Maximal Supergravity and Exceptional Symmetries

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Abstract

The group-theoretical analysis of symmetries proved to be a very useful tool in the understanding, description, and unification of various physical models. This thesis is devoted to exceptional symmetry groups that play a major role in the present picture of supergravity and string theory. Exceptional groups arise as global symmetries of dimensionally reduced supergravity. Furthermore, they occur as duality symmetries of M-Theory, the unified description of all superstring theories and supergravity. Since duality symmetries relate different sectors of the theory such as the strong and weak coupling regimes, they open a window to non-perturbative properties of the full quantum theory. In the present work, torus compactifications of $d=11$ supergravity and its global symmetries in various dimensions are reviewed. A possible incorporation of these symmetries into the eleven-dimensional theory is discussed. In two dimensions, where the theory becomes integrable, the infinite-dimensional symmetries are derived. Quantization of the resulting structure can be performed and yields a twisted Yangian double with central extension. A novel non-linear realization of all exceptional groups appearing in maximal supergravity is presented. This realization is quasiconformal in the sense that it leaves invariant a "generalized light-cone" in the representation space. Physical interpretations, possible applications and open problems are discussed in the last chapter.

Zusammenfassung

Die gruppentheoretische Untersuchung von Symmetrien ist ein wertvolles Hilfsmittel beim Verständnis, der Beschreibung und der Vereinheitlichung verschiedenster physikalischer Theorien. Die vorliegende Arbeit ist den Ausnahmegruppen gewidmet, die eine bedeutende Rolle für unser heutiges Bild von Supergravitation und Stringtheorie spielen. Ausnahmegruppen treten als globale Symmetrien in dimensional reduzierter Supergravitation auf. Außerdem beschreiben sie Dualitätssymmetrien von M-Theorie, der vereinheitlichten Beschreibung aller Superstringtheorien und Supergravitation. Da sie verschiedene Bereiche der Theorie, wie die starker und schwacher Kopplung, in Beziehung setzen, eröffnen sie einen Zugang zu störungstheoretisch nicht erfassbaren Eigenschaften der vollständigen Quantentheorie. In der vorliegenden Arbeit wird ein Überblick über Kompaktifizierungen von $d=11$ Supergravitation auf dem Torus und ihre globalen Symmetrien in den jeweiligen Dimensionen gegeben. Die Möglichkeit, diese Symmetrien in die elfdimensionale Theorie einzubinden, werden diskutiert. In zwei Dimensionen wird die Theorie integrabel und wir leiten ihre unendlich-dimensionale Symmetrie her. Die Symmetriestruktur kann quantisiert werden und führt zu einem getwisteten Yangian-Doppel mit zentraler Erweiterung. Eine neue nichtlineare Realisierung aller Ausnahmegruppen, die in maximaler Supergravitation auftreten, wird vorgestellt. Diese Realisierung ist quasikonform in dem Sinne, dass sie einen „verallgemeinerten Lichtkegel“ im Darstellungsraum invariant läßt. Das letzte Kapitel widmet sich physikalischen Interpretationen, Anwendungsmöglichkeiten und offenen Problemen.

To my parents

Contents

Chapter 1

Introduction

The quest for unification of gravitation and the standard model interactions into a quantum theory of gravity is nearly as old as the theories of quantum mechanics and general relativity themselves. In the seventies it was realized that string theory — originally developed in order to describe strong interactions — was a promising candidate for such a quantum theory of gravity as it contains a massless spin-2 particle that can be identified with the graviton.

During the "first superstring revolution", starting with the proof of the cancelation of anomalies in ten space-time dimensions [71], much progress was made in the understanding of perturbative string theory and its dimensional reduction to the four space-time dimensions we know from everyday experience. However, there were still many unsatisfactory aspects, one of them being the fact that perturbative string theory is not uniquely defined, but that there are five different consistent superstring theories in ten dimensions, leading to uncountably many different theories after reduction to four dimensions.

During the last six years much further progress in the understanding of string theory has been made. The starting point of this "second superstring revolution" was the discovery of so-called duality symmetries that relate different string theories or different limits (such as strong and weak coupling limit) of the same string theory. Using these dualities, it was realized that string theories have to contain not only strings but also extended objects of various dimensions, called D-branes.

The fact that all superstring theories are related by duality symmetries led to the conjecture of a more fundamental eleven-dimensional theory named M-Theory. Although M-Theory still is not really understood, it has been argued [166] that its low energy limit is described by eleven-dimensional supergravity [32]. This observation revived the interest in $d=11$ supergravity, a theory that has long been thought to play an important role in fundamental physics. In the past, interest in $d=11$ supergravity was motivated by its distinguished position as the highest-dimensional [138] supergravity theory that, after reduction to $d=4$, does not possess particles with spin greater than two. However, it suffered from its non-renormalizability and difficulties to extract realistic-looking theories in $d=4$ from it.

In this thesis we explore the symmetries of eleven dimensional supergravity as well as its counterparts in lower dimensions [151]. They are powerful tools to gain knowledge about supergravity and M-Theory. We will now discuss several aspects of these symmetries.

Duality symmetries

The concept of duality symmetries traces back to Dirac's [45] famous analysis of the interchange of electric and magnetic charge in four dimensions that exchanges strong and week coupling. An $SL(2,\mathbb{Z})$ symmetry has been conjectured acting in a similar way in type IIB string theory in ten dimensions [153] and on heterotic string theory in four dimensions [156] interchanging strong and week coupling, electrically and magnetically charged particles, and Bianchi identities with equations of motion. It has been named S-duality. Because of its non-perturbative nature a rigorous proof of S-duality is still beyond means, but several tests as the symmetry of the low energy effective action and the symmetry of the BPS charge spectrum have been performed.

Another discrete symmetry is the so-called T-duality that relates string theories in different backgrounds (see [70] and references therein). The simplest example is the fact that type IIB string theory, when compactified on a circle of radius R, is equivalent to the type IIB theory on a circle of radius α'/R (where α' is the sigma-model coupling constant). T-duality can be defined order by order in the string coupling constant and thus is a perturbative duality symmetry.

These different duality symmetries relate all the five pertubatively defined string theories. Another crucial observation was the fact that the $SL(2,\mathbb{Z})$ S-duality of type IIB string theory reminds on the modular symmetry symmetry known to arise in Kaluza-Klein [107, 116] type reductions on a torus. As mentioned before, the type IIB string theory is T-dual to the type IIA theory compactified to $d=9$ on a very small circle. This supported the conjecture that $d=9$ type IIA string theory is in fact an eleven-dimensional theory, reduced on a torus. This eleven-dimensional theory — named M-Theory— is believed to be fundamental and to reduce to the known superstring theories in certain limits [160, 166]. In the M-Theory picture the string coupling constant becomes geometrical, related to the length of the circle M-Theory is compactified on. In the strong coupling limit the circle decompactifies and string theory becomes eleven-dimensional supergravity [32]. In string perturbation theory where the string coupling constant is small, the eleventh coordinate is not visible.

Although it is not clear what M-Theory really is, it is widely believed that it should be a background independent pre-geometrical theory which gives rise to space-time geometry as a secondary concept. Furthermore it should poses a very rich symmetry structure related to the symmetries of eleven-dimensional supergravity.

Strong support of the deep relation between string theory and eleven-dimensional supergravity comes from their respective symmetry groups. It turned out that T-duality of compactified string theory and S-duality do not commute but generate the discrete so-called U-duality group [93]. In the case of type II string theory compactified to d dimensions on a $(11-d)$-torus the conjectured U-duality group is a discrete subgroup $E_{11-d(11-d)}(\mathbb{Z})$[1] of the global symmetry group of eleven-dimensional supergravity reduced to the same dimension.

Hidden symmetries

It is known for a long time that upon dimensional reduction of $d=11$ supergravity on a torus exceptional Lie algebras arise as so-called hidden symmetries [27, 28]. In the present work we will further investigate these hidden symmetries of supergravity, motivated by the idea that they are important for the understanding the symmetry structure of M-Theory itself.

The compactification of $d=11$ supergravity on an $(11-d)$-torus yields maximally extended supergravity in d dimensions. This theory in addition to the gravitational sector contains certain p-form fields $F^I_{(p)}$ and a set of scalar fields originating from the fields of $d=11$ supergravity after dimensional reduction.

[1] The real form $E_{11-d(11-d)}$ of the complex Lie algebra E_{11-d} is rigorously defined only for $d \leq 8$.

Its global symmetries are called "hidden" because they do not appear as symmetries acting on the original fields of the reduced Lagrangian. In order to make them explicit, certain fields of the theory have to be dualized[2]. In $d=3,...,10$ the equations of motion are symmetric under the hidden non-compact symmetry group G which is $E_{11-d(11-d)}$[3].

This symmetry can be made explicit by "maximal dualization" of all p-form fields of $F^I_{(p)}$, i.e. dualization of all differential forms to lowest possible rank. The p-form fields of rank $p < d/2$ then combine to yield irreducible representations of the global symmetry group G (acting on the index I that labels the fields $F^I_{(p)}$).

$$F_{(p)} \;\mapsto\; g\, F_{(p)}\,, \quad g \in G\,. \tag{1.1}$$

This way, the hidden symmetries can be realized as a global symmetry group of the Lagrangian in odd dimensions [28, 26, 99]. In even dimensions d it is not possible to implement the global symmetry on the level of the Lagrangian, because the $d/2$-form field strengths lie in the same irreducible multiplet of the hidden symmetry group as their Hodge-dual forms. However, regarding the $d/2$-forms and their duals as independent fields, one can introduce a so-called doubled Lagrangian and twisted self-duality constraint ensuring that these fields are dual to each other [31]. For example, in four dimensions the electric 2-form fields $F^I_{(2)}$ together with the magnetic fields $G^I_{(2)}$ defined by [28]

$$*G^I_{(2)} \;=\; -4\,\frac{\delta\mathcal{L}}{\delta F^I_{(2)}}\,, \quad (I = 1, \ldots, 28)\,, \tag{1.2}$$

form a vector $H_{(2)}$ in the **56** representation of $E_{7(7)}$

$$H_{(2)} \;\equiv\; \begin{pmatrix} F_{(2)} \\ G_{(2)} \end{pmatrix} \;\mapsto\; g\, H_{(2)}\,, \quad g \in G\,. \tag{1.3}$$

The scalar sector of the theory is governed by a non-linear σ-model G/H [16, 15]. The coset space G/H can be represented by a G-valued matrix $\mathcal{V}(x)$ containing all scalar degrees of freedom that transform under rigid G-transformations form the left and under local H-transformations from the right

$$\mathcal{V}(x) \;\mapsto\; g\, \mathcal{V}(x)\, h(x)\,, \quad g \in G\,, h(x) \in H\,. \tag{1.4}$$

The hidden symmetry group of supergravity is closely related to the U-duality group of string theory. In fact, the U-duality group of toroidally compactified type II string theory has been conjectured to be a discrete subgroup of the hidden symmetry group of supergravity [93]. The group $E_8(\mathbb{Z})$ appearing in three dimensions has been constructed recently in [137]. The construction of the groups $E_9(\mathbb{Z})$ and $E_{10}(\mathbb{Z})$ is still an open problem. These should be discrete subgroups of the infinite dimensional Lie groups E_9 and E_{10} conjectured to appear after reduction to two and one dimensions, respectively [99, 100, 140, 142].

[2] To be more precise, a Hodge duality operation has to be performed, which we will be denote by "dualization" to distinguish it from the S-, T-, and U-duality explained above.

[3] For $d > 5$ this is a classical group (see table 2.1).

Hidden symmetries in eleven dimensions

In addition to their appearance in dimensionally reduced supergravity and their beautiful mathematical structure there is one further reason why the hidden symmetry groups deserve attention. As it was shown a long time ago [40, 41, 140], already in eleven dimensions, supergravity permits reformulations where the tangent space symmetry $SO(1,10)$ is replaced by the local symmetries that would arise after reduction to four and three dimensions. In these constructions the dependence on all coordinates are retained and hence the symmetries are present not only in the dimensionally reduced theories but, at least in part, already in eleven dimensions. Thus, some part of the hidden symmetries of dimensionally reduced supergravity can be "lifted" to eleven dimensions thereby changing internal symmetries into spacetime symmetries. In this process the tensor gauge symmetry and the general coordinate invariance are unified into an exceptional symmetry acting on a "generalized vielbein" $e^m_{\mathcal{A}}$, defined in section 3.2, replacing the (inverse densitized) internal 8-bein ${e_a}^m$.

The generalized vielbeine are subject to algebraic constraints which follow from their explicit formulation in terms of the vielbein of $d=11$ supergravity [140, 133]. These constraints have no analogue in Riemannian geometry. We will solve these constraints in the case of $d=3$, $N=16$ supergravity making vast use of the special properties of the exceptional group $E_{8(8)}$. This gives additional evidence for an $E_{8(8)}$ structure in eleven dimensions.

Guided by the idea of the "lifting" of hidden internal symmetries in lower dimensions to the eleven-dimensional theory described above, which still is not understood completely, one might hope to proceed even further. After reduction of supergravity to two dimensions, the hidden symmetries become infinite dimensional [62, 100, 14] and one can try to lift them to eleven dimensions. If this were possible, it would certainly take us beyond $d=11$ supergravity as proposed in [143, 144]. The construction would involve the definition of an "unendlichbein" and the local gauge group $SO(16)^\infty$ would possibly contain and extend $N=16$ local supersymmetry.

Symmetries in two dimensions

The appearance of infinite dimensional symmetries in two dimensions is related to the fact that, in two dimensions, scalars are Hodge-dual to scalars[4]. Such symmetries were first encountered by Geroch [61] in Einstein gravity reduced to $d=2$. Later on, this was realized to be a more general property of gravity and supergravity theories [99, 97, 101]. In the case of maximal supergravity the resulting symmetry in $d=2$ is E_9 [97, 141].

The scalar sector of $d=2$, $N=16$ supergravity containing all 128 bosonic degrees of freedom is described by the σ-model $E_{8(8)}/SO(16)$ already present in $d=3$. The theory is integrable in the sense that the equations of motion are equivalent to the compatibility condition for a linear system of differential equations [141, 148]. The local gauge symmetry group extends to the infinite dimensional group $SO(16)^\infty$ that is the analogue of the maximal compact subgroup of E_9 (cf. [102]).

The canonical structure of this model and the Lie-Poisson realization of the associated infinite dimensional symmetry were analyzed only quite recently [121, 145]. As shown there, this structure is more involved than was originally thought. In particular, the affine Lie

[4] To be more precise, for all scalars ϕ in $d=2$, we have the relation $*d\phi = d*A$ with A again being a scalar field.

algebra seen at the level of the classical equations of motion is generated by a quadratic algebra of Yangian type in the canonical formulation.

One key feature of this result is that the quadratic algebra exhibited in [145], and therefore at least part of the model, can be quantized directly by replacing the Poisson algebra of charges with an exchange algebra involving a suitable R matrix, whereas a standard field theoretic quantization would appear to be prohibitively difficult.

Fortunately, the relevant R-matrix based on the exceptional group $E_{8(8)}$ has already been derived, namely in [23]. The structure that appears upon quantization is the Yangian $Y(\mathrm{e}_8)$[5]. As a consequence, the physical states of the quantized theory must belong to multiplets of $Y(\mathrm{e}_8)$ rather than multiplets of the affine algebra e_9 as one might have expected naively.

The Yangian of the exceptional algebra e_8 is distinguished from the Yangians of the classical Lie algebras by the fact that its fundamental representation is reducible over e_8, namely decomposes into $\mathbf{249} = \mathbf{1} \oplus \mathbf{248}$. The R-matrix associated with this representation has been given by Chari and Pressley in [23]. Using their result and the general analysis of Drinfeld [46] we obtain the RTT presentation of $Y(\mathrm{e}_8)$ which may be viewed [48], as the quantization of group-valued E_8 matrices endowed with the symplectic structure of dimensionally reduced gravity. The full quantum structure that appears upon quantization of the algebra of classical nonlocal charges is a centrally extended twisted version of the Yangian double, which reflects the $E_{8(8)}/SO(16)$ coset structure of the classical model.

The presence of this extra coset structure and its quantum consistency require further properties of the $Y(\mathrm{e}_8)$ R-matrix beyond those discussed in [23]. We explain these in detail in section 4.3. In particular, for a discrete set of values of the central extension, we find that the algebra $Y(\mathrm{e}_8)$ possesses nontrivial ideals which may be divided out to reduce the number of degrees of freedom. Somewhat surprisingly, there is only one among the altogether eight "exceptional" values of the central extension which admits a non-trivial ideal for which the quantum monodromy matrix becomes symmetric in the limit $\hbar \to 0$. The associated ideal can be divided out consistently to recover the classical coset space $E_{8(8)}/SO(16)$ of $N = 16$ supergravity. Remarkably, the relevant value of the central extension differs from the critical value for which the quantum algebra admits an additional infinite-dimensional center [150].

The U-duality conjecture

One of the main goals in the study of hidden symmetry groups of supergravity is their application in the context of string and M-Theory. Because of its non-perturbative nature the conjectured U-duality [93] symmetry of the full quantum string theory is hard to test. However, there are several predictions that can actually be tested in the low-energy regime. In the following, we will give some examples in which the conjectured U-duality can be tested.

First of all, as mentioned before, the low energy effective action for the massless modes of toroidally compactified type II string theory is given by maximal supergravity. Because the equations of motion are uniquely determined by the local symmetries, they cannot be changed by quantum corrections and have to be U-duality invariant from the beginning. Since the U-duality group is a discrete subgroup of the hidden symmetry group of supergravity, this is indeed the case.

[5]Here, and in the following, we denote the Lie algebra g of a Lie group G by lower case letters: e_8 is the Lie algebra of E_8.

Also, the allowed values for electric and magnetic charges that are restricted by the Dirac-Schwinger-Zwanziger [45, 154, 169] (DSZ) quantization condition should be U-duality invariant. Indeed, in $d=5$ where the electric and magnetic charges are parametrized by the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representation of $E_{6(6)}$, the DSZ quantization condition is given by the invariant combination in $\mathbf{27} \otimes \overline{\mathbf{27}}$ being an integer. In $d = 4$, where electric and magnetic charges together form the $\mathbf{56}$ representation of $E_{7(7)}$, the DSZ quantization condition demands the symplectic invariant of $E_{7(7)}$ (cf. eq. (5.40)) to take integer values.

The strongest tests of the U-duality conjecture concern certain states of the supergravity theory that satisfy a Bogomolny [13, 149] bound. The saturation of the so-called BPS condition causes a shortening of the supersymmetry multiplet and the corresponding states are believed to experience no quantum corrections [168], since renormalization should not change the representation theory.

BPS black holes

The supersymmetry algebra of extended supergravity permits central extensions [84] with central charges that correspond to electric and magnetic charges, as we will describe in the following. The most general super-Poincaré algebra in d dimensions even includes antisymmetric n-form charges $Z_{M_1,\dots,M_n}$ [162] that commute with all generators except the Lorentz generators. The objects that carry these charges are p-branes, extended objects with $(p+1)$-dimensional world volume [1]. In d-dimensional maximally extended supergravity all the n-form charges are fields coming from dimensional reduction of the eleven-dimensional supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} \;=\; (\Gamma^M C)_{\alpha\beta} P_M + \tfrac{1}{2!}(\Gamma^{MN} C)_{\alpha\beta} Z_{MN} + \tfrac{1}{5!}(\Gamma^{MNPQR} C)_{\alpha\beta} Z_{MNPQR}\,, \tag{1.5}$$

where $M, N, \dots = 0, \dots, 10$ are vector indices and $\alpha, \beta = 1, \dots, 32$ are spinor indices of $SO(11)$, and $C_{\alpha\beta}$ is the antisymmetric charge conjugation matrix [161]. The rank n of the central charges that appear depends on the dimension d.

The BPS condition relates the Arnowitt-Deser-Misner (ADM) mass [3] M_{ADM} of black hole solutions (or more generally: the tension of a p-brane) to the values of these charges [69, 37, 67] generalizing the bound obtained for Einstein-Maxwell theory [68, 65]. Of particular interest are black hole solutions in four and five dimensions. In $d=5$ there are 27 electric scalar charges $\tilde{Z}^{ij}$, 27 magnetic 1-form charges $\tilde{Z}^{ij}_\mu$ transforming in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ of $E_{6(6)}$, respectively, and another scalar central charge K originating from the 5-form in $d=11$. In $d=4$ there are 28 electric central charges Z^{ij} coming from $\tilde{Z}^{ij}$ and P_5 and 28 magnetic scalar central charges Z_{ij} coming from $\tilde{Z}^{ij}_5$ and K transforming under $\mathbf{28} \otimes \mathbf{28}$ of $SU(8)$. Therefore, we have the following central charge matrices [92] (Ω^{ij} is the symplectic invariant of $USp(8)$):

$$\begin{aligned} d = 5:&\quad Z^{ij} + K\Omega^{ij} &&\in \mathbf{27} \oplus \mathbf{1} \text{ of } USp(8)\,,\\ d = 4:&\quad Z^{ij} \pm \mathrm{i}\, Z_{ij} &&\in \mathbf{28} \oplus \mathbf{28} \text{ of } SU(8)\,. \end{aligned} \tag{1.6}$$

For supersymmetric solitonic solutions of maximal supergravity in these dimensions the ADM mass M_{ADM} is bounded from below by the moduli of the skew-eigenvalues $|z_n|$ ($n = 1, \dots, 4$) of the central charge matrix

$$M_{\mathrm{ADM}} \geq \max |z_n|\,. \tag{1.7}$$

The central charges Z depend on electric and magnetic charges defined as integrals of the field strengths over a hyper-surface Σ at spatial infinity and on the asymptotical values of the scalars of the theory at infinity. The asymptotic values of the scalar fields are called moduli as they parameterize the different vacua of the theory. The electric and magnetic charges p^I and q^I are defined as

$$\begin{aligned} d=5: \quad \mathcal{Z} \; &:= \oint_\Sigma F^I_{(2)} \; = \; (p^I)\,, \\ d=4: \quad \mathcal{Z} \; &:= \oint_\Sigma H^I_{(2)} \; = \; \begin{pmatrix} p^I \\ q^I \end{pmatrix}\,, \end{aligned} \tag{1.8}$$

with electrical 2-form field $F_{(2)}$ and and the combined electric-magnetic 2-form $H_{(2)}$ defined in (1.3). The charge vector $\mathcal{Z}$ has 27 components transforming under $E_{6(6)}$ in $d=5$ and 56 components transforming under $E_{7(7)}$ in $d=4$. The 28th component of the central charge matrix (1.6) can be shown [92] to correspond to the Taub-NUT charge as defined in [66]. Since this charge appears only for solutions that are not asymptotically flat, it is often neglected. However, we will point out that it has to be included in the analysis for $d<5$ in order to achieve U-duality invariance (see also the discussion in chapter 6). Whereas in $d=5$, the electrical charges and the scalar charge K (1.6) transform independently in the $\mathbf{27}\otimes\mathbf{1}$ representation of the U-duality group $E_{6(6)}$, in four dimensions they form a *single* irreducible **56** representation of the U-duality group $E_{7(7)}$. In three dimensions, these charges together with the charges corresponding to *all* the other physical scalar fields form an irreducible **248** representation of $E_{8(8)}$.

Inspection of the supersymmetry transformation shows that the 0-forms appearing as central charges of the supersymmetry algebra are not the bare charges $\mathcal{Z}$ as defined in (1.8), but the "dressed" charges $\overline{\mathcal{Z}}$ defined by

$$\overline{\mathcal{Z}} := \mathcal{V}_\infty^{-1}\mathcal{Z} \tag{1.9}$$

with $\mathcal{V}_\infty$ the asymptotic value of the group-valued matrix $\mathcal{V}$ of the scalar fields [93]. Since $\overline{\mathcal{Z}}$ is invariant under the action of the U-duality group, the same is true for the mass of states saturating the Bogomolny bound (1.7) as required by the U-duality conjecture.

Entropy of BPS black holes

The entropy of a black hole solution can be defined macroscopically by the Bekenstein-Hawking entropy formula [8, 88] and microscopically by the counting of micro-states using D-brane techniques (see [130] and references therein). It has been shown that the entropy of $N=2$ BPS black holes is independent of the moduli because the scalar fields flow towards the black hole horizon to fixed values given by rational functions in terms of the quantized charges [53, 158]. These fixed values correspond to the extremized value for the ADM mass in the moduli space [51, 52]. The entropy is given by the square of the ADM mass at this point in $d=4$ and by the ADM mass to the power 3/2 in $d=5$.

The entropy in all cases is given by an U-duality invariant expression that is independent of the moduli and only contains the electric and magnetic charges. In the case of $N=8$ supergravity in $d=4$, the entropy is found to correspond to the unique quartic E_7 invariant in its **56** representation (cf. section 5.2.4) [105] and in $d=5$ to the cubic E_6 invariant in the **27** representation (cf. section 5.1.3) [52].

$$\mathcal{I}_3 := \tilde{Z}^{ij}\tilde{Z}_{jk}\tilde{Z}^{kl}\Omega_{il} \tag{1.10}$$

$$\begin{aligned}\mathcal{I}_4 := & Z^{ij}Z_{jk}Z^{kl}Z_{li} - \tfrac{1}{4}Z^{ij}Z_{ij}Z^{kl}Z_{kl} + \tfrac{1}{96}\,\epsilon^{ijklmnpq}Z_{ij}Z_{kl}Z_{mn}Z_{pq} \\ & + \tfrac{1}{96}\,\epsilon_{ijklmnpq}Z^{ij}Z^{kl}Z^{mn}Z^{pq}\,.\end{aligned} \tag{1.11}$$

The dependence on the moduli (1.9) drops out in both cases, as expected.

A classification of BPS solutions preserving a different number of supersymmetries where given in an U-duality invariant form in [54]. The orbits under the U-duality group action $E_{7(7)}$ and $E_{6(6)}$ in $d=4$ and $d=5$, respectively, where given in [34, 50]. The $E_{6(6)}$ invariant in the **27** representation and the $E_{7(7)}$ invariant in the **56** representation provide the spaces of charges in $d=5$ and $d=4$ with a cubic and quartic norm, respectively. The resulting structure which in the $d=5$ case is the 27-dimensional exceptional Jordan algebra $J_3^{\mathbb{O}}$ is the starting point for the conformal and quasiconformal realizations of exceptional groups which we will describe in the following.

(Quasi)conformal realizations

Another important property of exceptional symmetry groups which we will discuss in this thesis is a novel realization as groups acting on certain vector spaces in a way that is analogous to the conformal group acting on Minkowski space.

It is an old idea to define generalized space-times by association with Jordan algebras J, in such a way that the space-time is coordinatized by the elements of J, and that its rotation, Lorentz, and conformal group can be identified with the automorphism, reduced structure, and the linear fractional group of J, respectively [73, 74, 75]. The aesthetic appeal of this idea rests to a large extent on the fact that key ingredients for formulating relativistic quantum field theories over four dimensional Minkowski space extend naturally to these generalized space times; in particular, the well-known connection between the positive-energy unitary representations of the four dimensional conformal group $SU(2,2)$ and the covariant fields transforming in finite dimensional representations of the Lorentz group $SL(2,\mathbb{C})$ [125, 124] extends to all generalized space-times defined by Jordan algebras [77].

All these conformal realizations possess a three-graded structure

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \tag{1.12}$$

corresponding to translations, Lorentz transformations with dilatations, and special conformal transformations, respectively. One exceptional example which we will work out in section 5.1.3 is the conformal realization of $E_{7(7)}$ on a 27-dimensional space. This representation space can be identified with $\mathfrak{g}^{-1}$, the space of grade -1. However, some of the algebras we are interested in, such as E_8 and G_2, do not admit a three-grading (1.12) but only a five-grading

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}\,. \tag{1.13}$$

Our main result, to be described in section 5.2, states that a “quasiconformal” realization is still possible on a space of dimension $\dim(\mathfrak{g}^1) + 1$ if the top grade spaces $\mathfrak{g}^{\pm 2}$ are one-dimensional. Five graded Lie algebras with this property are closely related to the so-called Freudenthal Triple Systems [57, 134], which were originally studied to provide alternative constructions of the exceptional Lie groups[6]. This relation will be made explicit in section 5.2.3. The novel realization of $E_{8(8)}$ which we will arrive at, together with its natural extension to $E_{8(-24)}$, contains various other constructions of exceptional Lie algebras by truncation, including the conformal realizations based on a three graded structure.

With the 56 coordinates interpreted as the electric and magnetic charges, the quartic norm could be interpreted as the entropy of BPS black hole solutions[105, 54]. The interpretation of the 57th coordinate cannot completely be settled on the level of charges without analyzing explicit solutions. However, we give some evidence in section 5.4 that it is related to the NUT charge [63]. We work out the explicit quasiconformal action on the physical fields in the truncation to $d=4$, $N=2$ [164] and also discuss the truncation to the simple $d=5$, $N=2$ Maxwell-Einstein supergravity theory [22].

It is natural from our point of view to define a distance in the charge space between any two black hole solutions using our eqs. (5.50), (5.51). If two black hole solutions are light-like separated in this space, they will remain so under the action of $E_{8(8)}$[7].

We have also worked out the detailed relation between the quartic $E_{7(7)}$ invariant in its $SU(8)$ and $SL(8,\mathbb{R})$ (cf. section 5.2.4). The $SU(8)$ basis is relevant for the central charges (1.6), which appear in the superalgebra via surface integrals at spatial infinity and determine the structure (and length) of BPS multiplets. By contrast, the 28 electric and 28 magnetic charges (1.8) carried by the solitons of $d=4$, $N=8$ supergravity transform separately under $SL(8,\mathbb{R})$ [28].

For applications to M-Theory, it would be important to obtain the exponentiated version of our realization. One might reasonably expect that modular forms with respect to a fractional linear realization of the arithmetic group $E_{8(8)}(\mathbb{Z})$ will play a role in this. We expect that our results will pave the way for the explicit construction of such modular forms. According to [95, 96] these would depend on 28+1 = 29 variables, such that the 57-dimensional Heisenberg subalgebra of $E_{8(8)}$ exhibited here would be realized in terms of 28 "coordinates" and 28 "momenta". Consequently, the 57 dimensions in which $E_{8(8)}$ acts might alternatively be interpreted as a generalized Heisenberg group, in which case the 57th component would play the role of a variable parameter $\hbar$.

The action of $E_{8(8)}(\mathbb{Z})$ on the 57 dimensional Heisenberg group would then constitute the invariance group of a generalized Dirac quantization condition. This observation is also in accord with the fact that the term modifying the vector space addition in $\mathbb{R}^{57}$ (cf. eq. (5.50)), which is required by $E_{8(8)}$ invariance, is just the cocycle induced by the standard canonical commutation relations on an (28+28)-dimensional phase space.

Using the quasiconformal representation, we can define $28+1$ oscillators corresponding to coordinates and the corresponding $28+1$ momenta. In terms of these operators we find a "bosonic construction" of $E_{8(8)}$ extending previous work that was limited to groups that possess a three-graded Jordan structure [81, 80].

[6]The more general Kantor-Triple-Systems for which $\mathfrak{g}^{\pm 2}$ have more than one dimension, will not be discussed in this thesis.

[7]For the exceptional $N=2$ Maxwell-Einstein supergravity [82, 83] defined by the exceptional Jordan algebra the U-duality groups in five and four dimensions are $E_{6(-26)}$ and $E_{7(-25)}$, respectively. The quasiconformal symmetry of the exceptional supergravity in four dimensions is hence $E_{8(-24)}$, with the maximal compact subgroup $E_7 \times SU(2)$.

Plan of the thesis

In chapter 2 we review the construction of maximal supergravity in various dimensions by successive toroidally compactification starting from eleven-dimensional supergravity. In particular, we discuss the global symmetries of the respective Lagrangians following the uniform presentation in [30]. These global symmetries can be enlarged by dualization of certain fields in the Lagrangian and this way one unveils the famous hidden exceptional symmetries of maximal supergravity in $d=5,4,3$.

In chapter 3 we first introduce our E_8 conventions. The decomposition of $E_{8(8)}$ with respect to $SO(16)$ and $SL(8)$ as well as operators projecting out the irreducible representations contained in the tensor product $\mathbf{248}\otimes\mathbf{248}$ are worked out explicitly. Then we use an alternative version of $d=11$ supergravity with local $SO(16)$ invariance [140] to realize, in eleven dimensions, parts of the $E_{8(8)}$ symmetry that up to now is known only in the reduction to three dimensions.

In chapter 4 we focus on the $N=16$ supergravity in two dimensions. Here, the global symmetries become infinite dimensional and they act on an infinite set of nonlocal charges that parameterize the phase space. After quantization we find a twisted Yangian double with central extension c which can be considered as a "quantum Geroch" acting on the space of physical states like the usual Geroch group acts on the space of physical solutions. The central charge is uniquely fixed by the correct classical limit.

In chapter 5 we first introduce the concept of conformal realizations by the example of the conformal group of Minkowski space. Then we give a novel non-linear quasiconformal realization of $E_{8(8)}$ on a space of 57 dimensions. We also give the possible truncation of our construction including all other exceptional Lie groups. We use the truncations to $SU(2,1)$ and $G_{2(2)}$, respectively, to illustrate the quasiconformal action in the physical context of supergravity. As another application, we construct a 29-dimensional oscillator-like representation of $E_{8(8)}$.

In chapter 6 we give a short summary of the results obtained and discuss some open questions. We also point out possible applications of the present work. We give some speculative physical interpretation of the peculiar properties of E_8 we observed in its quasiconformal construction.

Part of this research has been published in Refs. [119], [120], and [78].

Chapter 2

Exceptional Symmetries of Maximal Supergravity

Eleven dimensional supergravity [32] occupies a distinguished position among the various supergravity theories. First of all, $d=11$ is the maximal dimension in which a supergravity theory can be constructed that, after dimensional reduction, possesses no particle with spin greater than two in four dimensions [138]. Furthermore, it is believed to describe the low energy regime of M-Theory, which is argued to be the strong coupling limit of ten-dimensional type IIA string theory [166]. It gives rise to maximal supergravities by Kaluza-Klein type reduction to lower dimensions [152, 33].

One of the most remarkable facts about these dimensionally reduced supergravity theories is the emergence of hidden symmetries of exceptional type [28, 99, 26]. It is conjectured that, after quantization, discrete subgroups of these exceptional groups become the U-duality groups of toroidally compactified type II string theory [93]. In this chapter, we discuss how the exceptional groups $E_{6(6)}$, $E_{7(7)}$, and $E_{8(8)}$ appear as global symmetry groups of the bosonic sector of maximal supergravity theories in $d=5$, $d=4$, and $d=3$.

The scalar degrees of freedom form a nonlinear σ-model G/H [16]. The coset space G/H is a non-compact Riemannian symmetric space [91] with the non-compact group G acting as a global symmetry group and the maximal compact subgroup H of G acting as a local gauge group. In the final section, we review the basic properties of the reduction to $d=2$, where the global symmetry group becomes infinite dimensional.

We will see that the global symmetry groups $E_{11-d(11-d)}$ in d dimensions are manifest only after Hodge-dualization of certain gauge potentials [30]. In fact, all gauge potentials have to be dualized to forms of the lowest possible degree in order to render the equations of motion invariant under $E_{11-d(11-d)}$. In odd dimensions the Lagrangian is also invariant under the global symmetry group, whereas in even dimensions one has to introduce a doubled Lagrangian with a twisted self-duality constraint (cf. section 2.4.3 for the $d=4$ case).

Dualization exchanges Noether charges with topological charges, equations of motion with Bianchi identities, and internal rigid symmetries with gauge symmetries. In particular, the rigid symmetry group is not invariant under dualization. Therefore it has been suggested that the global symmetry group should be combined with all gauge symmetries into the so-called F(ull)-duality group [103, 98].

2.1 $N=1$ supergravity in $d=11$

In this section we briefly recall the definition and basic symmetry properties of eleven-dimensional supergravity [32] in our notation and conventions.

Field content

The field content of $d=11$ supergravity is impressively simple. It consists of the "elfbein" $E_M{}^A$ with flat index[1] A and curved index M, both running from 0 to 10, the Majorana spin-3/2 field ψ_M, and the fully antisymmetric gauge tensor field A_{MNP}. The eleven-dimensional metric

$$g_{MN} \; = \; E_M{}^A E_N{}^B \eta_{AB} \tag{2.1}$$

has signature $(-,+,\ldots,+)$ in our conventions.

Supersymmetry

The transformation laws under supersymmetry are given by

$$\begin{aligned}
\delta E_M{}^A \; &= \; \tfrac{1}{2}\bar{\varepsilon}\tilde{\Gamma}^A\psi_M \,, \\
\delta A_{MNP} \; &= \; -\tfrac{3}{2}\bar{\varepsilon}\tilde{\Gamma}_{[MN}\psi_{P]} \,, \\
\delta\psi_M \; &= \; 2\,D_M\varepsilon + \tfrac{1}{144}(\tilde{\Gamma}_M{}^{ABCD} - 8E_M{}^A\tilde{\Gamma}^{BCD})\varepsilon F_{ABCD} \,,
\end{aligned} \tag{2.2}$$

where $\tilde{\Gamma}$ are $d=11$ Γ-matrices which obey the Clifford algebra relation

$$\{\tilde{\Gamma}^A\,,\tilde{\Gamma}^B\} \; = \; 2\delta^{AB}\,\mathbb{1}\,, \quad \tilde{\Gamma}^{A_1\ldots A_{11}} = -\mathrm{i}\epsilon^{A_1\ldots A_{11}} \,, \tag{2.3}$$

and the covariant derivative D_M is defined by

$$D_M \; = \; \partial_M - \tfrac{1}{4}\omega_{MAB}\tilde{\Gamma}^{AB} \tag{2.4}$$

with the spin connection ω_{MAB}. The spin connection can be calculated from the coefficients of anholonomity Ω_{ABC}, defined as

$$\Omega_{ABC} \; = \; 2E_{[A}{}^M E_{B]}{}^N \partial_M E_{NC} \,,$$

using the formula

$$\omega_{MAB} \; = \; \tfrac{1}{2}E_M{}^C(\Omega_{ABC} - \Omega_{CAB} - \Omega_{BCA}) \,.$$

Lagrangian

The Lagrangian of $N=1$, $d=11$ supergravity is given by

$$\begin{aligned}
\mathcal{L} \; = \; & ER + \tfrac{1}{2}E\bar{\psi}_M\tilde{\Gamma}^{MNP}D_N\psi_P - \tfrac{1}{48}EF_{MNPQ}F^{MNPQ} \\
& + \tfrac{1}{(12)^4}\epsilon^{MNPQRSTUVWX}F_{MNPQ}F_{RSTU}A_{VWX} \\
& - \tfrac{1}{192}EF_{MNPQ}(\bar{\psi}_R\tilde{\Gamma}^{MNPQRS}\psi_S + 12\bar{\psi}^M\tilde{\Gamma}^{NP}\psi^Q)
\end{aligned} \tag{2.5}$$

[1] In the following, we will denote flat vector indices by $A,B,\ldots$ and curved world indices by $M,N,\ldots$. We hope that no confusion results from our double usage of $A,B,\ldots$ as both $SO(1,10)$ vector and $SO(16)$ spinor indices. It should always be clear from the context which is meant.

up to higher order fermionic terms. Here we have introduced the field strength F_{MNPQ} associated with the gauge field A_{MNP}

$$F_{MNPQ} \;=\; 4\,\partial_{[M}A_{NPQ]}\,, \tag{2.6}$$

which is subject to the Bianchi identity

$$\partial_{[M}F_{NPQR]} \;=\; 0\,, \tag{2.7}$$

and the determinant E of the elfbein $E_M{}^A$

$$E \;=\; \det E_M{}^A\,. \tag{2.8}$$

In the following we will confine ourselves to the bosonic sector of the theory and consistently set all fermionic fields to zero. The Lagrangian (2.5) then reads[2]

$$\mathcal{L} \;=\; ER - \tfrac{1}{48}EF_{(4)}^2 + \tfrac{1}{6} * (F_{(4)}\wedge F_{(4)}\wedge A_{(3)})\,, \tag{2.9}$$

where we denote by $F_{(n)}$ the n-form which is related to the components of the antisymmetric rank n tensor by

$$F_{(n)} \;=\; \tfrac{1}{n!}F_{M_1,\dots,M_n}dx^{M_1}\wedge\dots\wedge dx^{M_n}\,,$$

and the Hodge operator $*$ is defined by

$$*(dx^{M_1}\wedge\dots\wedge dx^{M_n}) \;=\; \tfrac{1}{(11-n)!}\epsilon_{N_1\dots N_{11-n}}{}^{M_1\dots M_n}dx^{N_1}\wedge\dots\wedge dx^{N_{11-n}}\,.$$

Symmetries of the action

The Lagrangian of $d=11$ supergravity exhibits several symmetries. First of all it is of course symmetric under local $N=1$ supersymmetry transformations (2.2), general $d=11$ coordinate transformations and local $SO(1,10)$ Lorentz transformations. As usual, the supersymmetry algebra closes on shell into general coordinate transformations and Lorentz transformations. General coordinate transformations with parameter ξ^M are given by

$$\begin{aligned}
\delta E_M{}^A &= \partial_M\xi^N E_N{}^A + \xi^N\partial_N E_M{}^A\,,\\
\delta A_{MNP} &= 3\,(\partial_{[M}\xi^Q)A_{NPQ]} + \xi^Q\partial_Q A_{MNP}\,,\\
\delta\psi_M &= \partial_M\xi^N\psi_N + \xi^N\partial_N\psi_M\,,
\end{aligned} \tag{2.10}$$

and the Lorentz transformations with parameter $\Omega_{AB}=-\Omega_{BA}$ are given by

$$\begin{aligned}
\delta E_M{}^A &= \Omega^A{}_B E_M{}^B\,,\\
\delta\psi_M &= \Omega_{AB}\tilde\Gamma^{AB}\psi_M\,.
\end{aligned} \tag{2.11}$$

[2]We write $F_{(4)}^2$ as an shortcut for the 0-form with coefficient $F_{MNPQ}F^{MNPQ}$. To be more precise, one should write $\frac{1}{2}E * (F_{(4)}\wedge *F_{(4)})$ for the term $-\frac{1}{48}EF_{(4)}^2$.

Furthermore the action is invariant under Abelian tensor gauge transformations with antisymmetric parameter ξ_{MN}, e.g.

$$\delta A_{MNP} \;=\; \partial_{[M}\xi_{NP]}\,. \tag{2.12}$$

This is not a symmetry of the Lagrangian because of the explicit appearance of A_{MNP} in the so-called Chern-Simons term and one has to use the Bianchi identity (2.7) to show that any gauge transformation only produces a total derivative which vanishes upon integration.

Note that there is another transformation

$$\begin{aligned}\delta g_{MN} \;&=\; \lambda^2 g_{MN}\,,\\ \delta A_{MNP} \;&=\; \lambda^3 A_{MNP}\,,\end{aligned} \tag{2.13}$$

which gives a homogeneous rescaling of the action and thus is a symmetry of the equations of motion.

2.2 Dimensional reduction of $d=11$ supergravity

In this section we discuss toroidal compactifications of 11-dimensional supergravity to d dimensions. One can directly discover a global $GL(N,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry and after dualization of certain fields one finds the exceptional symmetries which we are mainly interested in [30]. We proceed by successive compactifications on circles by the standard Kaluza-Klein [107, 116] procedure, closely following [123]. In order to reduce the $d+1$-dimensional theory to d dimensions, we split the coordinates $x^{\hat{M}}$ of the $(d+1)$-dimensional spacetime according to $x^{\hat{M}} = (x^M, z)$, where $\hat{M} = 0,\ldots,d$ and z is the coordinate of the extra dimension. Accordingly, the vielbein splits as

$$E_{\hat{M}}{}^{\hat{A}} = \begin{pmatrix} e^{\alpha\phi}E_M{}^A & e^{-(d-2)\alpha\phi}B_M \\ 0 & e^{-(d-2)\alpha\phi}\end{pmatrix}\,, \tag{2.14}$$

where $E_M{}^A$, ϕ, and $B_{(1)} = B_M dx^M$ are taken to be independent of the extra coordinate z and the constant α is given by $\alpha^{-2} = 2(d-1)(d-2)$. As we will see below, in this parameterization the action of supergravity takes a convenient form after dimensional reduction. For further simplification, we modify the definition of the field strength in lower dimensions to include so-called Kaluza-Klein corrections[3] and define

$$F_{(n)} \;:=\; dA_{(n-1)} - dA_{(n-2)}\wedge B_{(1)} \tag{2.15}$$

and analogously

$$\mathcal{F}_{(n)} \;:=\; dB_{(n-1)} - dB_{(n-2)}\wedge B_{(1)}\,. \tag{2.16}$$

In this convention the Einstein term of the action reduces to a pure Einstein action, a kinetic term for the dilaton ϕ, and a kinetic term for $\mathcal{F}_{(2)}$:

$$E^{(d+1)}R^{(d+1)} \;\to\; E^{(d)}R^{(d)} - \tfrac{1}{2}E^{(d)}(\partial\phi)^2 - \tfrac{1}{4}E^{(d)}e^{-2(d-1)\alpha\phi}\mathcal{F}^2_{(2)}\,, \tag{2.17}$$

[3] As mentioned in [30] this denotation is more precise than the term "Chern-Simons corrections", which is often used in this context, as they do not originate from the Chern-Simons term $F\wedge F\wedge A$ in higher dimensions.

where the upper index of R and E denotes the respective dimension. For ease of notation we will drop this extra index. The $(d+1)$-dimensional gauge potential $A_{(n)}$ reduces to the z-independent gauge potentials $A_{(n)}$ and $A_{(n-1)}$ according to

$$A_{(n)} \;\rightarrow\; A_{(n)} + A_{(n-1)} \wedge dz\,. \tag{2.18}$$

A kinetic term for the $(d+1)$-dimensional field strength $F_{(n)}$ reduces to kinetic terms for the d-dimensional field strengths $F_{(n)}$ and $F_{(n-1)}$ with the correct normalization:

$$\tfrac{1}{2n!}\,E\,F_{(n)}^2 \;\rightarrow\; \tfrac{1}{2n!}\,E\,e^{-2(n-1)\alpha\phi}F_{(n)}^2 - \tfrac{1}{2(n-1)!}\,E\,e^{2(d-n)\alpha\phi}F_{(n-1)}^2 \tag{2.19}$$

The above procedure can be applied successively to the 11-dimensional theory to get a maximal supergravity theory in d dimensions. The original eleven-dimensional fields $E_M{}^A$ and A_{MNP} then give rise to several fields in d dimensions:

$$\begin{aligned} E_M{}^A \;&\rightarrow\; E_\mu{}^\alpha\,,\ \vec{\phi}\,, \qquad B^m_{(1)}\,, \qquad B^m_{(0)n}\,, \\ A_{(3)} \;&\rightarrow\; A_{(3)}\,,\ A_{(2)m}\,,\ A_{(1)mn}\,,\ A_{(0)mnp}\,, \end{aligned} \tag{2.20}$$

where $\mu, \nu, \alpha = 0, \ldots, d-1$, and $m, n = d, \ldots, 10$. The gauge potentials $A_{(1)mn}$ and $A_{(0)mnp}$ are antisymmetric in their indices, whereas the Kaluza-Klein potentials $B^m_{(0)n}$ coming from successive reductions by one dimension are only defined for $n > m$. The $(11-d)$ scalar fields $\vec{\phi}$ coming from diagonal terms of the metric are dilatons. They are the moduli characterizing the size of the compactifying space. All other scalar fields, as e.g. $A_{(0)mnp}$, we denote as axions.

The expression for the n-form field strengths $F_{(n)}$ with Kaluza-Klein modifications according to (2.15) in terms of the pure exterior derivatives becomes increasingly complicated as one proceeds reducing to lower dimensions. Their explicit form in various dimensions is listed in [123]. In the following, we will need the field strength of the scalar fields $A_{(0)mnp}$ which takes the form

$$F_{(1)mnp} \;=\; \gamma^q{}_m \gamma^r{}_n \gamma^s{}_p dA_{(0)qrs}\,, \tag{2.21}$$

where $\gamma^m{}_n$ is the inverse of the matrix $\tilde{\gamma}^m{}_n = \delta^m_n + B^m_{(1)n}$ which can be consistently defined by the power series

$$\gamma^m{}_n \;=\; \delta^m_n - B^m_{(1)n} + B^m_{(1)p}B^p_{(1)n} - \cdots\,. \tag{2.22}$$

It terminates after finitely many terms because $B^m_{(1)n}$ is defined only for $m < n$.

The Kaluza-Klein corrected field strengths $\mathcal{F}_{(n)}$ for the Kaluza-Klein potentials $B^m_{(n-1)}$ according to (2.16) are defined by

$$\begin{aligned} \mathcal{F}^m_{(2)} \;&=\; dB^m_{(1)} - \gamma^n{}_p(dB^m_{(0)n}) \wedge B^p_{(1)}\,, \\ \mathcal{F}^m_{(1)n} \;&=\; \gamma^p{}_n\, dB^m_{(0)p}\,. \end{aligned} \tag{2.23}$$

The Chern-Simons term of the Lagrangian $\frac{1}{6} * (F_{(4)} \wedge F_{(4)} \wedge A_{(3)})$ reduces to terms of the form $*(F_{(p)} \wedge F_{(q)} \wedge A_{(r)})$ with $p+q+r = d$ in d dimensions. Here we have omitted all internal indices $m, n, \ldots$, which have to be contracted with an internal ϵ symbol. We will denote all terms coming from the reduction of the Chern-Simons term $F \wedge F \wedge A$ by $\mathcal{L}_{FFA}$. The explicit values for dimensions $2 \leq d < 11$ are given in [123].

Lagrangian

The bosonic part of the Lagrangian of 11-dimensional supergravity toroidally compactified to to d dimensions has the form [123]

$$\begin{aligned}\mathcal{L} \;=\; & ER - \tfrac{1}{2}E(\partial\vec{\phi})^2 - \tfrac{1}{48}Ee^{\vec{a}\cdot\vec{\phi}}F_{(4)}^2 - \tfrac{1}{12}E\sum_m e^{\vec{a}_m\cdot\vec{\phi}}F_{(3)m}^2 - \tfrac{1}{4}E\sum_{m<n} e^{\vec{a}_{mn}\cdot\vec{\phi}}F_{(2)mn}^2 \\ & -\tfrac{1}{4}E\sum_m e^{\vec{b}_m\cdot\vec{\phi}}(\mathcal{F}_{(2)}^m)^2 - \tfrac{1}{2}E\sum_{m<n<p} e^{\vec{a}_{mnp}\cdot\vec{\phi}}F_{(1)mnp}^2 - \tfrac{1}{2}E\sum_{m<n} e^{\vec{b}_{mn}\cdot\vec{\phi}}(\mathcal{F}_{(2)n}^m)^2 \\ & +\mathcal{L}_{FFA}\,, \end{aligned} \tag{2.24}$$

where the constants $\vec{a}$, $\vec{a}_m$, $\vec{a}_{mn}$, $\vec{a}_{mnp}$, $\vec{b}_m$, and $\vec{b}_{mn}$ are vectors with $(11-d)$ components characterizing the coupling of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. The vectors $\vec{a}$ describe the coupling to the various n-form field strengths originating from the gauge field strength F_{MNPQ}, whereas the vectors $\vec{b}$ describe the coupling to the field strengths originating from the vielbein $E_M{}^A$. They are given by [30]

$$\begin{aligned}&\vec{a} = -\vec{g}\,, \quad \vec{a}_m = \vec{f}_m - \vec{g}\,, \quad \vec{a}_{mn} = \vec{f}_m + \vec{f}_n - \vec{g}\,, \quad \vec{a}_{mnp} = \vec{f}_m + \vec{f}_n + \vec{f}_p - \vec{g} \\ &\vec{b}_m = -\vec{f}_m\,, \quad \vec{b}_{mn} = -\vec{f}_m + \vec{f}_n\,,\end{aligned} \tag{2.25}$$

where the $\vec{f}_m$ and $\vec{g}$ in terms of a $(11-d)$ dimensional orthonormal basis $\{\vec{e}_m\}$ are given by

$$\vec{f}_m \;=\; \sqrt{2}\,\vec{e}_m + \frac{3-\sqrt{d-2}}{11-d}\vec{g}\,, \qquad \vec{g} \;=\; \sqrt{\frac{2}{d-2}}\sum_{m=1}^{11-d}\vec{e}_m\,. \tag{2.26}$$

In the following section, we will cast the scalar part of the Lagrangian (2.24) into a form which is explicitly invariant under the global symmetry group G. It turns out that one has to Hodge-dualize some fields in order to get the maximal number of scalar fields and to find the exceptional symmetry groups $G = E_{11-d(11-d)}$. In this construction, the vectors of (2.25) that describe the coupling to scalar fields will correspond to the root lattice of the symmetry group G. The Lie algebra generators associated with the roots will then act as shift operators on the corresponding scalar fields.

2.3 Hidden symmetries

The $SL(11-d,\mathbb{R})$ part of the general coordinate transformations $GL(11,\mathbb{R})$ (2.10) along the internal directions ξ^m, $m = d,\ldots,10$ automatically becomes an internal symmetry of the reduced d-dimensional theory. It leaves the size of the internal torus fixed. The additional $\mathbb{R}$-part that is missing from $GL(11-d,\mathbb{R})$ changes the volume of the internal torus and is not a symmetry of the reduced theory. However, it can be combined with the scaling symmetry (2.13) to the full $GL(11-d,\mathbb{R})$ subgroup of $GL(11,\mathbb{R})$ as an internal symmetry.

Additionally, we have a shift symmetry acting on the 0-form potentials $A_{(0)mnp}$ as a remnant of the gauge symmetry (2.12). Since the scalars $A_{(0)mnp}$ and their transformations have to be independent of the internal coordinates, the original local gauge symmetry reduces to global shifts

$$\delta A_{(0)mnp} = c_{mnp} , \tag{2.27}$$

with arbitrary antisymmetric constant parameter c_{mnp}. These transformations commute since they originate from Abelian gauge symmetries and we have a $\mathbb{R}^q$ symmetry on the $q = \binom{11-d}{3}$ scalars. The shifts do not commute with the above internal coordinate transformations as $A_{(0)mnp}$ also transforms under $GL(11-d, \mathbb{R})$. Thus the resulting global symmetry in d dimensions is

$$GL(11-d, \mathbb{R}) \ltimes \mathbb{R}^q , \tag{2.28}$$

with $\ltimes$ denoting the semi-direct product of the groups.

The symmetries (2.28) are called "obvious" symmetries of d-dimensional supergravity as they are symmetries of the dimensionally reduced Lagrangian. However, it is a striking fact that these groups can be extended to symmetry groups E_{11-d} of the exceptional type for $3 \leq d \leq 5$. More precisely, we will realize the maximal non-compact real form $E_{11-d(11-d)}$[4].

In dimensions $d=6, \ldots, 10$ this is a genuine enhancement, as the group $GL(11-d, \mathbb{R}) \ltimes \mathbb{R}^q$ is a subgroup of the hidden symmetry group $E_{11-d(11-d)}$. One can see from Table 2.1 that at least the Abelian part $\mathbb{R}^q$ of the obvious symmetries is contained in the maximal Abelian subgroup $\mathbb{R}^p$ of the hidden symmetries[5]. Although the Lie group $E_{11-d(11-d)}$ is defined only

d	10	9	8	7	6
$E_{11-d(11-d)}$	$\mathbb{R}$	$GL(2, \mathbb{R})$	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SL(5, \mathbb{R})$	$O(5,5)$
max. $\mathbb{R}^p \subset E_{11-d(11-d)}$	$\mathbb{R}^1$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^6$	$\mathbb{R}^{10}$
$\mathbb{R}^q$	$\emptyset$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}^4$	$\mathbb{R}^{10}$

Table 2.1: Hidden symmetries in $d=6, \ldots, 10$

for $d=3,4,5$, in table 2.1 we extend the definition to the cases $d=6, \ldots, 10$.

In dimensions $d = 3,4,5$ the symmetry group $GL(11-d, \mathbb{R}) \ltimes \mathbb{R}^q$ is not contained in the group of hidden symmetries $E_{11-d(11-d)}$ as one can see from table 2.2. But also in these dimensions the group of hidden symmetries are much bigger than $GL(11-d, \mathbb{R}) \ltimes \mathbb{R}^q$ and they are of exceptional type. These are the cases we are mainly interested in, and in the following we will see how one can recover the exceptional symmetry groups by "maximal dualization".

2.4 Dualization

We have seen that the Lagrangian of d-dimensional supergravity has $GL(11-d, \mathbb{R}) \ltimes \mathbb{R}^q$ as a global symmetry. In this section, we will show how we can modify the Lagrangian by

[4] The groups $E_{11-d(11-d)}$ can consistently be defined for $2 < d < 10$ after dualization of certain fields in the Lagrangian (see Table 2.1 and Table 2.2).

[5] For each Lie algebra there are several maximal Abelian sub-algebras corresponding to different possible gradings of the algebra [128]. They are unique up to conjugation and we only list the dimension p of the largest of them here.

d	5	4	3
$E_{11-d(11-d)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
max. $\mathbb{R}^p \subset E_{11-d(11-d)}$	$\mathbb{R}^{16}$	$\mathbb{R}^{27}$	$\mathbb{R}^{36}$
$\mathbb{R}^q$	$\mathbb{R}^{20}$	$\mathbb{R}^{35}$	$\mathbb{R}^{56}$

Table 2.2: Hidden symmetries in $d=3,4,5$

dualization of certain fields to get the bigger symmetry group $E_{11-d(11-d)}$ of exceptional type. It turns out that all n-form fields in the Lagrangian which reduce their rank by dualization (i.e. $n > d/2$) have do be dualized. Only in dimensions 3, 4, and 5 one can produce additional scalars by dualization of $(d-2)$-form fields. And indeed, it can be shown, that for $d=6,\dots,10$ the scalar Lagrangian already possesses $E_{11-d(11-d)}$ symmetry with $GL(11-d,\mathbb{R}) \ltimes \mathbb{R}^q$ as a subgroup[6] [30].

The axions which are all the scalar fields other than the dilaton coming from dimensional reduction will correspond to the positive roots of $E_{11-d(11-d)}$. As we will describe, one can read off all dot products of axion vectors from the Dynking diagram

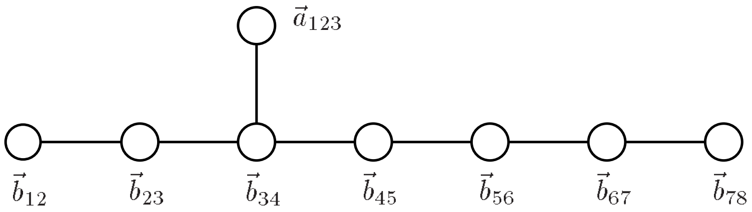

Figure 2.1: The Dynking diagram of $E_{11-d(11-d)}$ generated by axion fields

2.4.1 Maximal supergravity in $d>5$

In dimensions $5< d<8$ it is not possible to dualize higher order n-form fields to get additional scalars, so the scalar Lagrangian just contains the axions $B^m_{(0)n}$ and $A_{(0)mnp}$ originating from the dimensional reduction of the vielbein $E_M{}^A$ and the gauge field A_{MNP}, respectively. The associated dilaton vectors $\vec{b}_{mn}$ and $\vec{a}_{mnp}$ obey the relations

$$\vec{b}_{mn} + \vec{b}_{np} \;=\; \vec{b}_{mp}\,, \qquad \vec{a}_{mnp} + \vec{b}_{pq} \;=\; \vec{a}_{mnq}\,, \tag{2.29}$$

as one can easily see from (2.25). Since any vector $\vec{b}_{mn}$ and $\vec{a}_{mnp}$ can be obtained as an integer linear combination of $\vec{b}_{12}$, $\vec{b}_{23}, \dots, \vec{b}_{(10-d)(11-d)}$, and $\vec{a}_{123}$, we can regard the vectors $\vec{b}_{mn}$ and $\vec{a}_{mnp}$ as a root system of a Lie algebra with positive roots $\vec{b}_{12}$, $\vec{b}_{23}, \dots \vec{b}_{(10-d)(11-d)}$, and $\vec{a}_{123}$.

We define the generators $E_m{}^n$ and E^{mnp} associated with the roots $\vec{b}_{mn}$ and $\vec{a}_{mnp}$, respec-

[6] In $d=6,\dots,8$ this is a proper subgroup, whereas in dimensions $d=9,10$ the groups coincide.

tively. They obey the commutation relations

$$
\begin{aligned}
[E_m{}^n, E_p{}^q] &= \delta^n_p E_m{}^q - \delta^q_m E_p{}^n\,, &(2.30)\\
[E_m{}^n, E^{pqr}] &= -3\delta^{[p}_m E^{qr]n}\,, &(2.31)\\
[E^{mnp}, E^{qrs}] &= 0\,. &(2.32)
\end{aligned}
$$

Equation (2.30) is the defining relation for the Lie algebra $SL(11-d,\mathbb{R})$, whereas equations (2.31) and (2.32) state that the E^{mnp} are commuting generators transforming covariantly under $SL(11-d,\mathbb{R})$. We also define the generators $\vec{H}$ corresponding to shifts in the dilaton fields $\vec{\phi}$ which are vectors whose component are the $(11-d)$ Cartan generators of the Lie algebra defined above[7]. They obey the commutation relations

$$
[\vec{H}\,,\, E_m{}^n] = \vec{b}_{mn} E_m{}^n\,, \qquad [\vec{H}\,,\, E^{mnp}] = \vec{a}_{mnp} E^{mnp} \qquad \text{(no sum)}. \tag{2.33}
$$

The positive roots together with the generators of the Cartan subalgebra span the so-called Borel subalgebra[8] of any Lie algebra. This Borel subalgebra can be exponentiated to get the Borel subgroup of the corresponding Lie group. Now define an group-valued field $\mathcal{V}$ with values in the Borel subgroup defined above by $\mathcal{V} = \mathcal{V}_1\mathcal{V}_2\mathcal{V}_3$ with

$$
\begin{aligned}
\mathcal{V}_3 &= e^{\sum_{m<n<p} A_{(0)mnp} E^{mnp}}\,,\\
\mathcal{V}_2 &= \prod_{m<n} U_{mn} = U_{12}U_{13}U_{14}\cdots U_{23}U_{24}\cdots\,,\\
\mathcal{V}_1 &= e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\,,
\end{aligned} \tag{2.34}
$$

where U_{mn} is defined by

$$
U_{mn} = e^{B^m_{(0)n} E_m{}^n} \quad \text{(no sum)}\,,
$$

with the ordering of the U_{mn} indicated in the definition of $\mathcal{V}_2$ (2.34). Consider the corresponding Lie algebra element

$$
\mathcal{V}^{-1}d\mathcal{V} = \mathcal{V}_3^{-1}\mathcal{V}_2^{-1}(\mathcal{V}_1^{-1}d\mathcal{V}_1)\mathcal{V}_2\mathcal{V}_3 + \mathcal{V}_3^{-1}(\mathcal{V}_2^{-1}d\mathcal{V}_2)\mathcal{V}_3 + \mathcal{V}_3^{-1}d\mathcal{V}_3\,. \tag{2.35}
$$

We have

$$
\begin{aligned}
\mathcal{V}_2^{-1}(\mathcal{V}_1^{-1}d\mathcal{V}_1)\mathcal{V}_2 &= \sum_{m<n<p} F_{(1)mnp}E^{mnp}\,,\\
\mathcal{V}_2^{-1}d\mathcal{V}_2 &= \sum_{m<n} \mathcal{F}^m_{(1)n} E_m{}^n\,,\\
\mathcal{V}_3^{-1}d\mathcal{V}_3 &= \tfrac{1}{2}\vec{\phi}\cdot\vec{H}\,,
\end{aligned}
$$

[7]We note again that these shift have to be combined with the scaling transformation (2.13) to become a symmetry of the Lagrangian.

[8]The Borel subalgebra is defined as the maximal solvable subalgebra and, in a suitable basis, corresponds to upper triangular matrices.

where $F_{(1)mnp}$ and $\mathcal{F}^m_{(1)n}$ are the Kaluza-Klein corrected field strengths defined in (2.21) and (2.23), respectively. The corrections of $\mathcal{F}^m_{(1)n}$ can be calculated with the commutation relation (2.30) using the Baker-Campbell-Hausdorff (BCH) formula and paying attention to the ordering in the definition of $\mathcal{V}_2$. The corrections of $F_{(1)mnp}$ originate from the conjugation with $\mathcal{V}_2$ and can be calculated using the BCH formula and the commutation relation (2.31). Using the commutation relations (2.33) we finally arrive at

$$\mathcal{V}^{-1}d\mathcal{V} \;=\; \tfrac{1}{2}\vec{\phi}\cdot\vec{H} + \sum_{m<n} e^{\frac{1}{2}\vec{b}_{mn}\cdot\vec{\phi}}\mathcal{F}^m_{(1)n}E_m{}^n + \sum_{m<n<p} e^{\frac{1}{2}\vec{a}_{mnp}\cdot\vec{\phi}}F_{(1)mnp}E^{mnp}\,.$$

Scalar Lagrangian

The scalar part of the Lagrangian then can be rewritten as

$$\begin{aligned}\mathcal{L} \;&=\; ER + \tfrac{1}{4C}E\,\mathrm{tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})\\ &=\; ER - \tfrac{1}{4C}E\,\mathrm{tr}((\mathcal{V}^{-1}d\mathcal{V})^{\#}(\mathcal{V}^{-1}d\mathcal{V})^{\#}) - \tfrac{1}{4C}E\,\mathrm{tr}((\mathcal{V}^{-1}d\mathcal{V})(\mathcal{V}^{-1}d\mathcal{V}))\\ &\quad -\tfrac{1}{2C}E\,\mathrm{tr}(\mathcal{V}^{-1}d\mathcal{V}(\mathcal{V}^{-1}d\mathcal{V})^{\#})\,,\end{aligned} \tag{2.36}$$

where $\mathcal{M}$ is defined by

$$\mathcal{M} \;:=\; \mathcal{V}\mathcal{V}^{\#}\,, \tag{2.37}$$

as usual [16] and the involution $^{\#}$ is defined by

$$\mathcal{V}^{\#} \;:=\; \tau(\mathcal{V}^{-1})\,, \tag{2.38}$$

where τ is the Cartan involution which leaves invariant the compact generators of the group G. (Conversely, we could have defined the maximal compact subgroup H of G by $h^{\#}h = \mathbb{1}$ for all elements $h \in H$.) The Cartan-Killing form is given by

$$(g|h) = \tfrac{1}{C}\,\mathrm{tr}(gh^{\#})\,, \tag{2.39}$$

where the normalization constant C is eigenvalue of the quadratic Casimir operator in the adjoint representation of G. In this normalization we have

$$\begin{aligned}\mathrm{tr}(H_aH_b^{\#}) \;&\equiv\; \mathrm{tr}(H_aH^b) &&= C\,\delta_a^b\\ \mathrm{tr}(E_a{}^bE_c^{\#d}) \;&\equiv\; \mathrm{tr}(E_a{}^bE_d{}^c) &&= C\,\delta_a^c\delta_b^d\\ \mathrm{tr}(E^{abc}E^{\#def}) \;&\equiv\; \mathrm{tr}(E^{abc}E_{def}) &&= 6\,C\,\delta^d_{[a}\delta^e_b\delta^f_{c]}\end{aligned} \tag{2.40}$$

In order to get rid of the factor C in the following, we define the "renormalized" trace Tr by

$$\mathrm{Tr}(g) \;:=\; \tfrac{1}{C}\,\mathrm{tr}(g)\,. \tag{2.41}$$

Symmetries

From the Lagrangian in the form (2.36) one can observe its global symmetries. The Lagrangian is invariant under the multiplication of $\mathcal{V}$, which is itself an element of the Borel subgroup of G, by an arbitrary constant element $g \in G$ from the left hand side

$$\mathcal{V}(x) \;\mapsto\; g\mathcal{V}(x)\,. \tag{2.42}$$

Furthermore we have local gauge invariance under multiplication of $\mathcal{V}$ by an space-time dependent element $h(x)$ of the maximal compact subgroup $H \subset G$ from the right hand side.

$$\mathcal{V}(x) \;\mapsto\; \mathcal{V}(x)h(x)\,. \tag{2.43}$$

This transformation leaves $\mathcal{M}$ invariant and we can always use it to restore the Borel gauge of $\mathcal{V}$ after the transformation (2.42). This shows that the scalar sector of $d>5$ supergravity is given by the coset manifold G/H with internal metric $\mathcal{M}$.

Higher rank fields

We have restricted ourselves to the scalar part of the Lagrangian. For odd dimensions d it can be indeed shown that after dualization of all fields exceeding rank $d/2$, the entire Lagrangian is invariant under the respective global symmetry group [30]. In even dimensions the fields of rank $d/2$ together with their duals form an irreducible representation of the global symmetry group. In order to achieve invariance one has to introduce a so-called doubled Lagrangian containing all fields and their duals and a twisted self-duality condition [30, 31] (see also section 2.4.3).

2.4.2 $N=8$ supergravity in $d=5$

In $d=5$ we can dualize the 3-form gauge potential $A_{(3)}$ coming from dimensional reduction of the eleven-dimensional gauge field A_{MNP} to obtain an additional scalar field. The gauge potential $A_{(3)}$ appears in the Lagrangian (2.24) only in the form of its field strength $F_{(4)}$. We introduce a Lagrange multiplier φ to force the Bianchi identity $dF_{(4)} = 0$ to hold[9]. The additional term $\varphi\,, dF_{(4)}$ gives an equation equation of motion which can be solved for $F_{(4)}$ and yields

$$F_{(4)} \;=\; e^{-\vec{a}\cdot\vec{\phi}} * G_{(1)} \tag{2.44}$$

with the 1-form field strength $G_{(1)}$, which is dual to $F_{(4)}$, defined by

$$G_{(1)} \;=\; d\varphi - \tfrac{1}{72}A_{(0)mnp}dA_{(0)qrs}\epsilon^{mnpqrs}\,. \tag{2.45}$$

This way we trade the 3-form gauge field $A_{(3)}$ for its dual scalar φ and to the scalar Lagrangian is added a term of the form

$$\mathcal{L}_{\text{dual}} \;=\; -\tfrac{1}{2}Ee^{-\vec{a}\cdot\vec{\phi}}G_{(1)}^2\,. \tag{2.46}$$

[9]Actually, this equation is subject to correction because of the corrections (2.15) to $F = dA$, but we can neglect them here because we are only interested in the scalar part of the Lagrangian.

The shift transformation in the new axion field φ does not commute with the global shift of the gauge field $A_{(0)mnp}$ (2.27) because of the explicit appearance of $A_{(0)mnp}$ in the definition of $G_{(1)}$ (2.45). One finds

$$\vec{a}_{mnp} + \vec{a}_{qrs} \; = \; -\vec{a}\,, \tag{2.47}$$

when m, n, p, q, r, s are all different in accordance with (2.25) and (2.26). This relation defines a new generator D with commutation relations

$$\begin{aligned} [E^{mnp}, E^{qrs}] \; &= \; -\epsilon^{mnpqrs}\,, \quad [\vec{H}, D] \; = \; -\vec{a}D\,, \\ [E_m{}^n, D] \; &= \; 0\,, \quad [E^{mnp}, D] \; = \; 0\,, \end{aligned} \tag{2.48}$$

replacing relation (2.32).

Scalar Lagrangian

We introduce an extra factor $\mathcal{V}_4 = e^{\varphi D}$ to parameterize the most general field $\mathcal{V} = \mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\mathcal{V}_4$ with values in the Borel subgroup. The Lie algebra element $\mathcal{V}^{-1}d\mathcal{V}$ then gets extra Kaluza-Klein type corrections from the non-vanishing commutators in (2.48) and we have

$$\begin{aligned} \mathcal{V}^{-1}d\mathcal{V} \; = \; & \tfrac{1}{2}\vec{\phi}\cdot\vec{H} + \sum_{m<n} e^{\frac{1}{2}\vec{b}_{mn}\cdot\vec{\phi}}\mathcal{F}^m_{(1)n}E_m{}^n + \sum_{m<n<p} e^{\frac{1}{2}\vec{a}_{mnp}\cdot\vec{\phi}}F_{(1)mnp}E^{mnp} \\ & + e^{-\frac{1}{2}\vec{a}\cdot\vec{\phi}}G_{(1)}D\,. \end{aligned}$$

Inserted in formula (2.36) this gives exactly the Lagrangian of $d=5$ supergravity with the 4-form field strength $F_{(4)}$ dualized. The global symmetry of the Lagrangian is E_6 as we can read off from the dynking diagram. The coset manifold of the scalar sector of the theory is $E_6/USp(8)$.

Field content

The fields of the $d=5$, $N=8$ supergravity are

		spin	$USp(8)$ (local)	$E_{6(6)}$ (global)
the graviton:	$E_\mu{}^\alpha \in GL(5,\mathbb{R})/SO(4,1)$	2	**1**	**1**
8 gravitinos:	ψ^a_μ	3/2	**8**	**1**
27 vector fields:	A^{mn}_μ	1	**1**	**27**
48 spin-1/2 fermions:	χ^{abc}	1/2	**48**	**1**
42 scalars:	$\mathcal{V}_{mn}{}^{ab} \in E_{6(6)}/USp(8)$	0	**27**	$\overline{\mathbf{27}}$

where $\alpha, \mu = 1, ..., 5$ are flat and curved spacetime indices, respectively, a, b, c are (flat) $USp(8)$ indices, and mn is a (curved) double index labeling the **27** of $E_{6(6)}$.

For an extensive discussion of $d=5$, $N=8$ supergravity, the reader is referred to [26], where the full $E_{6(6)}$-invariant Lagrangian including higher rank fields and fermions, the equations of motion and truncations to the cases $N=6,4,2$ are discussed.

2.4.3 $N=8$ supergravity $d=4$

In the case of $d=4$ supergravity, the 3-form field strength $F_{(3)m}$ can be dualized to give additional scalars φ_m, $(m=1,\ldots,7)$. We can introduce them as seven Lagrange multipliers caring for the Bianchi identities for the the field strengths $F_{(3)m}$, which is given by

$$d(\tilde{\gamma}^m{}_n F_{(3)m}) = 0\,, \tag{2.49}$$

with the matrix $\tilde{\gamma}^m{}_n$ defined in (2.22) coming from Kaluza-Klein modifications and again up to further non-scalar Kaluza-Klein modifications we can neglect here. The additional equations of motion can be solved algebraically for $F_{(3)m}$ and give

$$F_{(3)} \;=\; e^{-\vec{a}_m\cdot\vec{\phi}} * G^m_{(1)} \tag{2.50}$$

with the 1-form field strength $G_{(1)}$ dual to $F_{(4)}$, defined by

$$G^m_{(1)} \;=\; \tilde{\gamma}^m{}_n(d\varphi^n + \tfrac{1}{72}A_{(0)mnp}dA_{(0)qrs}\epsilon^{mnpqrs})\,. \tag{2.51}$$

The scalar Lagrangian then gets an additional term of the form

$$\mathcal{L}_{\rm dual} \;=\; -\tfrac{1}{2}E\sum_m e^{-\vec{a}_m\cdot\vec{\phi}}(G^m_{(1)})^2\,, \tag{2.52}$$

For the shift transformation in the new axion fields φ^m we find

$$\vec{a}_{mnp} + \vec{a}_{qrs} \;=\; -\vec{a}_t\,, \tag{2.53}$$

where m,n,p,q,r,s,t are all different. This corresponds to (2.25) and (2.26) and now this relation replaces (2.47) and defines seven new generator D_m with commutation relations

$$\begin{aligned}
&[E^{mnp},E^{qrs}] \;=\; -\epsilon^{mnpqrst}D_t\,, \quad [\vec{H},D_m] \;=\; -\vec{a}_m D_m\,, \quad [E_m{}^n,D_p] \;=\; \delta^n_p D_m\,,\\
&\;[E^{mnp},D_q] \;=\; 0\,,
\end{aligned} \tag{2.54}$$

replacing relation (2.32).

Scalar Lagrangian

Again, we introduce an extra factor $\mathcal{V}_4$ to parameterize the most general field $\mathcal{V}=\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\mathcal{V}_4$ with values in the Borel subgroup as we did in $d=5$. Now, it is defined as $\mathcal{V}_4 = e^{\varphi^m D_m}$. The Lie algebra element $d\mathcal{V}\mathcal{V}^{-1}$ with all its Kaluza-Klein type corrections is then given by

$$\begin{aligned}
\mathcal{V}^{-1}d\mathcal{V} \;=\;& \tfrac{1}{2}\vec{\phi}\cdot\vec{H} + \sum_{m<n} e^{\frac{1}{2}\vec{b}_{mn}\cdot\vec{\phi}}\mathcal{F}^m_{(1)n}E_m{}^n + \sum_{m<n<p} e^{\frac{1}{2}\vec{a}_{mnp}\cdot\vec{\phi}}F_{(1)mnp}E^{mnp}\\
&+ e^{-\frac{1}{2}\vec{a}_m\cdot\vec{\phi}}G^m_{(1)}D_m\,.
\end{aligned}$$

Inserted in formula (2.36) this gives the Lagrangian of $d=4$ supergravity with all 3-form field strengths $F_{(3)i}$ dualized. The global symmetry of the Lagrangian is E_7 and the coset manifold of the scalar sector of the theory is $E_7/SU(8)$.

Doubled Lagrangian

The only additional fields of higher degree that appear are the 28 vector fields $A_{(1)mn}$ and $B^m_{(1)}$. They form together with their duals a **56** representation of $E_{7(7)}$ [28]. This is a general phenomenon in reductions to even dimensions and following [30] we will sketch how one can define a double Lagrangian invariant under the global symmetry group.

After dualization of the 21 vectors $A_{(1)mn}$ they form a **28** representation of $SL(8,\mathbb{R})$ together with the 7 vectors $B^m_{(1)}$. We denote their associated field strengths by $F^{ij}_{\mu\nu}$ where the indices i,j now run from 1 to 8. Furthermore we define 28 additional vectors $G^{ij}_{\mu\nu}$ by

$$*G^{ij}_{\mu\nu} \;=\; -4\,\frac{\delta\mathcal{L}}{\delta F^{ij}_{\mu\nu}}\,. \tag{2.55}$$

They are thus linear combinations of $F^{ij}_{\mu\nu}$ and $*F^{ij}_{\mu\nu}$ also containing the scalars of the theory. The Lagrangian is given by

$$\mathcal{L} \;=\; ER + \tfrac{1}{4}E\,\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) + \tfrac{1}{8}EF^{ij}_{\mu\nu} * G^{\mu\nu}_{ij}\,, \tag{2.56}$$

and its global invariance group is only $SL(8,\mathbb{R})$. As before, the matrix $\mathcal{M}$ parameterizes the $E_{7(7)}/SU(8)$ sigma model. At the level of the equations of motion the $SL(8,\mathbb{R})$ symmetry extends to an $E_{7(7)}$ symmetry under which the vectors $F^{ij}_{\mu\nu}$ and $F_{ij\,\mu\nu}$ together form a **56** representation. These vectors obey the so-called twisted self-duality condition [28, 30]

$$H_{(2)} \;=\; \Omega\mathcal{M} * H_{(2)}\,, \tag{2.57}$$

with

$$H_{(2)} \;=\; \begin{pmatrix} F_{(2)} \\ G_{(2)} \end{pmatrix} \quad \text{and} \quad \Omega \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{2.58}$$

Now we can regard the fields F and G as independent, only constrained by (2.57) and write down the $E_{7(7)}$-invariant doubled Lagrangian

$$\mathcal{L} \;=\; ER + \tfrac{1}{4}E\,\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) - \tfrac{1}{8}EH^T_{(2)}\mathcal{M} * H_{(2)}\,. \tag{2.59}$$

Field content

The fields of the $d=4$, $N=8$ supergravity are

		spin	$SU(8)$ (local)	$E_{7(7)}$ (global)
the graviton:	$E_\mu{}^\alpha \in GL(4,\mathbb{R})/SO(3,1)$	2	**1**	**1**
8 gravitinos:	ψ^a_μ	3/2	**8**	**1**
56 vector fields:	A^{mn}_μ, B^{mn}_μ	1	**1**	**28**
56 spin-$\frac{1}{2}$ fermions:	χ^{abc}	1/2	**56**	**1**
70 scalars:	$\mathcal{V}_{mn}{}^{ab} \in E_{7(7)}/SU(8)$	0	**56**	**56**

where $\alpha, \mu = 1, \dots, 4$ are flat and curved spacetime indices, respectively, $a, b, c = 1, \dots, 8$ are (flat) $SU(8)$ indices, and $mn = 1, \dots, 56$ is a (curved) double index labeling the **56** of $E_{7(7)}$. The fields A_μ^{mn} and B_μ^{mn} are the potentials of the electric and magnetic field strengths $F_{(2)}$ and $G_{(2)}$, respectively.

A detailed discussion of maximal $d = 4$ supergravity can be found in [28], where the hidden symmetries of exceptional type were recovered for the first time.

2.4.4 $N=16$ supergravity in $d=3$

In three dimensions, we can dualize both the 2-form field strengths $F_{(2)mn}$ and the 2-form field strengths $\mathcal{F}^m_{(2)}$ coming from dimensional reduction of the vielbein to get additional scalars λ^{mn} and λ_m, $(m, n = 1, \dots, 8,\ m < n)$. Again, we introduce them as Lagrange multipliers assuring the Bianchi identities for the field strengths $F_{(2)mn}$ and $\mathcal{F}^m_{(2)}$, which take the form

$$\begin{aligned} d(\tilde{\gamma}^m{}_p \tilde{\gamma}^n{}_q F_{(2)mn} - A_{(0)pqr}\tilde{\gamma}^r{}_s \mathcal{F}^s_{(2)}) &= 0\,, \\ d(\gamma^m_n \mathcal{F}^n_{(2)}) &= 0\,, \end{aligned} \tag{2.60}$$

with the matrices $\gamma^m{}_n$ $\tilde{\gamma}^m{}_n$ defined in (2.22) combing from Kaluza-Klein modifications. The additional equations of motion again can be solved algebraically an inserted in the Lagrangian. The scalar Lagrangian then gets two additional terms which can be combined with the Chern-Simons term and then read

$$\mathcal{L}_{\text{dual}} = -\tfrac{1}{2} E \sum_{m<n} e^{-\vec{a}_{mn}\cdot\vec{\phi}} (G^{mn}_{(1)})^2 - \tfrac{1}{2} E \sum_{m} e^{-\vec{b}_m\cdot\vec{\phi}} (G_{(1)m})^2 \tag{2.61}$$

with

$$\begin{aligned} G^{mn}_{(1)} &= \tilde{\gamma}^m{}_p \tilde{\gamma}^n{}_q (d\lambda^{pq} + \tfrac{1}{72} dA_{(0)rst} A_{(0)uvw} \epsilon^{pqrstuvw})\,, \\ G_{(1)n} &= \gamma^m{}_n (d\lambda_m - \tfrac{2}{A}{}_{(0)mpq} d\lambda^{pq} + \tfrac{1}{432} dA_{(0)npq} A_{(0)rst} A_{(0)uvm} \epsilon^{npqrstuv})\,. \end{aligned} \tag{2.62}$$

For the dilaton vectors $\vec{b}_m$ and $\vec{a}_{mnp}$ corresponding to the axionic scalars λ_m and λ^{mn} we find

$$\vec{a}_{mnp} + \vec{a}_{qrs} = -\vec{a}_{tu}\,, \quad \vec{a}_{mnp} + (-\vec{b}_{np}) = -\vec{b}_m\,, \tag{2.63}$$

where m, n, p, q, r, s, t, u are all different in the first equation. This corresponds to (2.25) and (2.26) and these relations replace (2.53). We define the 36 corresponding generators Z_m and Z^{mn} with commutation relations

$$\begin{aligned} [E^{mnp}, E^{qrs}] &= -\textstyle\sum_{t<u} \epsilon^{mnpqrstu} Z_{tu}\,, && \\ [E^{mnp}, Z_{qr}] &= -6\, \delta^{[mn}_{qr} Z^{p]}\,, & [E^{mnp}, Z^q] &= 0\,, \\ [E_m{}^n, Z_{pq}] &= 2\, \delta^n_p Z_{mq}\,, & [E_m{}^n, Z^p] &= -\delta^p_m Z^n\,, \\ [\vec{H}, Z_{mn}] &= -\vec{a}_{mn} Z_{mn}\,, & [\vec{H}, Z^m] &= -\vec{b}_m Z^m\,, \end{aligned}$$

replacing relation (2.32).

Lagrangian

Now we introduce two extra factors $\mathcal{V}_4$ and $\mathcal{V}_5$ to parameterize the Borel subgroup of E_8 with $\mathcal{V} = \mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\mathcal{V}_4\mathcal{V}_5$. They are defined by $\mathcal{V}_4 = e^{\lambda_m Z^m}$ and $\mathcal{V}_5 = e^{\sum_{m<n}\lambda^{mn}D_{mn}}$. The Lie algebra element $d\mathcal{V}\mathcal{V}^{-1}$ is given by

$$\begin{aligned}\mathcal{V}^{-1}d\mathcal{V} \;=\; & \tfrac{1}{2}\vec{\phi}\cdot\vec{H} + \sum_{m<n} e^{\frac{1}{2}\vec{b}_{mn}\cdot\vec{\phi}}\mathcal{F}^m_{(1)n}E_m{}^n + \sum_{m<n<p} e^{\frac{1}{2}\vec{a}_{mnp}\cdot\vec{\phi}}F_{(1)mnp}E^{mnp} \\ & + \sum_m e^{-\frac{1}{2}\vec{b}_m\cdot\vec{\phi}}G_{(1)m}Z^m \,. + \sum_{m<n} e^{-\frac{1}{2}\vec{a}_{mn}\cdot\vec{\phi}}G^{mn}_{(1)}Z_{mn}\,. \end{aligned} \tag{2.64}$$

This gives precisely the scalar and hence the full bosonic Lagrangian of $d=3$ supergravity with all 2-form field strengths dualized when inserted in formula (2.36). To be explicit, we get

$$\mathcal{L} \;=\; ER + \tfrac{1}{4}E\,\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) \tag{2.65}$$

with the matrix $\mathcal{M} = \mathcal{V}\mathcal{V}^{\#}$ containing all bosonic degrees of freedom. For $E_{8(8)}$ the involution $^{\#}$ is simply given by the transpose ^{T} and we have

$$\mathcal{M} = \mathcal{V}\mathcal{V}^{T}\,. \tag{2.66}$$

Symmetries

The global symmetry of the Lagrangian is $E_{8(8)}$ and the coset manifold of the bosonic sector of the theory is $E_{8(8)}/SO(16)$. To linearize the action of $E_{8(8)}$ and to make the coset space $E_{8(8)}/SO(16)$ explicit, we split the 248 E_8 generators according to the decomposition $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ under $SO(16)$ into 120 generators $X^{IJ} = -X^{JI}$ and Y^A with commutation relations given in section 3.1.1. The indices $I, J, \ldots = 1, \ldots, 16$, $A, B, \ldots = 1, \ldots, 128$ and $\dot{A}, \dot{B}, \ldots = 1, \ldots, 128$ label the vector representation, fundamental spinor representation and conjugate spinor representation, respectively.

Accordingly, the e_8 Lie algebra element $\mathcal{V}^{-1}d\mathcal{V}$ decomposes into the boson field P_μ parameterizing the coset space $E_{8(8)}/SO(16)$ and the $SO(16)$ gauge field Q_μ

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} \;=\; Q_\mu + P_\mu \;=\; \tfrac{1}{2}Q^{IJ}_\mu X^{IJ} + P^A_\mu Y^A\,. \tag{2.67}$$

Field content

We summarize the fields of $d=3$, $N=16$ supergravity. There are

		spin	$SO(16)$ (local)	$E_{8(8)}$ (global)
the graviton:	$E_\mu{}^\alpha \in GL(3,\mathbb{R})/SO(2,1)$	2	$\mathbf{1}$	$\mathbf{1}$
16 gravitinos:	ψ^a_μ	3/2	$\mathbf{16}$	$\mathbf{1}$
128 spin-$\frac{1}{2}$ fermions:	$\chi^{\dot{A}}$	1/2	$\overline{\mathbf{128}}$	$\mathbf{1}$
120 scalars:	$\mathcal{V}_{mn}{}^{ab} \in E_{8(8)}/SO(16)$	0	$\mathbf{120} \oplus \mathbf{128}$	$\mathbf{248}$

where $\underline{\alpha}, \mu = 0, 1, 2$ are flat and curved spacetime indices in three dimensions, respectively. The gravitino ψ_μ^I and the dreibein $E_\mu{}^{\underline{\alpha}}$ do not correspond to physical degrees of freedom.

The E_8 invariance of maximal $d = 3$ supergravity was originally shown in [101], while its complete Lagrangian and supersymmetry transformations were derived in [131]. In reference [135] the reduction from $d = 11$ to three and lower dimensions is explicitly carried out.

A classification of three-dimensional scalar cosets that arise in the reduction of $d = 4$ gravity coupled to matter was given in [16]. The corresponding supersymmetric models in $d = 3$ where studied in [43]. In many cases the four-dimensional theories can be obtained by dimensional reduction of higher dimensional supergravity theories coupled to matter. These so-called "oxidation endpoints" of the three-dimensional scalar coset models are studied in [29].

2.4.5 $N=16$ supergravity in $d=2$

The reduction to $d=2$ as described in [141] involves somewhat different features compared to higher dimensions. As in three dimensions the scalar sector is governed by an $E_{8(8)}/SO(16)$ nonlinear σ-model, but now the equations of motion admit a rigid non-compact E_9 symmetry. We will recall the basic properties of the two-dimensional theory. For further details, the reader is referred to [141, 148, 145].

The splitting of the vielbein as described in eq. (2.14) does not work for the reduction from three to two dimensions because α ill-defined in this case. Instead, we use local $SO(1, 2)$ invariance and $d=2$ diffeomorphism invariance to split the dreibein $E_\mu{}^{\underline{\alpha}}$ according to

$$E_\mu{}^{\underline{\alpha}} = \begin{pmatrix} \lambda \delta_{\hat{\mu}}{}^{\underline{\hat{\alpha}}} & \rho B_{\hat{\mu}} \\ 0 & \rho \end{pmatrix} , \tag{2.68}$$

where the indices carrying a hat are two-dimensional spacetime indices. The Kaluza-Klein field $B_{\hat{\mu}}$ is auxiliary in two dimensions and can be eliminated by solving its equations of motion. The dilaton ρ can be identified with one of the two remaining coordinates using the remaining diffeomorphism invariance and partly the $N=16$ supersymmetry. The conformal factor λ drops out of the matter part of the Lagrangian after certain redefinitions of the fermion fields.

Lagrangian

The bosonic Lagrangian of $d=2$, $N=16$ supergravity takes the form

$$\mathcal{L} \; = \; \rho E R + \tfrac{1}{4} \rho E \, \mathrm{Tr}(\partial \mathcal{M}^{-1} \partial \mathcal{M})$$

with the same matrix $\mathcal{M} = \mathcal{V}\mathcal{V}^T$ as in three dimensions, but now depending only on two spacetime coordinates.

Field content

We have the following fields in $d=2$, $N=16$ supergravity:

		spin	$SO(16)$ (local)	$E_{8(8)}$ (global)
the graviton:	$E_{\hat\mu}{}^{\hat\alpha} \in GL(2,\mathbb{R})/SO(1,1)$	2	**1**	**1**
the dilaton:	ρ	0	**1**	**1**
16 gravitinos:	$\psi^I_\pm$	3/2	**16**	**1**
16 dilatinos:	ψ^I_2	1/2	**16**	**1**
128 spin-$\frac{1}{2}$ fermions:	$\chi^{\dot A}$	1/2	$\overline{\mathbf{128}}$	**1**
128 scalars:	$\mathcal{V}_{mn}{}^{ab} \in E_{8(8)}/SO(16)$	0	$\mathbf{120} \oplus \mathbf{128}$	**248**

where $\hat\alpha, \hat\mu = 0, 1$ are flat and curved spacetime indices in two dimensions, respectively. The $\psi^I_\pm$ are Majorana-Weyl spinors. In the following we will switch to light-cone coordinates $x_\pm = x_0 \pm x_1$ whenever it makes the formulas simpler.

Integrability

The two-dimensional theory is integrable in the sense that it possesses a linear system [9, 126, 141]. The equations of motion are equivalent to the integrability condition of the linear system of differential equations

$$\hat{\mathcal{V}}^{-1}(x;\gamma)\,\partial_\pm \hat{\mathcal{V}}(x;\gamma) = L_\pm(x;\gamma)\,,$$

labeled by a complex parameter γ, which is called the spectral parameter. $\hat{\mathcal{V}}(\gamma)$ is an $E_{8(8)}$-valued matrix, whereas $L_\pm$ has values in the Lie algebra $\mathfrak{e}_8$ and is defined by

$$L_\pm \;=\; \tfrac{1}{2}\hat{Q}^{IJ}_\pm(\gamma) X^{IJ} + \hat{P}^A_\pm(\gamma) Y^A\,, \tag{2.69}$$

with

$$\hat{Q}^{IJ}_\pm(\gamma) \;=\; Q^{IJ}_\pm - \frac{2\mathrm{i}\gamma}{(1\pm\gamma)^2}\Big(8\psi^{[I}_{2\pm}\psi^{J]}_\pm \pm \Gamma^{IJ}_{\dot A\dot B}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\Big) - \frac{32\mathrm{i}\gamma^2}{(1\pm\gamma)^4}\psi^I_{2\pm}\psi^J_\pm \tag{2.70}$$

and

$$\hat{P}^A_\pm(\gamma) \;=\; \frac{1\mp\gamma}{1\pm\gamma}P^A_\pm + \frac{4\mathrm{i}\gamma(1\mp\gamma)}{(1\pm\gamma)^3}\Gamma^I_{A\dot B}\psi^I_{2\pm}\psi^{\dot B}_\pm\,. \tag{2.71}$$

In contrast to flat space integrable models, the spectral parameter γ is a function of the coordinates x. This is due to the explicit appearance of the dilaton ρ in the equations of motion. The spectral parameter γ fulfills the linear system of differential equations

$$\gamma^{-1}\partial_\pm\gamma \;=\; \frac{1\mp\gamma}{1\pm\gamma}\rho^{-1}\partial_\pm\rho\,,$$

which can be solved to get a function $\gamma(x;w)$ with an integration constant w which is called "constant spectral parameter". Actually, the function $\gamma(x;w)$ lives on a twofold covering of the complex w-plane, with the transition between the two sheets given by $\gamma \mapsto \frac{1}{\gamma}$.

Symmetries

As in three dimensions, the Lagrangian has a global $E_{8(8)}$ symmetry and scalars of the theory $\mathcal{V}$ form an $E_{8(8)}/SO(16)$ σ-model with the global action of $g \in E_{8(8)}$ and the local action of $h(x) \in SO(16)$ defined by

$$\mathcal{V}(x;\gamma) \;\mapsto\; g\mathcal{V}(x;\gamma)h(x)\,.$$

However, in the two-dimensional case the symmetries of the theory become infinite dimensional. These infinite dimensional symmetries hold on the level of the equations of motion and can be made explicit in the canonical formulation of supergravity. For this purpose, the theory can be reformulated in terms of an infinite set of nonlocal charges. The Poisson algebra of this complete set of charges can be explicitly computed and reveals the underlying symmetry structure. In chapter 4 we will review this symmetry structure and discuss its quantization.

Chapter 3

Hidden Symmetries in 11 Dimensions

In the last chapter we have seen how hidden symmetries of exceptional type arise in supergravity after dimensional reduction to lower dimensions. We will now study $d=11$ supergravity in formulations where the tangent space symmetry $SO(1,10)$ is replaced by the local symmetries that would arise in the reduction to four and three dimensions, i.e. $SO(1,3)\times SU(8)$ and $SO(1,2)\times SO(16)$, respectively [41, 140]. These formulations are equivalent on the level of equations of motion, but they differ off-shell. The crucial point is that part of the hidden symmetry group is already present in eleven dimensions. In these formulations, the lower dimensional supergravity theories with their exceptional symmetries can be obtained directly by dropping the dependence on the internal coordinates, without any dualizations. For this purpose, all bosonic fields of $d=11$ supergravity which would become scalar matter fields in the dimensional reduction are assigned to representations of the hidden symmetry groups $E_{7(7)}$ and $E_{8(8)}$, respectively. This is done by merging them into new objects, so-called "generalized vielbeine", with upper world indices running over the internal dimensions, and lower indices belonging to the **56** and **248** representations of $E_{7(7)}$ and $E_{8(8)}$, respectively.

In this chapter, we concentrate on the $SO(1,2)\times SO(16)$ version first presented in [140] and give further evidence that there is a hidden E_8 structure in eleven dimensional supergravity. We will show that the generalized vielbein is actually part of a full $E_{8(8)}$ matrix $\mathcal{V}$ (a "248-bein"), which now lives in eleven dimensions, and which incorporates the bosonic degrees of freedom of $d=11$ supergravity, with the exception of the dreibein remaining from the 3+8 split (which does not propagate in the dimensionally reduced theory). In particular the 3-form gauge potential $A_{(3)}$ that in previous work [41, 140, 133] appeared only via its field strength $F_{(4)} = dA_{(3)}$ is merged into the 248-bein, which therefore transforms also under tensor gauge transformations. We will discuss the algebraic relations obeyed by the generalized vielbein and their solution. Apart from the torus reduction, explicit solutions to this constraint were only known for the S^7 compactification of $d=11$ supergravity [42]. Furthermore, we will present an $E_{8(8)}$-covariant transformation acting on the 248-bein, suggesting a partial unification of internal coordinate and tensor gauge transformations.

In addition to the algebraic constraint which enforces the correct number of physical degrees of freedom, the 248-bein is subject to differential relations, which are the analog of the covariant constancy of the inverse densitized dreibein in $d=4$ canonical gravity. These relations are not discussed here.

In order to completely clarify the role of $E_{8(8)}$ in eleven-dimensional supergravity, several open problems need to be settled. One of them is the proper treatment of the tensor components $B_{\mu\nu m}$ and $B_{\mu\nu\rho}$, which cannot be gauged away using gauge transformations as one might naively think. Another ambitious task would be the construction of an invariant action in terms of the 248-bein $\mathcal{V}$ in eleven dimensions.

3.1 E_8 preliminaries

We review our conventions concerning $E_{8(8)}$ and some basic properties as well as its $SO(16)$ and $SL(8,\mathbb{R})$ decompositions, which played an important role also in [30]. The generators of the Lie algebra $\mathfrak{e}_8$ in the adjoint (and thus fundamental) representation are denoted by $X^{\mathcal{A}}$. The maximal compact subalgebra so(16) can be as the subalgebra invariant under the symmetric space involution

$$\tau(X^a) = -(X^a)^T \,. \tag{3.1}$$

3.1.1 The $SO(16)$ decomposition of $E_{8(8)}$

The adjoint representation of $\mathfrak{e}_8$ decomposes as $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ with respect to its maximal compact subalgebra so(16). Accordingly, we split $E_{8(8)}$ indices $\mathcal{A}, \mathcal{B}, \ldots$ as $([IJ], A), \ldots$, with antisymmetric pairs $[IJ]$, where $I,\, J = 1, \ldots, 16$ and $A = 1, \ldots, 128$. this corresponds to the adjoint and the fundamental spinor representation of so(16). The $E_{8(8)}$ Lie algebra generators $X^{\mathcal{A}} = (X^{IJ}, Y^A)$ obey

$$\begin{aligned} \left[X^{IJ}, X^{KL}\right] &= 4\,\delta^{\overline{I[K}} X^{\overline{L]J}} \,, \\ \left[X^{IJ}, Y^{A}\right] &= -\tfrac{1}{2}\Gamma^{IJ}_{AB} Y^B \,, \\ \left[Y^{A}, Y^{B}\right] &= \tfrac{1}{4}\Gamma^{IJ}_{AB} X^{IJ} \,, \end{aligned} \tag{3.2}$$

where the matrices Γ^{IJ}_{AB} are obtained from the $\mathfrak{so}(16)$ Γ-matrices in the standard fashion

$$\Gamma^I_{A\dot{A}}\Gamma^J_{\dot{A}B} = (\delta^{IJ})_{AB} + \Gamma^{IJ}_{AB}. \tag{3.3}$$

The totally antisymmetric structure constants $f^{\mathcal{ABC}}$ of $\mathfrak{e}_8$ therefore possess the non-vanishing components

$$f^{IJ,\,KL,\,MN} = -8\,\delta^{\overline{I[K}}\,\delta^{\overline{L]J}}_{MN} \,, \qquad f^{IJ,\,A,\,B} = -\tfrac{1}{2}\Gamma^{IJ}_{AB} \,, \tag{3.4}$$

The Cartan-Killing form is given by

$$\eta^{\mathcal{AB}} = \tfrac{1}{60}\,\mathrm{tr}(X^{\mathcal{A}}X^{\mathcal{B}}) = -\tfrac{1}{60} f^{\mathcal{A}}{}_{\mathcal{CD}} f^{\mathcal{BCD}} \tag{3.5}$$

with components $\eta^{AB} = \delta^{AB}$ and $\eta^{IJ\,KL} = -2\delta^{IJ}_{KL} = -\delta^i_k\delta^j_l + \delta^i_l\delta^j_k$. When summing over anti-symmetrized index pairs $[IJ]$, an extra factor of $\frac{1}{2}$ is always understood.

3.1.2 The $SL(8,\mathbb{R})$ decomposition of $E_{8(8)}$

To recover the $SL(8,\mathbb{R})$ basis of [30], we will further decompose the above representations into representations of the subgroup $SO(8) \equiv \big(SO(8)\times SO(8)\big)_{\mathrm{diag}} \subset SO(16)$. The indices corresponding to the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$, respectively, will be denoted by a, α and $\dot{\alpha}$. After a triality rotation the $SO(16)$ vector and spinor representations decompose as

$$
\begin{aligned}
\mathbf{16} &\leftrightarrow \mathbf{8_s} \oplus \mathbf{8_c}\,, \\
\mathbf{128_s} &\leftrightarrow (\mathbf{8_s}\otimes\mathbf{8_c}) \oplus (\mathbf{8_v}\otimes\mathbf{8_v}) = \mathbf{8_v} \oplus \mathbf{56_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}\,, \\
\mathbf{128_c} &\leftrightarrow (\mathbf{8_v}\otimes\mathbf{8_s}) \oplus (\mathbf{8_c}\otimes\mathbf{8_v}) = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{8_c} \oplus \mathbf{56_c}\,,
\end{aligned} \tag{3.6}
$$

respectively. We thus have $I = (\alpha, \dot\alpha)$ and $A = (\alpha\dot\beta, ab)$, and the $E_{8(8)}$ generators decompose as

$$
X^{[IJ]} \to (X^{[\alpha\beta]}, X^{[\dot\alpha\dot\beta]}, X^{\alpha\dot\beta})\,, \quad Y^A \to (Y^{\alpha\dot\alpha}, Y^{ab})\,. \tag{3.7}
$$

Next we regroup these generators as follows. The 63 generators

$$
E_a{}^b := \tfrac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} + \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{(ab)} - \tfrac{1}{8}\delta^{ab}Y^{cc}\,,
$$

for $1 \le a, b \le 8$ span an $SL(8,\mathbb{R})$ subalgebra of $E_{8(8)}$. The generator

$$
N := Y^{cc} \tag{3.8}
$$

extends this subalgebra to $GL(8,\mathbb{R})$. The remainder of the $E_{8(8)}$ Lie algebra then decomposes into the following representations of $SL(8,\mathbb{R})$:

$$
\begin{aligned}
Z^a &:= \tfrac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha})\,, \\
Z_{ab} &:= \tfrac{1}{8}\left(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}\right) + Y^{[ab]}\,, \\
E^{abc} &:= -\tfrac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha})
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
Z_a &:= -\tfrac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha})\,, \\
Z^{ab} &:= -\tfrac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{[ab]}\,, \\
E_{abc} &:= \tfrac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha})\,,
\end{aligned} \tag{3.10}
$$

The Cartan subalgebra is spanned by the diagonal elements $E_1{}^1, \ldots, E_7{}^7$ and N, or, equivalently, by $Y^{11}, \ldots, Y^{88}$. Obviously, the elements $E_a{}^b$ for $a < b$ (or $a > b$) together with the elements (3.9) (or (3.10)) for $a < b < c$ generate the Borel subalgebra of $E_{8(8)}$ associated with the positive (negative) roots of $E_{8(8)}$. Furthermore, these generators are graded w.r.t. the number of times the root α_8 (labelled by $\vec{a}_1 23$ in fig. 2.4 and corresponding to the element N in the Cartan subalgebra) appears, such that for any basis generator X we have $[N, X] = \deg(X) \cdot X$. The degree can be read off from

$$
\begin{array}{ll}
[N, Z^a] = 3\,Z^a & [N, Z_a] = -3\,Z_a \\
[N, Z_{ab}] = 2\,Z_{ab} & [N, Z^{ab}] = -2\,Z^{ab} \\
[N, E^{abc}] = E^{abc} & [N, E_{abc}] = -E_{abc} \\
[N, E_a{}^b] = 0 &
\end{array} \tag{3.11}
$$

The remaining commutation relations are given by

$$\begin{aligned} &[Z^a, Z^b] = 0 && [Z_a, Z_b] = 0 \\ &[Z_a, Z^b] = E_a{}^b - \tfrac{3}{8}\delta_a^b N \end{aligned} \tag{3.12}$$

$$\begin{aligned} &[Z_{ab}, Z^c] = 0 && [Z_{ab}, Z_c] = -E_{abc} \\ &[Z_{ab}, Z_{cd}] = 0 && [Z_{ab}, Z^{cd}] = 4\delta_{[a}^{[c} E_{b]}{}^{d]} + \tfrac{1}{2}\delta_{ab}^{cd} N \\ &[Z^{ab}, Z^c] = -E^{abc} && [Z^{ab}, Z_c] = 0 \end{aligned} \tag{3.13}$$

$$\begin{aligned} &[E^{abc}, Z^d] = 0 && [E_{abc}, Z^d] = 3\delta_{[a}^d Z_{bc]} \\ &[E^{abc}, Z_{de}] = -6\delta_{de}^{[ab} Z^{c]} && [E_{abc}, Z_{de}] = 0 \\ &[E^{abc}, E^{def}] = -\tfrac{1}{2}\epsilon^{abcdefgh} Z_{gh} && [E_{abc}, E_{def}] = \tfrac{1}{2}\epsilon_{abcdefgh} Z^{gh} \\ &[E^{abc}, Z_d] = 3\delta_d^{[a} Z^{bc]} && [E_{abc}, Z_d] = 0 \\ &[E^{abc}, Z^{de}] = 0 && [E_{abc}, Z^{de}] = 6\delta_{[ab}^{de} Z_{c]} \\ &[E^{abc}, E_{def}] = -18\delta_{[de}^{[ab} E_{f]}{}^{c]} - \tfrac{3}{4}\delta_{def}^{abc} N \end{aligned} \tag{3.14}$$

$$\begin{aligned} &[E_a{}^b, Z^c] = -\delta_a^c Z^b + \tfrac{1}{8}\delta_a^b Z^c && [E_a{}^b, Z_c] = \delta_c^b Z_a - \tfrac{1}{8}\delta_a^b Z_c \\ &[E_a{}^b, Z_{cd}] = -2\delta_{[c}^b Z_{d]a} - \tfrac{1}{4}\delta_a^b Z_{cd} && [E_a{}^b, Z^{cd}] = 2\delta_a^{[c} Z^{d]b} + \tfrac{1}{4}\delta_a^b Z^{cd} \\ &[E_a{}^b, E^{cde}] = -3\delta_a^{[c} E^{de]b} + \tfrac{3}{8}\delta_a^b E^{cde} && [E_a{}^b, E_{cde}] = 3\delta_{[c}^b E_{de]a} - \tfrac{3}{8}\delta_a^b E_{cde} \\ &[E_a{}^b, E_c{}^d] = \delta_c^b E_a{}^d - \delta_a^d E_c{}^b \end{aligned} \tag{3.15}$$

The elements $\{Z^a, Z_{ab}\}$ (or equivalently $\{Z_a, Z^{ab}\}$) span the maximal 36-dimensional abelian nilpotent subalgebra of $E_{8(8)}$ [128, 30].

Finally, the generators are normalized according to

$$\begin{aligned} \mathrm{tr}(NN) &= 60\cdot 8\,, \\ \mathrm{tr}(Z^a Z_b) &= 60\,\delta_b^a\,, \\ \mathrm{tr}(Z^{ab} Z_{cd}) &= 60\cdot 2!\,\delta_{cd}^{ab}\,, \\ \mathrm{tr}(E_{abc} E^{def}) &= 60\cdot 3!\,\delta_{abc}^{def}\,, \\ \mathrm{tr}(E_a{}^b E_c{}^d) &= 60\,\delta_a^d \delta_c^b - \tfrac{15}{2}\delta_a^b \delta_c^d\,, \end{aligned}$$

with all other traces vanishing.

3.1.3 Projectors onto irreducible representations of E_8

We will also need to deal with operators acting on the tensor product $\mathbf{248}\otimes\mathbf{248}$. The associated matrices will be denoted as $O_{\mathcal{AB}}{}^{\mathcal{CD}}$ where we refer to the indices $\mathcal{AB}$ as "incoming" and to the indices $\mathcal{CD}$ as "outgoing". The product of two such matrices O and P is consequently given by

$$(OP)_{\mathcal{AB}}{}^{\mathcal{CD}} := O_{\mathcal{AB}}{}^{\mathcal{EF}} P_{\mathcal{EF}}{}^{\mathcal{CD}} . \tag{3.16}$$

As with the generators above, the Cartan-Killing metric must be used whenever indices are raised or lowered from their "canonical" position on such matrices. We define

$$\overset{21}{O}_{\mathcal{AB}}{}^{\mathcal{CD}} := \overset{12}{O}_{\mathcal{BA}}{}^{\mathcal{DC}} . \tag{3.17}$$

To write down the projectors we need the operators $\mathbb{1}$, Π (i.e. the identity and the exchange operator, respectively), and $\tilde{\Pi}$, which are given by

$$\mathbb{1}_{\mathcal{AB}}{}^{\mathcal{CD}} = \delta^{\mathcal{C}}_{\mathcal{A}}\, \delta^{\mathcal{D}}_{\mathcal{B}}, \quad \Pi_{\mathcal{AB}}{}^{\mathcal{CD}} = \delta^{\mathcal{D}}_{\mathcal{A}}\, \delta^{\mathcal{C}}_{\mathcal{B}}, \quad \tilde{\Pi}_{\mathcal{AB}}{}^{\mathcal{CD}} = \eta_{\mathcal{AB}}\, \eta^{\mathcal{CD}}. \tag{3.18}$$

A further important operator is the symmetric Casimir element defined in the adjoint representation by

$$\begin{aligned} \Omega_{\mathrm{e}_8} &\equiv \eta_{\mathcal{AB}}\, X^{\mathcal{A}} \otimes X^{\mathcal{B}} \\ &= -\tfrac{1}{2}\, X^{IJ} \otimes X^{IJ} + X^{A} \otimes X^{A} \;\in\; \mathrm{so}(16) \otimes \mathrm{so}(16) + \mathfrak{k} \otimes \mathfrak{k} \, . \end{aligned} \tag{3.19}$$

In terms of the structure constants of $E_{8(8)}$, the Casimir element can be alternatively expressed as

$$(\Omega_{\mathrm{e}_8})_{\mathcal{AB}}{}^{\mathcal{CD}} = f^{\mathcal{E}}{}_{\mathcal{A}}{}^{\mathcal{C}}\, f_{\mathcal{EB}}{}^{\mathcal{D}} \, .$$

We will also need the twisted Casimir element $\Omega^{\tau}_{\mathrm{e}_8}$, defined by

$$\begin{aligned} \Omega^{\tau}_{\mathrm{e}_8} &\equiv \eta_{\mathcal{AB}} X^{\mathcal{A}} \otimes \tau(X^{\mathcal{B}}) \\ &= -\tfrac{1}{2}\, X^{IJ} \otimes X^{IJ} - X^{A} \otimes X^{A} \, , \end{aligned} \tag{3.20}$$

i.e. in indices:

$$(\Omega^{\tau}_{\mathrm{e}_8})_{\mathcal{AB}}{}^{\mathcal{CD}} = -(\Omega_{\mathrm{e}_8})_{\mathcal{CB}}{}^{\mathcal{AD}} = -f_{\mathcal{EA}}{}^{\mathcal{C}}\, f_{\mathcal{EB}}{}^{\mathcal{D}} \, . \tag{3.21}$$

The twisted Casimir element is obviously not $E_{8(8)}$ invariant, but only $SO(16)$ invariant.

The tensor product of two adjoint representations of $E_{8(8)}$ splits into its irreducible components[1] according to $\mathbf{248} \otimes \mathbf{248} = \mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{27000} \oplus \mathbf{30380}$. The corresponding projectors are given by:

$$\begin{aligned} \mathcal{P}_{1} &= \tfrac{1}{248}\, \tilde{\Pi}, \\ \mathcal{P}_{248} &= \tfrac{1}{60}\, (\Omega_{\mathrm{e}_8} \Pi - \Omega_{\mathrm{e}_8}) \, , \\ \mathcal{P}_{3875} &= \tfrac{1}{14} \left(\mathbb{1} - \tfrac{1}{4}\, \tilde{\Pi} + \Pi - \tfrac{1}{2} (\Omega_{\mathrm{e}_8} \Pi + \Omega_{\mathrm{e}_8}) \right) , \\ \mathcal{P}_{27000} &= \tfrac{1}{7} \left(3\, \mathbb{1} + \tfrac{3}{31}\, \tilde{\Pi} + 3\, \Pi + \tfrac{1}{4} (\Omega_{\mathrm{e}_8} \Pi + \Omega_{\mathrm{e}_8}) \right) , \\ \mathcal{P}_{30380} &= \tfrac{1}{2}\, \mathbb{1} - \tfrac{1}{2}\, \Pi + \tfrac{1}{60}\, (\Omega_{\mathrm{e}_8} - \Omega_{\mathrm{e}_8} \Pi) \, . \end{aligned} \tag{3.22}$$

[1] The irreducible component of a tensor product of representations and similar calculations can be conveniently computed using the LiE program package [25].

To verify that these operators indeed satisfy orthogonal projection relations, one needs the following relation

$$\Omega_{e_8}^2 = 12\,\mathbb{1} + 12\,\Pi + 12\,\tilde{\Pi} - 20\,\Omega_{e_8} + 10\,\Omega_{e_8}\Pi\,, \tag{3.23}$$

whose validity we have established with the help of a computer. In terms of the e_8 structure constants this relation becomes

$$f^{\mathcal{E}}{}_{\mathcal{A}\mathcal{G}} f_{\mathcal{B}\mathcal{E}\mathcal{H}} f^{\mathcal{G}\mathcal{I}\mathcal{C}} f_{\mathcal{I}}{}^{\mathcal{H}\mathcal{D}} = 24\delta^{\mathcal{C}}_{(\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B})} + 12\eta_{\mathcal{A}\mathcal{B}}\eta^{\mathcal{C}\mathcal{D}} - 20 f^{\mathcal{E}}{}_{\mathcal{A}}{}^{\mathcal{C}} f_{\mathcal{E}\mathcal{B}}{}^{\mathcal{D}} + 10 f^{\mathcal{E}}{}_{\mathcal{A}}{}^{\mathcal{D}} f_{\mathcal{E}\mathcal{B}}{}^{\mathcal{C}}\,. \tag{3.24}$$

In indices, the projectors read:

$$\begin{aligned}
(\mathcal{P}_1)_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} &= \tfrac{1}{248}\,\eta_{\mathcal{A}\mathcal{B}}\eta^{\mathcal{C}\mathcal{D}},\\
(\mathcal{P}_{248})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} &= -\tfrac{1}{60}\,f^{\mathcal{E}}{}_{\mathcal{A}\mathcal{B}} f_{\mathcal{E}}{}^{\mathcal{C}\mathcal{D}},\\
(\mathcal{P}_{3875})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} &= \tfrac{1}{7}\,\delta^{\mathcal{C}}_{(\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B})} - \tfrac{1}{56}\,\eta_{\mathcal{A}\mathcal{B}}\eta^{\mathcal{C}\mathcal{D}} - \tfrac{1}{14}\,f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}} f_{\mathcal{E}\mathcal{B}}{}^{\mathcal{D})},\\
(\mathcal{P}_{27000})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} &= \tfrac{6}{7}\,\delta^{\mathcal{C}}_{(\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B})} + \tfrac{3}{217}\,\eta_{\mathcal{A}\mathcal{B}}\eta^{\mathcal{C}\mathcal{D}} + \tfrac{1}{14}\,f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}} f_{\mathcal{E}\mathcal{B}}{}^{\mathcal{D})},\\
(\mathcal{P}_{30380})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} &= \delta^{\mathcal{C}}_{[\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B}]} + \tfrac{1}{60}\,f^{\mathcal{E}}{}_{\mathcal{A}\mathcal{B}} f_{\mathcal{E}}{}^{\mathcal{C}\mathcal{D}}\,.
\end{aligned} \tag{3.25}$$

All these projectors are manifestly symmetric w.r.t. interchange of the two subspaces, i.e. $\overset{12}{\mathcal{P}_j} = \overset{21}{\mathcal{P}_j}$. Furthermore, any E_8 matrix $\mathcal{V}$ obeys

$$\mathcal{P}_j\,\mathcal{V}\otimes\mathcal{V} = \mathcal{V}\otimes\mathcal{V}\,\mathcal{P}_j \tag{3.26}$$

which together with the normalization $\det\mathcal{V} = 1$ can be taken as defining relations for the group elements of E_8.

3.2 $SO(1,2)\times SO(16)$ invariant $d=11$ supergravity

We will first review $SO(1,2)\times SO(16)$ formulation of $d=11$ supergravity, referring readers to [140] for further details.

$SO(1,2)\times SO(16)$ invariant $d=11$ supergravity [140] is derived from the original version of [32] by first splitting up the fields in a way that would be appropriate for the reduction to three dimensions, but without dropping the dependence on any coordinates, and then reassembling the pieces into new objects transforming under local $SO(1,2)\times SO(16)$. This is achieved by first breaking the original tangent space symmetry $SO(1,10)$ down to $SO(1,2)\times SO(8)$ by a partial gauge choice for the elfbein, and then reenlarging it to $SO(1,2)\times SO(16)$ by the introduction of new gauge degrees of freedom. The construction thus requires a 3+8 split of the $d=11$ coordinates and tensor indices. The main task then is to identify the proper $SO(1,2)\times SO(16)$ covariant fields and to verify that all supersymmetry variations as well as the equations of motion can be entirely expressed in terms of the new fields.

In a first step one thus brings the elfbein into triangular form by (partial) use of local $SO(1,10)$ Lorentz invariance

$$E_M{}^A = \begin{pmatrix} \Delta^{-1}e_\mu{}^{\underline{\alpha}} & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \qquad \Delta := \det e_m{}^a\,. \tag{3.27}$$

Here curved $d\!=\!11$ indices decompose as $M=(\mu,m), N=(\nu,n),\dots$ with $\mu,\nu,\dots=0,1,2$ and $m,n,\dots=3,\dots,10$, and the associated flat indices are denoted by $\underline{\alpha},\underline{\beta},\dots$ and $a,b,\dots$, respectively (as in [140] the $SO(1,2)$ indices $\underline{\alpha},\underline{\beta},\dots$ are underlined to distinguish them from the $SO(8)$ spinor indices to be used below). The partially gauge fixed elfbein, whose form is preserved by the $SO(1,2)\times SO(8)$ subgroup of $SO(1,10)$, thus contains the Weyl rescaled dreibein $e_\mu{}^{\underline{\alpha}}$, the Kaluza-Klein vector $B_\mu{}^m$ and the achtbein $e_m{}^a$ yielding the scalar degrees of freedom living in the coset $GL(8,\mathbb{R})/SO(8)$.

The remaining bosonic degrees of freedom reside in the 3-index field A_{MNP}, which gives rise to various scalar and tensor fields upon performing a 3+8 split of the indices. First of all, there are 56 scalars A_{mnp} and 28 vector fields

$$B_{\mu mn} := A_{\mu mn} - B_\mu{}^p A_{mnp}\,. \tag{3.28}$$

If one were to reduce to three dimensions, the 8+28 vector fields $B_\mu{}^m$ and $B_{\mu mn}$ would be converted to 36 scalar degrees of freedom by means of a duality transformation. Here, they will be kept together with their dual scalars contained in the generalized vielbein, to which they are related by a nonlinear analog of the (linear) duality constraint of the reduced $d=3$ supergravity (cf. equation (3.70)).

In addition, A_{MNP} gives rise to (always in the 3+8 split)

$$\begin{aligned} B_{\mu\nu p} &:= A_{\mu\nu p} - 2B_{[\nu}{}^n A_{\mu]np} + B_\mu{}^m B_\nu{}^n A_{mnp}\,,\\ B_{\mu\nu\rho} &:= A_{\mu\nu\rho} - 3B_{[\mu}{}^m A_{\nu\rho]m} + 3B_{[\mu}{}^m B_\nu{}^n A_{\rho]mn}\\ &\qquad - B_\mu{}^m B_\nu{}^n B_\rho{}^p A_{mnp}\,. \end{aligned} \tag{3.29}$$

These fields are subject to the tensor gauge transformations

$$\delta A_{MNP} = 3\partial_{[M}\xi_{NP]}\,. \tag{3.30}$$

Under these, we have

$$\begin{aligned} \delta B_{\mu mn} &= \mathcal{D}_\mu \xi_{mn} + 2\partial_{[m}B_\mu{}^p \xi_{n]p} + 2\partial_{[m}\Lambda_{n]\mu}\,,\\ \delta B_{\mu\nu m} &= \partial_m \Lambda_{\mu\nu} + 2\partial_m B_{[\mu}{}^n \Lambda_{n\nu]} + 2\mathcal{D}_{[\mu}\Lambda_{\nu]m} + \mathcal{B}_{\mu\nu}{}^n \xi_{nm}\,,\\ \delta B_{\mu\nu\rho} &= 3\mathcal{D}_{[\mu}\Lambda_{\nu\rho]} - 3\mathcal{B}_{[\mu\nu}{}^m \Lambda_{\rho]m}\,, \end{aligned} \tag{3.31}$$

where

$$\mathcal{D}_\mu := \partial_\mu - B_\mu{}^m \partial_m \tag{3.32}$$

and

$$\mathcal{B}_{\mu\nu}{}^m := \mathcal{D}_\mu B_\nu{}^m - \mathcal{D}_\nu B_\mu{}^m\,. \tag{3.33}$$

The parameters $\Lambda_{\mu\nu}$ and $\Lambda_{\mu m}$ are defined by

$$\begin{aligned} \Lambda_{\mu\nu} &:= \xi_{\mu\nu} + 2B_{[\mu}{}^m \xi_{\nu]m} + B_\mu{}^m B_\nu{}^n \xi_{mn}\,,\\ \Lambda_{\mu m} &:= \xi_{\mu m} - B_\mu{}^n \xi_{nm}\,. \end{aligned} \tag{3.34}$$

It is easy to see that in the dimensionally reduced theory, one can make use of the parameter components $\Lambda_{\mu m}$ and $\Lambda_{\mu\nu}$ to set $B_{\mu\nu m} = B_{\mu\nu\rho} = 0$. Since we have not been able so far to cast the supersymmetry variations of these components into a completely $SO(16)$ covariant form, this gauge would also be very convenient in the present setting. However, there appears to be an obstacle to this gauge choice if the full $d = 11$ coordinate dependence is retained.

To identify the proper $SO(16)$ covariant bosonic fields, we must first explain how to rewrite the fermion fields. The $d = 11$ gravitino $\Psi_A \equiv (\Psi_{\underline{\alpha}}, \Psi_a)$ has 32 spinor components, which split as $\mathbf{2} \otimes (\mathbf{8}_s \oplus \mathbf{8}_c)$ under the $SO(1,2) \times SO(8)$ subgroup of $SO(1,10)$. Suppressing $SO(1,2)$ spinor indices, we then assign the resulting fields to the $SO(1,2) \times SO(16)$ fields ψ_μ^I and $\chi^{\dot{A}}$ via the following prescription [140]

$$\psi_\mu^I := \begin{cases} \Delta^{-1/2} e_\mu{}^{\underline{\alpha}} \left(\Psi_{\underline{\alpha}\alpha} + \gamma_{\underline{\alpha}} \Gamma^a_{\alpha\dot{\beta}} \Psi_{a\dot{\beta}} \right) & \text{if } I = \alpha\,, \\ \Delta^{-1/2} e_\mu{}^{\underline{\alpha}} \left(\Psi_{\underline{\alpha}\dot{\alpha}} - \gamma_{\underline{\alpha}} \Gamma^a_{\dot{\alpha}\beta} \Psi_{a\beta} \right) & \text{if } I = \dot{\alpha}\,, \end{cases} \tag{3.35}$$

and

$$\chi^{\dot{A}} := \begin{cases} \Delta^{-1/2} \left(\Gamma^b \Gamma^a\right)_{\alpha\beta} \Psi_{b\beta} & \text{if } \dot{A} = (a\alpha)\,, \\ -\Delta^{-1/2} \left(\Gamma^b \Gamma^a\right)_{\dot{\alpha}\dot{\beta}} \Psi_{b\dot{\beta}} & \text{if } \dot{A} = (\dot{\alpha}a)\,, \end{cases} \tag{3.36}$$

where I and $\dot{A}$ are $SO(16)$ vector and (conjugate) spinor indices, respectively (see section 3.1.2, and in particular (3.6) for the relevant $SO(8)$ decompositions).

The physical bosonic degrees of freedom, which correspond to the 128 propagating scalar degrees of freedom of maximal $d = 3$ supergravity, are fused into an appropriate generalized vielbein. The relevant expressions are found by proceeding from the following $SO(16)$ invariant ansatz for the supersymmetry variations of the vector fields in terms of the fermions (3.35) and (3.36) (in a suitable normalization):

$$\delta B_\mu{}^m = \tfrac{1}{2} e^m_{IJ}\, \bar{\varepsilon}^I \psi_\mu^J + e^m_A\, \Gamma^I_{A\dot{B}} \bar{\varepsilon}^I \gamma_\mu \chi^{\dot{B}} \tag{3.37}$$

$$\delta B_{\mu mn} = \tfrac{1}{2} e_{mnIJ}\, \bar{\varepsilon}^I \psi_\mu^J + e_{mnA} \Gamma^I_{A\dot{B}}\, \bar{\varepsilon}^I \gamma_\mu \chi^{\dot{B}} \tag{3.38}$$

The explicit expressions in a special $SO(16)$ gauge for the new bosonic quantities appearing on the r.h.s. of this equation can be found by comparing the above expressions with the ones obtained directly from $d = 11$ supergravity in the gauge (3.27). It is already known that [140]

$$(e^m_{IJ},\, e^m_A) := \begin{cases} \Delta^{-1} e_a{}^m \Gamma^a_{\alpha\dot{\beta}} & \text{if } [IJ] \text{ or } A = (\alpha\dot{\beta})\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.39}$$

again using the $SO(8)$ decompositions of section 3.1.2. By contrast, the objects $\left(e_{mnIJ},\, e_{mnA}\right)$ related to the gauge fields $B_{\mu mn}$ — hence with anti-symmetrized *lower* internal world indices — have been defined in [120] for the first time. Matching the r.h.s. of (3.38) with the variations obtained directly from the $d = 11$ supersymmetry variations of [32] we find

$$e_{mn\mathcal{A}} := \overset{\circ}{e}_{mn\mathcal{A}} + A_{mnp} e^p_{\mathcal{A}}\,, \tag{3.40}$$

where

$$\overset{\circ}{e}_{mnIJ}:=\begin{cases}\Delta^{-1}e_m{}^a e_n{}^b(\Gamma_{ab})_{\alpha\beta} & \text{if } [IJ]=[\alpha\beta]\,,\\ -\Delta^{-1}e_m{}^a e_n{}^b(\Gamma_{ab})_{\dot\alpha\dot\beta} & \text{if } [IJ]=[\dot\alpha\dot\beta]\,,\\ 0 & \text{if } [IJ]=[\alpha\dot\beta]\equiv-[\dot\beta\alpha]\,,\end{cases} \tag{3.41}$$

$$\overset{\circ}{e}_{mnA}:=\begin{cases}4\Delta^{-1}e_{\underline{ma}}e_{\underline{nb}} & \text{if } A=(ab)\,,\\ 0 & \text{if } A=(\alpha\dot\beta\,.)\end{cases} \tag{3.42}$$

Labeling the $E_{8(8)}$ indices $([IJ],A)$ collectively by $\mathcal{A},\mathcal{B},\dots=1,\dots,248$ as in section 3.1.1, the new objects $e^m_{\mathcal{A}}$ and $e_{mn\mathcal{A}}$ together form a rectangular $(8+28)\times 248$ matrix.

The new vielbeine are manifestly covariant w.r.t. local $SO(16)$, thereby enlarging the action of $SO(8)$ of the original theory. While the supersymmetry variation of $e^m_{\mathcal{A}}$ was already given in [140], the variation of $e_{mn\mathcal{A}}$ has been determined in [120] for the first time. In section 3.5 we show that

$$\delta e_{mnIJ}=-\tfrac{1}{2}\Gamma^{IJ}_{AB}\omega^A e_{mnB}\,,\qquad \delta e_{mnA}=\tfrac{1}{4}\Gamma^{IJ}_{AB}\omega^B e_{mnIJ}\,, \tag{3.43}$$

with the local $N=16$ supersymmetry parameter

$$\omega^A:=\tfrac{1}{4}\Gamma^I_{A\dot A}\bar\varepsilon^I\chi^{\dot A}\,. \tag{3.44}$$

In deriving this result, a compensating $SO(16)$ rotation must be taken into account to restore the triangular gauge. It is an important consistency check that this compensating rotation comes out to be the same for $e^m_{\mathcal{A}}$ and $e_{mn\mathcal{A}}$, and that the resulting transformation is exactly the same as for both fields. We can therefore combine the supersymmetry variations of the generalized vielbeine with local $SO(16)$ into the $E_{8(8)}$ covariant form

$$\delta e^m_{\mathcal{A}}=f_{\mathcal{AB}}{}^{\mathcal{C}}\omega^{\mathcal{B}}e^m_{\mathcal{C}}\,,\qquad \delta e_{mn\mathcal{A}}=f_{\mathcal{AB}}{}^{\mathcal{C}}\omega^{\mathcal{B}}e_{mn\mathcal{C}}\,. \tag{3.45}$$

We also note that the components $e_{mn\mathcal{A}}$ were not needed in [140] because they cannot appear in the supersymmetry variations of the fermionic fields (the latter depend on the 3-form potential via the 4-index field strengths only).

All fields transform under general coordinate transformations in eleven dimensions. Splitting the $d=11$ parameter as $\xi^M=(\xi^\mu,\xi^m)$, the transformations generated by ξ^μ take the standard form. For the remaining "internal" coordinate transformations with parameter ξ^m, we have

$$\begin{aligned}\delta B_\mu{}^m &= \mathcal{D}_\mu\xi^m+\xi^n\partial_n B_\mu{}^m\,,\\ \delta B_{\mu mn} &= 2\partial_{[m}\xi^p B_{\mu n]p}+\xi^p\partial_p B_{\mu mn}\,.\end{aligned} \tag{3.46}$$

Furthermore,

$$\begin{aligned}\delta e^m_{\mathcal{A}} &= \xi^p\partial_p e^m_{\mathcal{A}}-\partial_p\xi^m\, e^p_{\mathcal{A}}-\partial_p\xi^p\, e^m_{\mathcal{A}}\,,\\ \delta e_{mn\mathcal{A}} &= \xi^p\partial_p e_{mn\mathcal{A}}+2\partial_{[m}\xi^p\, e_{n]p\mathcal{A}}-\partial_p\xi^p\, e_{mn\mathcal{A}}\,.\end{aligned} \tag{3.47}$$

Whereas all the $SO(16)$ fields considered previously were inert under tensor gauge transformations $\delta A_{MNP} = 3\partial_{[M}\xi_{NP]}$, the non-invariance of $e_{mn\mathcal{A}}$ under such transformations, due to the appearance of the 3-index field A_{mnp} in its definition, is a new feature. Specifically, under tensor gauge transformations with parameter ξ_{mn} we have

$$\begin{aligned} \delta B_\mu{}^m &= 0\,, \\ \delta B_{\mu mn} &= \mathcal{D}_\mu \xi_{mn} - 2B_\mu{}^p \partial_{[m}\xi_{n]p}\,, \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} \delta e^m_{\mathcal{A}} &= 0\,, \\ \delta e_{mn\mathcal{A}} &= \partial_p \xi_{mn}\, e^p_{\mathcal{A}} + 2\partial_{[m}\xi_{n]p}\, e^p_{\mathcal{A}}\,. \end{aligned} \tag{3.49}$$

When combined with the previous coordinate transformations, these formulas are very suggestive of a unification of the internal coordinate transformations and tensor gauge transformations, based on combining the internal coordinate transformation parameters ξ^m with the residual tensor gauge parameters ξ_{mn} into a single set (ξ^m, ξ_{np}) of 36 parameters. In the remaining sections we will show how these local symmetries are related to the the maximal nilpotent commuting subalgebra of $E_{8(8)}$.

3.3 Solution of algebraic constraints on the generalized vielbein

From the expressions (3.39) – (3.42) one can deduce a number of algebraic constraints on the generalized vielbein. They are [133]

$$e^m_A e^n_A - \tfrac{1}{2} e^m_{IJ} e^n_{IJ} = 0\,, \tag{3.50}$$

$$\begin{aligned} \Gamma^{IJ}_{AB}\left(e^m_B e^n_{IJ} - e^n_B e^m_{IJ}\right) &= 0\,, \\ \Gamma^{IJ}_{AB} e^m_A e^n_B + 4 e^m_{K[I} e^n_{J]K} &= 0\,, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} e^{(m}_{IK} e^{n)}_{JK} - \tfrac{1}{16}\delta_{IJ} e^m_{KL} e^n_{KL} &= 0\,, \\ e^{(m}_{[IJ} e^{n)}_{KL]} + \tfrac{1}{24} e^m_A \Gamma^{IJKL}_{AB} e^n_B &= 0\,, \\ \Gamma^K_{\dot{A}B} e^{(m}_B e^{n)}_{KL} - \tfrac{1}{14}\Gamma^{IKL}_{\dot{A}B} e^{(m}_B e^{n)}_{KL} &= 0\,. \end{aligned} \tag{3.52}$$

For the new components, we find

$$e_{mnA} e_{pqA} - \tfrac{1}{2} e_{mnIJ} e_{pqIJ} = 0\,, \tag{3.53}$$

$$\begin{aligned} \Gamma^{IJ}_{AB}\left(e_{mnB} e_{pqIJ} - e_{pqB} e_{mnIJ}\right) &= 0\,, \\ \Gamma^{IJ}_{AB} e_{mnA} e_{pqB} + 4 e_{mnK[I} e_{pqJ]K} &= 0\,, \end{aligned} \tag{3.54}$$

and

$$e_{mnA}e^p_{\ A} - \tfrac{1}{2}e_{mnIJ}e^p_{\ IJ} \;=\; 0\,, \tag{3.55}$$

$$\begin{aligned}
\Gamma^{IJ}_{AB}\left(e_{mnB}e^p_{\ IJ} - e^p_{\ B}e_{mnIJ}\right) \;&=\; 0\,,\\
\Gamma^{IJ}_{AB}e_{mnA}e^p_{\ B} + 4e_{mnK[I}e^p_{\ J]K} \;&=\; 0\,,
\end{aligned} \tag{3.56}$$

whereas no analog of (3.52) exists for the components $e_{mn\mathcal{A}}$. All these relations are proved by decomposing the $SO(16)$ into $SO(8)$ and verifying the vanishing of their components case by case.

The above constraints can be elegantly rewritten in an $E_{8(8)}$ covariant form by means of the projectors onto the invariant subspaces of the tensor product of $E_{8(8)}$ representations $\mathbf{248}\otimes\mathbf{248}$ defined in section 3.1.3. The constrains (3.50), (3.51), (3.53) – (3.56) are equivalent to

$$\begin{aligned}
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e^n_{\mathcal{D}} \;&=\; 0\,,\\
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e_{pq\mathcal{D}} \;&=\; 0\,,\\
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e_{mn\mathcal{C}}e_{pq\mathcal{D}} \;&=\; ,0
\end{aligned} \tag{3.57}$$

for $j = 1$ and 248. It takes a little more work to verify that (3.52) can be expressed in the form[2]

$$(\mathcal{P}_{3875})_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e^n_{\mathcal{D}} = 0\,. \tag{3.58}$$

Observe the invariance of the constraints (3.57) and (3.58) under the combined general coordinate and tensor gauge transformations (3.47) and (3.49): the transformed generalized vielbein still obeys all constraints.

We next demonstrate that the $E_{8(8)}$ invariant algebraic relations on the generalized vielbein given above can be solved in terms of an $E_{8(8)}$ matrix $\mathcal{V}$. For the dimensionally reduced theory this is, of course, the expected result [101, 131]), but with the important difference that the dependence on all eleven coordinates is here retained. Thus, $E_{8(8)}$ is already present in eleven dimensions, though not a symmetry of the theory. The existence of $\mathcal{V}$ also clarifies why we end up with the correct number of physical degrees of freedom, a fact that cannot be directly ascertained by counting the constraints: being subject to local $SO(16)$ transformations, the matrix $\mathcal{V}$ possesses just the $128 = 248 - 120$ degrees of freedom of the coset $E_{8(8)}/SO(16)$. Thus, the counting works exactly as for the reduced theory.

To corroborate this claim, consider the $E_{8(8)}$ Lie algebra valued matrices

$$\begin{aligned}
\tilde{Z}^m \;&:=\; e^m_{\mathcal{A}}X^{\mathcal{A}} \equiv \tfrac{1}{2}e^m_{IJ}X^{IJ} + e^m_{A}X^{A}\,,\\
\tilde{Z}_{mn} \;&:=\; e_{mn\,\mathcal{A}}X^{\mathcal{A}} \equiv \tfrac{1}{2}e_{mn\,IJ}X^{IJ} + e_{mn\,A}X^{A}\,.
\end{aligned} \tag{3.59}$$

[2] A further relation given in [133]

$$e^{(m}_{\ A}e^{n)}_{\ B} + \tfrac{1}{32}(\Gamma^{IJ}\Gamma^{KL})_{AB}e^m_{\ IJ}e^n_{\ KL} - \tfrac{1}{16}\Gamma^{IJ}_{AC}e^{(m}_{\ C}e^{n)}_{\ D}\Gamma^{KL}_{DB} = 0$$

can be shown to be equivalent to (3.52) by means of a Fierz transformation and yields no new information. Therefore, there are no relations involving the projectors $\mathcal{P}_{27000}$ and $\mathcal{P}_{30380}$.

From the relations presented in the foregoing section we infer that these matrices commute (cf. (3.51), (3.54), (3.56))

$$[\tilde{Z}^m\,,\,\tilde{Z}^n] = [\tilde{Z}^m\,,\,\tilde{Z}_{pq}] = [\tilde{Z}_{mn}\,,\,\tilde{Z}_{pq}] = 0 \tag{3.60}$$

and are nilpotent (cf. (3.50), (3.53), (3.55))

$$\mathrm{Tr}\Big(\tilde{Z}^m\tilde{Z}^n\Big) = \mathrm{Tr}\Big(\tilde{Z}^m\tilde{Z}_{pq}\Big) = \mathrm{Tr}\Big(\tilde{Z}_{mn}\tilde{Z}_{pq}\Big) = 0 \tag{3.61}$$

(ensuring that any linear combination of these matrices has norm zero). Since they are linearly independent (as is most easily checked by setting $e_m{}^a = \delta^a_m$ in the original definition), they form a 36 dimensional abelian nilpotent subalgebra of $E_{8(8)}$. There is only one such algebra, which is unique up to conjugation [128, 30]. Consequently, there must exist an $E_{8(8)}$ matrix $\mathcal{V}$ such that

$$\tilde{Z}^m = \mathcal{V}^{-1}Z^m\mathcal{V}\,, \qquad \tilde{Z}_{mn} = \mathcal{V}^{-1}Z_{mn}\mathcal{V}\,, \tag{3.62}$$

where Z^m and Z_{mn} are the 8+28 nilpotent generators of $E_{8(8)}$ introduced in section 3.1.2. The assignment of the 8+28 vielbein components to these generators here is uniquely determined by $SL(8,\mathbb{R})$ covariance; its correctness will be confirmed below when we analyze (3.52). Thus,

$$\begin{aligned} e^m_{\mathcal{A}} &= \tfrac{1}{60}\,\mathrm{tr}(Z^m\mathcal{V}X_{\mathcal{A}}\mathcal{V}^{-1})\,,\\ e_{mn\,\mathcal{A}} &= \tfrac{1}{60}\,\mathrm{tr}(Z_{mn}\mathcal{V}X_{\mathcal{A}}\mathcal{V}^{-1})\,. \end{aligned} \tag{3.63}$$

By use of the relation

$$\tfrac{1}{60}\,\mathrm{tr}\left(X^{\mathcal{M}}\mathcal{V}X_{\mathcal{A}}\mathcal{V}^{-1}\right) = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \tag{3.64}$$

for the adjoint representation we can write

$$e^m_{\mathcal{A}} = \mathcal{V}^m{}_{\mathcal{A}}\,, \qquad e_{mn\,\mathcal{A}} = \mathcal{V}_{mn\,\mathcal{A}}\,, \tag{3.65}$$

whence the generalized vielbein $(e^m_{\mathcal{A}}, e_{mn\mathcal{A}})$ is actually a rectangular submatrix of $\mathcal{V}$. Let us emphasize once more that $\mathcal{V}$ still depends on eleven coordinates.

At this point, a remark concerning our use of indices is in order. In section 3.1.2, we used "flat" indices $a, b, c = 1, \ldots, 8$ to label the $E_{8(8)}$ generators in the $SL(8,\mathbb{R})$ decomposition. On the other hand, the index m appearing on the l.h.s. of (3.39) should be viewed as "curved" in the sense that it is acted on by internal coordinate transformations. Of course, there is no need for such a distinction in a flat background characterized by $e_m{}^a = \delta^a_m$ and $\mathcal{V} = \mathbb{1}$, whereas the two kinds of indices no longer coincide for curved backgrounds characterized by non-trivial $e_m{}^a$ and A_{mnp}. The nilpotent generators in (3.59) thus represent "curved" analogs of the "flat" generators Z^a and Z_{ab}, and the above relations tell us is that the transition from flat to curved configurations is entirely accounted for by the $E_{8(8)}$ matrix $\mathcal{V}$. This illustrates the enlargement of the geometry in comparison with the conventional description $d = 11$ supergravity, where the achtbein can only be deformed with a $GL(8,\mathbb{R})$ matrix.

To confirm the consistency of the above solution let us analyze the third set of constraints (3.52), which we have not yet discussed. Inspection reveals that the desired relation is equivalent to

$$\mathcal{P}_{3875}\big(\tilde{Z}^m \otimes \tilde{Z}^n\big) = 0 \tag{3.66}$$

Making use of the $E_{8(8)}$ invariance of $\mathcal{P}_{3875}$, we can replace curved by flat indices in this relation. This yields

$$(\mathcal{P}_{3875})_{\mathcal{A}\mathcal{B}}{}^{cd} \equiv \tfrac{1}{7}\,\delta_{\mathcal{A}}^{(c}\delta_{\mathcal{B}}^{d)} - \tfrac{1}{56}\,\eta_{\mathcal{A}\mathcal{B}}\eta^{cd} - \tfrac{1}{14}\,f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(c} f_{\mathcal{E}\mathcal{B}}{}^{d)} = 0\,. \tag{3.67}$$

By nilpotency, we have $\eta^{cd} = 0$, and contracting the remaining terms with $X^{\mathcal{B}}$ we see that (3.39) does satisfy (3.52), provided the following relation holds for all $\mathcal{A}$

$$\Big[[X_{\mathcal{A}}, Z^c]\,, Z^d\Big] = -2\delta_{\mathcal{A}}^{(c} Z^{d)}\,. \tag{3.68}$$

Since the algebra preserves the grading, the relation is trivially satisfied for all generators except $X_{\mathcal{A}} = Z_a$, which must be checked separately. A quick calculation, using the commutation relations listed in section 3.1.2, shows that the required relation is indeed satisfied. Let us emphasize once more that there is no analog of (3.67) for nilpotent elements conjugate to the Z_{mn}.

Having established the consistency of (3.63) it remains to investigate its uniqueness. It is easy to see that $\mathcal{V}$ is, in fact, *not* unique because (3.63) remains unchanged under the transformation

$$\mathcal{V} \longrightarrow n \cdot \mathcal{V} \qquad \text{for} \qquad n \in \mathcal{N} \tag{3.69}$$

where $\mathcal{N}$ is the Borel subgroup of $E_{8(8)}$ generated by Z^m and Z_{mn}. This remaining non-uniqueness can be fixed by invoking the differential relations

$$\begin{aligned} e^m_A P^A_\mu &= \varepsilon_\mu{}^{\nu\rho}\mathcal{B}_{\nu\rho}{}^m\,, \\ e_{mnA} P^A_\mu &= \varepsilon_\mu{}^{\nu\rho}\mathcal{B}_{\nu\rho mn}\,, \end{aligned} \tag{3.70}$$

where $\mathcal{B}_{\mu\nu}{}^m$ was already defined in (3.33), and

$$\mathcal{B}_{\mu\nu mn} := \mathcal{D}_\mu B_{\nu mn} - \mathcal{D}_\nu B_{\mu mn} + 4\partial_m B_{[\mu}{}^p B_{\nu]np} + 2\partial_{[m} B_{n]\mu\nu}\,. \tag{3.71}$$

The equation (3.70) thus relates the field strengths to (part of) the $E_{8(8)}$ connection P^A_μ, whose explicit expression in terms of the spin connection and the 4-index field strength is given in formula (22) of [140]. In the reduction to three dimensions these differential constraints become the linear duality relations that allow us to trade the 36 vector fields for their dual scalars.

3.4 The hidden $E_{8(8)}$ geometry of $d=11$ supergravity

The matrix $\mathcal{V}$ plays a role similar to the achtbein $e_m{}^a$, but also incorporates the tensor degrees of freedom from A_{mnp}, as well as the vector fields $B_\mu{}^m$ and $B_{\mu mn}$. We would now like to argue that $\mathcal{V} \in E_{8(8)}/SO(16)$ really *is* the appropriate vielbein encompassing the propagating degrees of freedom of $d = 11$ supergravity, in the same way as the ordinary vielbein, viewed as an element of $GL(d,\mathbb{R})/SO(d)$, describes the graviton degrees of freedom (with a Euclidean signature). To this aim, let us first recall that the internal part of the (inverse densitized) metric is recovered from the generalized vielbein via the $SO(16)$ invariant formula

$$\Delta^{-2} g^{mn} = \tfrac{1}{60} e^m_{\ \mathcal{A}} e^n_{\ \mathcal{A}} = \tfrac{1}{120}\left(e^m_{\ A} e^n_{\ A} + \tfrac{1}{2} e^m_{\ IJ} e^n_{\ IJ}\right) , \tag{3.72}$$

where the constraint (3.50) has been used. The summation on the r.h.s. breaks $E_{8(8)}$ to $SO(16)$ because of the plus sign in front of the second term; with a minus sign, the expression would vanish because of the constraints.

Just as for the standard vielbein, and having already introduced this terminology in the foregoing section, we now interpret the indices $\mathcal{M},\mathcal{N},\ldots$ and $\mathcal{A},\mathcal{B},\ldots$ appearing in (3.65) as "curved" and "flat", respectively. This immediately suggests the the following generalization of (3.72)

$$\mathcal{G}^{\mathcal{M}\mathcal{N}} := \tfrac{1}{120} \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \mathcal{V}^{\mathcal{N}}{}_{\mathcal{A}} \equiv \tfrac{1}{120}\left(\mathcal{V}^{\mathcal{M}}{}_{A} \mathcal{V}^{\mathcal{N}}{}_{A} + \tfrac{1}{2} \mathcal{V}^{\mathcal{M}}{}_{IJ} \mathcal{V}^{\mathcal{N}}{}_{IJ}\right) . \tag{3.73}$$

By construction, this metric is invariant under local $SO(16)$, which acts as

$$\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \longrightarrow \mathcal{V}^{\mathcal{M}}{}_{\mathcal{B}} \Sigma^{\mathcal{B}}{}_{\mathcal{A}} \tag{3.74}$$

on the 248-bein with Σ in the $\mathbf{120} \oplus \mathbf{128}$ representation of $SO(16)$ (and depending on all eleven coordinates). As we showed before, a "bosonized" version of local supersymmetry formally extends this to an action of a full local $E_{8(8)}$ acting from the right on $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ (which, however, does not leave $\mathcal{G}^{\mathcal{M}\mathcal{N}}$ inert any more).

Similarly, we now have a combined action of the internal coordinate and tensor gauge transformations on the 248-bein $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$, which is analogous to the action of general coordinate transformations on the standard vielbein. Namely, from the previous formulas (3.47) and (3.49) we can directly read off their action on the 36×248 submatrix of $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$:

$$\begin{aligned} \delta \mathcal{V}^m_{\mathcal{A}} &= \xi^p \partial_p \mathcal{V}^m_{\mathcal{A}} - \partial_p \xi^m\, \mathcal{V}^p_{\mathcal{A}} - \partial_p \xi^p\, \mathcal{V}^m_{\mathcal{A}} , \\ \delta \mathcal{V}_{mn\,\mathcal{A}} &= \xi^p \partial_p \mathcal{V}_{mn\,\mathcal{A}} - 2\partial_{[m} \xi^p\, \mathcal{V}_{n]p\,\mathcal{A}} - \partial_p \xi^p\, \mathcal{V}_{mn\,\mathcal{A}} , \end{aligned} \tag{3.75}$$

and

$$\begin{aligned} \delta \mathcal{V}^m_{\mathcal{A}} &= 0 , \\ \delta \mathcal{V}_{mn\,\mathcal{A}} &= \partial_p \xi_{mn}\, \mathcal{V}^p_{\mathcal{A}} + 2\partial_{[m} \xi_{n]p}\, \mathcal{V}^p_{\mathcal{A}} . \end{aligned} \tag{3.76}$$

This action can be extended to the full 248-bein via

$$\begin{aligned} \delta \mathcal{V} &= \xi^p \partial_p \mathcal{V} + \partial_p \xi^q \left(E_q{}^p - \tfrac{3}{8} \delta^p_q N\right) \mathcal{V} , \\ \delta \mathcal{V} &= \partial_p \xi_{qr} E^{pqr}\, \mathcal{V} \end{aligned} \tag{3.77}$$

by means of the $E_{8(8)}$ Lie algebra matrices given in section 3.1.2. It is important here that this action manifestly preserves $E_{8(8)}$: the transformed vielbein is still an element of $E_{8(8)}$. It is also straightforward to exponentiate the infinitesimal action to a full "diffeomorphism" generated by the pair (ξ^m, ξ_{mn}).

3.5 Supersymmetry variations of the generalized vielbein

The supersymmetry variations of the achtbein and the relevant components of the 3-form potential with our normalization read as follows in the $SO(8)$ basis:

$$\begin{aligned}\delta e_m{}^a &= \tfrac{1}{2}\left(\bar\varepsilon_\alpha\, \Gamma^a_{\alpha\dot\beta}\Psi_{m\dot\beta} - \bar\varepsilon_{\dot\alpha}\, \bar\Gamma^a_{\dot\alpha\beta}\Psi_{m\beta}\right) , \\ \delta A_{mnp} &= -\tfrac{3}{2}\left(\bar\varepsilon_\alpha\, (\Gamma_{[mn})_{\alpha\beta}\Psi_{p]\beta} - \bar\varepsilon_{\dot\alpha}\, (\bar\Gamma_{[mn})_{\dot\alpha\dot\beta}\Psi_{p]\dot\beta}\right) .\end{aligned} \tag{3.78}$$

(The relative minus signs in the second terms on the r.h.s. are due the fact that the Dirac conjugate spinors are appropriate to $d=3$ and differ from the ones in $d=11$ by an extra factor Γ^9). These formulas must now be compared with the $SO(16)$ covariant ones in terms of the generalized vielbein. The latter are most conveniently computed in terms of the matrices $\tilde Z^m$, $\tilde Z_{mn}$ from (3.59), making use of the $SL(8,\mathbb{R})$ decomposition of $E_{8(8)}$ described in section 3.1.2. In the special $SO(16)$ gauge (3.39), (3.41), (3.42) these matrices take the form

$$\begin{aligned}\tilde Z^m &= e^m{}_{\mathcal{A}}\, X^{\mathcal{A}} \;=\; 4\,\Delta^{-1} e_a{}^m\, Z^a , \\ \tilde Z_{mn} &= e_{mn\,\mathcal{A}}\, X^{\mathcal{A}} \;=\; 4\,\Delta^{-1} e_m{}^a\, e_n{}^b\, Z_{ab} + A_{mnp}\, \tilde Z^p ,\end{aligned} \tag{3.79}$$

in the upper Borel subalgebra. The $SO(16)$ covariant supersymmetry variations have been presented in (3.45) and can be written as

$$\delta\, \tilde Z^m = \left[\tilde Z^m\,,\,\omega\right] , \qquad \delta\, \tilde Z_{mn} = \left[\tilde Z_{mn}\,,\,\omega\right] , \tag{3.80}$$

with ω given by

$$\omega \;:=\; \tfrac{1}{4}\left(\Gamma^I_{A\dot A}\bar\varepsilon^I\chi^{\dot A}\, Y^A + \tfrac{1}{2}\,\omega^{IJ}_{\rm comp}\, X^{IJ}\right) , \tag{3.81}$$

where $\omega^{IJ}_{\rm comp}\; X^{IJ}$ is the compensating $SO(16)$ rotation to restore the triangular gauge of (3.79). Upon decomposition into the $SO(8)$ fields, the first term yields

$$\begin{aligned}\Gamma^I_{A\dot A}\bar\varepsilon^I\chi^{\dot A}\, Y^A \;=\;& 2\left(\bar\varepsilon_\alpha\Psi_{a\alpha} + \bar\varepsilon_{\dot\alpha}\Psi_{a\dot\alpha}\right)\,\left(Z^a{+}Z_a\right) \\ &+\left(\bar\varepsilon_\alpha\Gamma^{ab}_{\alpha\beta}\Psi_{b\beta} + \bar\varepsilon_{\dot\alpha}\bar\Gamma^{ab}_{\dot\alpha\dot\beta}\Psi_{b\dot\beta}\right)\,\left(Z^a{+}Z_a\right) \\ &-\tfrac{1}{2}\left(\bar\varepsilon_\alpha\Gamma^{abc}_{\alpha\dot\beta}\,\Psi_{c\dot\beta} + \bar\varepsilon_{\dot\alpha}\bar\Gamma^{abc}_{\dot\alpha\beta}\,\Psi_{c\beta}\right)\,\left(Z_{ab}{+}Z^{ab}\right) \\ &-\tfrac{1}{2}\left(\bar\varepsilon_\alpha\Gamma^{ab}_{\alpha\beta}\,\Psi_{c\beta} - \bar\varepsilon_{\dot\alpha}\bar\Gamma^{ab}_{\dot\alpha\dot\beta}\,\Psi_{c\dot\beta}\right)\,\left(E^{abc}{+}E_{abc}\right) \\ &-\left(\bar\varepsilon_\alpha\Gamma^{a}_{\alpha\dot\beta}\,\Psi_{b\dot\beta} - \bar\varepsilon_{\dot\alpha}\bar\Gamma^{a}_{\dot\alpha\beta}\,\Psi_{b\beta}\right)\,\left(E_a{}^b{+}E_b{}^a - \tfrac{3}{4}\delta^{ab}\,N\right) ,\end{aligned} \tag{3.82}$$

whereas $\omega^{IJ}_{\rm comp}X^{IJ}$ is determined in such a way as to rotate ω back into the upper Borel subalgebra. The resulting ω will then preserve the special gauge choice (3.79). Explicitly, it takes the form

$$
\begin{aligned}
\omega \;\equiv\;& \Gamma^I_{A\dot{A}} \bar{\varepsilon}^I \chi^{\dot{A}}\, Y^A + \tfrac{1}{2}\omega^{IJ}_{\text{comp}}\, X^{IJ} \\
=\;& 4\left(\bar{\varepsilon}_\alpha \Psi_{a\alpha} + \bar{\varepsilon}_{\dot{\alpha}} \Psi_{a\dot{\alpha}}\right) Z^a + 2\left(\bar{\varepsilon}_\alpha \Gamma^{ab}_{\alpha\beta} \Psi_{b\beta} + \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^{ab}_{\dot{\alpha}\dot{\beta}} \Psi_{b\dot{\beta}}\right) Z^a \\
&- \left(\bar{\varepsilon}_\alpha \Gamma^{abc}_{\alpha\dot{\beta}}\, \Psi_{c\dot{\beta}} + \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^{abc}_{\dot{\alpha}\beta}\, \Psi_{c\beta}\right) Z_{ab} \\
&- \left(\bar{\varepsilon}_\alpha \Gamma^{ab}_{\alpha\beta}\, \Psi_{c\beta} - \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^{ab}_{\dot{\alpha}\dot{\beta}}\, \Psi_{c\dot{\beta}}\right) E^{abc} \\
&- \left(\bar{\varepsilon}_\alpha \Gamma^{a}_{\alpha\dot{\beta}}\, \Psi_{b\dot{\beta}} - \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^{a}_{\dot{\alpha}\beta}\, \Psi_{b\beta}\right) \left(2E_a{}^b - \tfrac{3}{4}\delta^{ab}\, N\right)
\end{aligned}
\tag{3.83}
$$

From (3.80) we then find the supersymmetry variation:

$$
\begin{aligned}
\delta\, \tilde{Z}^m \;=\;& -2\Delta^{-1}\, Z^a \left(\bar{\varepsilon}_\alpha \Gamma^m_{\alpha\dot{\beta}}\, \Psi_{a\dot{\beta}} - \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^m_{\dot{\alpha}\beta}\, \Psi_{a\beta}\right) \\
&- 2\Delta^{-1} Z^a\, e_a{}^m \left(\bar{\varepsilon}_\alpha \Gamma^b_{\alpha\dot{\beta}}\, \Psi_{b\dot{\beta}} - \bar{\varepsilon}_{\dot{\alpha}} \bar{\Gamma}^b_{\dot{\alpha}\beta}\, \Psi_{b\beta}\right)
\end{aligned}
\tag{3.84}
$$

$$
\begin{aligned}
\delta\, \tilde{Z}_{mn} \;=\;& 4\Delta^{-1}\, Z_{ab}\, e_m{}^a \left(\bar{\varepsilon}_\alpha\, \Gamma^b_{\alpha\dot{\beta}} \Psi_{n\dot{\beta}} - \bar{\varepsilon}_{\dot{\alpha}}\, \bar{\Gamma}^b_{\dot{\alpha}\beta} \Psi_{n\beta}\right) \\
&- 2\Delta^{-1}\, Z_{ab}\, e_m{}^a e_n{}^b \left(\bar{\varepsilon}_\alpha\, \Gamma^c_{\alpha\dot{\beta}} \Psi_{c\dot{\beta}} - \bar{\varepsilon}_{\dot{\alpha}}\, \bar{\Gamma}^c_{\dot{\alpha}\beta} \Psi_{c\beta}\right) \\
&- \tfrac{3}{2}\, \tilde{Z}^p \left(\bar{\varepsilon}_\alpha\, (\Gamma_{[mn})_{\alpha\beta} \Psi_{p]\beta} - \bar{\varepsilon}_{\dot{\alpha}}\, (\bar{\Gamma}_{[mn})_{\dot{\alpha}\dot{\beta}} \Psi_{p]\dot{\beta}}\right) \\
&+ A_{mnp}\, \delta \tilde{Z}^p
\end{aligned}
\tag{3.85}
$$

and thus agreement with (3.78), since the one-but-last term is just $\delta A_{mnp} \tilde{Z}^p$.

Chapter 4

Yangian Symmetries in two Dimensions

In this chapter we discuss two-dimensional supergravity, where the hidden symmetries become infinite dimensional, generalizing the so-called Geroch group of general relativity [100, 14, 102].

The existence of infinite dimensional symmetries in these models is intimately linked to the integrability of the model. The model is integrable in the sense, that it is possible to define a linear system whose compatibility condition is equivalent to the equations of motion. This property has been discovered both for the bosonic models [126, 9, 14] and their locally supersymmetric extensions [141, 148]. In this way, the moduli space of classical solutions of the equations of motion can be viewed as an infinite dimensional quotient of the relevant affine Lie group by its (infinite dimensional) maximal compact subgroup. On this space the symmetry acts as a solution generating "isometry group", or as a group of "dressing transformations" [9, 155, 4, 10].

Our results underline the importance of quantum group structures for dimensionally reduced gravity and supergravity. The ultimate aim here is the identification of a "quantum Geroch group" which would act on the space of physical states in the same way as the classical Geroch group acts on the space of classical solutions. The relevance of these structures for string and M theory seems also quite obvious. After all, the resulting symmetries can be regarded as quantum deformations of the infinite dimensionally U-duality symmetries that have been conjectured to appear in compactified string and M theory. However, our results also indicate that some of the naive predictions put forward in this context may need to be revised. For instance, rather than with the arithmetic duality group $E_9(\mathbf{Z})$ one might end up with a discrete version of $Y(\mathrm{e}_8)$ in the full quantum theory.

4.1 $\boldsymbol{R}$-matrix and the Yangian $\boldsymbol{Y(\mathrm{e}_8)}$

The Yangian $Y(\mathrm{e}_8)$ [46] is recursively defined as the associative algebra with generators $\mathcal{X}^{\mathcal{A}}$ and $\mathcal{Y}^{\mathcal{A}}$ ($\mathcal{A} = 1, \ldots, 248$) and relations

$$\begin{aligned}
&[\mathcal{X}^{\mathcal{A}}, \mathcal{X}^{\mathcal{B}}] \;=\; \mathrm{i}\hbar f^{\mathcal{AB}}{}_{\mathcal{C}}\, \mathcal{X}^{\mathcal{C}} \,, \qquad [\mathcal{X}^{\mathcal{A}}, \mathcal{Y}^{\mathcal{B}}] \;=\; \mathrm{i}\hbar f^{\mathcal{AB}}{}_{\mathcal{C}}\, \mathcal{Y}^{\mathcal{C}} \,, \\
&[\mathcal{Y}^{\mathcal{A}}\,[\mathcal{Y}^{\mathcal{B}}, \mathcal{X}^{\mathcal{C}}]] - [\mathcal{X}^{\mathcal{A}}\,[\mathcal{Y}^{\mathcal{B}}, \mathcal{Y}^{\mathcal{C}}]] \;=\; -\hbar^2 L^{\mathcal{ABC}} \,, \\
&\text{with}\;\; L^{\mathcal{ABC}} = \tfrac{1}{24} f^{\mathcal{AD}}{}_{\mathcal{G}} f^{\mathcal{BE}}{}_{\mathcal{H}} f^{\mathcal{CF}}{}_{\mathcal{I}} f^{\mathcal{GHI}} \{\mathcal{X}^{\mathcal{D}}, \mathcal{X}^{\mathcal{E}}, \mathcal{X}^{\mathcal{F}}\} \,, \\
&\text{and}\;\; \{\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3\} = \sum_{\sigma} \mathcal{X}^{\sigma(1)} \mathcal{X}^{\sigma(2)} \mathcal{X}^{\sigma(3)} \,.
\end{aligned} \tag{4.1}$$

It admits a nontrivial coproduct and antipode structure whose explicit form will not be needed here, see e.g. Thm. 12.1.1 of [24] for details.

Due to the fact that $L^{\mathcal{ABC}}$ does not vanish when the $\mathcal{X}^{\mathcal{A}}$ are evaluated in the fundamental representation of e_8, it is not possible to lift this representation of e_8 to a representation of $Y(\mathrm{e}_8)$. Rather, the minimal representation of $Y(\mathrm{e}_8)$ is reducible over e_8 and contains an additional trivial representation of e_8 [46]. With respect to so(16) we thus have the decomposition

$$\mathbf{249} \to \mathbf{1} \oplus \mathbf{120} \oplus \mathbf{128}\,. \tag{4.2}$$

For compactness of notation, we will label the extra singlet by 0 and use hatted indices which run over all 249 dimensions, i.e. $0 \le \hat{\mathcal{A}}, \hat{\mathcal{B}}, \dots \le 248$.

The R-matrix associated with the fundamental representation of $Y(\mathrm{e}_8)$ is the solution $R(w)$ to the Quantum Yang-Baxter Equation ($\equiv$ QYBE)

$$\overset{12}{R}(u-v)\overset{13}{R}(u)\overset{23}{R}(v) = \overset{23}{R}(v)\overset{13}{R}(u)\overset{12}{R}(u-v) \tag{4.3}$$

or, with indices written out,

$$R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{G}}\hat{\mathcal{H}}}(u-v)R_{\hat{\mathcal{G}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{I}}}(u)R_{\hat{\mathcal{H}}\hat{\mathcal{I}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{R}}}(v) \;=\; R_{\hat{\mathcal{B}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{H}}\hat{\mathcal{I}}}(v)R_{\hat{\mathcal{A}}\hat{\mathcal{I}}}{}^{\hat{\mathcal{G}}\hat{\mathcal{R}}}(u)R_{\hat{\mathcal{G}}\hat{\mathcal{H}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{Q}}}(u-v)\,, \tag{4.4}$$

The classical limit is

$$R(w) = \mathbb{1} - \frac{\mathrm{i}\hbar}{w}\,\Omega_{\mathrm{e}_8} + \mathcal{O}\left(\tfrac{\hbar^2}{w^2}\right) \qquad \text{for} \quad w \to \infty\,. \tag{4.5}$$

where the definition of the Casimir element Ω_{e_8} is extended to $\mathbf{1}\oplus\mathbf{248}$ by a trivial (zero) action on the $\mathbf{1}$. We also impose the standard normalization condition

$$R(0) = \Pi\,. \tag{4.6}$$

Within the tensor product $\mathbf{249}\otimes\mathbf{249}$ we introduce in addition to the operators from (3.22) the projector $\mathcal{P}_0$ onto the one-dimensional space $\mathbf{1}\otimes\mathbf{1}$ and the projectors $\mathcal{P}_+$ and $\mathcal{P}_-$ onto the symmetric and onto the antisymmetric part of the space $(\mathbf{248}\otimes\mathbf{1})\oplus(\mathbf{1}\otimes\mathbf{248})$, respectively. Furthermore, there are e_8 invariant intertwining operators between subspaces of the same dimension, which we denote by $\mathcal{I}_{01}$, $\mathcal{I}_{10}$, $\mathcal{I}_{+248}$, and $\mathcal{I}_{248+}$. They are defined by

$$\begin{aligned} \mathcal{I}_{01}\mathcal{I}_{10} &= \mathcal{P}_0 \qquad \mathcal{I}_{10}\mathcal{I}_{01} = \mathcal{P}_1\,, \\ \mathcal{I}_{+248}\mathcal{I}_{248+} &= \mathcal{P}_+ \qquad \mathcal{I}_{248+}\mathcal{I}_{+248} = \mathcal{P}_{248}\,, \end{aligned} \tag{4.7}$$

respectively, up to a relative factor between the intertwiners which drops out in the above relations. Explicitly, the new projectors and intertwiners are given by

$$\begin{aligned}
(\mathcal{P}_0)_{00}{}^{00} &= 1\,,\\
(\mathcal{P}_+)_{\mathcal{A}0}{}^{\mathcal{B}0} &= (\mathcal{P}_+)_{\mathcal{A}0}{}^{0\mathcal{B}} = (\mathcal{P}_+)_{0\mathcal{A}}{}^{\mathcal{B}0} = (\mathcal{P}_+)_{0\mathcal{A}}{}^{0\mathcal{B}} = \tfrac{1}{2}\delta^{\mathcal{A}}_{\mathcal{B}}\,,\\
(\mathcal{P}_-)_{\mathcal{A}0}{}^{\mathcal{B}0} &= -(\mathcal{P}_-)_{\mathcal{A}0}{}^{0\mathcal{B}} = -(\mathcal{P}_-)_{0\mathcal{A}}{}^{\mathcal{B}0} = (\mathcal{P}_-)_{0\mathcal{A}}{}^{0\mathcal{B}} = \tfrac{1}{2}\delta^{\mathcal{A}}_{\mathcal{B}}\,,\\
\\
(\mathcal{I}_{01})_{00}{}^{\mathcal{AB}} &= \eta^{\mathcal{AB}}\,,\\
(\mathcal{I}_{10})_{\mathcal{AB}}{}^{00} &= \tfrac{1}{248}\eta_{\mathcal{AB}}\,,\\
(\mathcal{I}_{+248})_{0\mathcal{A}}{}^{\mathcal{BC}} &= (\mathcal{I}_{+248})_{\mathcal{A}0}{}^{\mathcal{BC}} = \tfrac{1}{120}f_{\mathcal{A}}{}^{\mathcal{BC}}\,,\\
(\mathcal{I}_{248+})_{\mathcal{AB}}{}^{\mathcal{C}0} &= (\mathcal{I}_{248+})_{\mathcal{AB}}{}^{0\mathcal{C}} = -f_{\mathcal{AB}}{}^{\mathcal{C}}\,,
\end{aligned} \tag{4.8}$$

with all other components vanishing. Again all operators are symmetric w.r.t. interchange of the two subspaces with the exception of the intertwiners $\mathcal{I}_{+248}$ and $\mathcal{I}_{248+}$, which obey

$$\overset{12}{\mathcal{I}}_{+248} = -\overset{21}{\mathcal{I}}_{+248}\,, \qquad \overset{12}{\mathcal{I}}_{248+} = -\overset{21}{\mathcal{I}}_{248+}\,. \tag{4.9}$$

For this reason, the R-matrix also fails to be symmetric under this interchange, i.e. $\overset{12}{R}(w) \neq \overset{21}{R}(w)$.

As shown in [23], the R-matrix associated to the fundamental representation of $Y(\mathrm{e}_8)$ in terms of these projectors and intertwiners is given by

$$\begin{aligned}
f^{-1}(w)R(w) &= \tfrac{w+\mathrm{i}\hbar}{w-\mathrm{i}\hbar}\,\mathcal{P}_{30380} + \mathcal{P}_{27000} + \tfrac{w^3+15w^2\mathrm{i}\hbar+44w(\mathrm{i}\hbar)^2+60(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{P}_{248}\\
&\quad + \tfrac{(w+\mathrm{i}\hbar)(w+6\mathrm{i}\hbar)}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)}\,\mathcal{P}_{3875} + \tfrac{w^3-15w^2\mathrm{i}\hbar+44w(\mathrm{i}\hbar)^2-60(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{P}_{+}\\
&\quad + \tfrac{w+\mathrm{i}\hbar}{w-\mathrm{i}\hbar}\,\mathcal{P}_{-} + \tfrac{w^4+30w^3\mathrm{i}\hbar+269w^2(\mathrm{i}\hbar)^2+660w(\mathrm{i}\hbar)^3+900(\mathrm{i}\hbar)^4}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{P}_{1}\\
&\quad + \tfrac{w^4-30w^3\mathrm{i}\hbar+269w^2(\mathrm{i}\hbar)^2-660w(\mathrm{i}\hbar)^3+900(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{P}_{0}\\
&\quad + \tfrac{w(\mathrm{i}\hbar)^3}{\alpha^2(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{I}_{01}\\
&\quad + \tfrac{248(60\alpha)^2w(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{I}_{10}\\
&\quad - \tfrac{60\sqrt{2}w(\mathrm{i}\hbar)^2}{\alpha(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{I}_{+248}\\
&\quad + \tfrac{30\sqrt{2}\alpha w(\mathrm{i}\hbar)^2}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{I}_{248+}\,.
\end{aligned}$$

We rewrite this in the form

$$f^{-1}(w)R(w) = \mathbb{1} + \sum_{j=1}^{4}\frac{\mathcal{R}_j}{w-w_j}\,, \tag{4.10}$$

where the poles are located at

$$w_1 = \mathrm{i}\hbar\,,\ w_2 = 6\mathrm{i}\hbar\,,\ w_3 = 10\mathrm{i}\hbar\,,\ w_4 = 15\mathrm{i}\hbar\,, \tag{4.11}$$

and the associated residues are

$$
\begin{aligned}
\mathcal{R}_1 &= 2\,\mathcal{P}_{30380} - \tfrac{14}{5}\,\mathcal{P}_{3875} + \tfrac{8}{3}\,\mathcal{P}_{248} - \tfrac{2}{3}\,\mathcal{P}_{+} + 2\,\mathcal{P}_{-} - \tfrac{62}{21}\,\mathcal{P}_{1} \\
&\quad - \tfrac{16}{21}\,\mathcal{P}_{0} - \tfrac{4\sqrt{2}}{3\alpha}\,\mathcal{I}_{+248} + \tfrac{2\sqrt{2}\alpha}{3}\,\mathcal{I}_{248+} - \tfrac{9920\alpha^2}{7}\,\mathcal{I}_{10} - \tfrac{1}{630\alpha^2}\,\mathcal{I}_{01}\;, \\
\mathcal{R}_2 &= \tfrac{84}{5}\,\mathcal{P}_{3875} - 54\,\mathcal{P}_{248} + 6\,\mathcal{P}_{+} + 124\,\mathcal{P}_{1} + 8\,\mathcal{P}_{0} + \tfrac{18\sqrt{2}}{\alpha}\,\mathcal{I}_{+248} \\
&\quad - 9\sqrt{2}\alpha\,\mathcal{I}_{248+} + 29760\alpha^2\,\mathcal{I}_{10} + \tfrac{1}{30\alpha^2}\,\mathcal{I}_{01}\;, \\
\mathcal{R}_3 &= \tfrac{250}{3}\,\mathcal{P}_{248} - \tfrac{10}{3}\,\mathcal{P}_{+} - \tfrac{1240}{3}\,\mathcal{P}_{1} - \tfrac{20}{3}\,\mathcal{P}_{0} - \tfrac{50\sqrt{2}}{3\alpha}\,\mathcal{I}_{+248} \\
&\quad + \tfrac{25\sqrt{2}\alpha}{3}\,\mathcal{I}_{248+} - 49600\alpha^2\,\mathcal{I}_{10} - \tfrac{1}{18\alpha^2}\,\mathcal{I}_{01}\;, \\
\mathcal{R}_4 &= \tfrac{2480}{7}\,\mathcal{P}_{1} + \tfrac{10}{7}\,\mathcal{P}_{0} + \tfrac{148800\alpha^2}{7}\,\mathcal{I}_{10} + \tfrac{1}{42\alpha^2}\,\mathcal{I}_{01}\;.
\end{aligned} \tag{4.12}
$$

The scalar function f is uniquely defined by its functional equation

$$
f(w)f(w-15\mathrm{i}\hbar) = \frac{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}{w(w-5\mathrm{i}\hbar)(w-9\mathrm{i}\hbar)(w-14\mathrm{i}\hbar)}\;, \tag{4.13}
$$

and its asymptotic behavior

$$
f(w) = 1 - \tfrac{2\mathrm{i}\hbar}{w} + \mathcal{O}\left(\tfrac{1}{w^2}\right) \quad \text{for} \quad w \to \pm\infty\;. \tag{4.14}
$$

It allows an explicit expression in terms of Γ-functions which however is not of particular interest for the following. Observe that (4.13) and (4.14) already imply the relations

$$
f(w)f(-w) = 1\;, \qquad f(w)^* = f(-w^*)\;. \tag{4.15}
$$

The free parameter α which appears in the solution of the QYBE is basically a consequence of the fact that the singlet in (4.2) may be rescaled with an arbitrary factor; two R matrices (4.10) with different values of α are related by conjugation with $\mathrm{diag}(\alpha_1\alpha_2^{-1}, \mathbb{1}_{120}, \mathbb{1}_{128}) \otimes \mathrm{diag}(\alpha_1\alpha_2^{-1}, \mathbb{1}_{120}, \mathbb{1}_{128})$. Without loss of generality we can thus fix the parameter α to

$$
60\alpha^2 := -1\;. \tag{4.16}
$$

For this value only, the R-matrix obeys the additional non-covariant relation

$$
R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w) = R_{\hat{\mathcal{C}}\hat{\mathcal{D}}}{}^{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w)\;, \tag{4.17}
$$

which is proved by inspection and by use of the special (non-covariant) property $f_{\mathcal{A}}{}^{\mathcal{BC}} = -f^{\mathcal{A}}{}_{\mathcal{BC}}$ of the $E_{8(8)}$ structure constants (3.4).

The following further properties of the R-matrix are easily verified:

$$
\overset{12}{R}(w)\,\overset{21}{R}(-w) = \mathbb{1}, \tag{4.18}
$$

$$
\overset{12}{R}(w)^* = \overset{21}{R}(-w^*)\;, \tag{4.19}
$$

where the second equation is only valid for imaginary α, which is compatible with our choice (4.16) above. In the context of two-dimensional scattering theory, these relations express the requirements of unitarity and hermiticity of the S-matrix, respectively. With indices written out they acquire the following explicit form

$$R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{G}}\hat{\mathcal{H}}}(w)R_{\hat{\mathcal{H}}\hat{\mathcal{G}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(-w) \;=\; \delta^{\hat{\mathcal{D}}}_{\hat{\mathcal{A}}}\delta^{\hat{\mathcal{C}}}_{\hat{\mathcal{B}}}\,, \tag{4.20}$$

$$\Big(R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w)\Big)^* \;=\; R_{\hat{\mathcal{B}}\hat{\mathcal{A}}}{}^{\hat{\mathcal{D}}\hat{\mathcal{C}}}(-w^*)\,. \tag{4.21}$$

The occurrence of poles at $w = w_j$ and relation (4.18) together imply that $R(w)$ is non-invertible at the points $w = -w_j$. More specifically, (4.18) yields the relations

$$\overset{12}{\mathcal{R}_j}\,\overset{21}{R}\,(-w_j) = 0 \tag{4.22}$$

From the formulas given above it is straightforward to check that the residue $\mathcal{R}_4$ at $w_4 = 15\mathrm{i}\hbar$ is singled out by its property of being proportional to a one-dimensional projector:

$$(\mathcal{R}_4)_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}} = \tfrac{10}{7}\,\eta_{\hat{\mathcal{A}}\hat{\mathcal{B}}}\eta^{\hat{\mathcal{C}}\hat{\mathcal{D}}}\,, \tag{4.23}$$

where $\eta_{\hat{\mathcal{A}}\hat{\mathcal{B}}}$ denotes the natural extension of the Cartan-Killing form into $\mathbf{249}\otimes\mathbf{249}$ given by the additional entry $\eta_{00} = 60\alpha^2 = -1$. Evaluating the QYBE (4.4) at $u-v = 15\mathrm{i}\hbar$ then gives rise to the following relation

$$(\mathcal{R}_4)_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{G}}\hat{\mathcal{H}}}R_{\hat{\mathcal{G}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{I}}}(u)R_{\hat{\mathcal{H}}\hat{\mathcal{I}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{R}}}(u-15\mathrm{i}\hbar) \;=\; \delta^{\hat{\mathcal{R}}}_{\hat{\mathcal{C}}}\;(\mathcal{R}_4)_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{Q}}}\,. \tag{4.24}$$

From these observations, we can deduce the crossing invariance property of the R-matrix:

$$R_{\hat{\mathcal{B}}}{}^{\hat{\mathcal{A}}\hat{\mathcal{D}}}{}_{\hat{\mathcal{C}}}(-w) \equiv \eta^{\hat{\mathcal{A}}\hat{\mathcal{G}}}R_{\hat{\mathcal{B}}\hat{\mathcal{G}}}{}^{\hat{\mathcal{D}}\hat{\mathcal{H}}}(15\mathrm{i}\hbar - w)\eta_{\hat{\mathcal{H}}\hat{\mathcal{C}}} = R_{\hat{\mathcal{C}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{A}}\hat{\mathcal{D}}}(w)\,. \tag{4.25}$$

The knowledge of the R-matrix associated to an irreducible representation of (4.1) now gives rise to another equivalent presentation of the Yangian algebra itself [46]. Consider the associative algebra with generators $(T_{(n)})_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}$, $(0 \le \hat{\mathcal{A}}, \hat{\mathcal{B}} \le 248)$, $n \in \mathbb{N}$ and defining relations

$$R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{E}}\hat{\mathcal{F}}}(u-v)T_{\hat{\mathcal{E}}}{}^{\hat{\mathcal{C}}}(u)T_{\hat{\mathcal{F}}}{}^{\hat{\mathcal{D}}}(v) \;=\; T_{\hat{\mathcal{B}}}{}^{\hat{\mathcal{F}}}(v)T_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{E}}}(u)R_{\hat{\mathcal{E}}\hat{\mathcal{F}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(u-v) \tag{4.26}$$

where $T_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}(u)$ denotes the formal series

$$T_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}(u) \;=\; \delta^{\hat{\mathcal{B}}}_{\hat{\mathcal{A}}} \;+\; \sum_{n=1}^{\infty}(T_{(n)})_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}\,u^{-n}\,. \tag{4.27}$$

The QYBE (4.4) ensures compatibility of the exchange relations (4.26) with associativity of the multiplication. Evaluation of these relations at $u - v = 15\mathrm{i}\hbar$ shows that there exists an invariant scalar quantity $q(T(u))$, the "quantum determinant", which is bilinear in the matrix entries of T:

$$q(T(u))\mathcal{R}_4 \;:=\; \mathcal{R}_4\,\overset{1}{T}(u+15\mathrm{i}\hbar)\,\overset{2}{T}(u) \;=\overset{2}{T}(u)\,\overset{1}{T}(u+15\mathrm{i}\hbar)\,\mathcal{R}_4\,. \tag{4.28}$$

Using (4.24) one checks that $q(T(u))$ lies in the center of the algebra (4.26). Thus, we may pass to the quotient of this algebra over the two-sided ideal generated by the central element by setting $q(T) = 1$ or equivalently

$$T_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{C}}}(u - 15\mathrm{i}\hbar)T_{\hat{\mathcal{B}}}{}^{\hat{\mathcal{D}}}(u)\eta_{\hat{\mathcal{C}}\hat{\mathcal{D}}} \;=\; \eta_{\hat{\mathcal{A}}\hat{\mathcal{B}}}\,. \tag{4.29}$$

It has been stated by Drinfeld [46] that this quotient is isomorphic to the Yangian $Y(\mathrm{e}_8)$ as defined at the beginning of this section (4.1).[1] The precise isomorphism requires knowledge of the universal R-matrix of $Y(\mathrm{e}_8)$ which is certainly beyond our scope here; one may however easily identify the generating elements $\mathcal{X}^a$ and $\mathcal{Y}^a$

$$\begin{aligned}\mathcal{X}^a &= \mathrm{tr}\left[X^a T_{(1)}\right]\,, \\ \mathcal{Y}^a &= \mathrm{tr}\left[X^a\left(T_{(2)} - \tfrac{1}{2}T_{(1)}T_{(1)}\right)\right]\,.\end{aligned} \tag{4.30}$$

We can further expand (4.26) around $u = \infty$ and use (4.5) to obtain the commutator

$$\Big[\overset{1}{T}_{(1)}\,,\,\overset{2}{T}(w)\Big] \;=\; \mathrm{i}\hbar\Big[\Omega_{\mathrm{e}_8}\,,\,\overset{2}{T}(w)\Big]\,, \tag{4.31}$$

which in particular reproduces the first two commutation relations of (4.1). We close the general discussion here with two well-known properties of the presentation (4.26) of the Yangian

- Any representation ρ of $Y(\mathrm{e}_8)$ defines a one-parameter family of representations ρ_a labeled by a complex number a:

  $$\rho_a(T(w)) := \rho(T(w - a))\,. \tag{4.32}$$

  The fundamental representation **249** in particular gives rise to the family $\mathbf{249}_a$:

  $$\rho_a^{\mathbf{249}}(T(w)) := R(w - a)\,, \tag{4.33}$$

  with the R-matrix from (4.10). Note that the relation (4.29) in these representations corresponds to the normalization (4.24) of the R-matrix.

- The coproduct of $Y(\mathrm{e}_8)$ takes the simple form:

  $$\Delta\left(T_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}(w)\right) = T_{\hat{a}}{}^{\hat{\mathcal{C}}}(w) \otimes T_{\hat{\mathcal{C}}}{}^{\hat{\mathcal{B}}}(w)\,. \tag{4.34}$$

4.2 Classical Yangian symmetries in $N = 16$ supergravity

The scalar sector of two-dimensional $N = 16$ supergravity is described by an $E_{8(8)}$-valued matrix $\mathcal{V}$ which transforms under a global $E_{8(8)}$ symmetry and a local $SO(16)$ gauge symmetry in the usual way

$$\mathcal{V}(x) \mapsto g\mathcal{V}(x)h(x)\,, \qquad g \in E_{8(8)}\,,\quad h(x) \in SO(16)\,. \tag{4.35}$$

[1] However, we presently cannot exclude the possibility that the center of (4.26) contains elements of higher degree in the T's which are not generated by $q(T)$.

Thus, its bosonic configuration space is given by the coset space $E_{8(8)}/SO(16)$. It may be parametrized by the symmetric $E_{8(8)}$-valued matrix

$$\mathcal{M} \equiv \mathcal{V}\mathcal{V}^T \ , \quad \text{i.e.} \quad \mathcal{M}_{\mathcal{A}\mathcal{B}} \equiv \mathcal{V}_{\mathcal{A}}{}^{\mathcal{C}}\mathcal{V}_{\mathcal{B}}{}^{\mathcal{C}} = \mathcal{M}_{\mathcal{B}\mathcal{A}} \ , \tag{4.36}$$

which is evidently gauge ($= SO(16)$) invariant. The symmetry of $\mathcal{M}$ may be characterized algebraically by the fact that it is annihilated by the antisymmetric projectors

$$(\mathcal{P}_{248})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ = \ 0 \ = \ (\mathcal{P}_{30380})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ , \tag{4.37}$$

whereas for the symmetric projectors from (3.25) one finds the following identities

$$\begin{aligned}
(\mathcal{P}_1)_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ &= \ \tfrac{1}{31}\eta_{\mathcal{A}\mathcal{B}} \quad \Longleftrightarrow \quad \eta^{\mathcal{A}\mathcal{B}}\mathcal{M}_{\mathcal{A}\mathcal{B}} = 8 \ , \\
(\mathcal{P}_{3875})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ &= \ 0 \ , \\
(\mathcal{P}_{27000})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ &= \ \mathcal{M}_{\mathcal{A}\mathcal{B}} - \tfrac{1}{31}\eta_{\mathcal{A}\mathcal{B}} \ .
\end{aligned} \tag{4.38}$$

To verify these relations one needs the $E_{8(8)}$-invariance of the projectors and the Cartan-Killing form:

$$(\mathcal{P}_j)_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}}\mathcal{M}_{\mathcal{C}\mathcal{D}} \ = \ \mathcal{V}_{\mathcal{A}}{}^{\mathcal{C}}\mathcal{V}_{\mathcal{B}}{}^{\mathcal{D}}(\mathcal{P}_j)_{\mathcal{C}\mathcal{D}}{}^{\mathcal{E}\mathcal{E}} \ , \qquad \mathcal{V}_{\mathcal{A}}{}^{\mathcal{C}}\mathcal{V}_{\mathcal{B}}{}^{\mathcal{D}} \ \eta_{\mathcal{C}\mathcal{D}} \ = \ \eta_{\mathcal{A}\mathcal{B}} \ . \tag{4.39}$$

The second relation in (4.38) requires the additional formula

$$f^{\mathcal{D}}{}_{\mathcal{A}\mathcal{C}}\, f_{\mathcal{D}\mathcal{B}\mathcal{C}} = \begin{cases} 8\delta^{IJ}_{KL} & \text{if } (\mathcal{A}\mathcal{B}) = (IJ, KL) \\ 0 & \text{otherwise} \end{cases} \tag{4.40}$$

where the summation over the $E_{8(8)}$ index $\mathcal{C}$ is with the "wrong metric" (i.e. with the $SO(16)$-covariant $\delta_{\mathcal{A}\mathcal{B}}$ rather than $\eta_{\mathcal{A}\mathcal{B}}$).

The scalar fields represented by the matrix $\mathcal{M}$ satisfy equations of motion which allow a Lax pair formulation [9, 126, 14, 140, 148]. In particular this allows the construction of an infinite family of nonlocal integrals of motion which are obtained from the transition matrices associated to the Lax pair [121, 145]. These integrals of motion are encoded in a symmetric $E_{8(8)}$-valued matrix $\mathcal{M}(w)$ obtained by integrating the Lax connection over certain space intervals and depending on a complex spectral parameter w. This matrix parameterizes the full scalar sector of the phase space in the sense that for real values of w the matrix $\mathcal{M}(w)$ coincides with the physical scalar fields $\mathcal{M}(x)$ evaluated on the particular axis in space-time where the dilaton field ρ vanishes

$$\mathcal{M}(w) = \mathcal{M}(x)\Big|_{\rho(x)=0,\tilde{\rho}(x)=w} \ . \tag{4.41}$$

We may further introduce its Riemann-Hilbert decomposition

$$\mathcal{M}_{\mathcal{A}\mathcal{B}}(w) \equiv U_+(w)_{\mathcal{A}}{}^{\mathcal{C}}\, U_-(w)_{\mathcal{B}}{}^{\mathcal{C}} \ , \tag{4.42}$$

into E_8-valued functions $U_\pm(w)$ which are holomorphic in the upper and the lower half of the complex w-plane, respectively. They are related by complex conjugation

$$(U_+(w))^* \ = \ U_-(w^*) \ . \tag{4.43}$$

In [121, 145] it was shown that these phase space quantities are subject to the following symplectic structure:

$$\begin{aligned}\{\mathcal{M}_{\mathcal{AB}}(v)\,,\,\mathcal{M}_{\mathcal{CD}}(w)\} \;\;&= \\ &\frac{1}{v-w}\Big((\Omega_{\mathrm{e}_8})_{\mathcal{AC}}{}^{\mathcal{MN}}\mathcal{M}_{\mathcal{MB}}(v)\mathcal{M}_{\mathcal{ND}}(w) + \mathcal{M}_{\mathcal{AM}}(v)\mathcal{M}_{\mathcal{CN}}(w)\,(\Omega_{\mathrm{e}_8})_{\mathcal{BD}}{}^{\mathcal{MN}} \\ &- \mathcal{M}_{\mathcal{AM}}(v)\,(\Omega^\tau_{\mathrm{e}_8})_{\mathcal{MC}}{}^{\mathcal{BN}}\mathcal{M}_{\mathcal{ND}}(w) - \mathcal{M}_{\mathcal{CM}}(w)\,(\Omega^\tau_{\mathrm{e}_8})_{\mathcal{AN}}{}^{\mathcal{MD}}\mathcal{M}_{\mathcal{NB}}(v)\Big)\,,\end{aligned} \tag{4.44}$$

with Ω_{e_8} and $\Omega^\tau_{\mathrm{e}_8}$ from (3.20) and (3.21), respectively. One may check that these Poisson brackets are covariant under $E_{8(8)}$ and compatible with the symmetry of $\mathcal{M}$ (4.37), as required for consistency. For the purpose of quantization to be addressed in the next section it is further convenient to decompose this structure according to (4.42) into the following brackets

$$\begin{aligned}\left\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\pm(w)\right\} \;&=\; \left[\frac{2\Omega_{\mathrm{e}_8}}{v-w}\,,\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\pm(w)\right]\,, \\ \left\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\mp(w)\right\} \;&=\; \frac{2\Omega_{\mathrm{e}_8}}{v-w}\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w)\frac{2\Omega^\tau_{\mathrm{e}_8}}{v-w}\,.\end{aligned} \tag{4.45}$$

In a theory with local symmetries, observables such as the conserved non-local charges contained in $U_\pm(w)$ must weakly commute with the associated canonical constraints. For the above charges this was shown to be the case in [145]. Namely, for the traceless components $T'_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho{}_\rho$ of the energy momentum tensor (generating local translations along the light-cone), we simply have

$$\left\{T'_{\mu\nu}(z)\,,\,U_\pm(w)\right\} = 0 \tag{4.46}$$

This relation expresses the invariance of the charges $U_\pm(w)$ under general coordinate transformations, which thus indeed constitute "observables" in the sense of Dirac.

In supergravity, we have in addition the constraints $S^I_\alpha(z)$ generating $N = 16$ local supersymmetry transformations (α is a spinor index in two dimensions). As shown in [145], the relations expressing the invariance of the charges $U_\pm(w)$ under local supersymmetry are considerably more complicated than (4.46). Recalling that the integrals of motion $U_\pm(w)$ are obtained from certain transition matrices $U(x, y; w)$ associated to the Lax pair of the model, we found that they obey Poisson bracket of the following type (for $x < z < y$):

$$\left\{U(x,y;w), S^I_\alpha(z)\right\} \;\;\sim\;\; U(x,z;w)\,X^{IJ}S^J_\alpha(z)\,U(z,y;w)\,, \tag{4.47}$$

which vanish indeed on the constraint surface. Apart from questions of operator ordering, it is clear from the form of (4.47) that the combined algebra of nonlocal charges and supersymmetry constraints does not close. It remains an open problem at this point whether one can arrive at a closed structure upon sufficient enlargement of the algebra. Quantization of this structure would entail the existence of a novel type of exchange relations between the conserved charges and the local supersymmetry constraints. The full algebra should then contain the Yangian charge algebra to be presented in the next section as well as a quantized version of the $N = 16$ superconformal algebra, into which the supersymmetry constraints close.

4.3 Quantization

We now wish to quantize the symplectic structure of the classical charge algebra by means of the R-matrix described above. This amounts to replacing the Poisson brackets (4.45) by quantum exchange relations, leading to a "twisted" Yangian double with central extension c. More precisely, we employ the construction (4.26) to replace the classically conserved non-local charges $U_\pm(w)$ (which by their definition are 248×248 matrices) by a corresponding set of 249×249 matrices $T_\pm(w)$ with operator-valued entries subject to the exchange relations

$$\overset{12}{R}(v-w)\,\overset{1}{T}_\pm(v)\,\overset{2}{T}_\pm(w) \;=\; \overset{2}{T}_\pm(w)\,\overset{1}{T}_\pm(v)\;\overset{12}{R}(v-w)\,, \tag{4.48}$$

$$\overset{12}{R}(v-w-\mathrm{i}\hbar c)\,\overset{1}{T}_-(v)\,\overset{2}{T}_+(w) \;=\; \overset{2}{T}_+(w)\,\overset{1}{T}_-(v)\;\overset{12}{Q}(v-w)\,, \tag{4.49}$$

where for the "twisted" R-matrix Q we require the classical expansion (cf. (4.5))

$$Q(w) = \mathbb{1} - \frac{\mathrm{i}\hbar}{w}\Omega^\tau_{\mathrm{e}_8} + \mathcal{O}\left(\frac{\hbar^2}{w^2}\right)\,, \tag{4.50}$$

and the compatibility relations

$$\begin{aligned}\overset{12}{Q}(u-v)\,\overset{13}{Q}(u)\,\overset{23}{R}(v) &= \overset{23}{R}(v)\,\overset{13}{Q}(u)\,\overset{12}{Q}(u-v)\,,\\ \overset{12}{R}(u-v)\,\overset{13}{Q}(u)\,\overset{23}{Q}(v) &= \overset{23}{Q}(v)\,\overset{13}{Q}(u)\,\overset{12}{R}(u-v)\,,\end{aligned} \tag{4.51}$$

whose derivation is completely analogous to (4.4). For $60\alpha^2=-1$, the solution is given by

$$Q_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w) \;:=\; R_{\hat{\mathcal{D}}\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}\hat{\mathcal{C}}}(-w) = R_{\hat{\mathcal{B}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{D}}\hat{\mathcal{A}}}(-w)\,, \tag{4.52}$$

i.e. by interchanging the two subspaces and taking the transpose of the original R-matrix in one of them. The interchange of subspaces here is necessary because $\overset{12}{R}\neq\overset{21}{R}$. It is easy to check that the above definition yields the correct first order term displayed in (4.50). Furthermore, although transposing the indices is a non-covariant operation, it turns out that all summations in (4.51) are again covariant, such that with a little algebra these relations can be reduced to the original QYBE (4.4).

We emphasize that the shift c (alias the central charge) in (4.49) is compatible with all of our requirements so far and therefore still arbitrary at this point. It is important here that the algebras for different c are not isomorphic; in particular, they may have different ideals. The central charge c will be fixed later by requiring symmetry of the quantum monodromy matrix. Note that a possible additional shift in the argument of Q in (4.52) has been absorbed into a redefinition of T_-.

As for the singular points, there is an important difference between (4.48) and (4.49): whereas the poles on the l.h.s. and r.h.s. of (4.48) always match, this is not so for (4.49) due to the shift. Thus, either some of the mixed operator products are singular, or the regularity on one side imposes the vanishing of certain residues on the other side. These questions as well as the proper quantum analogue of the classical holomorphy properties of the $T_\pm$'s may however only be addressed after specializing to a particular representation of (4.48), (4.49).

To be on the safe side here, we will use the exchange relations only at the generic points where the R-matrices are nonsingular.

In addition to these exchange relations we demand that the quantum determinant for both T_+ and T_- be equal to unity, namely (4.29)

$$T_\pm(w - 15\mathrm{i}\hbar)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{C}}}\, T_\pm(w)_{\hat{\mathcal{B}}}{}^{\hat{\mathcal{D}}}\, \eta_{\hat{\mathcal{C}}\hat{\mathcal{D}}} \;=\; \eta_{\hat{\mathcal{A}}\hat{\mathcal{B}}}\,. \tag{4.53}$$

Due to (4.13), Q in (4.52) is normalized such that the l.h.s. of this equation indeed lies in the center of the full algebra (4.48), (4.49). The hermiticity (4.19) of the R-matrix shows that the full quantum algebra is compatible with the following $*$-structure — suggested by the classical relation (4.43) —

$$\left(T_\pm(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}\right)^* = T_\mp(w^*)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}\,, \tag{4.54}$$

for the purely imaginary choice of the parameter α we have made.[2]

Let us show that the algebra (4.48) – (4.54) has the correct classical limit $\hbar \to 0$. If we embed the original non-local charges $U_\pm(w)$ by identifying them with the upper left 248×248 block of $T_\pm(w)$, the exchange relations (4.48), (4.49) reduce to the Poisson brackets (4.45) in the limit $\hbar \to 0$. The $U_\pm(w)$ being $E_{8(8)}$-valued matrices, the condition (4.53) can then be viewed as the quantum analog of the statement that any element of $E_{8(8)}$ also belongs to $SO(128,120)$. While this submatrix of $T_\pm$ is evidently the quantum analog of the classical charges, one may wonder about the significance of the extra components $T_0{}^a(w)$, $T_a{}^0(w)$ and the singlet $T_0{}^0(w)$. The exchange relations may be read in such a way, that the off-diagonal components can be solved to become functions of the 248 degrees of freedom originally present. In order to make the dependence explicit, we evaluate the defining relation (4.26) at the remaining poles $u - v = w_j$ $(j = 1,\ldots,3)$ (the residue at w_4 has already been exploited to derive (4.29)):

$$\overset{12}{\mathcal{R}}_j\,\overset{1}{T}(u)\,\overset{2}{T}(u-w_j) \;=\; \overset{2}{T}(u-w_j)\,\overset{1}{T}(u)\,\overset{12}{\mathcal{R}}_j\;. \tag{4.55}$$

After expansion around $u=\infty$ this equation can be solved order by order to get expressions for the off-diagonal components $(T_{(n)})_0{}^a$ and $(T_{(n)})_a{}^0$, respectively. In first order this yields

$$\overset{12}{\mathcal{P}}_j\,\left(\overset{1}{T}_{(1)}+\overset{2}{T}_{(1)}\right) \;=\; \left(\overset{2}{T}_{(1)}+\overset{1}{T}_{(1)}\right)\overset{12}{\mathcal{P}}_j\,, \tag{4.56}$$

for all projectors from (3.25) and (4.8). Hence,

$$(T_{(1)})_a{}^b \in \mathrm{e}_8\,, \quad \text{and} \quad (T_{(1)})_0{}^a = (T_{(1)})_a{}^0 = (T_{(1)})_0{}^0 = 0\,. \tag{4.57}$$

[2] It is helpful to note that like for $\mathfrak{g} = \mathrm{sl}_2$ [121] the algebra (4.48), (4.49) may in fact be mapped to the usual (untwisted) centrally extended Yangian double by the (non-covariant) map

$$T_+(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}} \mapsto T_+(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{C}}}\eta_{\hat{\mathcal{C}}\hat{\mathcal{B}}}\,, \quad T_-(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}} \mapsto T_-(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}}\,.$$

The additional relation (4.17) is required to show that this is indeed an automorphism of (4.48). With respect to (4.54) this map is, however, no $*$-isomorphism; the representation theory of (4.48), (4.49) will thus differ considerably from the one of the Yangian double $\mathcal{D}Y_c(\mathrm{e}_8)$. (Needless to say that even the latter is far from being developed.)

In second order we get the equations

$$\begin{aligned}
&\overset{12}{\mathcal{R}}_j \left(\overset{1}{T}_{(2)} - \tfrac{1}{2}(\overset{1}{T}_{(1)})^2 + \overset{2}{T}_{(2)} - \tfrac{1}{2}(\overset{2}{T}_{(1)})^2 + w_j \overset{2}{T}_{(1)} + \tfrac{1}{2}\left[\overset{1}{T}_{(1)}, \overset{2}{T}_{(1)}\right]\right) \\
&= \left(\overset{1}{T}_{(2)} - \tfrac{1}{2}(\overset{1}{T}_{(1)})^2 + \overset{2}{T}_{(2)} - \tfrac{1}{2}(\overset{2}{T}_{(1)})^2 + w_j \overset{2}{T}_{(1)} - \tfrac{1}{2}\left[\overset{1}{T}_{(1)}, \overset{2}{T}_{(1)}\right]\right) \overset{12}{\mathcal{R}}_j \ .
\end{aligned} \tag{4.58}$$

Together with (4.31), (4.29) and the explicit form of the residues of the R-matrix (4.12) these relations can be used to deduce

$$\left(T_{(2)} - \tfrac{1}{2} T_{(1)} T_{(1)}\right)_{\mathcal{A}}{}^{\mathcal{B}} \in \mathfrak{e}_8 \ , \tag{4.59}$$

$$(T_{(2)})_0{}^{\mathcal{A}} \ = \ \tfrac{\sqrt{2}}{\alpha}\, \mathrm{i}\hbar f^{\mathcal{AB}}{}_{\mathcal{C}}\, (T_{(1)})_{\mathcal{C}}{}^{\mathcal{B}} \ , \quad (T_{(2)})_0{}^0 \ = \ 0 \ , \ \text{etc.}$$

In this fashion one may in principle determine the components $T(w)_0{}^{\mathcal{A}}, T(w)_{\mathcal{A}}{}^0$ in all orders as functions of the $T(w)_{\mathcal{A}}{}^{\mathcal{B}}$ which vanish in the classical limit $\hbar \to 0$. Thus, we have consistently

$$T_{\pm}(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{B}}} \ \xrightarrow[\hbar \to 0]{} \ \left(\begin{array}{c|c} U_{\pm}(w)_{\mathcal{A}}{}^{\mathcal{B}} & 0 \\ \hline 0 & 1 \end{array} \right) . \tag{4.60}$$

Recall now that the classical phase space was parametrized by the symmetric $E_{8(8)}$-valued matrix $\mathcal{M}_{\mathcal{AB}}(w)$. On the quantum side we define this object in analogy to (4.42) as

$$\mathcal{M}_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w) \equiv T_+(w)_{\hat{\mathcal{A}}}{}^{\hat{\mathcal{C}}}\, T_-(w)_{\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}} \ , \tag{4.61}$$

where the operator ordering on the r.h.s. is fixed by this relation. The matrix entries of $\mathcal{M}$ are distinguished elements in the algebra (4.48), (4.49) in that they satisfy the exchange relations

$$\begin{aligned}
T_+(v)_{\hat{\mathcal{C}}}{}^{\hat{\mathcal{D}}}\, \mathcal{M}_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w) &= R_{\hat{\mathcal{A}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{K}}}(w - v)\mathcal{M}_{\hat{\mathcal{P}}\hat{\mathcal{Q}}}(w) R_{\hat{\mathcal{B}}\hat{\mathcal{K}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{L}}}(w - v - \mathrm{i}\hbar c)\, T_+(v)_{\hat{\mathcal{L}}}{}^{\hat{\mathcal{D}}} \ , \\
T_-(v)_{\hat{\mathcal{C}}}{}^{\hat{\mathcal{D}}}\, \mathcal{M}_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w) &= R_{\hat{\mathcal{A}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{K}}}(w - v + \mathrm{i}\hbar c)\mathcal{M}_{\hat{\mathcal{P}}\hat{\mathcal{Q}}}(w) R_{\hat{\mathcal{B}}\hat{\mathcal{K}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{L}}}(w - v)\, T_-(v)_{\hat{\mathcal{L}}}{}^{\hat{\mathcal{D}}} \ ,
\end{aligned} \tag{4.62}$$

as well as the closed algebra

$$\begin{aligned}
&R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{M}}\hat{\mathcal{N}}}(v - w)\, \mathcal{M}_{\hat{\mathcal{M}}\hat{\mathcal{K}}}(v)\, R_{\hat{\mathcal{C}}\hat{\mathcal{N}}}{}^{\hat{\mathcal{K}}\hat{\mathcal{L}}}(v - w - \mathrm{i}\hbar c)\, \mathcal{M}_{\hat{\mathcal{L}}\hat{\mathcal{D}}}(w) \ = \\
&\qquad \mathcal{M}_{\hat{\mathcal{B}}\hat{\mathcal{M}}}(w)\, R_{\hat{\mathcal{K}}\hat{\mathcal{A}}}{}^{\hat{\mathcal{M}}\hat{\mathcal{N}}}(w - v - \mathrm{i}\hbar c)\, \mathcal{M}_{\hat{\mathcal{N}}\hat{\mathcal{L}}}(v)\, R_{\hat{\mathcal{D}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{K}}\hat{\mathcal{L}}}(w - v) \ ,
\end{aligned} \tag{4.63}$$

which we hence view as the quantized version of (4.44).[3]

While the classical matrix $\mathcal{M}$ was manifestly symmetric (4.37), (4.38) this is not necessarily true for its quantum analog. Rather, we must now impose some quantum version of this

[3]Like its classical counterpart (4.44) the algebra (4.63) belongs to the general class of quadratic algebras which has been considered in [56].

condition in order to ensure that the number of degrees of freedom in the quantum structure matches the classical phase space. In other words, we still have to implement the quantum analogue of the classical coset structure. In algebraic language this amounts to dividing out another ideal from (4.48) – (4.49). Unlike the quantum determinant condition (4.53) (which we still assume to hold), this new condition will involve T_+ and T_- simultaneously.

To this end we return to the exchange algebra (4.48) – (4.49) with arbitrary, but fixed central charge c, and consider the set of elements

$$\phi_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w) \equiv \begin{cases} \mathrm{Res}\Big|_{v=\mathrm{i}\hbar c} R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(v)\,\mathcal{M}_{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w) & \text{if } R(\mathrm{i}\hbar c) \text{ is singular} \\ R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}}(\mathrm{i}\hbar c)\mathcal{M}_{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w) & \text{else}\,. \end{cases} \tag{4.64}$$

Use of the exchange relations (4.62) then yields in a first step

$$\begin{aligned} T_+(u)_{\hat{\mathcal{C}}}{}^{\hat{\mathcal{D}}}\;\phi_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w) \;&= \\ &= R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{Q}}}(\mathrm{i}\hbar c)R_{\hat{\mathcal{P}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{M}}\hat{\mathcal{K}}}(w-u)R_{\hat{\mathcal{Q}}\hat{\mathcal{K}}}{}^{\hat{\mathcal{N}}\hat{\mathcal{L}}}(w-u-\mathrm{i}\hbar c)\,\mathcal{M}_{\hat{m}\hat{\mathcal{N}}}(w)\,T_+(u)_{\hat{\mathcal{L}}}{}^{\hat{\mathcal{D}}}\,. \end{aligned} \tag{4.65}$$

The necessity of the choice $v = \mathrm{i}\hbar c$ in (4.64) becomes evident at this point: it is the only value of the argument of the R-matrix in (4.64) for which we can exploit the QYBE to re-arrange the indices (this would not be possible if the first factor on the r.h.s. were $R_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{Q}}}(v)$ with arbitrary argument v). Thus, by use of (4.4) we finally obtain

$$\begin{aligned} &= R_{\hat{\mathcal{B}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{K}}}(w-u-\mathrm{i}\hbar c)R_{\hat{\mathcal{A}}\hat{\mathcal{K}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{L}}}(w-u)R_{\hat{\mathcal{P}}\hat{\mathcal{Q}}}{}^{\hat{\mathcal{M}}\hat{\mathcal{N}}}(\mathrm{i}\hbar c)\,\mathcal{M}_{\hat{m}\hat{n}}(w)\,T_+(u)_{\hat{\mathcal{L}}}{}^{\hat{\mathcal{D}}} \\ &= R_{\hat{\mathcal{B}}\hat{\mathcal{C}}}{}^{\hat{\mathcal{Q}}\hat{\mathcal{K}}}(w-u-\mathrm{i}\hbar c)R_{\hat{\mathcal{A}}\hat{\mathcal{K}}}{}^{\hat{\mathcal{P}}\hat{\mathcal{L}}}(w-u)\;\phi_{\hat{p}\hat{q}}(w)\,T_+(u)_{\hat{\mathcal{L}}}{}^{\hat{\mathcal{D}}}\,. \end{aligned}$$

A similar calculation for $T_-(u)$ gives the same result. We conclude that the elements $\phi_{\hat{\mathcal{A}}\hat{\mathcal{B}}}(w)$ constitute the basis of a two-sided ideal of (4.48) – (4.49). Obviously, these ideals are non-trivial only if (4.64) does not contain $\mathcal{M}$ entirely, i.e. only if $R(\mathrm{i}\hbar c)$ or the relevant residue is singular or non-invertible. This happens only at the special values $c = \pm w_j$.[4]

There is (of course) a more group-theoretical interpretation of this construction: in view of (4.33) and (4.34), the exchange relations (4.62) express the fact, that under the adjoint action of $T_+(v)$ and $T_-(v)$, respectively, the matrix $\mathcal{M}(w)$ transforms in the tensor product $\mathbf{249}_w \otimes \mathbf{249}_{w-\mathrm{i}\hbar c}$. I.e. the existence of nontrivial ideals in $\mathcal{M}$ amounts to the reducibility of the tensor product $\mathbf{249}_{\mathrm{i}\hbar c} \otimes \mathbf{249}$ which is in correspondence with the singular points of the associated R-matrix [23] as we have explicitly seen here.

Returning to the problem of identifying the proper quantum analogue of the symmetry (4.37) of $\mathcal{M}$, let us now examine (4.64) for all critical choices of the central extension c with the desired conditions (4.37), (4.38). With the explicit form of (4.12) one confirms that there is a unique value of c such that the algebra (4.48) – (4.49) has an ideal which in the classical limit indeed reduces to (4.37), (4.38). The correct choice is

$$c = 1\,, \tag{4.66}$$

[4]Together with the fact (4.23) that $\mathcal{R}_4$ is proportional to a one-dimensional projector, (4.65) in particular shows the well-known infinite-dimensional enlargement of the center of the algebra (4.48) – (4.49) at the critical level $c = 15$ [150]. However, to achieve consistency with the classical coset structure $E_{8(8)}/SO(16)$ we need another value of the central extension here.

Dividing out the ideal corresponding to (4.64) now amounts to imposing the additional set of relations $\phi_{\hat{\mathcal{A}}\hat{\mathcal{B}}} = 0$, or

$$(\mathcal{R}_1)_{\hat{\mathcal{A}}\hat{\mathcal{B}}}{}^{\hat{\mathcal{C}}\hat{\mathcal{D}}} \mathcal{M}_{\hat{\mathcal{C}}\hat{\mathcal{D}}}(w) \;=\; 0\,. \tag{4.67}$$

The ensuing relations can be written more succinctly by splitting $\mathcal{R}_1$ into the $E_{8(8)}$-invariant projectors:

$$\begin{aligned}
(\mathcal{P}_{248})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} \mathcal{M}_{\mathcal{C}\mathcal{D}} &= -\tfrac{\alpha}{8}\, f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} (\mathcal{M}_{0\mathcal{C}} + \mathcal{M}_{\mathcal{C}0})\,, \\
(\mathcal{P}_{3875})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} \mathcal{M}_{\mathcal{C}\mathcal{D}} &= 0\,, \\
(\mathcal{P}_{30380})_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} \mathcal{M}_{\mathcal{C}\mathcal{D}} &= 0\,, \\
\mathcal{M}_{0\mathcal{C}} &= \mathcal{M}_{\mathcal{C}0}\,, \\
\mathcal{M}_{00} &= -\tfrac{1}{480\alpha^2}\, \eta^{\mathcal{A}\mathcal{B}} \mathcal{M}_{\mathcal{A}\mathcal{B}}\,.
\end{aligned} \tag{4.68}$$

Since the off-diagonal components $\mathcal{M}_{0\mathcal{A}}$ are of order $\mathcal{O}(\hbar)$ by (4.60), it now follows with our choice $60\alpha^2 = -1$ that the relations (4.68) indeed encompass the classical coset relations (4.37) and (4.38) in the limit $\hbar \to 0$.

Chapter 5

Conformal and Quasiconformal Realizations of Exceptional Groups

In this chapter, we present several nonlinear realizations of exceptional Lie algebras [78]. These realizations are called "conformal" if they are based on a three graded decomposition of the underlying Lie algebra (as e.g. $E_{7(7)}$) and "quasiconformal" if they are based on a five graded decomposition (as for $E_{8(8)}$).

The conformal realizations discussed here are based on Jordan algebras and closely resemble the well known realization of the conformal group of Minkowski spacetime. In particular, we give a completely explicit conformal Möbius-like nonlinear realization of $E_{7(7)}$ on the 27-dimensional space associated with the exceptional Jordan algebra $J_3^{\mathbb{O}S}$, with linearly realized subgroups $F_{4(4)}$ (the "rotation group") and $E_{6(6)}$ (the "Lorentz group"). Although in part this result is implicitly contained in the existing literature on Jordan algebras, the relevant transformations have not been exhibited explicitly so far, and are here presented in the basis that is also used in maximal supergravity theories.

The main result is a novel construction involving the maximally extended Lie group $E_{8(8)}$. This construction of $E_{8(8)}$ together with the corresponding construction of $E_{8(-24)}$ contain all previous examples of generalized space-times based on exceptional Lie groups, and at the same time goes beyond the framework of Jordan algebras. More precisely, we show that there exists a quasiconformal nonlinear realization of $E_{8(8)}$ on a space of 57 dimensions. This space may be viewed as the quotient of $E_{8(8)}$ by its maximal parabolic subgroup [95, 96]; there is no Jordan algebra directly associated with it, but it can be related to a certain Freudenthal triple system which itself is associated with the "split" exceptional Jordan algebra $J_3^{\mathbb{O}S}$ where $\mathbb{O}_S$ denote the split real form of the octonions $\mathbb{O}$ (see Ref. [5]). It furthermore admits an $E_{7(7)}$ invariant norm form $\mathcal{N}_4$, which gets multiplied by a (coordinate dependent) factor under the nonlinearly realized "special conformal" transformations. Therefore the light cone, defined by the condition $\mathcal{N}_4 = 0$, is actually invariant under the full $E_{8(8)}$, which thus plays the role of a generalized conformal group. By truncation we obtain quasiconformal realizations of other exceptional Lie groups.

We explicitly work out the truncations to $SU(2,1)$, which is the U-duality group of $d=4$, $N=2$ Maxwell-Einstein theory reduced to three dimensions, and to $G_{2(2)}$, which is the U-duality group of the simple $d=5$, $N=2$ supergravity [22] reduced to three dimensions. The action of the U-duality group can (at least in the first case) be identified with the symmetries of the Lagrangian. In the latter case, we identify certain known subalgebras of the symmetry group.

The quasiconformal realization of $E_{8(8)}$ gives rise to a novel oscillator-like representation of $E_{8(8)}$ on a space of 29 dimensions which can be used to study the representation theory of non-compact exceptional Lie groups.

5.1 Conformal Realizations

In this section we give an overview over the concept of conformal realizations. These are certain nonlinear realizations of Lie groups. We begin with the well known example of the conformal group of Minkowski space-time [125] in section 5.1.1. Then we reformulate the example in the language of Jordan algebras in section 5.1.2 generalizing the construction. The classification of all possible conformal realizations contains four infinite series and two exceptional cases corresponding to the Lie group E_6 and E_7. In section 5.1.3 we give an explicit realization of one of the exceptional cases, namely the conformal realization of the Lie algebra of $E_{7(7)}$.

5.1.1 The conformal group of Minkowski space-time

The concept of conformal realizations is best illustrated in terms of a simple and familiar example, namely the four dimensional Minkowski spacetime.

The conformal group of M_4 is defined as the group of automorphisms leaving invariant the light-cone

$$\{x^\mu \in M_4 \,|\, x^\mu x_\mu \;=\; 0\} \tag{5.1}$$

Its Lie algebra $\mathfrak{g} = \mathrm{su}(2,2)$ has a three graded structure

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1}\,, \tag{5.2}$$

where the grade 0 subspace is spanned by the Lorentz generators $M^{\mu\nu}$ and the generator D of dilatations. The latter are given by

$$x^\mu \;\mapsto\; D\, x^\mu\,. \tag{5.3}$$

The subspaces $\mathfrak{g}^{-1}$ and $\mathfrak{g}^{+1}$ correspond to generators of translations P^μ and special conformal transformations K^μ, respectively. The latter are given by

$$x^\mu \;\mapsto\; \frac{x^\mu - K^\mu x^2}{1 - 2\langle K, x\rangle + K^2 x^2}\,, \tag{5.4}$$

where $\langle x, y\rangle$ denotes the scalar product $x^\mu y_\mu$. The Möbius-like infinitesimal action of the special conformal transformations is

$$\delta x^\mu = 2\langle K, x\rangle x^\mu - \langle x, x\rangle K^\mu\,. \tag{5.5}$$

In addition to the Poincaré algebra we have the commutator relations

$$[P^\mu\,,\, K^\nu] \;=\; 2\,(\eta^{\mu\nu} D + M^{\mu\nu})\,, \qquad [K^\rho\,,\, M^{\mu\nu}] \;=\; \eta^{\rho\mu} K^\nu - \eta^{\rho\nu} K^\mu\,, \tag{5.6}$$

and the dilatations give the grading (5.2):

$$[D\,,\, P^\mu] \;=\; -P^\mu\,, \qquad [D\,,\, M^{\mu\nu}] \;=\; 0\,, \qquad [D\,,\, K^\mu] \;=\; K^\mu\,. \tag{5.7}$$

It is well known that the four-dimensional Minkowski space M_4 can be identified with the Jordan algebra $J_2^{\mathbb{C}}$ of 2×2 hermitean matrices with the commutative (but non-associative) product

$$\mathbf{x}\circ\mathbf{y} := \tfrac{1}{2}(\mathbf{x}\mathbf{y}+\mathbf{y}\mathbf{x}) \; . \tag{5.8}$$

The isomorphism between M_4 and $J_2^{\mathbb{C}}$ is given by $x^\mu \mapsto \mathbf{x} := x_\mu\sigma^\mu$ with Pauli matrices $\sigma^\mu \equiv (\mathbf{1},\vec{\sigma})$. The "norm form" on this algebra is just the ordinary determinant, i.e.

$$\mathcal{N}_2(x) := \det\mathbf{x} = x_\mu x^\mu \tag{5.9}$$

(it will be a higher order polynomial in the general case). Defining $\bar{x} := x_\mu\bar{\sigma}^\mu$ (where $\bar{\sigma}^\mu := (1,-\vec{\sigma})$) we introduce the Jordan triple product on $J_2^{\mathbb{C}}$:

$$\begin{aligned}\{a\,b\,c\} \;&:=\; (a\circ\bar{b})\circ c + (c\circ\bar{b})\circ a - (a\circ c)\circ\bar{b} \\ &=\; \tfrac{1}{2}(a\bar{b}c + c\bar{b}a) = \langle a,b\rangle c + \langle c,b\rangle a - \langle a,c\rangle b\end{aligned} \tag{5.10}$$

The automorphism group of $J_2^{\mathbb{C}}$, which is by definition compatible with the Jordan product, is just the rotation group $SU(2)$; the structure group, defined as the invariance of the norm form up to a constant factor, is the product $SL(2,\mathbb{C})\times\mathcal{D}$, i.e. the Lorentz group and dilatations. The conformal group associated with $J_2^{\mathbb{C}}$ is the group leaving invariant the light-cone $\mathcal{N}_2(x) = 0$.

The subspaces $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ can each be associated with the Jordan algebra $J_2^{\mathbb{C}}$, such that their elements are labeled by elements $a = a_\mu\sigma^\mu$ of $J_2^{\mathbb{C}}$. The precise correspondence is

$$U_a := a_\mu P^\mu \in \mathfrak{g}^{-1} \qquad \text{and} \qquad \tilde{U}_a := a_\mu K^\mu \in \mathfrak{g}^{+1} \; . \tag{5.11}$$

By contrast, the generators in $\mathfrak{g}^0$ are labeled by *two* elements $a,b\in J_2^{\mathbb{C}}$, namely

$$S_{ab} := a_\mu b_\nu (M^{\mu\nu} + \eta^{\mu\nu}D) \; . \tag{5.12}$$

All variations acquire a very simple form when expressed in terms of above generators: we have

$$\begin{aligned}U_a(x) \;&=\; a \, , \\ S_{ab}(x) \;&=\; \{a\,b\,x\} \, , \\ \tilde{U}_c(x) \;&=\; -\tfrac{1}{2}\{x\,c\,x\} \, ,\end{aligned} \tag{5.13}$$

where $\{...\}$ is the Jordan triple product introduced above. From these transformations it is elementary to deduce the commutation relations

$$\begin{aligned}[][U_a,\tilde{U}_b] \;&=\; S_{ab} \, , \\ [S_{ab},U_c] \;&=\; U_{\{abc\}} \, , \\ [S_{ab},\tilde{U}_c] \;&=\; \tilde{U}_{\{bac\}} \, , \\ [S_{ab},S_{cd}] \;&=\; S_{\{abc\}\,d} - S_{\{bad\}\,c} \; .\end{aligned} \tag{5.14}$$

(of course, these could have been derived directly from those of the conformal group). As one can also see, the Lie algebra $\mathfrak{g}$ admits an involutive automorphism ι exchanging $\mathfrak{g}^{-1}$ and $\mathfrak{g}^{+1}$ (hence, $\iota(K^\mu) = P^\mu$).

The above transformation rules and commutation relations can be defined for any Jordan algebra J and even for any Jordan triple system (JTS). The construction of a Lie algebra by means of the Jordan triple product has been known in the literature as the Tits-Kantor-Koecher construction [159, 108, 117]. The generalized linear fractional (Möbius) groups of Jordan algebras can be abstractly defined in an analogous manner [118], and shown to leave invariant certain generalized p-angles defined by the norm form of degree p of the underlying Jordan algebra [109, 76].

In the next section we summarize the classification and basic properties of Jordan algebras and Jordan triple systems and in section 5.1.3 we carry out the conformal realization of $E_{7(7)}$ on $\mathbb{R}^{27}$ in great detail. The commutation relations then have the same form as (5.14), except that $J_2^{\mathbb{C}}$ is replaced by the exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ over the split octonions $\mathbb{O}_S$.

While this construction works for the exceptional Lie algebras $E_{6(6)}$, and $E_{7(7)}$, as well as other Lie algebras admitting a three graded structure, it fails for $E_{8(8)}$, $F_{4(4)}$, and $G_{2(2)}$, for which a three grading does not exist. These algebras possess only a five graded structure

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} . \tag{5.15}$$

In [78] we developed a similar construction that can applied in this case. It will be presented in section 5.2

5.1.2 Jordan Triple Systems

Let us first give a classification of simple Jordan algebras. By definition these are algebras equipped with a commutative (but non-associative) binary product $a \circ b = b \circ a$ satisfying the Jordan identity

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2) . \tag{5.16}$$

There are four infinite series of simple complex Jordan algebras and one exceptional case: The algebras J_n^A of $n \times n$ matrices over the division algebras $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the algebra $\Gamma(d)$ of Dirac gamma matrices in d Euclidean dimensions give the infinite series of Jordan algebras. Additionally, there is an exceptional Jordan algebra $J_3^{\mathbb{O}}$ of dimension 27. All simple real Jordan algebras are real forms of these complex Jordan algebras.

In $d=3,4,6,10$ one can identify the Minkowski spacetime M_d with the Jordan algebras J_2^A with $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ using Isomorphisms

$$J_2^{\mathbb{R}} \simeq \Gamma(2), \quad J_2^{\mathbb{C}} \simeq \Gamma(3), \quad J_2^{\mathbb{H}} \simeq \Gamma(5), \quad J_2^{\mathbb{O}} \simeq \Gamma(9). \tag{5.17}$$

This is related to the existence of super Yang-Mills theories in $d=3,4,6,10$.

In any Jordan algebra we can define a so-called Jordan triple product

$$\{a\, b\, c\} = a \circ (\tilde{b} \circ c) + (a \circ \tilde{b}) \circ c - \tilde{b} \circ (a \circ c) ,$$

where $\tilde{}$ denotes a conjugation in J corresponding to the operation $\dagger$ in $\mathfrak{g}$. The triple product satisfies the identities

$$\begin{aligned} &\{a\,b\,c\} = \{c\,b\,a\}\,, \\ &\{a\,b\,\{c\,d\,x\}\} - \{c\,d\,\{a\,b\,x\}\} - \{a\,\{d\,c\,b\}\,x\} + \{\{c\,d\,a\}\,b\,x\} = 0\,, \end{aligned} \tag{5.18}$$

which can alternatively be taken as the defining identities of a Jordan triple system (JTS) with Jordan triple product $\{a\,b\,c\}$. Of particular interest are the hermitean Jordan triple systems for which the the triple product is linear in the first and in the las argument, but anti-linear in the second argument. As one can see from (5.13), the binary Jordan product does not enter the Tits-Kantor-Köcher (TKK) construction [159, 108, 117]. Only the Jordan triple product (5.18) is needed and the Jacobi identities of the constructed Lie algebra is satisfied if and only if the relations (5.18) are satisfied.

There are four infinite families of hermitean JTS's and two exceptional ones [139, 122]. They are:

type $I_{P,Q}$ generated by $P \times Q$ complex matrices $M_{P,Q}(\mathbb{C})$ with the triple product

$$\{a\,b\,c\} = ab^\dagger c + cb^\dagger a\,. \tag{5.19}$$

type II_N generated by anti-symmetric $N \times N$ complex matrices $A_N(\mathbb{C})$ and the triple product (5.19).

type III_N generated by symmetric $N \times N$ complex matrices $S_N(\mathbb{C})$ and the triple product (5.19).

type IV_N generated by Dirac gamma matrices $\Gamma(N)(\mathbb{C})$ in N dimensions with complex coefficients and the triple product (5.18).

type V generated by 1×2 octonionic matrices $M_{1,2}(\mathbb{O}_\mathbb{C})$ over the complex numbers as base field and the triple product

$$\{a\,b\,c\} = \{(a\bar{b}^\dagger)c + (\bar{b}a^\dagger)c - \bar{b}(a^\dagger c)\} + a \leftrightarrow b\,. \tag{5.20}$$

type VI generated by the exceptional Jordan algebra of 3×3 hermitean octonionic matrices $J_3^\mathbb{O}(\mathbb{C})$ over the complex numbers and the triple product (5.18).

The TKK construction associates every JTS with a 3-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1}\,, \tag{5.21}$$

satisfying the formal commutation relations:

$$\begin{aligned} [\mathfrak{g}^{+1}\,, \mathfrak{g}^{-1}] &= \mathfrak{g}^0\,, \\ [\mathfrak{g}^{+1}\,, \mathfrak{g}^{+1}] &= 0\,, \\ [\mathfrak{g}^{-1}\,, \mathfrak{g}^{-1}] &= 0\,. \end{aligned}$$

With the exception of the Lie algebras G_2, F_4, and E_8 every simple Lie algebra $\mathfrak{g}$ can be given a three graded decomposition with respect to a subalgebra $\mathfrak{g}^0$ of maximal rank.

By the TKK construction the elements U_a of the $\mathfrak{g}^{+1}$ subspace of the Lie algebra are labelled by the elements $a \in J$. Furthermore every such Lie algebra $\mathfrak{g}$ admits an involutive automorphism ι, which maps the elements of the grade $+1$ space onto the elements of the subspace of grade -1:

$$\iota(U_a) =: \tilde{U}_a \in \mathfrak{g}^{-1} \tag{5.22}$$

To get a complete set of generators of $\mathfrak{g}$ we define

$$\begin{aligned} [U_a, \tilde{U}_b] &= S_{ab} \,, \\ [S_{ab}, U_c] &= U_{\{abc\}} \,, \end{aligned} \tag{5.23}$$

where $S_{ab} \in \mathfrak{g}^0$ and $\{abc\}$ is the Jordan triple product under which the space J is closed. The remaining commutation relations are

$$\begin{aligned} [S_{ab}, \tilde{U}_c] &= \tilde{U}_{\{bac\}} \,, \\ [S_{ab}, S_{cd}] &= S_{\{abc\}d} - S_{c\{bad\}} \,, \end{aligned} \tag{5.24}$$

and the closure of the algebra under commutation follows from the defining identities of a JTS given above.

The algebra $\mathfrak{g}$ of the TKK construction is called the conformal algebra of the JTS J. The Lie algebra generated by S_{ab} is called the structure algebra, under which the elements of J transform linearly. The traceless elements of this action of S_{ab} generate the reduced structure algebra of J.

The different real forms of the exceptional JTS's J (type V and type VI) together with their corresponding conformal groups G and their structure group H are listed in the table below.

J	G	H
$M_{1,2}(\mathbb{O}_S)$	$E_{6(6)}$	$SO(5,5)$
$M_{1,2}(\mathbb{O})$	$E_{6(-14)}$	$SO(8,2)$
$J_3^{\mathbb{O}_S}$	$E_{7(7)}$	$E_{6(6)}$
$J_3^{\mathbb{O}}$	$E_{7(-25)}$	$E_{6(-26)}$

Here we are mainly interested in the real form $J_3^{\mathbb{O}_S}$, which corresponds to the split octonions $\mathbb{O}_S$ and has $E_{7(7)}$ and $E_{6(6)}$ as its conformal and reduced structure group, respectively.

5.1.3 Conformal Realization of $E_{7(7)}$

We will now give the explicit construction of the nonlinear conformal realization of $E_{7(7)}$ on a space of 27 dimensions, on which the $E_{6(6)}$ subgroup of $E_{7(7)}$ acts linearly. The 27-dimensional space is given by the real form of the exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ corresponding to the split octonions $\mathbb{O}_S$. The relevant three-graded decomposition of $E_{7(7)}$ w.r.t. $E_{6(6)}$ is given by

$$\mathbf{133} = \mathbf{27} \oplus (\mathbf{78} \oplus \mathbf{1}) \oplus \overline{\mathbf{27}} \,. \tag{5.25}$$

The Lie algebra $E_{6(6)}$ has $USp(8)$ as its maximal compact subalgebra. It is spanned by a symmetric tensor $\tilde{G}^{ij}$ in the adjoint representation **36** of $USp(8)$ and a fully antisymmetric symplectic traceless tensor $\tilde{G}^{ijkl}$ transforming under the **42** of $USp(8)$; indices $1 \leq i, j, \ldots \leq 8$ are $USp(8)$ indices and all tensors with a tilde transform under $USp(8)$. $\tilde{G}^{ijkl}$ is traceless with respect to the real symplectic metric $\Omega_{ij} = -\Omega_{ji} = -\Omega^{ij}$ (thus $\Omega_{ik}\Omega^{kj} = \delta_i^j$). The symplectic metric also serves to pull up and down indices, with the convention that this is always to be done from the left.

The remaining part of $E_{7(7)}$ is spanned by an extra dilatation generator $\tilde{H}$, translation generators $\tilde{E}^{ij}$ and the nonlinearly realized generators $\tilde{F}^{ij}$, transforming as **27** and $\overline{\mathbf{27}}$, respectively. Unlike for $E_{8(8)}$, there is no need here to distinguish the generators by the position of their indices, since the corresponding generators are linearly related by means of the symplectic metric.

The commutation relations are

$$
\begin{aligned}
[\tilde{G}^i{}_j \,,\, \tilde{G}^k{}_l] &= \delta_j^k \,\tilde{G}^i{}_l - \delta_l^i \,\tilde{G}^k{}_j \,, \\
[\tilde{G}^i{}_j \,,\, \tilde{G}^{klmn}] &= -4\,\delta_j^{[k}\,\tilde{G}^{lmn]i} - \tfrac{1}{2}\,\delta_j^i \tilde{G}^{klmn} \,, \\
[\tilde{G}^{ijkl} \,,\, \tilde{G}^{mnpq}] &= \tfrac{1}{36}\,\epsilon^{ijkls[mnp}\,\tilde{G}^{q]}{}_s \,,
\end{aligned}
$$

for the $E_{7(7)}$ subgroup and for the remaining part we have:

$$
\begin{aligned}
[\tilde{H} \,,\, \tilde{E}^{ij}] &= -\tilde{E}^{ij} \,, \qquad [\tilde{H} \,,\, \tilde{F}^{ij}] \;=\; \tilde{F}^{ij} \,, \\
[\tilde{G}^i{}_j \,,\, \tilde{E}^{kl}] &= \delta_j^k \,\tilde{E}^{il} - \delta_j^l \,\tilde{E}^{ik} - \tfrac{1}{4}\delta_j^i \,\tilde{E}^{kl} \,, \\
[\tilde{G}^i{}_j \,,\, \tilde{F}^{kl}] &= \delta_j^k \,\tilde{F}^{il} - \delta_j^l \,\tilde{F}^{ik} - \tfrac{1}{4}\delta_j^i \,\tilde{F}^{kl} \,, \\
[\tilde{G}^{ijkl} \,,\, \tilde{E}^{mn}] &= -\tfrac{1}{24}\,\epsilon^{ijklmnpq}\,\Omega_{pr}\Omega_{qs}\,\tilde{E}^{rs} \,, \\
[\tilde{G}^{ijkl} \,,\, \tilde{F}^{mn}] &= -\tfrac{1}{24}\,\epsilon^{ijklmnpq}\,\Omega_{pr}\Omega_{qs}\,\tilde{F}^{rs} \,, \\
[\tilde{E}^{ij} \,,\, \tilde{F}^{kl}] &= 12\,\tilde{G}^{ijkl} + 4\,\Omega^{n[k}\Omega^{l][i}\tilde{G}^{j]}{}_n + \tfrac{1}{8}\,\Omega^{ij}\Omega^{n[k}\tilde{G}^{l]}{}_n + \tfrac{1}{8}\,\Omega^{kl}\Omega^{n[i}\tilde{G}^{j]}{}_n \\
&\quad + (\Omega^{i[k}\Omega^{l]j} + \tfrac{1}{8}\Omega^{ij}\Omega^{kl})\,\tilde{H} \,.
\end{aligned}
$$

The fundamental **27** of $E_{6(6)}$ (on which we are going to realize a nonlinear action of $E_{7(7)}$) is given by the traceless antisymmetric tensor $\tilde{Z}^{ij}$ transforming as

$$
\begin{aligned}
\tilde{G}^i{}_j(\tilde{Z}^{kl}) &= 2\,\delta_j^{\overline{k}}\,\tilde{Z}^{\overline{il}} \,, \\
\tilde{G}^{ijkl}(\tilde{Z}^{mn}) &= \tfrac{1}{24}\epsilon^{ijklmnpq}\,\tilde{Z}_{pq} \,,
\end{aligned}
\tag{5.26}
$$

where

$$
\tilde{Z}_{ij} \;:=\; \Omega_{ik}\Omega_{jl}\tilde{Z}^{kl} = (\tilde{Z}^{ij})* \qquad \text{and} \qquad \Omega_{ij}\tilde{Z}^{ij} \;=\; 0 \,.
$$

Likewise, the $\overline{\mathbf{27}}$ representation transforms as

$$
\begin{aligned}
\tilde{G}^i{}_j(\bar{Z}^{kl}) &= 2\,\delta_j^{\overline{k}}\,\bar{Z}^{\overline{il}} \,, \\
\tilde{G}^{ijkl}(\bar{Z}^{mn}) &= -\tfrac{1}{24}\epsilon^{ijklmnpq}\,\bar{Z}_{pq} \,.
\end{aligned}
\tag{5.27}
$$

Because the product of two **27**'s contains no singlet, there exists no quadratic invariant of $E_{6(6)}$; however, there is a cubic invariant given by

$$\mathcal{N}_3(\tilde{Z}) := \tilde{Z}^{ij}\tilde{Z}_{jk}\tilde{Z}^{kl}\Omega_{il}\,. \tag{5.28}$$

We are now ready to give the conformal realization of $E_{7(7)}$ on the 27-dimensional space spanned by the $\tilde{Z}^{ij}$. As the action of the linearly realized $E_{6(6)}$ subgroup has already been given, we list only the remaining variations. The generator $\tilde{E}^{ij}$ acts by translations:

$$\tilde{E}^{ij}(\tilde{Z}^{kl}) = -\Omega^{i[k}\Omega^{l]j} - \tfrac{1}{8}\Omega^{ij}\Omega^{kl} \tag{5.29}$$

and $\tilde{H}$ by dilatations

$$\tilde{H}(\tilde{Z}^{ij}) = \tilde{Z}^{ij}\,. \tag{5.30}$$

The $\overline{\mathbf{27}}$ generators $\tilde{F}^{ij}$ are realized nonlinearly:

$$\begin{aligned}\tilde{F}^{ij}(\tilde{Z}^{kl}) \;:=\; & -2\,\tilde{Z}^{ij}(\tilde{Z}^{kl}) + \Omega^{i[k}\Omega^{l]j}(\tilde{Z}^{mn}\tilde{Z}_{mn}) + \tfrac{1}{8}\,\Omega^{ij}\Omega^{kl}(\tilde{Z}^{mn}\tilde{Z}_{mn})\\ & + 8\,\tilde{Z}^{km}\,\overline{\tilde{Z}_{mn}\Omega^{n[i}\Omega^{j]l}} - \Omega^{kl}(\tilde{Z}^{im}\Omega_{mn}\tilde{Z}^{nj})\end{aligned} \tag{5.31}$$

The norm form needed to define the $E_{7(7)}$ invariant "light cones" is now constructed from the cubic invariant of $E_{6(6)}$. Then $\mathcal{N}_3(\tilde{X}-\tilde{Y})$ is manifestly invariant under $E_{6(6)}$ and under the translations $\tilde{E}^{ij}$ (observe that there is no need to introduce a nonlinear difference unlike for $E_{8(8)}$). Under $\tilde{H}$ it transforms by a constant factor, whereas under the action of $\tilde{F}^{ij}$ we have

$$\tilde{F}^{ij}\Big(\mathcal{N}_3(\tilde{X}-\tilde{Y})\Big) = (\tilde{X}^{ij}+\tilde{Y}^{ij})\,\mathcal{N}(\tilde{X}-\tilde{Y})\,. \tag{5.32}$$

Thus the light cones in $\mathbb{R}^{27}$ with base point $\tilde{Y}$

$$\mathcal{N}_3(\tilde{X}-\tilde{Y}) = 0 \tag{5.33}$$

are indeed curved hyper-surfaces invariant under $E_{7(7)}$.

The connection to the Jordan Triple Systems of section 5.1.2 can now be made quite explicit. We first of all notice that we can define a triple product in terms of the $E_{6(6)}$ representations; it reads

$$\begin{aligned}\{\tilde{X}\,\tilde{Y}\,\tilde{Z}\}^{ij} \;=\; & 16\,\tilde{X}^{ik}\,\overline{\tilde{Z}_{kl}\tilde{Y}^{lj}} + 16\,\tilde{Z}^{ik}\,\overline{\tilde{X}_{kl}\tilde{Y}^{lj}} + 4\,\Omega^{ij}(\tilde{X}^{kl}\tilde{Y}_{lm}\tilde{Z}^{mn}\Omega_{kn})\\ & + 4\,\tilde{X}^{ij}\tilde{Y}^{kl}\tilde{Z}_{kl} + 4\,\tilde{Y}^{ij}\tilde{X}^{kl}\tilde{Z}_{kl} + 2\,\tilde{Z}^{ij}\tilde{X}^{kl}\tilde{Y}_{kl}\,.\end{aligned} \tag{5.34}$$

This triple product can be used to rewrite the conformal realization. Recalling that a triple product with identical properties exists for the 27-dimensional Jordan algebra $J_3^{\mathbb{O}\,s}$, we now now consider $\tilde{Z}$ as an element of $J_3^{\mathbb{O}\,s}$. Next we introduce generators labeled by elements of $J_3^{\mathbb{O}\,s}$, and end up with the variations

$$\begin{aligned}U_a(\tilde{Z}) &= a\,,\\ S_{ab}(\tilde{Z}) &= \{a\,b\,\tilde{Z}\}\,,\\ \tilde{U}_c(\tilde{Z}) &= = -\tfrac{1}{2}\{\tilde{Z}\,c\,\tilde{Z}\}\,,\end{aligned} \tag{5.35}$$

for $a, b, c \in J_3^{\mathbb{O}s}$. It is straightforward to check[1] that these reproduce the commutation relations (5.23) and (5.24).

5.2 Quasiconformal Realizations

The conformal realizations of the last section are based on the Tits-Kantor-Köcher construction and thus can only applied to Lie algebras admitting a three-grading (5.2). In this section we present a novel class of non-linear realizations which we will call "quasiconformal". It includes realizations of E_8, F_4, and G_2 all of which do not admit a three-graded decomposition. We will start with the maximal case, the exceptional Lie group $E_{8(8)}$, and its quasiconformal realization on $\mathbb{R}^{57}$, because this realization contains all others by truncation.

5.2.1 $E_{7(7)}$ decomposition of $E_{8(8)}$

Our results are based on the following five graded decomposition of $E_{8(8)}$ with respect to its $E_{7(7)} \times \mathcal{D}$ subgroup

$$\begin{array}{ccccccccc} \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2} \\ \mathbf{1} & \oplus & \mathbf{56} & \oplus & (\mathbf{133} \oplus \mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{1} \end{array} \tag{5.36}$$

with the one-dimensional group $\mathcal{D}$ consisting of dilatations. $\mathcal{D}$ itself is part of an $SL(2,\mathbb{R})$ group, and the above decomposition thus corresponds to the decomposition $\mathbf{248} \to (\mathbf{133},\mathbf{1}) \oplus (\mathbf{56},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3})$ of $E_{8(8)}$ under its subgroup $E_{7(7)} \times SL(2,\mathbb{R})$. The subspaces $\mathfrak{g}^i$ $(-2 \le i \le 2)$ form representations of the Lie algebra $\mathfrak{g}^0 \equiv E_7 \oplus \mathcal{D}$ with the dimensions given in the second line of (5.36). The grading of the decomposition (5.36) is given by the dilatation generator H of the algebra $\mathcal{D}$.

In order to write out the $E_{7(7)}$ generators, it is convenient to further decompose them w.r.t. the subgroup $SL(8,\mathbb{R})$ of $E_{7(7)}$. In this basis, the Lie algebra of $E_{7(7)}$ is spanned by the $SL(8,\mathbb{R})$ generators $G^i{}_j$, and the antisymmetric generators G^{ijkl}, transforming in the $\mathbf{63}$ and $\mathbf{70}$ representations of $SL(8,\mathbb{R})$, respectively. We also define

$$G_{ijkl} := \tfrac{1}{24}\epsilon_{ijklmnpq}\, G^{mnpq}$$

with $SL(8,\mathbb{R})$ indices $1 \le i, j, \ldots \le 8$. The commutation relations are

$$\begin{aligned} [G^i{}_j\,,\, G^k{}_l] &= \delta^k_j\, G^i{}_l - \delta^i_l\, G^k{}_j\,, \\ [G^i{}_j\,,\, G^{klmn}] &= -4\,\delta^{[k}_j\, G^{lmn]i} - \tfrac{1}{2}\,\delta^i_j G^{klmn}\,, \\ [G^{ijkl}\,,\, G^{mnpq}] &= \tfrac{1}{36}\,\epsilon^{ijkls[mnp}\, G^{q]}{}_s\,. \end{aligned}$$

The fundamental $\mathbf{56}$ representation of $E_{7(7)}$ is spanned by the two antisymmetric real tensors X^{ij} and X_{ij} and the action of $E_{7(7)}$ is given by

$$\begin{aligned} \delta X^{ij} &= \Lambda^i{}_k X^{kj} - \Lambda^j{}_k X^{ki} + \Sigma^{ijkl} X_{kl}\,, \\ \delta X_{ij} &= \Lambda^k{}_i X_{jk} - \Lambda^k{}_j X_{ik} + \Sigma_{ijkl} X^{kl}\,, \end{aligned} \tag{5.37}$$

[1] For these and similar calculations we used the FORM 3 program [163].

where

$$\Sigma_{ijkl} \;=\; \tfrac{1}{24}\epsilon_{ijklmnpq}\Sigma^{mnpq} \tag{5.38}$$

In order to extend $E_{7(7)}\times\mathcal{D}$ to the full $E_{8(8)}$, we must enlarge $\mathcal{D}$ to an $SL(2,\mathbb{R})$ with generators (E,F,H) in the standard Chevalley basis, together with 2×56 further real generators (E_{ij},E^{ij}) and (F_{ij},F^{ij}). Under hermitean conjugation, we have

$$E^{ij} \;=\; F^{\dagger}_{ij}\,, \qquad F^{ij} \;=\; -E^{\dagger}_{ij}\,, \quad \text{and} \quad E \;=\; -F^{\dagger}\,.$$

The grade $-2,-1,1$ and 2 subspaces in the above decomposition correspond to the subspaces $\mathfrak{g}^{-2}$, $\mathfrak{g}^{-1}$, $\mathfrak{g}^{1}$, and $\mathfrak{g}^{2}$ in (5.36), respectively:

$$E\oplus\{E^{ij},E_{ij}\}\oplus\{G^{ijkl},G^{i}{}_{j}\,;\,H\}\oplus\{F^{ij},F_{ij}\}\oplus F \tag{5.39}$$

The grading may be read off from the commutators with H

$$\begin{aligned}
[H\,,\,E] &= -2\,E\,, & [H\,,\,F] &= 2\,F\,,\\
[H\,,\,E^{ij}] &= -E^{ij}\,, & [H\,,\,F^{ij}] &= F^{ij}\,,\\
[H\,,\,E_{ij}] &= -E_{ij}\,, & [H\,,\,F_{ij}] &= F_{ij}\,.
\end{aligned}$$

The new generators (E_{ij},E^{ij}) and (F_{ij},F^{ij}) form two (maximal) Heisenberg subalgebras of dimension 28

$$[E^{ij}\,,\,E_{kl}] \;=\; 2\,\delta^{ij}_{kl}\,E\,, \qquad [F^{ij}\,,\,F_{kl}] \;=\; 2\,\delta^{ij}_{kl}\,F\,,$$

and they transform under $SL(8,\mathbb{R})$ as

$$\begin{aligned}
[G^{i}{}_{j}\,,\,E^{kl}] &= \delta^{k}_{j}\,E^{il}-\delta^{l}_{j}\,E^{ik}-\tfrac{1}{4}\delta^{i}_{j}\,E^{kl}\,,\\
[G^{i}{}_{j}\,,\,E_{kl}] &= \delta^{i}_{k}\,E_{lj}-\delta^{i}_{l}\,E_{kj}+\tfrac{1}{4}\delta^{i}_{j}\,E_{kl}\,,\\
[G^{i}{}_{j}\,,\,F^{kl}] &= \delta^{k}_{j}\,F^{il}-\delta^{l}_{j}\,F^{ik}-\tfrac{1}{4}\delta^{i}_{j}\,F^{kl}\,,\\
[G^{i}{}_{j}\,,\,F_{kl}] &= \delta^{i}_{k}\,F_{lj}-\delta^{i}_{l}\,F_{kj}+\tfrac{1}{4}\delta^{i}_{j}\,F_{kl}\,.
\end{aligned}$$

The remaining non-vanishing commutation relations are given by

$$[E,F]=H$$

and

$$\begin{aligned}
[G^{ijkl}\,,\,E_{mn}] &= -\delta^{[ij}_{mn}\,E^{kl]}\,, & [G^{ijkl}\,,\,E^{mn}] &= -\tfrac{1}{24}\,\epsilon^{ijklmnpq}\,E_{pq}\,,\\
[G^{ijkl}\,,\,F_{mn}] &= -\delta^{[ij}_{mn}\,F^{kl]}\,, & [G^{ijkl}\,,\,F^{mn}] &= -\tfrac{1}{24}\,\epsilon^{ijklmnpq}\,F_{pq}\,,
\end{aligned}$$

$$\begin{aligned}
[E^{ij}\,,\,F^{kl}] &= 12\,G^{ijkl}\,, & [E_{ij}\,,\,F_{kl}] &= -12\,G_{ijkl}\,,\\
[E^{ij}\,,\,F_{kl}] &= 4\,\delta^{[i}_{[k}G^{j]}{}_{l]}-\delta^{ij}_{kl}\,H, & [E_{ij}\,,\,F^{kl}] &= 4\,\delta^{[k}_{[i}G^{l]}{}_{j]}+\delta^{kl}_{ij}\,H,\\
[E\,,\,F^{ij}] &= -E^{ij}\,, & [E\,,\,F_{ij}] &= -E_{ij}\,,\\
[F\,,\,E^{ij}] &= F^{ij}\,, & [F\,,\,E_{ij}] &= F_{ij}\,.
\end{aligned}$$

To see that we are really dealing with the maximally split form of $E_{8(8)}$, let us count the number of compact generators: The antisymmetric part $(G^i{}_j - G^j{}_i)$ of $G^i{}_j$ and $(G^{ijkl} - G_{ijkl})$ correspond to the 63 generators of of the maximal compact subalgebra $SU(8)$ of $E_{7(7)}$ [28]. The remaining compact generators are the $28+28+1$ anti-hermitean generators $(E_{ij}+F^{ij})$, $(E^{ij}-F_{ij})$, and $(E+F)$ giving a total of 120 generators which close into the maximal compact subgroup $SO(16) \supset SU(8)$ of $E_{8(8)}$.

An important role is played by the symplectic invariant of two **56** representations. It is given by

$$\langle X, Y \rangle \;:=\; X^{ij}Y_{ij} - X_{ij}Y^{ij} \,. \tag{5.40}$$

The second structure which we need to introduce is the triple product. This is a trilinear map $\mathbf{56}\times\mathbf{56}\times\mathbf{56} \longrightarrow \mathbf{56}$, which associates to three elements X, Y and Z another element transforming in the **56** representation, denoted by (X,Y,Z), and defined by

$$\begin{aligned}
(X,Y,Z)^{ij} \;:=\;& -8\,X^{ik}Y_{kl}Z^{lj} - 8\,Y^{ik}X_{kl}Z^{lj} - 8\,Y^{ik}Z_{kl}X^{lj} \\
& -2\,Y^{ij}X^{kl}Z_{kl} - 2\,X^{ij}Y^{kl}Z_{kl} - 2\,Z^{ij}Y^{kl}X_{kl} \\
& +\tfrac{1}{2}\,\epsilon^{ijklmnpq}X_{kl}Y_{mn}Z_{pq} \\
(X,Y,Z)_{ij} \;:=\;& 8\,X_{ik}Y^{kl}Z_{lj} + 8\,Y_{ik}X^{kl}Z_{lj} + 8\,Y_{ik}Z^{kl}X_{lj} \\
& +2\,Y_{ij}Z^{kl}X_{kl} + 2\,X_{ij}Z^{kl}Y_{kl} + 2\,Z_{ij}X^{kl}Y_{kl} \\
& -\tfrac{1}{2}\,\epsilon_{ijklmnpq}X^{kl}Y^{mn}Z^{pq} \,.
\end{aligned} \tag{5.41}$$

A somewhat tedious calculation[2] shows that this triple product obeys the relations

$$\begin{aligned}
(X,Y,Z) \;=\;& (Y,X,Z) + 2\,\langle X,Y\rangle Z \,, \\
(X,Y,Z) \;=\;& (Z,Y,X) - 2\,\langle X,Z\rangle Y \,, \\
\langle (X,Y,Z),W\rangle \;=\;& \langle (X,W,Z),Y\rangle - 2\,\langle X,Z\rangle\langle Y,W\rangle \,, \\
(X,Y,(V,W,Z)) \;=\;& (V,W,(X,Y,Z)) + ((X,Y,V),W,Z) \\
& + (V,(Y,X,W),Z) \,.
\end{aligned} \tag{5.42}$$

We note that the triple product (5.41) could be modified by terms involving the symplectic invariant, such as $\langle X,Y\rangle Z$; the above choice has been made in order to obtain agreement with the formulas of [49].

While there is no (symmetric) quadratic invariant of $E_{7(7)}$ in the **56** representation, a real quartic invariant $\mathcal{I}_4$ can be constructed by means of the above triple product and the bilinear form; it reads

$$\begin{aligned}
\mathcal{I}_4(X^{ij},X_{ij}) \;:=\;& \tfrac{1}{48}\,\langle (X,X,X),X\rangle \\
\equiv\;& X^{ij}X_{jk}X^{kl}X_{li} - \tfrac{1}{4}X^{ij}X_{ij}X^{kl}X_{kl} \\
& +\tfrac{1}{96}\,\epsilon^{ijklmnpq}X_{ij}X_{kl}X_{mn}X_{pq} \\
& +\tfrac{1}{96}\,\epsilon_{ijklmnpq}X^{ij}X^{kl}X^{mn}X^{pq} \,.
\end{aligned} \tag{5.43}$$

[2] Which relies heavily on the Schouten identity $\varepsilon_{[ijklmnpq}X_{r]s} = 0$.

5.2.2 Quasiconformal nonlinear realization of $E_{8(8)}$

We will now present a nonlinear realization of $E_{8(8)}$ on the 57-dimensional real vector space with coordinates

$$\mathcal{X} := (X^{ij}, X_{ij}, x)\,,$$

where x is also real. While x is a $E_{7(7)}$ singlet, the remaining 56 variables transform linearly under $E_{7(7)}$. Thus $\mathcal{X}$ forms the $\mathbf{56} \oplus \mathbf{1}$ representation of $E_{7(7)}$. In writing the transformation rules we will omit the transformation parameters in order not to make the formulas (and notation) too cumbersome. To recover the infinitesimal variations, one can simply contract the formulas with the appropriate transformation parameters. The $E_{7(7)}$ subalgebra acts linearly as

$$\begin{aligned}
G^i{}_j(X^{kl}) &= 2\,\delta^k_j X^{il} - \tfrac{1}{4}\delta^i_j X^{kl}\,, & G^{ijkl}(X^{mn}) &= \tfrac{1}{24}\epsilon^{ijklmnpq} X_{pq}\,,\\
G^i{}_j(X_{kl}) &= -2\,\delta^i_k X_{jl} + \tfrac{1}{4}\delta^i_j X_{kl}\,, & G^{ijkl}(X_{mn}) &= \delta^{[ij}_{mn} X^{kl]}\,,\\
G^i{}_j(x) &= 0\,, & G^{ijkl}(x) &= 0\,,
\end{aligned} \tag{5.44}$$

H generates scale transformations

$$H(X^{ij}) = X^{ij}\,, \qquad H(X_{ij}) = X_{ij}\,, \qquad H(x) = 2\,x\,, \tag{5.45}$$

and the E generators act as translations; we have

$$E(X^{ij}) = 0\,, \qquad E(X_{ij}) = 0\,, \qquad E(x) = 1\,. \tag{5.46}$$

and

$$\begin{aligned}
E^{ij}(X^{kl}) &= 0\,, & E^{ij}(X_{kl}) &= \delta^{ij}_{kl}\,, & E^{ij}(x) &= -X^{ij}\,,\\
E_{ij}(X^{kl}) &= \delta^{kl}_{ij}\,, & E_{ij}(X_{kl}) &= 0\,, & E_{ij}(x) &= X_{ij}\,.
\end{aligned} \tag{5.47}$$

By contrast, the F generators are realized nonlinearly:

$$\begin{aligned}
F(X^{ij}) &= -\tfrac{1}{6}(X,X,X)^{ij} + X^{ij}\,x\\
&\equiv 4X^{ik}X_{kl}X^{lj} + X^{ij}X^{kl}X_{kl}\\
&\quad -\tfrac{1}{12}\epsilon^{ijklmnpq}X_{kl}X_{mn}X_{pq} + X^{ij}\,x\,,\\
F(X_{ij}) &= -\tfrac{1}{6}(X,X,X)_{ij} + X_{ij}\,x\\
&\equiv -4X_{ik}X^{kl}X_{lj} - X_{ij}X^{kl}X_{kl}\\
&\quad +\tfrac{1}{12}\epsilon_{ijklmnpq}X^{kl}X^{mn}X^{pq} + X_{ij}\,x\,,\\
F(x) &= 4\,\mathcal{I}_4(X^{ij}, X_{ij}) + x^2\\
&\equiv 4\,X^{ij}X_{jk}X^{kl}X_{li} - X^{ij}X_{ij}X^{kl}X_{kl}\\
&\quad +\tfrac{1}{24}\,\epsilon^{ijklmnpq}X_{ij}X_{kl}X_{mn}X_{pq}\\
&\quad +\tfrac{1}{24}\,\epsilon_{ijklmnpq}X^{ij}X^{kl}X^{mn}X^{pq} + x^2\,.
\end{aligned} \tag{5.48}$$

Observe that the form of the r.h.s. is dictated by the requirement of $E_{7(7)}$ covariance: $(F(X^{ij}), F(X_{ij}))$ and $F(x)$ must still transform as the **56** and **1** of $E_{7(7)}$, respectively. The action of the remaining generators is likewise $E_{7(7)}$ covariant:

$$
\begin{aligned}
F^{ij}(X^{kl}) &= -4\, X^{i[k} X^{l]j} + \tfrac{1}{4}\, \epsilon^{ijklmnpq} X_{mn} X_{pq}\,, \\
F^{ij}(X_{kl}) &= +8\, \delta^{[i}_{k} X^{j]m} X_{ml} + \delta^{ij}_{kl}\, X^{mn} X_{mn} + 2\, X^{ij} X_{kl} - \delta^{ij}_{kl}\, x\,, \\
F_{ij}(X^{kl}) &= -8\, \delta^{k}_{[i} X_{j]m} X^{ml} + \delta^{kl}_{ij}\, X^{mn} X_{mn} - 2\, X_{ij} X^{kl} - \delta^{kl}_{ij}\, x\,, \\
F_{ij}(X_{kl}) &= 4\, X_{ki} X_{jl} - \tfrac{1}{4}\, \epsilon_{ijklmnpq} X^{mn} X^{pq}\,, \\
F^{ij}(x) &= 4\, X^{ik} X_{kl} X^{lj} + X^{ij} X^{kl} X_{kl} \\
&\quad - \tfrac{1}{12}\, \epsilon^{ijklmnpq} X_{kl} X_{mn} X_{pq} + X^{ij}\, x\,, \\
F_{ij}(x) &= 4\, X_{ik} X^{kl} X_{lj} + X_{ij} X^{kl} X_{kl} \\
&\quad - \tfrac{1}{12}\, \epsilon_{ijklmnpq} X^{kl} X^{mn} X^{pq} - X_{ij}\, x\,.
\end{aligned}
\tag{5.49}
$$

Although $E_{7(7)}$ covariance considerably constrains the expressions that can appear on the r.h.s., it does not fix them uniquely: to the triple product (5.41) one could add further terms involving the symplectic invariant. However, all ambiguities are removed by imposing closure of the algebra, and we have checked by explicit computation that the above variations do close into the full $E_{8(8)}$ algebra in the basis given in the previous section. This is the crucial consistency check.

The term "quasiconformal realization" is motivated by the existence of a norm form that is left invariant up to a (possibly coordinate dependent) factor under all transformations. To write it down we must first define a nonlinear "difference" between two points $\mathcal{X} \equiv (X^{ij}, X_{ij}\,;\, x)$ and $\mathcal{Y} \equiv (Y^{ij}, Y_{ij}\,;\, y)$; curiously, the standard difference is *not* invariant under the translations (E^{ij}, E_{ij}). Rather, we must choose

$$
\delta(\mathcal{X}, \mathcal{Y}) := \left(X^{ij} - Y^{ij}, X_{ij} - Y_{ij}\ ;\ x - y + \langle X, Y\rangle\right). \tag{5.50}
$$

This difference still obeys $\delta(\mathcal{X}, \mathcal{Y}) = -\delta(\mathcal{Y}, \mathcal{X})$ and thus $\delta(\mathcal{X}, \mathcal{X}) = 0$, and is now invariant under (E^{ij}, E_{ij}) as well as E; however, it is no longer additive. In fact, with the sum of two vectors being defined as $\delta(\mathcal{X}, -\mathcal{Y})$, the extra term involving $\langle X, Y\rangle$ can be interpreted as the cocycle induced by the standard canonical commutation relations.

The relevant invariant is a linear combination of x^2 and the quartic $E_{7(7)}$ invariant $\mathcal{I}_4$, viz.

$$
\mathcal{N}_4(\mathcal{X}) \equiv \mathcal{N}_4(X^{ij}, X_{ij}; x) := 4\, \mathcal{I}_4(X) - x^2\,, \tag{5.51}
$$

In order to ensure invariance under the translation generators, we consider the expression $\mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}))$, which is manifestly invariant under the linearly realized subgroup $E_{7(7)}$. Remarkably, it also transforms into itself up to an overall factor under the action of the nonlinearly realized generators. More specifically, we find

$$\begin{aligned}
F\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) &= 2\,(x+y)\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y})) \\
F^{ij}\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) &= 2\,(X^{ij}+Y^{ij})\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y})) \\
H\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) &= 4\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))
\end{aligned}$$

Therefore, for every $\mathcal{Y}\in\mathbb{R}^{57}$ the "light cone" with base point $\mathcal{Y}$, defined by the set of $\mathcal{X}\in\mathbb{R}^{57}$ obeying

$$\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))=0\,, \tag{5.52}$$

is preserved by the full $E_{8(8)}$ group, and in this sense, $\mathcal{N}_4$ is a "conformal invariant" of $E_{8(8)}$. We note that the light cones defined by the above equation are not only curved hyper-surfaces in $\mathbb{R}^{57}$, but get deformed as one varies the base point $\mathcal{Y}$. As we will show in appendix B, the quartic invariant $\mathcal{I}_4$ can take both positive and negative values, but in the latter case eq. (5.52) does not have real solutions. However, we can remedy this problem by extending the representation space to $\mathbb{C}^{57}$ and using the same formulas to get a realization of the complexified Lie algebra $E_8(\mathbb{C})$ on $\mathbb{C}^{57}$.

The existence of a fourth order conformal invariant of $E_{8(8)}$ is noteworthy in view of the fact that no irreducible fourth order invariant exists for the linearly realized $E_{8(8)}$ group (the next invariant after the quadratic Casimir being of order eight).

5.2.3 Relation with Freudenthal Triple Systems

We will now rewrite the nonlinear transformation rules in another form in order to establish contact with the mathematical literature. Both the bilinear form (5.40) and the triple product (5.41) already appear in [49], albeit in a very different guise. In that work, the starting point is the set of 2×2 "matrices" of the form

$$A=\begin{pmatrix}\alpha_1 & x_1\\ x_2 & \alpha_2\end{pmatrix}\,, \tag{5.53}$$

where α_1,α_2 are real numbers and x_1,x_2 are elements of a simple Jordan algebra J of degree three. There are only four simple Jordan algebras J of this type, namely the 3×3 hermitean matrices over the four division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The associated matrices are then related to non-compact forms of the exceptional Lie algebras F_4, E_6, E_7, and E_8, respectively. For simplicity, let us concentrate on the maximal case $J_3^{\mathbb{O}\,s}$, when the matrix A carries 1+1+27+27 = 56 degrees of freedom. This counting suggests an obvious relation with the **56** of $E_{7(7)}$ and its decomposition under $E_{6(6)}$, but more work is required to make the connection precise. To this aim, [49] defines a symplectic invariant $\langle A,B\rangle$, and a trilinear product mapping three such matrices A,B and C to another one, denoted by (A,B,C). This triple system differs from a Jordan triple system in that it is not derivable from a binary product. The formulas for the triple product in terms of the matrices A,B and C given in [49] are somewhat cumbersome, lacking manifest $E_{7(7)}$ covariance. For this reason, instead of directly verifying that our prescription (5.41) and the one of [49] coincide, we have checked that they satisfy identical relations: a quick glance shows that the relations (T1)–(T4) [49] are indeed the same as our relations (5.42), which are manifestly $E_{7(7)}$ covariant.

To rewrite the transformation formulas we introduce Lie algebra generators U_A and $\tilde{U}_A$ labeled by the above matrices, as well as generators S_{AB} labeled by a pair of such matrices. For the grade ± 2 subspaces we would in general need another set of generators K_{AB} and $\tilde{K}_{AB}$ labeled by two matrices, but since these subspaces are one-dimensional in the present case, we have only two more generators K_a and $\tilde{K}_a$ labelled by one real number a. In the same vein, we reinterpret the 57 coordinates $\mathcal{X}$ as a pair (X, x), where X is a 2×2 matrix of the type defined above. The variations then take the simple form

$$\begin{aligned} K_a(X) &= 0\,, & \tilde{K}_a(X) &= -\tfrac{1}{6}\,a(X,X,X) + aXx\,, \\ K_a(x) &= 2\,a\,, & \tilde{K}_a(x) &= \tfrac{1}{6}\,a\langle (X,X,X), X\rangle + 2\,ax^2 \\ U_A(X) &= A\,, & \tilde{U}_A(X) &= \tfrac{1}{2}(X,A,X) - Ax\,, \\ U_A(x) &= \langle A, X\rangle\,, & \tilde{U}_A(x) &= -\tfrac{1}{6}\langle (X,X,X), A\rangle + \langle X, A\rangle x\,, \\ S_{AB}(X) &= (A,B,X)\,, & S_{AB}(x) &= 2\langle A, B\rangle x\,, \end{aligned} \tag{5.54}$$

From these formulas it is straightforward to determine the commutation relations of the transformations. To expose the connection with the more general Kantor triple systems we write

$$K_{AB} \equiv K_{\langle A,B\rangle} \tag{5.55}$$

in the formulas below. The consistency of this specialization is ensured by the relations (5.42). By explicit computation one finds

$$\begin{aligned} [U_A, \tilde{U}_B] &= S_{AB}\,, \\ [U_A, U_B] &= -K_{AB}\,, \\ [\tilde{U}_A, \tilde{U}_B] &= -\tilde{K}_{AB}\,, \\ [S_{AB}, U_C] &= -U_{(A,B,C)}\,, \\ [S_{AB}, \tilde{U}_C] &= -\tilde{U}_{(B,A,C)}\,, \\ [K_{AB}, \tilde{U}_C] &= U_{(A,C,B)} - U_{(B,C,A)}\,, \\ [\tilde{K}_{AB}, U_C] &= \tilde{U}_{(B,C,A)} - \tilde{U}_{(A,C,B)}\,, \\ [S_{AB}, S_{CD}] &= -S_{(A,B,C)D} - S_{C(B,A,D)}\,, \\ [S_{AB}, K_{CD}] &= K_{A(C,B,D)} - K_{A(D,B,C)}\,, \\ [S_{AB}, \tilde{K}_{CD}] &= \tilde{K}_{(D,A,C)B} - \tilde{K}_{(C,A,D)B}\,, \\ [K_{AB}, \tilde{K}_{CD}] &= S_{(B,C,A)D} - S_{(A,C,B)D} - S_{(B,D,A)C} + S_{(A,D,B)C}\,. \end{aligned} \tag{5.56}$$

For general K_{AB}, these are the defining commutation relations of a Kantor triple system, and, with the further specification (5.55), those of a Freudenthal triple system (FTS). Freudenthal introduced these triple systems in his study of the meta-symplectic geometries associated with exceptional groups [58]; these geometries were further studied in [2, 49, 134, 111]. A classification of FTS's may be found in [111], where it is also shown that there is a one-to-one correspondence between simple Lie algebras and simple FTS's with a non-degenerate bilinear form. Hence there is a quasiconformal realization of every Lie group acting on a generalized light-cone.

5.2.4 The quartic $E_{7(7)}$ invariant

In the $SL(8,\mathbb{R})$ basis $E_{7(7)}$ the quartic invariant is given by (5.43), which we here repeat for convenience

$$\begin{aligned}\mathcal{I}_4^{\mathrm{SL}(8,\mathbb{R})} &= X^{ij}X_{jk}X^{kl}X_{li} - \tfrac{1}{4}X^{ij}X_{ij}X^{kl}X_{kl} \\ &+\tfrac{1}{96}\,\epsilon^{ijklmnpq}X_{ij}X_{kl}X_{mn}X_{pq} \\ &+\tfrac{1}{96}\,\epsilon_{ijklmnpq}X^{ij}X^{kl}X^{mn}X^{pq}\,. \end{aligned} \tag{5.57}$$

Another very useful form of $E_{7(7)}$ makes the maximal compact subgroup $SU(8)$ manifest. The fundamental **56** representation then is spanned by the complex tensors Z_{AB} which are related to the $SL(8,\mathbb{R})$ basis by [28]

$$Z^{AB} = (Z_{AB})^* = \tfrac{1}{4\sqrt{2}}(X^{ij} - \mathrm{i}\,X_{ij})\Gamma^{ij}_{AB}\,, \tag{5.58}$$

where Γ^{ij}_{AB} are the $SO(16)$ gamma matrices. In this basis the quartic invariant takes the form

$$\begin{aligned}\mathcal{I}_4^{\mathrm{SU}(8)} &= Z^{AB}Z_{BC}Z^{CD}Z_{DA} - \tfrac{1}{4}Z^{AB}Z_{AB}Z^{CD}Z_{CD} \\ &+\tfrac{1}{96}\,\epsilon^{ABCDEFGH}Z_{AB}Z_{CD}Z_{EF}Z_{GH} \\ &+\tfrac{1}{96}\,\epsilon_{ABCDEFGH}Z^{AB}Z^{CD}Z^{EF}Z^{GH}\,. \end{aligned} \tag{5.59}$$

The precise relation between $\mathcal{I}_4^{\mathrm{SU}(8)}$ and $\mathcal{I}_4^{\mathrm{SL}(8,\mathbb{R})}$ has never been spelled out in the literature although it is claimed in [28] that they should be proportional. In fact, we have

$$\mathcal{I}_4^{\mathrm{SU}(8)} = -\mathcal{I}_4^{\mathrm{SL}(8,\mathbb{R})}\,. \tag{5.60}$$

To prove this claim, one needs the identities

$$\begin{aligned}\mathrm{Tr}(\Gamma^{ij}\Gamma^{kl}\Gamma^{mn}\Gamma^{pq}) &= -128\,\underbrace{\delta^{ij}_{p[k}\delta^{mn}_{l]q}} + 128\,\underbrace{\delta^{ij}_{p[m}\delta^{kl}_{n]q}} + 128\,\underbrace{\delta^{ij}_{k[m}\delta^{pq}_{n]l}} \\ &\quad + 96\,(\delta^{ij}_{kl}\delta^{mn}_{pq})_{\mathrm{sym}} \mp 8\,\epsilon^{ijklmnpq}\,, \end{aligned} \tag{5.61}$$

and

$$\begin{aligned}\epsilon^{ABCDEFGH}\Gamma^{ij}_{AB}\Gamma^{kl}_{CD}\Gamma^{mn}_{EF}\Gamma^{pq}_{GH} &= -128\,(12\,\delta^{ij}_{kl}\delta^{mn}_{pq} + 48\,\underbrace{\delta^{ij}_{p[k}\delta^{mn}_{l]q}})_{\mathrm{sym}} \\ &\quad \mp \epsilon^{ijklmnpq}\,, \end{aligned} \tag{5.62}$$

where $(\ldots)_{\mathrm{sym}}$ denotes symmetrization w.r.t. the pairs of indices (ij), (kl), (mn), (pq), and the signs $\mp$ depend on whether the spinor representation or the conjugate spinor representation of the gamma matrices is used:

$$\Gamma^{ijklmnpq} = \mp\epsilon^{ijklmnpq}$$

Using the local gauge symmetry $SU(8)$ the central charge matrix can be brought into the skew diagonal form [55]

$$Z_{AB} =: \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.63}$$

with complex parameters $z_1, \ldots, z_4$. For this configuration the quartic invariant becomes

$$\mathcal{I}_4^{\mathrm{SU}(8)} = \sum_{\alpha} |z_\alpha|^4 - 2 \sum_{\beta>\alpha} |z_\alpha|^2 |z_\beta|^2 + 4\, z_1 z_2 z_3 z_4 + 4\, z_1^* z_2^* z_3^* z_4^* \,. \tag{5.64}$$

Using this formula, one can easily see that both positive and negative values are possible for $\mathcal{I}_4$:

i) We find positive values for $\mathcal{I}_4$ when all but one parameter vanish:

$$\mathcal{I}_4^{\mathrm{SU}(8)} = |z_1|^4 > 0 \quad \text{for} \quad z_1 \neq 0,\, z_2 = z_3 = z_4 = 0$$

ii) $\mathcal{I}_4$ vanishes when all parameters take the same real (electric) or imaginary (magnetic) value:

$$\mathcal{I}_4^{\mathrm{SU}(8)} = 0 \quad \text{for} \quad z_1 = z_2 = z_3 = z_4 = M \text{ or } \mathrm{i}M\,,\ M \in \mathbb{R}$$

This is the example considered in [105] corresponding to maximally BPS black hole solutions in $d{=}4$, $N{=}8$ supergravity with vanishing entropy and vanishing area of the horizon.

iii) $\mathcal{I}_4$ is negative when all parameters take the same complex "dyonic" value. For instance,

$$\mathcal{I}_4^{\mathrm{SU}(8)} < 0 \quad \text{for} \quad z_1 = z_2 = z_3 = z_4 = \tfrac{1+\mathrm{i}}{\sqrt{2}} M\,,\ M \in \mathbb{R}\,,$$

corresponding to a maximally BPS multiplet with both electric *and* magnetic charges.

$\mathcal{I}_4$ can assume both positive and negative values [54].

5.3 Truncations of $E_{8(8)}$

For the lower rank exceptional groups contained in $E_{8(8)}$, we can derive similar conformal or quasiconformal realizations by truncation. In this section, we will give the list of quasiconformal realizations contained in $E_{8(8)}$. All simple Lie algebras (except for $SU(2)$) can be given a five graded structure (5.15) with respect to some subalgebra of maximal rank and associate a triple system with the grade +1 subspace [110, 6]. Conversely, one can construct every simple Lie algebra over the corresponding triple system.

The realization of E_8 over the FTS defined by the exceptional Jordan algebra can be truncated to the realizations of E_7, E_6, and F_4 by restricting oneself to subalgebras defined by quaternionic, complex, and real hermitean 3×3 matrices. Analogously the non-linear realization of $E_{8(8)}$ given in the previous section can be truncated to non-linear realizations of $E_{7(7)}$, $E_{6(6)}$, and $F_{4(4)}$. These truncations preserve the five grading. More specifically we find that the Lie algebra of $E_{7(7)}$ has a five grading of the form:

$$E_{7(7)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{32}} \oplus (SO(6,6) \oplus \mathcal{D}) \oplus \mathbf{32} \oplus \mathbf{1} \tag{5.65}$$

Hence this truncation leads to a nonlinear realization of $E_{7(7)}$ on a **33** dimensional space. Note that this is not a minimal realization of $E_{7(7)}$. Further truncation to the $E_{6(6)}$ subgroup preserving the five grading leads to:

$$E_{6(6)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{20}} \oplus (SL(6,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{20} \oplus \mathbf{1} \tag{5.66}$$

This yields a nonlinear realization of $E_{6(6)}$ on a **21** dimensional space, which again is not the minimal realization. Further reduction to $F_{4(4)}$ preserving the five grading

$$F_{4(4)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{14}} \oplus (Sp(6,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{14} \oplus \mathbf{1} \tag{5.67}$$

leads to a minimal realization of $F_{4(4)}$ on a fifteen dimensional space. One can further truncate F_4 to a subalgebra $G_{2(2)}$ while preserving the five grading

$$G_{2(2)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{4}} \oplus (SL(2,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{4} \oplus \mathbf{1}\,, \tag{5.68}$$

which then yields a nonlinear realization over a five dimensional space. One can go even further and truncate G_2 to its subalgebras $SU(2,1)$ and $SL(3,\mathbb{R})$

$$SL(3,\mathbb{R}) \;=\; \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus (SO(1,1) \oplus \mathcal{D}) \oplus \mathbf{2} \oplus \mathbf{1}\,, \tag{5.69}$$

$$SU(2,1) \;=\; \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus (SO(2) \oplus \mathcal{D}) \oplus \mathbf{2} \oplus \mathbf{1}\,, \tag{5.70}$$

which is the smallest simple Lie algebra admitting a five grading. We should perhaps stress that the nonlinear realizations given above are minimal for $G_{2(2)}$, $F_{4(4)}$, and $E_{8(8)}$ which are the only simple Lie algebras that do not admit a three grading and hence do not have unitary representations of the lowest weight type.

The above nonlinear realizations of the exceptional Lie algebras can also be truncated to subalgebras with a three graded structure, in which case our nonlinear realization reduces to the standard nonlinear realization over a JTS. This truncation corresponds to conformal realizations as described in section 5.1.

With respect to $E_{6(6)}$ the quasiconformal realization of $E_{8(8)}$ (5.36) decomposes as follows:

$$\begin{array}{ccccccccc}
\mathbf{1} & \oplus & \mathbf{56} & \oplus & (\mathbf{133} \oplus \mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{1} \\
 & & & & \mathbf{1} & & & & \\
 & & \mathbf{1} & & \oplus & & \mathbf{1} & & \\
 & & \oplus & & \mathbf{27} & & \oplus & & \\
 & & \mathbf{27} & & \oplus & & \mathbf{27} & & \\
\mathbf{1} & & \oplus & & \mathbf{78} & & \oplus & & \mathbf{1} \\
 & & \overline{\mathbf{27}} & & \oplus & & \overline{\mathbf{27}} & & \\
 & & \oplus & & \overline{\mathbf{27}} & & \oplus & & \\
 & & \mathbf{1} & & \oplus & & \mathbf{1} & & \\
 & & & & \mathbf{1} & & & &
\end{array}$$

The numbers in the first line are the dimensions of $E_{7(7)}$, whereas the remaining numbers correspond to representations of $USp(8)$ which is the maximal compact subgroup of $E_{6(6)}$. The **27** of grade -1 subspace and the $\overline{\mathbf{27}}$ of grade $+1$ subspace close into the $E_{6(6)} \oplus \mathcal{D}$ subalgebra of grade zero subspace and generate the Lie algebra of $E_{7(7)}$. Similarly $\overline{\mathbf{27}}$ of grade -1 subspace together with the **27** of grade $+1$ subspace form another $E_{7(7)}$ subalgebra of $E_{8(8)}$. Hence we have four different $E_{7(7)}$ subalgebras of $E_{8(8)}$:

i) $E_{7(7)}$ subalgebra of grade zero subspace which is realized linearly.

ii) $E_{7(7)}$ subalgebra preserving the 5-grading, which is realized nonlinearly over a 33 dimensional space

iii) $E_{7(7)}$ subalgebra that acts on the **27** dimensional subspace as the generalized conformal generators.

iv) $E_{7(7)}$ subalgebra that acts on the $\overline{\mathbf{27}}$ dimensional subspace as the generalized conformal generators.

Similarly for $E_{7(7)}$ under the $SL(6,\mathbb{R})$ subalgebra of the grade zero subspace the **32** dimensional grade $+1$ subspace decomposes as

$$\mathbf{32} = \mathbf{1} + \overline{\mathbf{15}} + \mathbf{15} + \mathbf{1}\,.$$

The **15** from grade $+1\,(-1)$ subspace together with $\overline{\mathbf{15}}$ (**15**) of grade $-1\,(+1)$ subspace generate a nonlinearly realized $SO(6,6)$ subalgebra that acts as the generalized conformal algebra on the **15** ($\overline{\mathbf{15}}$) dimensional subspace.

For $E_{6(6)}$, $F_{4(4)}$, $G_{2(2)}$, and $SU(8)(2,1)$ the analogous truncations lead to nonlinear conformal realizations of the subalgebras $SL(6,\mathbb{R})$, $Sp(6,\mathbb{R})$, $SO(2,2)$, and $SL(2,\mathbb{R})$, respectively.

5.4 Quasiconformal Realizations and U-Duality

In this section, we will discuss the physical interpretation of the quasiconformal realization of $E_{8(8)}$ in the context of maximal supergravity. As we have shown in chapter 2 the Lie group $E_{8(8)}$ is the hidden symmetry group of maximal supergravity in three dimensions. Furthermore, all the Lie groups we found as truncations in section 5.3 appear as symmetry groups of certain supergravity theories reduced to $d=3$ [16, 29].

To illustrate the quasiconformal group action, we will consider two different truncated models in $d=3$: $N=2$, $d=4$ supergravity [164] and the simple $N=2$, $d=5$ supergravity [22] reduced to three spacetime dimensions. Both theories are known to arise as Calabi-Yau compactifications of eleven-dimensional supergravity (see [20] and references therein).

The first example we will discuss is the $N=2$ theory in $d=4$, [164]. Dimensional reduction to $d=3$ yields a Lagrangian that is invariant under its U-duality group $SU(2,1)$ [87, 115]. The bosonic Lagrangian is the Einstein-Maxwell Lagrangian which of course is much simpler than the maximal supergravity theory with Lagrangian (2.5). However, several interesting aspects related to the physical interpretation of the quasiconformal realizations in the context of supergravity already appear in this simplified model. We give the explicit formulas for the global U-duality symmetry transformations and match them with the quasiconformal realization of $E_{8(8)}$ in its truncation to $SU(2,1)$. We identify the celebrated Ehlers-group $SL(2,\mathbb{R})$ [47] as a subgroup of our quasiconformal group and give an interpretation for the extra one-dimensional space $\mathfrak{g}^{-2}$ appearing in our construction (5.36) as the translation symmetry of the so-called twist or NUT potential [63].

The second example is $N=2$ supergravity in $d=5$ [22], which is known to resemble many features of $d=11$ supergravity. After reduction to three dimensions, it exhibits a global $G_{2(2)}$ symmetry [136]. The $G_{2(2)}$ symmetry group can also be obtained by truncation of the quasiconformal $E_{8(8)}$ realization. In this case we have in addition to the $SU(2,1)$ symmetry a "Matzner-Misner"-type $SL(2,\mathbb{R})$ [132] that combines with the Ehlers-$SL(2,\mathbb{R})$ into the $SL(3,\mathbb{R})$ group found in [127]. We identify both the $SU(2,1)$ and the $SL(3,\mathbb{R})$ subgroup of the quasiconformal realized $G_{2(2)}$.

5.4.1 $d=4$, $N=2$ supergravity reduced to $d=3$

We consider the truncation of the maximal supergravity theory discussed in chapter 2 to $N=2$. The Lagrangian (2.5) of the maximal supergravity theory compactified to $d=4$ dimensions reduces to the Einstein-Maxwell Lagrangian[3]

$$\mathcal{L} \;=\; E(R - \tfrac{1}{4}\,F_{(2)}^2) \tag{5.71}$$

with just a single two-form field strength $\mathcal{F}_{(2)}$.

We will now further compactify to $d=3$ dimensions. According to (2.20), the four-dimensional metric will give rise to a dilaton ϕ and a one-form $B_{(1)}$ that can be dualized to another scalar field ψ that is known as the twist or NUT potential. It is well known that these two scalar fields form a $SL(2)/SO(2)$ sigma model [61] and the nonlinearly acting $SL(2)$ is the Ehlers-group [47].

The two-form field strength $\mathcal{F}_{(2)}$ will give rise to a scalar field and a vector field in $d=3$. We denotes the "electric" scalar field by φ and dualize the vector field to an additional "magnetic" scalar field χ .

Lagrangian

We find the three-dimensional Lagrangian

$$\mathcal{L} = E\left[R - \tfrac{1}{2}\partial\phi\partial\phi - \tfrac{1}{2}e^{-4\phi}|\partial\psi - \varphi\partial\chi + \chi\partial\varphi|^2 + \tfrac{1}{2}e^{-2\phi}(|\partial\varphi|^2 + |\partial\chi|^2)\right]\,. \tag{5.72}$$

Symmetries

To recover the symmetries of the Lagrangian, we apply to the following transformations to the scalar fields

$$\begin{aligned} z &:= \tfrac{1}{2}(\varphi^2 + \chi^2 - e^{2\phi}) + i\psi \\ w &:= \varphi + i\chi \end{aligned} \tag{5.73}$$

[3] As in chapter 2 we use the 0-form Lagrangian. The volume form Lagrangian would be $\mathcal{L} = E * R - \frac{1}{2} * F_{(2)} \wedge F_{(2)}$

Then the scalar part of the Lagrangian (5.72) takes the form

$$\mathcal{L}_{\text{scalar}} = -\tfrac{1}{2}\,E\left[\frac{dw d\bar{w}}{z+\bar{z}-w\bar{w}} + \frac{(dz-\bar{w}dw)(d\bar{z}-wd\bar{w})}{(z+\bar{z}-w\bar{w})^2}\right] \tag{5.74}$$

This Lagrangian has the following global symmetries:

$$\begin{aligned}
(-2):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} z+ic \\ w \end{pmatrix} & (+2):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} \frac{z}{1+i\gamma\,z} \\ \frac{w}{1+i\gamma\,z} \end{pmatrix} \\
(-1):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} z+\frac{1}{2}A\bar{A}+\bar{A}\,w \\ w+A \end{pmatrix} & (+1):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} \frac{z}{1+\frac{1}{2}B\bar{B}\,z+\bar{B}\,w} \\ \frac{w+B\,z}{1+\frac{1}{2}B\bar{B}\,z+\bar{B}\,w} \end{pmatrix} \\
(0):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} d^2 z \\ d\,w \end{pmatrix} & (0'):\ \begin{pmatrix} z \\ w \end{pmatrix} &\mapsto \begin{pmatrix} z \\ e^{i\varepsilon}w \end{pmatrix}
\end{aligned} \tag{5.75}$$

with parameters $A := a+ib$, $B := \alpha+i\gamma$, and $a, b, c, d, \varepsilon, \alpha, \beta, \gamma \in \mathbb{R}$. The local form of these transformations is given by

$$\begin{aligned}
\delta_{-2}\begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} ic \\ 0 \end{pmatrix} \\
\delta_{-1}\begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} \bar{A}\,w \\ A \end{pmatrix} \\
\delta_{0}\begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} 2d\,z \\ d\,w+i\varepsilon\,w \end{pmatrix} \\
\delta_{1}\begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} -\bar{B}\,wz \\ B\,z-\bar{B}\,w^2 \end{pmatrix} \\
\delta_{2}\begin{pmatrix} z \\ w \end{pmatrix} &= \begin{pmatrix} -i\gamma\,z^2 \\ -i\gamma\,wz \end{pmatrix}
\end{aligned} \tag{5.76}$$

After transforming the fields back we find the following symmetries of the Lagrangian (5.72):

$$\begin{aligned}
\delta\phi &= d-\alpha\,\varphi-\beta\,\chi+\gamma\,\psi\,, \\
\delta\,\varphi &= a+d\,\varphi-\varepsilon\chi+\tfrac{\alpha}{2}(3\chi^2-\varphi^2-e^{2\phi})-\beta(\psi+2\varphi\chi) \\
&\quad +\gamma(\varphi\psi+\tfrac{1}{2}\chi(\varphi^2+\chi^2-e^{2\phi}))\,, \\
\delta\,\chi &= b+d\,\chi+\varepsilon\varphi+\alpha(\psi-2\varphi\chi)+\tfrac{\beta}{2}(3\varphi^2-\chi^2-e^{2\phi}) \\
&\quad +\gamma(\chi\psi-\tfrac{1}{2}\varphi(\varphi^2+\chi^2-e^{2\phi}))\,, \\
\delta\,\psi &= a\,\chi-b\,\varphi+c+2d\,\psi-(\alpha\,\varphi+\beta\,\chi)\psi-\tfrac{1}{2}(\alpha\,\chi-\beta\,\varphi)(\chi^2+\varphi^2-e^{2\phi}) \\
&\quad +\gamma(\psi^2-\tfrac{1}{4}(\varphi^2+\chi^2-e^{2\phi})^2)\,.
\end{aligned} \tag{5.77}$$

It is easy to verify that the eight transformations with parameters $(a, b, c, d, \varepsilon, \alpha, \beta, \gamma)$ form the Lie algebra of $SU(2,1)$. Generators with Latin letters correspond to compact generators and Greek letters to non-compact generators. The generators a, b, and c corresponding to shifts in the fields φ, χ, and ψ, respectively form a three-dimensional Heisenberg subalgebra and we have the 5-graded structure

$$\begin{array}{ccccccccc} \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2} \\ c & \oplus & \begin{pmatrix} a \\ b \end{pmatrix} & \oplus & \begin{pmatrix} d \\ \varepsilon \end{pmatrix} & \oplus & \begin{pmatrix} \alpha \\ \beta \end{pmatrix} & \oplus & \gamma \end{array} \tag{5.78}$$

In the following section we will identify these symmetry generators with some of the quasiconformal $E_{8(8)}$-generators.

5.4.2 Relation to Quasiconformal Transformations

In order to match the symmetry transformations (5.77) of the Lagrangian (5.72) with the quasiconformal transformations (5.44) – (5.49) we will bring the latter into a more similar form using explicit transformation parameter. We introduce parameters for the E_7 subalgebra in the $SL(8,\mathbb{R})$ basis, namely

$$\{\Lambda^i{}_j, \Sigma_{ijkl}\},$$

with Λ traceless ($\Lambda^i{}_i = 0$), Σ_{ijkl} fully antisymmetric, and Σ^{ijkl} defined by

$$\Sigma^{ijkl} := \tfrac{1}{24}\epsilon^{ijklmnpq}\Sigma_{mnpq}$$

as well as the real transformation parameters

$$\{\omega_{-2};\ \omega_{-1}^{ij}, \omega_{-1\,ij};\ \omega_0;\ \omega_1^{ij}, \omega_{1\,ij};\ \omega_2\}.$$

The most general infinitesimal quasiconformal transformation with these parameters then is given by the linear combination of $E_{8(8)}$ generators

$$\omega_{-2}E + \langle\omega_{-1}, E\rangle + \Sigma_{ijkl}G^{ijkl} + \Lambda^j{}_i G^i{}_j + \omega_0 H + \langle\omega_1, F\rangle + \omega_2 F \tag{5.79}$$

where we again have used the symplectic form (5.40) $\langle\omega_{-1}, E\rangle = \omega_{-1}^{ij}E_{ij} - \omega_{-1\,ij}E^{ij}$. The transformation of X^{ij}, X_{ij}, and x is given by

$$\begin{aligned} \delta(X^{ij}) &= \omega_{-1}^{ij} + \Lambda^i{}_k X^{kj} - \Lambda^j{}_k X^{ki} + \Sigma^{ijkl}X_{kl} + \omega_0 X^{ij} \\ &\quad +2\,\mathcal{D}_1\frac{\partial \mathcal{I}_4}{\partial X_{ij}} - X^{ij}\langle\omega_1, X\rangle - \omega_1^{ij}x - 2\,\omega_2\frac{\partial \mathcal{I}_4}{\partial X_{ij}} + \omega_2 X^{ij}x \\[1ex] \delta(X_{ij}) &= \omega_{-1\,ij} + \Lambda^k{}_i X_{jk} - \Lambda^k{}_j X_{ik} + \Sigma_{ijkl}X^{kl} + \omega_0 X_{ij} \\ &\quad -2\,\mathcal{D}_1\frac{\partial \mathcal{I}_4}{\partial X^{ij}} - X_{ij}\langle\omega_1, X\rangle - \omega_{1\,ij}x + 2\,\omega_2\frac{\partial \mathcal{I}_4}{\partial X^{ij}} + \omega_2 X_{ij}x \\[1ex] \delta(x) &= \omega_{-2} + \langle\omega_{-1}, X\rangle + 2\,\omega_0 x - 2\,\mathcal{D}_1\mathcal{I}_4 - \langle\omega_1, X\rangle\, x + \omega_2(4\,\mathcal{I}_4 + x^2)\,, \end{aligned} \tag{5.80}$$

with the differential operator $\mathcal{D}$ acting on (X^{ij}, X_{ij}) defined by

$$\mathcal{D}_1 = \omega_1^{ij}\frac{\partial}{\partial X^{ij}} + \omega_{1\,ij}\frac{\partial}{\partial X_{ij}}.$$

To show the invariance of (5.51) $\mathcal{N}_4 \equiv 4\,\mathcal{I}_4(X) - x^2$ we need the formula

$$\frac{\partial \mathcal{I}_4}{\partial X^{ij}}\mathcal{D}_1\frac{\partial \mathcal{I}_4}{\partial X_{ij}} - \frac{\partial \mathcal{I}_4}{\partial X_{ij}}\mathcal{D}_1\frac{\partial \mathcal{I}_4}{\partial X^{ij}} \;=\; \langle \omega_1, X\rangle\, \mathcal{I}_4 \tag{5.81}$$

The transformations (5.80) realize the Lie algebra of $E_{8(8)}$ on the space $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ corresponding to a maximal Heisenberg subalgebra, whereas the representation space of the transformations (5.77) contains an extra dilaton field φ on which the dilatations act as constant shifts. It turns out, that we can extend the transformations (5.80) to act also on the one-dimensional space $\mathcal{D}$ with generator z. The transformation of z can be defined by

$$\delta(z) \;=\; \omega_0 - \langle \omega_1, X\rangle + \omega_2\, z \tag{5.82}$$

The remaining transformations (5.80) have to be modified in order to close under commutation. We redefine the transformations to keep the form (5.80) but with a modified $\mathcal{I}$

$$\mathcal{I}_4' \;:=\; -(\sqrt{-\mathcal{I}_4} - \tfrac{1}{4}e^{2z})^2\,. \tag{5.83}$$

In fact, after this modifications the transformation (5.80) obey the same commutation relations as before and the formula (5.81) is still valid.

Note that here is a subtlety: Because of the square root in the definition (5.83) the transformations are in general not well-defined on the real representation space $\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\mathcal{D}$ but only on its complexification. But in the truncation considered in this section $\mathcal{I}_4$ is always negative and this problem does not occur.

The linear combination $\mathcal{N}_4' = 4\,\mathcal{I}_4' - x^2$ is again conformally invariant under all transformations of grade 0, 1, and 2. However, the formula (5.50) is not invariant under the "translations" of grade -1 and -2 and thus does not define a correct distance function of the new space $\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\mathcal{D}$ any more.

We will now truncate the transformations (5.80), (5.82) with 248 generators to a eight-dimensional $SU(2,1)$ subalgebra and identify a four-dimensional subspace of $\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\mathcal{D}$ corresponding to the physical fields of the Lagrangian (5.72). Let us start with the representation space. In the $SU(8)$ basis (5.58) we choose a basis vector Z_{AB} of the form (5.63) with $z_1 = \frac{1}{2}(\varphi + \mathrm{i}\chi)$ and all other components vanishing. Identifying x with ψ and z with φ we have the correspondence

$$\begin{aligned} X^{ij} &= \tfrac{1}{2\sqrt{2}}\,\varphi\,\Gamma^{ij}_{12}\,, & x &= \psi\,,\\ X_{ij} &= -\tfrac{1}{2\sqrt{2}}\,\chi\,\Gamma^{ij}_{12}\,, & z &= \phi\,. \end{aligned} \tag{5.84}$$

Using formula (5.64) we find

$$\mathcal{I}_4^{\mathrm{SL}(8,\mathbb{R})} \;=\; -\mathcal{I}_4^{\mathrm{SU}(8)} \;=\; -|z_1|^4 \;=\; -\tfrac{1}{16}(\varphi^2+\chi^2)^2\,, \tag{5.85}$$

that is always positive and (5.83) defines the real function

$$\mathcal{I}_4' \;=\; -\tfrac{1}{16}(\varphi^2+\chi^2-\tfrac{1}{4}e^{2z})^2\,. \tag{5.86}$$

We insert this equation into the transformations (5.80) and (5.82) and make the following choice with seven degrees of freedom for all the parameters ω:

$$\begin{aligned} \omega_{-2} &:= c\,, & \omega_{2} &:= \gamma\,, & \omega_0 &:= d \\ \omega_{-1}^{ij} &:= \tfrac{1}{2\sqrt{2}}\,a\,\Gamma_{12}^{ij}\,, & \omega_{1}^{ij} &:= \tfrac{1}{2\sqrt{2}}\,\beta\,\Gamma_{12}^{ij}\,, \\ \omega_{-1\,ij} &:= -\tfrac{1}{2\sqrt{2}}\,b\,\Gamma_{12}^{ij}\,, & \omega_{1\,ij} &:= -\tfrac{1}{2\sqrt{2}}\,\alpha\,\Gamma_{12}^{ij}\,, \end{aligned} \tag{5.87}$$

From the $E_{7(7)}$ part we choose the generator from its maximal compact subgroup $SU(8)$ that transforms the vector Z_{AB} defined in (5.63) by complex rotation $\delta z_i = \pm \mathrm{i}\varepsilon z_i$ with plus sign for $i=1,2$ and minus sign for $i=3,4$:

$$\Sigma_{ijkl} \;:=\; -\tfrac{1}{16}\varepsilon(\Gamma_{11}^{ijkl}+\Gamma_{22}^{ijkl}+\Gamma_{33}^{ijkl}+\Gamma_{44}^{ijkl}-\Gamma_{55}^{ijkl}-\Gamma_{66}^{ijkl}-\Gamma_{77}^{ijkl}-\Gamma_{88}^{ijkl}) \tag{5.88}$$

(see e.g. appendix of [28]). With this choice of parameters we recover exactly the symmetry transformations (5.77) of the Lagrangian (5.72).

5.4.3 $d\!=\!5$, $N\!=\!2$ supergravity reduced to $d\!=\!3$

The bosonic Lagrangian of simple $d\!=\!5$ supergravity [22] is given by[4]

$$\mathcal{L} \;=\; ER - \tfrac{1}{4}\,EF_{(2)}^2 - \tfrac{1}{3\sqrt{3}} * (F_{(2)}\wedge F_{(2)}\wedge A_{(1)})\,. \tag{5.89}$$

It is similar to the Lagrangian of eleven-dimensional supergravity (2.5) and mainly differs in the number of indices of the fields: We have an one-form field $A_{(1)}$ with field strength two-form $F_{(2)} = dA_{(1)}$. Because of its very similar group theoretical structure this model has been studied [22, 136, 137] as a toy model for $d\!=\!11$ supergravity.

After reduction to $d\!=\!3$ and dualization of all vector fields, the Lagrangian can be cast into form [136, 29]

$$\mathcal{L} \;=\; ER + \tfrac{1}{32}E\,\mathrm{tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})\,, \tag{5.90}$$

where the matrix $\mathcal{M} := \mathcal{V}\mathcal{V}^{\#}$ parameterizes a $G_{2(2)}/SO(4)$ sigma model. We will now discuss, how the rigid symmetries acting on the scalar fields in $d=3$ form $G_{2(2)}$. In particular, we want to associate the scalar field to the corresponding shift generators in the five-graded decomposition (5.68) of $G_{2(2)}$. For this purpose, we picture the generators of $G_{2(2)}$ in the following graphic: The numbered circles correspond to non-compact generators and the filled circles correspond to compact generators. As in the $SU(2,1)$ case, we can assign each physical scalar field in $d\!=\!3$ to a non-compact generator.

We will name the fields according to (2.20). In four dimensions we have a dilaton φ_4 and a vector $B_{(1)}^4$ coming from the metric and an axion $A_{(0)4}$ and another vector $A_{(1)}$ coming from the gauge field $A_{(1)}$ in five dimensions. The two scalar fields φ^4 and $A_{(0)4}$ transform under $SL(2,\mathbb{R})$ and we can identify them with the generators 1 and 2 in figure 5.4.3.

In $d=3$ we get from the metric another dilaton ϕ_3 which corresponds to the dilaton generator 3 and another vector $B_{(1)}^3$ which after dualization corresponds to the generator 8.

[4]The volume form Lagrangian is $\mathcal{L} = E * R - \frac{1}{2}E * F_{(2)}\wedge F_{(2)} + \frac{1}{3\sqrt{3}}F_{(2)}\wedge F_{(2)}\wedge A_{(1)}$ [29].

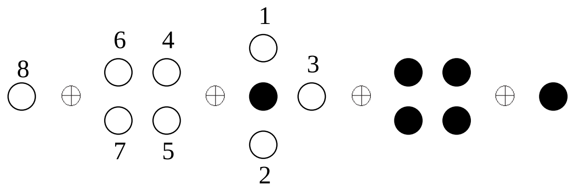

Figure 5.1: The 5-graded decomposition of $G_{2(2)}$

This assignment is obvious since the generators 3 and 8 together with the left-most compact generator in figure 5.4.3 again form an Ehlers-$SL(2,\mathbb{R})$.

The vector gauge field $A_{(1)}$ produces one "electric" scalar field $A_{(0)3}$ and one "magnetic" field by dualization of $A_{(1)}$. We assign the generators 4 and 5 to these fields. Together with the generators 3 and 8 and the corresponding compact generators we find the $SU(2,1)$ of the previous section. The vector $B^4_{(1)}$ also produces one "electric" scalar field $B^3_{(1)4}$ and a "magnetic" one by dualization of $B^4_{(1)}$. These correspond to the generators 6 and 7.

Consider only the fields originating from the metric, not that originating from the gauge field $A_{(1)}$. The corresponding generators 1, 3, 6, 7, 8 together with their compact counterparts form an $SL(3,\mathbb{R})$ in the decomposition (5.69). This $SL(3,\mathbb{R})$ has been found in [127].

We will now summarize how the U-duality group in $d=3$ emerges in its five-graded decomposition (5.15). The grade 0 subspace corresponds to the coset space of all scalars already present in four dimensions together with the dilaton. The grade 1 space corresponds to the vectors fields in four dimensions. Each vector in $d=4$ gives rise to one electric and one magnetic scalar field in three dimensions which together form a linear representation of the $d=4$ global symmetry group. All these scalars together with an additional axion corresponding to the grade 2 space form a Heisenberg algebra.

Already in this comparatively simple $G_{2(2)}$ model we can see how the three-dimensional U-duality group is generated by three different types of symmetry transformations:

- The $d{=}4$ "electric-magnetic" duality transformation in the grade 0 space.
- The Ehlers-symmetry corresponding to dilatations and grade ± 2 space.
- The grade 1 symmetry transformations corresponding to global shifts of electric and magnetic potentials.

A very interesting point here is the fact that global shifts of electric and magnetic potential do not commute but generate a shift in the NUT potential. This has implications for U-duality invariant quantities as the entropy of BPS black holes (cf. discussion in chapter 6).

5.5 Oscillator-like Representation of $E_{8(8)}$

We introduce 29 oscillators X^{ij} $(i,j=1,\ldots,8)$, z and the corresponding momenta $P_{ij} := -\frac{\partial}{\partial X^{ij}}$ and $p := -\frac{\partial}{\partial z}$ with commutator relations

$$[X^{ij}, P_{kl}] \;=\; \delta^{ij}_{kl}\,, \qquad [z,p] \;=\; 1\,.$$

By the following definitions we can realize the $E_{7(7)}$ Lie algebra:

$$\begin{aligned} G^i{}_j &:= 2\,X^{ik}P_{kj} + \tfrac{1}{4}X^{kl}P_{kl}\,\delta^i_j\,, \\ G^{ijkl} &:= -\tfrac{1}{2}X^{[ij}X^{kl]} + \tfrac{1}{48}\epsilon_{ijklmnpq}P_{mn}P_{pq}\,. \end{aligned}$$

We find the commutator relations

$$\begin{aligned} [G^i{}_j\,,\,G^k{}_l] &= \delta^k_j\,G^i{}_l - \delta^i_l\,G^k{}_j\,, \\ [G^i{}_j\,,\,G^{klmn}] &= -4\,\delta^{[k}_j\,G^{lmn]i} - \tfrac{1}{2}\,\delta^i_j G^{klmn}\,, \\ [G^{ijkl}\,,\,G^{mnpq}] &= \tfrac{1}{36}\,\epsilon^{ijkls[mnp}\,G^{q]}{}_s\,. \end{aligned}$$

We can extend this representation to an oscillator realization of $E_{8(8)}$ defining the following generators:

$$\begin{aligned} E &:= \tfrac{1}{2}z^2\,, \\ E^{ij} &:= z\,X^{ij}\,, \\ E_{ij} &:= z\,P_{ij}\,, \\ H &:= z\,p + c\,, \\ F^{ij} &:= -4\,z^{-1}X^{ik}P_{kl}X^{lj} - z^{-1}X^{ij}X^{kl}P_{kl} + \tfrac{1}{12}z^{-1}\epsilon^{ijklmnpq}P_{kl}P_{mn}P_{pq} \\ &\quad\; -X^{ij}p + (14-c)\,z^{-1}X^{ij}\,, \\ F_{ij} &:= 4\,z^{-1}P_{ik}X^{kl}P_{lj} + z^{-1}P_{ij}X^{kl}P_{kl} - \tfrac{1}{12}z^{-1}\epsilon_{ijklmnpq}X^{kl}X^{mn}X^{pq} \\ &\quad\; -P_{ij}p - (14+c)\,z^{-1}P_{ij}\,, \\ F &:= \tfrac{1}{2}p^2 + (c+\tfrac{1}{2})z^{-1}p + 2\,z^{-2}I_4(X,P)\,, \end{aligned}$$

with

$$\begin{aligned} I_4(X,P) &:= -X^{ij}P_{jk}X^{kl}P_{li} + \tfrac{1}{4}X^{ij}P_{ij}X^{kl}P_{kl} - \tfrac{7}{2}\,X^{ij}P_{ij} + (\tfrac{c^2}{4} + \tfrac{c}{2} + 15) \\ &\quad\; -\tfrac{1}{96}\,\epsilon^{ijklmnpq}P_{ij}P_{kl}P_{mn}P_{pq} - \tfrac{1}{96}\,\epsilon_{ijklmnpq}X^{ij}X^{kl}X^{mn}X^{pq}\,. \end{aligned}$$

The parameter c has not been fixed yet. It is straightforward to check the remaining commutations relations (5.45) – (5.49).

Chapter 6

Conclusion and Outlook

A short summary of the main results obtained in this thesis is as follows:

- In chapter 2 we reviewed the toroidal reductions of $d = 11$ supergravity to dimensions $d = 10, \ldots, 2$, paying special attention to those aspects important for the results obtained in the subsequent chapters: The derivation of the hidden symmetries listed in table 2.1 and table 2.2 from the bosonic part of the reduced Lagrangian (2.24) by maximal dualization of the fields, and the proof that the scalar part of the Lagrangian takes the form of a non-linear sigma model (2.36).

- We gave formulas for the decomposition of $E_{8(8)}$ with respect to $SL(8, \mathbb{R})$ and the projectors onto the irreducible representations of $E_{8(8)}$ contained in the tensor product $\mathbf{248} \otimes \mathbf{248}$. The latter are necessary for the calculation of the R matrix of the Yangian $Y(\mathrm{e}_8)$ and also proved useful for the determination of possible gauge groups of $d = 3$ maximal supergravity [147, 146].

- The alternative $SO(1, 2) \times SO(16)$ formulation [140] of $d = 11$ was discussed. The main ingredient was a generalized vielbein (3.40) subject to a number of algebraic constraints (3.50) – (3.56). We solved these constraints by identifying the generalized vielbein with a submatrix of a new $E_{8(8)}$-valued matrix (3.65) incorporating also tensor degrees of freedom coming from the 3-form field A_{MNP}. The supersymmetry variations on the new 248-bein were given (3.80) and a new group action (3.77) of $E_{8(8)}$ in $d = 11$ combining general coordinate transformations with tensor gauge transformations was discussed.

- The R matrix for the $E_{8(8)}$ sigma model coupled to gravity and supergravity was given explicitly in (4.10). We quantized the symplectic structure for $d = 2$ supergravity resulting in a twisted Yangian double (4.48), (4.49) with central extension. The central charge c was fixed by the classical limit (4.60).

- We illustrated the concept of non-linear conformal realizations of Lie algebras admitting a three-graded structure (5.2), using the example of the conformal group of Minkowski space and worked out the corresponding realization of $E_{7(7)}$. A novel, more general quasiconformal realization of Lie algebras admitting a five graded structure (5.15) was presented and applied to $E_{8(8)}$. The relation to Freudenthal triple systems was clarified. Quasiconformal representations of all the other exceptional Lie algebras were obtained by truncation, (5.65) – (5.68). We discussed their relevance in the context of U-duality and illustrated the quasiconformal action using the examples of $d = 4$ Einstein-Maxwell supergravity and simple $d = 5$, $N = 2$ supergravity reduced to three dimensions. A new realization of $E_{8(8)}$ in terms of 28 bosonic oscillators and the corresponding momenta was also given.

In the following we point out open questions and discuss possible applications of the structures investigated in the present work.

Towards the Symmetries of M-Theory

As long as M-Theory has not been rigorously formulated, we are forced to investigate those of its regimes that are accessible to us in the form of approximations. One of those is $d=11$ supergravity which is believed to be the low energy effective description of M-Theory. The exploration of the symmetries of supergravity is guided by the idea that at least part of them should be present in the full quantum theory.

One way to proceed is to incorporate as much symmetry as possible into $d=11$ supergravity, the low energy effective action of M-Theory. This approach was initiated by de Wit and Nicolai [39, 40, 41, 140] by reformulating $d=11$ supergravity in formulations explicitly invariant under $SO(1,3)\otimes SU(8)$ and $SO(1,2)\otimes SO(16)$ and later continued in [133] and in [120], the latter being one of the publications resulting from the thesis work presented here. The question of how to treat the remaining gauge freedom (3.34) and the field components $B_{\mu\nu m}$, $B_{\mu\nu\rho}$ (3.31) in a completely E_8-covariant manner could not be settled so far. It has been argued in [38] that the solution to this problem might require an extension of ordinary $d=11$ supergravity. Maybe a solution can be found more easily in the five-dimensional toy model [22, 136] where the question would be, whether or not one can find a $G_{2(2)}$ symmetry in five dimensions.

It would also be very interesting to perform an analogous analysis for the two-dimensional theory. Here, the global symmetries become the infinite-dimensional affine extension E_9 of E_8. The algebra E_9 acts on an infinite set of non-local conserved charges. However, so far it is not clear which part of these symmetries can be incorporated into the full $d=11$ theory.

Also further investigation of the two-dimensional theory, where quantization of the symmetry structure was achieved, would be of great interest. The main open problem which remains is the apparent incompatibility of the local supersymmetry constraints with the $Y(\mathrm{e}_8)$ charge algebra at the quantum level. So far, we have concentrated on the direct quantization of the algebra of nonlocal charges which are classically invariant under supersymmetry, i.e. Poisson commute weakly with the supersymmetry generators. A complete treatment should in addition contain a quantum version of the supersymmetry constraint algebra (a $N=16$ superconformal algebra) which could serve to define the physical states as its kernel. The Yangian structure presented in this thesis would then become a spectrum generating algebra for $N = 16$ supergravity. Let us emphasize, however, that the interplay between canonical constraints and non-local conserved charges in integrable field theories has so far not been studied at all at the quantum level, as the existing literature deals exclusively with flat space models rather than the generally covariant and locally supersymmetric models we are concerned with here.

Despite much progress, to get a better understanding of the symmetries of the full M-Theory it is still a long way to go. These symmetries might involve complicated, but intersting mathematical structures like hyperbolic Kac-Moody algebras — such as E_{10}— as proposed in [100, 140, 142], superalgebra extensions as proposed in [103, 31, 98, 165], quantum group deformations as proposed in [121, 119], or even more exotic algebras like loop extensions of infinite-dimensional Lie algebras such as $\hat{E}_9$ [44] which is not even of Kac-Moody Type.

Generating classical solutions

It is an old idea to use the global symmetry group of (super)gravity to generate new solutions by acting on known ones [47, 87, 61, 62, 115]. In recent years, the U-duality symmetries of supergravity theories have also been used to classify BPS black hole solutions in $d=5$ and $d=4$ according to their U-duality orbits [50, 54] and to generate the whole BPS spectrum by U-duality action on a generating solution [36, 35, 11]. Since our realization of $E_{8(8)}$ corresponds to further reduction by one dimension, it should be useful for a classification and generation of solutions in $d=3$.

Unitary irreducible representations of non-compact groups

The study of unitary irreducible representations (UIR) of non-compact groups is a very interesting topic in its own right. The general theory of UIRs of non-compact groups in a Fock space was given in [81, 80]. However, that analysis was restricted to Lie algebras admitting a three-graded Jordan structure. With the oscillator realization of section 5.5 it is now possible to study the UIRs of $E_{8(8)}$, and of all the groups contained therein as truncations along the same lines. Also, the (quasi)-superconformal extension might be of interest and can be studied in this framework [7, 12]. Some algebraic properties of these so-called minimal representations have been studied recently in the mathematical literature [112, 18, 17, 19].

Another interesting project would be the construction of unitary lowest weight representations (ULWR) on covariant quantum fields over generalized space-times defined by Jordan algebras along the lines of Ref. [77]. This generalizes the well-known construction of the four-dimensional conformal group $SU(2,2)$ [125, 124, 79]. Note that E_8 does not admit ULWRs, since a non-compact simple group only admits ULWRs if its quotient with respect to its maximal compact subgroup is an Hermitean symmetric space [85, 86]. However, one could repeat the conformal realization of E_7 of section 5.1.3 with the real form $E_{7(-25)}$ which has $E_6 \times U(1)$ as maximal compact subgroup. The corresponding ULWRs so far are not known and would be very interesting to construct.

Geometry of charge space

An interesting topic that deserves further elaboration is the geometry of the 57-dimensional space of generalized coordinates with the difference defined by (5.50) and the cones bounded by the "light-like" hyper planes (5.52). The classification of self-dual convex cones defined by a condition

$$\mathcal{N}(x) \; > \; 0 \tag{6.1}$$

is analogous to the classification of Jordan algebras with norm form $\mathcal{N}$. As shown in Ref. [60], the geometry of the cone in charge space has implications on the stability and the entropy of solutions carrying these charges. Since the 57-dimensional representation space is not a Jordan algebra, the analysis would have to be generalized to cover this case.

Higher dimensional origin of symmetries

The BPS condition for static, asymptotically flat black holes in $N=8$ supergravity coincides with the mass bound demanded by cosmic censorship [106]. However, for the $N=4$

truncation this is not always true. The extreme (in the sense of naked singularities) charged dilaton black hole embedded in $N=4$ supergravity has been found to break all supersymmetries [64, 59]. Shortly afterwards, it has been realized that a different embedding in the $N=4$ theory preserves at least $N=1$ supersymmetry.

In [114] this seemingly paradoxical situation of a different number of conserved supersymmetries for one and the same solution has been clarified. There, it has been shown that the different embeddings in the $N=4$ theory can be further embedded in the $N=8$ theory. They have the same number of preserved $N=8$ supersymmetries and are related by a $N=8$ U-duality transformation. Since the U-duality number of unbroken supersymmetries in the $N=4$ supergravity is not conserved.

However, there are solutions that do not fit into this scheme. The extreme dyonic Reissner-Nordström black hole has been shown to break all supersymmetries after embedding in both $N=4$ and $N=8$ supergravity [113]. Therefore any symmetry relating this solution to its pure electric and magnetic counterparts must be larger than $N=8$ U-duality. It has been speculated that this larger symmetry be related to a supergravity theory in higher dimension than $d=11$.

BPS states in three dimensions

It would be very interesting to find the correct BPS condition in the three-dimensional case. As we discussed in chapter 5, the quasiconformal realization of the U-duality group $E_{8(8)}$ suggests that the BPS bound in $d=3$ involves the Taub-NUT charge. This conjecture is supported by the analysis of central charges of the supersymmetry algebra [92, 161]. The central charge K (1.6) of the $d=5$ supersymmetry algebra is related to the NUT charge N carried by Kaluza-Klein monopole space-times [157, 72] which are not asymptotically Minkowski (AF) but only asymptotically locally flat (ALF).

Another indication comes from the mass bound demanded by cosmic censorship. It coincides with the BPS bound for static, asymptotically flat black holes in $N=8$ supergravity [106]. In the case of solutions including NUT charge this is generally not true. Therefore, a modification of the Bogomolny bound by a term quadratic in the NUT charge was proposed in [104]. By investigation of the central charges of the supersymmetry algebra one finds the same relation [92].

Entropy in three dimensions

It would be extremely interesting to discover an U-duality invariant expression for the entropy in $d=3$ where all physical degrees of freedom are given by scalars. The entropy formula is believed to involve the unique octic invariant $\mathcal{I}_8$ of E_8, which up to now is not known.

Also the entropy has been shown to get extra contributions originating from boundary terms in the Hamiltonian in the case of ALF spaces [94, 89]. For Taub-NUT spacetime the additional entropy is proportional to N^2. In the case of Taub-NUT-AdS which is asymptotically locally anti-de Sitter, attempts were made to relate the entropy to the partition function of a conformal field theory on the boundary of the space-time [90, 21] establishing an AdS/CFT correspondence [129, 167].

The difficulties in the computation of $\mathcal{I}_8$ is due to the fact that it is defined on the eightfold tensor product of the adjoint representation of E_8 which is of dimension $\mathbf{248}^8 =$

$\mathbf{14, 309, 137, 159, 611, 744, 256}$. With the quasiconformal representation on a 57-dimensional space we have found, a much more manageable representation of E_8 and the calculation of $\mathcal{I}_8$ might now be within reach.

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