level n and σ is a simple root. The root α has to be the lowest root in the complete ordering with this property. This complete ordering is used by the computer algebra package LambdaLie (section 2.7).

Observation 1 *The $n_{\alpha,\beta}$ can be determined with the following algorithm:*
Fix a complete ordering of the positive roots which refines the partial ordering given by the level of each root.
For any two roots $\rho,\eta \in \Phi$ $n_{\rho,\eta}$ is determined by the first of the following equations which is applicable. If the right hand side of the relevant equation contains other $n_{\alpha,\beta}$, these have to be determined by the same rule. This recursion ends after a finite number of steps.

$$n_{\rho,\eta} = 0 \qquad \textit{if } \rho+\eta \notin \Phi$$

(2.21) $\quad n_{\rho,\eta} = \epsilon\, n_{-\rho,-\eta}$ *if* ρ *and* η *are negative.*

(2.22) $\quad n_{\rho,\eta} = -\, n_{\eta,\rho}$ *if* ρ *is negative and* η *positive.*

(2.23) $\quad n_{\rho,\eta} = -\epsilon\, n_{-\eta,-\rho}$ *if* ρ *positive,* η *and* $\rho+\eta$ *negative.*

(2.24) $\quad n_{\rho,\eta} = \dfrac{\langle\rho,\eta\rangle B_\eta + n_{\rho-\eta,\eta}\, n_{-\eta,\rho}}{n_{-\eta,\rho+\eta}}$ *if* ρ *and* $\rho+\eta$ *positive,* η *negative.*

If ρ *and* η *are both positive, let* σ *be the simple root such that* $\rho+\eta-\sigma$ *is the minimal positive root with respect to the complete ordering. Then*

$n_{\rho,\eta} = -\, n_{\eta,\rho}$ *if* $\eta = \sigma$.

(2.25) $\quad n_{\rho,\eta} = \kappa\sqrt{-\frac{\epsilon}{2}\langle\rho,\rho\rangle\frac{B_\rho B_\eta}{B_{\rho+\eta}}q(p+1)}$ *if* $\rho = \sigma$.

(2.26) $\quad n_{\rho,\eta} = -\, n_{\eta,\rho}$ *if* $\eta-\sigma$ *is not a root,*
or if $\rho-\sigma$ *and* $\rho+\sigma$ *are roots.*

(2.27) $\quad n_{\rho,\eta} = \frac{1}{n_{\eta-\sigma,\sigma}}\left(n_{\sigma,\rho+\eta-\sigma}\, n_{\eta-\sigma,\rho} + n_{\eta-\sigma,\rho+\sigma}\, n_{\rho,\sigma}\right)$
if $\eta-\sigma$ *is a root and* $\rho-\sigma$ *or* $\rho+\sigma$ *is not a root.*

In these formulas any $n_{\alpha,\beta}$ *for* α *or* β *not a root is taken to be zero. In (2.25)* q *and* p *correspond to the* ρ*-string through* η. κ *may be chosen to be either* 1 *or* -1 *each time (2.25) is used. This is the case for the pairs* ρ, η *where* η *is positive,* ρ *simple, and* η *the minimal* β *in the complete ordering for which there is a simple root* α_i *such that* $\beta+\alpha_i = \rho+\eta$.

That the Jacobi identity holds for all basis vectors and that the recursion terminates has been checked for all types of Lie algebras up to rank 10 with the software LambdaLie, which will be presented in section 2.7 – in particular for all exceptional complex simple Lie algebras. There is no reason to believe, that the recursion does not terminate for some higher rank, since there is no fundamental structural difference between the root systems of the classical Lie algebras for rank $l > 5$. The general mechanism of the algorithm is explained in the following:

Equations (2.21), (2.22), and (2.23) follow from (2.16) and (2.20) and reduce all cases with at least one negative root to either the conditions of (2.24) or to two positive roots. Equation (2.24) is easily deduced from the Jacobi identity

$$[e_\rho,[e_\eta,e_{-\eta}]] + [e_{-\eta},[e_\rho,e_\eta]] + [e_\eta,[e_{-\eta},e_\rho]] = 0\ .$$

It contains two terms $n_{-\eta,\rho}$ and $n_{-\eta,\rho+\eta}$ which have positive roots in their indices. If $\rho-\eta$ is a root, the term $n_{\rho-\eta,\eta}$ is recursively obtained again with (2.24). This recursion terminates, because $\rho-k\eta$ is not a root for some positive integer k.

If both ρ and η are positive, the sign of $n_{\rho,\eta}$ can be chosen once for each root $\rho+\eta$ in (2.25), which is deduced from (2.19). With this, the vector $e_{\rho+\eta}$ is fixed. If $\rho \neq \sigma$, equation (2.25) is still correct, however, the normalization κ can not be chosen and has to be determined with (2.27). (2.25) can still be used as a consistency check. Equation (2.27) is deduced from the Jacobi identity

$$[e_\sigma, [e_{\eta-\sigma}, e_\rho]] + [e_\rho, [e_\sigma, e_{\eta-\sigma}]] + [e_{\eta-\sigma}, [e_\rho, e_\sigma]] = 0 \ ,$$

and most of the terms are unproblematic: $n_{\sigma,\rho+\eta-\sigma}$ is given by (2.25), $n_{\eta-\sigma,\rho}$ and $n_{\eta-\sigma,\sigma}$ include the root $\eta-\sigma$ of lower level compared to η in their indices. Their recursion will eventually terminate with (2.25). To show, that the recursion of the term $n_{\eta-\sigma,\rho+\sigma}$ terminates is difficult. In fact, as can be seen with LambdaLie, the condition that either $\rho-\sigma$ or $\rho+\sigma$ is not a root is necessary for the recursion to terminate for the Lie algebras $\mathfrak{c}_l$ and $\mathfrak{f}_4$. This condition avoids the appearance of non-zero $n_{\eta-\sigma,\rho+\sigma}$ as much as possible with equation (2.26).

It follows summary of the choices which can be made during the determination of a basis of the Lie algebra – or alternatively the structure constants:

$$\begin{aligned} &B_\alpha = B_{-\alpha} \in \mathbb{C} && \text{for each positive root } \alpha. \\ &\epsilon = \pm 1 && \\ &\kappa = \kappa(\rho,\eta) = \pm 1 && \text{if } \eta \text{ positive and } \rho = \sigma. \end{aligned} \tag{2.28}$$

If the complete ordering given by the construction of positive roots is used, the choice of κ is made for each positive non-simple root once. This is directly connected to the normalization of the ladder operators:

Fix the vectors e_{α_i} for the simple roots. Their normalizations are independent and can be chosen. Any other vector to a positive root is equal to a commutator of the form $\left[e_{\alpha_{i_n}}, \ldots \left[e_{\alpha_{i_2}}, e_{\alpha_{i_1}}\right] \ldots\right]$ multiplied by some chosen constant. These commutators are determined by the complete ordering. A different choice of order will lead to a different representation of each vector by such a commutator. This can be represented as a complex system [Mac97, dR01]: Each positive root is an object and each (directed) link represents a simple root which has to be added to the root represented by the source of the link in order to obtain the one represented by the target. Paths in this graph are equivalent if the commutator of the corresponding simple roots are equal.

2.2.4 The Weyl and the Chevalley Basis

For the following discussion it is necessary to make the choices mentioned above. For this, observe that if all B_α are chosen real and negative and $\epsilon = 1$

or if the B_α are chosen real and positive and $\epsilon = -1$ all the structure constants will be real. This follows from equation (2.25). There are several frequently used conventions, two of which are

- $B_\alpha = -1$ and $n_{\alpha,\beta} = n_{-\alpha,-\beta}$ – Weyl choice
- $B_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}$ and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ – Chevalley choice

Both have their advantages and disadvantages and we will use both, keeping them apart by denoting the ladder operators e_α and the structure constants $n_{\alpha,\beta}$ for the Weyl choice (corresponding to convention "A" in [Cor84]) and E_α and $N_{\alpha,\beta}$ for the Chevalley choice (corresponding to convention "C"). Between them, the following relations hold for α a positive root:

$$(2.29) \qquad \begin{aligned} E_\alpha &= \sqrt{\tfrac{2}{\langle\alpha,\alpha\rangle}}\, e_\alpha \\ E_{-\alpha} &= -\sqrt{\tfrac{2}{\langle\alpha,\alpha\rangle}}\, e_{-\alpha}\ . \end{aligned}$$

A similar relation between $n_{\alpha,\beta}$ and $N_{\alpha,\beta}$ involves several different cases.

The advantage of the Chevalley choice is that the structure constants are integers, as can be seen from a further simplification of (2.25) which follows from a case-by-case study [Sam69]:

$$N_{\alpha,\beta} = \kappa(p+1)$$

The ladder operators in the Chevalley convention are joined with the following basis of the Cartan subalgebra

$$(2.30) \qquad H_\alpha = \frac{2}{\langle\alpha,\alpha\rangle} h_\alpha$$

The commutators of this basis involve only integers.

The relations in the Weyl and the Chevalley basis are summarized by

Weyl basis	Chevalley basis
$\left[h_{\alpha_i}, h_{\alpha_j}\right] = 0$	$\left[H_{\alpha_i}, H_{\alpha_j}\right] = 0$
$[h_{\alpha_i}, e_\alpha] = \langle\alpha, \alpha_i\rangle e_\alpha$	$[H_{\alpha_i}, E_\alpha] = A_{ij}\alpha^j E_\alpha$
$[e_\alpha, e_{-\alpha}] = -\alpha^j h_{\alpha_j}$	$[E_\alpha, E_{-\alpha}] = H_\alpha$
$[e_\alpha, e_\beta] = n_{\alpha,\beta} e_{\alpha+\beta}$	$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$
$B(h_{\alpha_i}, h_{\alpha_j}) = \langle\alpha_i, \alpha_j\rangle$	$B(H_{\alpha_i}, H_{\alpha_j}) = \frac{2}{\langle\alpha_j,\alpha_j\rangle} A_{ij}$
$B(e_\alpha, e_{-\alpha}) = -1$	$B(E_\alpha, E_{-\alpha}) = \frac{2}{\langle\alpha,\alpha\rangle}$

In both cases the Killing form is not diagonal. An orthogonal basis is given by

Weyl basis	Chevalley basis
$f_\alpha = e_\alpha + e_{-\alpha}$	$F_\alpha = E_\alpha + E_{-\alpha}$
$g_\alpha = e_\alpha - e_{-\alpha}$	$G_\alpha = E_\alpha - E_{-\alpha}$
$B(f_\alpha, f_\alpha) = -2$	$B(F_\alpha, F_\alpha) = \frac{4}{\langle\alpha,\alpha\rangle}$
$B(g_\alpha, g_\alpha) = 2$	$B(G_\alpha, G_\alpha) = -\frac{4}{\langle\alpha,\alpha\rangle}$

And an orthogonal basis of the Cartan subalgebra. This can be obtained from an orthogonal basis b^j in the vector space E via H_{b^j}. For this we introduce the *fundamental weights* $\Lambda^i \in E$ for $i = 1, \ldots, l$ which are defined by the relation

$$A_{\alpha_i,\Lambda^k} = \frac{2\langle\alpha_i, \Lambda^k\rangle}{\langle\alpha_i, \alpha_i\rangle} = \delta_{ik} \ .$$

Therefore we have

$$\Lambda^k = \sum_{j=1}^{l} \alpha_j \left(A^{-1}\right)^{jk} \ .$$

The last fundamental weight Λ^l is clearly orthogonal to the remaining simple roots $\alpha_1, \ldots, \alpha_{l-1}$. They span a subspace E' of E which contains a subsystem Φ' of the root system Φ describing a subalgebra of the Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$. The fundamental weights of this subalgebra are

$$\Lambda^{k'} = \sum_{j=1}^{l-1} \alpha_j \left(A'^{-1}\right)^{jk}$$

where A' is the Cartan Matrix of the subalgebra obtained by deleting the last row and column from the original one. Clearly $\Lambda^{l-1'}$ is orthogonal to Λ^l as well as to $\alpha_2, \ldots, \alpha_{l-2}$. This scheme can be repeated until $\Lambda^l, \Lambda^{(l-1)'}, \Lambda^{(l-2)''}, \ldots,$ and $\Lambda^{1''\cdots'}$ is an orthogonal basis of E. An orthonormal basis is given by

$$b^k = \left(\left(A''^{\cdots\prime -1}\right)^{mm} \frac{\langle\alpha_m,\alpha_m\rangle}{2}\right)^{-\frac{1}{2}} \sum_{j=1}^{m} \alpha_j \left(A''^{\cdots\prime -1}\right)^{jm} \tag{2.31}$$

where $m = l + 1 - k$ and with $k - 1$ primes at each occurrence of the Cartan matrix. Note that since the simple roots can be permuted the procedure above heavily depends on the choice: b^1 is orthogonal to $\alpha_1, \ldots, \alpha_{l-1}$ whereas b^l is proportional to α_1 and not orthogonal to $\alpha_2, \ldots, \alpha_l$.

Another useful, but not orthogonal, basis of E is

$$\mu^j = \left(A^{-1}\right)^{ji} \alpha_i$$

because of $\left[H_{\mu^j}, E_\alpha\right] = \alpha^j E_\alpha$.

2.3 Automorphisms of Semi-Simple Lie Algebras

An automorphism θ of a real or complex Lie algebra $\mathfrak{g}$ is a bijective linear map $\mathfrak{g} \longrightarrow \mathfrak{g}$ which leaves the Lie product invariant:

$$\theta([X,Y]) = [\theta(X),\theta(Y)] \qquad \text{for } X,Y \in \mathfrak{g} \; .$$

The automorphisms of $\mathfrak{g}$ form a group, the *group of automorphisms* $\mathrm{Aut}(\mathfrak{g})$. An automorphism satisfying $\theta^\nu = \mathbb{1}$ for some positive number ν is said to be of order ν.

Let n be the dimension of the Lie algebra $\mathfrak{g}$. Because the map $\mathrm{ad}(X)$ is an endomorphism of the vector space $\mathfrak{g}$ the elements $X \in \mathfrak{g}$ are mapped via the matrix exponentiation $\mathrm{Exp}(\cdot)$ into the matrix group $Gl(n)$ by

$$X \mapsto \mathrm{Exp}(\mathrm{ad}(X)) \; .$$

The subgroup of $Gl(n)$ generated by the image of $\mathfrak{g}$ is the *adjoint group* $\mathrm{Int}(\mathfrak{g})$. In fact, it is a connected Lie group and $\mathfrak{g}$ its Lie algebra. Again because of the Jacobi identity $\mathrm{ad}(X)$ is a *derivation* of the Lie algebra:

$$\mathrm{ad}(X)\,[Y,Z] = [\mathrm{ad}(X)Y,Z] + [Y,\mathrm{ad}(X)Z] \qquad \text{for } X,Y,Z \in \mathfrak{g} \; .$$

This carries over to the exponentiation $\mathrm{Exp}(\mathrm{ad}(X))$ which therefore is an automorphism of the Lie algebra $\mathfrak{g}$. Automorphisms of this form are called *inner automorphisms*. The group $\mathrm{Int}(\mathfrak{g})$, also called the *group of inner automorphisms*, is the connected component of the identity $\mathbb{1}$ of $\mathrm{Aut}(\mathfrak{g})$ and therefore an invariant subgroup. $\mathrm{Aut}(\mathfrak{g})$ is therefore also a Lie group with Lie algebra $\mathfrak{g}$. All automorphisms in $\mathrm{Aut}(\mathfrak{g})$ which are not inner automorphisms are called *outer automorphisms*.

If for two subsets of a Lie algebra $A, B \subset \mathfrak{g}$ there is an inner automorphism θ such that $A = \theta(B)$, they are said to be *conjugate* to each other, while maps $\phi, \psi : \mathfrak{g} \to \mathfrak{g}$ are *conjugate* if there is an inner automorphism θ such that $\psi = \theta^{-1}\phi\theta$. It is natural to consider structures in Lie algebras only *up to conjugacy*.

For a fixed Cartan subalgebra $\mathfrak{h}$ Gantmacher [Gan39a] defines a *chief automorphism* θ_{chief} to be of the form

$$\theta_{\text{chief}} = \mathrm{Exp}(\mathrm{ad}(h))\theta_\vartheta \; , \tag{2.32}$$

where θ_ϑ is an extension of the Dynkin diagram symmetry ϑ defined in section 2.3.1 below and $h \in \mathfrak{h}$ is stable under θ_ϑ. He provides us with theorem 1:

Theorem 1 *Each automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ of a complex semi-simple Lie algebra $\mathfrak{g}$ may be decomposed into $\theta_0 e^{\psi}$ where θ_0 is a diagonalizable automorphism and ψ a derivation with only zero eigenvalues.*

Each diagonalizable automorphism θ_0 is conjugate to a chief automorphism θ_{chief}

$$\theta_0 = U^{-1}\theta_{\text{chief}}U \qquad \textit{where } U \in \mathrm{Int}(\mathfrak{g})\ . \tag{2.33}$$

A proof of the second part will be sketched at the end of the next section.

Let an automorphism θ of a Lie algebra $\mathfrak{g}$ have eigenvalues $\mu_1, \mu_2, \ldots$. $\mathfrak{g}$ decomposes into eigenspaces $\mathfrak{g} = \mathfrak{g}^{(\mu 1)} \oplus \mathfrak{g}^{(\mu_2)} \oplus \cdots$ with the commutation relations

$$\left[\mathfrak{g}^{(\mu_i)},\mathfrak{g}^{(\mu_j)}\right] \subset \mathfrak{g}^{(\mu_i\mu_j)}\ . \tag{2.34}$$

If $\mathfrak{g}$ is a real Lie algebra, the automorphism θ may be canonically extended to the complexification $(\mathfrak{g})_{\mathbb{C}}$ and the same relations hold. On the other hand, it is possible to restrict an automorphism of $(\mathfrak{g})_{\mathbb{C}}$ to an automorphism of $\mathfrak{g}$ if and only if θ leaves $\mathfrak{g}$ invariant, i.e. commutes with the conjugation σ in $(\mathfrak{g})_{\mathbb{C}}$ with respect to $\mathfrak{g}$ or, alternatively, if all eigenvalues are real and $\mathfrak{g}^{(\mu_i)} = \mathfrak{g} \cap (\mathfrak{g})_{\mathbb{C}}^{(\mu_i)}$ and $\mathfrak{g}$ is the direct sum of all those eigenspaces.

In the following section, automorphisms of the root system are introduced of which Dynkin diagram symmetries are a special case. After that we will have a closer look at automorphisms of finite order and conclude with a discussion of normal conjugations.

2.3.1 Automorphisms of the Root System

An *automorphism of the root system* ϑ is an automorphism of the real vector space E which maps a root system Φ into itself.

Theorem 2 *All Automorphisms ϑ of the root system Φ of a simple Lie algebra are isometries, i.e.*

$$\langle \vartheta(\alpha), \vartheta(\beta)\rangle = \langle \alpha, \beta\rangle \qquad \textit{for } \alpha, \beta \in \Phi\ .$$

Proof:

First we show $\langle \vartheta(\alpha), \vartheta(\alpha)\rangle = \langle \alpha, \alpha\rangle$. For a simply laced root system (only single lines between nodes) this is immediately clear since the roots are all of the same length. In the other cases there are two possible lengths. If α is long, $\vartheta(\alpha)$ cannot be short because of the following: Choose simple roots such that $\alpha_1 = \alpha$. Because ϑ is bijective and linear the simple roots are mapped to another set of possible simple roots. Inspection of the positive roots of B_l, C_l F_4 and G_2 shows that there is a root $\gamma = 1\alpha_1 + \sum_{i\in I} 2\alpha_i$ for some subset set I of $\{2,\ldots,l\}$ and that $\vartheta(\gamma)$ can only be a root if $\vartheta(\alpha_1)$ is a long root.

Similarly, if α is short, choose simple roots such that α is among them. γ is chosen maximal such that the long simple roots have odd coefficients and the short ones (among them is α) even. This is possible for all non-simply-laced Lie algebras. Now, $\vartheta(\gamma)$ can only be a root, if $\vartheta(\alpha)$ is short as can be seen by inspection of the positive roots.

If for two roots α, β it holds that $\langle\vartheta(\alpha), \vartheta(\beta)\rangle \neq \langle\alpha, \beta\rangle$, it follows from the above that $A_{\vartheta(\alpha),\vartheta(\beta)} \neq A_{\alpha,\beta}$ and therefore $p_\vartheta - q_\vartheta \neq p - q$. From $\alpha + q\beta \in \Phi$ follows $\vartheta(\alpha) + q\vartheta(\beta) \in \Phi$ and therefore $q_\vartheta \geq q$ and similarly $p_\vartheta \geq p$. If now $p_\vartheta = p$ it has to be $q_\vartheta > q$. This is a contradiction, because $\vartheta(\alpha + q_\vartheta\beta)$ would be a root, but $\alpha + q_\vartheta\beta$ not and ϑ is a bijective map of Φ onto Φ. Similarly for $q_\vartheta = q$.

End of Proof

Any linear or anti-linear automorphism θ of a complex semi-simple Lie algebra $\mathfrak{g}$ which leaves the Cartan subalgebra invariant $\theta(\mathfrak{h}) = \mathfrak{h}$ defines an automorphism of the root system ϑ *induced* by θ via $\vartheta(\alpha)(h) = \alpha(\theta^{-1}(h))$ or $\vartheta(\alpha)(h) = \overline{\alpha(\theta^{-1}(h))}$, respectively, for $\alpha \in \Phi$ and $h \in \mathfrak{h}$. The complex conjugation is superfluous if h is restricted to $\mathfrak{h}^*$, the real span of all h_{α_i}. Because of theorem 1 and because all Cartan subalgebras of a complex Lie algebra are conjugate, an $\mathfrak{h}$ satisfying $\theta(\mathfrak{h}) = \mathfrak{h}$ can be found for a diagonalizable automorphism θ.

On the other hand, a linear or anti-linear automorphism θ of the semi-simple Lie algebra $\mathfrak{g}$ with root system Φ is an *extension* of the automorphism of the root system ϑ, if $\vartheta(\alpha)(h) = \alpha(\theta^{-1}(h))$ or $\vartheta(\alpha)(h) = \overline{\alpha(\theta^{-1}(h))}$, respectively, for $\alpha \in \Phi$ and $h \in \mathfrak{h}$. The realization of such extensions is straight forward:

Theorem 3 *Let ϑ be an automorphism of the root system Φ of a complex semi-simple Lie algebra and choose arbitrary complex numbers ρ_{α_i} for $i = 1,\ldots,l$. The following relations define an (anti-)linear automorphism θ uniquely:*

$$\theta(h_{\alpha_i}) = h_{\vartheta(\alpha_i)}$$

$$\theta(e_\alpha) = \rho_\alpha e_{\vartheta(\alpha)} \qquad \text{where } \rho_\alpha \text{ is given by the } \rho_{\alpha_i} \text{ via}$$

$$\rho_{\alpha+\beta} = \frac{n_{\vartheta(\alpha),\vartheta(\beta)}}{n_{\alpha,\beta}}\rho_\alpha\rho_\beta \qquad \text{and} \tag{2.35}$$

$$\rho_{-\alpha} = \tfrac{B_\alpha}{B_{\vartheta\alpha}}\rho_\alpha^{-1}\,. \tag{2.36}$$

The fraction in the last line is equal to one in all used conventions. On the other hand, for each automorphism θ stabilizing $\mathfrak{h}$ the induced automorphism of the root system ϑ satisfies these relations.

If the ρ_{α_i} also satisfy

$$\rho_{-\alpha} = \overline{\rho_\alpha} \tag{2.37}$$

it holds that $\theta \circ \tau = \tau \circ \theta$ where τ is the anti-linear automorphism from (2.10).

If ϑ is of order ν, then θ is also of order ν if and only if the ρ_{α_i} satisfy

$$\begin{aligned} &\rho_{\vartheta^{\nu-1}(\alpha)}\rho_{\vartheta^{\nu-2}(\alpha)}\rho_{\vartheta^{\nu-3}(\alpha)}\cdots\rho_\alpha = 1 && \textit{for linear } \theta\ , \\ &\rho_{\vartheta^{\nu-1}(\alpha)}\overline{\rho_{\vartheta^{\nu-2}(\alpha)}}\rho_{\vartheta^{\nu-3}(\alpha)}\cdots\overline{\rho_\alpha} = 1 && \textit{for anti-linear } \theta,\ \nu \textit{ even, and} \\ &\rho_{\vartheta^{\nu-1}(\alpha)}\overline{\rho_{\vartheta^{\nu-2}(\alpha)}}\rho_{\vartheta^{\nu-3}(\alpha)}\cdots\rho_\alpha = 1 && \textit{for anti-linear } \theta,\ \nu \textit{ odd.} \end{aligned} \tag{2.38}$$

This is a combination and slight generalization of results in [Gan39a, Cor84, Ara62]. For most of the structures below it does not matter, whether θ is linear or anti-linear. In fact, one can always consider $\tau \circ \theta$ instead which is of the opposite type. However, care must be taken for θ of odd order. Below, both possibilities will always be assumed unless stated otherwise.

The relations (2.36) and (2.37) give in the used conventions

$$|\rho_\alpha| = 1\ . \tag{2.39}$$

Consider the case, where ϑ and θ are of the same finite order ν. For an automorphism of the root system ϑ to be extendible to θ of the same order it is not important, whether the extension σ is linear or anti-linear: The coefficients ρ_{α_i} can be chosen to be real and (2.35) and (2.36) will make sure the remaining ρ_α are real, as well.

If for $\alpha \in \Phi$ it is $\vartheta(\alpha) = -\alpha$, the equations (2.38) and (2.39) imply even more: For linear θ of odd order ν we have $\rho_\alpha = 1$, while for anti-linear θ of any finite order we get $\rho_\alpha = e^{2\pi i\xi/\nu}$, for some integer ξ.

The automorphisms of the root system form a finite group, the *automorphism group of the root system* $\mathrm{Aut}(\Phi)$. It has several interesting subgroups. First, there is the *Weyl group* $W(\Phi)$ which is defined to be generated by the Weyl reflections ω_α of (2.5) for all $\alpha \in \Phi$. It is easy to see that the Weyl reflections are automorphisms of the root system: $\omega_\alpha(\beta)$ is a root, since we know from (2.12) that $\beta - p\alpha$ for $p = A_{\alpha,\beta} + q$ is a root and $q \geq 0$. They are called reflections, because ω_α reflects E at the hyperplane perpendicular to α. It can also be shown [Jac62] that the ω_{α_i} already generate the whole Weyl group.

Because of linearity, identity of two products of Weyl reflections needs to be checked only on the simple roots:

$$\omega_1 = \omega_2 \quad \Leftrightarrow \quad \omega_1(\alpha_i) = \omega_2(\alpha_i) \text{ for all } i \, .$$

The Weyl group is an invariant subgroup of the group of automorphisms of the root system, because of

$$(\vartheta^{-1}\omega_\alpha\vartheta)(\beta) = \beta - A_{\alpha,\vartheta(\beta)}\vartheta^{-1}(\alpha) = \beta - A_{\vartheta^{-1}(\alpha),\beta}\vartheta^{-1}(\alpha) = \omega_{\vartheta^{-1}(\alpha)}(\beta) \, .$$

There is an important relation between the Weyl group and the group of inner automorphisms of the Lie algebra: An extension of an automorphism of the root system ϑ is an inner automorphism of the Lie algebra if and only if ϑ is an element of the Weyl group. In fact even more is true: The factor group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ is isomorphic to $\mathrm{Aut}(\Phi)/W(\Phi)$ for a complex or compact real Lie algebra $\mathfrak{g}$ (Theorem XI.5.4 of [Hel78]).

Since $W(\Phi)$ acts simply transitively on the set of fundamental systems of the root system Φ, the factor group $\mathrm{Aut}(\Phi)/W(\Phi)$ is isomorphic to the subgroup $D(\Psi)$ of $\mathrm{Aut}(\Phi)$ transforming one given fundamental system Ψ into itself (Theorem X.3.29 of [Hel78]). In other words, these automorphisms of the root system permute the simple roots in Ψ. Since this is only possible if the Cartan matrix is invariant under the corresponding simultaneous permutation of rows and columns and this corresponds to an invariance of the Dynkin diagram, elements of $D(\Psi)$ are called *Dynkin diagram symmetries*.

The fact that $D(\Psi)$ is a finite group clarifies the global structure of $\mathrm{Aut}(\mathfrak{g})$: It is a Lie group with $\mathrm{Int}(\mathfrak{g})$ as the connected component of unity and the $\theta\,\mathrm{Int}(\mathfrak{g})$ for all $\theta \in D(\Psi)$ are the connected components.

The proof of theorem 1 is now straight forward:

Proof of theorem 1 (Sketch): Let θ be a diagonalizable automorphism of a complex semi-simple Lie algebra. Because of the above, it is of the form $\theta = \mathrm{Exp}(\mathrm{ad}(X)) \circ \theta_\vartheta$ for some $X \in \mathfrak{g}$. Since $\mathrm{ad}(X)$ is diagonalizable, a Cartan subalgebra $\mathfrak{h}$ can be found containing it. All Cartan subalgebras of a complex semi-simple Lie algebra are conjugate. *End of Proof*

2.3.2 Automorphisms of Finite Order

As apparent from section 2.1.2 we will be mostly interested in *involutions* or *involutive* automorphisms which are defined to be of order 2.

Automorphisms of finite order are always diagonalizable. An eigenvalue μ of an automorphism θ of order ν has to satisfy $\mu^\nu = 1$ and therefore $|\mu| = 1$. All eigenvalues are of the form $(e^{2\pi i/\nu})^j$ with $j \in \mathbb{Z}_\nu$. Because of (2.34) there is a $\mathbb{Z}_\nu$-gradation (see section 2.6)

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}_\nu} \mathfrak{g}^{(j)} \quad \text{with} \quad \left[\mathfrak{g}^{(j)},\mathfrak{g}^{(k)}\right] = \mathfrak{g}^{(j+k)} \ . \tag{2.40}$$

where $\mathfrak{g}^{(j)}$ is an eigenspace of θ to the eigenvalue $(e^{2\pi i/\nu})^j$ and addition is modulo ν. On the other hand, for each $\mathbb{Z}_\nu$-gradation of $\mathfrak{g}$ of the form (2.40) the linear transformation of $\mathfrak{g}$ is given by multiplying the vectors of $\mathfrak{g}^{(j)}$ by $(e^{2\pi i/\nu})^j$.

Let ϑ be a Dynkin diagram symmetry ϑ of order ν_ϑ of a complex simple Lie algebra $\mathfrak{g}$. From theorem 3, it is clear that an extension θ_ϑ of ϑ of order ν_ϑ may be found: The relations (2.38) can always be satisfied by choosing e.g. $\rho_{\alpha_i} = 1$ for all simple roots α_i.

The θ_ϑ-stable subalgebra $\mathfrak{g}_\vartheta^{(0)}$ is a simple subalgebra of $\mathfrak{g}$. If H_{α_i} are the elements of a Chevalley basis of $\mathfrak{g}$ spanning the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then $H_{\alpha_i} + \cdots + H_{(\vartheta)^{(\nu_\vartheta-1)}(\alpha_i)}$ is a basis of a Cartan subalgebra $\mathfrak{h}^{(0)}$ of $\mathfrak{g}^{(0)}$. The set of all non-zero restrictions $\alpha|_{\mathfrak{h}^{(0)}}$ of roots α to this Cartan subalgebra is the *restricted root system*. As the name suggests it is a root system [Ara62], however not necessarily a reduced root system, as can easily be seen when considering the Dynkin diagram symmetry of A_2.

With an extension θ_ϑ of the Dynkin diagram symmetry ϑ fixed, we will now explore the possibilities for the first factor $\mathrm{Exp}(\mathrm{ad}(h))$ of a finite order chief automorphism (2.33). Since h is θ_ϑ-invariant, it is in the subalgebra $\mathfrak{g}_\vartheta^{(0)}$ which is stable under θ_ϑ. Choose the Chevalley basis H_{α_i}, E_α of $\mathfrak{g}_\vartheta^{(0)}$ which consists of sums of the form $X + \theta_\vartheta(X) + \cdots + (\theta_\vartheta)^{\nu_\vartheta-1}(X)$, where X is an element of the Chevalley basis of $\mathfrak{g}$, and express h as

$$h = \tfrac{2\pi i}{\nu}\theta^i H_{\alpha_i} \ , \tag{2.41}$$

and let for a root α be α^k, $k = 1, \ldots, l$ the coefficients in the expansion $\alpha = \sum \alpha^k \alpha_k$. The action on the basis of $\mathfrak{g}_\vartheta^{(0)}$ is then

$$\theta(H_{\alpha_j}) = H_{\alpha_j} \tag{2.42}$$

$$\theta(E_\alpha) = \exp(\tfrac{2\pi i}{\nu}\theta^i A_{ik}\alpha^k)E_\alpha \tag{2.43}$$

$$\begin{aligned}\theta(F_\alpha) &= \big(\cos(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k) + i\sin(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)\big)E_\alpha \\ &\quad + \big(\cos(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k) - i\sin(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)\big)E_{-\alpha} \\ &= \cos(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)F_\alpha + i\sin(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)G_\alpha\end{aligned} \tag{2.44}$$

$$\theta(G_\alpha) = \cos(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)G_\alpha + i\sin(\tfrac{2\pi}{\nu}\theta^i A_{ik}\alpha^k)F_\alpha \ . \tag{2.45}$$

For trivial ϑ it is $\mathfrak{g}_\vartheta^{(0)} = \mathfrak{g}$ and the automorphism θ is of order ν if for all roots $\alpha = \sum \alpha^i \alpha_i$ it holds

$$\exp(\tfrac{2\pi i}{\nu} \theta^i A_{ik} \alpha^k)^\nu = 1$$

i.e. $\theta^i A_{ik} \alpha^k$ has to be an integer modulo ν. Since α^k are all positive or all negative integers it suffices that

$$\theta_k = \theta^i A_{ik}$$

is an integer modulo ν. If ϑ is non-trivial, an expansion of h like (2.41) for the Chevalley basis of $\mathfrak{g}$ exists and the same reasoning applies.

From the above it appears as if the stable subalgebra $\mathfrak{g}^{(0)}$ of the chief automorphism $\mathrm{Exp}(\mathrm{ad}(h))\theta_\vartheta$ is the subalgebra of $\mathfrak{g}_\vartheta^{(0)}$ corresponding to the sub-diagram of the Dynkin diagram of $\mathfrak{g}_\vartheta^{(0)}$ consisting of the nodes k for which θ_k is zero. Even though this method is successful in many cases, it sometimes misses part of the stable subalgebra. The following result due to Kač [Kač69] and completely worked out in [Hel78] theorem X.5.15 and X.5.16 is based on the theory of affine Lie algebras:

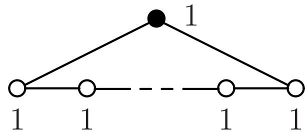
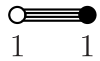
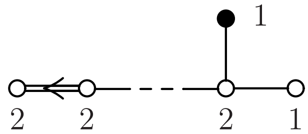
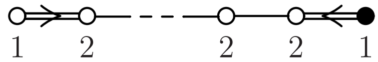
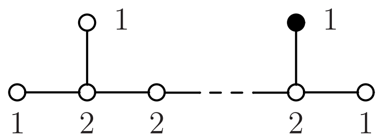
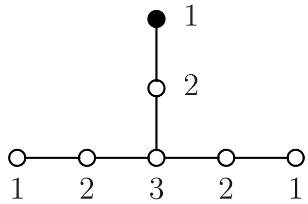
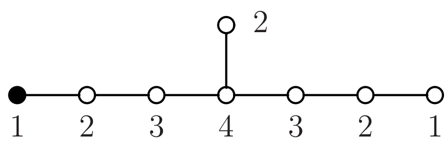
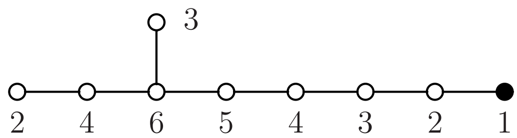
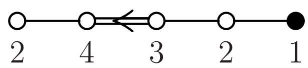
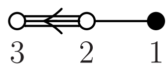
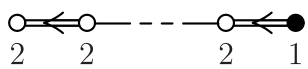
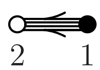
Theorem 4 *For a complex simple Lie algebra $\mathfrak{g}$ of rank l and type X_l, let ϑ be a Dynkin diagram symmetry of order ν_ϑ. Let $\mathfrak{g}_\vartheta^{(0)}$ be the simple subalgebra of rank n stable under the fixed extension θ_ϑ of ϑ. Let $a_0, a_1, \ldots, a_n$ be the Coxeter labels of the extended Dynkin diagram $X_l^{(\nu_\vartheta)}$ in table 2.2. For any choice of non-negative integers $\theta_0, \theta_1, \ldots, \theta_n$ without non-trivial common divisor there is a chief automorphism θ of $\mathfrak{g}$ of order*

$$\nu = \nu_\vartheta \sum_{i=0}^{n} a_i \theta_i$$

given by (2.33) and (2.41). These chief automorphisms exhaust all automorphisms of finite order of $\mathfrak{g}$ up to conjugacy.

Two such automorphisms θ and θ' are conjugate if and only if $\nu_\vartheta = \nu'_\vartheta$ and the sequences $\theta_0, \ldots, \theta_n$ can be transformed into $\theta'_0, \ldots, \theta'_n$ by a symmetry of the extended Dynkin diagram $X_l^{(\nu_\vartheta)}$.

The subalgebra $\mathfrak{g}^{(0)}$ stable under θ is reductive. Its semi-simple part has the Dynkin diagram obtained from the extended Dynkin diagram $X_l^{(\nu_\vartheta)}$ by deleting all nodes i for which $\theta_i \neq 0$, and the number of deleted nodes is one more than the complex dimension of the center of $\mathfrak{g}^{(0)}$.

Type	Φ	n	$\mathfrak{g}_\vartheta^{(0)}$
$A_n^{(1)}$		$2,\ldots$	$\mathfrak{a}_n$
$A_1^{(1)}$		1	$\mathfrak{a}_1$
$B_n^{(1)}$		$3,\ldots$	$\mathfrak{b}_n$
$C_n^{(1)}$		$2,\ldots$	$\mathfrak{c}_n$
$D_n^{(1)}$		$4,\ldots$	$\mathfrak{d}_n$
$E_6^{(1)}$		6	$\mathfrak{e}_6$
$E_7^{(1)}$		7	$\mathfrak{e}_7$
$E_8^{(1)}$		8	$\mathfrak{e}_8$
$F_4^{(1)}$		4	$\mathfrak{f}_4$
$G_2^{(1)}$		2	$\mathfrak{g}_2$
$A_{2n}^{(2)}$		$2,\ldots$	$\mathfrak{b}_n$
$A_2^{(2)}$		1	$\mathfrak{a}_1$

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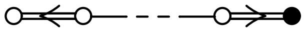
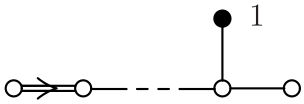
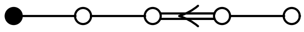
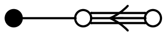
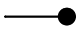
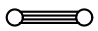
Type	Φ	n	$\mathfrak{g}_\vartheta^{(0)}$
$D_{n+1}^{(2)}$	 1 1 1 1	$2,\ldots$	$\mathfrak{b}_n$
$A_{2n-1}^{(2)}$	 1; 1 2 2 1	$3,\ldots$	$\mathfrak{c}_n$
$E_6^{(2)}$	 1 2 3 2 1	4	$\mathfrak{f}_4$
$D_4^{(3)}$	 1 2 1	2	$\mathfrak{g}_2$

Table 2.2: The extended Dynkin diagrams of all complex Lie algebras with Coxeter labels. Each diagram contains $n+1$ nodes.

The *extended Dynkin diagram* given in table 2.2 is the Dynkin diagram, as in table 2.1, of the stable subalgebra $\mathfrak{g}_\vartheta^{(0)}$ extended by a *zeroth node* which is denoted by . The remaining nodes form the Dynkin diagram of $\mathfrak{g}_\vartheta^{(0)}$ and are numbered corresponding to table 2.1 which is usually from left to right. The nodes in table 2.2 are labeled with the Coxeter labels mentioned in theorem 4.

In the context of this work it is not possible to give a full account of the theory of affine Lie algebras which leads to the definition and classification of extended Dynkin diagrams and a proof of theorem 4. The following paragraphs sketch the structure of the derivation as much as possible without explicit reference to affine Lie algebras:

The extended Dynkin diagrams are classified by the following conditions: Each sub-diagram has to be the Dynkin diagram of a simple Lie algebra (table 2.1) and $\det(A) = 0$, where A is the Cartan matrix described by the extended Dynkin diagram in the usual way. Two new types of links appear:



denotes $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and


denotes $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$.

Since $\det A = 0$, the columns of the Cartan matrix A are linearly dependent. The Coxeter labels are defined to be the coefficients of linear dependence.

This classification, however, does not provide an identification of the nodes of the extended Dynkin diagram with the simple roots of $\mathfrak{g}_\vartheta^{(0)}$. The meaning of the zeroth node is also not clear. In order to find these identifications, the Dynkin diagram symmetries have to be examined as follows.

Instead of the representation of the chief automorphism as (2.33) Helgason defines it by specifying its action on the generating set of vectors $E_i = E_{\alpha_i}$ for $i = 1, \ldots, n$ and $E_0 = E_\alpha$ where α is the lowest root of $\mathfrak{g}$ such that E_α is in $\mathfrak{g}_\vartheta^{(1)}$ and α_i are the simple roots of $\mathfrak{g}_\vartheta^{(0)}$. The action of θ is then given as $\theta(E_i) = (e^{2\pi i/\nu})^{\theta_i} E_i$. If a linearly dependent generating set H_i, $i = 0, \ldots, n$ of the Cartan subalgebra is defined accordingly, the Cartan matrix can be calculated and identified with one of the extended Dynkin diagrams. The zeroth node is then the node which corresponds to H_0. The connections with the other nodes in the extended Dynkin diagram are given by the Cartan matrix as usual.

All this becomes in fact simple, if a trivial automorphism of the root system ϑ is considered: $\mathfrak{g}_\vartheta^{(0)} = \mathfrak{g}_\vartheta^{(1)} = \mathfrak{g}$ and the root α such that $E_0 = E_\alpha$ is the lowest root $-\alpha_{\max}$. The Cartan matrix of the extended Dynkin diagram is the Cartan matrix of $\mathfrak{g}$ extended by the zeroth row $A_{0i} = A_{-\alpha_{\max},\alpha_i}$, the zeroth column $A_{j0} = A_{\alpha_j,-\alpha_{\max}}$, and $A_{00} = 2$. Because the α_j-string through $-\alpha_{\max}$ has $p = 0$, $A_{j0} \neq 0$ only if $-\alpha_{\max} + \alpha_j$ is a root. Then the zeroth node is connected to the jth node. The Coxeter labels are $a_0 = 1$ and a_i equal to the coefficient of α_i in the expansion of $\alpha_{\max}$:

$$\alpha_{\max} = \sum_{i=1}^{l} a_i \alpha_i \; . \tag{2.46}$$

Equation (2.43) tells us directly which elements of the Chevalley basis of $\mathfrak{g}$ span which eigenspace $\mathfrak{g}^{(i)}$ of θ. From (2.42) it is clear, that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is contained in the stable subalgebra $\mathfrak{g}^{(0)}$, which therefore has the same rank l as $\mathfrak{g}$. The intersection of $\mathfrak{h}$ with the semi-simple part $\left[\mathfrak{g}^{(0)},\mathfrak{g}^{(0)}\right]$ is a Cartan subalgebra $\mathfrak{h}^{(0)}$ of $\left[\mathfrak{g}^{(0)},\mathfrak{g}^{(0)}\right]$ and the center of $\mathfrak{g}^{(0)}$ is the orthogonal complement of $\mathfrak{h}^{(0)}$ within $\mathfrak{h}$.

If additionally to $\vartheta = \mathbb{1}$ it is $\theta_0 \neq 0$, the center of $\mathfrak{g}^{(0)}$ may be found easily if the numbering of the simple roots is changed such that $\theta_j \neq 0$ for $j = 0, \ldots, m$ and $\theta_j = 0$ for $j = m+1, \ldots, l$: The first m elements of the orthonormal basis H_{b^j} of (2.31) will span the center. Alternatively, the basis elements H_{μ^j} for $\theta_j \neq 0$ may be taken.

We will see below that in our study of space-time symmetries only the trivial automorphisms of the root system $\vartheta = \mathbb{1}$ and $\theta_0 \neq 0$ will appear and these simplifications can therefore be applied.

2.3.3 Normal Anti-Involutions or Conjugations

Finally, we will study an alternative way to construct *conjugations*, i.e. anti-linear involutions of a complex Lie algebra, which can be interpreted as conjugations with respect to a real form.

An involutive automorphism of the root system ϑ is *normal* if no $\vartheta(\alpha) - \alpha$ is a root and *anti-normal* if no $\vartheta(\alpha) + \alpha$ is a root for all roots α. The Dynkin diagram symmetries of order 2 from the previous section are a special class of normal involutions.

For a root system Φ with a normal involution ϑ define

$$\Phi^0 = \{\alpha \in \Phi \mid \vartheta(\alpha) = -\alpha\} .$$

Φ^0 is a root system. Let l^0 be its rank. Satake [Sat60] showed the following:

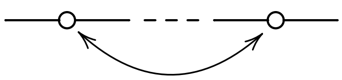
Theorem 5 *For a normal involutive automorphism ϑ of a root system Φ of rank l with Φ^0 of rank l^0 there is*
— a choice of simple roots $\alpha_1, \ldots, \alpha_{l-l^0}, \alpha_{l-l^0+1}, \ldots, \alpha_l$ and
— a permutation σ of oder 2 of the set of indices $\{1, \ldots, l-l^0\}$ such that

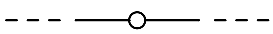
$$\begin{aligned} \vartheta(\alpha_i) &= \alpha_{\sigma(i)} + \sum_{j=l-l^0+1}^{l} c_j\alpha_j && \text{for } i = 1, \ldots, l-l^0 \text{ and } c_j \geq 0 \\ \vartheta(\alpha_i) &= -\alpha_i && \text{for } i = l-l^0+1, \ldots, l . \end{aligned}$$

From this it is obvious that Φ^0 spans the eigenspace $E^{(-)}$ of ϑ to the eigenvalue -1.

For the action of an involutive permutation σ there are only two possibilities: Either $\sigma(i) = i$ or $\sigma(i) = j$ and $\sigma(j) = i$. Therefore a normal involution of a root system can be visualized by the *Satake diagram*. It is the Dynkin diagram of Φ with two additional elements:

Type 1 Simple roots in Φ^0 which are mapped to its negative are denoted by a filled node in the Satake diagram: .

Type 2 Two simple roots α_i and α_j for which $\sigma(i) = j$ are connected by an arrow: 

Type 3 All simple roots which are not of type 1 or type 2 are denoted by a regular node: 

It turns out that this notation already fixes the normal involution, i.e. the coefficients c_j in theorem 5. These coefficients have to be the same for the roots α_i and $\alpha_{\sigma(i)}$ for ϑ to be involutive.

Let $\Phi^\Sigma = \Phi \setminus \Phi^0$. The projection of Φ^Σ onto the eigenspace $E^{(+)}$ of ϑ to the eigenvalue 1 is denoted by Σ. Araki [Ara62] showed that Σ is a root system, the *restricted root system*, albeit not necessarily a reduced root system. Elements of Σ are usually denoted by λ. Generally, more than one root $\alpha \in \Phi$ is projected onto one $\lambda \in \Sigma$. The rank of Σ is called the *split rank*.

Because the coefficients c_j in theorem 5 are the same for a simple root $\alpha_i \in \Phi^\Sigma$ and $\alpha_{\sigma(i)}$, the projections of α_i and $\alpha_{\sigma(i)}$ onto $E^{(+)}$ both are $\alpha_i + \alpha_{\sigma(i)} + \beta$ for the same $\beta \in \Phi^0$. Therefore, they are projected onto the same $\lambda \in \Sigma$. All the elements of Σ on which one or two simple roots of Φ are projected form a fundamental system of Σ. With this, and because the coefficients c_j are all non-negative, positive roots of Φ^+ are projected onto positive restricted roots in Σ^+ and negative onto negative.

In contrast to the Dynkin diagram symmetries, it is not always possible to extend a normal involution of Φ to an involution σ of $\mathfrak{g}$. Recall that, for this, it does not matter, whether the extension σ is linear or anti-linear. Since we are interested in conjugations, we will concentrate on the anti-linear case. Araki classified those Satake diagrams which describe extendible involutions by use of the relations (2.17) and (2.18) for the structure constants $n_{\alpha,\beta}$ and checking (2.38). He also obtains the coefficients c_i from theorem 5. The possible diagrams are given in table 2.3.

If on the other hand a conjugation of the complex Lie algebra $\mathfrak{g}$ induces a normal automorphism of the root system it is called a *normal* conjugation. The following result of Araki shows that the diagrams in table 2.3 already classify all normal conjugations up to conjugacy.

Theorem 6 *All normal conjugations of a complex Lie algebra $\mathfrak{g}$ which induce the same automorphism of the root system are conjugate under* $\mathrm{Int}(\mathfrak{g})$*.*

Let $\mathfrak{h}^+$ and $\mathfrak{h}^-$ be the the dual of $E^{(+)}$ and $E^{(-)}$, respectively. Their direct sum $\mathfrak{h}^+ \oplus \mathfrak{h}^-$ is the vector part of the Cartan subalgebra $\mathfrak{h}^*$ defined in section 2.2.2. All elements of Φ^0 are zero on $\mathfrak{h}^+$. The restricted root system Σ is usually defined to be the set of restrictions of roots in Φ^Σ to $\mathfrak{h}^+$ when starting from real Lie algebras.

The root spaces $\mathfrak{g}^\lambda$ for $\lambda \in \Sigma$ are defined to be

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g} \,|\, [h,X] = \lambda(h)X \text{ for all } h \in \mathfrak{h}^+\} \,. \tag{2.47}$$

The generalization of the root space decomposition (2.3) is

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{h}^+) \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}^\lambda \,, \tag{2.48}$$

where $\mathfrak{z}(\mathfrak{h}^+)$ is the centralizer of $\mathfrak{h}^+$ in $\mathfrak{g}$.

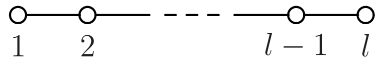
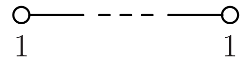
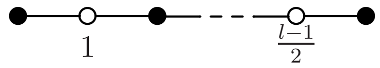
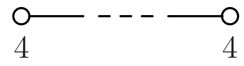
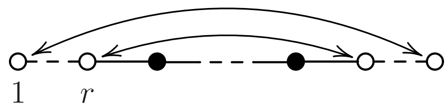
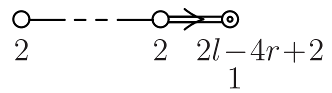
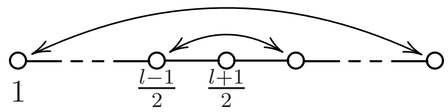
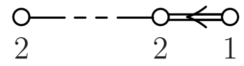
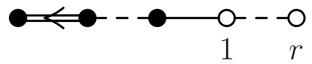
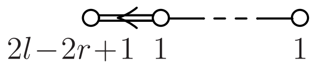
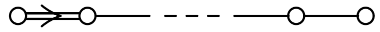
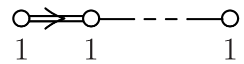
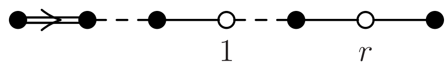
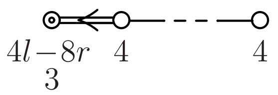
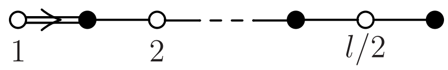
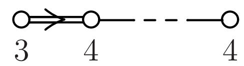
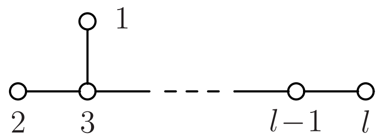
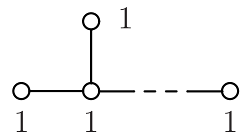
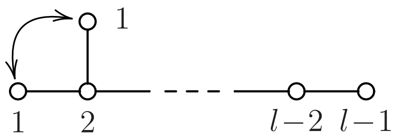
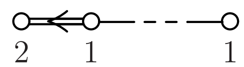
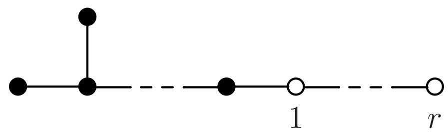
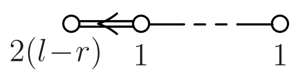
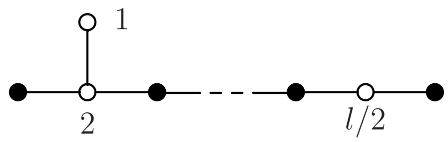
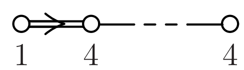
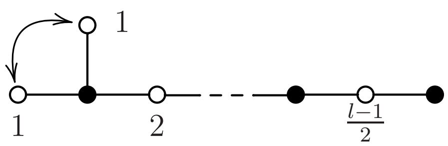
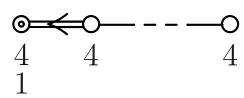
The root space $\mathfrak{g}^\lambda$ is not necessarily one-dimensional. It is a direct sum of $\mathfrak{g}^\alpha$ for all roots $\alpha \in \Phi^\Sigma$ which are projected onto λ. The number of these roots, i.e. the dimension of $\mathfrak{g}^\lambda$ is called the *multiplicity* $m(\lambda)$ of the restricted root λ. In table 2.3 the nodes of Σ are labeled with the multiplicities of the simple roots. If $2\lambda_i$ is a root, its multiplicity appears below the multiplicity of λ_i: Let λ_i and λ_j be simple restricted roots such that $2\lambda_i$ is a root while $2\lambda_j$ is not. Then these simple restricted roots will be denoted by

$$\underset{\substack{m(\lambda_i)\\ m(2\lambda_i)}}{---\odot---} \quad \text{and} \quad \underset{m(\lambda_j)}{---\circ---}$$

The multiplicities of the other roots can be inferred from these, since $m(\lambda)$ is invariant under the Weyl group.

In the diagram of Φ the nodes of type 2 and 3 are labeled with the number of the simple restricted roots in Σ. These are numbered from left to right as indicated in table 2.1.

For the distinction of the Satake diagrams corresponding to the same complex type a superscript was added to the usual type-rank notation. It contains a Roman numeral, which is based on the notation of Cartan, and the *split rank*, i.e. the dimension of $\mathfrak{h}^+$. Because Cartan's notation does not correspond one-to-one with the Satake diagrams, there are some discrepancies: Cartan's $A\,IV$ is $A_l^{(III,1)}$ in the new notation, $B\,II$ is $B_l^{(I,1)}$, and $D\,II$ is $D_l^{(Ib,r)}$. Also note that the compact forms included here are not considered in Cartan's notation and in the tables of Satake diagrams in [Ara62].

Type	Φ	Σ
$A_l^{(I,l)}$		
$A_l^{(II,(l-1)/2)}$		
$A_l^{(III,r)}$ $(0\leq r\leq \frac{l}{2})$		
$A_l^{(IIIa,(l+1)/2)}$		
$B_l^{(I,r)}$ $(0\leq r\leq l)$		
$C_l^{(I,l)}$		
$C_l^{(II,r)}$ $(0\leq r< \frac{l}{2})$		
$C_l^{(III,l/2)}$		
$D_l^{(I,l)}$		
$D_l^{(Ia,l-1)}$		
$D_l^{(Ib,r)}$ $(0\leq r\leq l-2)$		
$D_l^{(III,l/2)}$		
$D_l^{(IIIa,(l-1)/2)}$		

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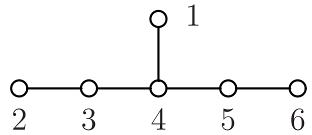
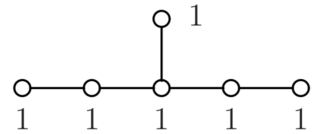
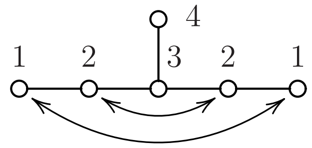
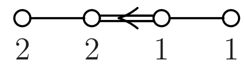
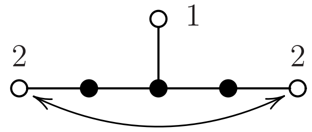
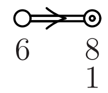
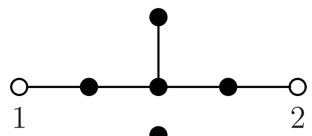
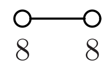
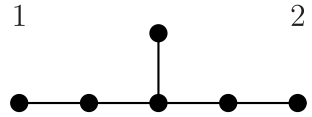
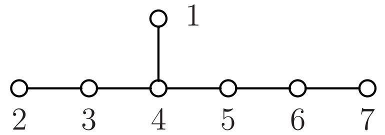
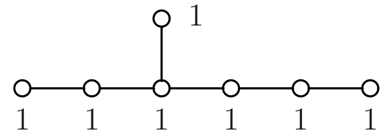
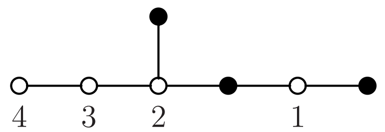
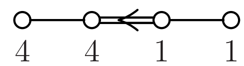
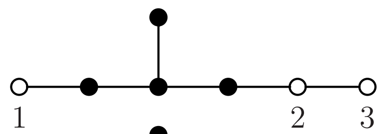
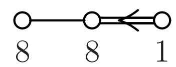
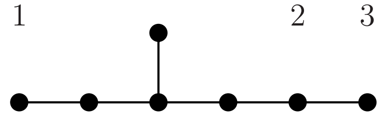
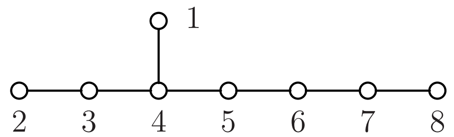
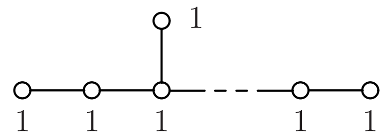
Type	Φ	Σ
$E_6^{(I,6)}$		
$E_6^{(II,4)}$		
$E_6^{(III,2)}$		
$E_6^{(IV,2)}$		
$E_6^{(compact,0)}$		
$E_7^{(V,7)}$		
$E_7^{(VI,4)}$		
$E_7^{(VII,3)}$		
$E_7^{(compact,0)}$		
$E_8^{(VIII,8)}$		

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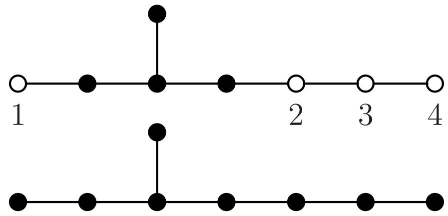
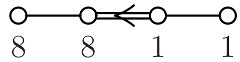
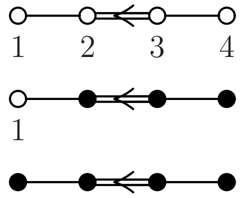
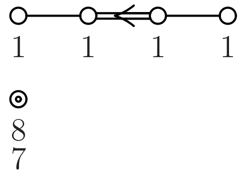
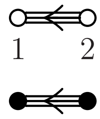
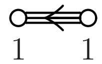
Type	Φ	Σ
$E_8^{(IX,4)}$ $E_8^{(compact,0)}$		
$F_4^{(I,4)}$ $F_4^{(II,1)}$ $F_4^{(compact,0)}$		
$G_2^{(I,2)}$ $G_2^{(compact,0)}$		

Table 2.3: All Satake diagrams and corresponding Dynkin diagrams of the restricted root systems.

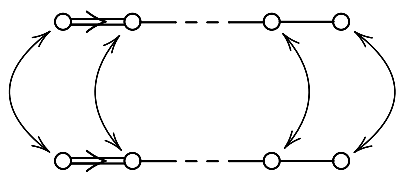
2.4 Simple Real Lie Algebras

In section 2.1.2 it was mentioned that two methods for the classification and construction of real Lie algebras exist. Both rely on the classification of involutive automorphisms described in the preceeding section. In section 2.4.1 the complex linear Cartan involutions θ will be classified while in section 2.4.2 the anti-linear conjugations σ will be studied. They are related by $\sigma = \theta \circ \tau$ (2.11).

The difference between these complementary approaches may be characterized by the type of Cartan subalgebra that is used for the structure theory of the real Lie algebra: While in a complex Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$ all Cartan subalgebras are conjugate to each other, this is not the case for real Lie algebras. Fix a Cartan subalgebra $\mathfrak{h}$ of ${}^{\mathbb{C}}\mathfrak{g}$. Two isomorphic real forms $\mathfrak{g}$ and $\mathfrak{g}'$ of ${}^{\mathbb{C}}\mathfrak{g}$ may have non-conjugate Cartan subalgebras $\mathfrak{h} \cap \mathfrak{g}$ and $\mathfrak{h} \cap \mathfrak{g}'$. Especially the summands in the decomposition $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p}$ may have varying dimensions, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The classification of the Cartan involution in the following section will involve a Cartan subalgebra $\mathfrak{h}$ for which $\mathfrak{h} \cap \mathfrak{k}$ has maximal dimension, i.e. $\mathfrak{h}$ is a *maximally compact* Cartan subalgebra, while the classification of conjugations will use *maximally non-compact* Cartan subalgebras for which $\mathfrak{h} \cap \mathfrak{p}$ is maximal. This latter type will from now on be denoted by $\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{p}}$ to distinguish it from the other type. This notation is in accordance with [War72].

In the literature these two methods are employed for the structure theory and classification of real Lie algebras. The older approach assumes a maximally compact Cartan subalgebra [Car14, Gan39a, Gan39b]. The structure is simpler and well suited for the description of the Cartan decomposition. For the description of the Iwasawa decomposition and parabolic subgroups it is better to assume a maximal non-compact one. This goes back to Satake [Sat60] and was used for a full classification by Araki [Ara62]. Unfortunately most text books only make use of one of the two. A notable exception is [Hel78], but even there, the second formalism appears only in the exercises.

In what follows, real forms of simple complex Lie algebras will be studied. One has to keep in mind, however, that this does not cover all simple real Lie algebras: As discussed in section 2.1.2 the complex Lie algebras ${}^{\mathbb{C}}\mathfrak{g}$ themselves viewed as real Lie algebras have to be considered separately, because their complexifications ${}^{\mathbb{C}}\mathfrak{g} \oplus {}^{\mathbb{C}}\mathfrak{g}$ are semi-simple complex Lie algebras. However, some of the results below apply when the obvious Dynkin diagram symmetry



of the semi-simple complexification is considered – here illustrated with C_l. Surprisingly, this is the only case, where the same involution of the root system leads to the same real form no matter which of the two methods is used. This case will sometimes not be mentioned explicitly.

2.4.1 Maximally Compact Cartan Subalgebras

Consider an involutive automorphism θ of the real Lie algebra $\mathfrak{g}$. From (2.40) we get the $\mathbb{Z}_2$-gradation $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ with the commutation relations

$$\left[\mathfrak{g}^{(0)},\mathfrak{g}^{(0)}\right] \subset \mathfrak{g}^{(0)}\ , \quad \left[\mathfrak{g}^{(0)},\mathfrak{g}^{(1)}\right] \subset \mathfrak{g}^{(1)}\ , \text{ and } \quad \left[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}\right] \subset \mathfrak{g}^{(0)}\ .$$

From these relations it can be seen immediately that the subspace

$$\mathfrak{g}^{\theta} := \mathfrak{g}^{(0)} \oplus i\mathfrak{g}^{(1)} \tag{2.49}$$

of the complexification $(\mathfrak{g})_{\mathbb{C}}$ of $\mathfrak{g}$ is a real Lie algebra and therefore a real form of $(\mathfrak{g})_{\mathbb{C}}$. The automorphism σ may be extended to $(\mathfrak{g})_{\mathbb{C}}$ and restricts to an automorphism of $\mathfrak{g}^{\theta}$. Note that $\mathfrak{g}^{\theta}$ may be isomorphic to $\mathfrak{g}$

Since for two real forms $\mathfrak{g}$ and $\mathfrak{g}'$ of a complex Lie algebra ${}^{\mathbb{C}}\mathfrak{g}$ and commuting complex conjugations σ and σ' with stability subalgebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, the involution $\theta = \sigma \circ \sigma'$ yields $\mathfrak{g}^{\theta} = \mathfrak{g}'$, all real forms of ${}^{\mathbb{C}}\mathfrak{g}$ may be found by (2.49) starting with one real form.

The natural starting-point is the compact form $\mathfrak{g}_{\text{cpt}}$ from equation (2.8). It is preferable over the normal form (2.9): Because of the linearity of the Killing form (2.2) it is immediately clear that $i\mathfrak{g}_{\text{cpt}}^{(1)}$ is non-compact while $\mathfrak{g}_{\text{cpt}}^{(0)}$ stays compact. Therefore, (2.49) immediately gives us the Cartan decomposition (2.7) of $\mathfrak{g}^{\theta}$ and θ is its Cartan involution. This is the classification of real forms via the Cartan involution mentioned as the first method in section 2.1.2.

Because we are only interested in real forms of ${}^{\mathbb{C}}\mathfrak{g}$ up to conjugacy, the Cartan involutions need only be classified up to conjugacy. The chief automorphisms described in theorem 4 are the obvious choice. Because of the equations (2.37), (2.38), (2.44), and (2.45) they leave the compact form $\mathfrak{g}_{\text{cpt}}$ invariant and therefore may be restricted to automorphisms of $\mathfrak{g}_{\text{cpt}}$. The following justifies the title of this section:

Proposition 1 *For a complex Lie algebra $\mathfrak{g}$ with fixed Cartan subalgebra $\mathfrak{h}$ the chief automorphism of theorem 4 leaves as much of $\mathfrak{h}$ invariant as any element of its conjugacy class in* $\mathrm{Aut}(\mathfrak{g})$.

Proof: This follows, because Gantmacher [Gan39a] shows that all elements of one connected component of $\mathrm{Aut}(\mathfrak{g})$ induce the same Dynkin diagram symmetry for a suitably chosen Cartan subalgebra and because conjugacy classes are subsets of connected components.

End of Proof

Given the coefficients $\theta_0, \ldots, \theta_n$ as in theorem 4, the compact part $\mathfrak{k}$ of the Cartan decomposition (2.49), i.e. the subalgebra $\mathfrak{g}^{(0)}$ of elements of $\mathfrak{g}_{\mathrm{cpt}}$ stable under θ can be read of the extended Dynkin diagram in table 2.2.

The rule for determining the possible coefficients θ_i can be simplified for involutive automorphisms: The only possibilities are $\theta_i = 0, 1$, and $\theta_i = 1$ amounts to deleting the corresponding node when determining $\mathfrak{k}$. There are only three cases: ($X = A, B, C, D, E, F$, or G)

(i) Delete 2 nodes with Coxeter label 1 of a diagram of type $X_l^{(1)}$.
$\Rightarrow \mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{u}(1)$, where $\mathfrak{k}_s$ is semi-simple.

(ii) Delete 1 node with label 2 of a diagram of type $X_l^{(1)}$.
$\Rightarrow \mathfrak{k}$ is semi-simple.

(iii) Delete 1 node labeled by 1 of a diagram of type $X_l^{(2)}$.
$\Rightarrow \mathfrak{k}$ is semi-simple.

In order to find only distinct real forms, it is necessary to bear in mind that two choices according to these rules which are are mapped into each other by asymmetry of the extended Dynkin diagram result in conjugate automorphisms (theorem 4). The corresponding real forms will be isomorphic. By inspections of the extended Dynkin diagrams of type $X_l^{(1)}$ this rule shows that for the first possibility one of the two deleted nodes may always be the zeroth. All the other possibilities are conjugate to this one by a symmetry of the extended diagram.

For the unique identification of the different real forms of one complex Lie algebra the *Killing index* δ is useful:

$$\delta = \dim \mathfrak{p} - \dim \mathfrak{k} \,, \tag{2.50}$$

i.e. the difference between the non-compact and the compact dimensions. For the exceptional groups it is customary to include it into the notation. E.g. $G_{2(2)}$ is the normal form of G_2 and $G_{2(-14)}$ the compact form. All non-compact real forms and their maximally compact subalgebra $\mathfrak{k}$ are given in table 2.4.

Type $\theta_i \neq 0$	$\mathfrak{k}$	Real Lie algebra	Conditions
$A_l^{(1)}$ $i = 0, n$	$\mathfrak{a}_{n-1} \oplus \mathfrak{a}_{l-n} \oplus \mathfrak{u}(1)$ $\mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(m, n)$ $\delta = 1-(m-n)^2$	$l + 1 = m + n$, $2 \leq n \leq l/2$, $l \geq 2$
$A_l^{(1)}$ $i = 0, 1$	$\mathfrak{a}_{l-1} \oplus \mathfrak{u}(1)$ $\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(1, l)$ $\delta = -l^2 + 2l$	$l \geq 2$
$A_1^{(1)}, 0, 1$	$\mathfrak{u}(1)$	$\mathfrak{su}(1, 1) \cong \mathfrak{sl}(2, \mathbb{R})$	$\delta = 1$
$A_{2n}^{(2)}$ $i = 0$	$\mathfrak{b}_n$ $\mathfrak{so}(2n + 1)$	$\mathfrak{sl}(2n + 1, \mathbb{R})$ $\delta = 2n$	$n \geq 2$, $l \geq 4$
$A_2^{(2)}, i = 0$	$\mathfrak{su}(2) \cong \mathfrak{so}(3)$	$\mathfrak{sl}(3, \mathbb{R})$	$\delta = 2$
$A_{2n-1}^{(2)}$ $i = 0$	$\mathfrak{c}_n$ $\mathfrak{sp}^{\text{cpt}}(n)$	$\mathfrak{su}^*(2n)$ $\delta = -2n - 1$	$n \geq 3$, $l \geq 5$
$A_{2n-1}^{(2)}$ $i = 1$	$\mathfrak{d}_n$ $\mathfrak{so}(2n)$	$\mathfrak{sl}(2n, \mathbb{R})$ $\delta = 2n - 1$	$n \geq 3$, $l \geq 5$
$B_l^{(1)}$ $i = 0, l$	$\mathfrak{b}_{l-1} \oplus \mathfrak{u}(1)$ $\mathfrak{so}(2l - 1) \oplus \mathfrak{so}(2)$	$\mathfrak{so}(2, 2l - 1)$ $\delta = -2l^2 + 7l - 4$	$l \geq 3$
$B_l^{(1)}$ $i = n$	$\mathfrak{b}_{n-1} \oplus \mathfrak{d}_m$ $\mathfrak{so}(2n - 1) \oplus \mathfrak{so}(2m)$	$\mathfrak{so}(2n - 1, 2m)$ $\delta = -2l^2 + (8n - 5)l - 8n^2 + 12n - 4$	$l + 1 = m + n$ $l \geq 3$, $1 \leq n \leq l-1$
$C_l^{(1)}$ $i = 0, 1$	$\mathfrak{a}_{l-1} \oplus \mathfrak{u}(1)$ $\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	$\mathfrak{sp}(l, \mathbb{R})$ $\delta = l$	$l \geq 2$
$C_l^{(1)}$ $i = n + 1$	$\mathfrak{c}_n \oplus \mathfrak{c}_{l-n}$ $\mathfrak{sp}^{\text{cpt}}(n) \oplus \mathfrak{sp}^{\text{cpt}}(l - n)$	$\mathfrak{sp}(n, l - n)$ $\delta = -4l^2 + (8n - 2)l - 2n^2 + 3n$	$l \geq 2$, $1 \leq n < l/2$
$D_l^{(1)}$ $i = 0, 1$	$\mathfrak{a}_{l-1} \oplus \mathfrak{u}(1)$ $\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	$\mathfrak{so}^*(2l)$ $\delta = -l$	$l \geq 4$

(continued on next page)

(continued from previous page)

Type $\theta_i \neq 0$	$\mathfrak{k}$	Real Lie algebra	Conditions
$D_l^{(1)}$ $i = 0, l$	$\mathfrak{d}_{l-1} \oplus \mathfrak{u}(1)$ $\mathfrak{so}(2l-2) \oplus \mathfrak{so}(2)$	$\mathfrak{so}(2, 2l-2)$ $\delta = -2(l-2)^2 + l$	$l \geq 5$
$D_l^{(1)}$ $i = n+1$	$\mathfrak{d}_n \oplus \mathfrak{d}_{l-n}$ $\mathfrak{so}(2n) \oplus \mathfrak{so}(2l-2n)$	$\mathfrak{so}(2n, 2l-2n)$ $\delta = -2(l-2n)^2 + l$	$l \geq 4,\ 2 \leq n \leq l/2$
$D_l^{(2)}$ $i = n+1$	$\mathfrak{b}_n \oplus \mathfrak{b}_{l-n-1}$ $\mathfrak{so}(2n+1) \oplus \mathfrak{so}(2l-2n-1)$	$\mathfrak{so}(2n+1, 2l-2n)$ $\delta = -2(l-2n-1)^2 + l$	$l \geq 3,\ 0 \leq n < l/2$
$E_6^{(1)}$ $i = 0, 2$	$\mathfrak{d}_5 \oplus \mathfrak{u}(1)$ $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$	$\mathfrak{e}_{6(-14)}$	$\delta = -14$
$E_6^{(1)}$, $i=1$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{e}_{6(2)}$	$\delta = 2$
$E_6^{(2)}$, $i=0$	$\mathfrak{f}_{4(-52)}$	$\mathfrak{e}_{6(-26)}$	$\delta = -26$
$E_6^{(2)}$, $i=4$	$\mathfrak{c}_4 \cong \mathfrak{sp}^{\text{cpt}}(4)$	$\mathfrak{e}_{6(6)}$	$\delta = 6$
$E_7^{(1)}$, $0, 7$	$\mathfrak{e}_{6(-78)} \oplus \mathfrak{u}(1)$	$\mathfrak{e}_{7(-25)}$	$\delta = -25$
$E_7^{(1)}$, 1	$\mathfrak{a}_7 \cong \mathfrak{su}(8)$	$\mathfrak{e}_{7(7)}$	$\delta = 7$
$E_7^{(1)}$ $i = 2$	$\mathfrak{d}_6 \oplus \mathfrak{a}_1$ $\mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$\mathfrak{e}_{7(-5)}$	$\delta = -5$
$E_8^{(1)}$, 2	$\mathfrak{d}_8 \cong \mathfrak{so}(16)$	$\mathfrak{e}_{8(8)}$	$\delta = 8$
$E_8^{(1)}$ $i = 8$	$\mathfrak{e}_7 \oplus \mathfrak{a}_1$ $\mathfrak{e}_{7(-133)} \oplus \mathfrak{su}(2)$	$\mathfrak{e}_{8(-24)}$	$\delta = -24$
$F_4^{(1)}$, 1	$\mathfrak{b}_4 \cong \mathfrak{so}(8)$	$\mathfrak{f}_{4(-20)}$	$\delta = -20$
$F_4^{(1)}$ $i = 4$	$\mathfrak{c}_3 \oplus \mathfrak{a}_1$ $\mathfrak{sp}^{\text{cpt}}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{f}_{4(4)}$	$\delta = 4$
$G_2^{(1)}$, 2	$\mathfrak{a}_1 \oplus \mathfrak{a}_1 \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{g}_{2(2)}$	$\delta = 2$

Table 2.4: All non-compact real forms of the simple complex Lie algebras. The nodes for which $\theta_i = 1$ are indicated by $i = \ldots$ in the first column.

In the study of space-time symmetries, Lie algebras with the compact part of the Cartan decomposition of the form $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$, where $\mathfrak{k}_s$ is a semi-simple and $\mathfrak{z}_\mathfrak{k}$ an Abelian ideal, i.e. those obtained in case (i), will be of principal interest. For these and for those obtained in case (ii), the Dynkin diagram symmetry is trivial: $\vartheta = \mathbb{1}$. In this case a basis of the real form is easily written down: The compact part $\mathfrak{k}$ is spanned by

$$iH_{\alpha_i}, \quad F_\alpha, \quad iG_\alpha$$

for all simple roots α_i and all positive roots $\alpha = \alpha^j \alpha_j$ for which $\theta_j \alpha^j$ is 0 or 2. Since only one of the coefficients θ_j for some $j = 1, \ldots, l$ is equal to one – recall that the second can always be chosen to be θ_0 – these are precisely the roots for which $\alpha^j = 0, 2$. A compact Cartan subalgebra is spanned by iH_{α_i}. The real forms obtained in case (i) and (ii) are precisely those which admit a compact Cartan subalgebra. The non-compact part $\mathfrak{p}$ it spanned by

$$iF_\alpha, \quad G_\alpha$$

for the remaining positive roots α, i.e. for those with $\alpha^j = 1$. Note that $\alpha^j = 0, 1, 2$ exhausts all positive roots, because the highest root $\alpha_{\max}$ has $\alpha^j_{\max} = 1, 2$ by the definition of the Coxeter labels.

If the Dynkin diagram symmetry is non-trivial (case (iii)), and the inner factor $\mathrm{Exp}(\mathrm{ad}\, h)$ of the involution is trivial ($h = 0$) then $\mathfrak{k}$ is spanned by

$$i(H_{\alpha_i} + H_{\vartheta(\alpha_i)}), \quad F_\alpha + \rho_\alpha F_{\vartheta(\alpha)}, \quad i(G_\alpha + \rho_\alpha G_{\vartheta(\alpha)})$$

while $\mathfrak{p}$ is spanned by

$$H_{\alpha_i} - H_{\vartheta(\alpha_i)}, \quad i(F_\alpha - \rho_\alpha F_{\vartheta(\alpha)}), \quad G_\alpha - \rho_\alpha G_{\vartheta(\alpha)}\ .$$

If additionally $h \neq 0$, a generator of this list corresponding to a root α has to be multiplied by i if the component λ^j of the restricted root λ on which α is projected is odd, where j denotes the deleted node from the extended Dynkin diagram.

2.4.2 Maximally Non-Compact Cartan Subalgebra

Starting from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (2.7) of a semi-simple real Lie algebra, $\mathfrak{a}_\mathfrak{p}$ is defined to be a maximal Abelian subalgebra of $\mathfrak{p}$. It can be completed to a Cartan subalgebra $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$ of $\mathfrak{g}$ with $\mathfrak{a}_\mathfrak{k} \in \mathfrak{k}$. It is customary to denote this *maximally non-compact* Cartan subalgebra by $\mathfrak{a}$ instead of $\mathfrak{h}$. The Weyl or Chevalley basis of the complexification $(\mathfrak{g})_\mathbb{C}$ of $\mathfrak{g}$ defined by $\mathfrak{a}$ will be used throughout this section. The Cartan subalgebra $(\mathfrak{a})_\mathbb{C}$ of $(\mathfrak{g})_\mathbb{C}$ certainly is stable under σ.

The key to the second method for the classification of real forms is the following result of Araki [Ara62]:

Theorem 7 *Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and σ the conjugation in $(\mathfrak{g})_\mathbb{C}$ with respect to $\mathfrak{g}$. Let Φ be the root system of $(\mathfrak{g})_\mathbb{C}$ with respect to $(\mathfrak{h})_\mathbb{C}$. The automorphism ϑ of the root system Φ induced by σ is normal if and only if $\mathfrak{h} \cap \mathfrak{p}$ is maximally Abelian in $\mathfrak{p}$.*

Since we have seen that each real semi-simple Lie algebra contains a Cartan subalgebra $\mathfrak{a}$ satisfying the condition in the theorem, the classification of normal conjugations of a complex semi-simple Lie algebra ${}^\mathbb{C}\mathfrak{g}$ in section 2.3.3 is in fact a classification of real forms of ${}^\mathbb{C}\mathfrak{g}$ up to conjugacy.

From section 2.3.1 we know that for roots $\alpha \in \Phi$ with $\vartheta(\alpha) = -\alpha$ it holds for the coefficient $\rho_\alpha = \pm 1$. As part of the proof of theorem 7 Araki shows that a maximal Abelian $\mathfrak{h} \cap \mathfrak{p}$ is equivalent to $\rho_\alpha = 1$ for $\vartheta(\alpha) = -\alpha$, where he assumes the e_α in theorem 3 to be a Weyl basis. From (2.29) it is clear that $\rho_\alpha = -1$ holds, if they are in a the Chevalley basis.

The dual space $\mathfrak{h}^+$ of $E^{(+)}$ defined in section 2.3.3 is in fact $\mathfrak{a}_\mathfrak{p}$ while $\mathfrak{h}^- = i\mathfrak{a}_\mathfrak{k}$. Therefore, $h \in \mathfrak{a}_\mathfrak{p}$ are precisely the elements of $\mathfrak{a}$ which are ad-diagonalizable within $\mathbb{R}$.

From the Weyl or Chevalley basis of the complexification $(\mathfrak{g})_\mathbb{C}$, a basis of $\mathfrak{g}$ is easily constructed to be

$$(2.51) \qquad h_{\alpha_k} + h_{\vartheta(\alpha_k)} \;, \quad i(h_{\alpha_k} - h_{\vartheta(\alpha_k)}) \;, \quad e_\alpha + \rho_\alpha e_{\vartheta(\alpha)} \;, \quad i(e_\alpha - \rho_\alpha e_{\vartheta(\alpha)})$$

for all $\alpha \in \Phi$ and α_k the simple roots in Φ. In the following we will only consider the Weyl basis. Because of the linear relation between h_α and its dual α, the first type of basis vector $h_{\alpha_k} + h_{\vartheta(\alpha_k)}$ is only non-zero if $\vartheta(\alpha_k) \neq -\alpha_k$. This is the case for the simple roots corresponding to white nodes in the Satake diagram. Such simple root α_k connected to another simple root $\alpha_{\sigma(k)}$ by an arrow in the Satake diagram was called of type 2. For such a simple root there is a basis vector of $\mathfrak{a}_\mathfrak{p}$, namely

$$(2.52) \qquad h_{\alpha_k} + h_{\vartheta(\alpha_k)} \quad = \quad h_{\alpha_k} + h_{\alpha_{\sigma(k)}} + \sum c_j h_{\alpha_j} \;,$$

where the sum runs over all simple roots α_j in Φ^0 as is clear from theorem 5. Because for α_k and $\alpha_{\sigma(k)}$ the coefficients c_j are the same, this vector is the same for both. For these two roots there also is a basis vector

$$(2.53) \qquad i(h_{\alpha_k} - h_{\vartheta(\alpha_k)}) = i(h_{\alpha_k} - h_{\alpha_{\sigma(k)}} - \sum c_j h_{\alpha_j})$$

of $\mathfrak{a}_\mathfrak{k}$. For the remaining simple roots corresponding to white nodes (type 3) the vector (2.53) is zero and therefore there is only one basis element (2.52) of $\mathfrak{a}_\mathfrak{p}$. Finally, for the black nodes (type 1) the expression (2.52) is zero and (2.53) simplifies to ih_{α_k}, which is a basis vector of $\mathfrak{a}_\mathfrak{k}$.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. Its complexification is spanned by $(\mathfrak{a}_\mathfrak{k})_\mathbb{C}$ and all e_α for $\alpha \in \Phi^0$, because these roots are precisely those which are zero on $\mathfrak{a}_\mathfrak{p}$. With this and (2.51) it is apparent that

$$ih_{\alpha_k} \quad f_\alpha \quad ig_\alpha$$

is a basis of a semi-simple ideal of $\mathfrak{m}$, where α_k are the simple roots in Φ^0 and $\alpha \in \Phi^0$. Since $\mathfrak{a}_\mathfrak{k} \subset \mathfrak{m}$, the vectors $i(h_{\alpha_k} - h_{\vartheta(\alpha_k)})$ are additional basis elements of $\mathfrak{m}$ for α_k and $\alpha_{\sigma(k)}$ a pair of simple roots of type 2. In this case $\alpha_k - \vartheta(\alpha_k)$ (and also $\alpha_k - \alpha_{\sigma(k)}$) lies in the eigenspace $E^{(-)}$ of ϑ to the eigenvalue -1 and is linear independent of, but not orthogonal to Φ^0. Therefore Φ^0 spans a true subspace of $E^{(-)}$. With a suitable vector $\xi \in E^{(-)}$ which is orthogonal to Φ^0, there is a basis element $ih_\xi \in \mathfrak{a}_\mathfrak{k}$ of the Abelian ideal of $\mathfrak{m}$, which therefore is reductive. An explicit construction of ξ can be accomplished with the methods developed at the end of section 2.2.4.

If the precise basis of $\mathfrak{m}$ is not of interest, the structure can be read of directly from the Satake diagram: The semi-simple part has the Dynkin diagram obtained by deleting all white nodes and the dimension of the Abelian ideal is equal to the number of arrows.

Since a root $\alpha \notin \Phi^0$ and its image $\vartheta(\alpha)$ are both projected to the same restricted root λ, the last two types of vectors in the basis (2.51) all lie in some restricted root space $(\mathfrak{g})^\lambda_\mathbb{C}$ as defined in (2.47), even in the same one. Note that this is not the case for the root spaces $(\mathfrak{g})^\alpha_\mathbb{C}$. The restricted root spaces can therefore be defined for the real Lie algebra by $\mathfrak{g}^\lambda = (\mathfrak{g})^\lambda_\mathbb{C} \cap \mathfrak{g}$. The multiplicity $m(\lambda)$ gives their dimension: Consider all pairs $\alpha, \beta \in \Phi^\Sigma$ of roots with $\alpha \neq \beta$, $\vartheta(\alpha) = \beta$, and which are restricted to λ. The vectors

$$e_\alpha + \rho_\alpha e_{\vartheta(\alpha)}\ , \quad i(e_\alpha - \rho_\alpha e_{\vartheta(\alpha)}) \tag{2.54}$$

are both non-zero. Otherwise α and β would be in Φ^0. For a root with $\vartheta(\alpha) = \alpha$ one of these vectors is zero – depending on ρ_α. Therefore, for each root projected to λ there is a basis element of the basis (2.51) in $\mathfrak{g}^\lambda$.

The generalized root space decomposition (2.55) also exists:

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{a}_\mathfrak{p}) \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}^\lambda\ , \tag{2.55}$$

where $\mathfrak{z}(\mathfrak{a}_\mathfrak{p})$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{g}$, which is $\mathfrak{z}(\mathfrak{a}_\mathfrak{p}) = \mathfrak{m} \oplus \mathfrak{a}_\mathfrak{p}$, because $\mathfrak{a}_\mathfrak{p}$ is maximally Abelian in $\mathfrak{p}$ and $\mathfrak{m}$ its centralizer in $\mathfrak{k}$.

With the definitions

$$\mathfrak{n}^{\pm} = \bigoplus_{\lambda \in \Sigma^{\pm}} \mathfrak{g}^{\lambda}$$

it is now possible to write down the *Bruhat decomposition* of real semi-simple Lie algebras:

$$\mathfrak{g} = \mathfrak{n}^{+} \oplus \mathfrak{m} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}^{-} \ .$$

For each basis vector of the type given in (2.54) for positive $\alpha \in \Phi^{\Sigma}$ there is a corresponding vector for $-\alpha$. These vectors span $\mathfrak{n}^{+}$ and $\mathfrak{n}^{-}$ respectively, which therefore have equal dimension. With these pairs a basis of $\mathfrak{n}^{+} \oplus \mathfrak{n}^{-}$ can be found in which the Killing form $B(\cdot,\cdot)$ is diagonal:

$$\begin{aligned} &f_{\alpha} + \rho_{\alpha} f_{\vartheta(\alpha)} \ , \quad i(g_{\alpha} - \rho_{\alpha} g_{\vartheta(\alpha)}) \quad \in \mathfrak{k} \quad \text{and} \\ &g_{\alpha} + \rho_{\alpha} g_{\vartheta(\alpha)} \ , \quad i(f_{\alpha} - \rho_{\alpha} f_{\vartheta(\alpha)}) \quad \in \mathfrak{p} \ . \end{aligned} \tag{2.56}$$

Recall that for $\vartheta(\alpha) = \alpha$ one of each of these pairs is zero. Therefore $(\mathfrak{n}^{+} \oplus \mathfrak{n}^{-}) \cap \mathfrak{k}$ and $(\mathfrak{n}^{+} \oplus \mathfrak{n}^{-}) \cap \mathfrak{p}$ always have the same dimension. Because $\mathfrak{m} \subset \mathfrak{k}$ and $\mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{p}$, the Killing index δ defined in (2.50) is given by

$$\delta = \dim \mathfrak{a}_{\mathfrak{p}} - \dim \mathfrak{m} \ .$$

Finally, the well known Iwasawa decomposition for Lie algebras is now obvious:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}^{+} \ .$$

Type	$\mathfrak{m}$	$\dim \mathfrak{n}^{\pm}$	Real Lie algebra
$A_l^{(I,l)}$	$\{0\}$	$\frac{1}{2}(l^2+l)$	$\mathfrak{sl}(l+1,\mathbb{R}),\ \delta = l$
$A_l^{(II,(l-1)/2)}$	$\mathfrak{su}(2)^{(l+1)/2}$	$\frac{1}{2}(l^2-1)$	$\mathfrak{su}^*(l+1),\ \delta = -l-2$
$A_l^{(III,r)}$	$\mathfrak{su}(l-2r+1) \oplus \mathfrak{u}(1)^r$	$r-2r^2+2rl$	$\mathfrak{su}(m,n)$
	$\delta = 1-(l+1-2r)^2,\quad n = r,\quad l+1 = m+n,\quad n < m$		
$A_l^{(IIIa,(l+1)/2)}$	$\mathfrak{u}(1)^{(l-1)/2}$	$\frac{1}{2}(l^2+l)$	$\mathfrak{su}(\frac{l+1}{2},\frac{l+1}{2}),\ \delta = 1$
$B_l^{(I,r)}$	$\mathfrak{so}(2(l-r)+1)$	$2lr-r^2$	$\mathfrak{so}(r,2l+1-r)$
	$\delta = -2l^2+(4r-1)l-2r^2+2r,\quad r = 2n-1$		
$C_l^{(I,l)}$	$\{0\}$	l^2	$\mathfrak{sp}(l,\mathbb{R}),\ \delta = l$
$C_l^{(II,r)}$	$\mathfrak{sp}^{\mathrm{cpt}}(l-2r) \oplus \mathfrak{su}(2)^r$	$4lr-4r^2-r$	$\mathfrak{sp}(r,l-r)$
	$\delta = -2(l-2r)^2-l,\quad 0 \le r < l/2$		
$C_l^{(III,l/2)}$	$\mathfrak{su}(2)^{l/2}$	$l^2-\frac{1}{2}l$	$\mathfrak{sp}(l/2,l/2),\ \delta = -l$
$D_l^{(I,l)}$	$\{0\}$	l^2-l	$\mathfrak{so}(l,l),\ \delta = l$
$D_l^{(Ia,l-1)}$	$\mathfrak{u}(1)$	l^2-l	$\mathfrak{so}(l-1,l+1),\delta = l-2$
$D_l^{(Ib,r)}$	$\mathfrak{so}(2(l-r))$	$2lr-r^2-r$	$\mathfrak{so}(r,2l-r)$
	$\delta = -2l^2+(4r+1)l-2r^2,\quad 0 \le r \le l-2$		
$D_l^{(III,l/2)}$	$\mathfrak{su}(2)^{l/2}$	$l^2-\frac{3}{2}l$	$\mathfrak{so}^*(2l),\ \delta = -l$
$D_l^{(IIIa,(l-1)/2)}$	$\mathfrak{su}(2)^{(l-1)/2} \oplus \mathfrak{u}(1)$	$l^2-\frac{3}{2}l+\frac{1}{2}$	$\mathfrak{so}^*(2l),\ \delta = -l$
$E_6^{(I,6)}$	$\{0\}$	36	$\mathfrak{e}_{6(6)},\ \delta = 6$
$E_6^{(II,4)}$	$\mathfrak{u}(1)^2$	36	$\mathfrak{e}_{6(2)},\ \delta = 2$
$E_6^{(III,2)}$	$\mathfrak{su}(4) \oplus \mathfrak{u}(1)$	30	$\mathfrak{e}_{6(-14)},\ \delta = -14$
$E_6^{(IV,2)}$	$\mathfrak{so}(8)$	24	$\mathfrak{e}_{6(-26)},\ \delta = -26$
$E_6^{(compact,0)}$	$\mathfrak{e}_{6(-78)}$	0	$\mathfrak{e}_{6(-78)},\ \delta = -78$
$E_7^{(V,7)}$	$\{0\}$	63	$\mathfrak{e}_{7(7)},\ \delta = 7$

(continued on next page)

(continued from previous page)

Type	$\mathfrak{m}$	$\dim \mathfrak{n}^{\pm}$	Real Lie algebra
$E_7^{(VI,4)}$	$\mathfrak{su}(2)^3$	60	$\mathfrak{e}_{7(-5)}$, $\delta = -5$
$E_7^{(VII,3)}$	$\mathfrak{so}(8)$	51	$\mathfrak{e}_{7(-25)}$, $\delta = -25$
$E_7^{(compact,0)}$	$\mathfrak{e}_{7(-133)}$	0	$\mathfrak{e}_{7(-133)}$, $\delta = -133$
$E_8^{(VIII,8)}$	$\{0\}$	120	$\mathfrak{e}_{8(8)}$, $\delta = 8$
$E_8^{(IX,4)}$	$\mathfrak{so}(8)$	108	$\mathfrak{e}_{8(-24)}$, $\delta = -24$
$E_8^{(compact,0)}$	$\mathfrak{e}_{8(-248)}$	0	$\mathfrak{e}_{7(-248)}$, $\delta = -248$
$F_4^{(I,4)}$	$\{0\}$	24	$\mathfrak{f}_{4(4)}$, $\delta = 4$
$F_4^{(II,1)}$	$\mathfrak{so}(7)$	15	$\mathfrak{f}_{4(-20)}$, $\delta = -20$
$F_4^{(compact,0)}$	$\mathfrak{f}_{4(-52)}$	0	$\mathfrak{f}_{4(-52)}$, $\delta = -52$
$G_2^{(I,2)}$	$\{0\}$	6	$\mathfrak{f}_{4(4)}$, $\delta = 4$
$G_2^{(compact,0)}$	$\mathfrak{g}_{2(-14)}$	0	$\mathfrak{g}_{2(-14)}$, $\delta = -14$

Table 2.5: Classification of the real forms of all simple Lie algebras and the structure and dimension of the several subalgebras related to the minimal parabolic subalgebra and the Killing index. The possible values for the rank l and restricted rank r are as in table 2.3.

For the classical Lie algebras the identification of the Satake diagrams involves some simple but technical arguments which follow for the sake of completeness:

With $l+1 = m+n$, the Killing index of $\mathfrak{su}(m,n)$ is $\delta = 1-(l+1-2n)^2 = -l^2+(4n-2)l+4(n-n^2)$. This makes it easy to identify $A_l^{(III,r)}$ for $n=r$ and $A_l^{(IIIa,(l+1)/2)}$ for $m=n=(l+1)/2$ as candidates.

But $A_l^{(I,l)}$ would be possible, too, if $\delta = l$. This is solved by $l = 2n-3/2 \pm \sqrt{-2n+9/4}$. It should be $-2n+9/4 \geq 0$, i.e. $n \leq 9/8$. The only possibilities are $n=1$, $l = 1/2 \pm 1/2$ and $n=0$, $l=-3/2 \pm 3/2$ which contradict $m,n \geq 1$ except for $n,l=1$. But in this case the Satake diagram is the same as $A_1^{(IIIa,1)}$ in table 2.3. $\mathfrak{su}(n,m)$ could be $A_l^{(I,l)}$ if $-4n^2+4ln+4n-l^2-2l = l$ which is solved by $2l = 4n-3 \pm \sqrt{9-8n}$, i.e. $n=1$ and $l=1$, which is covered by the isomorphy $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$.

$A_l^{(II,(l-1)/2)}$ would require $\delta = -l-2$ which is solved by $l = 2n-1/2 \pm \sqrt{5/4+2n}$. It thus has to exist some odd integer x such that $x^2 = 5+8n$. I.e. x^2-5 has to be divisible by 8. However, x^2-1 is divisible by 8 for all odd integer $x \geq 3$: The factors in $(x+1)(x-1)$ are even for x odd and one of them is divisible by 4. Since the difference $5-1=4$ is not divisible by 8, x^2-5 will not be divisible by 8 for all odd x.

$\mathfrak{su}^*(2n)$ could be $A_l^{(I,l)}$ if $l = -2n-1$ which is impossible.

$\mathfrak{sl}(l,\mathbb{R})$ could be $A_l^{(III,r)}$ if $2l = 4r-3 \pm \sqrt{9-8r}$ which we have already discussed for $\mathfrak{su}(n,m)$ above.

For $\mathfrak{su}^*(l+1)$ and $A_l^{(III,r)}$ we get the equation $2l = 4r-1 \pm \sqrt{9-8r}$, i.e. $r=1$ and $l=2$, but l has to be odd.

Comparing δ of $\mathfrak{so}(2,D)$ for odd D with δ of $B_l^{(I,r)}$ gives $r=2$.

Of the three Satake diagrams of type C_l only $C_l^{(I,l)}$ has positive Killing index and describes therefore $\mathfrak{sp}(l,\mathbb{R})$. It could describe $\mathfrak{sp}(n,l-n)$ if $l = -2(l-2n)^2-l$ which is only possible for negative l. On the other hand, $C_l^{(II,r)}$ can only describe $\mathfrak{sp}(l,\mathbb{R})$ if $l = -2(l-2r)^2-l$, which is again not possible.

The Satake diagram $D_l^{(I,l)}$ could also describe $\mathfrak{so}^*(2l)$ if $-2l^2+9l-8 = l$, which solves to $l=2$, since $\mathfrak{so}^*(4)$ and $\mathfrak{so}(2,2)$ are isomorphic.

For $\mathfrak{so}(2,D)$ with even D, we get an agreement again for $r=2$ with $D_l^{(Ib,2)}$ for $l \geq 4$. For $D_l^{(Ia,l-1)}$ solving $\delta_{\mathfrak{so}(2,D)} = l-2$ gives $l=1,3$, but D_1 is in fact $\mathfrak{u}(1)$. For $D_l^{(I,l)}$ solving $\delta_{\mathfrak{so}(2,D)} = l$ gives $l=2$. In fact, this case is the semi-simple conformal group in two dimensions isomorphic to $A_1^{(IIIa,1)} \oplus A_1^{(IIIa,1)}$. For types III and $IIIa$ solving $\delta_{\mathfrak{so}(2,D)} = -l$ gives $l = 5/2 \pm 3/2 = 4,1$. For

$l = 4$ the Satake diagram is identical to $D^{(4)}_{Ib,2}$ as expected from the isomorphy between $\mathfrak{so}(6,2)$ and $\mathfrak{so}^*(8)$. $l = 1$ is again $\mathfrak{u}(1)$. On the other hand, the correspondence between $\mathfrak{so}^*(2l)$ and these Satake diagrams is obvious. That $\mathfrak{so}^*(2l)$ cannot correspond to $D_l^{(Ib,r)}$ cannot be seen just by comparing the Killing index, because $-2l^2 + (4r+1)l - 2r^2 = -l$ has solutions. This is one of the few exceptions to the rule that the different real forms of a complex simple Lie algebra can be distinguished by the Killing index.

Because $\mathfrak{a}_\mathfrak{p}$ is a maximal Abelian subalgebra whose elements are ad-diagonalizable within $\mathbb{R}$, it would be the Cartan subalgebra of a real Lie algebra if the definition for the complex field were generalized to the real field by requiring diagonalizable and not only semi-simple endomorphisms. The restricted root system Σ would be the corresponding root system.

Continuing this direct way of generalization, the restricted root spaces $(\mathfrak{g})^\lambda_\mathbb{C}$ for $\lambda \in \Sigma$ defined in equation (2.47) would be the generalizations of the complex root spaces $(\mathfrak{g})^\alpha_\mathbb{C}$ for $\alpha \in \Phi$. However, while the latter are all one-dimensional, the former are direct sums of the $(\mathfrak{g})^\alpha_\mathbb{C}$ for those α which are projected on λ. Since the multiplicity $m(\lambda)$ is not necessarily one, $(\mathfrak{g})^\lambda_\mathbb{C}$ may be higher dimensional. Therefore, it is useful to stick to the root system Φ, the Cartan subalgebra $(\mathfrak{a})_\mathbb{C}$, and the root spaces $(\mathfrak{g})^\alpha_\mathbb{C}$ of the complexification $(\mathfrak{g})_\mathbb{C}$. As done before, the root system and Cartan subalgebra of a real Lie algebra will always be the root system Φ of the complexification and the intersection of the Cartan subalgebra of the complexification with the real Lie algebra.

2.5 Parabolic Subalgebras

In the study of space-time symmetries in chapter 3 the generalization of conformal symmetry is based on the notion of parabolic subgroups. The following developments in the corresponding Lie algebra theory are based on [War72].

Parabolic subalgebras $\mathfrak{b}_\Theta$ are defined to be subalgebras of a real semi-simple Lie algebra $\mathfrak{g}$ containing a *minimal* parabolic subalgebra $\mathfrak{b}_0$. These are defined to be all subalgebras conjugate to the *standard minimal parabolic subalgebra*

$$\mathfrak{b}_0 = \mathfrak{m} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}^- \ ,$$

where the summands are defined as in the previous section with respect to a suitable Cartan subalgebra $\mathfrak{a}$ and a fundamental system such that the automorphism of the root system induced by the conjugation in $(\mathfrak{g})_\mathbb{C}$ with respect to $\mathfrak{g}$ satisfies theorem 5.

Let Θ be a subset of the simple restricted roots in the restricted root system Σ, define $\mathfrak{a}_\mathfrak{p}(\Theta) \subset \mathfrak{a}_\mathfrak{p}$ to be spanned by h_{λ_j} for all simple restricted roots $\lambda_j \in \Theta$.

The subset $\mathfrak{a}_{\mathfrak{p}_\Theta}$ of $\mathfrak{a}_\mathfrak{p}$ contains all $h \in \mathfrak{a}_\mathfrak{p}$ such that $\lambda_j(h)$ is zero for all $\lambda_j \in \Theta$. These spaces are clearly orthogonal with respect to $B(\cdot,\cdot)$ and $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}(\Theta) \oplus \mathfrak{a}_{\mathfrak{p}_\Theta}$. The basis elements of the form $h_{\alpha_k} + h_{\vartheta(\alpha_k)}$ from (2.52) for a simple root α_k which is projected on to an element of Θ span $\mathfrak{a}_\mathfrak{p}(\Theta)$. A basis for $\mathfrak{a}_{\mathfrak{p}_\Theta}$ may be found by orthogonalizing the remaining basis elements with respect to $\mathfrak{a}_\mathfrak{p}(\Theta)$.

Let $\Sigma(\Theta)$ be the set of restricted roots in Σ which lie in the span of Θ. It is the sub-system of Σ described by the sub-diagram of the restricted Dynkin diagram consisting of the nodes corresponding to the elements of Θ. Correspondingly, $\Phi(\Theta)$ is a sub-system of Φ consisting of Φ^0 and all roots in Φ^Σ which are projected onto an element of $\Sigma(\Theta)$. It is described by a sub-diagram of the Satake diagram. Also, let $\Sigma_\Theta = \Sigma \setminus \Sigma(\Theta)$, and let $\Sigma(\Theta)^\pm$ and $\Sigma_\Theta^\pm$ be the corresponding sets of positive and negative roots. Define the following subalgebras of $\mathfrak{n}^\pm$:

$$\mathfrak{n}^\pm(\Theta) = \bigoplus_{\lambda \in \Sigma(\Theta)^\pm} \mathfrak{g}^\lambda \qquad \text{and} \qquad \mathfrak{n}_\Theta^\pm = \bigoplus_{\lambda \in \Sigma_\Theta^\pm} \mathfrak{g}^\lambda \ .$$

They form a direct sum $\mathfrak{n}^\pm = \mathfrak{n}^\pm(\Theta) \oplus \mathfrak{n}_\Theta^\pm$. With this, the generalization of $\mathfrak{m}$ is defined by its Bruhat decomposition

$$\mathfrak{m}_\Theta = \mathfrak{n}^+(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a}_\mathfrak{p}(\Theta) \oplus \mathfrak{n}^-(\Theta) \ .$$

It is reductive and its semi-simple ideal corresponds to the root system $\Phi(\Theta)$. A basis of the Abelian ideal may be found in the lines described in the previous section. The compact part $\mathfrak{m}_\Theta \cap \mathfrak{k}$ of the subalgebra $\mathfrak{m}_\Theta$ will also be used and is denoted by $\mathfrak{m}_\Theta(K)$.

As long as only the structure of $\mathfrak{m}_\Theta$ is needed, they may be classified directly from the Satake diagram: Any sub-diagram which contains all black nodes, arbitrary pairs of type 2, and arbitrary single nodes of type 3 describes the semi-simple part of one $\mathfrak{m}_\Theta$. The number of remaining pairs of type 2 is the dimension of the Abelian ideal. Then, the dimension of $\mathfrak{a}_{\mathfrak{p}_\Theta}$ is given by the same number of remaining pairs of type 2 plus the number of remaining nodes of type 3.

Let $\mathfrak{l}_\Theta$ be the centralizer of $\mathfrak{a}_{\mathfrak{p}_\Theta}$ in $\mathfrak{g}$. It is given by $\mathfrak{l}_\Theta = \mathfrak{a}_{\mathfrak{p}_\Theta} \oplus \mathfrak{m}_\Theta$. The generalized Bruhat decomposition of $\mathfrak{g}$ is now immediate:

$$\mathfrak{g} = \mathfrak{n}_\Theta^+ \oplus \mathfrak{m}_\Theta \oplus \mathfrak{a}_{\mathfrak{p}_\Theta} \oplus \mathfrak{n}_\Theta^- \ . \tag{2.57}$$

Since $\mathfrak{n}_\Theta^\pm$ are isomorphic nilpotent subalgebras, their dimension can be calculated with the help of this direct sum.

A *standard parabolic subalgebra* is defined to be

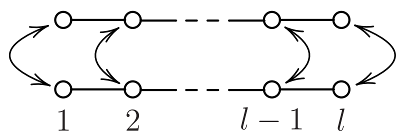
$$\mathfrak{b}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_{\mathfrak{p}_\Theta} \oplus \mathfrak{n}_\Theta^- = \mathfrak{m}_\Theta(K) \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}^- \ , \tag{2.58}$$

which clearly contains $\mathfrak{b}_0$. The first of the decompositions in (2.58) is usually called the *Langlands decomposition*, and the second one we will call the *compact Langlands decomposition*. The subalgebra $\mathfrak{l}_\Theta$ of $\mathfrak{b}_\Theta$ is called a *Levi subalgebra*. As the name suggests, these subalgebras exhaust all parabolic subalgebras up to conjugacy:

Theorem 8 *Let $\mathfrak{g}$ be a real semi-simple Lie algebra and $\mathfrak{b}_0$ a standard minimal parabolic subalgebra as defined above. The set of all $\mathfrak{b}_\Theta$, where Θ runs through all subsets of the set of simple restricted roots, are the only subalgebras containing $\mathfrak{b}_0$ and no two of them are conjugate. Any parabolic subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is conjugate to exactly one of them.*

In [War72] this is proved directly for parabolic subgroups. An axiomatic definition for general groups is given by the definition of Tits systems. This definition contains the standard minimal parabolic subgroup explicitly.

Given a set Θ of simple restricted roots, the structure of $\mathfrak{m}_\Theta$ is obtained from the Satake diagram as follows: Delete each white node which corresponds to a simple restricted root not in Θ from the Satake diagram. The resulting Satake diagram describes the semi-simple part of $\mathfrak{m}_\Theta$. As for the structure of M, the number of deleted pairs of type 2, i.e. nodes connected by an arrow, is equal to the dimension of the center of $\mathfrak{m}_\Theta$. After the deletion, the diagram is not necessarily connected. Each connected component is a Satake diagram of table 2.3 and describes a simple ideal of $\mathfrak{m}_\Theta$ with one exception: Some components may be of the form



From section 2.4 we know that this Satake diagram describes $\mathfrak{sl}(l+1, \mathbb{C})$ viewed as a real Lie algebra.

We will show for $\mathfrak{su}(m, n)$, how all $\mathfrak{m}_\Theta$ are obtained:
Deleting any number of pairs connected by an arrow from the Satake diagrams $A_l^{(III,r)}$ or $A_l^{(IIIa,(l+1)/2)}$ always results in one connected diagram of the same type and with the same number of black nodes. This number is $m - n - 1$ (where -1 may be used to denotes the diagram $A_{l'}^{(IIIa,(l'+1)/2)}$). Therefore, one simple ideal is $\mathfrak{su}(m', n')$, where $m' - n' = m - n$, $1 \le m' \le m$, and $1 \le n'$. The remaining simple ideals are $\mathfrak{sl}(n_i, \mathbb{C})$. For d deleted nodes we get

$$\text{(2.59)} \qquad \mathfrak{m}_\Theta \cong \mathfrak{u}(1)^d \oplus \mathfrak{su}(m', n') \oplus \bigoplus_{i=1}^{N} \mathfrak{sl}(n_i, \mathbb{C}) \quad \text{where } N \le d \text{ and}$$

$$\sum n_i = \tfrac{1}{2}(m+n-m'-n') - d + N \quad \text{and } m-n = m'-n' \; .$$

In the diagram $A_{l'}^{(IIIa,(l'+1)/2)}$ the middle node can also be deleted. This corresponds to replacing the summand $\mathfrak{su}(m',m')$ above by $\mathfrak{sl}(m',\mathbb{C})$.

2.6 Graded Structures of Lie Algebras

Graded structures of Lie algebras will appear at several places in the following chapter. Therefore it is useful to discuss the general structure here. First recall the following definitions (e.g. [Hel78, FS97]): A *gradation* of a Lie algebra $\mathfrak{g}$ by an Abelian semigroup S is a vector-space decomposition

$$\mathfrak{g} = \bigoplus_{j\in S} \mathfrak{g}^{(j)} \quad \text{such that} \quad \left[\mathfrak{g}^{(j)},\mathfrak{g}^{(k)}\right] = \mathfrak{g}^{(j+k)} \; .$$

For symmetric spaces $\mathbb{Z}_2$-gradations play an important role. They are in one-to-one correspondence with involutive automorphisms of $\mathfrak{g}$. A $\mathbb{Z}$-gradation with the additional property $\mathfrak{g}^{(j)} = \{0\}$ for $|j| > \frac{n-1}{2}$ for odd n is called an *n-grading* – the number of subspaces $\mathfrak{g}^{(j)}$ which are non-zero is n.

It is straight forward to define n-gradings with the help of root systems: Consider a root space decomposition of a Lie algebra $\mathfrak{g} = \bigoplus_{\alpha\in\overline{\Phi}} \mathfrak{g}^{\alpha}$ with respect to any root system Φ. This can be either the root space decomposition (2.3) with respect to the root system of the complex Lie algebra or its generalization (2.55) with respect to the restricted root system. The set $\overline{\Phi}$ is the union of the root system and $\{0\}$, and $\mathfrak{g}^0$ is the centralizer of the Cartan subalgebra $\mathfrak{h}$ on which Φ is defined.

Choose a subset Θ of the simple roots of Φ. Fix an ordering of the simple roots such that $\Theta = \{\alpha_{m+1},\ldots,\alpha_l\}$ while the remaining simple roots are $\alpha_1,\ldots,\alpha_m$. For a root $\alpha = \sum_{i=1}^{l} \alpha^i\alpha_i$ define the *level* to be $\ell_\Theta(\alpha) = \sum_{i=1}^{m} \alpha^i$ which is an integer. This level function for $\Theta = \{\}$ already had an appearance in the construction of the root system out of the simple roots and the Cartan matrix in section 2.2.1. An n-grading of $\mathfrak{g}$ is now defined by

$$\mathfrak{g}^{(j)} = \bigoplus_{\ell_\Theta(\alpha)=j} \mathfrak{g}^{\alpha} \; .$$

From $\left[\mathfrak{g}^{\alpha},\mathfrak{g}^{\beta}\right] \subset \mathfrak{g}^{\alpha+\beta}$ and $\ell_\Theta(\alpha+\beta) = \ell_\Theta(\alpha)+\ell_\Theta(\beta)$ it is obvious that this really defines an n-grading. If $\alpha_{\max}$ is the highest root in Φ this will be an $(2\ell_\Theta(\alpha_{\max})+1)$-grading:

Proposition 2 *All the subspaces* $\mathfrak{g}^{(j)}$ *are non-zero, i.e.* $\mathfrak{g}^{(j)} \neq \{0\}$, *for the levels* $j = -\ell_\Theta(\alpha_{max}), \dots, \ell_\Theta(\alpha_{max})$.

Proof: For each root $\alpha \in \Phi$ the root space $\mathfrak{g}^\alpha$ is non-zero by definition. With the highest root $\alpha_{\max}$, there is also a root $-\alpha_{\max}$ which has level $-\ell_\Theta(\alpha_{\max})$. For any positive root α there is a simple root α_i such that $\alpha - \alpha_i$ is a positive root or zero (e.g. [Cor84] p. 526, below example 13.7.V). Now there are two cases: If $\alpha_i \in \Theta$ it is $\ell_\Theta(\alpha - \alpha_i) = \ell_\Theta(\alpha)$ and otherwise $\ell_\Theta(\alpha - \alpha_i) = \ell_\Theta(\alpha) - 1$. In the first case there is another simple root α_j such that $\alpha - \alpha_i - \alpha_j$ is a positive root or zero and the same two cases apply for $\alpha - \alpha_i - \alpha_j$. This procedure is repeated until a positive or zero root β is found such that

$$\ell_\Theta(\beta) = \ell_\Theta(\alpha) - 1 \qquad \text{and} \qquad \ell_\Theta(\beta + \alpha_k) = \ell_\Theta(\alpha)\ ,$$

where α_k is the last simple root found. This happens in a finite number of steps, because the positive roots are linear combinations of the simple roots with non-negative integer coefficients and $\ell_\Theta(0) = 0$. Thus, for each positive root α there is such a root β. If $\mathfrak{g}^\alpha \subset \mathfrak{g}^{(j)}$ is non-zero, the non-zero space $\mathfrak{g}^\beta$ is a subspace of $\mathfrak{g}^{(j-1)}$. Thus, $\mathfrak{g}^{(j)}$ is non-zero for $j = \ell_\Theta(\alpha_{\max}), \dots, 0$. With an analogous reasoning it can be shown that the spaces $\mathfrak{g}^{(-j)}$ are non-zero.

End of Proof

With the same method as presented in the proof it can also be shown

Proposition 3 *For* $j = 1, \dots, \ell_\Theta(\alpha_{max})$ *the space* $\left[\mathfrak{g}^{(j)}, \mathfrak{g}^{(-1)}\right]$ *is non-zero.*

Proof: As in the previous proof find a root β with $\ell_\Theta(\beta) = j$ and a simple root α_i such that $\alpha - \alpha_k$ is a root and $\ell_\Theta(\beta - \alpha_k) = j - 1$ and therefore $\ell_\Theta(-\alpha_k) = -1$. From theorem 13.6.I in [Cor84] it is then known that $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha_i}]$ is non-zero.

End of Proof

If the restricted root system Σ and the set of simple restricted roots Θ, which appears in the definition of standard parabolic subalgebras in section 2.5, are used to define a grading, the following identifications are found:

$$\mathfrak{n}_\Theta^\pm = \bigoplus_{j=\pm 1}^{\pm \ell_\Theta(\alpha_{\max})} \mathfrak{g}^{(j)}\ , \qquad \mathfrak{l}_\Theta = \mathfrak{g}^{(0)}\ . \tag{2.60}$$

Since Σ is a restriction of Φ to $\mathfrak{a}_\mathfrak{p}$, this n-grading can also be considered to be defined via Φ. The corresponding set Θ^Φ contains all the simple roots of Φ which are restricted to elements of Θ or to zero.

2.7 LambdaLie: Explicit Construction of Real Lie Algebras

The computer algebra package LambdaLie [dR04] developed by the author is an implementation of the structures of complex and real simple Lie algebras which are discussed in the previous sections. The constructive approach in our presentation corresponds in most cases to the implementation in the software.

LambdaLie is capable of calculating the structure constants of finite dimensional complex simple Lie algebras if its type and rank are given. Automorphisms which are identity on the Cartan subalgebra can be specified. Based on this, the real forms with compact Cartan subalgebra can be obtained in form of a basis in the complex Lie algebra.

Automorphisms of the root system can be searched for by specifying several properties. For normal automorphisms this can be done by giving the Satake diagram. Extensions of these to automorphisms of the complex Lie algebra can be obtained. Since these extensions are defined on the basis vectors, it can be interpreted as a linear or anti-linear automorphism.

An anti-linear involution can be interpreted as a conjugation with respect to a real form of the complex Lie algebras. A basis of this real form can be found. Therefore, it is possible to specify type, rank, and Satake diagram and directly obtain the real Lie algebra $\mathfrak{g}$ and its maximally non-compact Cartan subalgebra $\mathfrak{a}$ described by it.

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the Killing index $\delta = (\dim \mathfrak{p} - \dim \mathfrak{k})$ is implemented and an ideal of $\mathfrak{k}$ can be found, if it exists.

All standard parabolic subalgebras $\mathfrak{b}_\Theta$ and the subalgebras $\mathfrak{m}_\Theta$, $\mathfrak{a}_{\mathfrak{p}_\Theta}$, $\mathfrak{n}^-_\Theta$, and $\mathfrak{n}^+_\Theta$ can be found by specifying the set Θ of simple restricted roots. The structure of $\mathfrak{m}_\Theta$ can be analyzed.

An efficient implementation of a polynomial algebra is also contained in LambdaLie. It can be used to find invariants of representations. It was developed in order to find the light cones of generalized conformal space-time symmetries.

LambdaLie is implemented in Common Lisp and was developed in the CMU Common Lisp environment. It is closely related to the package LambdaTensor [Fis02] by T. Fischbacher, which was developed for solving problems within three dimensional supergravity. Many features of LambdaTensor are used within LambdaLie.

2.7.1 Introduction to the Syntax

LambdaLie is written in Common Lisp and unlike most other computer algebra programs uses the language structure of its original language, Lisp. The same applies for the package LambdaTensor, which makes both packages easy to use simultaneously. It is also very easy to write algorithms using the results of LambdaLie and LambdaTensor, since Common Lisp is a well developed and documented programming language. Throughout this document, a few language features of Lisp and of LambdaTensor are used: A function of name `name` is called with the parameters `par1 par2` by entering `(name par1 par2)`. Lists are denoted by `'(1 2 3 4)`. Nested lists only need one apostrophe: `'(a b (c d) e)`. `#(1 2 3 4)` denotes a vector. When entering a root vector, one can use `(root 1 2 2 2 2)`. This restricts the type of the entries to integer.

One of the main features of LambdaTensor is an efficient implementation of sparse arrays. A tensor or array is sparse if most of its entries are zero. Then it usually is more efficient to store only the entries different from zero and their places in the array. Tensor multiplication can be performed much more efficiently. See the documentation of LambdaTensor for further details.

LambdaLie uses a special kind of structure for storing and constructing information about mathematical objects. It is an extension of the standard Lisp structure type, which is similar to structure types in other computer languages. A structure of a given type is a list of fixed slots which can be accessed via names. The most commonly used structure type in LambdaLie is called `simple-lie-algebra` or short `sla`. A structure of this type may be created and assigned to the variable `alg` with

```
* (setf alg (make-sla :rank 3))
```

This sets the slot `rank` to 3. The slots can be accessed with `(sla-rank alg)`, i.e. for each slot of a structure type there is a function for accessing that slot. The function name is composed of the structure type name and the slot name connected by a dash. This function can also be used to set a slot:

```
* (setf (sla-type alg) :a)
```

sets the slot `type` of the structure stored in the variable `alg` to the value `:a`.

So far, this has been standard Common Lisp syntax. The extended type of structure used in LambdaLie knows two special values each slot may have: `:unavailable` and `:unaccessible`. The first one denotes, that a slot is not set to any value, but it might be possible to construct a value from other slots in the structure. `:unaccessible` means, that it is not possible to construct a value. When accessing a slot, the result will be either a valid value or `:unaccessible`. On the other hand, it is sometimes useful to set a slot value to `:unavailable`, because a new value is to be constructed from the other slots in the structure. There are some additional functions defined for each slot: `(sla-type-avp alg)` returns `T` (denoting "true"), if the slot is already set to a value. Otherwise `NIL` (denoting "false"). `(sla-type-acp alg)` returns `T`, if a value may be deduced from the other slots. It is just checked, whether there is an appropriate set of functions defined for this deduction, but does not actually call these functions. `(sla-type-err alg)` and `(sla-type-thr alg)` cause an error instead of returning `:unaccessible`, if the slot cannot be deduced. They is mainly useful internally in the definition of functions. This short introduction is concluded with a small example:

```
* (setf alg (make-sla :type :g :rank 2))
Warning:  Declaring ALG special.

#< extended structure ABSTRACT-SIMPLE-LIE-ALGEBRA >
* (sla-rank-acp alg)
T
* (sla-rank-avp alg)
T
* (sla-cartan-avp alg)
NIL
* (sla-cartan-acp alg)
T
* (sla-cartan alg)
#2A((2 -3) (-1 2))
* (sla-cartan-avp alg)
T
* (setf (sla-type alg) :unavailable)
:UNAVAILABLE
* (sla-type-avp alg)
NIL
* (sla-type-acp alg)
T
* (sla-type alg)
(:B 1 1 1/3)
:G
* (sla-automorphism-acp alg)
NIL
* (sla-automorphism alg)
:UNACCESSIBLE
```

Here, `sla-rank` and `sla-type` of a simple Lie algebra have been set. The Cartan matrix `sla-cartan` is `:unavailable`, but "accessible". It is accessed, which implicitly calls a builder function which deduces the Cartan matrix from the type and the rank. After that, `sla-cartan` is "available". Now, `sla-type` is explicitly set to `:unavailable`. Since there is a builder function defined, which can deduce the type from the Cartan matrix and the rank, it stays"accessible". Once it is accessed, this builder is called. In contrast, there is no builder function, which can deduce the slot `sla-automorphism` from the type and the rank – we know, that G_2 has more than one automorphism. Finally, an example which shows what happens, when things go wrong:

```
* (setf (sla-type alg) :unavailable)
:UNAVAILABLE
* (setf (aref (sla-cartan alg) 0 1) -4)
-4
* (sla-cartan alg)
#2A((2 -4) (-1 2))
* (sla-type-acp alg)
T
* (sla-type alg)
Warning:  CLASSIFY-SIMPLE-LIE-ALGEBRA-FROM-CARTAN gives (:B . 2),
 but CARTAN->TYPE gives :UNACCESSIBLE
(:B 1 1 1/4)
:UNACCESSIBLE
:BUILDER-ERROR
```

The `sla-type` is set again to `:unavailable` and the second element of the first row of the Cartan matrix is set to -4. Since there is a builder function, which can construct the type from the Cartan matrix, the type is "accessible". But when we try to access it, the result is `:UNACCESSIBLE` and `:BUILDER-ERROR`, because this Cartan matrix does not describe a root system. The first three lines of output are additional information of that specific builder function and not important

2.7.2 Root Systems

Simple root systems are stored in the Lisp structure `simple-root-system` or short `srs`. Such a structure is created with `(make-srs)` or, when some of the slots are to be set with e.g. `(make-srs :type :a :rank 3 :satake-diagram '(2 o o))`. The important slots are

`type` The type of the root system. It has to be one of `:a :b :c :d :e :f :g`. It may be deduced from `srs-cartan, srs-rank`.

`rank` The rank of the root system can be deduced from either `cartan` or `simp-roots`. It will not be checked, whether `type` and `rank` together describe a valid root system.

`cartan` The Cartan matrix A of the root system. It is built from either `type, rank` corresponding to the Dynkin diagrams in table 2.1 or from `rank, metric, simp-roots`. It is defined as in equation (2.6). If it is set to a Cartan matrix not belonging to a root system, no error is thrown. If then the root system is generated by e.g. `(srs-v-roots sys)` LambdaLie will enter into an infinite loop, because it will try to generate the whole weight system of an infinite-dimensional Lie algebra, e.g. by `(srs-v-roots (make-srs :type :e :rank 9))`.

`simp-roots` A vector containing the simple roots α_i as vectors. Normally these will be `#(#(1 0 ... 0) ... #(0 0 ... 0 1))`. They can be built from `v-roots` and `num-pos-roots`, if `v-roots` is sorted such that the positive roots come first.

`v-roots` A vector containing all roots α as vectors.

`h-roots` A hash, relating the roots to numbers starting at zero:
`(hv (srs-h-roots sys) 4)` gives the fifth root and
`(hv (srs-h-roots sys) #(1 2 1))` gives the number of the root `#(1 2 1)`. This is the index in the vector `v-roots`.

`levels` This is filled, when the root system is generated from the Cartan matrix by the algorithm described in section 2.2.1. It is a list containing the number of the first root in each level of positive roots starting from the highest level. I.e. the first element `(first (srs-level sys))` will always be the number of the highest root, which is equal to the number of positive roots minus one. The last element `(last (srs-level sys))` will always be zero. If the root system is obtained in any other way, this slot will be useless.

`num-pos-roots` The number of positive roots, i.e. half the number of elements of `v-roots`. `(aref (srs-v-roots sys) (srs-num-pos-roots sys))` is the first negative root.

`2inv-root-squares` An array of the same length as `(srs-v-roots sys)`. Each element is equal to twice the inverse square $\frac{2}{\langle\alpha,\alpha\rangle}$ of the corresponding root α. This is always an integer. This array is built from the Cartan matrix via (2.13), (2.14), and (2.15).

`metric` The Euclidean metric $\langle\cdot,\cdot\rangle$ in the real vector space spanned by the roots as given by (2.15). It is given as a matrix in the basis of the simple roots. `2inv-root-squares` is a prerequisite for calculating this from the Cartan matrix.

The remaining slots of the structure type `simple-root-system` describe an automorphism of the root system. Both the root system and the automorphism describe a real Lie algebra based on a maximally non-compact Cartan subalgebra as explained in section 2.4.2.

`satake-diagram` A list or a vector of length `rank` describing a Satake diagram. White nodes are denoted by an `o` and black nodes by `*`. In many cases that uniquely fixes the diagram and the corresponding automorphism will be found. If this is not the case, the automorphism can be fixed by specifying the image of some of the simple roots. This only makes sense for the white nodes, because a simple root corresponding to a black node is mapped to its negative. An example, where this is necessary is A_3: `'(o o o)` might describe the Satake diagram $A_3^{(I,3)}$ as well as $A_3^{(IIIa,2)}$. The first one can be specified by `'(0 o o)` and the second by `'(2 o o)`.

`automorphism` A vector of length `rank` describing an automorphism of the root system. Each simple root is mapped to the root given by the number in this vector. It is best, not to specify it directly. but to specify `satake-diagram` or use the function `find-automorphism` described below.

`restricted-root-sys` A structure of type `simple-root-system` describing the restricted root system. The restricted roots are represented in the outer base given by the non-restricted simple roots and the metric is the restriction of the regular metric.

In addition to the builder-functions, which are responsible for deducing one slot from other slots of the structure, there are some other functions which might be useful when dealing with root systems:

`(print-sqmat mat)` Nice output of square matrices like the Cartan matrix.

`(srs-complete-root-list sys subset)` Generates a subsystem of roots which is generated by the subset `subset`. This is a list of roots which can be given either by their number or by the vector in the expansion into simple roots:
`(srs-complete-root-list sys-b4 ‘( 1 2 3 ,(root 2 2 2 1)))`
will generate a subsystem of type D_4 of a root system of B_4. The output is a list of root numbers.

`(srs-root*root sys r1 r2)` Scalar product of two roots which are given either by their number or as a vector.

`(srs-find-automorphism sys ... )` Searches for an automorphism of the root system with certain properties which can be specified by additional parameters:
`(srs-find-automorphism b4 :start #(1 1 2) :end #(17 17 17))`
will find the first automorphism which maps the simple roots onto the roots given by numbers between the entries of these two vectors. With '`:order 2`', '`:normal t`', '`:anti-normal t`', and '`:diag-sym t`' only involutive, normal, anti-normal automorphisms, and Dynkin diagram symmetries are found, respectively. With '`:intm #b011`' the third simple root is kept fixed (to the value specified by '`:start`') and only the first and second are varied. '`#b011`' is binary notation for the integer 3. The bits are numbered from right to left and each bit denotes, whether the corresponding root should be kept fixed or be varied. The result is a vector which can be assigned to `srs-automorphism`.

`(output-srs-find-automorphism sys ...)` This is a comfortable wrapper for `srs-find-automorphism` and accepts the same parameters. This version finds all automorphisms with the specified properties.

`(srs-nice-rs-aut sys)` Prints the images of the simple roots under the automorphism `(srs-automorphism sys)` as vectors and not only as root numbers.

`(srs-satake-arrow sys simp-root)` If the automorphism `(srs-automorphism sys)` is described by a Satake diagram, this function results in the simple root which is connected to '`simp-root`' by an arrow in the diagram.

2.7.3 Lie Algebras

The structure type `abstract-simple-lie-algebra` (short `sla`) is used for storing information about simple Lie algebras. This type inherits all the slots from the structure type `simple-root-system`. All the access functions for these slots exist in two versions: One with the prefix `srs-` and one with `sla-`. A structure of type `sla` can be created with
`(make-srs :type :a :rank 3 :satake-diagram '(2 o o))`.
The basis of the complex Lie algebra is implicitly assumed to consist of root vectors E_α for the roots α in the order given by the slot `v-roots` inherited from the root system and of the basis of the Cartan subalgebra H_{α_i} for the simple roots α_i. In addition to the slots of `simple-root-system` these are the important slots:

`dim` The dimension of the Lie algebra obtained directly from the root system with `dim` = `rank` + 2 · `num-pos-roots`.

`nab-convention` This can be set to either `:weyl` or `:chevalley` and fixes the conventions as described in section 2.2.4. The default is `:chevalley`.

`sp-killing-form` A square matrix stored in a sparse array and describing the killing form on the basis of the complex Lie algebra.

`sp-nab` A square matrix stored in a sparse array which contains the structure constants $N_{\alpha,\beta}$. For two roots α and β which are stored in the ith and jth element of `v-roots`, respectively, $N_{\alpha,\beta}$ is given by `(sp-ref (sla-sp-nab alg) i j)`.

`sp-fabc` A rank three tensor containing the structure constants of the complex Lie algebra. This can be used as an argument to many functions of the software package LambdaTensor.

`compact-csa-intmask` This is a bit vector specifying an inner involutive automorphism of the Lie algebra which is identity on the Cartan subalgebra. The bits of `(sla-compact-csa-intmask alg)` are the coefficients $\theta_(i)$ for $i = 1, \ldots, l$ as described in theorem 4. There is no check on the conditions stated in the theorem. Therefore, several such bit vectors result in equivalent real Lie algebras.

`automorphism-rho` A vector containing the coefficients ρ_α for an automorphism specified by `(sla-automorphism alg)` and all roots α.

`v-sp-base` A basis of the real form described by either the involution given by `compact-csa-intmask` or the conjugation given by `automorphism-rho` stored in a vector. The individual elements of the basis are sparse arrays.

`sp-base-transf` A rank two sparse array describing the basis transformation between the basis of the complex Lie algebra and that given by `v-sp-base`.

`sp-real-fabc` The structure constants of the real Lie algebra. This can be used as an argument to many functions of the software package LambdaTensor.

`v-sp-cartan-compact` A basis of the maximally compact subalgebra $\mathfrak{k}$ in a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

`v-sp-cartan-c-der` The derived subalgebra $[\mathfrak{k},\mathfrak{k}]$ of $\mathfrak{k}$. This is a semi-simple subalgebra $\mathfrak{k}_s$ of $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$ since $\mathfrak{k}$ is reductive.

`v-sp-cartan-c-der-comm` A basis of the commutant of the derived subalgebra. This is the center $\mathfrak{z}_{\mathfrak{k}}$ of $\mathfrak{k}$.

`v-sp-cartan-non-compact` The non-compact part $\mathfrak{p}$ of a Cartan decomposition.

`killing-index` The Killing index δ of the real form.

2.7.4 Parabolic Subalgebras

Parabolic subalgebras $\mathfrak{b}_\Theta$ are stored in the structure type `parabolic-subalgebra`, or short `psa`. It has the following slots, the first two of which should be set upon creation, e.g. by

```
* (make-psa :lie-algebra alg :restricted-intmask #b01001),
```

`lie-algebra` The Lie algebra of which it is a parabolic subalgebra. Given as a structure of type `abstract-simple-lie-algebra`.

`restricted-intmask` A bit vector describing the subset Θ of the set of simple restricted roots. Note that the bits are given from right to left.

`sub-m` A basis of the subalgebra $\mathfrak{m}_\Theta$ given as a vector of sparse arrays.

`sub-a` A basis of $\mathfrak{a}_{\mathfrak{p}_\Theta}$.

`sub-n` A basis of $\mathfrak{n}_\Theta^+$.

`sub-n-` A basis of $\mathfrak{n}_\Theta^-$.

`sub-ma` A basis of $\mathfrak{a}_\mathfrak{p}(\Theta) \oplus \mathfrak{a}_\mathfrak{k}$.

`sub-map` A basis of $\mathfrak{a}_\mathfrak{p}(\Theta)$.

`sub-m-sla` A structure of type `abstract-simple-lie-algebra` describing $\mathfrak{m}_\Theta$.

2.8 Global Structure of Lie Groups

This section summarizes the facts about Lie groups and their relation to Lie algebras which are essential for the developments in the following chapters. For the analysis of space-time symmetries it is essential that the global structure of simple Lie groups is not completely fixed by the Lie algebra: Two Lie groups with the same Lie algebra may have non-isomorphic discrete centers. The standard reference for the presented material is [Hel78].

A *real Lie group* G is a real analytic manifold with a group multiplication which is an analytic function $G \times G \to G$. The Lie algebra $\mathfrak{g}$ corresponding to G is the tangent space at the identity $T_{\mathbf{e}}G$ with the commutator defined in the following way: For $X \in T_{\mathbf{e}}G$ let $\lambda(t)$ be a curve with tangent vector X at $t = 0$. Define the *left invariant vector field* $\widetilde{X}$ induced by X to be the derivation which maps $f \in C^\infty(G)$ to $g \mapsto \frac{d}{dt} f(g\lambda(t))|_{t=0}$. The commutator is now defined by the commutator of vector fields evaluated at the identity: $[X,Y] = [\widetilde{X}, \widetilde{Y}]_{\mathbf{e}} = (\widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X})_{\mathbf{e}}$.

The Lie algebra of a Lie subgroup of G is a subalgebra of $\mathfrak{g}$. On the other hand, each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected subgroup of G, which is called the *connected subgroup* of G *corresponding to* the subalgebra. A Lie group is called *semi-simple* or *simple* if its Lie algebra is of the respective type.

For each $X \in \mathfrak{g}$ there is a unique analytic homomorphism of the group $(\mathbb{R}, +)$ into G given by the integral curve λ_X of $\widetilde{X}$ through $\mathbf{e}$ with $\lambda_X(0) = \mathbf{e}$. The *exponential map* $\exp\colon \mathfrak{g} \to G$ is then defined by $\exp(X) = \lambda_X(1)$. There exists an open neighborhood of $0 \in \mathfrak{g}$ which is mapped diffeomorphically onto an open neighborhood of $\mathbf{e} \in G$.

For a fixed $g \in G$ let $\kappa(g)\colon G \to G$ be given by $\kappa(g)\colon h \mapsto ghg^{-1}$. A subgroup of G which is invariant under $\kappa(g)$ for $g \in G$ is called an *invariant* or *normal* subgroup. The Lie subalgebra corresponding to an invariant subgroup is invariant.

The *adjoint representation* $\mathrm{Ad}\colon G \to Gl(\mathfrak{g})$ of G is defined to be the derivative of $\kappa(g)$ at the identity: $\mathrm{Ad}(g) = T_{\mathbf{e}}\,\kappa(g)\colon \mathfrak{g} \to \mathfrak{g}$. For a connected Lie group the image of Ad turns out to be the adjoint group $\mathrm{Int}(\mathfrak{g})$ and its kernel is equal to the center Z of G. In fact Ad is an analytic isomorphism of G/Z onto $\mathrm{Int}(\mathfrak{g})$. If $\mathfrak{g}$ has trivial center, $\mathrm{Int}(\mathfrak{g})$ has trivial center, as well. Note that this is not the case, if $\mathfrak{g}$ has center different from $\{0\}$, as the example $[X,Y] = Z$, $[X,Z] = 0$, $[Y,Z] = 0$ shows.

However, a reductive Lie algebra has non-trivial center. This case is covered by the following situation: If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}_{\mathfrak{g}}$ is a direct sum of ideals where $\mathfrak{z}_{\mathfrak{g}}$ is the center, $\mathrm{Int}(\mathfrak{g})$ does have trivial center and is isomorphic to $\mathrm{Int}(\mathfrak{g}_0)$.

The relationship between the adjoint representation Ad of a Lie group and the adjoint group $\mathrm{Int}(\mathfrak{g})$ of its Lie algebra is made explicit by

$$\mathrm{Ad}(\exp(X)) = \mathrm{Exp}(\mathrm{ad}(X)) ,$$

which holds for all $X \in \mathfrak{g}$ and Exp denotes the matrix exponentiation.

Let $\mathfrak{g}$ be a Lie algebra such that $\mathrm{Int}(\mathfrak{g})$ is a compact Lie group. This is the case if and only if $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{z}_\mathfrak{g}$ is reductive and its semi-simple part $\mathfrak{g}_s$ is a compact Lie algebra (i.e. the Killing form is negative definite). Under the same condition $\exp\colon \mathfrak{g} \to G$ is surjective for any connected Lie group G with Lie algebra $\mathfrak{g}$. Note that G is not necessarily compact. The prominent example of this case is the “compact” factor of the Cartan decomposition of a semi-simple Lie group discussed in the next section. With these results and the following theorem each compact semi-simple connected Lie group decomposes into a direct product of invariant simple subgroups:

Theorem 9 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\exp\colon \mathfrak{g} \to G$ surjective. Each connected subgroup H of G whose Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ is an invariant subgroup of G.*

Proof: All elements of G may be written as $\exp(X)$ with $X \in \mathfrak{g}$. For $g = \exp(X) \in G$ and $h = \exp(Y) \in H$ it holds

$$\kappa(g)h = \exp(\mathrm{Ad}(g)Y) = \exp(\mathrm{Exp}(\mathrm{ad}(X))Y) \in H , \tag{2.61}$$

since $\mathrm{Exp}(\mathrm{ad}(X))Y \in \mathfrak{h}$.

End of Proof

Finally, a connected semi-simple Lie group with compact Lie algebra is always compact as a manifold.

2.8.1 Semi-Simple Lie Groups

Let $\widetilde{\mathrm{Int}}(\mathfrak{g})$ be the universal covering group of $\mathrm{Int}(\mathfrak{g})$ for $\mathfrak{g}$ semi-simple. The center Z of $\widetilde{\mathrm{Int}}(\mathfrak{g})$ is discrete and isomorphic to the fundamental group of $\mathrm{Int}(\mathfrak{g})$. Any connected Lie group G with Lie algebra $\mathfrak{g}$ is isomorphic to $\widetilde{\mathrm{Int}}(\mathfrak{g})/\mathsf{Z}_0$, where Z_0 is some subgroup of Z. The center of G is isomorphic to Z/Z_0. This leads to

Theorem 10 *A connected semi-simple Lie group is determined up to isomorphy by its Lie algebra and center.*

Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the semi-simple Lie algebra $\mathfrak{g}$ of a connected Lie group G. Let K be the connected subgroup corresponding to the subalgebra $\mathfrak{k}$. The *Cartan decomposition* of G is given by the following

Theorem 11 *The mapping $\mathfrak{p} \times K \to G$ given by $(X, k) \mapsto \exp(X)k$ is a diffeomorphism.*

The image of $(\mathfrak{p}, \mathbf{e})$ is the connected subgroup of G corresponding to $\mathfrak{p}$ and is denoted by P. The Cartan decomposition is written concisely $G = PK = KP$. Of course this does not mean that K and P commute. Since $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$ is reductive and its semi-simple ideal $\mathfrak{k}_s$ compact, the results before theorem 9 apply: $\mathrm{Int}(\mathfrak{k})$ is compact, and the map $\exp\colon \mathfrak{k} \to K$ is surjective. The Cartan decomposition gives $G = \exp(\mathfrak{k})\exp(\mathfrak{p})$.

The center Z of G is contained in K. This means that the center of the universal covering is the fundamental group of $K_{\mathrm{Int}(\mathfrak{g})}$, the connected subgroup of $\mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{k}$. A classification of the fundamental groups of all possible $K_{\mathrm{Int}(\mathfrak{g})}$ completes the classification of real semi-simple Lie algebras to that of all connected semi-simple Lie groups. This is equivalent to the classification of the centers of all their universal coverings.

Because $\mathrm{Int}(\mathfrak{k})$ is compact, K decomposes into a direct product of invariant subgroups $K = Z_\mathfrak{k} \times K_s$ corresponding to $\mathfrak{k} = \mathfrak{z}_\mathfrak{k} \oplus \mathfrak{k}_s$, where $Z_\mathfrak{k}$ is the center of K and K_s is semi-simple. The latter factor K_s decomposes into a direct product of invariant simple subgroups. The center Z of G is therefore also a direct product of subgroups each of which lies in the center of one of the factors of K. The center of the universal covering of G is thus the direct product of the centers of the universal coverings of the simple components of K_s and of a discrete subgroup of $Z_\mathfrak{k}$. The centers of the universal coverings of all simple compact connected Lie groups are given in table 2.6 [Hel78].

$\mathfrak{g}$	$\mathfrak{a}_l$	$\mathfrak{b}_l$	$\mathfrak{c}_l$	$\mathfrak{d}_{2n+1}$	$\mathfrak{d}_{2n}$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
$\mathsf{Z}_{\widetilde{\mathrm{Int}}(\mathfrak{g})}$	$\mathbb{Z}_{l+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$

Table 2.6: Centers of the universal coverings of all simple compact connected Lie groups.

It is possible to give an explicit representation of the centers of the compact simple Lie groups by $\exp(2\pi i h)$, with ih elements of a Cartan subalgebra for which $\alpha(h)$ is an integer. Some of these actually give the same element of the center. This is the case if they are represented by the same operator in every irreducible representation. Since every representation of a compact Lie algebra gives upon exponentiation a representation of the universal covering group, this can be checked only with reference to the Lie algebra. In [Cor84] appendix H generators of the center of all universal covering compact semi-simple Lie groups are given.

On the other hand, an Abelian Lie group of dimension d is isomorphic to $U(1)^n \times \mathbb{R}^{d-n}$. The Abelian ideal $Z_{\mathfrak{k}}$ may now have this form. If G is simply connected, i.e. isomorphic to the universal covering $\widetilde{\mathrm{Int}}(\mathfrak{g})$, $Z_{\mathfrak{k}}$ is completely non-compact ($\mathbb{R}^d$) and contains a subgroup of the center of G isomorphic to $\mathbb{Z}^d$. This implies that K is not necessarily compact.

2.8.2 Parabolic Subgroups

The following facts can be found in [War72]. Recall the definitions from section 2.5. As for Lie algebras, *parabolic subgroups* are defined to be conjugate to *standard* parabolic subgroups with respect to some choice of maximally non-compact Cartan subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of some real semi-simple Lie group. For some subset Θ of the simple restricted roots let M_Θ, $M_\Theta(K)$, L_Θ, $A_{\mathfrak{p}\Theta}$, and $N_\Theta^\pm$ be the connected subgroups of G with Lie algebras $\mathfrak{m}_\Theta$, $\mathfrak{m}_\Theta(K)$, $\mathfrak{l}_\Theta$, $\mathfrak{a}_{\mathfrak{p}\Theta}$, and $\mathfrak{n}_\Theta^\pm$ respectively. Let $\mathsf{M}_\Theta(K)$ and L_Θ be the centralizer of $\mathfrak{a}_{\mathfrak{p}\Theta}$ in G and K respectively. Define $\mathsf{M}_\Theta = \mathsf{M}_\Theta(K) M_\Theta$. It holds $\mathsf{L}_\Theta = \mathsf{M}_\Theta A_{\mathfrak{p}\Theta}$. The connected subgroups M_Θ, $M_\Theta(K)$, and L_Θ are the connected components of M_Θ, $\mathsf{M}_\Theta(K)$, and L_Θ, respectively. The standard parabolic subgroup corresponding to Θ is given by

$$\mathsf{B}_\Theta = \mathsf{M}_\Theta A_{\mathfrak{p}\Theta} N_\Theta^- = \mathsf{M}_\Theta(K) A_{\mathfrak{p}} N^- \ . \tag{2.62}$$

As for Lie algebras, these two decompositions will be called the *(compact) Langlands decomposition*. The subgroups L_Θ, M_Θ, and $\mathsf{M}_\Theta(K)$ are not necessarily connected. The component structure is given by $\mathsf{M}_\Theta(K) = \mathsf{Z}^{\mathsf{M}} M_\Theta(K)$ where $M_\Theta(K)$ is the connected subgroup corresponding to the Lie algebra $\mathfrak{m}_\Theta(K)$, which is the connected component of $\mathsf{M}_\Theta(K)$ containing the identity, and Z^{M} is a discrete group. The group Z^{M} may be independent of Θ and is therefore usually defined with reference to the minimal parabolic subgroup and the centralizer M of $\mathfrak{a}_{\mathfrak{p}}$ as discussed below. Warner [War72] uses the non-Abelian discrete group

$$\mathsf{Z}^{\mathsf{M}} = \mathsf{Z}(A_{\mathfrak{p}}) = \mathrm{Ad}_G^{-1}(\mathrm{Ad}_G(K) \cap \mathrm{Exp} \circ \mathrm{ad}(i\mathfrak{a}_{\mathfrak{p}}))\ ,$$

while Knapp [Kna02] defines an Abelian discrete group F. Neither of $\mathsf{Z}(A_{\mathfrak{p}})$ and F is claimed to be the group of connected components of any of the $\mathsf{M}_\Theta(K)$. Knapp also shows for G which have a complexification that F is generated by all elements of the form $\exp(2\pi i H_\alpha)$, where H_α is defined as equation (2.30) and α is a *real* root, which is defined to have real values on $\mathfrak{a}$ (i.e. it is zero on $\mathfrak{a}_{\mathfrak{k}}$).

Since the center Z_G of G is contained in K and L_Θ is the centralizer of $\mathfrak{a}_{\mathfrak{p}\Theta}$ in G and, the center Z_G is a subgroup of the center of $\mathsf{L}_\Theta \cap K = \mathsf{M}_\Theta(K)$. Knapp also shows that F is contained in the center of $\mathsf{M}_\Theta(K)$ for G which have a complexification.

The Cartan decomposition of M_Θ is $\mathsf{M}_\Theta(K)\exp(\mathfrak{p}\cap\mathfrak{m}_\Theta)$. The connected factors in (2.62) can be represented by

$$A_{\mathfrak{p}\Theta} = \exp(\mathfrak{a}_{\mathfrak{p}\Theta}) \qquad \text{and} \qquad N_\Theta^\pm = \exp(-h)\exp(h - \mathfrak{n}_\Theta^\pm)\ , \tag{2.63}$$

where $h \in \mathfrak{a}$ such that no root is zero on h. These are in fact analytic diffeomorphisms of the Lie algebras onto the Lie groups. The exponential map from $\mathfrak{n}_\Theta^\pm$ into $N_\Theta^\pm$ is surjective.

Of special interest is the minimal case. As for the subalgebras, the index Θ will be dropped for $\Theta = \{\}$. Since $\mathfrak{a}_\mathfrak{p}$ is maximal Abelian in $\mathfrak{p}$, its centralizer L in G is of the form $\mathsf{L} = \mathsf{M}\mathfrak{a}_\mathfrak{p}$, where M is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in K.

2.8.3 The Weyl Group

In this section we summarize some facts about the Weyl group $W = W(\Sigma)$ of the restricted root system Σ as defined in section 2.3.1.

Let M^* be the normalizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. It has the same Lie algebra $\mathfrak{m}$ as M. The Weyl group W is isomorphic to the discrete group M^*/M. Therefore, each element of the Weyl group ω may be represented by an element m_ω of M^*. For a fixed $\omega \in W$ define

$$\mathfrak{n}_\omega^- = \mathfrak{n}^- \cap \mathrm{Ad}(m_\omega)\mathfrak{n}^- \qquad \mathfrak{u}_\omega^- = \mathfrak{n}^- \cap \mathrm{Ad}(m_\omega)\mathfrak{n}^+\ .$$

As the notation implies, these definitions depend only on ω and not on the choice of m_ω. Let $N_\omega^- = \exp(\mathfrak{n}_\omega^-)$ and $U_\omega^- = \exp(\mathfrak{u}_\omega^-)$. These are simply connected subgroups of N^+ and

$$N_\omega^- \cap U_\omega^- = \{\mathbf{e}\} \quad \text{and} \quad N^+ = N_\omega^- U_\omega^-\ .$$

Consider a subset Θ of the simple restricted roots. To it corresponds a decomposition of the Weyl group: Let $\langle\Theta\rangle$ be the subsystem of the restricted root system Σ consisting of linear combinations of Θ. W_s is defined to be the subgroup of W generated by the Weyl reflections ω_{α_i} for $\alpha_i \in \Theta$, i.e. the Weyl group of $\langle\Theta\rangle$. Let W_u be the set of all $\omega \in W$ which do not map any negative root on a positive root in $\langle\Theta\rangle$. For each $\omega \in W$ there is a unique decomposition $\omega = \omega_s\omega_u$ with $\omega_s \in W_s$ and $\omega_u \in W_u$.

The Weyl group acts simply transitively on the set of fundamental systems of the root system, i.e. for two fundamental systems Ψ_1 and Ψ_2 of Σ there is exactly one element $\omega \in W$ such that $\omega(\Psi_1) = \Psi_2$. Denote the element of the Weyl group, which maps a fixed fundamental system $\Psi = \{\lambda_1, \ldots, \lambda_r\}$ to its negative $-\Psi = \{-\lambda_1, \ldots, -\lambda_r\}$ by $\mathring{\omega}$. With this we have $\mathfrak{u}^-_{\mathring{\omega}} = \mathfrak{n}^-$ and $U^-_{\mathring{\omega}} = N^-$. Let $\mathring{\omega} = \mathring{\omega}_s\,(\mathring{\omega}_u)^{-1}$ be the decomposition of $\mathring{\omega}$ with respect to Θ. Then $\mathfrak{u}^-_{\mathring{\omega}_u} = \mathfrak{n}^-_\Theta$ and $U^-_{\mathring{\omega}_u} = N^-_\Theta$.

2.8.4 Decompositions of Real Semi-Simple Lie Groups

The Cartan decomposition was already given in theorem 11. It is the basis of all studies of real semi-simple Lie algebras. Some more decompositions will be of importance to this study.

Let the *fundamental Weyl chamber* $\mathfrak{a}_\mathfrak{p}^+$ be the subset of $\mathfrak{a}_\mathfrak{p}$ of all $h \in \mathfrak{a}_\mathfrak{p}$ such that $\lambda(h)$ is positive for all simple restricted roots λ. Define $A_\mathfrak{p}^+ = \exp(\mathfrak{a}_\mathfrak{p}^+)$ and let $\overline{A_\mathfrak{p}^+}$ be its closure in G. The Weyl chambers are conjugate under the Weyl group of the restricted root system. The following result is referred to as the *KAK decomposition* and by some authors [Hel78] also as the Cartan decomposition.

Theorem 12 *Let G be a connected semi-simple Lie group, then, with the definitions as above*

$$G = K\overline{A_\mathfrak{p}^+}K\ .$$

For each $g \in G$ there is a unique $a \in \overline{A_\mathfrak{p}^+}$ such that $g = k_1 a k_2$ for $k_1, k_2 \in K$.

The *Iwasawa decomposition* is given by the following result:

Theorem 13 *A connected semi-simple Lie group G may be written in the form*

$$G = KA_\mathfrak{p}N^-\ .$$

For each $g \in G$ there are unique $k \in K$, $h \in \mathfrak{a}_\mathfrak{p}$, and $n \in N^-$ such that $g = k\exp(h)n$.

Note that by specifying an Iwasawa decomposition of a Lie group the Cartan decomposition, the non-compact part of the Cartan subalgebra, and the fundamental system are already fixed. Fixing an Iwasawa decomposition fixes therefore all other choices which can be made in the theory.

With the preparations from section 2.8.3 the *Bruhat decomposition* is given by

Theorem 14 *Let G be a connected semi-simple Lie group, $\mathsf{B}_0 = \mathsf{M}A_{\mathfrak{p}}N^-$ a standard minimal parabolic subgroup, and W the Weyl group of the restricted root system. G is the disjoint union*

$$G = \bigcup_{\omega\in W} \mathsf{B}_0 m_\omega \mathsf{B}_0 = \bigcup_{\omega\in W} N^- m_\omega \mathsf{B}_0 = \bigcup_{\omega\in W} U_\omega^- m_\omega \mathsf{B}_0 \ .$$

Moreover, the subgroup U_ω^- parameterizes the orbit $\mathsf{B}_0 m_\omega \mathsf{B}_0 = N^- m_\omega \mathsf{B}_0$ of the coset $m_\omega \mathsf{B}_0 = [m_\omega]_{\mathsf{B}_0}$ in G/B_0, and the map $U_\omega^- \to [\mathsf{B}_0 m_\omega]_{\mathsf{B}_0}$, $u \mapsto u m_\omega \mathsf{B}_0$ is a regular bijection.

Recall that a regular map $\phi\colon M \to N$ between analytic manifolds is analytic and its derivative $T_p\phi\colon T_pM \to T_{\phi(p)}N$ at any $p \in M$ is injective.

For the non-minimal parabolic subgroups, there is a similar result, the *generalized Bruhat decomposition*:

Theorem 15 *Let G be a connected semi-simple Lie group, Θ some subset of the simple restricted roots, $\mathsf{B}_\Theta = \mathsf{M}_\Theta A_{\mathfrak{p}\Theta} N_\Theta^-$ a standard parabolic subgroup, and W_u subset of the Weyl group of the restricted root system as described in section 2.8.3. G is the disjoint union*

$$G = \bigcup_{\omega^{-1}\in W_u} N^- m_\omega \mathsf{B}_\Theta = \bigcup_{\omega^{-1}\in W_u} U_\omega^- m_\omega \mathsf{B}_\Theta \ . \tag{2.64}$$

Moreover, the subgroup U_ω^- parameterizes the orbit $N^- m_\omega \mathsf{B}_\Theta$ of the coset $m_\omega \mathsf{B}_\Theta$ in G/B_Θ, and the map $U_\omega^- \to [N^- m_\omega]_{\mathsf{B}_\Theta}$ given by $u \mapsto u m_\omega \mathsf{B}_\Theta$ is a regular bijection.

The N^--orbits in G/B_Θ are called *Schubert cells*. Recall the generalized Bruhat decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{n}_\Theta^+ \oplus \mathfrak{b}_\Theta$. The corresponding product $N_\Theta^+ \mathsf{B}_\Theta$ is not the whole G. The discussion of the element of the Weyl group $\mathring{\omega}_u$ in the previous section shows that this product is just one of the sets in the disjoint union (2.64):

Theorem 16 *The subgroup $N_\Theta^+ \mathsf{B}_\Theta$ is a dense open submanifold of G whose complement has zero Haar measure. [War72]*

This dense subgroup appears in the generalized Bruhat decomposition in the disjoint union for $\mathring{\omega}_u \in {W_u}^{-1}$, because $N_\Theta^+ = {m_{\mathring{\omega}_u}}^{-1} U_\omega^- m_{\mathring{\omega}_u}$.

In the following chapter it will be shown that G/B_Θ can in certain cases be interpreted as space-time which is symmetric under G. It is the generalization of conformally compactified Minkowski space-time. The subspace $N_\Theta^+\mathsf{B}_\Theta/\mathsf{B}_\Theta$ is clearly isomorphic to the nilpotent Lie group N_Θ^+, which is isomorphic to the Lie algebra $\mathfrak{n}_\Theta^+$ by (2.63). It is the generalization of the ordinary Minkowski space-time.

Chapter 3

Simple Space-Time Symmetries

In this chapter the main results of this work will be presented. A program to study quantum field theories with simple Lie groups describing the symmetry of space-time is introduced. The classical examples are conformal field theories in any space-time dimension. Quantum field theories are required to satisfy conventional principles as required for example by the Wightman axioms [SW64]. These include assumptions about the representation on Hilbert space of the symmetry group of space-time, which is the Poincaré group in general quantum field theory and the conformal group $SO(2, D)$ in conformal field theory. While the latter is a simple Lie group, the former is not even semi-simple. However, the Poincaré group is a Lie subgroup of the conformal group, which is the smallest simple Lie group containing the Poincaré group.

In general quantum field theory the space-time symmetry group G has to act as a unitary representation on Hilbert space. Since we require the spectrum condition to hold, there has to be a generator H_0 such that iH_0 is in the Lie algebra of G with non-negative spectrum in the infinitesimal version of this representation. Since iH_0 is interpreted as the generator of time translations, H_0 is the energy operator. Therefore, such representations of G are called *positive energy representations*.

The fields in a quantum field theory are required to satisfy a locality principle, which can only be defined if there exists a notion of relatively spacelike points in the space-time $\mathbb{M}$ given by a globally causal structure. The property that two points are relatively spacelike has to be preserved by the action of G on $\mathbb{M}$. Therefore, the space-time $\mathbb{M}$ has to admit a globally causal structure which is G-invariant.

Conformal symmetry on Minkowski space-time $\mathbb{M} = \mathbb{R}^D$ does not admit such a causal structure: $\mathbb{M}$ has to be compactified to $\mathbb{M}^c$ in order to properly define the special conformal transformations. Then, spacelike pairs of points are transformed into timelike. This has been resolved by Lüscher and Mack [LM75] by going to the universal covering $\widetilde{\mathbb{M}}^c$ of $\mathbb{M}^c$.

In a homogenous setting it is natural to assume that the space-time symmetry group G acts transitively on $\mathbb{M}$. Otherwise, $\mathbb{M}$ would decompose into G-orbits, which are possibly governed by differing physical laws and may be treated independently. Thus, $\mathbb{M}$ is a homogenous space $\mathbb{M} = G/H$, where H is the stability group of an arbitrary point $\mathbf{o} \in \mathbb{M}$. In the conformal case, the universal covering space $\widetilde{\mathbb{M}}^c$ is in fact a homogenous space of the universal covering of $SO(2, D)$ and H has to be simply connected.

In addition to these general requirements which are due to well known principles of quantum field theory we will retain two additional properties of conformal symmetry:

The stability group H in conformally symmetric space-time is the identity component B_Θ of a parabolic subgroup B_Θ (section 2.8.2): The factors of its Langlands decomposition $B_\Theta = M_\Theta A_{\mathfrak{p}\Theta} N_\Theta^-$ are the Lorentz group M_Θ, the group of dilatations $A_{\mathfrak{p}\Theta}$, and the group of special conformal transformations N_Θ^-. The group of translations is N_Θ^+ and is not part of the stability group. In this chapter we will consider simple Lie groups and their action on homogenous spaces, where the stability group of a point is the identity component of a parabolic subgroup.

The second property of conformal symmetry which we will retain is the following. The special conformal transformations act trivially on the tangent space at $\mathbf{o}$. From this follows that the the translations commute with each other.

After these assumptions are summarized in section 3.1 the action of the generalized special conformal transformations is analyzed in section 3.2. With these preparations G-invariant infinitesimal causal structures are discussed in section 3.3 and with corollary 1 a criterion for their existence is derived. In which situations it is a global causal structure and the implications on the global structure of G and $\mathbb{M}$ are discussed in section 3.4. It turns out that the existence of positive energy representations is already implied by the conditions on the global structure of $\mathbb{M}$ which provide for global causality. This will be seen in section 3.5.

In section 3.6 the results are used to classify all simple Lie groups G and corresponding space-times $\mathbb{M}$ which satisfy the assumptions. The chapter is concluded with section 3.7 in which it is tested, whether a time reflection exists in the possible space-time symmetry groups. Surprisingly, it turns out that not all of the groups which satisfy our assumptions possess a time reflection.

The results presented in this chapter will be published in [MdR04].

3.1 Generalizing Conformal Space-Time Symmetry

For a Lie group G to describe a space-time symmetry in a quantum field theory with spectrum condition, it is necessary to have a non-trivial continuous unitary representation U with a selfadjoint generator H_0 with a purely positive spectrum. More precisely, such a generator is the image under the representation $\mathsf{u} = \partial\mathsf{U}$ of an element of the Lie algebra $iH_0 \in \mathfrak{g}$ divided by the imaginary unit. Positivity of its spectrum is denoted by $\frac{1}{i}\mathsf{u}(iH_0) \geq 0$ or short $\mathsf{u}(H_0) \geq 0$. H_0 is an element of the complexification of the Lie algebra $\mathfrak{g}$. Representations with these properties will be called *positive energy representations*. Since we are interested in groups, which allow an interpretation as space-time symmetries in the framework of quantum field theory satisfying the spectrum condition, we will make in the following the

Assumption 1 *G is a connected simple Lie group which possesses positive energy representations.*

The Lie groups satisfying this assumption have been classified long ago [Lüs75, Ol'82]. A review will be given in section 3.5. Note that there is no reference to the action of G on the space-time $\mathbb{M}$ in assumption 1. Therefore, the groups satisfying it may act on any kind of space-time manifold. The action on $\mathbb{M}$ is constrained by the

Assumption 2 *G acts transitively and differentiably on the space-time manifold $\mathbb{M}$. The space-time $\mathbb{M}$ admits a globally causal structure which is invariant under this action.*

These two are the assumptions stemming from principles of quantum field theory. The structure of the conformal group $SO(2, D)$ and of its action on the space-time $\widetilde{\mathbb{M}}^c$ motivate the remaining two assumptions:

Assumption 3 *The stability group H of any point under the action is the identity component B_Θ of a parabolic subgroup B_Θ. Let $H = B_\Theta = M_\Theta A_{\mathfrak{p}_\Theta} N_\Theta^-$ be its Langlands decomposition. The reductive subgroup M_Θ is the generalized Lorentz group, the elements of $A_{\mathfrak{p}_\Theta}$ are the generalized dilatations, and the nilpotent subgroup N_Θ^- is the group of generalized special conformal transformations.*

With this assumption, the global structure of $\mathbb{M}$ can be analyzed: Recall the compact Langlands decomposition $\mathsf{B}_\Theta = \mathsf{M}_\Theta(K) A_\mathfrak{p} N^-$ of a parabolic subgroup from equation (2.62). The corresponding decomposition of the identity component is $B_\Theta = M_\Theta(K) A_\mathfrak{p} N^-$. The subgroup $M_\Theta(K)$ is the intersection of K

with M_Θ. Comparing this with the Iwasawa decomposition of G from theorem 13 gives

$$G/\mathsf{B}_\Theta = KA_\mathfrak{p}N^-/M_\Theta(K)A_\mathfrak{p}N^- = K/M_\Theta(K)\,. \tag{3.1}$$

At first sight this seems to be a compact space. However, we have seen in section 2.8.1 that K is not necessarily compact. In fact, it will be shown in section 3.4 that it has to be non-compact.

It will be useful to have a decomposition of $\mathfrak{k}$ which corresponds to the quotient in (3.1). For this, choose a subspace $\mathfrak{q}_\mathfrak{k}$ of $\mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_\mathfrak{k}$. With this we also have

$$\mathfrak{g} = \mathfrak{b}_\Theta \oplus \mathfrak{q}_\mathfrak{k}\,. \tag{3.2}$$

The subspace $\mathfrak{q}_\mathfrak{k}$ can be identified with the tangent space of the space-time $K/\mathsf{M}_\Theta(K)$ at $[\mathbf{e}]_{\mathsf{M}_\Theta(K)}$. One such subspace $\mathfrak{q}_\mathfrak{k}$ can be constructed with the vectors

$$f_\alpha + \rho_\alpha f_{\vartheta(\alpha)}\,, \quad i(g_\alpha - \rho_\alpha g_{\vartheta(\alpha)})$$

for α not in the span of Θ. These were already given in (2.56) as basis vectors of $(\mathfrak{n}+\oplus\mathfrak{n}-)\cap\mathfrak{k}$. The subspace $\mathfrak{q}_\mathfrak{k}$ constructed above is $\mathfrak{q}_\mathfrak{k} = (\mathfrak{n}_\Theta^+\oplus\mathfrak{n}_\Theta^-)\cap\mathfrak{k}$, which follows from the generalized Bruhat decomposition (2.57) of Lie algebras.

Finally, the last

Assumption 4 *The generalized special conformal transformations in N_Θ^- act trivially on the tangent space of $\mathbb{M}$ at the point $\mathbf{o} = [\mathbf{e}]_{B_\Theta}$.*

3.2 Special Conformal Transformations

We start with an analysis of the mathematical implications from assumption 4.

Let a Lie group G act differentiably on $\mathbb{M}$ and call the action ρ. Let the derivative of $\rho(g)$ at x be $\sigma^x(g) = T_x(\rho(g))\colon T_x\mathbb{M} \to T_{\rho(g)x}\mathbb{M}$. For $X \in T_x\mathbb{M}$ and a curve $\lambda(t) \in \mathbb{M}$ which has tangent vector X at $t = 0$ this derivative $\sigma^x(g)X$ it defined to be the tangent vector of the curve $\rho(g)\lambda(t)$ at $t = 0$.

From the transitivity of the action required in assumption 2 it follows that the space-time manifold $\mathbb{M}$ is a homogenous space $\mathbb{M} \cong G/H$. Choose a fixed element $\mathbf{o} \in \mathbb{M}$ and identify it with $[\mathbf{e}]_H$, where $\mathbf{e} \in G$ is the identity element. The stability subgroup of any $x \in \mathbb{M}$ is equal to $\kappa(g)H = gHg^{-1}$, where $x = \rho(g)\mathbf{o} = [g]_H$. Since $\kappa(g)$ is an automorphism of G, all stability subgroups are isomorphic.

Because it holds $\sigma^x(gh) = \sigma^{\rho(h)x}(g) \circ \sigma^x(h)$, the restriction of σ^x to the stability subgroup gHg^{-1} is a representation of gHg^{-1} on $T_x\mathbb{M}$. For $h \in H$ it is $\sigma^x(ghg^{-1}) = \sigma^{\mathbf{o}}(g)\sigma^{\mathbf{o}}(h)\sigma^x(g^{-1})$ and $\sigma^x(g^{-1}) = \sigma^{\mathbf{o}}(g)^{-1}$. The representations σ^x and $\sigma^{\mathbf{o}}$ of H are therefore equivalent with intertwiner $\sigma^{\mathbf{o}}(g)$.

It suffices to study the representation $\sigma^{\mathbf{o}}$ and we will therefore drop the "$\mathbf{o}$" in the notation were appropriate: $\sigma = \sigma^{\mathbf{o}}$.

We will now have a closer look at the action σ. It will turn out to be purely Lie algebraic. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the subalgebra corresponding to the subgroup H. Choose any subspace $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, and define the map $P_{\mathfrak{q}}$ on $\mathfrak{g}$ given by $P_{\mathfrak{q}}(X + Y) = Y$ if $X \in \mathfrak{h}$ and $Y \in \mathfrak{q}$. We call this map the *projection* on $\mathfrak{q}$, even though there is no scalar product. For the remainder of this chapter the symbol "Ad" will always denote the adjoint map $G \to \mathrm{Int}(\mathfrak{g})$, even if it is considered only on a subgroup of G. The following result is a generalization of a result used in the theory of causal symmetric spaces ([FÓ95]):

Lemma 1 *For any subspace $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the map*

$$\sigma_{\mathfrak{q}} : g \mapsto P_{\mathfrak{q}} \circ \mathrm{Ad}(g)$$

defines a representation of H on $\mathfrak{q}$ which is equivalent to the representation σ on the tangent space $T_{\mathbf{o}}\mathbb{M}$.

Proof: The map $\sigma_{\mathfrak{q}}$ is a representation, because $\mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$ for $g \in H$.
Let $X \in T_{\mathbf{e}}G$ and $\lambda(t) \in G$ be a curve with tangent vector X at $\mathbf{e}$. The curve $[\lambda(t)]_H \in \mathbb{M}$ has tangent vector $Y = [X]_{\mathfrak{h}} \in \mathfrak{g}/\mathfrak{h} \cong T_{\mathbf{o}}\mathbb{M}$. The action ρ of an element $h \in H$ satisfies $\rho(h)[g]_H = [\kappa(h)g]_H$. The action of its derivative $\sigma(h)$ at $\mathbf{o}$ on the vector Y is tangent to the curve

$$\rho(h)[\lambda(t)]_H = [\kappa(h)\lambda(t)]_H \ .$$

It has tangent vector $[\mathrm{Ad}(h)X]_{\mathfrak{h}}$, which we will also call $\mathrm{Ad}(h)X$. This shows that the representations σ and Ad of H on $T_{\mathbf{o}}\mathbb{M} \cong \mathfrak{g}/\mathfrak{h}$ are equivalent. Now, the intertwiner between $\sigma_{\mathfrak{q}}$ and Ad is $\mathfrak{q} \to \mathfrak{g}/\mathfrak{h}$, $X \mapsto [X]_{\mathfrak{h}}$. *End of Proof*

Consider the generalization of conformal symmetry described in section 3.1: Let $H = \mathsf{B}_\Theta$ be some standard parabolic subgroup.

There are two useful choices for $\mathfrak{q}$: One possibility is the subalgebra $\mathfrak{n}_\Theta^+$ as can be seen from the generalized Bruhat decomposition (2.57). The other possibility is the suitably chosen subspace $\mathfrak{q}_{\mathfrak{k}} \in \mathfrak{k}$ with $\mathfrak{k} = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_{\mathfrak{k}}$ (3.2).

The following lemma will give us the mathematical version of assumption 4. The proof will make use of the relations $\left[\mathfrak{n}_\Theta^\pm,\mathfrak{l}_\Theta\right] = \mathfrak{n}_\Theta^\pm$ which is a feature of parabolic subalgebras as follows from the n-graded structure of $\mathfrak{g}$ described in section 2.6.

Lemma 2 *Assumption 4 is equivalent to the requirement that*

$$\left[\mathfrak{n}_\Theta^-,\mathfrak{n}_\Theta^+\right] \subset \mathfrak{l}_\Theta \,. \tag{3.3}$$

Proof: Any generalized special conformal transformation $n \in N_\Theta^-$ is of the form $n = \exp(Y)$, where $Y \in \mathfrak{n}_\Theta^-$. It acts trivially on $T_\mathbf{o}\mathbb{M} \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{n}_\Theta^+ \ni X$ if $P_{\mathfrak{n}_\Theta^+} \circ \mathrm{Ad}(\exp Y)X$ is equal to X. This is the case, if $\mathrm{Exp}(\mathrm{ad}\, Y)X = X + X'$ and $X' \in \mathfrak{n}_\Theta^- \oplus \mathfrak{l}_\Theta$. Since $\left[\mathfrak{n}_\Theta^-,\mathfrak{l}_\Theta\right] \subset \mathfrak{n}_\Theta^-$, this follows from (3.3). On the other hand, let $n(t) = \exp(tY) \in N_\Theta^-$. For small enough t $[\mathrm{Exp}(\mathrm{ad}(Y))X]_\mathfrak{h}$ can only be equal to $[X]_\mathfrak{h}$ (which would follow from assumption 4), if $[Y,X] \in \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^-$. Because the structure constants are symmetric with respect to a change of basis which interchanges positive and negative roots, it follows from $\left[\mathfrak{n}_\Theta^-,\mathfrak{n}_\Theta^+\right] \subset \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^-$ that the "reflected" relation $\left[\mathfrak{n}_\Theta^+,\mathfrak{n}_\Theta^-\right] \subset \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^+$ holds, as well. This gives us (3.3).

End of Proof

At the same time, the representation $\sigma_{\mathfrak{n}_\Theta^+}$ of B_Θ on $\mathfrak{q} = \mathfrak{n}_\Theta^+$ defined in lemma 1 obtains a simple form:

Lemma 3 *Under assumption 4 the representation $\sigma_{\mathfrak{n}_\Theta^+}$ simplifies as follows: For $b = nl \in N_\Theta^- \mathsf{L}_\Theta = \mathsf{B}_\Theta$ it is $\sigma_{\mathfrak{n}_\Theta^+}(b) = \mathrm{Ad}(l)$.*

Proof: It is $\sigma_{\mathfrak{n}_\Theta^+}(b)X = P_{\mathfrak{n}_\Theta^+} \circ \mathrm{Ad}(n) \circ \mathrm{Ad}(l)X$ for $X \in \mathfrak{n}_\Theta^+$. Let $X' = \mathrm{Ad}(l)X$, which is in $\mathfrak{n}_\Theta^+$ because of $\left[\mathfrak{l}_\Theta,\mathfrak{n}_\Theta^+\right] \subset \mathfrak{n}_\Theta^+$, and $n = e^Y$ with $Y \in \mathfrak{n}_\Theta^-$.

Then we have $\mathrm{Ad}(n)X' = \mathrm{Exp}(\mathrm{ad}(Y))X' = X' + X''$ with $X'' \in \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^-$, because of (3.3) and $\left[\mathfrak{n}_\Theta^-,\mathfrak{l}_\Theta\right] \subset \mathfrak{n}_\Theta^-$. Thus, $P_{\mathfrak{n}_\Theta^+} \circ \mathrm{Ad}(n)$ acts as the identity.

End of Proof

The triviality of the action of N_Θ^- carries over to the representation on $\mathfrak{q}_\mathfrak{k}$, but the projection is still necessary: $\sigma_{\mathfrak{q}_\mathfrak{k}}(b) = P_{\mathfrak{q}_\mathfrak{k}} \circ \mathrm{Ad}(l)$. A little more can be said about the representation $\sigma_{\mathfrak{q}_\mathfrak{k}}$: Since $\mathfrak{q}_\mathfrak{k} \subset \mathfrak{n}_\Theta^+ \oplus \mathfrak{n}_\Theta^-$ and $\mathrm{Ad}(l)\mathfrak{n}_\Theta^\pm \subset \mathfrak{n}_\Theta^\pm$, the projection will be only "along" (i.e. by addition of an element of) $\mathfrak{n}_\Theta^-$. For $l \in \mathsf{M}_\Theta(K)$ the projection will be trivial. The intertwiner between these two representations is the projection with a suitable coefficient. In the remainder of this section the representation space $\mathfrak{n}_\Theta^+$ will be used, while in section 3.4 the space $\mathfrak{q}_\mathfrak{k}$ plays the more important role.

3.3 Infinitesimal Causal Structure

The global causal structure needed can be described infinitesimally by a specification of tangent vectors which non-spacelike curves are allowed to have. These lie in cones which are non-trivial in a certain way described below and invariant under the assumed symmetry. In this section we will develop criteria for the existence of such an infinitesimal causal structure.

3.3.1 Definitions

We will summarize a few definitions and facts about cones and cone-fields in homogenous spaces [Seg76, FÓ95]:

A *cone* C is a subset of a vector space which contains λv for all $v \in C$ and $\lambda > 0$. A cone is *proper* if $C \cap -C = \{0\}$ and *generating* if its linear span is V. A convex, proper, generating, and closed cone is called *regular* cone.

An *infinitesimal causal structure* or *cone-field* $\mathcal{C}$ on a d-dimensional manifold $\mathbb{M}$ is an assignment of a regular cone $\mathcal{C}(x)$ in the tangent space $T_x\mathbb{M}$ to each $x \in \mathbb{M}$ such that there are an open covering $\{U_i\}_{i\in I}$ of $\mathbb{M}$, a cone $C \in \mathbb{R}^d$, and (smooth or analytic) maps $\varphi_i : U_i \times \mathbb{R}^d \to T\mathbb{M}$ with $\varphi_i(x, C) = \mathcal{C}(x)$.

As before, consider a Lie group G which acts differentiably on $\mathbb{M}$ with the action ρ and consider again the derivative $\sigma^x(g)$ of $\rho(g)$ at x. A causal structure is G-*invariant* if for each $g \in G$ and $x \in \mathbb{M}$ the derivative $\sigma^x(g)$ maps the cone-field into itself:

$$\sigma^x(g)\mathcal{C}(x) \subset \mathcal{C}(\rho(g)x)\,.$$

3.3.2 General Results

If the action of G is transitive, i.e. $\mathbb{M} \cong {}^{G}\!/\!_{H}$, there is a bijection from the set of regular H-invariant cones C in $T_{\mathbf{o}}\mathbb{M}$ with $\mathbf{o} = [\mathbf{e}]_H$ onto the set of G-invariant infinitesimal causal structures on $\mathbb{M}$ [FÓ95]:

$$C \mapsto (\mathcal{C} : [g]_H \mapsto \sigma^{\mathbf{o}}(g)C)\,. \tag{3.4}$$

This is another reason, why only the action $\sigma = \sigma^{\mathbf{o}}$ of H on $T_{\mathbf{o}}\mathbb{M}$ has to be studied. In particular, we are interested in the cones invariant under this representation.

If H is trivial, i.e. if $\mathbb{M} \cong G$, then the H-invariant regular cones in $T_{\mathbf{e}}G$ are just all regular cones. Therefore, there always exist G-invariant infinitesimal causal structures on G.

Some general results on invariant cones in arbitrary vector spaces follow. They can be found in [Pan81, Vin82, Seg76] in a slightly different form.

Let G be any connected semi-simple real Lie group and V a real finite-dimensional G-module. K is defined to be a maximal subgroup of G such that the image in $\mathrm{Gl}(V)$ is compact. Note that this is not necessarily the compact factor of a Cartan decomposition of G. This notation is fixed for the following three lemmata.

Lemma 4 *There is a K-stable $v_K \in V$, $v_K \neq 0$ if and only if there exists a G-invariant closed convex cone C in V with $C \neq C \cap -C$. In this case there will always be a proper cone of this type.*

Proof: [Pan81] For simplicity set G and K equal to their images in $\mathrm{Gl}(V)$. Let C be a closed convex cone in V. Because of $C \neq C \cap -C$ there exists a linear functional f in V which is non-negative and non-trivial on C. (§17.1 and §16.3 in [Köt66]) Select $v \in C$ with $f(v) > 0$, then

$$v_K = \int_K k \cdot v \, d\mu(k)$$

is K-invariant and $f(v_K) > 0$. It follows from linearity that $v_K \neq 0$.

Let on the other hand v_K be K-invariant. Choose a scalar product $(\cdot,\cdot)$ in the complexification $(V)_{\mathbb{C}}$ of V which is left invariant by the compact form of the complexification $(G)_{\mathbb{C}}$ of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. $X \in \mathfrak{p}$ is Hermitian and e^X positive-definite Hermitian. Because of the invariance of v_K and the Cartan decomposition $G = PK$, the G-orbit of v_K is equal to the P-orbit. For u, w in this orbit we have $(u, w) = (e^X v_K, e^Y v_K) = (e^Y e^X v_K, v_K)$. With the Cartan decomposition $e^Y e^X = e^Z k$ and the positive-definiteness of e^Z it is $(u, w) > 0$. This extends by linearity to the convex cone generated by this orbit. This convex cone is G-invariant, and by continuity we have $(u, w) \geq 0$ for u, w in the closure C. If v and $-v$ are in C, $(v, -v) \geq 0$ i.e. $v = 0$, thus C is proper. *End of Proof*

Irreducibility will be needed for the cones to be regular:

Lemma 5 *If V is an irreducible G-module and contains a non-trivial G-invariant convex closed cone C, C is regular.*

Proof: Both the linear span of C and the subspace $C \cap -C$ are invariant linear subspaces of V which can only be V or $\{0\}$. If the linear span of C is $\{0\}$ then C is trivial. If $C \cap -C$ is V then C is V. *End of Proof*

The irreducibility will be important for global causality, because the following result of [Vin82] will be needed:

Lemma 6 *If V is irreducible and contains a non-zero K-stable vector v_K then v_K is unique up to scalar multiplication with respect to this property.*

3.3.3 Generalized Conformal Symmetry

Finding the invariant infinitesimal causal structures on $\mathbb{M}$ is by lemma 3 reduced to finding the L_Θ-invariant regular cones in $\mathfrak{n}_\Theta^+$. Since L_Θ is not semisimple, lemma 4 cannot be applied directly. The factor $A_{\mathfrak{p}_\Theta}$ in $L_\Theta = M_\Theta A_{\mathfrak{p}_\Theta}$ has to be treated separately, which is achieved by the following

Theorem 17 *Under the assumptions 3 and 4 the representation* Ad *of $L_\Theta = M_\Theta A_{\mathfrak{p}_\Theta}$ on $\mathfrak{n}_\Theta^+$ has the following two properties:*

(i) The elements of $A_{\mathfrak{p}_\Theta}$ are represented as positive constant times identity.

(ii) The representation of M_Θ on $\mathfrak{n}_\Theta^+$ is irreducible.

With this, all M_Θ-invariant cones will be L_Θ-invariant, because multiplication with a positive constant stays in any cone per definition. Theorem 17 tells us that elements of $A_{\mathfrak{p}_\Theta}$ act on tangent vectors to $\mathbb{M}$ at $\mathbf{o}$ by dilatation. In this sense, $A_{\mathfrak{p}_\Theta}$ is a generalization of the dilatation subgroup of the conformal group.

Lemma 4 and lemma 5 can now be applied, to the M_Θ-module $\mathfrak{n}_\Theta^+$. The maximal subgroup 'K' in the lemma, which is defined to be represented compactly, is then equal to $M_\Theta \cap K = M_\Theta(K)$, because the considered representation is obtained from the adjoint representation of G, and the 'compact' factor K in the Cartan decomposition of a semi-simple Lie group is compact in the adjoint representation as discussed in section 2.8.1.

It can be summarized:

Corollary 1 *Under the assumptions 3 and 4 consider the case that M_Θ is semi-simple. There is a M_Θ-invariant regular cone in $\mathfrak{n}_\Theta^+$ if and only if there is a $M_\Theta(K)$-stable vector. If this is the case, there exists an G-invariant infinitesimal causal structure on $\mathbb{M}$.*

The question arises whether it is a global causal structure in one of the senses defined in [HE73]. In section 3.4 it will be shown that it is globally hyperbolic.

For the proof of theorem 17 a rather technical result is needed wich will be presented in the next section. It will also be applied in section 3.6.2 for the classification of the possible symmetry groups and homogenous spaces which allow an interpretation as space-time. The proof will follow in section 3.3.5. The reader who wishes to reach the results quickly may now skip to section 3.4 at first reading.

3.3.4 Graded Structure of $\mathfrak{g}$

Consider the graded structure defined by the restricted root system Σ and the set of simple restricted roots Θ as described in section 2.6. With

$$\mathfrak{n}_\Theta^\pm = \bigoplus_{j=\pm 1}^{\pm \ell_\Theta(\alpha_{\max})} \mathfrak{g}^{(j)} , \qquad \mathfrak{l}_\Theta = \mathfrak{g}^{(0)} . \tag{3.5}$$

as given in (2.60) we get the

Lemma 7 *Let $\mathfrak{g}$ be a simple Lie algebra.*
Assumption 4 implies that $\mathfrak{g} = \mathfrak{n}_\Theta^- \oplus \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^+$ is a 3-grading of $\mathfrak{g}$.

Proof: In section 2.6 it has been shown that $\mathfrak{g}$ possesses an n-grading. With (3.5) it remains to show that $\ell_\Theta(\alpha_{\max}) = 1$. If it were $\ell_\Theta(\alpha_{\max}) > 1$, it would hold, for example, $\left[\mathfrak{g}^{(2)},\mathfrak{g}^{(-1)}\right] \subset \mathfrak{g}^{(1)}$. From proposition 3 we know that $\left[\mathfrak{g}^{(2)},\mathfrak{g}^{(-1)}\right]$ is non-zero.

On the other hand, $\left[\mathfrak{g}^{(2)},\mathfrak{g}^{(-1)}\right] \subset \left[\mathfrak{n}_\Theta^+,\mathfrak{n}_\Theta^-\right]$, which is by (3.3) in $\mathfrak{l}_\Theta = \mathfrak{g}^{(0)}$. Because this has zero intersection with $\mathfrak{g}^{(1)}$, the commutator $\left[\mathfrak{g}^{(2)},\mathfrak{g}^{(-1)}\right]$ has to be zero. A contradiction. *End of Proof*

Recall from section 2.5 that $\Sigma_\Theta \subset \Sigma$ consists of the restricted roots, which are not in the linear span $\Sigma(\Theta)$ of Θ. The restricted roots in Σ can further be restricted to $\mathfrak{a}_{\mathfrak{p}_\Theta}$. Note that the set of these restrictions $\Sigma^\Theta = \{\lambda|_{\mathfrak{a}_{\mathfrak{p}_\Theta}} \text{ for } \lambda \in \Sigma_\Theta\}$ in general does not fulfill the axioms of root systems anymore ([War72] p. 71, p. 9). Since $\Sigma(\Theta)$ is precisely the set of roots which are zero on $\mathfrak{a}_{\mathfrak{p}_\Theta}$, no element of Σ^Θ is equal to zero.

With the following lemma the results of theorem 17 are easily obtained:

Lemma 8 *Excluding the trivial case, the 3-grading of lemma 7 implies that Σ^Θ is a root system of type A_1. Explicitly this means: Θ contains all but one simple restricted root λ_e. It has coefficient $a_e = -1, 0,$ or 1 in the expansion of any restricted root $\lambda = \sum_{i=1}^{l} a_i\lambda_i$. Accordingly, Θ^Φ, the set of simple roots in Φ which are projected onto Θ, contains all but one simple root α_e with the same property for the expansion of any root α. Especially, α_e is the only simple root whose restriction to $\mathfrak{a}_\mathfrak{p}$ is λ_e.*

Proof: First, the following result is needed: Since the restricted root system Σ of a simple Lie algebra has a connected Dynkin diagram, it contains a root $\lambda = \sum_{i=1}^{l} \lambda_i$: For root systems of type A_l this is the highest root. For the other types recall the rules for constructing the root system presented in section 2.2.1, especially equation (2.12)

$$p - q = A_{\lambda_i,\lambda} \, .$$

If $\lambda = \sum_{\lambda_j \in \Xi} \lambda_j$ is a root, where Ξ is a subset of the simple roots, and if λ_i is not in Ξ and is connected to an element of Ξ in the Dynkin diagram, then p is zero and $A_{\lambda_i,\lambda} < 0$. Therefore q is greater zero and $\lambda + \lambda_i$ is a root. Starting with any simple root, induction shows that $\lambda = \sum_{i=1}^{l} \lambda_i$ is a root.

Assume that Θ does not contain the two distinct simple restricted roots λ_j and λ_k. The restricted root λ constructed above is of level $\ell_\Theta(\lambda) \geq 2$, which contradicts the 3-grading. Let λ_e be the unique simple root which is not in Θ. Any root $\lambda = \sum_{i=1}^{l} a_i \lambda_i$ with $a_e \in \{2, -2, 3, -3, \dots\}$ will be of level $\ell_\Theta(\lambda) = a_e$, again contradicting the 3-grading. Therefore $a_e \in \{-1, 0, 1\}$

There still could be two distinct simple roots α_j and α_k in Φ which are projected on λ_e. Then, the same reasoning as above can be applied to show that there is a root α of level $\ell_{\Theta^\Phi}(\alpha) = 2$, which is equal to the level of the projection of α.

End of Proof

Corollary 2 *The* 3*-grading of lemma 7 implies that* $\mathfrak{a}_{\mathfrak{p}_\Theta}$ *is one-dimensional.*

For the proof of theorem 17 the following general result about root systems is needed. It is closely related to the graded structure.

Lemma 9 *Let* $\alpha = \sum \alpha^i \alpha_i$ *be the expansion of a root* α *into simple roots. All positive roots* α *in* Φ *with the property that the coefficient* α^k *is non-zero can be constructed by successively adding simple roots starting with* α_k *such that every intermediate sum is a root. (This may be viewed as part of the construction of the root system described in section 2.2.1.)*

Proof: From the construction of the root system described in section 2.2.1 it is known that each positive root is obtained by successively adding simple roots where each intermediate sum is also a positive root. It remains to be shown that this construction can be started with α_k.

For a positive root α define T_α to be the subdiagram of the Dynkin diagram of Φ consisting of the simple roots α_j for which $\alpha^j \neq 0$ in the expansion. The following three facts are needed:

(i) The subdiagram T_α for a positive root α is a connected Dynkin diagram.

(ii) If α is a positive root and α_i a simple root for which $\alpha^i = 0$ and α_i is connected to the subdiagram T_α within the Dynkin diagram of Φ then $\alpha + \alpha_i$ is a positive root.

(iii) Assume that in the construction a sequence like $(\alpha + \alpha_i) + \alpha_k$ appears, where α has coefficients $\alpha^i \neq 0$, $\alpha^k = 0$, and $\alpha + \alpha_i$ is a root. Then $\alpha + \alpha_k$ is also a root, and the construction may be reorderd $(\alpha + \alpha_k) + \alpha_i$.

We will first show that the statement of lemma 9 follows from (i), (ii), and (iii): When using (iii) repeatedly to reorder a given construction of some positive root α, in the resulting construction appears the root $\alpha' = \sum 1\, \alpha_i$, where it is summed only over simple roots α_i in T_α. Because of (i) and (ii) this root α' may be constructed starting from α_k and adding all the other simple roots in T_α by moving along the connected graph.

To show (i), (ii), and (iii) the equation (2.12) describing root strings is needed:

$$q = p - A_{\alpha_i,\alpha}\,.$$

Statement (ii) follows since $p = 0$ and $A_{\alpha_i,\alpha}$ is negative in that situation. If on the other hand α_i is not connected to T_α, then $A_{\alpha_i,\alpha}$ is zero and consequently $\alpha + \alpha_i$ is no root. This justifies (i). Statement (iii) can be seen because $\alpha + \alpha_k$ is a root, which follows from (ii).

End of Proof

3.3.5 Proof of Theorem 17

Proof of theorem 17 part (i): Since the simple restricted roots span the dual of $\mathfrak{a}_\mathfrak{p}$, it follows from lemma 8 that $\mathfrak{a}_{\mathfrak{p}\Theta}$ is one-dimensional. Let h be some element of $\mathfrak{a}_{\mathfrak{p}\Theta}$. Any restricted root λ gives $\lambda(h) = \sum a_i \lambda_i(h) = a_e \lambda_e(h)$. $\mathfrak{n}_\Theta^+$ is spanned by ladder vectors e_λ with $a_e = 1$ and therefore $[h,e_\lambda] = \lambda_e(h) e_\lambda$. Thus, h acts on the whole $\mathfrak{n}_\Theta^+$ by scalar multiplication with the constant $\lambda_e(h)$.

End of Proof

Proof of theorem 17 part (ii): Consider the 3-grading of the root system Φ in lemma 7. The root system of $\mathfrak{m}_\Theta$ consists of the roots α with $\ell_{\Theta^\Phi}(\alpha) = 0$ and span a subspace $E_{\mathfrak{m}_\Theta}$. The complexification of $\mathfrak{n}_\Theta^+$ is spanned by the root vectors e_α for positive roots α with $\ell_{\Theta^\Phi}(\alpha) = 1$. The weight system of the representation of $\mathfrak{m}_\Theta$ on $\mathfrak{n}_\Theta^+$ is the projection of these roots onto $E_{\mathfrak{m}_\Theta}$ with respect to the canonical Euclidean product $\langle \cdot, \cdot \rangle$ defined by the Killing form of of the complexification $(\mathfrak{g})_\mathbb{C}$ of $\mathfrak{g}$. Since the root spaces in $(\mathfrak{g})_\mathbb{C}$ are one-dimensional, all these weights are simple.

Clearly α_e from lemma 8 is the only simple root with $\ell_{\Theta^\Phi}(\alpha_e) = 1$. From lemma 9 we know that all roots with $\ell_{\Theta^\Phi}(\alpha) = 1$ can be constructed from α_e by considering the α_i-strings, where $\ell_{\Theta^\Phi}(\alpha_i) = 0$ — i.e. belonging to $\mathfrak{m}_\Theta$. Let Λ^- be the projection of α_e on $E_{\mathfrak{m}_\Theta}$. It is the lowest weight of the representation on $\mathfrak{n}^+_\Theta$. The same strategy applied to the root system for constructing the positive roots described in section 2.2.1 can be applied to the weight system of a representation: The α_i-string of weights containing the weight λ is $\lambda + k\alpha_i$ for $-p \leq k \leq q$ and

$$p - q = \frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \,.$$

Because λ is an orthogonal projection of α on the space $E_{\mathfrak{m}_\Theta}$ spanned by the α_i, it is $\langle \lambda, \alpha_i \rangle = \langle \alpha, \alpha_i \rangle$. Comparing this with (2.12) shows that this α_i-string of weights coincides with a α_i-string of roots and therefore the system of weights constructed from Λ^- is identical to the projection of the set of roots α with $\ell_{\Theta^\Phi}(\alpha) = 1$.

For any representation with weight Λ^- these constructed weights are at least contained in its weight system. This is also true for an irreducible representation containing Λ^-. And therefore the irreducible representation containing Λ^- has the same weight system as $\mathfrak{n}^+_\Theta$. Since all weights of $\mathfrak{n}^+_\Theta$ are simple, it is equal to this irreducible representation.

End of Proof

3.4 Global Causal Structure

We recall a few definitions: A differentiable curve $\lambda\colon I \to \mathbb{M}$, where $I \subset \mathbb{R}$ is an interval, is (future-directed) *non-spacelike* with respect to an infinitesimal causal structure $\mathcal{C}$, if the tangent vector at $\lambda(\tau)$ is in the cone $\mathcal{C}(\lambda(\tau)) \setminus \{0\}$ for all $\tau \in I$. For $p \in \mathbb{M}$ the *causal future* resp. *past* $J^\pm(p)$ of p is the set of points $q \in \mathbb{M}$ with a non-spacelike curve λ such that $\lambda(0) = p$ and $\lambda(\pm\tau) = q$ with $\tau \geq 0$.

An infinitesimal causal structure $p \mapsto \mathcal{C}(p)$ is *strongly causal*, if every neighbourhood of a point $p \in \mathbb{M}$ contains a neighbourhood of p which no non-spacelike curve intersects more than once, i.e. the neighbourhood is mapped to an interval by λ^{-1}. An infinitesimal causal structure is *globally hyperbolic* if it is strongly causal and if for $p, q \in \mathbb{M}$ the set $J^+(p) \cap J^-(q)$ is compact.

A set $S \in \mathbb{M}$ is called *acausal*, if for $p \in S$ no $q \in S$, $p \neq q$ is in $J^+(p)$.

The *edge* of an acausal set $S \in \mathbb{M}$ is the set of all $p \in \overline{S}$ such that in every neighbourhood of p there are points q in the future and r in the past of p which can be joined by a timelike curve within the neighbourhood without intersecting S. If S is a submanifold without boundary, the edge is empty, because the boundary clearly contains the edge.

A set $\Sigma \in \mathbb{M}$ is called a *Cauchy surface* for $\mathbb{M}$, if it is acausal, has no edge, and each inextendible non-spacelike curve in $\mathbb{M}$ intersects Σ. A criterion for global hyperbolicity is given by the

Proposition 4 *[HE73] $\mathbb{M}$ is globally hyperbolic if and only if it has a Cauchy surface. Then $\mathbb{M}$ is homeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a manifold of codimension 1 and each $\{a\} \times \Sigma$, $a \in \mathbb{R}$ is a Cauchy surface for $\mathbb{M}$.*

Let us turn to the generalization of conformal symmetry described above. Consider the homogenous space $\mathbb{M} = {}^{K}\!/\!_{M_\Theta(K)}$ from equation (3.1). If K is compact $\mathbb{M}$ certainly is and cannot have the topology described above. Recall from section 2.8 that in a Cartan decomposition of a simple Lie group $G = KP$ the "compact" factor K can be non-compact if the Lie algebra $\mathfrak{k}$ of K has a non-trivial center $\mathfrak{z}_\mathfrak{k}$. With the assumptions that $K = K_s \times Z_\mathfrak{k}$, $Z_\mathfrak{k}$ non-compact, and $M_\Theta(K) \subset K_s$ we get the desired structure of space-time:

$$\mathbb{M} = {}^{K_s}\!/\!_{M_\Theta(K)} \times Z_\mathfrak{k}\,.$$

If $M_\Theta(K)$ is not a subgroup of K_s, but its identity component $M_\Theta(K)$ is, we can still go to the universal covering of this quotient, by just dividing through this identity component:

$$\mathbb{M} = {}^{K_s}\!/\!_{M_\Theta(K)} \times Z_\mathfrak{k}\,.$$

$Z_\mathfrak{k}$ is isomorphic to $\mathbb{R}$ and we have to show that the quotient $\Sigma = {}^{K_s}\!/\!_{M_\Theta(K)}$ is a Cauchy surface.

Lemma 10 *Let $t : \Sigma \times Z_\mathfrak{k} \to Z_\mathfrak{k}$ be the component map. For any $a \in Z_\mathfrak{k} \cong \mathbb{R}$ the submanifold $\Sigma \times \{a\}$ is a Cauchy surface if the derivative of t along each vector in $\mathcal{C}(p)$ for all $p \in \mathbb{M}$ is strictly positive (or negative for all vectors).*

Proof: Let $\lambda : I \to \mathbb{M}$ be any non-spacelike curve and choose any $\tau \in I$. Because the tangent vector $T_\tau\lambda(1)$ of λ at τ lies in $\mathcal{C}(\lambda(\tau))$, it follows by the assumption in the lemma that $\frac{d}{d\tau} t \circ \lambda(\tau) > 0$ and thus $t \circ \lambda$ is a strongly monotonous function.

Thus, λ has exactly one intersection with $\Sigma \times \{t \circ \lambda(\tau)\}$. Since this is true for all non-spacelike curves, any $\Sigma \times \{a\}$ is acausal.

It also follows that $\Sigma \times \{a\}$ has no edge: For some $p \in \Sigma \times \{a\}$ let any two points q and r be in the future and past of p, respectively. The points q and

r are in some submanifolds $\Sigma \times \{a^q\}$ and $\Sigma \times \{a^r\}$, respectively, for some a^r and a^q with $a^r < a < a^q$. For any timelike curve λ connecting q and r the function $t \circ \lambda$ takes on the values a^r and a^p and because of the intermediate value theorem also a, i.e. λ intersects $\Sigma \times \{a\}$, which therefore has no edge.

An inextendible non-spacelike curve λ has to intersect each $\Sigma \times \{a\}$ because otherwise it would approach one $\Sigma \times \{a\}$ arbitrarily closely and not have a limit point. This would mean that its tangent vector would approach the tangent bundel of $\Sigma \times \{a\}$. Since the $\mathcal{C}(p)$ are closed cones which contain this vector, the tangent space of $\Sigma \times \{a\}$ would actually intersect some $\mathcal{C}(p)$. A curve with tangent vectors in this intersection would clearly have zero derivative of t. A contradiction. *End of Proof*

Corresponding to the group level, the assumptions read on the Lie algebra level $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$ (sum of ideals) and $\mathfrak{m}_\Theta(K) \subset \mathfrak{k}_s$. The suitably chosen subspace $\mathfrak{q}_\mathfrak{k}$ introduced in (3.2) in section 3.1, which defined to satisfy $\mathfrak{k} = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_\mathfrak{k}$ may then contain $\mathfrak{z}_\mathfrak{k}$, and thus $\mathfrak{k}_s = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_s$, and $\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_s \oplus \mathfrak{z}_\mathfrak{k}$.

We finally end up with the following Lie algebraic criterion:

Lemma 11 $\mathbb{M}$ *carries a globally hyperbolic causal structure, if there is a B_Θ-invariant regular cone C in $\mathfrak{q}_\mathfrak{k}$ which has intersection $\{0\}$ with $\mathfrak{q}_s$.*

Proof: Since $\mathfrak{q}_s$ has codimension 1 in $\mathfrak{q}_\mathfrak{k}$, C lies in one halfspace with boundary $\mathfrak{q}_s$. The decomposition $\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_s \oplus \mathfrak{z}_\mathfrak{k}$ corresponds via the equivalence established in lemma 1 to the decomposition of the tangent space $T_\mathbf{o}\mathbb{M} = T_\mathbf{o}\Sigma \oplus T_\mathbf{o}Z_\mathfrak{k}$.

Consider the bijection (3.4) between H-invariant regular cones in $T_\mathbf{o}\mathbb{M}$ and the infinitesimal causal structures in $\mathbb{M}$. For $\mathbb{M} \ni p = [g]$ we can choose the representative g to be in K. Then $\sigma(g)$ maps $T_\mathbf{o}Z_\mathfrak{k}$ to $T_pZ_\mathfrak{k}$ and $T_\mathbf{o}\Sigma$ to $T_p\Sigma$. C is now mapped to a regular cone into a halfspace of $T_p\mathbb{M}$ with boundary $T_p\Sigma$ which intersects $T_p\Sigma$ only at zero. Identify $Z_\mathfrak{k}$ – and with it all $T_pZ_\mathfrak{k}$ – with $\mathbb{R}$ such that the $T_pZ_\mathfrak{k}$-component of all elements in $\sigma(g)(C)$ is positive via this identification. This is then by continuity and connectedness of G the case for all $\sigma(g)(C)$ for $g \in K$. The derivative of the function t in direction of a vector with positive $T_pZ_\mathfrak{k}$-component is clearly positive. *End of Proof*

We are now in a position to formulate the criteria for the existence of a G-invariant infinitesimal causal structure on $\mathbb{M}$ that turns $\mathbb{M}$ into a globally hyperbolic manifold:

Theorem 18 *Let G be a connected simple Lie group and B_Θ the identity component of a parabolic subgroup satisfying assumption 3 and 4.* $\mathbb{M} = G/B_\Theta$ *carries an infinitesimal causal structure such that* $\mathbb{M}$ *is globally hyperbolic if* *(i)* $\mathfrak{k}$ *has non-trivial center* $\mathfrak{z}_{\mathfrak{k}}$ *and* $Z_{\mathfrak{k}}$ *is non-compact,*

(ii) $\mathfrak{z}_{\mathfrak{k}}$ *is not contained in* $\mathfrak{m}_\Theta(K)$*, and*

(iii) $\mathfrak{m}_\Theta$ *is semi-simple.*

Proof: As discussed above, (i) and (ii) imply that $\mathbb{M}$ can cary an infinitesimal causal structure which turns it into a globally hyperbolic manifold.

The conditions (i) and (ii) also lead to the existence of a $M_\Theta(K)$-stable vector in $\mathfrak{q}_{\mathfrak{k}} = \mathfrak{q}_s \oplus \mathfrak{z}_{\mathfrak{k}}$: Any $iH_0 \in \{0\} \oplus \mathfrak{z}_{\mathfrak{k}}$ is such a vector. Fix iH_0 in the following and choose the identification of $Z_{\mathfrak{k}}$ with $\mathbb{R}$ such that the derivative of t in direction of iH_0 is positive.

As for corollary 1 we know with (iii) and theorem 17 from lemma 4 and lemma 5 that an M_Θ-invariant regular cone exists in $\mathfrak{n}_\Theta^+$ and therefore there also exists an invariant cone C in the equivalent module $\mathfrak{q}_{\mathfrak{k}}$. Lemma 3 and theorem 17 show that it is invariant under the action of B_Θ.

Lemma 11 shows that the infinitesimal causal structure obtained from C turns $\mathbb{M}$ into a globally hyperbolic manifold: If the intersection of $\mathfrak{q}_s$ and C were not $\{0\}$, this intersection would be a M_Θ-invariant regular cone in $\mathfrak{q}_s$. With (iii) and lemma 4 there would be an $M_\Theta(K)$-stable non-zero vector in $\mathfrak{q}_s$ – clearly not proportional to iH_0. This contradicts lemma 6. *End of Proof*

The classification which will be performed in section 3.6 will rely on assumption 4 and condition (i) in theorem 18. It turns out that for all cases M_Θ will be simple, apart from the well known case of $G = Sl(2, \mathbb{R})$, and that iH_0 will lie outside of $\mathfrak{m}_\Theta(K)$. Since condition (i) is equivalent to assumption 1, as we will see in the next section, it will be a complete classification.

3.5 Unitary Positive Energy Representations

The following is claimed to be due to Ol'šanskiĭ [Ol'82] in the mathematical literature, but Lüscher [Lüs75] already lists all groups allowing positive energy representations. It turns out that the existence of positive energy representations and causal structures are closely related:

Let G be a connected simple Lie group with Lie algebra $\mathfrak{g}$ which satisfies assumption 1. Let u be an infinitesimal positive energy representation and $iH_0 \in \mathfrak{g}$ a vector such that $\frac{1}{i}\mathsf{u}(iH_0) \geq 0$. Consider the $\mathrm{Ad}(G)$-invariant closed convex cone $C \in \mathfrak{g}$ generated by iH_0, i.e. the closure of the set of all linear combinations with non-negative coefficients of elements of the form $\mathrm{Ad}(g)iH_0$, where $g \in G$.

Theorem 19 *For all $X \in C$ it holds $\frac{1}{i}\mathsf{u}(X) \geq 0$ and C is a regular cone.*

Proof: Let $h(t)$ be a curve in G with tangent vector iH_0 at the identity $\mathbf{e}$. Then $\mathrm{Ad}(g)iH_0$ is the tangent vector of $gh(t)g^{-1}$ at $\mathbf{e}$ and clearly has positive spectrum, as well:

$$\tfrac{1}{i}\tfrac{d}{dt}(\Psi, \mathsf{U}(gh(t)g^{-1})\Psi)|_{t=0} = \tfrac{1}{i}\tfrac{d}{dt}(\mathsf{U}(g^{-1})\Psi, \mathsf{U}(h(t))\mathsf{U}(g^{-1})\Psi)|_{t=0} \geq 0\,.$$

By linearity of u this extends to linear combinations of such vectors with non-negative coeffients and by continuity to the closure C of this set.

If $C = \mathfrak{g}$ it would be $\frac{1}{i}\mathsf{u}(Y) \geq 0$ and $\frac{1}{i}\mathsf{u}(-Y) \geq 0$ for any $Y \in \mathfrak{g}$, thus $\mathsf{u}(Y) = 0$ and $\mathsf{U}(g) = 1$ for all $g \in G$. But U is required to be non-trivial. C certainly contains iH_0 and therefore cannot be empty. Therefore C is non-trivial and by lemma 5 regular.

End of Proof

Now we can apply lemma 4 and lemma 5:

Corollary 3 *A Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponding to a simple connected Lie group G contains a* $\mathrm{Ad}(G)$*-invariant regular cone C if and only if $\mathfrak{k}$ has non-zero center $\mathfrak{z}^{\mathfrak{k}}$.*

Proof: The adjoint action of a simple group is irreducible. It is only left to show that the K-stable vector v_K implied by lemma 4 lies in K. Let $v_K = v_K^{\mathfrak{k}} + v_K^{\mathfrak{p}}$ be the Cartan decomposition of v_K. $\mathrm{Ad}(k)$ for $k \in K$ commutes with θ — the Cartan involution of $\mathfrak{g}$ — and therefore $v_K^{\mathfrak{k}}$ and $v_K^{\mathfrak{p}}$ are separately $\mathrm{Ad}(K)$-invariant. Because of $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ it is $\mathrm{ad}(v_K^{\mathfrak{p}})(\mathfrak{k} + \mathfrak{p}) \subset \mathfrak{k}$ and thus, $\mathrm{ad}(v_K^{\mathfrak{p}})^2 = 0$. Since $\mathrm{ad}(v_K^{\mathfrak{p}})$ is diagonalizable, it is $\mathrm{ad}(v_K^{\mathfrak{p}}) = 0$ and $v_K^{\mathfrak{p}} = 0$.

End of Proof

Hence, there are no additional requirements to theorem 18. Condition (i) of theorem 18 is the well known neccessary and sufficient criterion for positive energy representations to exist.

Recall the reductive structure and explicit form of iH_0 mentioned in the proof of theorem 18: $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$ (sum of ideals) where $\mathfrak{k}_s$ is compact and semi-simple and $\mathfrak{z}_{\mathfrak{k}}$ is one-dimensional. The generalization of the conformal Hamiltonian H_0 is now chosen such that $\mathfrak{z}^{\mathfrak{k}} = \mathbb{R}iH_0$.

3.6 Classification of the Space-Time Symmetry Groups

For the classification of the Lie groups satisfying the four assumptions in section section 3.1 we need, on the one hand, the structure of $\mathfrak{k}$ to check (i) in theorem 18, i.e. the existence of positive energy representations. On the other hand the structure of parabolic subgroups and their Langlands decompositions is needed to check (ii), (iii), the 3-grading required by lemma 7, and clarify the structure of the space-time manifold $\mathbb{M}$. Therefore it is needed to employ both approaches presented in section 2.4 for the following classification.

3.6.1 Maximally Compact Cartan Subalgebra

Since this method is suitable for determining $\mathfrak{z}_{\mathfrak{k}}$, it will lead by corollary 3 to a classification of all simple Lie groups which allow positive energy representations.

The real forms with non-trivial center $\mathfrak{z}_{\mathfrak{k}}$ of the factor $\mathfrak{k}$ in the Cartan decomposition are precisely those obtained by the first method of the list in section 2.4.1: Choose two nodes in the extended Dynkin diagram (table 2.2) of type $X_l^{(1)}$ with Coxeter label 1, where $X = A, B, C, D, E, F, G$. The subdiagram obtained by deleting these nodes is the Dynkin diagram of the semi-simple part $\mathfrak{k}_s$ of $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$. Since one of these two nodes can always be chosen to be the zeroth node due to symmetries of the extended Dynkin diagrams, this is equivalent to choosing one node with Coxeter label 1 in the ordinary Dynkin diagram in table 2.1. Let α_j be the corresponding simple root. A basis of the center $\mathfrak{z}_{\mathfrak{k}}$ may be found as explained at the end of section 2.3.2. Since $\mathfrak{z}_{\mathfrak{k}}$ is one-dimensional, this simplifies even more: Let Λ^j be the fundamental weight corresponding to α_j. For a root $\alpha = \sum \alpha^i \alpha_i$ of $\mathfrak{k}_s$ (i.e. $\alpha^i = 0$ for $i = j$) the root vector e_α commutes with h_{Λ^j}:

$$[h_{\Lambda^j}, e_\alpha] = \sum_{i=1}^{l} \alpha^i \langle \Lambda^j, \alpha_i \rangle e_\alpha = 0 \quad \text{if } \alpha^i = 0 \text{ for } i = j \ ,$$

since $\langle \Lambda^j, \alpha_i \rangle = \delta_{ji}$. Because iH_0 must commute with $\mathfrak{k}_s$, the center $\mathfrak{z}_{\mathfrak{k}}$ of $\mathfrak{k}$ is given by

$$\mathfrak{z}_{\mathfrak{k}} = \mathbb{R} i h_{\Lambda^j} = \mathbb{R} i H_0 \ . \tag{3.6}$$

From table 2.2 or table 2.4 the Lie algebras of Lie groups which can have positive energy representations are obtained and listed in table 3.1.

Name	$\mathfrak{k}_s$	$\gamma\Lambda^i$
$\mathfrak{su}(m,n)$	$\mathfrak{su}(m)\times\mathfrak{su}(n)$	$((1)n,\ldots,(m-1)n,mn,m(n-1),\ldots,m(1))$
$\mathfrak{so}(2,d),\ B_l$	$\mathfrak{so}(d)$	$(1,\ldots,1)$
$\mathfrak{sp}(l,\mathbb{R})$	$\mathfrak{su}(l)$	$(\frac{1}{2}l,l-1,\ldots,2,1)$
$\mathfrak{so}(2,d),\ D_l$	$\mathfrak{so}(d)$	$(\frac{1}{2},\frac{1}{2},1,\ldots,1)$
$\mathfrak{so}^*(2l)$	$\mathfrak{su}(l)$	$(\frac{1}{2}l,\frac{1}{2}l-1,l-2,\ldots,1)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)$	$(3,2,4,6,5,4)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)}$	$(3,2,4,6,5,4,3)$

Table 3.1: All simple real Lie algebras with positive energy representations, semi-simple part of maximally compact subgroup, and up to a factor the fundamental weight Λ^i such that ih_{Λ^i} is the time translation generator in the basis corresponding to a maximally compact Cartan subalgebra. γ is some real number.

Keep in mind the isomorphisms in the low dimensions, which follow from the isomorphies of Dynkin diagrams of low rank. Note that the extended Dynkin diagrams do not generalize properly to ranks lower than the ones listed in table 2.2. Via the isomorphies of Dynkin diagrams an extended diagram can be found. For example, the extended diagram $D_3^{(1)}$ does not exist. However, the diagram D_3 is isomorphic to A_3, for which there exists the diagram $A^{(1)}$. The equivalences for the Lie algebras appearing in table 3.1 are

$$\begin{aligned}
\mathfrak{so}(1,2) &\cong \mathfrak{su}(1,1)(\cong \mathfrak{sl}(2,\mathbb{R})) \cong \mathfrak{sp}(1,\mathbb{R})\\
\mathfrak{so}(3,2) &\cong \mathfrak{sp}(2,\mathbb{R})\\
\mathfrak{so}(4,2) &\cong \mathfrak{su}(2,2)\\
\mathfrak{so}(6,2) &\cong \mathfrak{so}^*(8)\\
\mathfrak{so}^*(6) &\cong \mathfrak{su}(3,1)
\end{aligned}$$

The first four lines are well known in conformal field theory. Two other isomorphisms reveal that the group is in fact not simple: This is due to the Dynkin diagram of D_2 which is of the form $A_1 \times A_1$:

$$\begin{aligned}
\mathfrak{so}(2,2) &\cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})\\
\mathfrak{so}^*(4) &\cong \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R})
\end{aligned}$$

The first one is the well known two-dimensional conformal group, but the second case might be interesting to study – especially, because the host of existing material for $\mathfrak{sl}(2,\mathbb{R})$ can immediately be applied. Also easy to treat should be the smallest ones: $\mathfrak{su}(2,1)$ (rank 2), $\mathfrak{sp}(3,\mathbb{R})$, $\mathfrak{su}(3,1)$ (rank 3). Certainly the most interesting cases are the two exceptional groups.

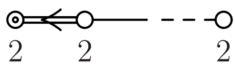
3.6.2 Maximally Non-Compact Cartan Subalgebra

For checking the remaining three conditions, the 3-graded structure necessary to satisfy assumption 4 (see lemma 7) and conditions (ii) and (iii) of theorem 18, we will now employ the structure theory obtained by considering maximally non-compact Cartan subalgebras and the classification of parabolic subgroups based on it.

From lemma 8 we know that there must be one simple restricted root λ_e such that any restricted root λ has coefficient $a_e = -1, 0,$ or 1 in the expansion $\lambda = \sum_{i=1}^{l} a_i \lambda_i$, where λ_i are the simple restricted roots. If this is the case for the highest restricted root $\lambda_{\max}$, it is certainly true for all roots. If the restricted root system Σ is a reduced root system, the coefficients a_i of the highest root are the same as if Σ were a (non-restricted) roots system. Let X_l be the type of the restricted root system. Recall from equation (2.46) that these coefficients a_i for the highest root $\lambda_{\max}$ are the Coxeter labels of the non-zero nodes in the extended Dynkin diagram of type $X_l^{(1)}$, where $X = A, B, C, D, E, F, G$ (i.e. for a trivial Dynkin diagram symmetry). Therefore lemma 8 requires that the Coxeter label of one of the non-zero nodes in the extended Dynkin diagram $X_l^{(1)}$ is equal to 1. (The Coxeter label a_0 of the zeroth node —● in the extended diagram is not allowed.)

However, we know that some restricted root systems are non-reduced. For these the Coxeter labels are directly defined to be the coefficients of the highest restricted root $\lambda_{\max}$. From the non-reduced root system BC_l one obtains a reduced root system of type C_l when dropping all the shortest roots, especially the short simple root λ_1. This shows that the Coxeter label of the long simple roots $\lambda_2, \ldots, \lambda_l$ is 2, because they are the short simple roots in C_l. The long simple root $2\lambda_1$ of C_l has Coxeter label 1 and therefore λ_1 has Coxeter label 2:

BC_l



We see that a Lie algebra with restricted root system of type BC_l does not have the 3-grading property described in lemma 8. The other cases are easily checked by inspection of table 2.3 and table 2.2. Also, Θ is fixed by the condition that it contains all simple restriced roots but one, called λ_e, with Coxeter label 1.

The generator iH_0 of $\mathfrak{z}_\mathfrak{k}$ in this basis can be found with the help of the software LambdaLie [dR04] presented in section 2.7.

It turns out that for $\mathfrak{su}(m,n)$ with $m < n$, $\mathfrak{so}^*(2l)$ for odd l, and $\mathfrak{e}_{6(-14)}$ there is no suitable choice for λ_e, because the corresponding diagrams are non-reduced and thus have Coxeter label 2 for all nodes. For the remaining cases there is only one possibility for λ_e – with the exception of the semi-simple $\mathfrak{so}(2,2)$. The result can be found in table 3.2 together with the simple restricted root λ_e wich is not in Θ, the type of $\mathfrak{m}_\Theta$, and dimension of $\mathfrak{n}_\Theta^\pm$. The latter actually is the dimension of the space-time $\mathbb{M} = G/\mathsf{B}_\Theta$ and is calculated with

$$\dim \mathfrak{n}_\Theta^\pm = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{m}_\Theta - \dim \mathfrak{a}_{\mathfrak{p}\Theta}) \ ,$$

which follows from (2.57). The cases of $\mathfrak{so}(2,2)$ and $\mathfrak{so}^*(4)$ are excluded from the table, because they are semi-simple and thus some of the results of section 3.3 do not apply. The proper treatment of these cases is to factorize into the simple components as is well known from the 2-dimensional conformal case.

Observe that the Dynkin diagram of Σ always is of type B_r and correspondingly λ_e is the unique long simple restricted root. This is a surprise, because there appears to be no reason, why there should not be more than one possibility for λ_e. Observe also that λ_e corresponds to a node of type 3 in the Satake diagram. This means that it is only the projection of one simple root α_e as we have seen in lemma 8. This makes it easy to determine the generalized Lorentz group $\mathfrak{m}_\Theta$: Its Satake diagram is obtained by removing α_e from the Satake diagram of $\mathfrak{g}$. This also implies that $\mathfrak{m}_\Theta$ is semi-simple.

Type	λ_e	$\mathfrak{m}_\Theta$	$\dim \mathbb{M}$
$\mathfrak{su}(m,m)$	m	$\mathfrak{sl}(m,\mathbb{C})$	m^2
$\mathfrak{so}(2,D)$	2	$\mathfrak{so}(1,D-1)$	D
$\mathfrak{sp}(l,\mathbb{R})$	1	$\mathfrak{sl}(l,\mathbb{R})$	$\frac{1}{2}l(l+1)$
$\mathfrak{so}^*(4n)$	1	$\mathfrak{su}^*(2n)$	$2n(n-1)$
$\mathfrak{e}_{7(-25)}$	3	$\mathfrak{e}_{6(-26)}$	27

Table 3.2: The possible parabolic subalgebras satisfying assumption (3.3). $m \geq 1$, $D \geq 3$, $l \geq 2$, $n \geq 2$

It remains to check that iH_0 is not contained in $\mathfrak{m}_\Theta(K)$. It turns out that for the considered algebras this is the case. As already mentioned, iH_0 is obtained using computer algebra methods. The results are listed in table 3.3.

Group	iH_0
$\mathfrak{su}(m,m)$	$f_{\alpha_m} + f_{\alpha_{m-1}+\alpha_m+\alpha_{m+1}} + \cdots + f_{\alpha_1+\cdots+\alpha_l}$
$\mathfrak{so}(2,D)$ B_l	$f_{\alpha_1} + f_{\alpha_{\max}}$
$\mathfrak{sp}(l,\mathbb{R})$	$f_{\alpha_1} - f_{\alpha_1+2\alpha_2} + f_{\alpha_1+2\alpha_2+2\alpha_3} + \cdots \pm f_{\alpha_{\max}}$
$\mathfrak{so}(2,D)$ D_l	$f_{\alpha_l} + f_{\alpha_{\max}}$
$\mathfrak{so}^*(4n)$	$f_{\alpha_1} + f_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4} + f_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + f_{\alpha_{\max}}$
$\mathfrak{e}_{7(-25)}$	$f_{\alpha_7} + f_{\alpha_1+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7} + f_{\alpha_{\max}}$

Table 3.3: iH_0 in the Cartan-Weyl basis with respect to the maximally non-compact Cartan subalgebra, where $f_\alpha = e_\alpha + e_{-\alpha}$ as in section 2.2.4.

An interesting observation can be made in table 3.3: iH_0 always is a sum of f_α, where α are highest roots of subalgebras of $\mathfrak{g}$ which also belong to our list. This can be visualized by the following chains:

$$\begin{aligned}
\mathfrak{su}(1,1) \subset \mathfrak{su}(2,2) \subset \cdots \subset &\mathfrak{su}(m,m)\\
\mathfrak{sp}(1,\mathbb{R}) \subset \mathfrak{sp}(2,\mathbb{R}) \subset \cdots \subset &\mathfrak{sp}(l,\mathbb{R})\\
\mathfrak{so}(1,2) \subset &\mathfrak{so}(2,D)\\
\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{so}^*(8) \subset \mathfrak{so}^*(12) \subset \cdots \subset &\mathfrak{so}^*(4n)\\
\mathfrak{so}(1,2) \subset \mathfrak{so}(2,10) \subset &\mathfrak{e}_{7(-25)}
\end{aligned}$$

The subalgebras are obtained by taking a subdiagram of the Satake diagram of $\mathfrak{g}$. There are always two of these subalgebras: $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}$ itself. The infinite classical series $\mathfrak{su}(m,m)$, $\mathfrak{sp}(l,\mathbb{R})$, and $\mathfrak{so}^*(4n)$ form chains, because there are no suitable subdiagrams. Surprisingly, this is not the case for the classical conformal case $\mathfrak{so}(2,D)$. The only way to extend the chain $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{so}(2,D)$ is by $\mathfrak{e}_{7(-25)}$. Not also that $\mathfrak{so}^*(4)$ does not appear in the chain of $\mathfrak{so}^*(4n)$.

Since conditions (i) and (ii) of theorem 18 are automatically satisfied, since (i) is neccessary and sufficient for positive energy representations to exist, and since the 3-grading condition is neccessary and sufficient for assumption 4 to be satisfied, we get the final result

Theorem 20 *The assumptions of section 3.1 are satisfied only by the simple Lie groups G with infinite center and Lie algebra $\mathfrak{su}(m,m)$, $\mathfrak{so}(2,D)$, $\mathfrak{sp}(l,\mathbb{R})$,*

$\mathfrak{so}^*(4n)$, *or* $\mathfrak{e}_{7(-25)}$ *and the corresponding homogenous space* $G/_{\mathsf{B}_\Theta}$ *for* Θ *as given by table 3.2.*

3.7 Existence of a Time Reflection

We will characterize the time reversal by its properties in the space-time symmetry group G. Therefore, this result is relevant not only for the generalization of conformal field theory, where the stability group H is a parabolic subgroup. The space-time manifold $\mathbb{M}$ is not required to have any specific properties.

Let us assume a time reversal T exists. It acts on $\mathbb{M}$ in such a way that this induces an automorphism of G of the form $\rho(g) \mapsto T\rho(g)T^{-1}$ and of $\mathfrak{g}$ ($\mathrm{Ad}(T)$) with the following properties: Of course T reflects time, $\mathrm{Ad}(T)iH_0 = -iH_0$. Since T should act trivially on space Σ, it has to be $\mathrm{Ad}(T)X = X$ for all $X \in \mathfrak{k}_s$.

Recall from Section 3.6.1 that $iH_0 = ih_{\Lambda^j}$, where j is the number of the marked node. This clearly is a linear combination of the h_{α_i} with purely imaginary coefficients.

Lemma 12 *Write the generator of time translations in the form*

$$iH_0 = r\Big(h_{\alpha_j} + \tfrac{1}{2}\sum_{i\neq j} \mu_i h_{\alpha_i}\Big)\ . \tag{3.7}$$

with purely imaginary r.

An automorphism $\mathrm{Ad}(T)$ *of* $\mathfrak{g}$ *such that* $\mathrm{Ad}(T)iH_0 = -iH_0$ *and* $\mathrm{Ad}(T)h_{\alpha_i} = h_{\alpha_i}$ *for* $i \neq j$ *can only exist if* μ_i *are integers.*

Proof: That iH_0 can be written in the form (3.7) follows, because it is orthogonal to all h_{α_i} for $i \neq j$. Since the h_{sri} for all simple roots α_i form a basis of the Cartan subalgebra, the coefficient r of α_j in an expansion of iH_0 into this basis has to be non-zero. That r is imaginary follows, because the fundamental weights are real linear combinations of the simple roots.

The automorphism $\mathrm{Ad}(T)$ of $\mathfrak{g}$ maps the Cartan subalgebra $\mathfrak{h}$ into itself and must therefore define an automorphism of the root system. We will also call it T. In particular, $T(\alpha_j)$ must be a root, $T(\Lambda^i) = -\Lambda^i$, because of (3.6), and $T(\alpha_j) = \alpha_j$ for $j \neq i$.

From (3.7) it follows that

$$T(\alpha_i) = \Big(-\alpha_i - \sum_{j\neq i} \mu_j \alpha_j\Big)\,.$$

Roots are sums of integer multiples of simple roots, therefore $T(\alpha_i)$ is not a root if any of the μ_j are non-integer.

End of Proof

With this result we can determine, which space-time symmetry groups allow for a time reversal, by studying table 3.1: As is well known and can immediately be seen in the table, $\mathfrak{so}(2, D)$ always allows a time reversal. For $\mathfrak{su}(m, n)$ the following ratios have to be integers: $\frac{2p}{m}$ for $0 < p < m$ and $\frac{2q}{n}$ for $0 < q < n$. Therefore $m, n \leq 2$. The only case not covered by an isomorphic conformal group is $\mathfrak{su}(1, 2)$. For $\mathfrak{sl}(l, \mathbb{R})$ $\frac{4p}{l}$ has to be integer for $0 < p < l$, which is true only for $l = 1, 2, 4$ where only the last case is not isomorphic to a conformal group. For $\mathfrak{so}^*(2l)$ $\frac{2l-4}{l}$ and $\frac{4p}{l}$ for $0 < p < l - 1$ have to be integers. This is true for $l = 2, 4$, but $\mathfrak{so}^*(4)$ is not simple and $\mathfrak{so}^*(8)$ is again isomorphic to a conformal group. For $\mathfrak{e}_{6(-14)}$ and $\mathfrak{e}_{7(-25)}$ we have $\mu = (\frac{3}{2}, 1, 2, 3, \frac{5}{2}, 1)$ and $\mu = (2, \frac{2}{3}, \frac{4}{3}, 2, \frac{5}{3}, \frac{4}{3}, 1)$ respectively. Thus, both do not allow a time reversal.

For the conformal symmetry groups $SO(2, D)$ it is well known that a time reversal exists. The only other cases, which are not excluded by the arguments above are $\mathfrak{su}(1, 2)$ and $\mathfrak{sp}(4, \mathbb{R})$. For $\mathfrak{sp}(4, \mathbb{R})$ it has been shown that no automorphism of the root system exists with the software package LambdaLie described in section 2.7 with the following code

---

```
* (setf alg (make-sla :type :c :rank 4))
#< extended structure ABSTRACT-SIMPLE-LIE-ALGEBRA >
* (output-srs-find-automorphism alg :start #(0 1 2 3)
                              :end #(31 1 2 3) :intm 1 :order 2)
1 #(0 1 2 3)
     #(1 0 0 0)
     #(0 1 0 0)
     #(0 0 1 0)
     #(0 0 0 1)

Altogether: 1 automorphisms
```

For $\mathfrak{su}(1, 2)$ an involutive automorphism of the root system is found with

```
* (setf alg (make-sla :type :a :rank 2))
#< extended structure ABSTRACT-SIMPLE-LIE-ALGEBRA >
* (output-srs-find-automorphism alg :start #(0 1) :end #(5 1)
                                                  :intm 1 :order 2)
1 #(0 1)
     #(1 0)
     #(0 1)
2 #(5 1)
     #(-1 -1)
     #(0 1)

Altogether: 2 automorphisms
```

The following code verifies that it is extendible to an automorphism of the Lie algebra. For this it is important that the order is explicitly set to 2, because only then the resulting automorphism is checked for being involutive.

```
* (setf (sla-automorphism alg)
        (make-array '(2) :element-type 'fixnum
                         :initial-contents #(5 1)))
#(5 1)
* (setf (sla-automorphism-order alg) 2)
2
* (sla-automorphism-rho alg)
#(-1 -1 -1 -1 -1 -1)
```

This is summarized in the following

Theorem 21 *[Nonexistence of time reflection] Of all simple Lie groups that allow an interpretation as a space-time symmetry only* $\mathfrak{su}(1,2)$ *and the conformal groups* $\mathfrak{so}(2,D)$ *admit an automorphism with the properties of a time reflection.*

Chapter 4

Simple Space-Time Symmetries in Physics

This chapter will briefly describe the next steps in the program to study quantum field theory with a simple Lie group as space-time symmetry group. In the first section, results of conformal field theory will be presented. It will be indicated in which way they can be generalized and what remains to be done. The second section is a brief review of other research fields in physics and mathematics related to our studies. Some of the results and methods developed in these fields might find applications in the future development of generalized conformal space-time symmetries.

4.1 General Quantum Field Theory with Simple Space-Time Symmetry

The results presented in chapter 3 for the simple Lie groups which allow an interpretation as space-time symmetries, as explained there, globally describe the space-time manifold and its symmetries as has been done for the classical conformal group $SO(2, D)$ in [LM75]. A lot of the results of conformal field theory should generalize to the simple Lie groups listed in table 3.2. In this section we will discuss these and indicate, what needs to be done for this generalization.

4.1.1 Unitary Positive Energy Representations

All irreducible unitary positive energy representations U of the universal covering group G of $SO(4, 2)$ are classified in [Mac77a]. Let $G = KP$ be a Cartan decomposition. A representation U decomposes into a Hilbert sum of finite dimensional irreducible representations of $K = K_s Z_{\mathfrak{k}}$. This is a standard result if G has finite center [War72] and $Z_{\mathfrak{k}}$ is therefore compact. It has been shown in [Lüs75] that this also holds in our case, where $Z_{\mathfrak{k}} \cong \mathbb{R}$ (see section 3.4). Let iH_0 be the generator of the center $Z_{\mathfrak{k}}$ of K. It has been shown by Mack that any irreducible unitary positive energy representation U of G possesses a lowest weight $\lambda = (d, \Lambda^-)$, where d is the lowest eigenvalue of H_0 and Λ^- is a lowest weight of one of the representations of K_s in the decomposition of U. Two such representations with the same lowest weight λ are unitarily equivalent. A restriction on the weight λ may be derived by requiring the scalar product to be positive semi-definite. Explicit realizations of these representations are given by Rühl [Rüh73a, Rüh73b]; however, these have no easy generalization to the other space-time symmetry groups. Otherwise, these results are easily generalized.

For physical applications, more explicit realizations of the representations are needed. In [Mac77a] the unitary positive energy representations of G are constructed as induced representations: The representation spaces are sections φ of vector bundles on $G/\mathsf{Z}^{\mathsf{M}} B_\Theta$, where Z^{M} is the factor of the center of G which is not in the Lorentz group and $B_\Theta = M_\Theta A_{\mathfrak{p}_\Theta} N_\Theta^-$ the identity component of a parabolic subgroup. Equivalently, these sections may be regarded as infinitely differentiable functions φ on G with values in a finite dimensional representation space E^λ of a representation D^λ of the parabolic subgroup $\mathsf{B}_\Theta = \mathsf{Z}^{\mathsf{M}} B_\Theta$.

The functions φ have to satisfy the covariance property

$$\varphi(gp) = D^\lambda(p)^{-1} \varphi(g) . \tag{4.1}$$

On this function space $g \in G$ is represented as

$$(\mathsf{U}(g)\varphi)(g') = \varphi(g^{-1}g') .$$

Recall from theorem 16 that $N_\Theta^+ \mathsf{B}_\Theta$ is a dense open submanifold of G. Because of (4.1) and continuity, the functions φ are completely determined by their values on N_Θ^+. For almost all $g \in G$, $x \in N_\Theta^+$ the element $g^{-1}x$ is in $N_\Theta^+ \mathsf{B}_\Theta$ and may be decomposed as $g^{-1}x = x'p$, where $x' \in N_\Theta^+$ and $p \in \mathsf{B}_\Theta$. With this, the representation of g on φ may be written as

$$(\mathsf{U}(g)\varphi)(x) = D^\lambda(p)^{-1} \varphi(x') .$$

The representation of the special conformal transformations N_Θ^- by D^λ is required to be trivial. Therefore an irreducible D^λ is completely described by $\lambda = (c, \Lambda_M^-)$ where Λ_M^- is the lowest weight of the representation of the identity component of the Lorentz group M_Θ and c is the *dimension* which specifies an irreducible representation a^c of the dilatations $a \in \mathfrak{a}_{\mathfrak{p}_\Theta}$. The definition of the scalar product below involves an integral over the group of translations N_Θ^+, which is of dimension D. With this it can be shown that the lowest eigenvalue d of $\mathsf{U}(H_0)$ and the conformal dimension c are related by $c = d - D/2$.

For the universal covering of $SO(4,2)$, the discrete group Z^M is a subgroup of the center Z_G of G and isomorphic to $\mathbb{Z}$. Let Z^M be the set of all γ^N and it follows from Schur's lemma that γ has to be represented by a multiple of the identity and γ^N has to be represented as multiplication with $e^{\pi i c N}$.

For $SO(4,2)$ the generator γ of Z^M is essentially given by $e^{2\pi i H_0}$, where H_0 is the conformal Hamiltonian. This explains the relation between the lowest weight and the representation of the Lorentz group.

The scalar product on the space of functions φ may be constructed with an intertwiner which is given by a generalized Kunze Stein formula [KS67]:

$$(\Delta_+^\lambda \varphi)(g) = n_+(\lambda) \int_{N_\Theta^+} dx\, \varphi(g\, \hat{\mathring{\omega}}_u\, x) ,$$

where $\hat{\mathring{\omega}}_u$ is (a representation of) the element of the Weyl group mapping $\mathfrak{n}_\Theta^+$ onto $\mathfrak{n}_\Theta^-$ defined in section 2.8.3. The map $\Delta_+^\lambda \varphi$ is an element of the dual space of the space of the φ (i.e. a distribution), and the scalar product may be defined as

$$(\varphi_1, \varphi_2) = \langle \varphi_1, \Delta_+^\lambda \varphi_2 \rangle .$$

It has to be shown, that this scalar product is well defined and positive on all functions φ. For the conformal group in 4 space-time dimensions this is done in [Mac77a].

A unitary irreducible representation U of a semi-simple Lie group G with countably infinite center Z is said to belong to the *interpolated discrete series* if the integral

$$\int_{G/\mathsf{Z}} dg\, |(\varphi_1, \mathsf{U}(g)\varphi_2)|^2$$

if finite for some non-zero vectors φ_1 and φ_2 in the representation space. dg is the Haar measure on the group G/Z. We have seen above that an element $z \in \mathsf{Z}$ of the center is represented under an irreducible representation by a constant $\lambda(z)$ times identity. Unitarity implies that $|\lambda(z)| = 1$, which shows that the integrand is invariant under $g \to gz$, and the integral is therefore well-defined. Most of the irreducible unitary positive-energy representations of the conformal group in 4 space-time dimensions belong to the interpolated discrete series.

4.1.2 Quantum Field Theory

The Hilbert space $\mathcal{H}$ of states of a quantum field theory with generalized conformal space-time symmetry is a direct sum of such irreducible unitary positive energy representation spaces with a unique vector Ω spanning the trivial representation – the vacuum vector. Quantum fields ϕ_λ are introduced as distributions on $\mathbb{M}$ whose values are operators on $\mathcal{H}$. They transform under the irreducible representation labeled by λ:

$$\mathsf{U}(g)\phi_\lambda(x)\mathsf{U}^{-1}(g) = D^\lambda(p)^{-1}\phi_\lambda(x') \ ,$$

with $x' \in N_\Theta^+$ and $p \in \mathsf{B}_\Theta$ uniquely given by $g^{-1}x = x'p$ as above. Let f, g be test functions on which the fields can be evaluated. The vectors obtained by the action of the field operators on the vacuum $\phi_\lambda(f)\Omega$ span the irreducible unitary positive energy representation spaces in $\mathcal{H}$ described above. The two-point function $(\Omega, \phi^*_{\lambda'}(g)\phi_\lambda(f)\Omega)$ equals the scalar product of this vector with $\phi_{\lambda'}(g)\Omega$.

The main ingredients for the generalization of the results for $SO(4,2)$ mentioned in this section are the structure of the center Z_G of G and of the discrete group Z^M, their relation, and their representation. The two views at the unitary positive energy representations correspond to the two choices of Cartan subalgebras and the corresponding structure theories of real Lie algebras in section 2.4. An easy transition between these two views would be useful when working with the positive energy representations. In fact, the computer algebra package LambdaLie, introduced in section 2.7, will be useful in this task.

4.1.3 Operator Product Expansions

Conformal space-time symmetry imposes strong constraints on the quantum field theories invariant under it. For 4-dimensional conformal field theory this is exploited by Mack [Mac77b] by analyzing operator product expansions [Wil69]

$$\phi_i(x)\phi_j(y)\Omega = \sum_k C^{ijk}(x-y)\phi_k(x) \ , \tag{4.2}$$

which are assumed to be asymptotic expansions for small enough $x - y$. The fields ϕ_k in the sum are all local fields and their derivatives. Mack shows that a partially resummed version of this operator product expansion converges: The expansion has the form

$$\phi_i(x)\phi_j(y)\Omega = \sum_k \int_{N_\Theta^+} dz\ B^{ijk}(xy;z)\phi_k(z)\Omega \ . \tag{4.3}$$

When smeared with test functions $f(x, y)$, it converges strongly. The sum is over non-derivative fields. The coefficients B_{ijk} are $\mathbb{R}$-valued distributions.

The convergence comes from the fact that this expansion is essentially a partial wave expansion (harmonic analysis) of the state vector $\phi_i(x)\phi_j(y)\Omega$ into states which transform irreducibly under the space-time symmetry group G.

With the expansion all N-point Wightman functions $(\Omega, \phi_{i_1}(x_1) \ldots \phi_{i_N}(x_N)\Omega)$ can be constructed from the two- and three-point functions of lower order.

The asymptotic expansion (4.2) can be derived in conformally invariant quantum field theory under some additional assumptions with dynamical methods [Mac73].

4.1.4 N-Point Functions

The two-, three- and maybe even four-point functions are determined up to some coupling constants by the conformal symmetry if that the space-time symmetry group G acts transitively on relatively space-like pairs and triples (and maybe 4-tuples) of points. These properties of the symmetry are analyzed as follows.

As we required in assumption 2 the group G acts transitively on the space-time manifold $\mathbb{M}$. Choose any point x_1' in $\mathbb{M}$. It is mapped to any fixed point $gx_1' = x_1 \in \mathbb{M}$ by some $g \in G$. This element g is only fixed up to multiplication with an element h of the stability group H^1 of x_1 since $hgx_1' = x_1$. For any point $x_2' \in \mathbb{M}$ relatively space-like to x_1', the points gx_2' and hgx_2' are relatively space-like to x_1, since G leaves the causal structure invariant. If any point gx_2' relatively space-like to x_1 can be mapped to a fixed $hgx_2' = x_2$ by some $h \in H^1$, space-like pairs of points are transitive under G. Let $H^{12} \subset H^1$ be the stability subgroup of x_1 and x_2. The same line of arguments can be used to show that if any point hgx_3' relatively space-like to the fixed points x_1, x_2 can be mapped to a fixed x_3 by the stability group H^{12}, then G acts transitively on triples of relatively space-like points. This scheme continues in the obvious way. We will outline the general structure used to analyze the transitivity properties for the conformal group as is done in [Mac77b].

By assumption 3 the space-time manifold is a homogenous space $\mathbb{M} = {}^{G}\!/_{B_\Theta}$, where $B_\Theta = M_\Theta A_{\mathfrak{p}\Theta} N_\Theta^-$ is the identity component of a parabolic subgroup B_Θ of G. Recall the generalized Bruhat decomposition from theorem 15. To simplify the notation, we will identify the element of the Weyl group $\omega \in W(\Sigma)$ with its representative $m_\omega \in \mathsf{M}^*$ in the normalizer of $\mathfrak{a}_\mathfrak{p}$ in K. The special element $\mathring{\omega}_u$ introduced in section 2.8.3 will be denoted by $\mathring{\omega}$. Let Z^M be the group of connected components of the parabolic subgroup $\mathsf{B}_\Theta = \mathsf{Z}^\mathsf{M} B_\Theta$.

Note, that the two discrete groups describing the component structure of B_Θ introduced in section 2.8.2 are not necessarily isomorphic to this group.

Multiplying the generalized Bruhat decomposition with $\mathring{\hat{\omega}}$ gives

$$G = \bigcup_{\omega^{-1} \in W_u, z \in \mathsf{Z}^{\mathsf{M}}} \mathring{\hat{\omega}}\, U^-_\omega \omega z B_\Theta \ .$$

This disjoint union is suitable for describing the global structure of $\mathbb{M}$. The subset in this decomposition given by $z = \mathbf{e}$ and $\omega = \mathbf{e}$ is $\mathring{\hat{\omega}}\, B_\Theta$. This is a single point in $\mathbb{M}$ and we choose the first fixed point x_1 to be this point. Its stability subgroup is $\mathring{\hat{\omega}} B_\Theta \mathring{\hat{\omega}}^{-1} = M_\Theta A_{\mathfrak{p}\Theta} N^+_\Theta$. Another subset in the generalized Bruhat decomposition given by $z = \mathbf{e}$ and $\omega = \mathring{\hat{\omega}}^{-1}$ is $N^+_\Theta B_\Theta$. This the identity component of the dense submanifold $N^+_\Theta \mathsf{B}_\Theta$ of G in theorem 16. Since N^+_Θ is the subgroup of translations, the subset $[N^+_\Theta]_{B_\Theta}$ of $\mathbb{M}$ is the generalization of Minkowski space-time. (Recall, that $\mathbb{M}$ is the generalization of the universal covering of the conformal compactification of Minkowski space-time.)

In the conformal case $G = SO(4,2)$ the set of points relatively space-like to $x_1 = [\mathring{\hat{\omega}}]_{B_\Theta}$, which is the unique point at *space-like infinity*, is precisely the set $[N^+_\Theta]_{B_\Theta}$. This is expected to be a general result. The second fixed point x_2 may be chosen to be $[\mathbf{e}]_{B_\Theta}$. Since its stability subgroup is B_Θ, the stability group of x_1 and x_2 is $H^{12} = B_\Theta \cap M_\Theta A_{\mathfrak{p}\Theta} N^+_\Theta = M_\Theta A_{\mathfrak{p}\Theta}$ – the group of Lorentz transformations and dilatations. This group has to act transitively on the points, relatively space-like to x_2, in the generalized Minkowski space $[N^+_\Theta]$. This may be checked on the Lie algebra level: Choose such a point x_3. It may be represented by $x_3 = [\exp(X_3)]$ for $X_3 \in \mathfrak{n}^+_\Theta$. Elements of $M_\Theta A_{\mathfrak{p}\Theta}$ close to the identity $\mathbf{e}$ act as

$$\begin{aligned} &\exp(tX^K)\exp(tX^M)\exp(tX^K)\exp(X_3)B_\Theta \\ &\qquad = \exp(X_3 + t(\left[X^K, X_3\right] + \left[X^M, X_3\right] + \left[X^A, X_3\right]) + O(t^2))B_\Theta \ , \end{aligned}$$

for $X^K \in \mathfrak{m}_\Theta(K)$, $X^M \in \mathfrak{m}_\Theta \cap \mathfrak{p}$, and $X^A \in \mathfrak{a}_{\mathfrak{p}\Theta}$. The three commutators can be simplified to $[X, X_3]$ with $X \in \mathfrak{m}_\Theta \oplus \mathfrak{a}_{\mathfrak{p}\Theta}$. If the linear map $\mathfrak{m}_\Theta \oplus \mathfrak{a}_{\mathfrak{p}\Theta} \to \mathfrak{n}^+_\Theta$ given by $X \mapsto [X, X_3]$ is surjective, x_3 is mapped to a small open neighborhood of itself by multiplication with elements of $M_\Theta A_{\mathfrak{p}\Theta}$. If it is the case for any possible x_3, this may be lifted to a global property. The software package LambdaLie introduced in section 2.7 contains an algorithm to find all X_3 which have this local property. The result obtained there is a set of equations describing the points, where the map given above is not surjective. In the conformal case, this is precisely the light-cone. Unfortunately, the main memory of todays computers is not large enough to apply this algorithm to the theory symmetric under the exceptional group $E_{7(-25)}$.

In conformal theory of more than two dimensions the fixed point x_3 may be chosen to be $x_3 = (0, 0, \ldots, 1)$, whose stability group H^{123} in $M_\Theta A_{\mathfrak{p}_\Theta}$ is the group of rotations around the last axis. Since the set of space-like points relative to x_1, x_2, x_3 is non-compact, it is not possible to find a fourth point x_4 which is conjugate to all these under the compact group of rotations around the last axis.

With the constraints on the two- and three-point functions it is possible to restrict the conformal operator product expansion (4.3): The coefficients B_{ijk} are finite linear combinations of distributions which are given by the representation theory. The coefficients appearing in this linear combination determine the theory completely. In general, there will be countably many coupling constants.

4.1.5 Dimensional Reduction

Eventually, it will be desirable to construct theories with generalized conformal space-time symmetry which are models of the physical world. Part of these models will be a dimensional reduction of the space-time manifold to a 4-dimensional manifold. The generalized conformal symmetry group G will be broken down to a 4-dimensional conformal or Poincaré group G^4, which will be a subgroup of G. The commutant U of G^4 in G will give rise to an internal symmetry, part of which might be turned into a local symmetry. For $G = E_{7(-25)}$ there are only pairs of the form $G^4 = SO(4,2)$, $U = SU(4) \times Sl(2,\mathbb{R})$, as has been obtained with LambdaLie (section 2.7). This is not enough to accommodate the gauge group $SU(3) \times SU(2) \times U(1)$ of standard model of elementary physics. In place of $SU(2)$, its non-compact brother $Sl(2,\mathbb{R})$ appears. $SO(4) \supset SO(3) \times U(1)$ might be identified with the Pati-Salam GUT. It is very strange that a non-compact internal symmetry group appears.

However, the commutant of a Poincaré subgroup might have the standard model gauge group as a subgroup. Another approach to accommodate the gauge group would use the exceptional group $E_{8(-24)}$ as combined gauge and space-time group. It contains a subgroup $E_{7(-25)} \times Sl(2,\mathbb{R})$ and the commutant of a $SO(4,2)$ subgroup is $U = SO(8,2)$, a non-compact version of $SO(10)$, which is used as gauge group in grand unification. While there are no positive energy representations of $E_{8(-24)}$, the decomposition of unitary representations of $E_{8(-24)}$ into representations of the subgroup $E_{7(-25)}$ might contain positive energy representations.

Of course, these are only qualitative arguments indicating the directions which are worth while exploring. Definite results can only be obtained by performing the reduction to a 4-dimensional manifold with Poincaré or conformal symmetry. There may be several non-equivalent ways of doing this.

Related to this is the determination [Mac77a] of the Poincaré content (m, s) in the irreducible representations of the conformal group. The mathematical generalization of this would be the decomposition of the irreducible representations of the generalized conformal group G into irreducibles of the generalized Poincaré group which is given by the generalizations of the translations and Lorentz transformations: $N^+_\Theta \mathsf{M}_\Theta$. The physical analog is the decomposition into irreducibles of the 4-dimensional Poincaré group which is determined by the specific way of dimensional reduction.

Methods developed in the context of hidden symmetries in supergravity might be useful in the construction of models with generalized conformal space-time symmetries and their dimensional reduction. This will be discussed in section 4.2.3.

4.2 Related Research Fields

In the following, four recent fields of mathematical and physical research are mentioned which are related to our studies. The relation will be indicated and how their results and tools might be useful in the future developments of simple space-time symmetries. The two mathematical fields are concerned with causal symmetric spaces and Euclidean Jordan algebras. In physics there is a close relation to supergravity and some aspects of the holographic principle.

4.2.1 Causal Symmetric Spaces

Symmetric spaces are homogenous spaces G/H of a semi-simple Lie group G with an involution θ such that the subgroup of elements which are left invariant under θ is the stability subgroup H. Causal structures on symmetric spaces have been studied by mathematicians for several reasons: Paneitz [Pan81] built on concepts from cosmology propagated by [Seg76], Vinberg [Vin82] studied the automorphism group of convex cones, and Ol'šanskiĭ [Ol'82, Ol'91] was interested in unitary representations of certain infinite-dimensional Lie groups.

Ol'šanskiĭ was the first to classify causal symmetric spaces and showed how they could play an important role in harmonic analysis and the Gel'fand-Gindikin program [GG78], which aims at the realization of families of similar unitary representations of Lie groups in a unified geometric way. For the study of unitarizable highest weight representations, causal symmetric spaces are of particular importance [Óla00, BP02]. There is also a relation to Jordan Algebras [FK94], whose relation to generalized conformal symmetry is described in the next section. Introductions to the subject of causal symmetric spaces may be found in [HÓ97, FK94, FÓ95, Óla98, Hil95].

The relation to generalized conformal symmetry is as follows. Recall the structure of the homogenous spaces satisfying our assumptions in section 3.1, which are given in table 3.2: They are of the form G/B_Θ where $\mathsf{B}_\Theta = \mathsf{M}_\Theta A_{\mathfrak{p}\Theta} N_\Theta^-$ is a parabolic subgroup. $A_{\mathfrak{p}\Theta}$ is a certain Abelian subgroup of the non-compact part P of the Cartan decomposition $G = KP$. The reductive Lie group $\mathsf{L}_\Theta = \mathsf{M}_\Theta A_{\mathfrak{p}\Theta}$ is defined to be its centralizer in G and N_Θ^- the connected subgroup with nilpotent Lie algebra $\mathfrak{n}_\Theta^-$.

For the homogenous spaces G/B_Θ given in table 3.2, which generalize conformal symmetry, the homogenous spaces G/L_Θ are precisely all irreducible causal symmetric spaces of Cayley type. These are defined to be the self-dual spaces for a duality relation which maps compactly causal onto non-compactly causal symmetric spaces. For all remaining irreducible causal symmetric spaces G/H, which are not self-dual, the stability group H is semi-simple. Another interesting fact is that for the compactly causal symmetric spaces the semi-simple Lie group G admits positive energy representations. (Thus, it is a direct product of the groups listed in table 3.1.)

This identification may be useful in the study of unitary positive energy representations of G and harmonic analysis on G/B_Θ. For this it is essential that positive energy representations are lowest weight representations [Mac77a], which are equivalent to highest weight representations as can be seen by interchanging the positive and negative roots.

4.2.2 Euclidean Jordan Algebras

Closely related to causal symmetric spaces are Jordan algebras [FK94]. They are defined to be a vector space V with a bilinear product $V \times gV \to V, (x, y) \mapsto xy$ which is Abelian $xy = yx$ and satisfies $x((xx)y) = (xx)(xy)$. In a Euclidean Jordan algebra there is a scalar product $(\cdot, \cdot)$ which is associative, i.e. $(xy, z) = (y, xz)$. Euclidean Jordan algebras are in one-to-one correspondence with symmetric cones, which are regular cones in a Euclidean vector space and at the same time Riemannian symmetric spaces. Symmetric cones play a role in the representation theory and harmonic analysis on causal symmetric spaces.

In [Ber00] a conformal Lie algebra and corresponding conformal Lie group of a Jordan triple system (which is a generalization of Jordan algebras) is defined. The definition presented there involves so called twisted symmetric spaces which are the global objects corresponding to Jordan algebras in a similar way as Lie groups are the global objects corresponding to Lie algebras. A conformal completion of a Jordan algebra is defined to be the quotient of the conformal group by the stability subgroup of zero, which can be identified with a parabolic subgroup. If the conformal group is a classical Lie group these completions are determined rather explicitly.

The conformal Lie algebras of Euclidean Lie algebras are precisely the Lie algebras listed in in table 3.1 and the conformal completion mentioned above have the corresponding manifolds $\mathbb{M}$ as universal coverings. The Jordan structure might be exploited to find explicit realizations of the space-time manifold and the conformal group. For example the exceptional Lie group $E_{7(-25)}$ is the conformal group of the Euclidean Jordan algebra $Herm(3, \mathbb{O})$ of Hermitian 3×3 matrices with entries in the octonions. The software package LambdaTensor by T. Fischbacher [Fis02], on which LambdaLie (section 2.7) is based, implements the octonions and can be used to implement these Jordan algebras.

4.2.3 Hidden Symmetries in Supergravity

When compactifying maximal $d = 11$ supergravity on a torus of $11 - d$ dimensions, the resulting Lagrangean is symmetric under a Lie group of type $E_{11-d(11-d)}$ which acts on the scalar fields in d dimenions together with the n-form fields which are Hodge-dualized to scalars. In [Koe01, GKN01a] a non-linear conformal realization on a 27-dimenional space of the exceptinal Lie algebra $\mathfrak{e}_{7(7)}$ is described, which is based on the relation to Jordan tripel systems described in the previous section [Gün93, GS82]. This is done with the aid of a 3-grading similar to the one used we used in chapter 3 and which is obtained by decomposing $\mathfrak{e}_{7(7)}$ with respect to a $\mathfrak{e}_{6(6)}$ subalgebra:

$$\mathbf{133} = \mathbf{27} \oplus (\mathbf{78} \oplus \mathbf{1}) \oplus \overline{\mathbf{27}} \,,$$

where the $\mathbf{78}$ is the $\mathfrak{e}_{6(6)}$ subalgebra, which is now further decomposed with respect to its maximal compact subalgebra $\mathfrak{sp}^{\mathrm{cpt}}(8)$ as $\mathbf{78} = \mathbf{36} \oplus \mathbf{42}$. ($\mathbf{27}$ is irreducible under $\mathfrak{sp}^{\mathrm{cpt}}(8)$.) These representations are then given as tensors of several different ranks with additional properties such as symmetric, antisymmetric, or traceless with respect to a symplectic metric. With these tensor realizations the commutation relations and the realization of $\mathfrak{e}_{7(7)}$ can be given in a relatively concise form.

It is also possible to describe scalar invariants, which are the analog of $x_\mu x^\mu$ in Minkowski space. It can be seen quite easily that there is no such quadratic invariant, because the decomposition of $\mathbf{27} \otimes \mathbf{27}$ into irreducibles contains no trivial representation. However, there is a qubic invariant $\mathcal{N}_3$ of the $\mathfrak{e}_{6(6)}$ subalgebra, and the hyper-surface described by $\mathcal{N}_3 = 0$ is invariant under $\mathfrak{e}_{7(7)}$.

This is the generalization of the light cone $x_\mu x^\mu = 0$ which is invariant under ordinary conformal symmetry.

With tensor realizations such as those described above, commonly used techniques in Lagrangean quantum field theory could be used when dealing with generalized conformal space-time symmetries with Lie algebra $\mathfrak{e}_{7(-25)}$. When trying to realize $\mathfrak{e}_{7(-25)}$ in a similar way as $\mathfrak{e}_{7(7)}$, the decomposition is necessarily more complicated because the maximal compact subalgebra of $\mathfrak{e}_{6(-26)}$ is $\mathfrak{f}_{4(-52)}$ which is still exceptional and would have to be decomposed further. A maximal classical subalgebra of $\mathfrak{f}_{4(-52)}$ is $\mathfrak{so}(9)$. Instead, one could decompose with respect to a non-compact subalgebra, e.g. a non-compact form of $\mathfrak{sp}^{\mathrm{cpt}}(8)$. This would preserve the simple decomposition structure, but the restrictions of the tensors would be less natural. Such a realization might also be obtained directly from the one given for $\mathfrak{e}_{7(7)}$ by multiplying the right generators with the imaginary unit i. A cubic invariant corresponding to $\mathcal{N}_3$ described above would be related to the G-invariant causal structure described in chapter 3. It should turn out that the light cone $\mathcal{N}_3 = 0$ is the boundary of the cone described there.

In [Koe01, GKN01a] a quasi-conformal realization of $\mathfrak{e}_{8(8)}$ is presented. A conformal realization is not possible because $\mathfrak{e}_8$ does not admit a 3-grading. This quasi-conformal realization is based on a 5-grading of $\mathfrak{e}_8$ and can be truncated to quasi-conformal realizations of all the other exceptional Lie algebras $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{6(6)}$, $\mathfrak{f}_{4(4)}$, and $\mathfrak{g}_{2(2)}$. These are related to Freudental tripel systems, a generalization of Jordan tripel systems.

Based on the quasi-conformal realization of $\mathfrak{e}_{8(8)}$ an oscillator-like representation of $\mathfrak{e}_{8(8)}$ is presented in [GKN01b]. This is a generalization of the singleton and doubleton realization of $\mathfrak{sl}(2,\mathbb{R})$. The singleton realization is given by creation and anihilation operators a and a^+ satisfying $[a,a^+] = 1$ and $[a,a] = [a^+,a^+] = 0$. The three generators of $\mathfrak{sl}(2,\mathbb{R})$ are given by $e_+ = a^+a^+$, $e_- = aa$, and $h = a^+a + aa^+$. Unitary lowest weight representations are then found by requiring $a\Omega = 0$ and constructing the representation space by successive action of the creation operator a^+. This representation can also be realized on a space of functions (Schrödinger representation) by a coordinate and a momentum operator $x = a + a^+$, $p = i(a - a^+)$ which are represented by multiplication and derivative. $\mathfrak{e}_{8(8)}$ can be represented by such phase space variables with 29 coordinate and 29 momentum operators. The generators of $\mathfrak{e}_{8(8)}$ are rational expressions in these 58 operators. The subalgebra $\mathfrak{e}_{7(7)}$ is generated by polynomial in $28+28$ operators up to order 4. Since $\mathfrak{e}_{8(8)}$ and $\mathfrak{e}_{7(7)}$ do not posses unitary lowest weight representations, requirements like $a\Omega = 0$ will not lead to consistent representations. However, if such an oscillator-like representation can be found for $\mathfrak{e}_{7(-25)}$ this should lead to a nice, direct construction of unitary lowest weight representations.

Since the hidden symmetries appear after compactification, it arises the question whether these symmetries are in fact present in the 11-dimensional supergravity theory. They would have to be symmetries of space-time and necessarily link spinor and tensor degrees of freedom. This seems to be in contradiction to the spin-statistics theorem [SW64]. In [Ram03] an attempt to generalize 11-dimensional supergravity is discussed which is supposed to circumvent this difficulty. The same obstruction will be encountered when trying to connect the generalized conformal symmetry presented in this work to conventional quantum field theory.

Another connection to supergravity is given by the computer algebra tool LambdaLie developed for analysis of the vacuum structure of gauged maximal supergravities [FNS02, Fis03]. The package LambdaLie presented in section 2.7 is based on these developments.

4.2.4 Holographic Principle

A version of the much discussed holographic principle is formulated by Rehren [Reh00] in the following way: Local quantum field theories on AdS space-time in $D+1$ dimensions are in one-to-one correspondence with conformally invariant quantum field theories on conformally compactified Minkowski space-time in D dimensions. The state spaces of corresponding theories coincide. AdS space-time may be given as $SO(2,D)/SO(1,D)$ and we know from chapter 3 that the conformal compactification is $SO(2,D)/\mathsf{B}_\Theta$ where B_Θ is a maximal parabolic subgroup. Rehren proves the holographic principle, where the essential ingredient is the equality of the symmetry group of both theories. This corresponds to an equivalence of the harmonic analysis on AdS space-time and conformally compactified space-time. Since the former space-time is in fact a causal symmetric space, the correspondence of the harmonic analysis on causal symmetric spaces and on generalized conformal space-times suggested in section 4.2.1 might provide further insight into the holographic principle.

Chapter 5

Non-Compact Gauge Symmetries

The theory of parabolic subalgebras and subgroups presented in section 2.5 and 2.8.2 has an application in gauge theories with non-unitary parallel transporters. Admitting not only unitary parallel transporters is motivated by the belief that fundamental theories should be based on as few a priori assumptions as possible [Mac01]. Parallel transporters are elements of the holonomy group, which is a Lie group and in conventional theories equal to the compact gauge group. When admitting non-unitary parallel transporters, the holonomy group will be non-compact and larger than the gauge group. General relativity is a special kind of gauge theory, whose parallel transporters take values in $Sl(2,\mathbb{C})$, which is non-compact [Neu03].

The natural setting for studying such generalizations of gauge theories includes a discrete space-time. There, it is even possible to allow parallel transporters which are not invertible. If unitarity is not required and the theory is required to be symmetric only under gauge transformations of a unitary group, the parallel transporters decompose into a unitary part and a part which can be interpreted as a regular field in the theory, which has a natural interpretation as a Higgs field [AMOP04, Ang03]. This Higgs field may break the unitary gauge symmetry. The requirements on the Higgs potential can be analyzed with our methods. For a detailed treatment of the discrete case and examples see [Leh04]. In this work renormalization group flow in such theories has also been studied (see also [LM03]). It is possible to derive the quark masses and Kobayashi Maskawa matrix from a similar model [LMP03].

The adjoint $U(\mathcal{C})^*$ of a parallel transporter $U(\mathcal{C})$ along a path $\mathcal{C}$ is usually equal to the parallel transporter $U(-\mathcal{C})$ along the inverse path $-\mathcal{C}$. In conventional gauge theories it is always assumed that $U(\mathcal{C})^* = U(\mathcal{C})^{-1}$. In a discrete setting it is natural to drop this assumption and therefore obtain non-unitary parallel transporters. However, in the conventional principal bundle setting of continuous gauge theories, this is not possible. For the central result of this chapter, it is essential that the holonomy group is a non-compact semi-simple Lie group with finite center and the gauge group its maximally compact subgroup. We can retain the relation $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$ and therefore consider the continuous setting in the conventional framework.

Although the adjoint operation is not needed for the central result, we will discuss its relation to automorphisms of Lie algebras in section 5.2. Non-unitarity always implies $U(\mathcal{C})^* \neq U(\mathcal{C})^{-1}$. Since we will always assume $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$, we have $U(-\mathcal{C}) \neq U(\mathcal{C})^*$. In the discrete setting one may retain the relation $U(-\mathcal{C}) = U(\mathcal{C})^*$ while abandoning $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$.

Let $H = KP$ be the Cartan decomposition of the holonomy group and $G = K$ the gauge group. The theory of parabolic subgroups may be used to determine the unbroken subgroup of the gauge group when the Higgs field assumes a vacuum expectation value different from zero in the Lie algebra $\mathfrak{p}$ of P. For this, the commutant of any element $\Xi_{min} \in \mathfrak{p}$ has to be determined. Any $\Xi_{min} \in \mathfrak{p}$ is conjugate to an element of a maximally Abelian subalgebra $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$, and by symmetry under the Weyl group even to an element of the closure of the fundamental Weyl chamber $\overline{\mathfrak{a}_\mathfrak{p}^+}$. The commutant of any $\Xi_{min} \in \mathfrak{a}_\mathfrak{p}$ turns out to be the intersection $\mathsf{M}_\Theta(K)$ of some parabolic subgroup B_Θ with the gauge group $G = K$. The subgroup $\mathsf{M}_\Theta(K)$ is not necessarily connected.

5.1 The Holonomy Group

Consider a Lagrange field theory on a $D+1$-dimensional space-time manifold $\mathbb{M}$, such that a field ψ is a section of a vector bundle $\mathcal{V}$ over $\mathbb{M}$. The vector bundle $\mathcal{V}$ defines for each point $x \in \mathbb{M}$ a vector space V_x, the *fiber* at *base point* x, and these vector spaces are isomorphic for any two points $x, y \in \mathbb{M}$: $V_x \cong V_y$. The field ψ assigns to each point $x \in \mathbb{M}$ a vector in the corresponding vector space $\psi(x) \in V_x$.

Even though two such vector spaces V_x and V_y are isomorphic, there is no unique way to identify them and therefore no unique differentiation of fields is defined. Differentiation of the fields is only defined by specifying a *connection* ∇ as follows: The connection ∇ may be given by specifying *parallel transports* $U(\mathcal{C})$ along all differentiable paths $\mathcal{C}$ in $\mathbb{M}$. If the path $\mathcal{C}$ starts at x and ends at y, the parallel transport $U(\mathcal{C})$ is a linear map $V_x \to V_y$, which is usually taken to be bijective. A parallel transport can be used to identify the two vector spaces V_x and V_y. It is important to note that the parallel transport between two points in general depends on the path between these two points. Therefore, a connection generally does not provide a unique identification of fibers. The parallel transports defining a connection ∇ satisfy the following conditions:

$$U(\varnothing) = \mathbb{1} \,, \qquad \text{where } \varnothing \text{ is the trivial path,}$$
$$U(\mathcal{C}_2) \circ U(\mathcal{C}_1) = U(\mathcal{C}_2 \circ \mathcal{C}_1) \,,$$

where the starting point of the path $\mathcal{C}_2$ is the endpoint of $\mathcal{C}_1$ and the composition $\mathcal{C}_2 \circ \mathcal{C}_1$ is defined to be the path obtained by first traversing $\mathcal{C}_1$ and after that $\mathcal{C}_2$. The relation $U(\mathcal{C})^{-1} = U(-\mathcal{C})$, which was mentioned in the introduction and will hold during the following sections, amounts to identifying paths with the relation $-\mathcal{C} \circ \mathcal{C} \cong \varnothing$.

In a discrete field theory – e.g. lattice gauge theory – the manifold is replaced by a discrete set of points with nearest-neighbor relationships. This can be modelled by a graph, whose vertices are then the space-time points. An edge between two points indicates that these two points are nearest neighbors. Paths $\mathcal{C}$ move along the edges. Therefore, each parallel transporter $U(\mathcal{C})$ is composed of parallel transporters which map between neighboring points: $U(\mathcal{C}) = U(x_{n-1}, x_n) \circ \cdots \circ U(x_1, x_2)$, where $U(x, y)$ is the unique parallel transport from x to a neighboring vertex y along the edge between them. The connection is completely specified by giving these *fundamental* parallel transporters. In the discrete setting it is natural to drop the requirement $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$ (which we assume in the following). This corresponds to considering directed edges between the vertices in the graph and an independent assignment of a parallel transport to each directed edge.

The *holonomy group* H of the connection ∇ is the group of transformations of V_x into itself given by all parallel transports $U(\mathcal{C})$ for a fixed $x_0 \in \mathbb{M}$ and all closed paths with start- and endpoint at x_0 [KN63]. The holonomy group H is a Lie group and does not depend on the choice of x_0. The parallel transport along an arbitrary path $\mathcal{C}$ can now be viewed as an element of H: Define a *moving frame* (Vielbein) to be a differentiable assignment of a basis in V_x to each point $x \in \mathbb{M}$. This fixes an isomorphism between V_{x_0} and V_x by identifying the basis vectors. Any parallel transport $U(\mathcal{C})$ may be viewed as a map from V_{x_0} to V_{x_0} and can be shown to be an element of the holonomy group. If in the discrete case the relation $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$ does not hold, the holonomy group is directly defined to be the set of all these maps and their inverses.

More accurately, the action of the parallel transport $U(\mathcal{C})$ on V_x, which is uniquely identified with V_{x_0} as above, is a representation of the holonomy group H on V_x (or V_{x_0} or any other fiber). This is important, when considering several fields ψ_i: The fibers V_x^i are irreducible representation spaces of H and the corresponding representations are not necessarily faithful. The above discussion now applies to the direct sum of the vector bundles $\mathcal{V} = \bigoplus \mathcal{V}^i$ whose fibers carry a faithful reducible representation of H.

The differentiation given by the connection ∇ can now be defined: Let $X \in T_x\mathbb{M}$ be a tangent vector at the point $x \in \mathbb{M}$ and $\mathcal{C}\colon I \to \mathbb{M}$ a path with $\mathcal{C}(0) = x$ and tangent vector X at zero. ($I \subset \mathbb{R}$ is an interval which contains zero.) The *covariant derivative* of the field ψ in direction of the tangent vector X at the point x is now given by

$$\nabla_X \psi(x) = \tfrac{d}{dt}\big(U(-\mathcal{C}_t)\psi(\mathcal{C}(t))\big)|_{t=0} \ , \tag{5.1}$$

where $\mathcal{C}_t$ is the segment of the path $\mathcal{C}$ between $\mathcal{C}(0)$ and $\mathcal{C}(t)$. This derivative is well defined, because all the values $\psi(\mathcal{C}(t)) \in V_{\mathcal{C}(t)}$ of the field ψ at different points $\mathcal{C}(t) \in \mathbb{M}$ are transported back into V_x with $U(-\mathcal{C}_t)$. Therefore $U(-\mathcal{C}_t)\psi(\mathcal{C}(t)) \in V_x$ for all t, and the derivative of a map $\mathbb{R} \to V_x$ is the usual one.

The covariant derivative $\nabla_X \psi(x)$ may be described infinitesimally, if a moving frame in a neighborhood of x is given. Then $U(-\mathcal{C}_t)$ may be viewed as a map $V_x \to V_x$ and may be expanded for small t:

$$U(-\mathcal{C}_t) = \mathbb{1} - tB_X + o(t^2) \ , \tag{5.2}$$

where B_X is an element of the Lie algebra $\mathfrak{h}$ of the holonomy group H for each tangent vector $X \in T_x\mathbb{M}$. Again, more accurately, B_X is a representation matrix of a Lie algebra element, where the representation of the Lie algebra corresponds to the representation of H described above. B_X is the *vector potential* at x of the connection ∇. If this expansion is applied to (5.1) we get

$$\begin{aligned}\nabla_X \psi(x) &= \tfrac{d}{dt}\psi(\mathcal{C}(t))|_{t=0} - \tfrac{d}{dt}\big(t(B_X\psi)(\mathcal{C}(t))\big)|_{t=0} \\ &= \partial_X\psi(x) - B_X\psi(x) \ .\end{aligned}$$

This argument may be applied to the covariant derivative ∇_X at any point x in the neighborhood, where the moving frame is defined. Thus, the vector potential $B_X(x)$ is a function of x in this neighborhood.

If local coordinates x_μ, $\mu = 0, \ldots, D$ of $\mathbb{M}$ are chosen and X_μ is the tangent vector in direction of the coordinate x_μ, the vector potential may be given by its components $B_\mu(x) = B_{X_\mu}(x)$, which are again elements of the Lie algebra H. The covariant derivative obtains the familiar form

$$\nabla_\mu \psi(x) = \nabla_{X_\mu} \psi(x) = \partial_\mu\psi(x) - B_\mu(x)\psi(x) \ .$$

A *gauge transformation* is given by a map $S\colon \mathbb{M} \to H$ and the transformation of the field ψ and parallel transport $U(\mathcal{C})$, where $\mathcal{C}$ is a path from x to y, given by

$$\psi(x) \to S(x)\psi(x) \qquad U(\mathcal{C}) \to S(y)U(\mathcal{C})S^{-1}(x) \ . \tag{5.3}$$

The gauge transformation of the vector potential B_X can be derived from this, when gauging the parallel transport along an infinitesimal path as (5.2):

$$S(\mathcal{C}(0))U(-\mathcal{C}_t)S^{-1}(\mathcal{C}(t)) \\ = \mathbb{1} + tS(x)\,\partial_X S^{-1}(x) - tS(x)B_X(x)S^{-1}(x) + o(t^2)\ ,$$

where $\mathcal{C}(0) = x$ and X is the tangent vector to $\mathcal{C}$ at zero, and we used $S(\mathcal{C}(t)) = S(x) + t\,\partial_X S(x) + o(t^2)$. This shows that the vector potential transforms as

$$B_X \to SB_XS^{-1} - S\,\partial_X S^{-1}\ . \tag{5.4}$$

Since the vector potential has values in the Lie algebra $\mathfrak{h}$, the transformed expression should be in the Lie algebra, as well. In fact, B_X is a representation of a Lie algebra element and S a representation of a Lie group element. The first summand in (5.4) is then a representation of the Lie algebra element $\mathrm{Ad}(S)B_X$. The second summand should be interpreted as the tangent vector at $t = 0$ of $\lambda(t) = S(\mathcal{C}(0))S^{-1}(\mathcal{C}(t))$ which is a path in H with $\lambda(0) = \mathbf{e}$, the identity of H. The tangent vector at $t = 0$ is an element of the Lie algebra.

5.2 Generalized Star Operation

In this section we will briefly mention the relation between automorphisms of the Lie algebra and Lie group and an abstractly defined adjoint operation on the parallel transporters and vector potentials. Conventionally, this is given by a scalar product in the fibers V_x. The basis in V_x given by a moving frame is required to be orthogonal with respect to this scalar product. The adjoint of a parallel transport $U(\mathcal{C})^*$ is given by the adjoint with respect to the scalar product, if the spaces V_x are identified via such a moving frame.

In order to be able to write down a Lagrangian density such an *adjoint* or *star operation* on the Lie algebra $\mathfrak{h}$ is usually needed, but no direct reference to a scalar product is needed. In conventional gauge theories a star operation is given by $B^* = -B$. This implies that the elements of H are unitary, i.e. $U^* = U^{-1}$. The star operation has to be an involutive *anti-automorphism*:

$$[A,B]^* = [B^*,A^*] = -\,[A^*,B^*]\ .$$

Anti-automorphisms are not to be confused with anti-linear automorphisms of a complex Lie algebra discussed in section 2.3.1. It turns out that the classification of linear automorphisms θ already classifies the anti-automorphisms, since $-\theta$ is an anti-automorphism. Therefore, let θ be an involution of the Lie algebra $\mathfrak{h}$ and define the star operation to be

$$X^* = -\theta(X)\ .$$

If H is a simply connected semi-simple Lie group, an automorphism of the Lie algebra $\mathfrak{h}$ induces an automorphism $\exp(X) \mapsto \exp(\theta(X))$ of H ([Hel78] proof of theorem VI.1.1). Then there also is a star operation on H:

$$(\exp(X))^* = \exp(-\theta(X)) = \exp(\theta(X))^{-1} ,$$

which satisfies $(gh)^* = h^* g^*$.

The Lagrangian density is invariant only under the "unitary" gauge transformations, i.e. under those $S \in H$ for which $S^* = S^{-1}$. These elements form the stability subgroup G of the automorphism of the Lie group and is called the *gauge group*. If H is a connected Lie group, it is the connected subgroup corresponding to the subalgebra of $\mathfrak{h}$ which is stable under the automorphism θ.

5.3 Gauge Transformations

We will now consider the case, where H is a connected semi-simple Lie group and θ is the Cartan involution of its Lie algebra $\mathfrak{h}$. If $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and $H = KP$ are the Cartan decompositions of $\mathfrak{h}$ and H, respectively, then the gauge group is $G = K$ and its Lie algebra $\mathfrak{g} = \mathfrak{k}$.

The vector potential $B_X(x) \in \mathfrak{h}$ decomposes according to the Cartan decomposition of the Lie algebra $\mathfrak{h}$:

$$B_X(x) = A_X(x) + \Xi_X(x) , \qquad A_X(x) \in \mathfrak{g} , \qquad \Xi_X(x) \in \mathfrak{p} .$$

$\Xi_X(x)$ may be viewed as a tangent vector to the symmetric space H/K. If H is simple, this is an irreducible Riemannian symmetric space.

If S is a gauge transformation with $S(x) \in G$ for all $x \in \mathbb{M}$, then the second summand in equation (5.4) will be an element of the subalgebra $\mathfrak{g}$ of $\mathfrak{h}$. Therefore the two fields A_X and Ξ_X transform according to

$$A_X \;\rightarrow\; S A_X S^{-1} - S\, \partial_X S^{-1} , \qquad \Xi_X \;\rightarrow\; S \Xi_X S^{-1} = \mathrm{Ad}_H(S)\Xi_X .$$

It is important that G acts on $\Xi_X \in \mathfrak{p}$ via the adjoint representation Ad_H of the holonomy group H. It is a representation of G which is not necessarily irreducible and need not be its adjoint representation. In fact, if it is the adjoint representation of G, then $\mathfrak{g}$ and $\mathfrak{p}$ are isomorphic as vector spaces and $\mathfrak{h}$ has the structure $\mathfrak{h} = \mathfrak{g} \oplus i\mathfrak{g}$ and is therefore the complexification of $\mathfrak{g}$, i.e. a complex Lie algebra viewed as real.

If $\mathbb{M}$ is replaced by a discrete graph, the analog of the infinitesimal vector potential B_X are the fundamental parallel transporters $U(x, y)$ between two neighboring points x and y along the edges in the graph. Consider the gauge transformation of its Cartan decomposition

$$U(x,y) = U_G U_P \longrightarrow S(y)\,U(x,y)\,S^{-1}(x) = S(y) U_G U_P S^{-1}(x)\ .$$

Either U_G or U_P should now be interpreted as a parallel transport from x to y. In order to retain a conventional gauge theory with gauge group G, this should be U_G, i.e. $U(x,y) = U_G(x,y) U_P^y(x)$. The notation $U_P^y(x)$ indicates that U_P transforms as a field at x but also depends on the direction of the parallel transport $U(x,y)$: If an identity $\mathbb{1} = S^{-1}(x)S(x)$ is inserted in the gauge transformation above, we get

$$U_G(x,y) \;\to\; S(y)\,U_G(x,y)\,S^{-1}(x) \qquad U_P^y(x) \;\to\; S(x)\,U_P^y(x)\,S^{-1}(x)\ .$$

While U^G and the infinitesimal A_X can be interpreted as parallel transporters and the corresponding vector potential with gauge group G, the transformation laws of U^P and Ξ_X are those of an ordinary vector field: $U_P^y(x)$ transforms under the gauge symmetry at one point $x \in \mathbb{M}$ and Ξ_X transforms under a representation of G without the derivative term characteristic for vector potentials of connections. This is closely related to polar decompositions of complex systems [dR01]. In fact, the Cartan decomposition is a kind of polar decomposition of group elements.

5.4 Higgs Potential

Consider a Lagrange density in a continuum theory which contains a potential term V which only depends on the field Ξ_X. It is well known from conventional gauge theories that there cannot be a potential term depending on the gauge field A_X. Since we assume that the theory is invariant under gauge transformations in G, the V potential has to be invariant:

$$V(\Xi_X) = V(\mathrm{Ad}_H(S)\Xi_X) \qquad \text{for all } S \in G\ .$$

Let $\mathfrak{a}_\mathfrak{p}$ be a maximally Abelian subalgebra of $\mathfrak{p}$ as in section 2.4.2. Each element of $\mathfrak{p}$ is conjugate to an element of $\mathfrak{a}_\mathfrak{p}$ under the adjoint action of $G = K$ (Theorem 6.51 in [Kna02]). As discussed in section 2.8.4, the closures of the Weyl chambers in $\mathfrak{a}_\mathfrak{p}$ are conjugate under the Weyl group $W(\Sigma)$ of the restricted root system Σ and each element $\omega \in W(\Sigma)$ may be represented by an $m_\omega \in \mathsf{M}^* \subset K = G$. Therefore, each element of $\mathfrak{p}$ is conjugate to an element of the closure of the fundamental Weyl chamber $\overline{\mathfrak{a}_\mathfrak{p}^+}$. I.e. for each $\Xi_X = \Xi_X(x) \in \mathfrak{p}$ there is an $S = S(x) \in G$ such that

$$\mathrm{Ad}_H(S)\,\Xi_X \in \overline{\mathfrak{a}_\mathfrak{p}^+}\ .$$

The potential V can therefore be viewed as a function on $\overline{\mathfrak{a}_\mathfrak{p}^+}$. It can also be viewed as a function on the vector space $\mathfrak{a}_\mathfrak{p}$ which is invariant under the action of the Weyl group $W(\Sigma)$ as done in [Leh04].

Since X is a tangent vector to the space-time manifold $\mathbb{M}$, the field Ξ_X is a vector field and has $D{+}1$ components. Above, we considered a potential, which depends on only one of these components. It can immediately be generalized to the case, where all components $\Xi_\mu(x)$ at one $x \in \mathbb{M}$ are proportional: In this situation V is a function of the whole $\mathfrak{a}_\mathfrak{p}$. If they are proportional with positive proportionality constants (which may depend on x), V is a function on the subset $\overline{\mathfrak{a}_\mathfrak{p}^+}$.

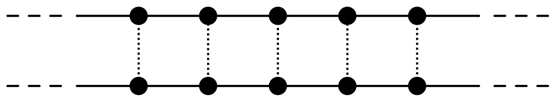
An example is a gauge field B_X which takes values in G except on a D dimensional submanifold ($D-1$-brane). On this brane, $B_X \in G$, if X is tangential to the brane. The potential is then a function of a scalar field $\Xi_{X(x)}(x)$ for x on the brane, where $X(x)$ is the normal vector of the brane at x. One assumes that the field $\Xi_{X(x)}(x)$ has values in $\overline{\mathfrak{a}_\mathfrak{p}^+}$, since a basis of V_x can be chosen, which contains $X(x)$ and otherwise elements tangent to the brane. This brane could be interpreted as the locus of a space-time defect [Leh04].

In the discrete case, the action functional may contain a potential term V which depends on the parallel transporters $U(x,y)$ for any pair of neighboring points x, y. It should be invariant under the gauge transformations (5.3):

$$V(U(x,y)) = V(S(y)U(x,y)S^{-1}(x)\ .$$

Since $S(y)$ and $S^{-1}(x)$ are independent elements of G, the analogous result to that obtained in the continuum follows from the KAK decomposition in theorem 12: The potential V is a function on $\overline{A_\mathfrak{p}^+}$. Since the exponential mapping $\exp() : \mathfrak{a}_\mathfrak{p} \to A_\mathfrak{p}$ is an isomorphism, it can also be viewed as a function on $\overline{\mathfrak{a}_\mathfrak{p}^+}$.

A similar obstruction as in the continuum case appears, if one considers two such parallel transporters $U(x,y)$ and $U(x,z)$ which have the same starting point x (similarly for the end points or one start and one end). If the gauge transformation $S(x)$ is used to transform $U(x,y)$ into $\overline{A_\mathfrak{p}^+}$, it is fixed and cannot be used to transform $U(x,z)$ anymore. It gets even worse, if there is yet another $U(x,u)$, which would be natural in a lattice gauge theory in two or more dimensions. This problem can be avoided by studying theories of the following type [Ang03, Leh04]. Consider a by-layered lattice like



where the parallel transporters along the solid lines take values only in G and only the ones along the dotted lines take values in the whole holonomy group H.

5.5 Symmetry Breaking

Consider a potential term $V\colon \overline{\mathfrak{a}_{\mathfrak{p}}^{+}} \to \mathbb{R}$ which assumes its minimum at points different from zero. The field Ξ_X assumes some vacuum expectation value $\Xi_{min} \in \overline{\mathfrak{a}_{\mathfrak{p}}^{+}}$ in the vacuum state, where Ξ_{min} is one of the minimum points of V. By this, the gauge symmetry is spontaneously broken. The unbroken gauge group U is the subgroup of G, which commutes with $\Xi_{min} \in \overline{\mathfrak{a}_{\mathfrak{p}}^{+}}$, i.e. the centralizer of Ξ_{min} in G. If H is a simple Lie group, it is classified by the following theorem:

Theorem 22 *Let H be a simple connected Lie group with Lie algebra $\mathfrak{h}$ and Cartan decompositions $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and $H = KP$. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. Let λ_i be a choice of simple restricted roots with respect to $\mathfrak{a}_{\mathfrak{p}}$ and h_{λ_i} a dual basis of $\mathfrak{a}_{\mathfrak{p}}$ as defined in section 2.4.2. For any $\Xi_{min} \in \mathfrak{a}_{\mathfrak{p}}$ let Θ be the set of simple restricted roots λ_i which are non-zero on Ξ_{min}. Then, the centralizer of Ξ_{min} in K is $\mathsf{M}_\Theta(K)$, which was defined in section 2.8 as the commutant in K of the subalgebra $\mathfrak{a}_{\mathfrak{p}_\Theta}$ of $\mathfrak{a}_{\mathfrak{p}}$ on which the elements of Θ are zero.*

Proof: With these definitions it easy to see that the elements of $\mathfrak{n}^{\pm}(\Theta)$ commute with Ξ_{min} and all elements of $\mathfrak{n}^{\pm}_{\Theta}$ do not. Therefore, the centralizer of Ξ_{min} in $\mathfrak{h}$ is $\mathfrak{l}_\Theta = \mathfrak{m} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}^{+}(\Theta) \oplus \mathfrak{n}^{-}(\Theta)$ and its intersection with $\mathfrak{g} = \mathfrak{k}$ is $\mathfrak{m}_\Theta(K)$. Accordingly, the centralizer in $G = K$ is $\mathsf{M}_\Theta(K)$. (See also [War72] p. 78)

End of Proof

With this result, the classification of parabolic subalgebras described in section 2.5 can be used, to find all possible unbroken subgroups U of the gauge group G which is the maximal compact subgroup of any simple holonomy group H. Let us consider the example $H = SU(m,n)$, for which $\mathfrak{m}_\Theta$ for all possible Θ was already given in (2.59). With $\mathfrak{m}_\Theta(K) = \mathfrak{m}_\Theta \cap \mathfrak{k}$ we get

$$\mathfrak{m}_\Theta(K) \cong \mathfrak{u}(1)^d \oplus \mathfrak{su}(m') \oplus su(n') \oplus \bigoplus_{i=1}^{N} \mathfrak{su}(n_i) \quad \text{where } N \leq d \text{ and}$$
$$\sum n_i = \tfrac{1}{2}(m+n-m'-n') - d + N \quad \text{and } m-n = m'-n' \,.$$

Let $M_\Theta(K)$ be the connected subgroup with Lie algebra $\mathfrak{m}_\Theta(K)$ as given above, then $\mathsf{M}_\Theta(K) = \mathsf{Z}^{\mathsf{M}} M_\Theta(K)$ gives the global structure of the unbroken gauge group, where Z^{M} is the subgroup of the center of H consisting of elements which are not in $M_\Theta(K)$. For Z^{M} the realization described in section 2.8.2 may be used.

Chapter 6

Conclusion

The general theory of complex and real Lie algebras was presented in chapter 2. In the presentation constructive aspects were emphasized. The main difficulty in the explicit construction of complex simple Lie algebras was the derivation of the structure constants $n_{\alpha,\beta}$ for the commutators $[e_\alpha,e_\beta] = n_{\alpha,\beta}e_{\alpha+\beta}$ such that the Jacobi identity is satisfied. An algorithm was derived with which they can be obtained in an efficient way.

The simple real Lie algebras are determined by automorphisms of the simple complex Lie algebras. (The simple complex Lie algebras viewed as real are also simple real Lie algebras, which are not obtained in this way.) Automorphisms were reviewed thoroughly and it was shown how they can be constructed. Two complementary ways to describe the structure of simple real Lie algebras by use of automorphisms of complex Lie algebras were presented. They are related to a maximally compact and a maximally non-compact Cartan subalgebra, respectively. The former is directly linked to the Cartan decomposition, the latter to the Iwasawa and Bruhat decompositions and to parabolic subalgebras.

The computer algebra package LambdaLie, developed by the author, is an implementation of the structure theory of complex and real simple Lie algebras. It is based on the abstract theory presented in this work. Starting from the novel algorithm for the determination of the structure constants of complex Lie algebras it is capable of representing subalgebras, automorphisms, real Lie algebras and parabolic subalgebras in a way which is useful for further studies within computer algebra. One application is the determination of invariants of the action of a Levi subgroup L_Θ in the quotient space G/B_Θ, where G is a simple real Lie group and B_Θ one of its parabolic subgroups.

The global structure of Lie groups was briefly discussed. The Iwasawa, KAK and Bruhat decompositions were presented. Of particular importance for the study of generalized conformal space-time symmetries is a generalized version of the Bruhat decomposition.

As the central result, the pairs of Lie groups G and space-time manifolds $\mathbb{M}$ which generalize conformal symmetry were classified in chapter 3 based on the following assumptions: The group G admits unitary positive energy representations and the manifold $\mathbb{M}$ admits a global causal structure invariant under the action of G. With these properties, local quantum field theories on $\mathbb{M}$ which are covariant under the action of G and satisfy the spectrum condition can be defined.

The remaining assumptions generalize conformal symmetry: The manifold $\mathbb{M}$ is isomorphic to a quotient G/H, where $H = B_\Theta$ is the identity component of a parabolic subgroup of G. The elements of the factors M_Θ, $A_{\mathfrak{p}\Theta}$, and N_Θ^- of the Langlands decomposition $B_\Theta = M_\Theta A_{\mathfrak{p}\Theta} N_\Theta^-$ are identified as generalizations of Lorentz transformations, dilatations, and special conformal transformations, respectively. Finally, it was assumed that the special conformal transformations act trivially on the tangent space of the base point $\mathbf{o} = [\mathbf{e}]_{B_\Theta}$, where $\mathbf{e}$ is the unit of G.

The only simple real Lie groups satisfying these assumptions are the universal coverings of

$$SO(2,D), \quad SO^*(4n), \quad SU(m,m), \quad Sp(l,\mathbb{R}), \quad \text{and} \quad E_{7(-25)}\ .$$

The parabolic subgroups satisfy a maximality condition and were be described explicitly with the tools provided. The existence of an automorphism of G which allows an interpretation as time reflection was studied. It turned out that only $SO(2,D)$ possesses such an automorphism. In all other cases space-time has an intrinsic chiral structure. An Euclidean version of the symmetry group cannot be a real form of the same complex Lie group unless the time reflection automorphism exists.

These generalized conformal symmetries are of interest, because non-perturbative results in conformal field theory which are based on the rich symmetry structure should transfer to quantum field theories with generalized conformal space-time symmetry. In chapter 4 the next steps in this direction were indicated. Related research fields in mathematics and physics which might give useful insights were also presented.

Finally, the techniques of Lie theory presented in chapter 2 were applied to study the symmetry breaking structure in a generalization of gauge theories in chapter 5. In these theories parallel transporters have values in a non-compact holonomy group H, and the action functional is symmetric only under gauge transformations in the maximally compact subgroup $G = K$ of H. A potential for the parallel transporters is allowed and has to satisfy a bi-invariance property. This can be exploited with the help of the KAK decomposition. It was shown that the unbroken gauge group is the intersection $\mathsf{M}_\Theta(K)$ of a parabolic subgroup B_Θ with the compact gauge group G. Our methods can be used for a complete classification of possible unbroken gauge groups.

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