Dimensional Regularization in Position Space and a Forest Formula for Regularized Epstein-Glaser Renormalization

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Zusammenfassung

In der vorliegenden Arbeit werden die Methoden von Dimensionaler Regularisierung (DimReg) und Minimaler Subtraktion (MS) konsistent im Minkowsiki-Ortsraum formuliert, und in den Rahmen der perturbativen Algebraischen Quantenfeldtheorie (pAQFT) implementiert. Die entwickelten Kozepte werden benutzt, um die Rekursion von Epstein und Glaser für die Konstruktion zeitgeordneter Produkte in allen Ordnungen der kausalen Störungstheorie zu lösen. Es wird eine geschlossene Lösung in Form einer Waldformel à la Zimmermann angegeben. Eine Verbindung zu dem alternativen Zugang zur Renormierungstheorie über Hopf-Algebren wird hergestellt.

Abstract

The present work contains a consistent formulation of the methods of dimensional regularization (DimReg) and minimal subtraction (MS) in Minkowski position space. The methods are implemented into the framework of perturbative Algebraic Quantum Field Theory (pAQFT). The developed methods are used to solve the Epstein-Glaser recursion for the construction of time-ordered products in all orders of causal perturbation theory. A solution is given in terms of a forest formula in the sense of Zimmermann. A relation to the alternative approach to renormalization theory using Hopf algebras is established.

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Introduction

Yet knowing how way leads on to way,
I doubted if I should ever come back.
Robert Frost: The Road Not Taken

In 1957 Bogoliubov and Parasiuk introduced an inductive method for the solution of the renormalization problem of perturbative Quantum Field Theory, that is the problem of constructing the terms in the perturbative expansion of the scattering matrix ($\boldsymbol{\mathcal{S}}$-matrix) [BP57, BS59]. It was later shown by Hepp "that the subtraction rules of Bogoliubov an Parasiuk lead to well-defined renormalized Green's distributions" [Hep66]. From this common origin the method of Bogoliubov, Parasiuk, and Hepp (BPH) evolved mainly along two different roads. The *BPHZ method* induced by the solution of the BPH recursion in terms of Zimmermann's forest formula in momentum space on the one hand side [Zim69], and *causal perturbation theory* induced by Epstein and Glaser's rigorous solution of the renormalization problem in position space on the other [EG73]. Both methods are rigorous incarnations of BPH, however, they have played quite different roles in the development of perturbative Quantum Field Theory (pQFT).

Causal perturbation theory has proven to be superior to the so-called "standard approach" to renormalization in momentum space when it comes to more conceptual questions of perturbative renormalization, and it is widely accepted as the landmark with which one has to test new approaches to renormalization (see, e.g., [FHS10]). What is more, Epstein-Glaser renormalization is the only renormalization method which has been successfully formulated on more general, physical backgrounds [BF00a]. Induced by the development of Quantum Field Theory on curved spacetimes [Rad96, BFK96, BF00a][1] and along with the successful formulation of the renormalization group in generic, globally hyperbolic spacetimes by Hollands and Wald [HW01, HW02, HW03], Brunetti, Dütsch, and Fredenhagen started a program on the structural analysis of perturbative Quantum Field Theory in the algebraic approach [DF99, DF01b, DF01a, DF03, Hol04, DF04, BF04, DF07, BF07]. One of the main results of this program was the precise formulation and proof of what Popineau and Stora called the *Main Theorem of Perturbative Renormalization* [PS82]. This is the fact that the definition of the $\boldsymbol{\mathcal{S}}$-matrix of pQFT involves a freedom described by the Stückelberg-Petermann renormalization group

[1] See also [BF09] for a selfcontained treatment of the topic.

[SP53]. A milestone of the algebraic approach to perturbative Quantum Field Theory was reached with the formulation of *perturbative Algebraic Quantum Field Theory (pAQFT)*, which has been shown to give a common basis to the different other incarnations of the renormalization group in literature [BDF09].

Despite these deep results in perturbative renormalization, causal perturbation theory has its weak point when it comes to concrete predictions, say, for scattering amplitudes in collision processes of elementary particles. Here the standard approach to pQFT in momentum space and in particular the method of dimensional regularization (DimReg) and minimal subtraction (MS) [BG72a, tHV72] combined with Zimmermann's forest formula has proven to be efficient in its application and to produce predictions which are in astonishing accordance with measurements in accelerator experiments. The proof that DimReg+MS is compatible with the combinatorics described by BPHZ was given in [BM77a, BM77b, BM77c], and in particular the compatibility with gauge theories has contributed to the success of dimensional regularization in favor of other analytic renormalization techniques in elementary particles phenomenology [BRS75]. A seemingly forgotten part of this road is that Zimmermann realized in 1970 and proved in 1975 that the additional subtractions found in his formula in comparison to BPH do not contribute in the limit where the regularization is removed [Zim70, Zim76].[2]

In recent years great interest in the mathematical community for the renormalization method of DimReg+MS combined with BPHZ has been triggered by Kreimer's discovery of a Hopf algebra structure underlying the BPHZ renormalization method [Kre98]. Connes and Kreimer pointed out various relations of this discovery to fields of research in pure mathematics, such as Number Theory and Noncommutative Geometry [CK98, CK00, CK01]. Consequently, by now the field has grown to a research area of considerable extent between the poles of more mathematically oriented research in Algebraic Geometry and Number Theory [BEK06, CM07] and applications in the computation of higher order contributions to the perturbative expansion of the $\mathcal{S}$-matrix [Wei06, BW09, BW10]. Shortly after Kreimer's discovery Gracia-Bondía and Lazzarini observed, and Pinter showed that also Epstein-Glaser renormalization exhibits a Hopf algebra structure of the Connes-Kreimer type [GBL00, Pin00a, Pin00b]. Thus the Hopf algebra structure was observed to be a remnant of the common origin of the two roads in perturbative Quantum Field Theory briefly outlined above. It was shown that the Hopf algebra structure of BPHZ renormalization is invariant under certain partial summations of graphs in the perturbative expansion [BF00b, BF01, Fra07, vS07a]. Furthermore, many results on the occurrence of different Hopf algebras in perturbative QFT have been obtained in recent years and it was found that the Faà di Bruno Hopf algebra plays a distinguished role among them [FGB05].

[2] I want to thank José Gracia-Bondía for directing my attention to these references. See also reference [FHS10] in this respect.

The aim of this thesis is to "combine the good parts" of both roads to renormalization in perturbative Quantum Field Theory. That is, to incorporate the effective methods of dimensional regularization and minimal subtraction combined with Zimmermann's forest formula in the conceptually clear setting of causal perturbation theory. After a brief introduction to the theory of extension of distributions, which is the main tool in modern formulations of causal perturbation theory, in the first chapter, I will describe the setting of perturbative Algebraic Quantum Field Theory in the special case of flat Minkowski spacetime in Chapter II. Following the arguments in an appendix of the pAQFT article (loc. cit.) I will show in Chapter III how one can use a modification of the Bessel parameter in a representation of the Wightman two point function in Minkowski space to construct a dimensionally regularized analytic (Hadamard-) two point function in flat spacetime which depends smoothly on the mass parameter m^2. As shown by Hollands, smooth mass dependence is a suitable requirement for a covariant treatment of renormalization [Hol04]. In Chapter IV I will then construct the dimensionally regularized position space amplitude to any graph Γ in scalar quantum field theory as a distribution in $\mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ ($\mathbb{M}$ denotes Minkowski spacetime and $|V(\Gamma)|$ the number of vertices of Γ). I have to remark here that Bollini and Giambiagi already gave a formulation of dimensional regularization in position space by Fourier transforming the regularized momentum space amplitude to position space and found a modification in the Bessel parameter of the corresponding two point function [BG96]. Conversely, a Fourier transformation of the amplitudes constructed in this work (which are different from the ones found by Bollini and Giambiagi) to momentum space is not possible in general, since the condition of smoothness in m^2 will select a propagator which is not in Schwartz space. As a result of this chapter, I define the position space dimensionally regularized $\boldsymbol{\mathcal{S}}$-matrix, $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$, which fulfills the conditions of the main theorem of perturbative renormalization as proven in [DF04, BDF09]. In Chapter V I will show how minimal subtraction can be applied to the dimensionally regularized position space amplitudes in a graph by graph manner, and will test the method by reproducing the result of Zimmermann that so-called "pure BPHZ subgraphs" do not contribute to the forest formula in the limit where the regularization is removed [Zim76]. The last chapter of my thesis will use $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$ as an example for an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix, but does not depend on the way it was constructed. In this sense, the results of the last chapter are independent of the formulation of dimensional regularization in position space summarized above, and consequently they can be applied in a much wider range. I will show in Chapter VI of the present thesis that a forest formula for regularized Epstein-Glaser renormalization can be derived directly from the main theorem of perturbative renormalization. I will give the formula and prove locality of the MS counterterms. Furthermore I will show that the Hopf algebra structure observed in perturbative renormalization theory can be understood as a direct consequence

of the main theorem. However, in contrast to the Connes-Kreimer theory of renormalization, the Feynman rules will emerge naturally from the construction, and it will be shown that the commutative Hopf algebra of graphs introduced by Connes and Kreimer is not enough for an algebraic construction of the counterterms found in pAQFT. Another difference to the original Hopf algebra approach is that the Hopf algebraic structure found in the construction of pAQFT counterterms will correspond to sums of graphs rather than individual ones, however, the correspondence to the Hopf algebra of graphs is established by linearity; in accordance with the results of [BF00b, BF01, Fra07, vS07a].

In order to be precise and prevent confusion, I want to remark that I mean by "regularization" in this thesis always parametric regularization, i.e., the introduction or modification of a parameter, which makes the extensions of the regularized distributions unique. "Dimensional regularization" is one example. "Renormalization", on the other hand, I want to use as a synonym for the extension of distributions, as it is widely used in the terms "Epstein-Glaser renormalization" or "BPHZ renormalization". Observe, however, that the extension of its time-ordered products is only a necessary but by no means sufficient prerequisite for a quantum field theory to be renormalizable by power counting. That is to say, we are not concerned with the number of counterterms that are to be introduced at each order of perturbation theory, but only with the fact that this number is finite. Neither will we treat the question whether the counterterms can be absorbed in a redefinition of the parameters in a Lagrangian of the theory.

A last remark I want to make is that the extension of the time-ordered products to the total diagonal, which will be treated in some detail below and corresponds to the elimination of ultraviolet (UV) divergences in the standard approach, suffices for the perturbative definition of the quantum field theory under investigation in the algebraic adiabatic limit. This was shown in [BF00a, DF01b, HW03]. The algebraic adiabatic limit is a way to remove the explicit spacetime dependence of the interaction without introducing so-called infrared (IR) divergences. IR divergences typically appear in the standard approach if one removes the cutoff at small momenta (or large distances) in theories with long range interactions. Such divergences appear also in the causal approach of Epstein and Glaser in the strong and in the weak adiabatic limit. Neither strong nor weak adiabatic limit will be treated in this thesis, and as much as the algebraic adiabatic limit is concerned I cannot add anything new to the discussion in [BDF09, Chap. 6].

CHAPTER I

Mathematical Preliminaries: Extension of Distributions

The main tool in renormalization in position space is the extension of distributions, thus we want to summarize here the basic definitions and main results of this part of distribution theory. We will first give the general result on the existence of extensions of distributions with the same scaling degree and will indicate how such extensions are constructed. In the second section we will review the special case of homogeneous distributions; homogeneity being a suitable condition for the existence of a unique extension. We will generalize the uniqueness result on homogeneous extensions to the case of heterogeneous distributions in the third section. The fourth section will be devoted to the definition of an (analytic) regularization of a distribution. We will derive some direct consequences to be used in later chapters. A general reference for this chapter, and a guidance for mathematical questions throughout the thesis is the book of Hörmander [Hör03].

We generally use the notation of Laurent Schwartz for the function spaces, $\mathscr{E}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$ of smooth functions, and $\mathscr{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support (test functions) with their respective standard topologies; and $\mathscr{E}'$, respectively $\mathscr{D}'$ for their dual spaces.

I.1. Extensions and Steinmann Scaling Degree

DEFINITION I.1 (Extension). Let $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ be a distribution defined for all test functions supported in the complement of the origin. We call $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ an *extension* of u, if

$$\forall f \in \mathscr{D}(\mathbb{R}^d \setminus \{0\}) : \quad \dot{u}(f) = u(f) . \tag{I.1}$$

Not every distribution $u \in \mathscr{D}(\mathbb{R}^d \setminus \{0\})$ has an extension, and if there is one it is not unique. However, by (I.1) two extensions of u differ by a distribution supported at the origin. By [Hör03, Thm. 2.3.4] any distribution supported at the origin is a polynomial in the derivatives of Dirac's δ-distribution. We call such distributions *local* and denote the space of all local distributions by $\mathscr{E}'_{\text{Dirac}}$. One way to restrict the freedom in the extension procedure is to require that the extension should have the same scaling degree, cf. [Ste71, BF00a].

DEFINITION I.2 (Steinmann Scaling Degree). Let

$$\begin{array}{rccl} \Lambda: & \mathbb{R}_+ \times \mathscr{D} & \to & \mathscr{D} \\ & (\rho, \phi) & \mapsto & \phi^\rho := \rho^{-d}\phi(\rho^{-1}\cdot) \end{array} \tag{I.2}$$

be the action of the positive reals on test functions in $\mathscr{D} \in \left\{\mathscr{D}(\mathbb{R}^d), \mathscr{D}(\mathbb{R}^d \setminus \{0\})\right\}$. This induces, via the pullback, the action on distributions. For $u \in \mathscr{D}'$ we define

$$\forall \phi \in \mathscr{D}: \quad u_\rho(\phi) := u(\phi^\rho)\,.$$

The *scaling degree* $\mathrm{sd}(u)$ of a distribution u with respect to the origin is defined to be

$$\mathrm{sd}(u) := \inf\left\{\omega \in \mathbb{R} : \lim_{\rho\to 0^+} \rho^\omega u_\rho = 0 \in \mathscr{D}'\right\}\,.$$

EXAMPLE I.3. Dirac's δ-distribution has scaling degree $\mathrm{sd}(\delta) = d$, since

$$\lim_{\rho\to 0^+} \rho^\omega \langle \delta, \phi^\rho\rangle = \lim_{\rho\to 0^+} \rho^{\omega-d}\phi(0)\,.$$

Furthermore, a similar argument shows that any smooth function has scaling degree smaller than or equal to zero. The basic properties of the scaling degree are summarized in the following

LEMMA I.4 (cf. [BF00a, Lem. 5.1]). *Let $u \in \mathscr{D}'(\mathbb{R}^d)$, $v \in \mathscr{D}'(\mathbb{R}^k)$ and let $\alpha \in \mathbb{N}^d$ be a multiindex, then*

(a) $\mathrm{sd}(\partial^\alpha u) \leqslant \mathrm{sd}(u) + |\alpha|$
(b) $\mathrm{sd}(x^\alpha u) \leqslant \mathrm{sd}(u) - |\alpha|$
(c) $\forall f \in \mathscr{E}(\mathbb{R}^d) : \mathrm{sd}(fu) \leqslant \mathrm{sd}(u)$
(d) $\mathrm{sd}(u \otimes v) = \mathrm{sd}(u) + \mathrm{sd}(v)$

For later reference we also define the related concept of degree of divergence of a distribution.

DEFINITION I.5 (Degree of Divergence). Let $u \in \mathscr{D}' \in \left\{\mathscr{D}'(\mathbb{R}^d), \mathscr{D}'(\mathbb{R}^d \setminus \{0\})\right\}$, then we define the degree of divergence of u

$$\mathrm{div}(u) := \mathrm{sd}(u) - d\,.$$

Observe that the scaling degree of a product of distributions $u, v \in \mathscr{D}'(\mathbb{R}^d)$, if it exists, is given by the scaling degree of the tensor product $u \otimes v \in \mathscr{D}'(\mathbb{R}^{2d})$,

$$\mathrm{sd}(uv) = \mathrm{sd}(u \otimes v)\,,$$

whereas the degree of divergence of the product is greater

$$\mathrm{div}(uv) = \mathrm{div}(u \otimes v) + d\,.$$

Although this observation follows directly from Lemma I.4(d) the following theorem shows that it reflects the freedom involved in the definition of the product of distributions. Recall that, if it exists, the pointwise product $uv \in \mathscr{D}'(\mathbb{R}^d)$ is defined

as the pullback of $u \otimes v \in \mathscr{D}'(\mathbb{R}^{2d})$ via the diagonal map

$$\mathrm{diag} : \mathbb{R}^d \ni x \mapsto (x, x) \in \mathbb{R}^{2d} ,$$

cf. [Hör03, Thm. 8.2.10]). One often encounters the situation, that the pullback $uv = \mathrm{diag}^*(u \otimes v)$ defines the product only in the complement of the origin, even if $u, v \in \mathscr{D}'(\mathbb{R}^d)$. This is the case, e.g., in perturbative renormalization theory and hence one is naturally lead to the problem of finding extensions of certain (products of) distributions [BP57]. This lead Epstein and Glaser to their constructive extension procedure by "distribution splitting" [EG73]. The mathematically quite involved inductive procedure carried out by Epstein and Glaser may be called the first rigorous construction of extensions of distributions in position space. It was Steinmann who introduced the concept of scaling degree in the discussion related to the construction of extensions of certain distributions [Ste71]. There are later works contributing to this topic, such as [Est98a], and it is treated by now in several text books [Hör03], however, the most general result known to the author is the theorem to be cited below. It was to my best knowledge first proven in [BF00a, Thms. 5.2 & 5.3].

THEOREM I.6 (Extension of Distributions). *Let $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ have scaling degree $\mathrm{sd}(u)$ with respect to the origin. Let*

- $\mathrm{sd}(u) < d$. *Then there exists a unique extension $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ of u, which has the same scaling degree, $\mathrm{sd}(\dot{u}) = \mathrm{sd}(u)$.*
- $d \leqslant \mathrm{sd}(u) < \infty$. *Then there exist several extensions $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ with $\mathrm{sd}(\dot{u}) = \mathrm{sd}(u)$. They are uniquely defined by their values on a finite set of test functions.*

For completeness we remark that u has no extension, if $\mathrm{sd}(u) = \infty$, the distribution $f \mapsto \int dx\, e^{\frac{1}{x}} f(x)$ is a standard example of this case. Furthermore we remark that the scaling degree of the extension $\dot{u}$ cannot be smaller than that of u. Thus the condition that $\dot{u}$ should have the same scaling degree as u is a condition of minimal scaling degree or "maximal smoothness" at the origin.

SKETCH OF PROOF OF THM. I.6. Let first $\mathrm{sd}(u) < d$. Uniqueness follows immediately from the fact that two extensions $\dot{u}, \ddot{u} \in \mathscr{D}'(\mathbb{R}^d)$ differ by a polynomial $P(\delta)$ in derivatives of Dirac's δ-distribution, which has scaling degree $\mathrm{sd}(P(\delta)) \geqslant d$, cf. Lemma I.4(a).

Let $\vartheta \in \mathscr{E}(\mathbb{R}^d)$, $0 \leqslant \vartheta \leqslant 1$, such that $\vartheta(x) = 0$ for $|x| < 1$ and $\vartheta(x) = 1$ for $|x| \geqslant 2$ and set $\vartheta_\rho(x) = \vartheta(\rho x)$, then

$$\dot{u} := \lim_{\rho \to \infty} \vartheta_\rho u \tag{I.3}$$

converges in $\mathscr{D}'(\mathbb{R}^d)$, i.e.,

$$\forall g \in \mathscr{D}(\mathbb{R}^d) : \quad \lim_{\rho \to \infty} \langle u, \vartheta_\rho g \rangle \in \mathbb{C} ,$$

and $\dot{u}$ defines an extension of u with the same scaling degree, cf. [BF00a].

Now regard the case $d \leqslant \mathrm{sd}(u) < \infty$. We define the space $\mathscr{D}_\lambda(\mathbb{R}^d)$ of functions, which vanish up to order $\lambda > 0$ at the origin,

$$\mathscr{D}_\lambda(\mathbb{R}^d) = \left\{ f \in \mathscr{D}(\mathbb{R}^d) | \quad \forall\, |\alpha| \leqslant \lambda : \, (\partial^\alpha f)\,(0) = 0 \right\}. \tag{I.4}$$

Then u is uniquely defined on functions, which vanish up to the order given by the degree of divergence of u, i.e., u has a unique extension $\tilde{u} \in \mathscr{D}'_\lambda(\mathbb{R}^d)$, $\lambda = \mathrm{div}(u)$, with the same scaling degree. Any function $f \in \mathscr{D}_\lambda(\mathbb{R}^d)$ can be written in the form

$$f(x) = \sum_{|\alpha| = \lfloor \lambda \rfloor + 1} x^\alpha g_\alpha(x)\,, \quad g_\alpha \in \mathscr{D}(\mathbb{R}^d)\,, \tag{I.5}$$

where $\lfloor \lambda \rfloor$ denotes Gauß's floor function, i.e., the largest integer smaller or equal λ. We define $\tilde{u}$ by

$$\langle \tilde{u}, f \rangle := \sum_{|\alpha| = \lfloor \lambda \rfloor + 1} \left\langle (x^\alpha u)^{\cdot}, g_\alpha \right\rangle\,,$$

where the extensions $(x^\alpha u)^{\cdot}$ on the right hand side are unique by Lemma I.4(b) and the first part of the theorem. They can be computed as weak limits of the form (I.3), and thus the limit exist for each term separately. We have

$$\sum_{|\alpha| = \lfloor \lambda \rfloor + 1} \lim_{\rho \to \infty} \left\langle x^\alpha u, \vartheta_\rho g_\alpha \right\rangle = \lim_{\rho \to \infty} \left\langle u, \vartheta_\rho f \right\rangle\,,$$

which shows that the extension $\tilde{u}$ does not depend on the chosen representation (I.5) of $f \in \mathscr{D}_\lambda(\mathbb{R}^d)$, cf. [DF04].

Regard now the ambiguity left in the extension to $\mathscr{D} = \mathscr{D}(\mathbb{R}^d)$. $\mathscr{D}_\lambda \subset \mathscr{D}$ is a closed subspace, hence there are projections $W : \mathscr{D} \to \mathscr{D}_\lambda$, one for each choice of the complement $\mathscr{C}$,

$$\mathscr{D} = \mathscr{D}_\lambda \oplus \mathscr{C}\,.$$

where $\mathscr{D}_\lambda \equiv \mathrm{ran}(W)$ and $\mathscr{C} \equiv \mathrm{ran}(1 - W)$. This split of $\mathscr{D}$ induces a split of the dual space $\mathscr{D}'$ according to the following diagram, cf. [DS67, Lem. VI.3.3],

$$\begin{tikzcd} \mathscr{C} & \mathscr{D} \arrow[l, "1-W"'] \arrow[r, "W"] \arrow[d, dashed, "\text{duality}"] & \mathscr{D}_\lambda \\ \mathscr{D}_\lambda^\perp & \mathscr{D}' \arrow[l, "1-W'"'] \arrow[r, "W'"] & \mathscr{C}^\perp \end{tikzcd} \qquad \mathscr{D}' = \mathscr{D}_\lambda^\perp \oplus \mathscr{C}^\perp \tag{I.6}$$

where

$$\mathscr{D}_\lambda^\perp := \left\{ u \in \mathscr{D}' | \quad \forall f \in \mathscr{D}_\lambda : \langle u, f \rangle = 0 \right\},$$

and

$$\mathscr{C}^\perp := \left\{ v \in \mathscr{D}' | \quad \forall g \in \mathscr{C} : \langle v, g \rangle = 0 \right\}.$$

The dual projections are induced by

$$\forall u \in \mathscr{D}',\, \forall f \in \mathscr{D} : \quad \left\langle W' u, f \right\rangle := \langle u, W f \rangle\,.$$

Any dual basis $\left\{ w_\alpha \in \mathscr{D} : \left\langle \delta^{(\beta)}, w_\alpha \right\rangle = \delta_\alpha^\beta \right\}$ of the basis $\left\{ \delta^{(\alpha)} : |\alpha| \leqslant \lambda \right\}$ of $\mathscr{D}_\lambda^\perp$ spans a complement $\mathscr{C} = \mathscr{D} \ominus \mathscr{D}_\lambda$ and thus defines a projection W. As a consequence we have the following characterization of projections $W : \mathscr{D} \to \mathscr{D}_\lambda$.

LEMMA I.7 (cf. [DF04, Lem. B.1]). *There is a one-to-one correspondence between families of functions*

$$\left\{ w_\alpha \in \mathscr{D} \mid \quad \forall\, |\beta| \leqslant \lambda : \left(\partial^\beta w_\alpha\right)(0) = \delta_\alpha^\beta,\ |\alpha| \leqslant \lambda \right\} \tag{I.7}$$

and projections $W : \mathscr{D} \to \mathscr{D}_\lambda$. The set (I.7) defines a projection W by

$$Wf := f - \sum_{|\alpha| \leqslant \lambda} f^{(\alpha)}(0)\, w_\alpha\,.$$

Conversely a set of functions of the form (I.7) is given by any basis of $\mathscr{C} \equiv \operatorname{ran}(1-W)$ dual to the basis $\left\{ \delta^{(\alpha)} : |\alpha| \leqslant \lambda \right\}$ of $\mathscr{D}_\lambda^\perp \subset \mathscr{D}'$.

We reach the conclusion that for each projection $W : \mathscr{D} \to \mathscr{D}_\lambda$ there is a unique extension $W'\tilde{u} \in \mathscr{C}^\perp$ of $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$. The most general extension $\dot{u}$ of u, fulfilling the assumptions of the theorem, can be read off from the split of $\mathscr{D}'$ (I.6),

$$\dot{u} = W'\tilde{u} + \sum_{|\alpha| \leqslant \operatorname{div}(u)} C_\alpha \delta^{(\alpha)}\,, \tag{I.8}$$

where $C_\alpha \in \mathbb{C}$ are free constants. Observe, however, that a particular extension is fixed by a choice of its values on $\mathscr{C}$, namely $\langle \dot{u}, w_\alpha \rangle = C_\alpha$. □

Although (I.8) gives the most general extension of u with the same scaling degree, it is important to note that the second term in (I.8) does not introduce an additional freedom, but only reflects the freedom in the choice of the projection W.

LEMMA I.8 (cf. [DF04, Lem. B.2]). *Let $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ be an extension of $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ with $\operatorname{div}(\dot{u}) = \operatorname{div}(u) = \lambda$. Then there exists a complementary space $\mathscr{C}$ of $\mathscr{D}_\lambda$ in $\mathscr{D}$ such that*

$$\dot{u}\big|_{\mathscr{C}} = 0\,,$$

i.e., $C_\alpha = 0$ in (I.8).

That is, any extension $\dot{u}$ of u can be written as a *W-extension*, $\dot{u} = W'\tilde{u}$ with a suitably chosen projection W.

Despite its wide applicability ($\operatorname{sd}(u)$ is defined for any $u \in \mathscr{D}'$), the scaling degree is often too rough a tool for describing the behavior of distributions at the origin. A refinement of the notion of scaling degree is the degree of homogeneity defined only for homogeneous distributions. We will see in the next section that this refinement leads to a stronger result regarding the uniqueness of extensions.

I.2. Homogeneous Distributions

DEFINITION I.9 (Homogeneous Distribution, cf. [Hör03, Def. 3.2.2]). A distribution $u \in \mathscr{D}' \in \left\{ \mathscr{D}'(\mathbb{R}^d), \mathscr{D}'(\mathbb{R}^d \setminus \{0\}) \right\}$ is called *homogeneous* of degree $D \in \mathbb{C}$, if

$$\forall \rho > 0 : \quad \langle u, \phi \rangle = \rho^{-D} \left\langle u_\rho, \phi \right\rangle \qquad \forall \phi \in \mathscr{D}\,, \tag{I.9}$$

where $\left\langle u_\rho, \phi \right\rangle := \langle u, \phi^\rho \rangle$ with $\phi^\rho(x) := \rho^{-d}\phi(\rho^{-1}x)$ as in (I.2). We will sometimes write $D = \operatorname{homog}(u)$ for the homogeneity degree of a distribution u.

REMARK I.10. Observe that a distribution u, homogeneous of degree $D \in \mathbb{C}$, has scaling degree $\mathrm{sd}(u) = -\mathrm{Re}(D)$,

$$\mathrm{sd}(u) = \inf\left\{\omega \in \mathbb{R} : \lim_{\rho\to 0^+} \rho^{\omega} \langle u, \phi^{\rho}\rangle = \lim_{\rho\to 0^+} \rho^{\omega+\mathrm{Re}(D)}\rho^{i\,\mathrm{Im}(D)} = 0\right\} = -\mathrm{Re}(D)\,.$$

In this sense the homogeneity degree "homog" is a refinement of the scaling degree "sd", and Consequently we will get a stronger statement for the extendability of homogeneous distributions in Theorem I.12 below. However, before citing this result, let us regard an alternative characterization of homogeneity.

THEOREM I.11 (Euler). *A distribution $u \in \mathscr{D}' \in \left\{\mathscr{D}'(\mathbb{R}^d), \mathscr{D}'(\mathbb{R}^d\setminus\{0\})\right\}$ is homogeneous of degree $D \in \mathbb{C}$ if and only if*

$$\langle (x\cdot\partial_x - D)\, u, \phi\rangle = 0 \qquad \forall \phi \in \mathscr{D}\,, \tag{I.10}$$

where $x\cdot\partial_x = \sum_{i=1}^d x^i \frac{\partial}{\partial x_i}$ denotes the radial vector field or "Euler operator".

PROOF. By (I.9) we infer that

$$\left(\rho\partial_\rho\right)\rho^{-D}\left\langle u_\rho, \phi\right\rangle = \left(\rho\partial_\rho\right)\langle u, \phi\rangle = 0\,.$$

Computing the derivative gives:

$$\begin{aligned}\rho\partial_\rho\left(\rho^{-D}\left\langle u_\rho, \phi\right\rangle\right) &= -D\rho^{-D}\left\langle u_\rho, \phi\right\rangle + \rho^{-D}\left\langle (x\cdot\partial_x)\, u_\rho, \phi\right\rangle \\ &= \rho^{-D}\left\langle (x\cdot\partial_x - D)\, u_\rho, \phi\right\rangle\,,\end{aligned}$$

hence, evaluating at $\rho = 1$ gives (I.10). Conversely let $\varphi(\rho) := \langle u, \phi^\rho\rangle$, then

$$\begin{aligned}\rho\varphi'(\rho) &= \rho\partial_\rho\left\langle u(x), \rho^{-d}\phi(\rho^{-1}x)\right\rangle \\ &= -d\,\langle u, \phi^\rho\rangle - \langle u(x), (x\cdot\partial_x)\,\phi^\rho(x)\rangle \\ &= -d\,\langle u, \phi^\rho\rangle + d\,\langle u, \phi^\rho\rangle + \langle (x\cdot\partial_x)\, u(x), \phi^\rho(x)\rangle \\ &= D\,\langle u, \phi^\rho\rangle = D\varphi(\rho)\end{aligned}$$

Hence we have the differential equation

$$\frac{\varphi'(\rho)}{\varphi(\rho)} = \frac{D}{\rho}$$

which is solved by $\varphi(\rho) = C\rho^D$, i.e., $C = \varphi(1)$. This means

$$\left\langle u_\rho, \phi\right\rangle = \rho^D\,\langle u, \phi\rangle\,. \qquad \square$$

Observe that u is a (weak) eigenvector of $x\cdot\partial_x$ to the (weak) eigenvalue D.

THEOREM I.12 (cf. [Hör03, Thm. 3.2.3]). *Let $u \in \mathscr{D}'(\mathbb{R}^d\setminus\{0\})$ scale homogeneously of degree $D \in \mathbb{C}$ and let $-D \notin \mathbb{N}_0 + d$, then u has a unique extension $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ which is homogeneous of degree D. The map*

$$\mathscr{D}'(\mathbb{R}^d\setminus\{0\}) \ni u \mapsto \dot{u} \in \mathscr{D}'(\mathbb{R}^d)$$

is continuous.

A proof of the theorem can be found in the book of Hörmander. Observe, however, that the existence of a (not necessarily homogeneous) extension already follows from Theorem I.6. And if there is a homogeneous extension the uniqueness follows from the proof of the same theorem and the fact that the derivatives of Dirac's δ-distribution have integer scaling degree greater or equal d. Observe that any homogeneous extension $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ of a homogeneous distribution $u \in \mathscr{D}(\mathbb{R}^d \setminus \{0\})$ in particular has the same scaling degree, $\mathrm{sd}(\dot{u}) = \mathrm{sd}(u)$ by Remark I.10. Thus Theorem I.12 really is a refinement of the previous result (Theorem I.6) for the special case of homogeneous distributions.

I.3. Heterogeneous Distributions

A straight forward generalization of Theorem I.12 to the case when the distribution is not homogeneous, but is given as a finite sum of homogeneous parts will be important for the construction of the dimensionally regularized amplitude in Section IV.2.

DEFINITION I.13 (Heterogeneous Distribution). A distribution $u \in \mathscr{D}'$ is called *heterogeneous of order* $k \in \mathbb{N}$ *and multidegree* $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ $(i \neq j \Leftrightarrow \alpha_i \neq \alpha_j)$, if

$$\prod_{j=1}^{k} \left(x \cdot \partial_x - \alpha_j\right) u = 0 \,. \tag{I.11}$$

LEMMA I.14. *Heterogeneous distributions of finite order have a unique decomposition into their homogeneous components.*

PROOF. Let u be a heterogeneous distribution of order k and multidegree $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$, i.e., u fulfills (I.11). Then

$$P_i := \prod_{j \neq i} \frac{x \cdot \partial_x - \alpha_j}{\alpha_i - \alpha_j}$$

projects u onto the eigenspace of $x \cdot \partial_x$ to eigenvalue α_i, since $(x \cdot \partial_x - \alpha_i)\, P_i u = 0$ by assumption (I.11), and

$$P_i^2 u = \prod_{j \neq i} \frac{x \cdot \partial_x - \alpha_j}{\alpha_i - \alpha_j} P_i u = \prod_{j \neq i} \frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_j} P_i u = P_i u \,.$$

Thus $u_i := P_i u$ is homogeneous of degree α_i, and u can be uniquely decomposed into eigenvectors of $x \cdot \partial_x$,

$$u = \sum_{i=1}^{k} u_i \,. \qquad \square$$

COROLLARY I.15 (Ext. of Heterogeneous Distributions). *Let* $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ *be a heterogeneous distribution of multidegree* $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$. *Let furthermore*

$$-\alpha_j \in \mathbb{C} \setminus \mathbb{N}_0 \quad \forall j \in \{1, \ldots, k\} \,.$$

Then u *has a unique heterogeneous extension* $\dot{u} \in \mathscr{D}'(\mathbb{R}^d)$ *of the same multidegree.*

PROOF. Uniqueness. Let $\ddot{u} \in \mathscr{D}'(\mathbb{R}^d)$ be a second extension of u, then $\ddot{u}$ differs from $\dot{u}$ by a distribution supported at $\{0\}$,

$$\ddot{u} - \dot{u} = \sum_{|\alpha| \leqslant \rho} C_\alpha \delta^{(\alpha)} .$$

However, any term in the sum on the right hand side has integer degree of homogeneity smaller or equal to $-d$. Hence the sum in the above expression is not annihilated by $\prod_{j=1}^{k} \left(x \cdot \partial_x - \alpha_j\right)$ with $-\alpha_j \notin \mathbb{N}_0$, whereas $\dot{u}$ is annihilated by assumption. Thus $\ddot{u}$ is not annihilated by $\prod_{j=1}^{k} \left(x \cdot \partial_x - \alpha_j\right)$ and hence is not heterogeneous of multidegree $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$.

Existence. By Lemma I.14 above, u has a unique decomposition into homogeneous parts, $u_i \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$, each of which has a unique homogeneous extension $\dot{u}_i$ by Theorem I.12. Hence

$$\dot{u} = \sum_{i=1}^{k} \dot{u}_i$$

is an extension of u which is heterogeneous of order k and multidegree $\boldsymbol{\alpha}$. □

If the eigenvalues in the product (I.11) coincide, $\alpha_i = \alpha \ \forall i \in \{1, \ldots, k\}$, we get to the notion of *almost homogeneous distributions*, which are homogeneous up to a polynomial of order $k-1$ in $\ln(\rho)$, where $\rho > 0$ is the scaling parameter. Hollands and Wald proved a uniqueness result for the extension also in this case [HW02, Lem. 4.1], see also [DF04, Prop. 3.3]. However, the distributions we will analyze in this work are at most heterogeneous of finite order, and thus we will not need the lemma of Hollands and Wald. This is due to the fact that we regard only regularized distributions, a concept to be defined in the next section. We want to remark, however, that in the limit where the regularization is removed we will get back almost homogeneous distributions in the generic case.

I.4. Regularization of Distributions

DEFINITION I.16 (Regularization). Let $u \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ be a distribution with degree of divergence $\operatorname{div}(u) = \lambda$. Let $\tilde{u} \in \mathscr{D}'_\lambda(\mathbb{R}^d)$ be the unique extension of u with the same degree of divergence. A family $\left\{u^\zeta\right\}_{\zeta \in \Omega \setminus \{0\}}$ of distributions $u^\zeta \in \mathscr{D}'(\mathbb{R}^d)$, where $\Omega \subset \mathbb{C}$ is a neighborhood of the origin, is called a *regularization of* u, if

$$\forall g \in \mathscr{D}_\lambda(\mathbb{R}^d): \quad \lim_{\zeta \to 0} \left\langle u^\zeta, g \right\rangle = \langle \tilde{u}, g \rangle \ . \tag{I.12}$$

The regularization $\left\{u^\zeta\right\}$ is called *analytic*, if for any function $f \in \mathscr{D}(\mathbb{R}^d)$ the map

$$\zeta \mapsto \left\langle u^\zeta, f \right\rangle \tag{I.13}$$

is analytic for $\zeta \in \Omega \setminus \{0\}$, possibly with a pole of finite order at the origin, i.e., (I.13) is a meromorphic function. We speak of a *finite regularization*, if

$$\forall f \in \mathscr{D}(\mathbb{R}^d): \quad \lim_{\zeta \to 0} \left\langle u^\zeta, f \right\rangle \in \mathbb{C} ,$$

in this case $\lim_{\zeta \to 0} u^\zeta \in \mathscr{D}'(\mathbb{R}^d)$ is a renormalization, or extension of u.

Given a regularization $\{u^\zeta\}$ of u we have for all $f \in \mathscr{D}(\mathbb{R}^d)$ and any projection $W : \mathscr{D} \to \mathscr{D}_\lambda$ that

$$\langle \tilde{u}, Wf \rangle = \lim_{\zeta \to 0} \left\langle u^\zeta, Wf \right\rangle . \tag{I.14}$$

According to Lemma I.7 for any W-projection there exists a family of functions $\{w_\alpha \in \mathscr{D} : (\partial_\beta w_\alpha)(0) = \delta_{\alpha\beta}\}$ such that

$$W = 1 - \sum_{|\alpha| \leqslant \lambda} (-1)^{|\alpha|} w_\alpha \delta^{(\alpha)} ,$$

and since $u^\zeta \in \mathscr{D}'(\mathbb{R}^d)$ we can write (I.14) as

$$\langle \tilde{u}, Wf \rangle = \lim_{\zeta \to 0} \left[\left\langle u^\zeta, f \right\rangle - \sum_{|\alpha| \leqslant \lambda} \left\langle u^\zeta, w_\alpha \right\rangle f^{(\alpha)}(0) \right] . \tag{I.15}$$

In the generic case the limit on the right hand side cannot be split, since the limits of the individual terms might not exist. However, if $\{u^\zeta : \zeta \in \Omega \setminus \{0\}\}$ is an analytic regularization, the individual terms can be expanded in Laurent series around $\zeta = 0$, and since the overall limit is finite the principal parts (pp) of these Laurent series have to coincide,

$$\forall f \in \mathscr{D} : \quad \mathrm{pp} \left\langle u^\zeta, f \right\rangle = \mathrm{pp} \left(\sum_{|\alpha| \leqslant \lambda} \left\langle u^\zeta, w_\alpha \right\rangle f^{(\alpha)}(0) \right) = \sum_{|\alpha| \leqslant \lambda} f^{(\alpha)}(0) \, \mathrm{pp} \left\langle u^\zeta, w_\alpha \right\rangle .$$

We conclude that the principal part of any analytic regularization $\{u^\zeta\}$ is a polynomial in derivatives of Dirac's δ-distribution up to order $\lambda = \mathrm{div}(u)$,

$$\mathrm{pp}(u^\zeta) = \sum_{|\alpha| \leqslant \mathrm{div}(u)} C_\alpha(\zeta) \delta^{(\alpha)} \in \mathscr{E}'_{\mathrm{Dirac}} , \tag{I.16}$$

where we set $C_\alpha(\zeta) = \mathrm{pp} \langle u^\zeta, w_\alpha \rangle$. That is, $\mathrm{pp}(u^\zeta)$ is local for all $\zeta \in \Omega \setminus \{0\}$ and vanishes if $\mathrm{div}(u) < 0$. The fact that the principal part of an analytic regularization is a local distribution will be crucial for the discussion in Chapter VI of the present thesis. In particular this implies that we can fix an extension $\dot{u}_{\mathrm{MS}} \in \mathscr{D}'$ of u by setting

$$\langle \dot{u}_{\mathrm{MS}}, f \rangle := \lim_{\zeta \to 0} \left[\left\langle u^\zeta, f \right\rangle - \mathrm{pp}(\left\langle u^\zeta, f \right\rangle) \right] . \tag{I.17}$$

This way of choosing a renormalization of u is called *minimal subtraction* (MS). By construction $\dot{u}_{\mathrm{MS}}$ has the same scaling degree as u, and thus minimal subtraction can be implemented as a W-extension, cf. Lemma I.8. We choose a projection

$$W^{\mathrm{MS}} : \mathscr{D} \to \mathscr{D}_\lambda , \quad \lambda = \mathrm{div}(u) ,$$

which fulfills

$$\forall f \in \mathscr{D} : \quad \langle \dot{u}_{\mathrm{MS}}, f \rangle = \left\langle \tilde{u}, W^{\mathrm{MS}} f \right\rangle . \tag{I.18}$$

Let us regard this projection for finite $\zeta \in \Omega \setminus \{0\}$. The regular part of $\langle u^\zeta, f \rangle$ is

given by

$$\begin{aligned}\left\langle \mathrm{rp}(u^\zeta), f\right\rangle &= \left\langle u^\zeta - \mathrm{pp}(u^\zeta)\,,\, W^{\mathrm{MS}} f + \sum_{|\alpha|\leqslant \mathrm{div}(u)} w_\alpha^{\mathrm{MS}} f^{(\alpha)}(0)\right\rangle \\ &= \left\langle u^\zeta, W^{\mathrm{MS}} f\right\rangle + \sum_{|\alpha|\leqslant \mathrm{div}(u)} \left\langle \mathrm{rp}(u^\zeta), w_\alpha^{\mathrm{MS}}\right\rangle f^{(\alpha)}(0)\,,\end{aligned}$$

since $\mathrm{pp}\left\langle u^\zeta, W^{\mathrm{MS}} f\right\rangle = 0$ by (I.12). The first term on the right hand side, as well as the left hand side of this equation tend to $\langle \dot{u}_{\mathrm{MS}}, f\rangle$ as $\zeta \to 0$, cf. (I.17)/(I.18),

$$\lim_{\zeta\to 0}\left\langle \mathrm{rp}(u^\zeta), f\right\rangle = \lim_{\zeta\to 0}\left\langle u^\zeta, W^{\mathrm{MS}} f\right\rangle = \langle \dot{u}_{\mathrm{MS}}, f\rangle\,. \tag{I.19}$$

Hence the sum on the right hand side has to vanish in this limit. Since it is the regular part of some Laurent series we infer that it is at least of order one in ζ,

$$\forall \zeta \in \Omega\setminus\{0\}: \quad \left\langle \mathrm{rp}(u^\zeta), f\right\rangle = \left\langle u^\zeta, W^{\mathrm{MS}} f\right\rangle + \mathcal{O}(\zeta)\,. \tag{I.20}$$

Hence for finite regularization parameter, $\zeta \in \Omega\setminus\{0\}$, minimal subtraction can be expressed as a W-projection up to a contribution which vanishes identically in the limit $\zeta \to 0$. This fact will become important in the discussion of minimal subtraction on the level of graph amplitudes in Chapter V and in particular for the proof of Proposition V.4.

The coefficients $C_\alpha(\zeta)$ in (I.16) are called *counterterms*. They are local in the sense that they are the coefficients of a local distribution. In particular they do not depend on the chosen W-projection. The $C_\alpha(\zeta)$ are often referred to as infinite counterterms, since they do not possess a limit as $\zeta \to 0$ and the way of introducing them by splitting the W-projection before taking the limit in (I.15) was also discussed in [KTV96].

In Fourier space the $C_\alpha(\zeta)$ are the coefficients of a polynomial in (external) momenta p,

$$\mathfrak{F}\Big(\sum_{|\alpha|\leqslant\lambda} C_\alpha(\zeta)\delta^{(\alpha)}\Big)(p) = \sum_{|\alpha|\leqslant\lambda} \frac{i^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} C_\alpha(\zeta) p^\alpha\,.$$

In this sense the counterterms are invariant under Fourier transform and thus provide a basis on which one can compare the position space approach to dimensional regularization and minimal subtraction ($\mathrm{DimReg}^x + \mathrm{MS}$), to be discussed in the present work, with the standard approach in momentum space.

CHAPTER II

The Setting of Perturbative Algebraic Quantum Field Theory

We want to analyze the methods of analytic regularization (in particular dimensional regularization) and minimal subtraction, introduced on the level of distributions in the previous chapter, in the algebraic approach to perturbative Quantum Field Theory. More specifically, we will use the framework of *perturbative Algebraic Quantum Field Theory (pAQFT)* [BDF09]. Although the methods of pAQFT apply in a much more general framework we want to restrict ourselves in the present work to the case of d-dimensional Minkowski spacetime. The aim of this chapter is to introduce the main concepts and the basic constructions of pAQFT, which will be used in the main part of this work.

II.1. Classical Field Theory and Deformation Quantization

In 1990 Dito showed how the formalism of deformation quantization can be applied to field theory [Dit90]. He constructed the algebra of the free scalar field without reference to an underlying Hilbert space. In his work he used the earlier analysis of the deformation of algebras by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [BFF$^+$78]. Dito also related his approach to the (despite its mathematical problems) widely known and used Feynman path integral approach to quantum field theory. The work of Brunetti, Dütsch, and Fredenhagen [DF01b, DF01a, DF03, DF04, BF07, BDF09] showed that the star product approach of Dito can be extended to a purely algebraic formulation of perturbative Quantum Field Theory (pQFT) in general and perturbative renormalization theory in particular. To give a motivation for the deformation view point in field theory, we briefly review in this first section the structure of the algebra of classical field theory and define its Poisson structure. The construction of the algebra of observables of pQFT will then be carried out in full detail in the next section for the case of flat Minkowski spacetime.

Let $\mathbb{M}$ denote the d-dimensional Minkowski spacetime with metric tensor $\eta = (1, -1, \ldots, -1)$ on the diagonal. Timelike (spacelike) vectors fulfill $x^2 > 0$ ($x^2 < 0$) and the set of all timelike vectors is called the open lightcone $\mathcal{V} = \mathcal{V}^+ \dot\cup \mathcal{V}^-$. It is the disjoint union of two connected components, which we refer to as the forward and backward lightcones, $\mathcal{V}^\pm = \{x \in \mathbb{M} \,|\, x^2 > 0,\ \pm x^0 > 0\}$. We denote by $\overline{\mathcal{V}^\pm}$ and $\partial\mathcal{V}^\pm$ the closure and boundary of these sets, respectively.

The configuration space of classical field theory is the space of smooth functions

$$\varphi : \mathbb{M} \to \mathbb{C}\,, \quad \varphi \in \mathscr{E}(\mathbb{M})\,,$$

and the observables are (not necessarily linear) functionals on this space.

DEFINITION II.1 (Smooth Functional). A functional

$$\begin{array}{rccc} F: & \mathscr{E}(\mathbb{M}) & \to & \mathbb{C} \\ & \varphi & \mapsto & F(\varphi)\,, \end{array}$$

is called *smooth*, if for any $\varphi \in \mathscr{E}(\mathbb{M})$ and for all $n \in \mathbb{N}$ its nth functional derivative,

$$\left\langle F^{(n)}(\varphi), h_1 \otimes \cdots \otimes h_n \right\rangle := \frac{d^n}{d\lambda_1 \cdots d\lambda_n} F(\varphi + \sum_{i=1}^{n} \lambda_i h_i)\Big|_{\lambda_1 = \cdots = \lambda_n = 0}\,, \quad h_i \in \mathscr{E}(\mathbb{M})\,,$$

exists as a symmetric distribution with compact support, $F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{M}^n)$. Occasionally we will write $F^{(n)}(\varphi) = \frac{\delta^n F}{\delta \varphi^n}$. The triangular brackets denote the dual pairing (of $\mathscr{E}'(\mathbb{M}^n)$ with $\mathscr{E}(\mathbb{M}^n)$ here). We denote the space of smooth functionals by $\widetilde{\mathcal{F}}(\mathbb{M})$.

DEFINITION II.2 (Support of a Functional). Let $F \in \widetilde{\mathcal{F}}(\mathbb{M})$, then we define the support of F implicitly by the equivalence, $h \in \mathscr{E}(\mathbb{M})$,

$$\operatorname{supp}(F) \cap \operatorname{supp}(h) = \varnothing \quad \Leftrightarrow \quad \forall \varphi \in \mathscr{E}(\mathbb{M}) : F(\varphi + h) = F(\varphi)\,.$$

Smooth functionals form a (commutative) algebra, $\left(\widetilde{\mathcal{F}}(\mathbb{M}), \cdot\right)$, with respect to the pointwise product, $\forall F, G \in \widetilde{\mathcal{F}}(\mathbb{M})$, $\forall \varphi \in \mathscr{E}(\mathbb{M})$,

$$(F \cdot G)\,(\varphi) := F(\varphi) G(\varphi)\,.$$

Typical examples of such functionals are,

$$\text{(II.1)} \qquad \text{(i)} \quad F(\varphi) = \int f(x)\, \varphi(x)^3\, dx \quad , \quad \text{(ii)} \quad G(\varphi) = \int g(x)\, [\partial \varphi(x)]^2\, dx\,,$$

$$\text{or} \quad \text{(iii)} \quad K(\varphi) = \int k(x,y)\, \varphi(x)\, \varphi(y)\, dx\, dy\,,$$

where $f, g \in \mathscr{D}(\mathbb{M})$, $k \in \mathscr{D}(\mathbb{M}^2)$ are test functions of compact support, and the integral is taken over the whole spacetime $\mathbb{M}$, or over $\mathbb{M}^2$, respectively. The field itself is represented as a linear evaluation functional, $\forall f \in \mathscr{D}(\mathbb{M})$,

$$\varphi \mapsto \varphi(f) = \int f(x)\, \varphi(x)\, dx\,.$$

It is not possible (in the framework presented here) to deform the whole algebra of smooth functionals $\left(\widetilde{\mathcal{F}}(\mathbb{M}), \cdot\right)$. However, we can restrict ourselves to a suitably chosen subalgebra of functionals $\mathcal{F}(\mathbb{M}) \subset \widetilde{\mathcal{F}}(\mathbb{M})$, which will have a quantized counterpart in deformation quantization. This "deformable algebra" is defined by imposing conditions on the *wave front set* of the functional derivatives $F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{M}^n)$, $F \in \widetilde{\mathcal{F}}(\mathbb{M})$. The wave front set of a distribution $u \in \mathscr{D}'(\mathbb{M}^n)$, roughly speaking, is a conic subset of the cotangent bundle $\mathrm{WF}(u) \subset \dot{T}^*(\mathbb{M}^n) = T^*(\mathbb{M}^n) \setminus \{0\}$, where the first component gives the singular support, $\operatorname{singsupp}(u)$,

and the second gives the directions in which the Fourier transform $\mathfrak{F}(u)$ does not decrease rapidly. The precise definition of this mathematical tool, as well as the large number of deep implications of the wave front set properties of distributions can be found in the book of Hörmander, cf. [Hör03, Def. 8.1.2].

DEFINITION II.3 (Deformable Algebra). A smooth functional $F \in \widetilde{\mathcal{F}}(\mathbb{M})$ is an element of the *deformable algebra* $(\mathcal{F}(\mathbb{M}), \cdot)$, if for all $n \in \mathbb{N}$ the wave front set of $F^{(n)}(\varphi)$ does not meet the n-fold product of the closed forward or backward light cone,

$$\forall \varphi : \quad \mathrm{WF}(F^{(n)}(\varphi)) \cap \left(\mathrm{supp}(F^{(n)}(\varphi)) \times \left[\overline{\mathcal{V}^+}^n \cup \overline{\mathcal{V}^-}^n\right]\right) = \varnothing .$$

We refer to elements of the deformable algebra as *deformable functionals*.

Since $\mathrm{WF}(u) \subset \dot{T}^*(\mathbb{M}^n)$, the forward and backward lightcone are here to be understood as subsets of the cotangent space, $\overline{\mathcal{V}^+}^n, \overline{\mathcal{V}^-}^n \subset T^*_x(\mathbb{M}^n)$.

The deformable algebra $(\mathcal{F}(\mathbb{M}), \cdot)$ can be made into a Poisson algebra by using the Peierls bracket [DF03]. The Poisson structure is defined by

$$\lfloor F, G \rceil (\varphi) := i \left\langle F^{(1)}(\varphi), \Delta * G^{(1)}(\varphi) \right\rangle . \tag{II.2}$$

where $*$ denotes convolution, and Δ is defined as the difference of the unique retarded and advanced fundamental solutions $\Delta_{\mathrm{ret}}, \Delta_{\mathrm{adv}} \in \mathscr{D}'(\mathbb{M})$ of the Klein-Gordon operator, cf. [DF03, Eq. (28)]. Regard these fundamental solutions in more detail, we have

$$\left(\Box + m^2\right) \Delta_{\substack{\mathrm{ret}\\ \mathrm{adv}}} = \delta , \quad \text{with} \qquad \mathrm{supp}(\Delta_{\substack{\mathrm{ret}\\ \mathrm{adv}}}) = \overline{\mathcal{V}^{\pm}}$$
$$\Delta := (\Delta_{\mathrm{ret}} - \Delta_{\mathrm{adv}}) .$$

We also define the corresponding "causal two point function" or "commutator function",

$$\forall f, g \in \mathscr{D}(\mathbb{M}) : \quad \Delta(f, g) := \langle f, \Delta * g \rangle ,$$

and it should be clear from the context, which of the two interpretations of the symbol Δ is to be understood. The wave-front set of the commutator function is given by

$$\mathrm{WF}(\Delta) = \left\{ (x, y, k, -k) \in \dot{T}^*(\mathbb{M}^2) | \, (x-y)^2 = 0, \, k \| (x-y), \, k^2 = 0 \right\} . \tag{II.3}$$

And by [Hör03, Thm. 8.2.10] the pointwise product of the distributions in

$$\lfloor F, G \rceil = \int \Delta(x, y) \cdot F^{(1)}(x) \, G^{(1)}(y) \, dx \, dy , \quad F, G \in \mathcal{F}(\mathbb{M}) ,$$

is well-defined, if the covectors in the second component cannot add up to zero. The wave front set of the tensor power $\left(F^{(1)} \otimes G^{(1)}\right)(\varphi) \in \mathscr{E}'(\mathbb{M}^2)$ is given by,

cf. [Hör03, Thm. 8.2.9],

$$\begin{aligned}\mathrm{WF}(F^{(1)} \otimes G^{(1)}) \subset \mathrm{WF}(F^{(1)}) \times \mathrm{WF}(G^{(1)}) \\ \cup \left\{ [\mathrm{supp}(F) \times \{0\}] \times \mathrm{WF}(G^{(1)}) \right\} \\ \cup \left\{ \mathrm{WF}(F^{(1)}) \times [\mathrm{supp}(G) \times \{0\}] \right\} ,\end{aligned}$$

and by comparing this with (II.3) we see that the covectors cannot add up to zero, if $F, G \in \mathcal{F}(\mathbb{M})$. For $(x, y, k_x, k_y) \in \mathrm{WF}(\Delta)$ we clearly have $(k_x, k_y) \in \overline{\mathcal{V}} \times \overline{\mathcal{V}}$, and since $k_f, k_g \notin \overline{\mathcal{V}}$ for $(x, k_f) \in \mathrm{WF}(F^{(1)}(\varphi))$ and $(y, k_g) \in \mathrm{WF}(G^{(1)}(\varphi))$ neither one of the equations

$$k_x + k_f = 0 \qquad \text{or} \qquad k_y + k_g = 0$$

has a solution. By a similar argument we also have $\lfloor F, G \rceil \in \mathcal{F}(\mathbb{M})$ in that case. It was proven that, besides linearity, antisymmetry and the Leibniz rule, the bracket $\lfloor \cdot, \cdot \rceil$ fulfills also the Jacobi identity, and thus defines a genuine Poisson structure on $\mathcal{F}(\mathbb{M})$ [DF03]. As was discussed there a (formal) quantization of $(\mathcal{F}(\mathbb{M}), \lfloor \cdot, \cdot \rceil, \cdot)$ can be understood as a map

$$(\mathcal{F}(\mathbb{M}), \lfloor \cdot, \cdot \rceil, \cdot) \to (\mathcal{F}(\mathbb{M})[[\hbar]], [\cdot, \cdot]_\star, \star) ,$$

to a non-commutative, associative algebra, such that

$$F \star G \xrightarrow{\hbar \to 0} F \cdot G \quad \text{and} \quad \tfrac{1}{\hbar} [F, G]_\star \xrightarrow{\hbar \to 0} \lfloor F, G \rceil . \tag{II.4}$$

In particular we have for the field itself

$$[\varphi(f), \varphi(g)]_\star = i\hbar \Delta(f, g) . \tag{II.5}$$

The product $\star$ of the quantized algebra is not to be confused with the notation $*$ for convolution.

II.2. The Algebra of Observables

As the name suggests the subalgebra $(\mathcal{F}(\mathbb{M}), \cdot)$ can be deformed to give the algebra of perturbative Quantum Field Theory. To construct this algebra, let $H \in \mathscr{D}'(\mathbb{M})$ be a *Hadamard distribution*. That is, cf. [Rad96], H is a (weak) solution of the Klein Gordon equation

$$\left(\Box + m^2 \right) H = 0 \tag{II.6}$$

and the corresponding two point distribution, defined by

$$H(f, g) := \langle f, H * g \rangle , \tag{II.7}$$

satisfies the causality condition

$$\forall f, g \in \mathscr{D}(\mathbb{M}) : \quad H(f, g) - H(g, f) = i\Delta(f, g) \tag{II.8}$$

and has the "positive frequency" - or Hadamard wave front set

$$\mathrm{WF}(H) = \Big\{ (x,y,k,-k) \in \dot{T}^*(\mathbb{M}^2) \,|\, \text{if } x \neq y : \ (x-y)^2 = 0,\, k \| (x-y),\, k^0 > 0\,; \\ \text{if } x = y : \ k^2 = 0,\, k^0 > 0 \Big\}. \tag{II.9}$$

One example of such a Hadamard distribution is the Wightman function Δ_+, i.e. the positive frequency part of the commutator function $i\Delta$, $\Delta_+ = \frac{i}{2}\Delta + \Delta_1$. However, any other distribution H, which differs from Δ_+ by a smooth, symmetric, Lorentz invariant solution of the Klein-Gordon equation will also fulfill the defining equations (II.6), (II.8), and (II.9), and thus be a valid Hadamard function for the construction below. The algebraic properties are completely independent of this choice, the analytic properties, however, will change significantly for different choices of H. A Hadamard solution, especially well-suited for our purposes, will be constructed explicitly in Chapter III. Equation (II.8) fixes the antisymmetric part of H to be $\frac{i}{2}\Delta$, which implies that the Poisson structure (II.2) can be induced by the bidifferential operator

$$\Gamma_H := \int dx\, dy\, H(x,y) \frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\varphi(y)} \tag{II.10}$$

in the following sense

$$\lfloor F, G \rceil = M \circ \Gamma_H \left(F \otimes G - G \otimes F \right) .$$

By means of this differential operator a formal quantization of the Poisson algebra $(\mathcal{F}(\mathbb{M}), \lfloor \cdot, \cdot \rceil, \cdot)$ can be given in form of the following

PROPOSITION II.4 (Deformed Algebra). *Let $\mathcal{F}(\mathbb{M})[[\hbar]]$ be the space of formal power series in $\hbar$ with coefficients in the deformable algebra $\mathcal{F}(\mathbb{M})$ and regard $F, G \in \mathcal{F}(\mathbb{M})[[\hbar]]$. Then*

(II.11)

$$\begin{tikzcd} \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \arrow[rr, "\exp(\hbar\Gamma_H)"] \arrow[dr, dashed, "\star"'] & & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2}, \arrow[dl, "\cdot"] \\ & \mathcal{F}(\mathbb{M})[[\hbar]] & \end{tikzcd} \qquad F \star G := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(k)}, H^{\otimes k} G^{(k)} \right\rangle$$

defines a (non-commutative) associative product on $\mathcal{F}(\mathbb{M})[[\hbar]]$ which fulfills the quantization condition (II.4). We call $(\mathcal{F}(\mathbb{M})[[\hbar]], \star)$ the algebra of pQFT.

PROOF. First we want to argue that $F \star G$ is well-defined for any pair $F, G \in \mathcal{F}(\mathbb{M})[[\hbar]]$. The kernel representation of the kth term of (II.11) is given by

$$\left\langle F^{(k)}, H^{\otimes k} G^{(k)} \right\rangle = \int d\vec{x}\, d\vec{y}\, F^{(k)}(x^1, \ldots, x^k) \prod_{i=1}^{k} H(x^i, y^i)\, G^{(k)}(y^1, \ldots, y^k) .$$

The wave front set of $H^{\otimes k}$ is given by, cf. [Hör03, Thm. 8.2.9],

$$\mathrm{WF}(H^{\otimes k}) \subset \bigcup_{\sigma \in \mathrm{Perm}(k)} \bigcup_{\substack{i+j=k \\ i \geqslant 1}} \sigma \left\{ \mathrm{WF}(H)^i \times [(\mathrm{supp}(H) \times \{0\})]^j \right\} ,$$

where $\mathrm{Perm}(k)$ denotes the symmetric group in k variables. Hence all covectors in the x-components, k_{x^i}, are elements of the closed forward lightcone $\overline{\mathcal{V}^+}$, cf. (II.9), and equivalently for the y-components we have $\forall i$: $k_{y^i} \in \overline{\mathcal{V}^-}$. Using [Hör03, Thm. 8.2.10] we conclude that the pointwise product of distributions above is well-defined, if

$$\mathrm{WF}(F^{(k)}) \cap \left(\overline{\mathcal{V}^-}\right)^k = \varnothing = \mathrm{WF}(G^{(k)}) \cap \left(\overline{\mathcal{V}^+}\right)^k ,$$

which is automatically fulfilled for $F, G \in \mathcal{F}(\mathbb{M})[[\hbar]]$.

Associativity of $\star$ follows directly from the associativity of the pointwise product $M\,(F \otimes G) = F \cdot G$ and the form of (II.11). The detailed argument can be found in [Wal07, Sec. 6.2.4]; it is summarized in the following. Let

$$\Gamma_H^{(1,2)}\,(A \otimes B \otimes C) := \Gamma_H\,(A \otimes B) \otimes C\,,$$

$$\Gamma_H^{(1,3)}\,(A \otimes B \otimes C) := \left\langle H, A^{(1)} \otimes B \otimes C^{(1)} \right\rangle ,$$

and

$$\Gamma_H^{(2,3)}\,(A \otimes B \otimes C) := A \otimes \Gamma_H\,(A \otimes B)\ .$$

Then the Leibniz rule in the second argument implies

$$\Gamma_H \circ [\mathrm{id} \otimes M] = [\mathrm{id} \otimes M] \circ \left(\Gamma_H^{(1,2)} + \Gamma_H^{(1,3)}\right) ,$$

and analogously

$$\Gamma_H \circ [M \otimes \mathrm{id}] = [M \otimes \mathrm{id}] \circ \left(\Gamma_H^{(1,3)} + \Gamma_H^{(2,3)}\right) .$$

These formulas generalize to the exponential of the bidifferential operators Γ_H and $\Gamma_H^{(i,j)}$ by linearity of the tensor product. Hence for the star product one gets

$$\begin{aligned}
A \star (B \star C) &= M \circ e^{\hbar\Gamma_H} \circ \left(\mathrm{id} \otimes M \circ e^{\hbar\Gamma_H}\right)(A \otimes B \otimes C) \\
&= M \circ e^{\hbar\Gamma_H} \circ (\mathrm{id} \otimes M) \circ e^{\hbar\Gamma_H^{(2,3)}}\,(A \otimes B \otimes C) \\
&= M \circ (\mathrm{id} \otimes M) \circ e^{\hbar\left(\Gamma_H^{(1,2)} + \Gamma_H^{(1,3)} + \Gamma_H^{(2,3)}\right)}\,(A \otimes B \otimes C) \\
&= M \circ (M \otimes \mathrm{id}) \circ e^{\hbar\left(\Gamma_H^{(1,2)} + \Gamma_H^{(1,3)} + \Gamma_H^{(2,3)}\right)}\,(A \otimes B \otimes C) \\
&= (A \star B) \star C
\end{aligned}$$

by associativity of the pointwise product M. □

An especially important subclass of $(\mathcal{F}(\mathbb{M})[[\hbar]], \star)$ for the description of interactions in QFT is the class of local functionals.

DEFINITION II.5 (Local Functional). Let $\mathrm{Diag}(\mathbb{M}^n) = \{\vec{x} \in \mathbb{M}^n : x_1 = \cdots = x_n\}$ denote the thin diagonal in $\mathbb{M}^n$. A functional $F \in \mathcal{F}(\mathbb{M})$ is called local, if for all $n \in \mathbb{N}$:

[LF-1] the nth order functional derivative is supported on the thin diagonal,

$$\forall n \in \mathbb{N}: \quad \mathrm{supp}(F^{(n)}(\varphi)) \subset \mathrm{Diag}(\mathbb{M}^n)\,,$$

and

[LF-2] the wave front set of the nth order functional derivative lies transversal to the thin diagonal,

$$\forall n \in \mathbb{N}: \quad \mathrm{WF}(F^n(\varphi)) \subset [T\mathrm{Diag}(\mathbb{M}^n)]^\perp .$$

We denote the space of local functionals by $\mathcal{F}_{\mathrm{loc}}(\mathbb{M})$.

The first two functionals in (II.1) are examples of local functionals, the third is not local. Observe that

$$[T\mathrm{Diag}(\mathbb{M}^n)]^\perp \cap \left(\mathbb{M}^n \times \left[\left(\overline{\mathcal{V}^+}\right)^n \cup \left(\overline{\mathcal{V}^-}\right)^n\right]\right) = \varnothing ,$$

and hence the wave front set condition for local functionals [LF-2] implies the wave front set condition of Definition II.3; local functionals are deformable. However, they do not form a subalgebra of $(\mathcal{F}(\mathbb{M})[[\hbar]], \star)$, since the product of two local functionals is not local in the generic case. Furthermore we want to remark that, in contrast to the definition of deformability, the definition of locality does not depend on the underlying Minkowski signature, a fact which is a major ingredient in the Euclidean formulation of Epstein-Glaser renormalization in [Kel09].

From the viewpoint of microlocal analysis the wave front set condition [LF-2] implies that, if $F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})$, the distributions $F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{M}^n)$ can be pulled back to surfaces which lie transversal to the thin diagonal [Hör03, Thm. 8.2.4]. This implies that their distributional part depends only on relative coordinates, i.e. the Schwartz kernel of the functional derivative of any local functional can be written as

(II.12)
$$F^{(n)}(\varphi)(x_1, \dots, x_n) = \sum_k f_\varphi^{n,k}(x)\, P_k(\partial_{\boldsymbol{r}})\delta(\boldsymbol{r}) , \qquad f_\varphi^{n,k} \in \mathscr{D}(\mathrm{Diag}(\mathbb{M}^n)) \cong \mathscr{D}(\mathbb{M})$$

where $x = \frac{1}{n}\sum_{i=1}^n x_i$ is the "center of mass"-coordinate, $\boldsymbol{r} = (r_1, \dots, r_{n-1})$ are relative coordinates, and P_k are homogeneous, symmetric polynomials in $n-1$ spacetime variables. In flat spacetime the thin diagonal is the coordinate space of the center of mass, and the relative coordinates are defined in a transversal surface. In this sense [LF-2] is a microlocal remnant of translation invariance.

The causal partition of unity, which was used in [BF00a] for the distributional construction of time-ordered products in curved spacetime, can generally not be used in the functional framework introduced here. It is replaced by the following result.

LEMMA II.6 ([BDF09, Lem. 3.2]). *Any local functional can be written as a finite sum of local functionals of arbitrarily small supports. That is*

$$F = \sum_i \sigma_i F_i , \quad \sigma_i \in \{\pm 1\} ; F, F_i \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] , \tag{II.13}$$

where $\mathrm{supp}(F_i) \subset B_i$, *with* B_i *a ball of arbitrarily small radius* $\varepsilon > 0$.

The proof of this Lemma in the given reference uses a different definition of local functionals than the one given above. However, both definitions can be shown

to be equivalent; see [BFLR10] and [Kel09, App. C].

Combining (II.12) with (II.13), we have that the functional derivative of any local functional can be expressed as

$$F^{(n)}(\varphi)(x_1,\dots,x_n) = \sum_k \sum_i f_\varphi^{n,k,i}(x)\, P_k(\partial_{\boldsymbol{r}})\delta(\boldsymbol{r})\,, \quad F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]\,, \tag{II.14}$$

where the support of $f_\varphi^{n,k,i} \in \mathscr{D}(\mathbb{M})$ can be chosen arbitrarily small.

II.3. The Time Ordered Product

We want to regard the situation where the time evolution of the interacting theory is solved perturbatively, i.e., in terms of formal power series. A convenient way to describe this evolution is by introduction of a time ordering prescription for the interaction functionals. Given the theory can be described as a free theory at asymptotic times [Haa58, Rue62], one can encode the information on the transition probabilities in collision processes of elementary particles in the so-called $\boldsymbol{\mathcal{S}}$-matrix [LSZ55, LSZ57],

$$\boldsymbol{\mathcal{S}}(F) = \exp_{\cdot_{\boldsymbol{\mathcal{T}}}}(F)\,. \tag{II.15}$$

We will not consider the problem of defining the $\boldsymbol{\mathcal{S}}$-matrix for non-compactly supported interaction functionals, commonly referred to as the infrared problem (IR-problem). Instead we will stay within the algebra $(\mathcal{F}(\mathbb{M})[[\hbar]], \star)$ and discuss the definition of $\boldsymbol{\mathcal{S}}$ as a map

$$\boldsymbol{\mathcal{S}} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}(\mathbb{M})[[\hbar]]$$

defined on local functionals, i.e., the ultraviolet problem (UV-problem). The observation, which lies at the very heart of (perturbative) renormalization theory is that this map cannot be defined in a unique way, but that such a definition necessarily introduces a freedom into the theory, commonly described in terms of the Stückelberg-Petermann renormalization group [SP53]. We will come to that point in greater detail in Chapter VI.

In this section we will define the time ordered product and introduce the notion of its partial algebra of functionals. In the framework of pAQFT the time ordered product, despite its significantly different properties, can be introduced in much the same way as the star product was defined in the last section. Regard the second order functional differential operator

$$\Gamma'_{H_F} := \frac{1}{2}\int dx\, dy\, H_F(x,y)\, \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}\,, \tag{II.16}$$

where $H_F \in \mathscr{D}'(\mathbb{M}^2)$ denotes a Feynman propagator. That is, there is a fundamental solution of the Klein-Gordon operator, $H_F \in \mathscr{D}'(\mathbb{M})$,

$$\left(\Box + m^2\right) H_F = \delta\,, \tag{II.17}$$

such that $\forall f, g \in \mathscr{D}(\mathbb{M})$:

$$H_F(f,g) = \langle f, H_F * g \rangle \quad \text{and} \quad H_F(f,g) = H_F(g,f)\,, \tag{II.18}$$

and the wave front set of the propagator H_F is given by

$$\begin{aligned} \mathrm{WF}(H_F) = \Big\{ (x,y,k,-k) \in \dot{T}^*(\mathbb{M}^2) \,|\, & \text{for } x \neq y : \ (x-y)^2 = 0,\ k \| (x-y), \\ & k \in \partial \mathcal{V}^{\pm} \text{ if } x \in \partial \mathcal{V}_y^{\pm}; \\ & \text{for } x = y : \ k \in \dot{T}_x^*(\mathbb{M}) \ \Big\}\,, \end{aligned} \tag{II.19}$$

where we understand $\overline{\mathcal{V}_y^{\pm}}$ as the causal future / past of $y \in \mathbb{M}$. Feynman propagators, $H_F \in \mathscr{D}(\mathbb{M}^2)$, have to be carefully distinguished from the *two point functions*, which are (bi-) solutions of the Klein-Gordon equation (II.6) and have positive frequency wave front sets (II.9).

We have that $u = H_F(f,\cdot) = H_F(\cdot, f)$ is a solution of the inhomogeneous equation

$$\left(\Box + m^2\right) u = f\,,$$

what already implies that the wave front set of H_F necessarily contains $\mathrm{WF}(\delta)$, or more generally [Hör03, (8.1.11), Thm. 8.3.1],

$$\mathrm{WF}(\delta) = \mathrm{WF}((\Box + m^2)H_F) \subset \mathrm{WF}(H_F) \subset \mathrm{WF}(\delta) \cup \mathrm{Char}(\Box + m^2)\,, \tag{II.20}$$

and thus a definition of powers of H_F which uses only wave front set properties is impossible, cf. [Hör03, Thm. 8.2.10]. Such a definition will involve an extension procedure as described in the previous chapter, i.e., renormalization, and it will be helpful to note that one can read off the scaling degree of H_F directly from (II.17), cf. Lemma I.4,

$$\mathrm{sd}(H_F) = d - 2\,, \quad d = \dim(\mathbb{M})\,. \tag{II.21}$$

As in the case of the star product, there is a freedom in the definition of H_F. Observe, however, that H_F is fixed once we have chosen a Hadamard function H which determines the star product,

$$H_F = H + i\Delta_{\mathrm{adv}}\,. \tag{II.22}$$

This is also reflected by the fact that H and H_F both can be defined as certain boundary values of the same analytic function. We will exploit the freedom in the choice of the pair H, H_F in Chapter III for the construction of a propagator which is especially well-suited for the discussion of dimensional regularization in position space. That is, we will add a smooth, symmetric, Lorentz invariant solution of the Klein-Gordon equation to the Wightman function, such that the so defined Hadamard function and corresponding Feynman propagator will have desirable additional properties. However, let us now come to the definition of the time-ordered product and its partial algebra.

PROPOSITION II.7 (Partial Algebra of $\cdot_{\boldsymbol{\mathcal{T}}}$). *Let $H_F \in \mathscr{D}'(\mathbb{M}^2)$ be a Feynman propagator in the sense of (II.17)-(II.19) and let Γ_{H_F} be defined by (II.16). Then the product induced by*

$$\begin{tikzcd} \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \arrow[r, "\boldsymbol{\mathcal{T}}^{\otimes 2}"] \arrow[d, "\cdot"'] & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \arrow[d, dashed, "\cdot_{\boldsymbol{\mathcal{T}}}"] \\ \mathcal{F}(\mathbb{M})[[\hbar]] \arrow[r, "\boldsymbol{\mathcal{T}}"] & \mathcal{F}(\mathbb{M})[[\hbar]] \end{tikzcd} \qquad F \cdot_{\boldsymbol{\mathcal{T}}} G = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(k)}, H_F^{\otimes k} G^{(k)} \right\rangle , \tag{II.23}$$

where

$$\boldsymbol{\mathcal{T}} := \exp(\hbar \Gamma'_{H_F}) \qquad \left(\boldsymbol{\mathcal{T}}^{-1} := \exp(-\hbar \Gamma'_{H_F})\right)$$

denotes the time-ordering (anti-time-ordering) operator, makes $(\mathcal{F}(\mathbb{M})[[\hbar]], \cdot_{\boldsymbol{\mathcal{T}}})$ into a partial algebra. That is $F \cdot_{\boldsymbol{\mathcal{T}}} G$ is defined for all pairs $F, G \in \mathcal{F}(\mathbb{M})[[\hbar]]$ with

$$\operatorname{supp}(F) \cap \operatorname{supp}(G) = \varnothing ,$$

and $\cdot_{\boldsymbol{\mathcal{T}}}$ is associative for any three functionals with pairwise disjoint supports.

REMARK II.8. First. Observe that equation (II.22) implies that $\cdot_{\boldsymbol{\mathcal{T}}}$ really is the time-ordered product for $\star$. We have for the scalar field

$$\varphi(f) \cdot_{\boldsymbol{\mathcal{T}}} \varphi(g) = \varphi(f)\varphi(g) + \hbar \langle f, H_F\, g \rangle$$

$$\varphi(f) \star \varphi(g) = \varphi(f)\varphi(g) + \hbar \langle f, H\, g \rangle \ .$$

Assume that the support of f is *later than* the support of g, $\operatorname{supp}(f) \gtrsim \operatorname{supp}(g)$, i.e., $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ can be separated by a Cauchy surface Σ, such that $\operatorname{supp}(f)$ lies in the future and $\operatorname{supp}(g)$ in the past of Σ. Then we infer from $\operatorname{supp}(\Delta_{\mathrm{adv}}) \subset \overline{\mathcal{V}^-}$ that $\langle f, \Delta_{\mathrm{adv}} * g \rangle = 0$ and hence

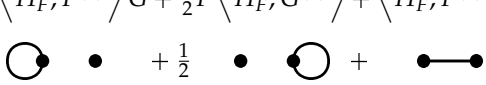
$$\langle f, H_F\, g \rangle = \langle f, H\, g \rangle \ .$$

We thus see that

$$\varphi(f) \cdot_{\boldsymbol{\mathcal{T}}} \varphi(g) = \varphi(f) \star \varphi(g) \qquad \text{if} \quad \operatorname{supp}(f) \gtrsim \operatorname{supp}(g) .$$

Second. The action of the second order differential operator Γ'_{H_F} on functionals can be interpreted directly in terms of graphs. Let $F, G \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})$ be interaction functionals, then the Leibniz rule implies

$$\begin{aligned} \Gamma'_{H_F}(F \cdot G) &= \tfrac{1}{2} \left\langle H_F, F^{(2)} \right\rangle G + \tfrac{1}{2} F \left\langle H_F, G^{(2)} \right\rangle + \left\langle H_F, F^{(1)} \otimes G^1 \right\rangle \\ &= \tfrac{1}{2} \ + \ \tfrac{1}{2} \ + \end{aligned} \tag{II.24}$$



The first two terms in this sum are tadpoles, i.e., graphs with lines connecting one vertex with itself. Observe that they are removed, by applying $\left(1 - \Gamma'_{H_F}\right)$ to the interaction functionals before pointwise multiplication,

$$\left(1 + \Gamma'_{H_F}\right) \left[\left(1 - \Gamma'_{H_F}\right) F \cdot \left(1 - \Gamma'_{H_F}\right) G\right] = \ + \ .$$

See the proof below for the general case.

PROOF OF PROPOSITION II.7. Before proving the main part of the proposition, we want to show that the diagram in (II.23) is equivalent to the given formula. This is the same as showing that there are no tadpole terms in the graphical expansion of $F \cdot_{\boldsymbol{\mathcal{T}}} G$ and higher oder products. The Leibniz rule (II.24) can be written as a coproduct rule for Γ'_{H_F} ,

$$\Delta\Gamma'_{H_F} = \Gamma'_{H_F} \otimes \mathrm{id} + \mathrm{id} \otimes \Gamma'_{H_F} + \Gamma_{H_F} .$$

Writing M for pointwise multiplication and abbreviating $\Gamma_{H_F} \equiv \Gamma$, we read off the diagram

$$\begin{aligned} F \cdot_{\boldsymbol{\mathcal{T}}} G &= e^{\hbar\Gamma'} \circ M \left(e^{-\hbar\Gamma'} F \otimes e^{-\hbar\Gamma'} G \right) \\ &= M \circ e^{\hbar\Delta\Gamma'} \left(e^{-\hbar\Gamma'} F \otimes e^{-\hbar\Gamma'} G \right) \\ &= M \circ e^{\hbar\Gamma} \left(F \otimes G \right) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(k)}, H_F^{\otimes k} G^{(k)} \right\rangle . \end{aligned}$$

And we see that the tadpoles drop out of the expansion. The graph expansion of the time-ordered will be discussed in more detail in Section IV.1.

One infers from (II.22) that

$$F \cdot_{\boldsymbol{\mathcal{T}}} G = \begin{cases} F \star G & \text{if } \mathrm{supp}(F) \gtrsim \mathrm{supp}(G) \\ G \star F & \text{if } \mathrm{supp}(G) \gtrsim \mathrm{supp}(F) , \end{cases} \tag{II.25}$$

where, as before, $\mathcal{O} \gtrsim \mathcal{U}$ for two regions $\mathcal{O}, \mathcal{U} \subset \mathbb{M}$ denotes that $\mathcal{O}$ is later than $\mathcal{U}$. Observe that if $\mathcal{O} \gtrsim \mathcal{U}$ and $\mathcal{U} \gtrsim \mathcal{O}$ then $\mathcal{O}$ and $\mathcal{U}$ are *causally disjoint*. With (II.25) it follows from Proposition II.4 that $F \cdot_{\boldsymbol{\mathcal{T}}} G$ is well-defined as long as the functionals have disjoint supports. Associativity follows from the same proposition, but can also be proven directly. Let $A, B, C \in \mathcal{F}(\mathbb{M})[[\hbar]]$ be three deformable functionals with pairwise disjoint supports, then the twofold product is defined, and we have

$$\begin{aligned} A \cdot_{\boldsymbol{\mathcal{T}}} (B \cdot_{\boldsymbol{\mathcal{T}}} C) &= \boldsymbol{\mathcal{T}} \left[\boldsymbol{\mathcal{T}}^{-1} A \cdot \boldsymbol{\mathcal{T}}^{-1} \boldsymbol{\mathcal{T}} \left(\boldsymbol{\mathcal{T}}^{-1} B \cdot \boldsymbol{\mathcal{T}}^{-1} C \right) \right] \\ &= \boldsymbol{\mathcal{T}} \left[\boldsymbol{\mathcal{T}}^{-1} A \cdot \boldsymbol{\mathcal{T}}^{-1} B \cdot \boldsymbol{\mathcal{T}}^{-1} C \right] = (A \cdot_{\boldsymbol{\mathcal{T}}} B) \cdot_{\boldsymbol{\mathcal{T}}} C \qquad \square \end{aligned}$$

We want to remark that there is a subalgebra $\mathcal{F}_0(\mathbb{M})[[\hbar]] \subset \mathcal{F}(\mathbb{M})[[\hbar]]$, where besides the pointwise, $\cdot$, and the star, $\star$, also the time-ordered product, $\cdot_{\boldsymbol{\mathcal{T}}}$, can be defined as a full product. Namely, $\mathcal{F}_0(\mathbb{M})$ is given as the algebra of functionals, such that for any element all functional derivatives are smooth, compactly supported functions,

$$\forall F \in \mathcal{F}_0(\mathbb{M}),\ \forall n \in \mathbb{N} : \quad F^{(n)}(\varphi) \in \mathscr{D}(\mathbb{M}^n) .$$

The third example given in (II.1) is an element of $\mathcal{F}_0(\mathbb{M})$. The field equation, $\left(\Box + m^2\right) \varphi = 0$, generates an ideal in $(\mathcal{F}_0(\mathbb{M}), \star)$,

$$\mathcal{J} = \left\{ F \in \mathcal{F}_0(\mathbb{M}) : F(\varphi) = \sum_a G_a(\varphi)\, \partial^a (\Box + m^2)\varphi,\ G_a \in \mathcal{F}_0(\mathbb{M}) \right\} ,$$

$a \in \mathbb{N}_0^d$, $d = \dim(\mathbb{M})$. And following [DF03, Footnote 5] we want to assume that $\mathcal{J}$ is the set of functionals, which vanish on the space of (smooth) solutions of the Klein-Gordon equation. Let $F \in \mathcal{J}$ and $G \in \mathcal{F}_0(\mathbb{M})$, one easily checks that $F \star G \in \mathcal{J}$. However, the same is not true for the time-ordered product, $F \cdot_{\boldsymbol{\mathcal{T}}} G \notin \mathcal{J}$, in the region where $\cdot_{\boldsymbol{\mathcal{T}}}$ and $\star$ are different, since H_F (in contrast to H) is not a solution of the Klein-Gordon equation. Since we do not want to deal with this and related issues, it is more convenient to work with fields, which are not required to satisfy the field equation, so-called *off-shell fields*. As shown in [DF03] a restriction to the space of solutions is always possible. See also [BD08] for an explicit construction of the maps involved.

II.4. The Renormalization Problem

Associativity of the time-ordered product makes it possible to speak of n-fold products

$$\begin{array}{rccc} \boldsymbol{\mathcal{T}}_n : & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes n} & \to & \mathcal{F}(\mathbb{M})[[\hbar]] \\ & F_1 \otimes \cdots \otimes F_n & \mapsto & F_1 \cdot_{\boldsymbol{\mathcal{T}}} \cdots \cdot_{\boldsymbol{\mathcal{T}}} F_n \,, \end{array}$$

which are well-defined if the supports of the functionals $F_1, \ldots, F_n \in \mathcal{F}(\mathbb{M})[[\hbar]]$ are pairwise disjoint,

$$\operatorname{supp}(F_i) \cap \operatorname{supp}(F_j) = \varnothing \quad \forall i, j \in \{1, \ldots, n\} \,,\, i \neq j \,.$$

The aim of perturbative QFT, however, is to define the terms of the $\boldsymbol{\mathcal{S}}$-matrix (II.15), which are time ordered products of the same interaction functional

$$\boldsymbol{\mathcal{S}}(F) = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\mathcal{T}}_n (F \otimes \cdots \otimes F) \,.$$

Hence one has to extend the definition of $\boldsymbol{\mathcal{T}}_n$ towards functionals with overlapping supports. In the present formalism such an extension is only possible for local functionals $F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$. One way to extend the maps $\boldsymbol{\mathcal{T}}_n$ to local functionals with overlapping supports is the inductive procedure of Epstein and Glaser [EG73]. See also [BF00a] and [BDF09] for modern generalizations of the original treatment. We want to remark that the recursive construction of Epstein-Glaser can be performed without reference to the star-product structure of pQFT, and hence is suitable also for a discussion of the renormalization problem on Euclidean space, [Kel09].

We will show in the last chapter that the inductive procedure of Epstein and Glaser can be solved, by implementing analytic regularization and minimal subtraction, which gives preferred extensions in each step of the induction. An analytic regularization, which has been proven to have especially nice properties when it comes to gauge theories, is dimensional regularization [BRS75]. And the next two chapters will be devoted to the implementation of this method into the framework of perturbative Algebraic Quantum Field Theory, although we will restrict ourselves to the study of scalar quantum field theory, only.

CHAPTER III

The Dimensionally Regularized Analytic Hadamard Function

Despite its rigorous formulation in [BG72a, tHV72] dimensional regularization has always been a somewhat shady or almost mystic concept, since the idea of a complex spacetime dimension is quite obscure from a conceptual point of view. Hence, the insight of Bollini and Giambiagi that dimensional regularization can be implemented in position space as a modification of the Bessel parameter in the two point function was, although passing largely unnoticed, an important one for the mathematical physicist interested in a conceptually clear formulation of perturbative quantum field theory [BG96]. We will follow the detailed argument of [BDF09, App. A], which, however, contains a small flaw, to show how a modification of the Bessel parameter leads to the notion of a "dimensionally regularized" two point function for arbitrary, integer spacetime dimensions. This two point function will then be used in the next chapter to define a dimensionally regularized time-ordered product and the corresponding $\boldsymbol{\mathcal{S}}$-matrix.

In a series of articles Hollands and Wald developed a description of the renormalization group flow on globally hyperbolic spacetimes by investigating the behavior of renormalizable theories under rescalings of the metric [HW01, HW02, HW03]. A major ingredient of the approach is their "scaling expansion" of time-ordered products around the thin diagonal. This expansion has the property that the scaling degree of the individual terms becomes smaller and smaller as one goes to higher and higher orders in the expansion. In scalar QFT on Minkowski spacetime such a scaling expansion can be introduced as an expansion in the mass parameter m^2 [Hol04, DF04]. This requires, however, that the two point function depends smoothly on m^2. The Wightman two point function, Δ_+, in even dimensions, exhibits a logarithmic dependence on the mass parameter, and thus cannot be used in this framework. However, we can take advantage of the freedom involved in the choice of a (Hadamard) two point function, briefly discussed in Sections II.2 and II.3, and add to Δ_+^m a smooth, symmetric, Lorentz invariant solution of the Klein-Gordon equation, which establishes a smooth dependence on m^2 for the sum. In flat spacetime the requirement of smoothness in mass fixes the two point function uniquely in odd dimensions, and up to a parameter μ of mass dimension one in even dimensions [DF04]. Hence, the algebra and the time-ordering are fixed (up to the parameter μ) by this smoothness condition.

III.1. Odd Dimensions

The Wightman distribution Δ_+^m on d-dimensional Minkowski spacetime can be expressed, for spacelike arguments $x \in \mathbb{M}$ (i.e., $x^2 < 0$ in our choice of the metric) in terms of modified Bessel functions (see e.g. [BS59]),

$$\Delta_+^m(x) = (2\pi)^{1-\nu}\, m^\nu \left(-x^2\right)^{-\frac{\nu}{2}} K_\nu(\sqrt{-m^2x^2})\,, \quad \nu = \tfrac{d}{2} - 1\,. \tag{III.1}$$

The right hand side of this equation is a function of one (real) variable x^2, which is parametrized by the complex order ν of the modified Bessel function. In the case where $\nu \in \frac{1}{2}\mathbb{N}_0$, and only then, this function has the physical interpretation of the Wightman two point function on a spacetime of dimension $d = 2\,(\nu + 1)$. The Wightman two point function, as well as its generalizations for arbitrary $\nu \in \mathbb{C}$ do not scale smoothly in m^2. However, starting from (III.1) we can construct a Hadamard two point function H_m, which scales smoothly in $m^2 \in \mathbb{R}$, by adding to Δ_+^m a smooth, Lorentz invariant solution F of the Klein-Gordon equation, $H_m = \Delta_+^m + F$. Any such Lorentz invariant solution F has the form

$$F(x) = \left(-x^2\right)^{-\frac{\nu}{2}} G_\nu(\sqrt{-m^2x^2})\,, \tag{III.2}$$

for spacelike arguments x, where G_ν is a solution of the modified Bessel equation of order ν. For non-integer order, $\nu \in \mathbb{C}\backslash\mathbb{N}_0$, (e.g., odd dimensions d) G_ν is a linear combination of the modified Bessel functions of first kind $\{I_\nu, I_{-\nu}\}$. For integer order, $n \in \mathbb{N}_0$, (i.e., even dimensions d) it is a linear combination of $\{I_n, K_n\}$, where K_n is the modified Bessel function of second kind, see Appendix A for details.

Requiring smoothness at $x = 0$ implies for arbitrary order $\nu \in \mathbb{C}$ that

$$F(x) \sim \left(-x^2\right)^{-\frac{\nu}{2}} I_\nu(\sqrt{-m^2x^2})\,. \tag{III.3}$$

For $\nu \in \mathbb{C}\backslash\mathbb{N}_0$ the modified Bessel functions are related by

$$K_\nu = \frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu} - I_\nu\right]\,, \tag{A.4}$$

hence, using this together with (III.3) we reach

$$\begin{aligned} H_m^\nu(x) &= \Delta_+^m(x) + F(x) \\ &= (2\pi)^{1-\nu} m^\nu \left(-x^2\right)^{-\frac{\nu}{2}} \left[K_\nu(\sqrt{-m^2x^2}) + a\cdot I_\nu(\sqrt{-m^2x^2})\right] \\ &= (2\pi)^{1-\nu} m^\nu \left(-x^2\right)^{-\frac{\nu}{2}} \left[\tfrac{\pi}{2\sin(\nu\pi)} I_{-\nu}(\sqrt{-m^2x^2}) + \left(a - \tfrac{\pi}{2\sin(\nu\pi)}\right) I_\nu(\sqrt{-m^2x^2})\right], \end{aligned} \tag{III.4}$$

where $a \in \mathbb{C}$ is a free parameter yet to be specified. In order to fix the parameter a we regard the scaling behavior in m^2 of the two terms in (III.4). The (modified) Bessel functions are of the form

$$I_\nu(y) = y^\nu f_\nu(y^2)\,.$$

with an entire analytic function f_ν. Thus, in the first term the factor $m^{-\nu}$ in $I_{-\nu}$

cancels with the prefactor m^ν leaving a smooth function of m^2 behind. In the second term we get an overall factor $m^{2\nu}$, which is not a smooth function of m^2, unless $\nu \in \mathbb{N}_0$ (which is excluded). Hence the free parameter a has to be chosen in such a way that the term proportional to I_ν cancels, $a = \frac{\pi}{2\sin(\nu\pi)}$. Summarizing the above, we have found, for non-integer order $\nu \in \mathbb{C}\backslash\mathbb{N}_0$, and in particular for odd dimensions d, a *unique* Hadamard two point function, which depends smoothly on m^2, it is given by

$$H^\nu_m(x) = \frac{(2\pi)^{2-\nu}\, m^\nu}{4\sin(\nu\pi)} \left(-x^2\right)^{-\frac{\nu}{2}} I_{-\nu}(\sqrt{-m^2x^2}), \quad x^2 < 0, \quad \nu \in \mathbb{C}\backslash\mathbb{N}_0. \tag{III.5}$$

III.2. Analytic continuation

We have already seen that $H^\nu_m(x)$ scales smoothly in the mass parameter m^2 for spacelike $x \in \mathbb{M}$, and want to discuss now the analytic properties of H^ν_m. It is a fundamental result of complex analysis that the analytic continuation of H^ν_m is unique in the region where it exists, so let $\mathcal{H}^\nu_m : \mathbb{M}^{\mathbb{C}} \to \mathbb{C}$ be this continuation, defined as a function on the complexified Minkowski space, $\mathbb{M}^{\mathbb{C}} := \mathbb{M} \otimes_{\mathbb{R}} \mathbb{C}$. The modified Bessel functions are defined for arbitrary complex arguments and writing $I_{-\nu}(\sqrt{m^2z^2}) = \left(\sqrt{m^2z^2}\right)^{-\nu} f_\nu(m^2z^2)$, with an entire analytic function f_ν, we see that the analytic continuation of (III.5) can be written as

$$\mathcal{H}^\nu_m(z) = (2\pi)^{1-\nu} \frac{\pi}{2\sin(\nu\pi)} \left(-z^2\right)^{-\nu} f_\nu(m^2\, z^2), \quad z \in \mathbb{M}^{\mathbb{C}}, \tag{III.6}$$

from which the smoothness in m^2 is obvious. Using the series representation of the modified Bessel function (A.3) we immediately get a series expansion of $\mathcal{H}^\nu_m$ in m^2,

$$\mathcal{H}^\nu_m(z) = \frac{\pi^{2-\nu}}{\sin(\nu\pi)} \left(-z^2\right)^{-\nu} \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(-\nu+s+1)} \left(\frac{-z^2}{4}\right)^s \left(m^2\right)^s. \tag{III.7}$$

Since this formula contains the power of a complex number, $\left(-z^2\right)^{-\nu} = e^{-\nu\,\mathrm{Log}(-z^2)}$, we have to choose a branch of the logarithm in order to make $\mathcal{H}^\nu_m$ single-valued. As we shall see below, choosing the principal branch, Log, of the complex logarithm,

$$\begin{array}{rccl} \mathrm{Log}: & \mathbb{C}\backslash\{0\} & \to & \mathbb{R} \oplus i\,(-\pi,\pi] \\ & r\,e^{i\vartheta} & \mapsto & \mathrm{Log}(r\,e^{i\vartheta}) = \ln(r) + i\vartheta\,, \quad \vartheta \in (-\pi,\pi] \cong \mathbb{R}/2\pi\,. \end{array}$$

gives a single-valued function $\mathcal{H}^\nu_m$ with the analytic properties of the Wightman function. The principal branch has a discontinuity along the negative real axis ($\vartheta = \pi$), resulting in the fact that the function on the right hand side of (III.7), regarded as a function of one complex variable

$$z^2 = (x-iy)^2 = x^2 - y^2 - 2i\,xy\,,$$

i.e., $\mathcal{H}^\nu_m(z) = \hbar^\nu_m(z^2)$, is analytic in the cut plane $\mathbb{C}\backslash\mathbb{R}^0_+$, see Figure 1(a). The condition $z^2 \notin \mathbb{R}^0_+$ is fulfilled if the complex four vector[1] $z \in \mathbb{M}^{\mathbb{C}}$ lies in the so-called *future* or *past tube*

$$\mathbb{T}^\pm = \left\{x - iy : \, x \in \mathbb{M}, \, y \in \mathcal{V}^\pm\right\} \subset \mathbb{M}^{\mathbb{C}} \,,$$

where, as before, $\mathcal{V}^\pm = \left\{y \in \mathbb{M} : \, \pm y^0 > \|\boldsymbol{y}\|\right\}$ denote the open forward and backward light cones. Observe that $\mathrm{Im}(z^2) = 0$ implies $z^2 < 0$, if $z = x - iy \in \mathbb{T}^\pm$:

$$x^0 y^0 = \boldsymbol{x} \cdot \boldsymbol{y} \leqslant \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| < \pm \, \|\boldsymbol{x}\| \, y^0 \,,$$

hence $x^2 < 0$ and $z^2 < 0$. We reach the conclusion that the function $\mathcal{H}^\nu_m : \mathbb{M}^{\mathbb{C}} \to \mathbb{C}$, given by (III.7), is analytic in the future and past tubes $\mathbb{T}^\pm$.[2] Furthermore we see from the explicit formula (III.7) that the growth of $|\mathcal{H}^\nu_m(x - iy)|$ is bounded by an inverse polynomial as y approaches zero from within the forward or backward light cone. Hence, by [Hör03, Thm. 3.1.15], the boundary values of $\mathcal{H}^\nu_m$ from inside the future and past tube exist as distributions in $\mathscr{D}'(\mathbb{M})$.

Hence we can define the Hadamard distribution $H^\nu_m \in \mathscr{D}'(\mathbb{M})$ to be the boundary value of $\mathcal{H}^\nu_m$ as the real subspace $\mathbb{M} \subset \mathbb{M}^{\mathbb{C}}$ is approached from the future tube, $y = \left(\frac{\varepsilon}{2}, \boldsymbol{0}\right) \in \mathcal{V}^+$,

$$\langle H^\nu_m, f\rangle := \lim_{\varepsilon \to 0^+} \int_{\mathbb{M}} dx \, f(x) \, \hbar^\nu_m(x^2 - ix^0\varepsilon) \,, \quad \nu \in \mathbb{C}\backslash\mathbb{N}_0 \,. \tag{III.8}$$

The wave front set of $H^\nu_m \in \mathscr{D}'(\mathbb{M})$ lies within the dual cone of $\mathcal{V}^+$, cf. [Hör03, Thm. 8.1.6], which is the closed cone $\overline{\mathcal{V}^+}$, and hence we have

$$\mathrm{WF}(H^\nu_m) \subset \mathbb{M} \times \left(\overline{\mathcal{V}^+} \backslash \{0\}\right) \,, \qquad \nu \in \mathbb{C}\backslash\mathbb{N}_0 \,.$$

Thus we have found a parametrized Hadamard distribution, which can be used to define a star product of functionals as described in the previous chapter, cf. Proposition II.4.

The corresponding Feynman fundamental solution can be defined in the same way, it is the time-ordered version of H^ν_m,

$$H_F^{m,\nu}(x) := \theta(x^0) H^\nu_m(x) + \theta(-x^0) H^\nu_m(-x) \,,$$

where θ is the Heaviside step function. Rephrased in the language of complex analysis, for $x^0 > 0$ the Feynman fundamental solution is the boundary value of the analytic Hadamard function $\mathcal{H}^\nu_m : \mathbb{M}^{\mathbb{C}} \to \mathbb{C}$ from inside the future tube, i.e. $H_F^{m,\nu}(x) = \lim_{\varepsilon \to 0^+} \hbar^\nu_m(x^2 - ix^0\varepsilon)$, and for $x^0 < 0$ it is the boundary value of the same analytic function from inside the past tube, $H_F^{m,\nu}(x) = \lim_{\varepsilon \to 0^+} \hbar^\nu_m(x^2 + ix^0\varepsilon)$.

[1] By abuse of terminology we will use the term (complex) "four vector" for elements of (complex) Minkowski space of arbitrary dimension $d \geqslant 2$.

[2] It is well-known that the analyticity domain of the (Hadamard) two point function is bigger than just the future and past tubes. By Lorentz invariance, the extended tube, and by permutation symmetry even the so-called *permuted extended tubes* are part of the analyticity domain of the corresponding analytic n point functions [HW57]. However, for our purposes it will suffice to consider the subsets $\mathbb{T}^\pm$ of the analyticity domain of $\mathcal{H}^\nu_m$.

Hence in both cases it results

$$\forall f \in \mathscr{D}(\mathbb{M}\setminus\{0\}) : \left\langle H_F^{m,\nu}, f\right\rangle = \lim_{\varepsilon\to 0^+} \int dx\, f(x)\, \mathcal{H}_m^\nu(x^2 - i\varepsilon)\,. \tag{III.9}$$

The distribution $H_F^{m,\nu}$ is given as the boundary value of the analytic two point function $\mathcal{H}_m^\nu$ from two disjoint areas of its analyticity domain in the cases $x^0 > 0$ and $x^0 < 0$. For $x^0 = 0$ and $\boldsymbol{x} \neq 0$ we are in the analyticity domain of $\mathcal{H}_m^\nu$. For $x = 0$ a definition of $H_F^{m,\nu}$ as boundary value of $\mathcal{H}_m^\nu$ is not possible, hence we have defined the Feynman propagator as a distribution $H_F^{m,\nu} \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$. Observe, however, that $H_F^{m,\nu}$ has a unique extension to $\mathscr{D}'(\mathbb{M})$ with the same scaling degree. The scaling degree of $H_F^{m,\nu}$ can be read off directly from (III.7),

$$\mathrm{sd}(H_F^{m,\nu}) = 2\nu\,. \tag{III.10}$$

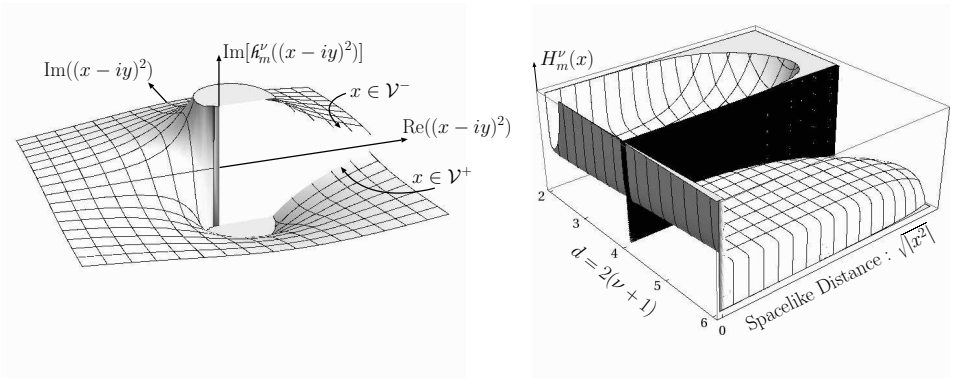
Hence for half-integer Bessel order $\nu = \frac{d}{2} - 1$, we have that $\mathrm{sd}(H_F^{m,d}) = d - 2$ and hence $H_F^{m,d} \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$ has a unique extension $\dot{H}_F^{m,d} \in \mathscr{D}'(\mathbb{M})$. Observe that the terms proportional to $\left(m^2\right)^s$ in the expansion (III.7) are homogeneous of degree $D = 2\,(s - \nu)$, cf. Definition I.9. Hence these terms have unique homogeneous extensions for $2\,(\mathrm{Re}(\nu) - s) \in \mathbb{N}_0 + d$, cf. Remark I.10 and Theorem I.12, and in particular for $\nu \notin \frac{1}{2}\mathbb{N}_0$. This observation is the basis for the discussion undertaken in the next section and the following chapter.

Hence we have found a unique analytic Hadamard function $\mathcal{H}_m^\nu$, which depends smoothly on m^2 for Bessel order $\nu \in \mathbb{C}\setminus\mathbb{N}_0$, and hence in particular for odd dimensions. Before turning to the more intricate case of even dimensions, let us briefly discuss the properties of the analytic Hadamard function by visualizing $\mathcal{H}_m^\nu$ in the two pictures of Fig. 1. In particular observe that the Hadamard boundary value $H_m^\nu(x)$ grows exponentially in spacelike directions, a fact which makes a direct comparison in terms of the full time-ordered products of our formulation to the well-established formulation of dimensional regularization in momentum space difficult, if not impossible. A comparison of the counterterms, as described briefly in Section I.4, however, should be possible.

III.3. Even dimensions

The fact that (III.5) or respectively (III.7) fixes the analytic two point function uniquely for any complex parameter $\nu \in \mathbb{C}\setminus\mathbb{N}_0$ suggests to construct the corresponding two point function for $\nu \in \mathbb{N}_0$, i.e., for even dimensions $d = 2\,(\nu + 1)$, by a limiting procedure. We introduce a regularization parameter $\zeta \in \mathbb{C}$ and set

$$\nu \mapsto \frac{(d + \zeta)}{2} - 1\,, \quad d \in 2\mathbb{N}, \quad 0 < |\zeta| < 2\,,$$



(a) (b)

Figure 1: Generalizations of the Hadamard distribution.
(a) to complex arguments. $\hbar((x-iy)^2)$ is analytic in the cut plane $\mathbb{C}\backslash\mathbb{R}_+^0$, which implies that $\mathcal{H}_m^\nu(x-iy)$ is analytic in the future and past tube $\mathbb{T}^\pm$. The values of $H_m^\nu(x)$ for timelike vectors x are the boundary values of this function, as y approaches zero from inside the forward light cone. For the plot $\nu = \frac{1}{2}$ and $m^2 = 1$ was chosen.
(b) to "complex dimensions". The plot shows the qualitative behavior of the Hadamard function $H_m^\nu(x)$ for spacelike distances, $x^2 < 0$, in dependence of the parameter $\nu \in \mathbb{C}$. One sees the (simple) poles at integer values $\nu = \frac{d}{2} - 1 \in \mathbb{N}_0$, cf. (III.7), and that the local singularity structure at $x = 0$ does not change as ν varies. Observe also the (alternating) behavior for large spacelike distances, the absolute value of H_m^ν grows exponentially, hence it cannot be the kernel of a Schwartz distribution, cf. also (III.7). In the plot ν varies over the reals from 0 to 2. We have chosen $m^2 = 1$ for the plot.

in (III.5). The resulting parametrization of $\mathcal{H}_m^\nu$ we denote by

(III.11)

$$\widetilde{\mathcal{H}}_m^{\mu,\zeta}(z) := \frac{(2\pi)^{1-\frac{(d+\zeta)}{2}}\,\mu^{-\zeta} m^{\frac{(d+\zeta)}{2}-1}}{4\sin(\left(\frac{(d+\zeta)}{2}-1\right)\pi)}\left(-z^2\right)^{\frac{2-(d+\zeta)}{4}} I_{1-\frac{(d+\zeta)}{2}}(\sqrt{-m^2z^2})$$
$$= (-1)^{\frac{d}{2}-1}\left(\frac{1}{2\pi}\right)^{\frac{d}{2}} m^{d-2}\frac{\pi}{2\sin(\zeta\frac{\pi}{2})}\left(\frac{m}{\mu}\right)^{\zeta}\left(\sqrt{-m^2z^2}\right)^{1-\frac{(d+\zeta)}{2}} I_{1-\frac{(d+\zeta)}{2}}(\sqrt{-m^2z^2})\,.$$

We had to introduce a parameter μ of mass dimension one, in order to get the right mass dimension for the two point function, $\mathrm{md}(\widetilde{\mathcal{H}}_m^{\mu,\zeta}) = d-2$. Apart from the dependence on the free parameter μ, $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ has for $0 < |\zeta| < 2$ the same properties as $\mathcal{H}_m^\nu$ for $|\nu|$ between two integers. In particular $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ is analytic in future and past tube $\mathbb{T}^\pm$ and depends smoothly on the mass parameter m^2.

In order to get an expression for even dimensions, we want to perform the limit $\zeta \to 0$. This limit can not be performed directly, since $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ diverges as the parameter ζ tends to zero. The aim of this section is to construct a dimensionally regularized analytic two point function $\mathcal{H}_m^{\mu,\zeta}$ (without the tilde), which is a smooth

function of m^2, and solves the Klein-Gordon equation in the limit $\zeta \to 0$. This is done by exploiting the freedom in the choice of the Hadamard distribution H mentioned earlier. We will see that $\mathcal{H}_m^{\mu,\zeta}$ differs from $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ by an analytic, Lorentz invariant function, which is a smooth function of m^2 and solves the Klein-Gordon equation "in $d+\zeta$ dimensions". The exact meaning of this assertion will become clear in the construction to be carried out now.

For $0 < |\zeta| < 2$ we can express $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ in terms of the modified Bessel functions, cf. (A.4),

(III.12)

$$\begin{aligned}\widetilde{\mathcal{H}}_m^{\mu,\zeta}(z) &= (2\pi)^{-\frac{(d+\zeta)}{2}}\,\mu^{-\zeta} m^{\frac{(d+\zeta)}{2}-1}\left(-z^2\right)^{\frac{2-(d+\zeta)}{4}}\cdot \\ &\qquad\cdot\left[K_{\frac{(d+\zeta)}{2}-1}(\sqrt{-m^2z^2}) + \frac{\pi}{2\sin(\left(\frac{(d+\zeta)}{2}-1\right)\pi)} I_{\frac{(d+\zeta)}{2}-1}(\sqrt{-m^2z^2})\right] \\ &= \mathcal{W}_m^{\mu,\zeta}(z) + \widetilde{\mathcal{B}}_m^{\mu,\zeta}(z)\,,\end{aligned}$$

where we set $\mathcal{W}_m^{\mu,\zeta}$ for the term proportional to $K_{\frac{(d+\zeta)}{2}-1}$ and $\widetilde{\mathcal{B}}_m^{\mu,\zeta}$ for the $I_{\frac{(d+\zeta)}{2}-1}$-term. The first term in (III.12), $\mathcal{W}_m^{\mu,\zeta}(z)$, is well defined for $\zeta = 0$, it is (the analytic continuation of) the Wightman function (III.1) in d dimensions. The second term,

$$\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z) = \\ (-1)^{\left(\frac{d}{2}-1\right)}\,(2\pi)^{-\frac{(d+\zeta)}{2}}\,\mu^{-\zeta} m^{\frac{(d+\zeta)}{2}-1}\left(-z^2\right)^{\frac{2-(d+\zeta)}{4}}\cdot\frac{\pi}{2\sin(\zeta\frac{\pi}{2})} I_{\frac{(d+\zeta)}{2}-1}(\sqrt{-m^2z^2})\,,$$

is a meromorphic function in ζ. More precisely, for any fixed $d \in 2\mathbb{N}$ the map $\zeta \mapsto \widetilde{\mathcal{B}}_m^{\mu,\zeta}$ is analytic in the punctured disk $\{0 < |\zeta| < 2\}$ and has a simple pole at $\zeta = 0$. That is, $\zeta\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z)$ has a removable singularity at $\zeta = 0$, or equivalently $\lim_{\zeta\to 0}\zeta^2\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z) = 0$, cf. [Con78, Def. 1.6 and Thm. 1.2]. Using the fact that $\nu \mapsto I_\nu(y)$ is an entire analytic function [Wat22, § 3·13], [WW02], and abbreviating

$$f(\zeta) = (2\pi)^{-\frac{(d+\zeta)}{2}}\,\mu^{-\zeta} m^{\frac{(d+\zeta)}{2}-1}\left(-z^2\right)^{\frac{2-(d+\zeta)}{4}} I_{\frac{(d+\zeta)}{2}-1}(\sqrt{-m^2z^2})\,,$$

we compute for $d \in 2\mathbb{N}$ (using l'Hôspital's rule),

$$\lim_{\zeta\to 0}\frac{\pi}{2\sin(\left(\frac{(d+\zeta)}{2}-1\right)\pi)}\left[\zeta^2 f(\zeta)\right] = \left\{\frac{1}{\cos(\left(\frac{(d+\zeta)}{2}-1\right)\pi)}\left[\zeta^2 f'(\zeta) + 2\zeta f(\zeta)\right]\right\}_{\zeta=0} = 0\,.$$

Given these properties, $\zeta \mapsto \widetilde{\mathcal{B}}_m^{\mu,\zeta}(z)$ can be expanded in a Laurent series [Con78, 1.11],

$$\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z) = \sum_{n=-1}^{\infty} a_n(x)\,\zeta^n = \tfrac{1}{\zeta}\mathrm{Res}(\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z), \zeta = 0) + \mathcal{G}_m^{\mu,\zeta}(z)\,, \tag{III.13}$$

where $\zeta \to \mathcal{G}_m^{\mu,\zeta}(x)$ is analytic in the full disk $\{|\zeta| < 2\}$. For finite ζ the function $\widetilde{\mathcal{B}}_m^{\mu,\zeta}$ is (the analytic continuation of) a smooth, Lorentz invariant solution of the Klein-Gordon equation "in $d+\zeta$ dimensions", i.e., $\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z) \sim \left(-z^2\right)^{-\frac{\nu}{2}} I_\nu(\sqrt{-m^2z^2})$, where I_ν is the modified Bessel function of first kind of order $\nu = \frac{d+\zeta}{2} - 1$, cf. (III.2).

Furthermore $\widetilde{\mathcal{H}}_m^{\mu,\zeta}(z) = \mathcal{W}_m^{\mu,\zeta}(z) + \widetilde{\mathcal{B}}_m^{\mu,\zeta}(z)$ is a smooth function of m^2. It is the whole purpose of this derivation, to maintain as many of these properties as possible as ζ tends to zero. Subtracting just the pole part of (III.13), as suggested in the original treatment [BDF09, App. A], although preserving smoothness in z and m^2, does not lead to a solution of the Klein-Gordon equation. The reason for this is that the residue does not solve the Klein-Gordon equation in $d+\zeta$, but in d dimensions,

$$a_{-1}(x) = \mathrm{Res}(\widetilde{\mathcal{B}}_m^{\mu,\zeta}(z), \zeta = 0) = \lim_{\zeta\to 0} \zeta \widetilde{\mathcal{B}}_m^{\mu,\zeta}(z)$$

$$= (-1)^{\left(\frac{d}{2}-1\right)} (2\pi)^{-\frac{d}{2}} m^{\frac{d}{2}-1} \left(-z^2\right)^{\frac{2-d}{4}} I_{\frac{d}{2}-1}(\sqrt{-m^2z^2}) . \tag{III.14}$$

In order to maintain also the solution property in the limit $\zeta \to 0$, we need to subtract from $\widetilde{\mathcal{B}}_m^{\mu,\zeta}$ a smooth, Lorentz invariant solution of the Klein-Gordon equation "in $d+\zeta$ dimensions". Furthermore, in order not to spoil the smoothness of $\widetilde{\mathcal{H}}_m^{\mu,\zeta}(z)$ in the mass parameter m^2, the subtraction has to be a smooth function of m^2. We conclude that the subtraction is a scalar multiple of

$$\mathcal{S}_m^{\zeta}(z) = (-1)^{\left(\frac{d}{2}-1\right)} m^{d-2} \left(\sqrt{-m^2z^2}\right)^{1-\frac{d+\zeta}{2}} I_{\frac{d+\zeta}{2}-1}(\sqrt{-m^2z^2}) . \tag{III.15}$$

Observe that $\mathcal{S}_m^{\zeta}$ is an entire analytic function of z^2, since $I_\nu(z) = z^\nu f_\nu(z^2)$ with f_v entire analytic. A possible subtraction is given by

$$\mathcal{B}_m^{\mu,\zeta}(z) = \widetilde{\mathcal{B}}_m^{\mu,\zeta}(z) - \alpha \tfrac{\pi}{2\sin(\zeta\frac{\pi}{2})} \mathcal{S}_m^{\zeta}(z)$$

$$= \left[(2\pi)^{-\frac{\zeta}{2}} \left(\tfrac{m}{\mu}\right)^{\zeta} \tfrac{\pi}{2\sin(\zeta\frac{\pi}{2})} - \tfrac{\pi}{2\sin(\zeta\frac{\pi}{2})}\right] \cdot$$

$$\cdot (-1)^{\left(\frac{d}{2}-1\right)} (2\pi)^{-\frac{d}{2}} m^{\frac{d}{2}-1} \left(-z^2\right)^{\frac{2-d}{4}} \left(\sqrt{-m^2z^2}\right)^{-\frac{\zeta}{2}} I_{\frac{d+\zeta}{2}-1}(\sqrt{-m^2z^2}) ,$$

where we had to set the factor in front of the subtraction to be $\alpha = (2\pi)^{-\frac{d}{2}}$ in order to get a well-defined limit. Adding this to the analytic Wightman function $\mathcal{W}_m^{\mu,\zeta}$ defines the *dimensionally regularized analytic Hadamard function*

$$\mathcal{H}_m^{\mu,\zeta} := \mathcal{W}_m^{\mu,\zeta} + \mathcal{B}_m^{\mu,\zeta} \equiv \widetilde{\mathcal{H}}_m^{\mu,\zeta} - (2\pi)^{-\frac{d}{2}} \tfrac{\pi}{2\sin(\zeta\frac{\pi}{2})} \mathcal{S}_m^{\zeta}(z) ,$$

which has the explicit form(s)

$$\mathcal{H}_m^{\mu,\zeta}(z) = (2\pi)^{-\frac{d}{2}} m^{d-2} \left(m\sqrt{-z^2}\right)^{1-\frac{d+\zeta}{2}} \cdot$$

$$\cdot (-1)^{\frac{d}{2}-1} \frac{\pi}{2\sin(\zeta\frac{\pi}{2})} \left[\left(\frac{m}{\mu}\right)^{\zeta} I_{1-\frac{d+\zeta}{2}}(m\sqrt{-z^2}) - I_{\frac{d+\zeta}{2}-1}(m\sqrt{-z^2})\right] \tag{III.16}$$

$$= (2\pi)^{-\frac{d}{2}} m^{d-2} \left(m\sqrt{-z^2}\right)^{1-\frac{d+\zeta}{2}} \cdot$$

$$\cdot \left[\left(\frac{m}{\mu}\right)^{\zeta} K_{\frac{d+\zeta}{2}-1} + (-1)^{\frac{d}{2}-1} \frac{\pi}{2\sin(\zeta\frac{\pi}{2})} \left\{\left(\frac{m}{\mu}\right)^{\zeta} - 1\right\} I_{\frac{d+\zeta}{2}-1}\right] . \tag{III.17}$$

From the representation (III.16) we can directly derive a series expansion of

the dimensionally regularized analytic two point function in powers of m^2 by inserting the power series expansion (A.3) for the modified Bessel functions, $d \in 2\mathbb{N}$,

(III.18)

$$\mathcal{H}_m^{\mu,\zeta}(z) = (-1)^{\frac{d-2}{2}} (2\pi)^{-\frac{d}{2}} 2^{\frac{2-(d+\zeta)}{2}} \frac{\pi}{2\sin(\zeta\frac{\pi}{2})} \cdot \\ \cdot \left[\sum_{s=0}^{\infty} \left\{ \left(\frac{2}{\mu\sqrt{-z^2}} \right)^{\zeta} \frac{1}{s!\,\Gamma(-\frac{\zeta}{2} - \frac{d-4}{2} + s)} - \frac{\theta(s - \frac{d-3}{2})}{\left(s - \frac{d-2}{2}\right)!\,\Gamma(\frac{\zeta}{2} + s + 1)} \right\} \cdot \right. \\ \left. \cdot \left(\frac{\sqrt{-z^2}}{2} \right)^{2s-(d-2)} \left(m^2\right)^s \right].$$

Observe that the second term in curly brackets is a smooth function of z^2, since it only contributes for $s > \frac{d-3}{2}$, i.e. $2s - (d-2) \geqslant 0$.

Since $\mathcal{H}_m^{\mu,\zeta}$ differs from $\widetilde{\mathcal{H}}_m^{\mu,\zeta}$ by an entire analytic function, cf. (III.15), it is also analytic in the future and past tube. Furthermore $\zeta \mapsto \mathcal{H}_m^{\mu,\zeta}(z)$ is analytic in $\{|\zeta| < 2\}$ by construction, however, for completeness we want to give an explicit argument here. Regard (III.17), $K_{\frac{d+\zeta}{2}-1}$ and $I_{\frac{d+\zeta}{2}-1}$ are entire analytic functions of ζ by general properties of the (modified) Bessel functions (see e.g. [AS70, Sec. 9.6]). The analyticity domain of $\zeta \mapsto \mathcal{H}_m^{\mu,\zeta}(z)$ is thus determined by the factor $\alpha(\zeta) = \frac{\pi}{2\sin(\zeta\frac{\pi}{2})} \left\{ \left(\frac{m}{\mu}\right)^{\zeta} - 1 \right\}$, which obviously is analytic in $\{0 < |\zeta| < 2\}$. One can show differentiability in $\zeta = 0$ by computing the differential quotient directly,

$$\lim_{\zeta \to 0} \frac{1}{\zeta} \left[\alpha(\zeta) - \alpha(0)\right] = \frac{1}{2} \left[\ln\left(\frac{m}{\mu}\right) \right]^2 .$$

Hence $(z, \zeta) \mapsto \mathcal{H}_m^{\mu,\zeta}(z)$ is analytic for $z \in \mathbb{T}^{\pm}$ and $|\zeta| < 2$ as asserted above. The limit $\zeta \to 0$ of $\mathcal{H}_m^{\mu,\zeta}$ exists and defines the analytic Hadamard function in even dimensions

$$\mathcal{H}_m^{\mu}(z) := \lim_{\zeta \to 0} \mathcal{H}_m^{\mu,\zeta}(z), \quad d \in 2\mathbb{N},$$

it can be read off directly from (III.17), $d \in 2\mathbb{N}$,

$$\mathcal{H}_m^{\mu}(z) = \frac{m^{\frac{d}{2}-1} \left(-z^2\right)^{\frac{2-d}{4}}}{(2\pi)^{\frac{d}{2}}} \left[K_{\frac{d}{2}-1}(\sqrt{-m^2z^2}) + \frac{(-1)^{\frac{d}{2}}}{2} \ln\left(\frac{\mu^2}{m^2}\right) I_{\frac{d}{2}-1}(\sqrt{-m^2z^2}) \right].$$

By construction $\mathcal{H}_m^{\mu}$, as well as its regularization $\mathcal{H}_m^{\mu,\zeta}$, $0 \leqslant |\zeta| < 2$, is a smooth function of m^2. The boundary values of $\mathcal{H}_m^{\mu}(z) = \mathcal{h}_m^{\mu}(z^2)$ define the Hadamard distribution and Feynman fundamental solution,

(III.19) $$\left\langle H_m^{\mu}, f \right\rangle = \lim_{\varepsilon \to 0^+} \int dx\, f(x)\, \mathcal{h}_m^{\mu}(x^2 - ix^0\varepsilon), \qquad H_m^{\mu} \in \mathscr{D}'(\mathbb{M}),$$

and

(III.20) $$\left\langle H_F^{m,\mu}, g \right\rangle = \lim_{\varepsilon \to 0^+} \int dx\, f(x)\, \mathcal{h}_m^{\mu}(x^2 - i\varepsilon), \quad H_F^{m,\mu} \in \mathscr{D}'(\mathbb{M} \setminus \{0\}).$$

Within the analyticity domain of $\mathcal{H}_m^{\mu,\zeta}(z)$ the limit $\zeta \to 0$ can be exchanged with taking boundary values, resulting in regularizations of these distributions. What seems artificial at this stage, since H_m^μ needs no regularization at all and $H_F^{m,\mu}$ already has a unique extension with the same scaling degree by Theorem I.6, will prove to be useful for the regularization of higher time-ordered products in the next chapter. Hence we define the dimensionally regularized Hadamard distribution, as well as the corresponding Feynman fundamental solution as boundary values of $\mathcal{H}_m^{\mu,\zeta}(z) = \hbar_m^{\mu,\zeta}(z^2)$,

$$\left\langle H_m^{\mu,\zeta}, f \right\rangle = \lim_{\varepsilon\to 0^+} \int dx\, f(x)\, \hbar_m^{\mu,\zeta}(x^2 - ix^0\varepsilon)\,, \quad H_m^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M})\,; \tag{III.21}$$

and

$$\left\langle H_F^{m,\mu,\zeta}, g \right\rangle = \lim_{\varepsilon\to 0^+} \int dx\, g(x)\, \hbar_m^{\mu,\zeta}(x^2 - i\varepsilon)\,, \quad H_F^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}\setminus\{0\})\,. \tag{III.22}$$

The scaling degree of $H_F^{m,\mu,\zeta}$ can be read off directly from (III.18),

$$\mathrm{sd}(H_F^{m,\mu,\zeta}) = d + \mathrm{Re}(\zeta) - 2\,. \tag{III.23}$$

It is smaller than d for $\mathrm{Re}(\zeta) < 2$ and we infer again from Theorem I.6 that $H_F^{m,\mu,\zeta}$ has a unique extension in this case. For $\zeta \notin \mathbb{R}$, observe that the singular term of the expansion (III.18) is homogeneous of degree $D = 4 - (d + \zeta)$ and hence we get a unique extension by means of Theorem I.12.

This unique extension $\dot{H}_F^{m,\mu,\zeta} \in \mathscr{D}'(\mathbb{M})$ is a regularization of $H_F^{m,\mu} \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$ in the sense of Definition I.16, since by what was said above we have

$$\forall f \in \mathscr{D}(\mathbb{M}\setminus\{0\}): \quad \lim_{\zeta\to 0} \left\langle \dot{H}_F^{m,\mu,\zeta}, f \right\rangle = \lim_{\zeta\to 0} \left\langle H_F^{m,\mu,\zeta}, f \right\rangle = \left\langle H_F^{m,\mu}, f \right\rangle\,.$$

And since the distribution on the right hand side also has a unique extension $\dot{H}_F^{m,\mu}$, we even have

$$\forall f \in \mathscr{D}(\mathbb{M}): \quad \lim_{\zeta\to 0} \left\langle \dot{H}_F^{m,\mu,\zeta}, f \right\rangle = \left\langle \dot{H}_F^{m,\mu}, f \right\rangle\,. \tag{III.24}$$

Observe that $H_F^{m,\mu,\zeta}$ has a unique extension by means of homogeneity, and not, as $H_F^{m,\mu}$, by a pure scaling degree argument. We will see in the next chapter how this leads to an analytic regularization of arbitrary time-ordered products.

CHAPTER IV

Dimensional Regularization in Position Space

The work of Bollini and Giambiagi on dimensional regularization in position space, mentioned previously, focused on the Fourier transform of this regularization method between momentum space and position space [BG96]. In contrast to their work, our analysis will be formulated exclusively in position space, and a direct translation to momentum space will generally not be possible. However, an advantage of the method presented here is that all expressions will depend smoothly on the mass parameter m^2, which makes it possible to apply the covariant framework of [Hol04], see also [DF04]. We will analyze the graph structure of the time-ordered product in the first section and use this in the second section to construct for any graph a unique dimensionally regularized amplitude. We will define the dimensionally regularized time-ordered product and the corresponding scattering matrix. This dimensionally regularized $\boldsymbol{\mathcal{S}}$-matrix will then be used as an example in the solution of the Epstein-Glaser recursion, and to establish the relation to Connes-Kreimer theory of renormalization in the last chapter.

IV.1. Graph structure of the Time-Ordered Product

Before turning to the time-ordered product we want to introduce very briefly the basic notions connected with the definition of a graph. An oriented graph Γ is a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$ together with maps

$$\mathfrak{s}, \mathfrak{t} : E(\Gamma) \to V(\Gamma),$$

which give source and target of an edge $e \in E(\Gamma)$, respectively. Furthermore we give Γ an *orientation* by assigning to any pair $(e, v) \in E(\Gamma) \times V(\Gamma)$ the value

$$(e : v) := \begin{cases} +1 & \text{if } \mathfrak{t}(e) = v \\ -1 & \text{if } \mathfrak{s}(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

We call e *adjacent to* v if $(e : v) \neq 0$. A graph for which the orientation map $(e, v) \mapsto (e : v)$ is multi-valued we call *tadpole*. However, the definition of the time-ordered product in (II.23) implies that there are no tadpoles occurring in its graph expansion, i.e., Equation (IV.2) below; see the proof of Proposition II.7 and the preceding Remark II.8. In particular this implies that we will only need to consider graphs for which

$$\forall e \in E(\Gamma) : \quad \mathfrak{s}(e) \neq \mathfrak{t}(e). \tag{IV.1}$$

Furthermore we remark that for scalar QFT the orientation of a given (Feynman-) graph can be chosen freely, one speaks of an *unoriented graph* in this case. Let $\mathcal{G}$ denote the set of all unoriented graphs Γ for which the orientation map $(e, v) \mapsto (e : v)$ is single-valued, i.e. (IV.1) holds.

Consider the n-fold time-ordered product introduced in Section II.4 as a map

$$\begin{array}{rccl} \boldsymbol{\mathcal{T}}_n : & \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} & \to & \mathcal{F}(\mathbb{M})[[\hbar]] \\ & F_1 \otimes \cdots \otimes F_n & \mapsto & F_1 \cdot_{\boldsymbol{\mathcal{T}}} \cdots \cdot_{\boldsymbol{\mathcal{T}}} F_n \,. \end{array}$$

It was defined with the help of a second order, symmetric functional differential operator (II.16), which can be written as

$$\Gamma_{H_F}(F \cdot G) = \left\langle H_F, F^{(1)} \otimes G^{(1)} \right\rangle ,$$

due to the absence of tadpoles. On the level of graphs this operation is represented by drawing one line, H_F, between the interaction vertices, F and G. We can split the time-ordered product, $\boldsymbol{\mathcal{T}}_n$, in a similar way into two parts [Fre09]; a differential operator,

$$\boldsymbol{\delta}^\alpha : F_1 \otimes \cdots \otimes F_n \mapsto F_1^{(\alpha_1)} \otimes \cdots \otimes F_n^{(\alpha_n)} , \quad \alpha \in \mathbb{N}^n ,$$

where α_i is the number of lines adjacent to the vertex with interaction F_i,[1] and a distribution,

$$S_\Gamma : F_1^{(\alpha_1)} \otimes \cdots \otimes F_n^{(\alpha_n)} \mapsto \left\langle S_\Gamma, F_1^{(\alpha_1)} \otimes \cdots \otimes F_n^{(\alpha_n)} \right\rangle ,$$

containing the information as to which vertices of the graph Γ are connected by a line. The n-fold time-ordered product can then be written as

$$F_1 \cdot_{\boldsymbol{\mathcal{T}}} \cdots \cdot_{\boldsymbol{\mathcal{T}}} F_n = \sum_{\alpha \in \mathbb{N}^n} \sum_{\Gamma \in \mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \left\langle S_\Gamma, \boldsymbol{\delta}^\alpha \left(F_1 \otimes \cdots \otimes F_n\right) \right\rangle , \tag{IV.2}$$

where $\mathcal{G}_\alpha$ is the set of (non-tadpole) graphs with $n = \dim(\alpha)$ vertices and $\frac{|\alpha|}{2}$ lines such that there are α_i lines joining at vertex i. $\mathrm{Sym}(\Gamma) \in \mathbb{N}$ is the symmetry factor of the graph Γ to be defined below. Observe that in the case of polynomial interactions, e.g., $F_i(\varphi) = \left\langle \varphi^{k_i}, f \right\rangle$, $i \in \{1, \ldots, n\}$, and fixed $n \in \mathbb{N}$ only finitely many of the functional derivatives $\boldsymbol{\delta}^\alpha$ give non-vanishing contributions to (IV.2). For arbitrary interactions the limiting parameter is the order in $\hbar$ up to which one wants to compute. One can generate a dependence on the loop number $\ell(\Gamma)$ for connected graphs Γ by absorbing one factor $\hbar$ in each interaction functional,

$$\ell(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1 ,$$

a well known identity from graph theory [GY03b].

EXAMPLE IV.1. As an example regard the threefold time-ordered product of (not necessarily local) functionals $F, G, H \in \mathcal{F}(\mathbb{M})[[\hbar]]$. Using Cauchy's product formula and the Leibniz rule one derives from the power series expansion (II.23)

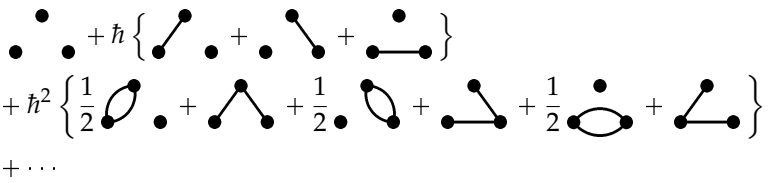
[1] Since we do not want to restrict ourselves to any particular type of interaction, the number of edges adjacent to a given vertex is not fixed a priori.

the following expression for the threefold time-ordered product

$$F \cdot_{\boldsymbol{\mathcal{T}}} G \cdot_{\boldsymbol{\mathcal{T}}} H = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} \binom{m}{k} \left\langle F^{(k+m-k)} G_{(k)}^{(n-m)} H_{(n-m)(m-k)} \right\rangle , \tag{IV.3}$$

where we used the abbreviation $G_{(k)} := H_F^{\otimes k} * G^{(k)}$, cf. [Kel09]. The first terms of this expansion are given by

$$\begin{aligned} F \cdot_{\boldsymbol{\mathcal{T}}} G \cdot_{\boldsymbol{\mathcal{T}}} H = FGH &+ \hbar \left\{ \left\langle F^{(1)} G_{(1)} H \right\rangle + \left\langle F^{(1)} G H_{(1)} \right\rangle + \left\langle F G^{(1)} H_{(1)} \right\rangle \right\} \\ &+ \hbar^2 \left\{ \tfrac{1}{2} \left\langle F^{(2)} G_{(2)} H \right\rangle + \left\langle F^{(2)} G_{(1)} H_{(1)} \right\rangle + \tfrac{1}{2} \left\langle F^{(2)} G H_{(2)} \right\rangle \right. \\ &\qquad \left. + \left\langle F^{(1)} G^{(1)} H_{(1)(1)} \right\rangle + \tfrac{1}{2} \left\langle F G^{(2)} H_{(2)} \right\rangle + \left\langle F^{(1)} G^{(1)}_{(1)} H_{(1)} \right\rangle \right\} \\ &+ \cdots \end{aligned}$$



Observe that the prefactor of each graph is given by its symmetry factor, $\mathrm{Sym}(\Gamma)^{-1} = \frac{1}{n!}\binom{n}{m}\binom{m}{k} = \frac{1}{(n-m)!k!(m-k)!}$, where for a general graph, $\Gamma \in \mathcal{G}$, $\mathrm{Sym}(\Gamma)$ is the product of the number of possible permutations of edges which join the same two vertices in Γ.

The terms in (IV.3) can equivalently be expressed as a composition of the maps S_Γ and δ^α above,

$$\begin{array}{c} F \otimes G \otimes H \\ \Big\downarrow \delta^\alpha \\ F^{(m)} \otimes G^{(n-m+k)} \otimes H^{(n-k)} \\ \Big\downarrow \frac{1}{\mathrm{Sym}(\Gamma)} S_\Gamma \\ \frac{1}{n!}\binom{n}{m}\binom{m}{k} \left\langle F^{(k+m-k)} G_{(k)}^{(n-m)} H_{(n-m)(m-k)} \right\rangle , \end{array}$$

hence we can write (IV.3) equivalently as graph expansion,

$$F \cdot_{\boldsymbol{\mathcal{T}}} G \cdot_{\boldsymbol{\mathcal{T}}} H = \sum_{\Gamma \in \mathcal{G}} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \left\langle S_\Gamma, \delta^\alpha (F \otimes G \otimes H) \right\rangle , \quad \alpha \in \mathbb{N}^3, \quad |\alpha| = 2\,|E(\Gamma)| .$$

We will properly define these maps, δ^α and S_Γ, in the sequel.

In the case of local functionals $F_v \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ we have that the functional derivative can be written in the form, cf. Eq. (II.14),

$$F_v^{(\alpha_v)}(\varphi)(x_1, \ldots, x_{\alpha_v}) = \sum_k \sum_i f_\varphi^{v,k,i}(x_v)\, P_k(\partial_{\boldsymbol{r}_v}) \delta(\boldsymbol{r}_v) \in \mathscr{D}(\mathbb{M}) \otimes \mathscr{E}'_{\mathrm{Dirac}}(\mathbb{M}^{\alpha_i - 1}) ,$$

where $x_v = \frac{1}{\alpha_v}\sum_{k=1}^{\alpha_v} x_k$ is the center of mass coordinate and $\boldsymbol{r}_v$ are relative coordinates at vertex $v \in V(\Gamma)$. P_k are homogeneous polynomials of order k in $\alpha_v - 1$ variables, $\mathscr{E}'_{\text{Dirac}}$ denotes the space of distributions supported at zero, and $\operatorname{supp}(f_\varphi^{v,k,i})$ can be chosen arbitrarily small. We want to introduce the short hand notation

$$\mathscr{D}_{\text{loc}}(\mathbb{M}^{\alpha_v}) := \mathscr{D}(\mathbb{M}) \otimes \mathscr{E}'_{\text{Dirac}}(\mathbb{M}^{\alpha_v - 1})$$

for the space of the α_vth functional derivative of a local functional, $F^{(\alpha_v)}(\varphi) \in \mathscr{D}_{\text{loc}}(\mathbb{M}^{\alpha_v})$ and call

$$\begin{array}{rlcl} \delta^\alpha\big|_\varphi : & \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]]^{\otimes |V(\Gamma)|} & \to & \bigotimes_{v\in V(\Gamma)} \mathscr{D}_{\text{loc}}(\mathbb{M}^{\alpha_v}) \\ & F_1 \otimes \cdots \otimes F_n & \mapsto & F_1^{(\alpha_1)}(\varphi) \otimes \cdots \otimes F_n^{(\alpha_n)}(\varphi) \quad , n = |V(\Gamma)| \end{array}$$

the *adjacency differential operator*.

While the definition of δ^α can be done purely algebraically, the construction of the distribution S_Γ on the other hand involves renormalization, i.e., an extension procedure for distributions. We start from the tensor power

$$\widetilde{S_\Gamma} = \bigotimes_{e\in E(\Gamma)} H_F(e)\,, \tag{IV.4}$$

containing one factor $H_F \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$ for every edge e in Γ. Hence $\widetilde{S_\Gamma}$ is a well-defined distribution in $\mathscr{D}'((\mathbb{M}\setminus\{0\})^{|E(\Gamma)|})$ that can be uniquely extended to $\mathscr{D}'(\mathbb{M}^{|E(\Gamma)|})$, since the Feynman fundamental solution $H_F \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$ has a unique extension $\dot{H}_F \in \mathscr{D}'(\mathbb{M})$ with the same scaling degree.

The renormalization problem is now to find a restriction S_Γ of the tensor distribution $\widetilde{S_\Gamma}$ to the space

$$\bigotimes_{v\in V(\Gamma)} \mathscr{D}_{\text{loc}}(\mathbb{M}^{\alpha_v}) = \mathscr{D}(\mathbb{M}^{|V(\Gamma)|}) \otimes \bigotimes_{v\in V(\Gamma)} \mathscr{E}'_{\text{Dirac}}(\mathbb{M}^{\alpha_v - 1})\,.$$

The space $\mathscr{E}'_{\text{Dirac}}$ is spanned by the δ-distribution and its derivatives [Hör03, Thm. 2.3.4], thus the tensor product

$$\mathscr{V} = \bigotimes_{v\in V(\Gamma)} \mathscr{E}'_{\text{Dirac}}(\mathbb{M}^{\alpha_v - 1})$$

is graded by the number of derivatives in front of the δ-distributions.

$$\mathscr{V} = \bigoplus_{|\vec{k}|} \mathscr{V}_{|\vec{k}|}\,, \quad \left|\vec{k}\right| = \sum_{v\in V(\Gamma)} k_v\,.$$

Regard the application of $\widetilde{S_\Gamma}$ to the image of $\delta^\alpha\big|_\varphi$,

$$\begin{aligned} \left\langle \widetilde{S_\Gamma}, \bigotimes_{v\in V(\Gamma)} F_v^{(\alpha_v)} \right\rangle &= \left\langle \widetilde{S_\Gamma}, \bigotimes_{v\in V(\Gamma)} \sum_{k_v}\sum_{I_v} f_\varphi^{v,k_v,I_v}\, P_{k_v}(\partial_{\boldsymbol{r}_v})\delta(\boldsymbol{r}_v) \right\rangle \\ &= \left\langle \widetilde{S_\Gamma}, P_{\vec{k}}(\partial_{\vec{\boldsymbol{r}}}) \bigotimes_{v\in V(\Gamma)} \sum_{k_v}\sum_{I_v} f_\varphi^{v,k_v,I_v}\, \delta(\boldsymbol{r}_v) \right\rangle \end{aligned}$$

where $\vec{k} = (k_v)_{v\in V(\Gamma)}$ and $\vec{\boldsymbol{r}} = (\boldsymbol{r}_v)_{v\in V(\Gamma)}$. We dualize the application of $P_{\vec{k}}(\partial_{\vec{\boldsymbol{r}}})$ and

get

$$\left\langle \widetilde{S_\Gamma}, \bigotimes_{v\in V(\Gamma)} F_v^{(\alpha_v)} \right\rangle = \left\langle \widetilde{S_{(\Gamma,\vec{k})}}, \bigotimes_{v\in V(\Gamma)} \sum_{k_v} \sum_{I_v} f_\varphi^{v,k_v,I_v}\, \delta(\boldsymbol{r}_v) \right\rangle$$

where according to Lemma I.4 this will increase the scaling degree of the distribution by $\left|\vec{k}\right|$,

$$\mathrm{sd}(\widetilde{S_{(\Gamma,\vec{k})}}) = \mathrm{sd}(\widetilde{S_\Gamma}) + \left|\vec{k}\right| . \tag{IV.5}$$

The multiindex $\vec{k}$ thus encodes the derivative couplings (i.e., the interaction functionals containing derivatives of the fields) in the graph Γ. In the framework of Connes-Kreimer Hopf algebras, or Feynman graphs in general, $\vec{k}$ sometimes is called the "external structure of the graph", see [CM07] for instance.

The remaining restriction of $\widetilde{S_{(\Gamma,\vec{k})}}$ to a distribution in $\mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ can conveniently be described by the (simplicial) cohomology of the graph Γ. For ease of presentation, we will forget about the external structure $\vec{k}$ for the time being. The algebraic structure to be presented below can be developed to a large extend without recourse to the external structure. We will reintroduce $\vec{k}$ by replacing $\Gamma \mapsto \left(\Gamma, \vec{k}\right)$, where we find it to be relevant for the understanding.

IV.1.1. Simplicial cohomology of a graph and choice of relative coordinates. The presentation in this subsection is very much inspired by [BBK09, Sec. 2.1]. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a field. We define the (simplicial) cohomology $H^1(\Gamma, \mathbb{K})$ with coefficients in $\mathbb{K}$ of a connected graph $\Gamma \in \mathcal{G}$ by the exact sequence

$$0 \longrightarrow \mathbb{K} \overset{c}{\hookrightarrow} \mathbb{K}^{|V(\Gamma)|} \overset{d}{\longrightarrow} \mathbb{K}^{|E(\Gamma)|} \overset{o}{\twoheadrightarrow} H^1(\Gamma, \mathbb{K}) \longrightarrow 0 \ . \tag{IV.6}$$

Let $\{a_v : v \in V(\Gamma)\}$ be a basis of $\mathbb{K}^{|V(\Gamma)|}$ and $\{b_e : e \in E(\Gamma)\}$ a basis of $\mathbb{K}^{|E(\Gamma)|}$. The maps in (IV.6) are then defined as the "center of mass",

$$c : x \mapsto x \sum_{v\in V(\Gamma)} a_v \,,$$

and

$$d : a_v \mapsto \sum_{e\in E(\Gamma)} (e : v)\, b_e \,.$$

One immediately checks that

$$\forall x \in \mathbb{K} : \quad (d \circ c)\,(x) = x \sum_{e\in E(\Gamma)} \sum_{v\in V(\Gamma)} (e : v)\, b_e = 0 \,.$$

Furthermore, $o \circ d = 0$ is equivalent to

$$H^1(\Gamma, \mathbb{K}) = \mathrm{coker}(d) = \mathbb{K}^{|E(\Gamma)|}/\mathrm{im}(d) \,,$$

an alternative definition of $H^1(\Gamma, \mathbb{K})$. The dimension of this cohomology is called the first Betti number and gives the number of independent loops of the graph Γ, $\dim(H^1(\Gamma, \mathbb{K})) = |E(\Gamma)| - |V(\Gamma)| + 1$.

Let us regard the map d. The image of a general element $\vec{x} = \sum_{v\in V(\Gamma)} x^v a_v$ is

given by

$$\mathscr{d}(\vec{x}) = \sum_{e \in E(\Gamma)} \left(x^{\mathfrak{t}(e)} - x^{\mathfrak{s}(e)} \right) b_e \,.$$

Thus $\mathscr{d}$ expresses the coordinates of a given edge $e \in E(\Gamma)$ in terms of the coordinates of the adjacent vertices, $r^e = x^{\mathfrak{t}(e)} - x^{\mathfrak{s}(e)}$.

EXAMPLE IV.2. Regard the very simple graph with two vertices and one edge,

$$\gamma = \underset{v}{\bullet} \overset{e}{\text{———}} \underset{w}{\bullet} \,.$$

Let $v = \mathfrak{s}(e)$ and $w = \mathfrak{t}(e)$, and choose a basis $\{a_v, a_w\}$ of $\mathbb{K}^{|V(\gamma)|}$. Then $x\, a_v + y\, a_w \in \mathbb{K}^{|V(\gamma)|}$ and we have

$$\mathscr{d}(x\, a_v + y\, a_w) = (y - x)\, b_e \,.$$

Thus the pullback of $\mathscr{d} : \mathbb{K}^{|V(\gamma)|} \to \mathbb{K}^{|E(\gamma)|} \equiv \mathbb{K}$ maps a function $f \in \mathscr{D}(\mathbb{K})$ to

$$(\mathscr{d}^* f)\,(x, y) = f(y - x) \,.$$

Consequently, a distribution $u \in \mathscr{D}'(\mathbb{K})$ will be mapped to $\mathscr{d}^* u \in \mathscr{D}'(\mathbb{K}^2)$, with

$$(\mathscr{d}^* u)\,(f \otimes g) = \int_{\mathbb{K}^2} dx\, dy\, u(y - x)\, f(x)\, g(y) = \langle f, u * g \rangle \,,$$

where $*$ denotes the convolution product and the pullback is defined in the sense of [Hör03, Thm. 6.1.2].

This construction can be lifted to any $\mathbb{K}$-vector space V, by forming the tensor product $\mathbb{K} \otimes V$. We are interested here in the lift to Minkowski spacetime $\mathbb{M} = \mathbb{R} \otimes \mathbb{M}$. Thus we have, $\mathbb{M}^n = \mathbb{R}^n \otimes \mathbb{M}$,

$$0 \longrightarrow \mathbb{M} \overset{\hat{\iota}}{\hookrightarrow} \mathbb{M}^{|V(\Gamma)|} \overset{\hat{\mathscr{d}}}{\longrightarrow} \mathbb{M}^{|E(\Gamma)|} \twoheadrightarrow H^1(\Gamma, \mathbb{M}) \longrightarrow 0 \,,$$

i.e., one short exact sequence for each component of $z \in \mathbb{M}$.

EXAMPLE IV.2 (revisited). In terms of this cohomology the Hadamard two point function is the pullback of the Hadamard solution $H \in \mathscr{D}'(\mathbb{M})$ by $\mathscr{d}_\gamma$,

$$\left(\hat{\mathscr{d}}_\gamma^* H \right)(f, g) = \langle f, H * g \rangle \,,$$

analogously the Feynman propagator is the pullback of the Feynman fundamental solution $H_F \in \mathscr{D}'(\mathbb{M} \setminus \{0\})$,

$$\left(\hat{\mathscr{d}}_\gamma^* H_F \right)(f, g) = \langle f, H_F * g \rangle \,,$$

$\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$.

Also translation invariance can be formulated very conveniently in this cohomological framework. The image of c gives all possible translations of the vertex coordinates by a given vector $a \in \mathbb{K}$. Hence the orbits of these translations are the elements of the cokernel $\operatorname{coker}(c) = \mathbb{K}^{|V(\Gamma)|}/\operatorname{im}(c)$. We can fix a basis of $\operatorname{coker}(c)$ by choosing the coordinates of a vertex v_0 and setting $V_0 := V(\Gamma) \setminus \{v_0\}$. This

provides us with a projection

$$\pi_\Gamma : \mathbb{K}^{|V(\Gamma)|} \to \mathbb{K}^{|V_0|} \equiv \mathbb{K}^{|V(\Gamma)|-1}$$

and an isomorphism ϕ between $\mathbb{K}^{|V_0|}$ and $\operatorname{coker}(c)$,

$$\begin{array}{rccl} \phi : & \mathbb{K}^{|V_0|} & \to & \operatorname{coker}(c) \\ & a_v & \mapsto & a_v + \operatorname{im}(c)\,. \end{array}$$

All translation invariant functions in $\mathbb{K}^{|V(\Gamma)|}$ can be seen as generic functions on $\operatorname{coker}(c)$, or $\mathbb{K}^{|V_0|}$ respectively. They are related by the pullback via π_Γ, e.g. for smooth functions,

$$\begin{array}{rccl} \pi_\Gamma^* : & \mathscr{E}(\mathbb{K}^{|V_0|}) & \to & \mathscr{E}_{\text{tr.inv.}}(\mathbb{K}^{|V(\Gamma)|}) \\ & f & \mapsto & (\pi_\Gamma^* f) = f \circ \pi_\Gamma\,. \end{array}$$

We define the *choice of relative coordinates* in $\mathbb{K}^{|E(\Gamma)|}$ by

$$\begin{array}{rccl} \iota_\Gamma := \mathscr{d}_\Gamma \circ \phi : & \mathbb{K}^{|V_0|} & \to & \mathbb{K}^{|E(\Gamma)|} \\ & \sum_{v\in V_0} x^v a_v & \mapsto & \sum_{e\in E(\Gamma)} r^e(\vec{x})\, b_e\,, \end{array}$$

where $r^e(\vec{x}) = \sum_{v\in V_0} (e:v)\, x^v$ is computed to be

$$r^e(\vec{x}) = \begin{cases} x^{\mathfrak{t}(e)} - x^{\mathfrak{s}(e)} & \text{if } v_0 \notin \{\mathfrak{s}(e), \mathfrak{t}(e)\} \\ x^{\mathfrak{t}(e)} & \text{if } v_0 = \mathfrak{s}(e) \\ -x^{\mathfrak{s}(e)} & \text{if } v_0 = \mathfrak{t}(e)\,, \end{cases}$$

giving the "coordinates of the edges" relative to v_0. In Minkowski spacetime we define correspondingly,

$$\hat{\iota}_\Gamma := \hat{\mathscr{d}}_\Gamma \circ \hat{\phi} : \mathbb{M}^{|V_0|} \to \mathbb{M}^{|E(\Gamma)|}$$

as the choice of relative coordinates in $\mathbb{M}^{|E(\Gamma)|}$.

We now want to define $S_\Gamma \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ as the pullback of $\widetilde{S_\Gamma}$ via $\hat{\mathscr{d}}_\Gamma$,

$$S_\Gamma = \hat{\mathscr{d}}_\Gamma^* \, \widetilde{S_\Gamma}\,.$$

Let us regard the case of the unextended amplitude $\widetilde{S_\Gamma} \in \mathscr{D}'((\mathbb{M}\setminus\{0\})^{|E(\Gamma)|})$. Each edge corresponds to a Feynman propagator H_F and any set of edges joining the same two vertices will have the same coordinate $r^e(\vec{x})$. This introduces powers of H_F, which are well-defined distributions only outside the origin, $(H_F)^k \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$. As a consequence the pullback S_Γ is a well-defined distribution only outside the large diagonal

$$\text{DIAG} = \left\{ \vec{x} \in \mathbb{M}^{|V(\Gamma)|} \,|\, \exists v, w \in V(\Gamma), v \neq w : \, x_v = x_w \right\}\,,$$

$S_\Gamma \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|}\setminus\text{DIAG})$. As remarked before a restriction of $\widetilde{S_\Gamma}$ by means of a wave front set argument, i.e. by applying [Hör03, Thm. 8.2.4], is not possible due to the wave front set of H_F. A restriction of $\widetilde{S_\Gamma}$, or equivalently an extension of $\hat{\mathscr{d}}_\Gamma^* \widetilde{S_\Gamma}$ to $\mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$, will involve renormalization. In the case of even dimensions,

$d \in 2\mathbb{N}$, the amplitude is a tensor power of the Feynman propagator (III.20),

$$\widetilde{S^{\mu}_{\Gamma}} := \bigotimes_{e \in E(\Gamma)} H^{\mu}_{F}(e) \in \mathscr{D}'((\mathbb{M} \setminus \{0\})^{|E(\Gamma)|}),$$

and hence depends on an additional parameter μ of mass dimension one.

We review briefly the Epstein-Glaser induction for constructing the extension S_Γ in order to discuss the renormalization freedom in the cohomological framework advertised here.

IV.1.2. The Epstein-Glaser induction. Having defined what we mean by a graph $\Gamma \in \mathcal{G}$, we define an *Epstein-Glaser subgraph (EG subgraph)* $\gamma \subseteq \Gamma$ to be a subset of the set of vertices $V(\gamma) \subseteq V(\Gamma)$ together with all lines in Γ connecting them,

$$E(\gamma) = \{e \in E(\Gamma) : \{\mathfrak{s}(e), \mathfrak{t}(e)\} \subset V(\gamma)\}.$$

The orientation of γ is inherited from Γ. The first step of the Epstein-Glaser induction is to choose extensions for all EG subgraphs with two vertices, $|V(\gamma)| = 2$. In this case we have translation invariant distributions $\hat{d}^*_\gamma \widetilde{S_\gamma} \in \mathscr{D}'(\mathbb{M}^2 \setminus \mathrm{Diag})$ ($\mathrm{Diag} = \left\{\vec{x} \in \mathbb{M}^{|V(\gamma)|} \,|\, \forall v, w \in V(\Gamma) : \, x_v = x_w\right\}$ denotes the thin diagonal), which correspond to generic distributions $\hat{\iota}^*_\gamma \widetilde{S_\gamma} \in \mathscr{D}'(\mathbb{M} \setminus \{0\})$. The scaling degree of these distributions is given by their number of lines $\mathrm{sd}(\hat{\iota}^*_\gamma \widetilde{S_\gamma}) = |E(\gamma)|\,(d-2)$, and we can choose a (possibly unique) extension according to Theorem I.6. By translation invariance this gives extensions $S_\gamma \in \mathscr{D}'(\mathbb{M}^2)$ of the distributions $\hat{d}^*_\gamma S_\gamma \in \mathscr{D}'(\mathbb{M}^2 \setminus \mathrm{Diag})$. By causality, i.e. relation (II.25) of time-ordered and algebra product, these extensions define the (translation-invariant) restrictions of all EG subgraphs with three vertices up to the thin diagonal.

For a generic EG subgraph $\gamma \subseteq \Gamma$ we make the assumption that the restrictions of all EG subgraphs of γ with less vertices have already been chosen (induction hypothesis). The causality condition then gives a translation invariant distribution $\hat{d}^*_\gamma \widetilde{S}_\gamma \in \mathscr{D}'(\mathbb{M}^{|V(\gamma)|} \setminus \mathrm{Diag})$ which corresponds to a generic distribution $\hat{\iota}^*_\gamma \widetilde{S_\gamma} \in \mathscr{D}'(\mathbb{M}^{|V(\gamma)|-1} \setminus \{0\})$. The scaling degree and hence the degree of divergence of this distribution is completely fixed by the structure of the graph, cf. (II.21),

$$\mathrm{div}(\gamma) = \mathrm{div}(\hat{\iota}^*_\gamma \widetilde{S_\gamma}) = |E(\gamma)|\,(d-2) - (|V(\gamma)| - 1)\,d\,, \qquad d = \dim(\mathbb{M})\,. \tag{IV.7}$$

We call γ superficially convergent if $\mathrm{div}(\gamma) < 0$, logarithmically divergent if $\mathrm{div}(\gamma) = 0$ and divergent of degree $\mathrm{div}(\gamma)$ otherwise. Again by Theorem I.6 there is a choice to be made in the extension of $\hat{\iota}^*_\gamma \widetilde{S_\gamma}$ in the case $\mathrm{div}(\gamma) \geqslant 0$. The inductive procedure of Epstein-Glaser will thus lead to an extension $S_\Gamma \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ of $\hat{d}^*_\Gamma \widetilde{S_\Gamma} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|} \setminus \mathrm{DIAG})$. As suggested above, we will refer to any such extension of $\hat{d}^*_\Gamma \widetilde{S_\Gamma}$ as *restriction* of $\widetilde{S_\Gamma} \in \mathscr{D}'((\mathbb{M} \setminus \{0\})^{|E(\Gamma)|})$.

In the case of couplings which involve derivatives of the fields, also the external structure of γ has to be taken into account, cf. (IV.5),

$$\mathrm{div}\left(\gamma, \vec{k}\right) = \mathrm{div}(\gamma) + \left|\vec{k}\right|\,.$$

This introduces an additional freedom in the choice of the extension in each step, but does otherwise not change the inductive procedure.

The combination of all choices involved in the inductive construction of a restriction $S_\Gamma \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ of $\widetilde{S_\Gamma}$ make up the Stückelberg-Petermann renormalization group acting on local functionals, cf. [BDF09]. We will see in the next section that the freedom in this construction is considerably restricted, if we replace in (IV.4) the Feynman propagator by its dimensionally regularized counterpart.

IV.2. The Regularized Amplitude

The aim of this section is to construct a regularization of the above defined amplitude $\hat{a}_\Gamma^* \widetilde{S_\Gamma^\mu} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|} \backslash \mathrm{DIAG})$ by applying the Epstein-Glaser reduction procedure to

$$\widetilde{S_\Gamma^{\mu,\zeta}} := \bigotimes_{e \in E(\Gamma)} H_F^{m,\mu,\zeta}(e)\,. \tag{IV.8}$$

We will see in the sequel that $\widetilde{S_\Gamma^{\mu,\zeta}}$ has a unique restriction $S_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ by means of Corollary I.15. This will provide a regularization of the original amplitude $\hat{a}_\Gamma^* \widetilde{S_\Gamma^\mu}$ outside the large diagonal.

Regard the dimensionally regularized Feynman fundamental solution $H_F^{m,\mu,\zeta} \in \mathscr{D}'(\mathbb{M} \backslash \{0\})$ constructed in Chapter III. The expansion of $H_F^{m,\mu,\zeta}$ in powers of the mass parameter m^2 follows directly from the expansion of the analytic Hadamard function (III.18),

$$\begin{aligned} H_F^{m,\mu,\zeta}(x) &= (-1)^{\frac{d-2}{2}} (2\pi)^{-\frac{d}{2}} 2^{\frac{2-(d+\zeta)}{2}} \frac{\pi}{2\sin(\zeta\frac{\pi}{2})} \cdot \\ &\cdot \left[\sum_{s=0}^{\infty} \left\{ \left(\frac{2}{\mu\sqrt{-x^2+i0}} \right)^{\zeta} \frac{1}{s!\,\Gamma(-\frac{\zeta}{2} - \frac{d-4}{2} + s)} - \frac{\theta(s - \frac{d-3}{2})}{\left(s - \frac{d-2}{2}\right)!\,\Gamma(\frac{\zeta}{2} + s + 1)} \right\} \cdot \right. \\ &\qquad \left. \cdot \left(\frac{\sqrt{-x^2+i0}}{2} \right)^{2s-(d-2)} \left(m^2\right)^s \right], \\ &=: \sum_{s=0}^{\infty} H_F^{s,\mu,\zeta}(x) \left(m^2\right)^s , \end{aligned} \tag{IV.9}$$

where we used the common shorthand $f(x^2 - i0) = \lim_{\varepsilon \to 0^+} f(x^2 - i\varepsilon)$. The coefficients of this series, $H_F^{s,\mu,\zeta} \in \mathscr{D}'(\mathbb{M} \backslash \{0\})$, are sums of a distributional and a smooth part, both of which are homogeneous, but of different degree. The distributional part is homogeneous of degree $2s - (d + \zeta - 2)$, whereas the smooth part is identically zero for $s < \frac{d-3}{2}$ and homogeneous of degree $2s - (d-2)$ otherwise.

Regard now the finite tensor powers of the dimensionally regularized Feynman distribution. The expansion of $\left(H_F^{m,\mu,\zeta}\right)^{\otimes k}$, $k = |E(\Gamma)|$, in m^2 follows directly

from (IV.9) and Cauchy's product formula,

$$\text{(IV.10)} \qquad \left(H_F^{m,\mu,\zeta}\right)^{\otimes k} = \sum_{s_k=0}^{\infty} \left(m^2\right)^{s_k} \sum_{s_{k-1}=0}^{s_k} \cdots \sum_{s_1=0}^{s_2} H_F^{(s_k-s_{k-1}),\mu,\zeta} \otimes \cdots \otimes H_F^{s_1,\mu,\zeta}$$
$$=: \sum_{s_k=0}^{\infty} \left(m^2\right)^{s_k} \left(H_F^{\mu,\zeta}\right)^{\otimes \vec{s}},$$

where $\vec{s} = (s_k - s_{k-1}, \ldots, s_1) \in \mathbb{N}^k$. The scaling degree of the coefficient of $\left(m^2\right)^{s_k}$ is given by the sum of the scaling degrees of the individual factors, cf. Lemma I.4 and Remark I.10,

$$\text{(IV.11)} \qquad \mathrm{sd}(\left(H_F^{\mu,\zeta}\right)^{\otimes \vec{s}}) = k\,(d + \mathrm{Re}(\zeta) - 2) - 2s_k\,.$$

Hence the scaling degree of the coefficients become arbitrarily small as one regards higher powers of m^2. Thus according to Theorem I.6 these coefficients will have unique restrictions to arbitrary subdiagonals.[2] In other words, only a finite number of coefficients in (IV.10) need renormalization. This is one of the advantages of the concept of a "scaling expansion" introduced in [HW02]; see also [Hol04], [DF04, p. 1310ff]. The coefficients in (IV.10) which need renormalization when restricted to subdiagonals are in general not homogeneous, since the coefficients $H_F^{s,\mu,\zeta}$ in (IV.9) are not homogeneous. However, since we regard graphs with a finite number of edges, they are certainly heterogeneous of finite order. If we assume $\zeta \notin \mathbb{Q}_+$ their multidegree contains no integer number. We can construct a restriction of $\widetilde{S_\Gamma^{\mu,\zeta}}$ by the same procedure described in Section IV.1.2, with the only difference that we have a preferred choice of the extension at each order according to Corollary I.15, namely the extensions which are heterogeneous of the same multidegree. Thus we are lead to a unique restriction $S_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$.

PROPOSITION IV.3 (Regularization outside DIAG). *The restriction of $\widetilde{S_\Gamma^{\mu,\zeta}}$ is uniquely defined by the above homogeneity condition and gives a distribution $S_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$. $S_\Gamma^{\mu,\zeta}$ is a regularization of $\hat{a}_\Gamma^* \widetilde{S_\Gamma^{\mu}} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|} \backslash \mathrm{DIAG})$, in the sense that*

$$\forall f \in \mathscr{D}(\mathbb{M}^{|V(\Gamma)|} \backslash \mathrm{DIAG}) : \quad \lim_{\zeta \to 0} \left\langle S_\Gamma^{\mu,\zeta}, f \right\rangle = \left\langle \hat{a}_\Gamma^* \widetilde{S_\Gamma^{\mu}}, f \right\rangle .$$

We will refer to $S_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ as the dimensionally regularized amplitude of Γ. By translation invariance it naturally corresponds to a unique dimensionally regularized amplitude in relative coordinates, $s_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|-1})$, which is the unique extension of $\hat{\imath}_\Gamma^* \widetilde{S_\Gamma^{\mu,\zeta}} \in \mathscr{D}'((\mathbb{M} \backslash \{0\})^{|V(\Gamma)|-1})$.

PROOF. The first part follows from the construction above. By the discussion at the end of the previous chapter we have that the unique extensions $\dot{H}_F^{m,\mu,\zeta} \in \mathscr{D}'(\mathbb{M})$ is a regularization of $H_F^{m,\mu} \in \mathscr{D}'(\mathbb{M} \backslash \{0\})$, cf. (III.24). And by continuity of

[2] Keep in mind that the product of distributions can be defined as the restriction of their tensor product to subdiagonals, cf. [Hör03, Thm. 8.2.10].

the maps involved we get the assertion,

$$\hat{d}_\Gamma^* \left(\bigotimes_{e\in E(\Gamma)} H_F^{m,\mu}(e) \right) = \hat{d}_\Gamma^* \left(\bigotimes_{e\in E(\Gamma)} \lim_{\zeta\to 0} H_F^{m,\mu,\zeta}(e) \right)$$
$$= \lim_{\zeta\to 0} \hat{d}_\Gamma^* \left(\bigotimes_{e\in E(\Gamma)} H_F^{m,\mu,\zeta}(e) \right)$$

Observe that the extension map $\hat{d}_\Gamma^* \widetilde{S_\Gamma^{\mu,\zeta}} \mapsto S_\Gamma^{\mu,\zeta}$ is also continuous by Theorem I.12. □

We have that $S_\Gamma^{\mu,\zeta}$ is a regularization of $\hat{d}_\Gamma^* \widetilde{S_\Gamma^{\mu}}$ in a broader sense of the word, since $\hat{d}_\Gamma^* \widetilde{S_\Gamma^{\mu}}$ is defined only in the complement of the large diagonal. However, the regularization $S_\Gamma^{\mu,\zeta} \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|})$ comes with a natural renormalization prescription, defined at any order of (causal) perturbation theory: minimal subtraction (MS). This has already been introduced on the conceptual level in Section I.4, and we will see in the next chapter, how minimal subtraction is to be applied to the regularized graph amplitudes $s_\Gamma^{\mu,\zeta}$ and $S_\Gamma^{\mu,\zeta}$, respectively. The complete renormalization of the graph amplitudes will then be discussed in Chapter VI and it will be useful, for the derivation of the underlying combinatorial structure to collect all the different contributions to the perturbative expansion in the definition of a unique dimensionally regularized $\boldsymbol{\mathcal{S}}$-matrix, defined as a map on local functionals, cf. Section II.3.

DEFINITION IV.4 (Dimensionally Regularized $\boldsymbol{\mathcal{S}}$-matrix). Let

$$\Gamma'_{H_F^{m,\mu,\zeta}} := \frac{1}{2} \int dx\, dy\, H_F^{m,\mu,\zeta}(x,y) \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}\,,$$

be the dimensionally regularized Feynman bidifferential operator. Define the regularized time-ordering operator

$$\boldsymbol{\mathcal{T}}_{\mu,\zeta} := \exp(\hbar\Gamma'_{H_F^{m,\mu,\zeta}})\,,$$

and the dimensionally regularized time-ordered product on local functionals $F, G \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$

$$F \cdot_{\boldsymbol{\mathcal{T}}_{\mu,\zeta}} G := \boldsymbol{\mathcal{T}}_{\mu,\zeta} \left(\boldsymbol{\mathcal{T}}_{\mu,\zeta}^{-1} F \cdot \boldsymbol{\mathcal{T}}_{\mu,\zeta}^{-1} G \right)\,.$$

Then we define the dimensionally regularized $\boldsymbol{\mathcal{S}}$-matrix as

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta}(F) := \exp_{\cdot_{\boldsymbol{\mathcal{T}}_{\mu,\zeta}}}(F) = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\mathcal{T}}_{\mu,\zeta}^n (F^{\otimes n})\,, \qquad F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]\,,$$

where $\boldsymbol{\mathcal{T}}_{\mu,\zeta}^n$ denotes the uniquely extended regularized n-fold time-ordered product constructed (graph by graph) by Epstein-Glaser induction.

Inserting (IV.2) we can write the regularized $\boldsymbol{\mathcal{S}}$-matrix also in terms of a graph

expansion

(IV.12)

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta}(F) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\mathbb{N}^n} \sum_{\Gamma\in\mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \left\langle S_\Gamma^{\mu,\zeta}, \boldsymbol{\delta}^\alpha \left(F^{\otimes n}\right)\right\rangle , \qquad F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]].$$

This expansion is often referred to as the perturbative expansion of the $\boldsymbol{\mathcal{S}}$-matrix. And we want to remark that the sum over all graphs at a fixed order n of causal perturbation theory is finite, if we assume that $F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ is a polynomial interaction functional. This remains valid, if F contains derivatives of the field. The order of causal perturbation theory is given by the number of vertices of the graphs contributing to (IV.12), irrespective of the fact if they are connected or not. Conversely, the sum is finite at each order $\mathcal{O}(\hbar^{|E(\Gamma)|})$, and we repeat the remark that this is in essence the "loop order", if we regard only graphs with a fixed number of connected components $c(\Gamma)$. The order is given by the Betti number of the graph [GY03b],

$$\ell(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + c(\Gamma),$$

if we "hide" one power of $\hbar$ in the interaction functional F.

CHAPTER V

Minimal Subtraction

Minimal subtraction (MS) in combination with dimensional regularization (DimReg) and Zimmermann's forest formula as a renormalization technique has earned wide acclaim in the standard approach to perturbative renormalization in momentum space. After having constructed the dimensionally regularized position space amplitude to any graph $\Gamma \in \mathcal{G}$, we want to extend the notion of minimal subtraction given in Section I.4 also to graph amplitudes and products thereof. As a matter of fact, we will find that minimal subtraction can be formulated independently of the graph expansion and the representation (position- or momentum space). This is to say that we can define a minimal subtraction operator which acts directly on the prepared, dimensionally regularized time-ordered product, regarded as a linear map between functional spaces,

$$\boldsymbol{\mathcal{T}}^{n}_{\mu,\zeta,\mathrm{prep}} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}(\mathbb{M})[[\hbar]] \,.$$

The fact that this leads to local counterterms will be the crucial observation which makes the abstraction in the next chapter possible, and the presented forest formula for Epstein-Glaser renormalization applicable in any chosen representation.

What will be said in this chapter relies on the fact that we dispose of a *prepared amplitude*. This will be defined in the first section, and we will implement the graph structure in the second. In the third section we will define minimal subtraction at subgraphs, and we will test our method by rederiving the result of Zimmermann that only Epstein-Glaser subgraphs contribute to nested projections in the limit where the regularization is removed [Zim76]. The independence on the representation will be discussed in the fourth section and as a result we will define the minimal subtraction operator on prepared time-ordered products.

V.1. Prepared Amplitude

DEFINITION V.1 (Prepared Amplitude). A regularization $\left\{ s^{\mu,\zeta}_{\Gamma,\mathrm{prep}} : \zeta \in \Omega \setminus \{0\} \right\}$ (in the strict sense of Definition I.16) is called *prepared amplitude* of $\left\{ s^{\mu,\zeta}_{\Gamma} \right\}$, if it is a regularization of $\hat{\iota}^{*}_{\Gamma} \widetilde{S^{\mu}_{\Gamma}} \in \mathscr{D}'((\mathbb{M} \setminus \{0\})^{|V(\Gamma)|-1})$ outside the large diagonal in the sense of Proposition IV.3, i.e.,

$$\forall f \in \mathscr{D}((\mathbb{M} \setminus \{0\})^{|V(\Gamma)|-1}) : \qquad \lim_{\zeta \to 0} \left\langle s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}, f \right\rangle = \left\langle \hat{\iota}^{*}_{\Gamma} \widetilde{S^{\mu}_{\Gamma}}, f \right\rangle ,$$

and $s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}$ is heterogeneous of finite, non-integer order in $\Omega \setminus \{0\}$.

Observe that for $|V(\Gamma)| = 2$ the regularization outside the large diagonal is already a regularization in the strict sense of Definition I.16, and thus a prepared amplitude. Hence minimal subtraction can be applied and leads to a finite regularization. In the logical framework of Epstein-Glaser one would then define the prepared amplitude of the third order and subtract the counterterm, and so on to the order one chooses to compute. At each step minimal subtraction is applied to a prepared amplitude, but if we want to define the subtraction performed on the unrenormalized amplitude, these subtractions will be nested. One aim of this chapter is to analyze these nested subtractions. They will be used in Chapter VI to solve the recursion of Epstein-Glaser. A closed expression for the prepared amplitude will then follow immediately from the solution. Thus we can assume here that we dispose of a prepared amplitude $s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}$. Since $s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}$ is a regularization by assumption, we can directly apply the analysis of Section I.4 and have that the principal part of its Laurent series is a local distribution,

$$\mathrm{pp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}) \in \mathscr{E}'_{\mathrm{Dirac}}(\mathbb{M}^{|V(\Gamma)|-1}), \tag{V.1}$$

where we denoted by $\mathscr{E}'_{\mathrm{Dirac}}$ the space of distributions supported at the origin. We infer that

$$\left\{\mathrm{rp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}) : \zeta \in \Omega\setminus\{0\}\right\}$$

is a finite regularization of $\hat{\iota}^*_\Gamma \widetilde{S^\mu_\Gamma}$ and hence

$$s^\mu_{\Gamma,\mathrm{ren}} := \lim_{\zeta\to 0} \mathrm{rp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}) \in \mathscr{D}'(\mathbb{M}^{|V(\Gamma)|-1}).$$

is a renormalization. To have a name for it, we call $\mathrm{pp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}})$ and $\mathrm{rp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}})$ the *projected* prepared amplitudes. Nested projections will lead to projections in different parts of the same graph. The different components of a graph needed for the discussion later on will be defined in the following section.

V.2. Subgraphs and Complements

Since the method we are analyzing here was originally formulated in momentum space, where the edges of the graphs carry as label the "momentum flowing through this line", it is natural to consider as subgraphs all graphs, which are given by a subset of the set of edges. Given a graph Γ, we call a *BPHZ subgraph* any subgraph $\gamma \subseteq \Gamma$ given by a subset of the set of edges, $E(\gamma) \subseteq E(\Gamma)$, and all adjacent vertices,

$$V(\gamma) = \{v \in V(\Gamma) | \exists e \in E(\gamma) : (e : v) \neq 0\}.$$

The orientation is inherited from Γ. See, e.g., [CK82] for a description of the BPHZ procedure within dimensional regularization and minimal subtraction in momentum space.

The set of BPHZ subgraphs of a graph $\Gamma \in \mathcal{G}$ is a superset to the set of Epstein-Glaser subgraphs defined in Section IV.1.2, and we can associate to any BPHZ

subgraph a unique Epstein-Glaser subgraph. Given a graph Γ with BPHZ subgraph $\gamma \subseteq \Gamma$, we define the *full vertex part* $\overline{\gamma}$ of γ to be the graph with the same set of vertices, $V(\overline{\gamma}) = V(\gamma)$, and all lines in Γ connecting them,

$$E(\overline{\gamma}) = \{e \in E(\Gamma) : \mathfrak{s}(e), \mathfrak{t}(e) \in V(\Gamma)\}\,.$$

$\overline{\gamma}$ obviously is an Epstein-Glaser subgraph. Any BPHZ subgraph, which is not a full vertex part, we call a *pure BPHZ subgraph*.

For the definition of products of (projected) amplitudes corresponding to different parts of the same graph Γ it is important to have the notion of a complement of a subgraph. Observe, however, that there are two natural ways to define this complement, and both will be of relevance in the sequel.

DEFINITION V.2 (Complements of a graph). Let $\Gamma \in \mathcal{G}$ be a graph and $G \subset \Gamma$ be a subgraph. We define the *line complement* $\Gamma \boxminus G$ of G in Γ to be the graph with

$$V(\Gamma \boxminus G) = V(\Gamma) \qquad \text{and} \qquad E(\Gamma \boxminus G) = E(\Gamma) \backslash E(G)\,.$$

Furthermore we define the *vertex complement* $\Gamma \oslash G$ to be the full vertex part with vertex set

$$V(\Gamma \oslash G) = V(\Gamma) \backslash V(G)\,,$$

i.e.,

$$E(\Gamma \oslash G) = \{e \in E(\Gamma) | \mathfrak{s}(e), \mathfrak{t}(e) \in V(\Gamma \oslash G)\}\,.$$

Observe that, while the vertex complement $\Gamma \oslash G$ is a full vertex part by definition, the line complement is not a full vertex part in the generic case. For the line complement the number of lines is preserved in the sense that

$$E(\Gamma) = E(G) \dot{\cup} E(\Gamma \boxminus G)\,.$$

For the vertex complement, on the other hand, the number of lines is not preserved; $E(G) \dot{\cup} E(\Gamma \oslash G)$ will be a subset of $E(\Gamma)$ in general, because the lines connecting G with $\Gamma \oslash G$ are not considered. We have $E(\Gamma) = E(G) \dot{\cup} E(\Gamma \oslash G)$ if and only if Γ is multiply connected with G one of its (possibly also multiply connected) components. However, the vertex set is preserved for the vertex complement,

$$V(\Gamma) = V(G) \dot{\cup} V(\Gamma \oslash G)\,,$$

a fact that will be of importance in the discussion of Chapter VI.

EXAMPLE V.3. Regard the graph Γ and subgraph $G \subset \Gamma$,

$\Gamma =$, $G =$.

Then the two complements of Definition V.2 are depicted by

$\Gamma \boxminus G =$ and $\Gamma \oslash G =$,

respectively.

We will now use the line complement for the definition of minimal subtractions at BPHZ subgraphs, and meet the vertex complement again in the next chapter.

V.3. MS at Subgraphs and Redundant Projections

Let $\gamma \subset \Gamma$ be a proper BPHZ subgraph. Then γ has less edges than Γ, $E(\gamma) \subsetneq E(\Gamma)$, and it may have less vertices, $V(\gamma) \subseteq V(\Gamma)$. Let

$$\varpi_{\Gamma,\gamma} : \mathbb{M}^{|V(\Gamma)|-1} \to \mathbb{M}^{|V(\gamma)|-1} \tag{V.2}$$

be the induced projection ($\varpi_{\Gamma,\gamma} = \mathrm{id}$, if $V(\Gamma) = V(\gamma)$). Then the pullback $\varpi^*_{\Gamma,\gamma} s^{\mu,\zeta}_\gamma$ exists as a distribution in $\mathscr{D}'(\mathbb{M}^{|V(\Gamma)|-1})$, cf. [Hör03, Thm. 6.1.2], and we have that

$$s^{\mu,\zeta}_\Gamma = s^{\mu,\zeta}_{\Gamma\setminus\gamma} \cdot \varpi^*_{\Gamma,\gamma} s^{\mu,\zeta}_\gamma \,, \quad \zeta \notin \mathrm{Q}_+ \,, \tag{V.3}$$

where by the expression on the right hand side we understand the unique heterogeneous extension of the pointwise product of the distributions, as constructed in Section IV.2. Let us now regard the same product (V.3), when the subgraph part is replaced by a projected prepared amplitude,

$$\left(1 - T^\gamma_{\mathrm{MS}}\right) s^{\mu,\zeta}_\Gamma := s_{\Gamma\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}) \,, \quad \text{or} \quad T^\gamma_{\mathrm{MS}} s^{\mu,\zeta}_\Gamma := s_{\Gamma\setminus\gamma} \cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}) \,, \tag{V.4}$$

respectively. Where the pullback via the projection (V.2) is understood, but not explicitly written to improve readability. Then we define the product on the respective right hand sides term by term in the Laurent expansion, i.e.,

$$s_{\Gamma\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}) = \sum_{n=0}^{\infty} \zeta^n \frac{1}{2\pi i} \oint_C d\xi \, \frac{1}{\xi^{n+1}} s^{\mu,\zeta}_{\Gamma\setminus\gamma} \cdot \varpi^*_{\Gamma,\gamma} s^{\mu,\xi}_\gamma \,, \tag{V.5}$$

and

$$s_{\Gamma\setminus\gamma} \cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}) = \sum_{n=-\infty}^{-1} \zeta^n \frac{1}{2\pi i} \oint_C d\xi \, \frac{1}{\xi^{n+1}} s^{\mu,\zeta}_{\Gamma\setminus\gamma} \cdot \varpi^*_{\Gamma,\gamma} s^{\mu,\xi}_\gamma \,, \tag{V.6}$$

where $C \subset \Omega \setminus \{0\}$ is a small circle around the origin. The product of the distributions under the complex line integral is defined by (V.3) for an appropriate parameter value ζ and almost all values of ξ. Observe also that the extension map ($u \mapsto \dot{u}$) is continuous for a homogeneous distribution, in the case the map is uniquely defined, cf. Theorem I.12. The fact that we regard finite sums of homogeneous distributions (i.e., heterogeneous distributions of finite order) does not spoil this continuity, and hence the extension of the distribution under the integral above commutes with the integration over one of its parameters.

In Zimmermann's forest formula [Zim69, Thm. 3.3], if one reads it as if it was formulated in position space with the above definitions, there occur nested projections of the form

$$\left(1 - T^{\mathrm{MS}}_G\right) \left(1 - T^{\mathrm{MS}}_\gamma\right) s^{\mu,\zeta}_{G,\mathrm{prep}} = \mathrm{rp}\left[s^{\mu,\zeta}_{G\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})\right] , \quad \gamma \subset \mathrm{G}. \tag{V.7}$$

Shortly after the publication in 1969 Zimmermann himself realized that not all

nested projections of the above form contribute in the limit where the regularization is removed, i.e., $\zeta \to 0$ [Zim70]. The projection $\left(1 - T_\gamma^{\mathrm{MS}}\right)$ in (V.7) is redundant, if G and γ have the same set of vertices. This, in turn, leads to the fact that only Epstein-Glaser graphs contribute to the forest formula. Zimmermann used the Pauli-Villars regularization method to prove this fact in [Zim76]. In the momentum space version of dimensional regularization and minimal subtraction the canceling of spurious terms in the limit has also been observed by Falk, Häußling, and Scheck by calculating explicit examples. Consequently the authors proposed an alternative renormalization method in momentum space, which takes into account the spurious subtractions [FHS10].

We want to use Zimmermann's observation as a test of our position space dimensional regularization method, and the prescription for minimal subtraction. We will see that in position space, i.e., for the nested projection (V.7) with the definitions given above, Zimmermann's result is a direct consequence of the fact that we can write the projection to the regular part, rp, as a W-projection on test functions (up to a term of $\mathcal{O}(\zeta)$), cf. Equation (I.20).

PROPOSITION V.4 (Redundant Projections). *Let $\gamma \subsetneq G$ be two BPHZ subgraphs of $\Gamma \in \mathcal{G}$ with the same vertex set, i.e., γ is a pure BPHZ subgraph,*

$$V(\gamma) = V(G)\,, \quad \textit{and} \quad E(\gamma) \subsetneq E(G)\,.$$

The contribution of the pure BPHZ subgraph $\gamma \subsetneq \Gamma$ to

$$\left(1 - T_G^{\mathrm{MS}}\right)\left(1 - T_\gamma^{\mathrm{MS}}\right) s^{\mu,\zeta}_{G,\mathrm{prep}} = \mathrm{rp}\left[s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})\right]$$

vanishes identically in the limit $\zeta \to 0$. That is, $\forall f \in \mathscr{D}(\mathbb{M}^{|V(\overline{\gamma})|-1})$:

$$\lim_{\zeta\to 0}\left\langle \mathrm{rp}(s^{\mu,\zeta}_{G,\mathrm{prep}}), f\right\rangle = \lim_{\zeta\to 0}\left\langle \mathrm{rp}\left[s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})\right], f\right\rangle. \tag{V.8}$$

PROOF. The argument of the limit on the right hand side of (V.8) can be rewritten using (I.20),

$$\begin{aligned}\left\langle \mathrm{rp}\left[s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})\right], f\right\rangle &= \left\langle s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{rp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}), W_G^{\mathrm{MS}} f\right\rangle + \mathcal{O}(\zeta) \\ &= \left\langle s^{\mu,\zeta}_{G,\mathrm{prep}}, W_G^{\mathrm{MS}} f\right\rangle - \left\langle s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}}), W_G^{\mathrm{MS}} f\right\rangle + \mathcal{O}(\zeta)\,,\end{aligned} \tag{V.9}$$

and we will show in the sequel that the second term in this expression vanishes identically for finite ζ.

The principal part $\mathrm{pp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})$ is a local distribution, $\mathrm{supp}(\mathrm{pp}(s^{\mu,\zeta}_{\gamma,\mathrm{prep}})) = \{0\}$, cf. (V.1). Hence also the product in the second term of (V.9) is supported at the origin,

$$\mathrm{supp}(s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma})) = \{0\}\,,$$

and thus local, $s^{\mu,\zeta}_{G\setminus\!\!\setminus\gamma} \cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma}) \in \mathscr{E}'_{\mathrm{Dirac}}$. The degree of divergence of this local distribution can be inferred directly from the scaling degrees of the individual

lines, cf. (I.16) and (III.23),

$$\begin{aligned}\mathrm{sd}(s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma})) &= |E(G\boxminus\gamma)|\,(d+\mathrm{Re}(\zeta)-2)+|E(\gamma)|\,(d-2)\ ,\\ \mathrm{div}(s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma})) &= |E(G)|\,(d-2)-|V(G)|\,(d-1)+|E(G\boxminus\gamma)|\,\mathrm{Re}(\zeta)\\ &= \mathrm{div}(G)+|E(G\boxminus\gamma)|\,\mathrm{Re}(\zeta)\,.\end{aligned}$$

Hence we infer that

$$s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma}) = \sum_{|\alpha|\leqslant\left\lfloor \mathrm{div}(s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma}))\right\rfloor} C_\alpha(\zeta)\delta^{(\alpha)}\,,$$

where $\lfloor\cdot\rfloor$ denotes, as before, Gauß's floor function. Given $\left\lfloor \mathrm{div}\left(s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma})\right)\right\rfloor = \mathrm{div}(G)$, which is the case if $\mathrm{Re}(\zeta) < \frac{1}{|E(G\boxminus\gamma)|}$, we have that

$$\forall g\in\mathscr{D}_{\mathrm{div}(G)}:\quad \left\langle s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma}), g\right\rangle = 0$$

and hence, by the uniqueness property of analytic functions,

$$\forall f\in\mathscr{D}:\quad \left\langle s^{\mu,\zeta}_{G\boxminus\gamma}\cdot \mathrm{pp}(s^{\mu,\zeta}_{\gamma}), W^{\mathrm{MS}}_G f\right\rangle = 0$$

for ζ in a neighborhood of the origin. □

As a matter of fact Proposition V.4 implies only that all forests containing the same set of full vertex parts give the same contribution to the sum. Hence it could happen that the contributions add up to give multiple contributions to the forest formula. However, one can show that of all forests with the same set of full vertex parts only one contributes to Zimmermann's formula. The combinatorial argument is also given in Zimmermann's proof in [Zim76]. We don't want to repeat it at this point, since the result is implied by the forest formula for regularized Epstein-Glaser renormalization we will prove in the next chapter. Motivated by these results we drop the cumbersome distinction between Epstein-Glaser - and BPHZ subgraphs and define a subgraph to be what we called to this point an Epstein-Glaser subgraph or full vertex part.

DEFINITION V.5 (Subgraph). Let $\Gamma\in\mathcal{G}$ be a graph. We define a subgraph $\gamma\subseteq\Gamma$ to be given by a subset of the set of vertices $V(\gamma)\subseteq V(\Gamma)$ and all lines in Γ connecting them,

$$E(\gamma) = \{e\in E(\Gamma) : \{\mathfrak{s}(e),\mathfrak{t}(e)\}\subset V(\gamma)\}\,. \tag{V.10}$$

We explicitly allow single vertices as subgraphs, and since there are no tadpoles in $\mathcal{G}$ (cf. Section IV.1), these one vertex subgraphs will have no lines. Observe that also $\Gamma\subseteq\Gamma$, trivially, is a subgraph.

V.4. MS for the Time-ordered Product

Regard the set $\mathcal{G}_V\subset\mathcal{G}$ of all graphs with the same set of vertices V,

$$\mathcal{G}_V = \{\Gamma\in\mathcal{G} : V(\Gamma) = V\}\,.$$

This set gives all graph contributions to the order $|V|$ of causal perturbation theory, and $\mathcal{G}_V(|E|) = \mathcal{G}_\alpha$, $\dim(\alpha) = |V|$, $|\alpha| = 2\,|E|$, is finite if one regards only the contributions up to a given order $|E|$ in $\hbar$, cf. Section IV.1 and Equation (IV.12). Assume that we have a prepared amplitude $s^{\mu,\zeta}_{\Gamma,\mathrm{prep}} \in \mathscr{D}'(\mathbb{M}^{|V|-1})$ for all graphs $\Gamma \in \mathcal{G}_V(|E|)$ at any order $|E|$ of $\hbar$. Let $S^{\mu,\zeta}_{\Gamma,\mathrm{prep}} \in \mathscr{D}'(\mathbb{M}^{|V|})$ be the corresponding translation invariant amplitude defined for local functionals in the sense of formal power series in $\hbar$. Then we can write the minimal subtraction operator at order $|V|$ of causal perturbation theory on the level of graph amplitudes as,

(V.11)

$$R_V(\boldsymbol{\mathcal{T}}^{|V|}_{\mu,\zeta,\mathrm{prep}}) = \sum_{\Gamma\in\mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)}\, R_V\Big\langle S^{\mu,\zeta}_{\Gamma,\mathrm{prep}}, \boldsymbol{\delta}^\alpha\Big\rangle \quad \text{with} \quad R_V = \begin{cases} \mathrm{id} & \text{if } |V| = 1 \\ -\mathrm{pp} & \text{if } |V| > 1\,. \end{cases}$$

Here $\mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}})$ is the translation invariant analogue of the local distribution $\mathrm{pp}(s^{\mu,\zeta}_{\Gamma,\mathrm{prep}}) \in \mathscr{E}'_{\mathrm{Dirac}}(\mathbb{M}^{|V(\Gamma)|-1})$ defined above, $\mathrm{supp}(\mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}})) \subset \mathrm{Diag}(\mathbb{M}^{|V(\Gamma)|})$. We want to apply the corresponding term in the above sum to a tensor product of local functionals. Analogous to the discussion in Section IV.1 we get

(V.12)

$$\Big\langle -\mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}}), \boldsymbol{\delta}^\alpha\Big\rangle \big(\bigotimes_{v\in V} F_v\big) = \Big\langle -\mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}}), \bigotimes_{v\in V(\Gamma)} f^v_\varphi\, \delta(\boldsymbol{r}_v)\Big\rangle\,, \quad f^v_\varphi \in \mathscr{D}(\mathbb{M}),$$

where $f^v_\varphi \in \mathscr{D}(\mathbb{M})$ is a sum of pointwise products of test functions with the field φ. Since $-\mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}})$ is supported on the thin diagonal, all functions f^v_φ are evaluated at the same point and the expression on the right hand side of (V.12) gives a local functional. We have

$$\Big\langle \mathrm{pp}(S^{\mu,\zeta}_{\Gamma,\mathrm{prep}}), \boldsymbol{\delta}^\alpha\Big\rangle : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes|V|} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]\,.$$

The fact which establishes the independence of the presented formalism on the chosen representation, is that the projection to the principal part, pp, is an operation with respect to the parameter ζ, and can be performed outside the brackets " $\langle\cdot\rangle$ ". Actually it was defined like that in Section I.4. Although the evaluation of these brackets might look very different, depending on the chosen representation. Thus minimal subtraction is really an operation which can be performed directly on time-ordered products,and it is sensible to define

DEFINITION V.6 (Minimal Subtraction Operator on Subsets). For any vertex set V, we define the *minimal subtraction operator (MS operator) on subsets* as

$$R_V(\boldsymbol{\mathcal{T}}^{|V|}_{\mu,\zeta,\mathrm{prep}}) := \begin{cases} \mathrm{id} & \text{if } |V| = 1 \\ -\mathrm{pp}(\boldsymbol{\mathcal{T}}^{|V|}_{\mu,\zeta,\mathrm{prep}}) & \text{if } |V| > 1\,, \end{cases}$$

where $\mathrm{id} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ is the identity map on local functionals, and

$$-\mathrm{pp}(\boldsymbol{\mathcal{T}}^{|I|}_{\mu,\zeta,\mathrm{prep}}) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes|I|} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$$

is the local counterterm at order $|I|$ of causal perturbation theory.

CHAPTER VI

The Epstein-Glaser Forest Formula

'What's the sandwich scenario, Mo?'
'Ham and cheese; ham and tomato; cheese and tomato.'
'And ham, cheese and tomato.'
'How did you know?'
'You've never noticed how you group sandwiches into Venn diagrams?'
'Do I?'

David Mitchell: Ghostwritten

It was the principle of covariance, understood as the axiom that all physically relevant concepts must have an analogue in (globally hyperbolic) curved spacetime, which brought to light the more profound structures of perturbative renormalization theory in the investigation undertaken by Brunetti, Dütsch, Fredenhagen, Hollands, and Wald (see references in the introduction).[1] As already said in the introduction of this thesis one of the main results of their program was the formulation of perturbative Algebraic Quantum Field Theory (pAQFT), briefly introduced in Chapter II. In this last chapter we will show that the tools of pAQFT and in particular the precise statement of Stora's main theorem of perturbative renormalization, augmented by the results on analytic regularization we have gained in the previous chapters will make it possible to solve the recursive procedure of Epstein-Glaser renormalization and to prove a forest formula in the sense of Zimmermann for Epstein-Glaser renormalization. The result will be independent of the chosen representation and will in particular be applicable in momentum and in position space. The main theorem of renormalization, written in termwise form by using Faà di Bruno's formula for the n-fold chain rule [FdB55], implies a recursion relation for the minimally subtracted counterterms to an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix. This recursion relation will be crucial for the proof of the forest formula.

In 1982 Joni and Rota introduced a bialgebra related to Faà di Bruno's formula [JR82]. We will use this bialgebra to derive (a summed up version of) the Connes-Kreimer Hopf algebra of graphs directly from the main theorem of renormalization. However, in contrast to the Connes-Kreimer approach the Feynman

[1] The covariance principle was made precise in [BFV03]. And we want to use this footnote to remark that despite its reputation of being conceptually clear but "too far from reality" to have predictive power for experiments the algebraic approach and in particular perturbative Algebraic Quantum Field Theory has lead to falsifiable predictions in cosmology [DFP08].

rules will emerge naturally and are not assumed to be characters into the commutative ring of Laurent series with scalar coefficients. The emergent Feynman rules will rather produce linear maps between spaces of (local) functionals. On the space of linear maps the construction induces two products, a symmetrized tensor product, and a non-commutative product, which is given as the composition of linear maps. Both products, as well as the coproduct, need to be reflected in the Hopf algebra (of graphs) in order to encode the algebraic structure of the recursive construction of counterterms. By giving this derivation we will establish the relation of the pAQFT formalism to the "Hopf algebra school" which was not present in the original pAQFT article (cf. [BDF09, p. 45]).

After some preliminary remarks on the differential calculus used in this chapter, we will cite the main theorem of renormalization from [BDF09] in the second section. The third section will be devoted to the derivation of a forest formula for regularized Epstein-Glaser renormalization from Stora's main theorem. The above described Hopf algebra will be constructed in the fourth section of this chapter.

VI.1. Preliminaries on differential calculus

We take the elevator in the hierarchy of differential calculi one floor up and want to regard functional derivatives of the $\boldsymbol{\mathcal{S}}$-matrix, regarded as a map between spaces of (local) functionals,

$$\boldsymbol{\mathcal{S}} \equiv \exp_{\cdot_{\mathcal{T}}} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}(\mathbb{M})[[\hbar]] \,,$$

and of the renormalization group transformations $Z \in \mathcal{R}$ to be defined below as maps,

$$Z : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \,.$$

The n-fold derivative of $\boldsymbol{\mathcal{S}}$ at the origin

$$\boldsymbol{\mathcal{S}}^{(n)}(0)(F^{\otimes n}) = \frac{d^n}{d\lambda^n}\boldsymbol{\mathcal{S}}(\lambda F)\Big|_{\lambda=0}$$

gives the n-fold time-ordered product, i.e. the nth coefficient in the series expansion of $\boldsymbol{\mathcal{S}}$, cf. Equation (IV.12). The n-fold derivative of $Z \in \mathcal{R}$ gives the counterterm at order n of causal perturbation theory. We will equivalently use $\boldsymbol{\mathcal{S}}^{(n)} = \boldsymbol{\mathcal{S}}^{(n)}\big|_0 \equiv \boldsymbol{\mathcal{S}}^{(n)}(0)$, and likewise for Z, wherever there is no risk of confusion.

The mathematically precise definition of such a differential calculus is quite involved and a focus of research in analysis [Ham82, KM97, Nee05] (taking the stairs here, might be very hard). However, it is enough for our purposes to assume that a calculus can be defined in such a way that the corresponding differential,

$$\frac{\delta}{\delta F} : \boldsymbol{\mathcal{S}} \mapsto \boldsymbol{\mathcal{S}}^{(1)}(F) \,,$$

fulfills the chain - and the Leibniz rule in the sense below. A calculus fulfilling the chain rule was defined for locally convex spaces in [Nee05]. And as shown in [BDF09, Sec. 3.1] $\mathcal{F}(\mathbb{M})$ can be endowed with a locally convex topology, defined

as the initial topology of the Hörmander topology on spaces of distributions with conic wave front set.

Let Z be differentiable at F, and $\boldsymbol{\mathcal{S}}$ be differentiable at $Z(F)$, then we want to assume that the derivative of their composition is given by the chain rule

$$(\boldsymbol{\mathcal{S}} \circ Z)^{(1)}(F) = \left(\boldsymbol{\mathcal{S}}^{(1)} \circ Z\right)(F) \cdot \left(Z^{(1)}(F)\right) \equiv \left.\frac{\delta \boldsymbol{\mathcal{S}}}{\delta F'}\right|_{Z(F)} \cdot \left.\frac{\delta Z}{\delta F'}\right|_{F}, \tag{VI.1}$$

where on the right hand side we have a composition of linear maps, generally denoted by " $\cdot$ " in this chapter. For the iteration of the chain rule and the proof of the n-fold chain rule in Lemma VI.4 we will also need that the derivative fulfills the Leibniz rule in the following sense,

$$(Z \otimes Z)^{(1)} = Z^{(1)} \otimes Z + Z \otimes Z^{(1)}. \tag{VI.2}$$

We call a map

$$\boldsymbol{\Psi} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}(\mathbb{M})[[\hbar]]$$

analytic (at F), if the nth functional derivative exists for all $n \in \mathbb{N}$ as a totally symmetric, linear map

$$\boldsymbol{\Psi}^{(n)}(F) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}(\mathbb{M})[[\hbar]],$$

and

$$\boldsymbol{\Psi}^{(n)}(F) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]], \quad \text{if } \mathrm{im}(\boldsymbol{\Psi}) \subset \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]. \tag{VI.3}$$

VI.2. The Main Theorem of Renormalization

An important insight in perturbative renormalization theory is the fact that the freedom in the definition of the $\boldsymbol{\mathcal{S}}$-matrix can be described in terms of the Stückelberg-Petermann renormalization group [SP53]. Popineau and Stora termed this fact the "main theorem of perturbative renormalization theory" [PS82]. One can find it, although not under this name, already in the early literature of renormalization theory [GML54, BS59]. Modern versions are included in [Pin01, Gri01b]. The precise statement and proof of this theorem in the algebraic approach to perturbative QFT [DF04, DF07] made it possible to show that the renormalization group of Stückelberg and Petermann provides a common basis also to other renormalization groups found in literature [BDF09]. We will give here a minimalistic review of the basic definitions needed to formulate the main theorem of renormalization in pAQFT. A more detailed summary, including a sketch of the proof is contained in Section 4.1 of [BDF09].

The $\boldsymbol{\mathcal{S}}$-matrix

$$\begin{array}{rccl} \boldsymbol{\mathcal{S}} : & \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] & \to & \mathcal{F}(\mathbb{M})[[\hbar]] \\ & F & \mapsto & \boldsymbol{\mathcal{S}}(F) = \exp_{\cdot_{\mathcal{T}}}(F) \end{array}$$

is analytic at the origin, where its derivatives are given by the n-fold time ordered products. However, $\boldsymbol{\mathcal{S}}$ is not unique, but needs to be defined perturbatively by renormalization. As shown in [BDF09] the prerequisites needed for a definition of

$\boldsymbol{\mathcal{S}}$ within causal perturbation theory can be expressed directly in terms of properties of the $\boldsymbol{\mathcal{S}}$-matrix itself. Let $A, B, F \in \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]]$, then $\boldsymbol{\mathcal{S}}$ is required to fulfill the following conditions,

[C1] Causality. $\boldsymbol{\mathcal{S}}(A+B) = \boldsymbol{\mathcal{S}}(A) \star \boldsymbol{\mathcal{S}}(B)$, if $\text{supp}(A) \gtrsim \text{supp}(B)$.

[C2] Starting Element. $\boldsymbol{\mathcal{S}}(0) = 1$, $\boldsymbol{\mathcal{S}}^{(1)}(0) = \text{id}$.

[C3] φ-Locality. The value of $\boldsymbol{\mathcal{S}}(F) \in \mathcal{F}(\mathbb{M})[[\hbar]]$ at a given field configuration φ_0 depends only on the Taylor expansion of $F \in \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]]$ around φ_0,

$$\boldsymbol{\mathcal{S}}(F)(\varphi_0) = \boldsymbol{\mathcal{S}}(F^{[N]}_{\varphi_0})(\varphi_0) + \mathcal{O}(\hbar^{N+1}),$$

where $F^{[N]}_{\varphi_0}(\varphi) = \sum_{n=0}^{N} \frac{1}{n!} \left\langle F^{(n)}(\varphi_0), (\varphi - \varphi_0)^{\otimes n} \right\rangle$ denotes the Taylor expansion of F up to order N.

[C4] Field Independence. $\boldsymbol{\mathcal{S}}$ depends only implicitly, i.e. via the interaction F, on the field configuration,

$$\forall \psi : \left\langle \frac{\delta \boldsymbol{\mathcal{S}}(F)}{\delta \varphi}, \psi \right\rangle = \boldsymbol{\mathcal{S}}^{(1)}(0)(F) \left\langle \frac{\delta F}{\delta \varphi}, \psi \right\rangle.$$

While [C1] and [C2] are directly related to the inductive procedure of Epstein-Glaser, condition [C3] implies that only finitely many terms will contribute if one cuts the perturbative expansion of the $\boldsymbol{\mathcal{S}}$-matrix at a given order in $\hbar$, see also the discussion at the end of Chapter IV. This makes it possible to regard also more general, and in particular non-polynomial interactions $F \in \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]]$ in pAQFT. Furthermore [C3] implies together with the fourth condition [C4] the Wick expansion formula for the time-ordered product of Epstein and Glaser [EG73]. This is needed to reduce the problem of renormalizing $\boldsymbol{\mathcal{S}}$ to an extension problem for distributions. See the discussion in [BDF09, Sec. 4.1] and also [Kel09, Sec. 4.B].

The freedom in the definition of the $\boldsymbol{\mathcal{S}}$-matrix is described by the Stückelberg-Petermann renormalization group $\mathcal{R}$. In the framework of perturbative Algebraic Quantum Field Theory $\mathcal{R}$ is the group of analytic maps

$$Z : \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]],$$

with composition as group operation, and $Z \in \mathcal{R}$ having the following properties,

[RG1] $Z(0) = 0$

[RG2] Starting Element. $Z^{(1)}(0) = \text{id}$

[RG3] $Z = \text{id} + \mathcal{O}(\hbar)$

[RG4] Locality. Let $A, B, C \in \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]]$ with $\text{supp}(A) \cap \text{supp}(C) = \emptyset$, then

$$Z(A+B+C) = Z(A+B) - Z(B) + Z(B+C)$$

[RG5] φ-Locality. $Z(F)(\varphi_0) = Z(F^{[N]}_{\varphi_0})(\varphi_0) + \mathcal{O}(\hbar^{N+1})$

[RG6] Field Independence. Z depends only implicitly on the field φ,

$$\forall \varphi \in \mathscr{E}(\mathbb{M}) : \frac{\delta Z}{\delta \varphi} = 0.$$

With these definitions at hand, we can now formulate

THEOREM VI.1 (Main Theorem of Renormalization, cf. [BDF09, Thm. 4.1]). *Given two $\boldsymbol{\mathcal{S}}$-matrices $\boldsymbol{\mathcal{S}}$ and $\widehat{\boldsymbol{\mathcal{S}}}$ satisfying the conditions Causality, Starting Element, φ-Locality, and Field Independence, [C1]-[C4], there exists a unique $Z \in \mathcal{R}$ such that*

$$\widehat{\boldsymbol{\mathcal{S}}} = \boldsymbol{\mathcal{S}} \circ Z\,. \tag{VI.4}$$

Conversely, given an $\boldsymbol{\mathcal{S}}$-matrix $\boldsymbol{\mathcal{S}}$ satisfying [C1]-[C4] and a $Z \in \mathcal{R}$, then (VI.4) defines a new $\boldsymbol{\mathcal{S}}$-matrix satisfying conditions [C1]-[C4].

We will be interested in this chapter mainly in a special class of scattering matrices, which we define now.

DEFINITION VI.2 (Analytically Regularized $\boldsymbol{\mathcal{S}}$-matrix). Any scattering matrix, $\boldsymbol{\mathcal{S}}_\kappa$, which fulfills the conditions [C1]-[C4] and depends analytically on an additional parameter $\kappa \in \Omega\setminus\{0\} \subset \mathbb{C}$, such that for all $n \in \mathbb{N}$, $n \geqslant 2$, the n-fold functional derivative,

$$\boldsymbol{\mathcal{S}}_\kappa^{(n)}(0) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}(\mathbb{M})[[\hbar]]\,,$$

is the analytic regularization of a time-ordered product outside the large diagonal in the sense of Proposition IV.3, we want to call an *analytically regularized $\boldsymbol{\mathcal{S}}$-matrix.*

Observe that the definition implies that the second derivative $\boldsymbol{\mathcal{S}}_\kappa^{(2)}(0)$ corresponds to an analytic regularization in the strict sense of Definition I.16. In the functional framework this implies

$$\mathrm{pp}(\boldsymbol{\mathcal{S}}_\kappa^{(2)}(0)) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes 2} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]\,,$$

cf. Section V.4. An example for such an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix is the unique dimensionally regularized $\boldsymbol{\mathcal{S}}$-matrix $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$ of Definition IV.4. This follows directly from its construction, since it was defined using the methods of Epstein-Glaser renormalization. However, that $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$ fulfills [C1]-[C4] is also readily seen from its perturbative expansion (IV.12). And we will take $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$ as an example, wherever it is necessary to introduce a regularization in the discussion below.

By the above theorem, the Stückelberg-Petermann renormalization group acts transitively on all $\boldsymbol{\mathcal{S}}$-matrices fulfilling [C1]-[C4]. Thus, if we want to find a finitely regularized $\boldsymbol{\mathcal{S}}$-matrix $\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}}$ which also fulfills [C1]-[C4] we will have to construct an element $Z_{\mu,\zeta}$ of the Stückelberg-Petermann renormalization group, such that

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}} = \boldsymbol{\mathcal{S}}_{\mu,\zeta} \circ Z_{\mu,\zeta}\,. \tag{VI.5}$$

has a limit $\zeta \to 0$ in the set of $\boldsymbol{\mathcal{S}}$-matrices. That is

$$\boldsymbol{\mathcal{S}}_{\mu,\mathrm{ren}} := \lim_{\zeta\to 0} \left(\boldsymbol{\mathcal{S}}_{\mu,\zeta} \circ Z_{\mu,\zeta}\right) \tag{VI.6}$$

exists in the sense of formal power series in $\hbar$ term by term in the perturbative expansion; see also [BDF09, Sec. 5.2]. In Epstein-Glaser renormalization the construction of these local counterterms, i.e., the perturbative definition of the map Z

has to be done recursively, i.e., term by term in the perturbative expansion starting with the counterterm $Z^{(2)}$ for $\boldsymbol{\mathcal{S}}^{(2)}(0)$. There will be a choice involved in each step of this recursion and hence it is impossible to express it in an algorithm which computes, say, the n'th counterterm. However, in the case we dispose of a regularization, $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$, the second term, $\boldsymbol{\mathcal{S}}^{(2)}_{\mu,\zeta}$, is a regularization in the strict sense, and hence we have a preferred choice for the local counterterm,

$$Z^{(2)}_{\mu,\zeta} = -\mathrm{pp}(\boldsymbol{\mathcal{S}}^{(2)}_{\mu,\zeta}) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes 2} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \,.$$

which renormalizes the time ordered product,

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mu,\zeta,\mathrm{ren}} := \mathrm{rp}(\boldsymbol{\mathcal{S}}^{(2)}_{\mu,\zeta}) = (1-\mathrm{pp})\,\boldsymbol{\mathcal{S}}^{(2)}_{\mu,\zeta} \,.$$

It will be show in the following section that this preferred choice can be done at all orders of perturbation theory in a consistent way, i.e., with local counterterms at all orders. This, in turn, makes it possible to solve the recursive renormalization procedure of Epstein-Glaser in quite the same way as it was done by Zimmermann in 1969 for BPH in momentum space. We will derive from Equation (VI.5) a forest formula for Epstein-Glaser renormalization which solves the recursive construction of counterterms to all orders in causal perturbation theory. We want to remark that the choices at all orders are unique in the minimal subtraction scheme, such that this leads to a recursive procedure, which, in principle, can also be taught to a computer - in contrast to the original Epstein-Glaser method.

The relation of the presented method to the modern formulation of renormalization in terms of Hopf algebras will be given in Section VI.4.

VI.3. A Forest Formula for Epstein-Glaser Renormalization

Since we will stay in the functional framework throughout the derivation of the forest formula, the result will be valid independent of the chosen representation, in particular it holds for momentum space as well as position space, whatever is the best suited representation for the regularization. Furthermore, it is formulated without regard to the graph expansion of the time-ordered product. Partitions will take the place of graphs as the basic combinatorial objects. However, analogous to the discussion in Section V.4, the forest formula for Epstein-Glaser renormalization also holds in a graph by graph manner, and then implies Zimmermann's forest formula of [Zim69] enhanced by his discussion on spurious subtractions in [Zim76]; see also Proposition V.4 and the discussion thereafter. However, the combinatorial structures used here will make the role of forests in Zimmermann's formula even more transparent.

That a version of Zimmermann's forest formula should also exist in position space was observed before. And the assertion is natural considering the common origin of BPHZ and Epstein-Glaser renormalization. It has been shown that Zimmermann's Taylor subtractions with respect to external momenta of the graphs correspond to the W-projections in the Epstein-Glaser framework [Pra99, Pra00].

Gracia-Bondía and Lazzarini gave a direct translation of this "Taylor surgery" (GB) to position space by considering a more general test function space for the "infrared regulators", i.e. the test functions w_α of Lemma I.7, in fact they allowed the w_α to be distributions of the Cesàro type [GB03, GBL03]. A translation of the complete forest formula to position space was given by Steinmann in the case of QED [Ste00]. However, Steinmann's treatment was unsatisfactory in two points. First, Steinmann's formulation involves the differentiation of (generalized) functions at singular points. This was recognized by the author himself and is due to the fact that the implicit regularization[2] of the momentum space framework, namely the fact that Zimmermann performs his manipulations on the integral kernel of the convolution rather than the integral itself, has no counterpart in position space. Momentum space convolution corresponds to the pointwise product (of distributions) in position space and it is partly due to this implicit regularization that momentum space integrals were introduced in perturbative quantum field theory in the first place [BP57]. Second, Steinmann regards Quantum Electro Dynamics (QED). The fact that QED has only one basic vertex of valence three implies that there are no graphs with less lines but the same set of vertices so that the spurious subtractions do not occur in QED and other theories "of graphical φ^3-type". Consequently, Steinmann's version of the forest formula cannot be considered as a complete translation of Zimmermann's forest formula (which treats general graphs in $\mathcal{G}$) to position space. Observe that Zimmermann implements a preferred choice for the extension at all orders in perturbation theory by performing his Taylor subtractions always at zero external momentum. In order to define this rigorously he has to introduce additional maps which conceal part of the underlying pattern. However, as already remarked above and as will be clear from the construction below, such a choice of extension at all orders of perturbation theory is indispensable for the solution of the recursive procedure of Bogoliubov, Parasiuk and Hepp, or Epstein and Glaser, respectively. We start by exploring the termwise structure of the main theorem (Theorem VI.1), by applying the n-fold derivative to (VI.4). The Faà di Bruno formula arises naturally.

VI.3.1. Faà di Bruno's formula. In 1855 Francesco Faà di Bruno proved a formula for the n-fold chain rule [FdB55]. And it is quite appealing that this old formula, when applied to Equation (VI.4) gives a termwise version of the main theorem of perturbative renormalization (Theorem VI.1). Considering the time since its first proof, there are quite a few versions of Faà di Bruno's formula in the literature today. However, in order to keep the relation to causal perturbation theory and BPHZ renormalization visible at all steps in our calculation, a set partition version of the form found in [Joh02, p. 219] seems to be the most appropriate. We prove here an adjusted version. But let us first give an easy definition, mainly to fix notation.

[2] I hope this is the only spot in the thesis where I use the word "regularization" only in the sense of "making things well-defined".

DEFINITION VI.3 (Partition, Blocks). By a *partition* $\mathcal{P}$ of a finite set V we mean any set of non-empty, disjoint subsets $V_i \subset V$, $i \in I$, such that

$$V = \dot{\bigcup_{i\in I}} V_i\,, \qquad \text{that is,} \quad \mathcal{P} = \{V_i : i \in I\}\,,$$

where $\dot{\cup}$ denotes disjoint union. We refer to the non-empty, disjoint subsets V_i as *blocks* of $\mathcal{P}$, and denote the set of all partitions of V by $\mathrm{Part}V$.

We generally consider partitions of the set which corresponds to the set of vertices, $V(\Gamma)$, in the graphical representation, and in most cases it will be more convenient to regard instead the set of numbers $\{1, \ldots, n\}$. However, this implicit numbering of vertices, is irrelevant for the derivation due to the symmetry of the functional derivative briefly introduced in Section VI.1.

LEMMA VI.4 (Main Theorem - termwise). *Let $\boldsymbol{\mathcal{S}} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}(\mathbb{M})[[\hbar]]$ be an $\boldsymbol{\mathcal{S}}$-matrix fulfilling conditions [C1]-[C4] and let $Z : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ be an element of the Stückelberg-Petermann renormalization group, $Z \in \mathcal{R}$. Then the nth term in the perturbative expansion of the transformed $\boldsymbol{\mathcal{S}}$-matrix, $\widehat{\boldsymbol{\mathcal{S}}} = \boldsymbol{\mathcal{S}} \circ Z$, is given by*

$$(\boldsymbol{\mathcal{S}} \circ Z)^{(n)}(0) = \sum_{\mathcal{P}\in \mathrm{Part}\{1,\ldots,n\}} \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}(0) \cdot \left(\bigotimes_{I\in\mathcal{P}} \left[Z^{(|I|)}(0) \right] \right), \tag{VI.7}$$

where the sum is over all partitions $\mathcal{P}$ of the index set $\{1, \cdots, n\}$ into $|\mathcal{P}|$ blocks, and " $\cdot$ " denotes the composition of linear maps,

$$\mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \xrightarrow{\bigotimes_{I\in\mathcal{P}} Z^{(|I|)}(0)} \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes|\mathcal{P}|} \xrightarrow{\boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}(0)} \mathcal{F}(\mathbb{M})[[\hbar]]. \tag{VI.8}$$

PROOF. We prove (VI.7) by induction following [Joh02]. For $n = 1$ we have,

$$(\boldsymbol{\mathcal{S}} \circ Z)^{(1)}\Big|_F = \boldsymbol{\mathcal{S}}^{(1)}\Big|_{Z(F)} \cdot Z^{(1)}\Big|_F .$$

However, evaluating at $F = 0$ gives an empty assertion (id = id) due to the starting element conditions [C2], [RG1], and [RG2]. The first non-trivial contribution is from the second derivative, which we compute explicitly for illustration,

$$\begin{aligned}(\boldsymbol{\mathcal{S}} \circ Z)^{(2)}\Big|_F &= \left(\boldsymbol{\mathcal{S}}^{(1)}\Big|_{Z(F)} \cdot Z^{(1)}\Big|_F \right)^{(1)}\Big|_F \\ &= \boldsymbol{\mathcal{S}}^{(2)}\Big|_{Z(F)} \cdot Z^{(1)}\Big|_F \otimes Z^{(1)}\Big|_F + \boldsymbol{\mathcal{S}}^{(1)}\Big|_{Z(F)} \cdot Z^{(2)}\Big|_F ,\end{aligned}$$

where the two terms correspond to the two partitions $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$ of $\{1, 2\}$. Evaluating at $F = 0$ gives, again by using [C2], [RG1], and [RG2],

$$(\boldsymbol{\mathcal{S}} \circ Z)^{(2)}(0) = \boldsymbol{\mathcal{S}}^{(2)}(0) + Z^{(2)}(0)\,,$$

and $Z^{(2)} = (\boldsymbol{\mathcal{S}} \circ Z)^{(2)} - \boldsymbol{\mathcal{S}}^{(2)}$ is found to be the counterterm at second order.

For the induction step regard the derivative of (VI.7). By (VI.1)/(VI.2) we get,

$$\left[\boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}\Big|_{Z(F)}\left(\bigotimes_{I\in\mathcal{P}}\left[Z^{(|I|)}\Big|_F\right]\right)\right]^{(1)} = \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|+1)}\Big|_{Z(F)}\left(Z^{(1)}\Big|_F\otimes\bigotimes_{I\in\mathcal{P}}\left[Z^{(|I|)}\Big|_F\right]\right)$$

$$+\sum_{I'\in\mathcal{P}}\boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}\Big|_{Z(F)}\left(Z^{(|I'|+1)}\Big|_F\otimes\bigotimes_{I\in\mathcal{P}\setminus\{I'\}}\left[Z^{(|I|)}\Big|_F\right]\right). \tag{VI.9}$$

Any partition $\widetilde{\mathcal{P}}$ of $\{1,\dots,n+1\}$ can be written in terms of a partition $\mathcal{P}$ of $\{1,\dots,n\}$ by either adjoining $\{n+1\}$ as a block of its own, or by adding $\{n+1\}$ to one of the blocks in $\mathcal{P}$, i.e.,

$$\widetilde{\mathcal{P}} = \mathcal{P}\cup\{\{n+1\}\} \qquad \text{or} \qquad \widetilde{\mathcal{P}} = \left(\mathcal{P}\setminus\{I'\}\right)\cup\left\{I'\cup\{n+1\}\right\},$$

for some block $I'\in\mathcal{P}$. Thus equation (VI.9) contains all partitions of $\{1,\dots,n+1\}$ which can be obtained from $\mathcal{P}$. Evaluating at $F=0$ gives the result. □

Equation (VI.7) describes the action of the Stückelberg-Petermann group on time-ordered products. This action followed directly from the main theorem by applying Faà di Bruno's formula. Since Equation (VI.7) is not the most cited version of Faà di Bruno's formula, we want to show that it reduces to the more prevalent versions if we evaluate $(\boldsymbol{\mathcal{S}}\circ Z)^{(n)}(0)$ at the n-fold tensor power of one and the same interaction functional, $F^{\otimes n}\in\mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n}$. Due to the symmetry of the functional derivative partitions with identical block sizes will give the same contribution to $(\boldsymbol{\mathcal{S}}\circ Z)^{(n)}(0)(F^{\otimes n})$. So the question is: How many of them are there? Depending on how one chooses to sort these partitions, one gets the different versions of Faà di Bruno's formula. As an example we give one of the derivations.

Let $\mathcal{P}\in\mathrm{Part}\{1,\dots,n\}$ be a partition with $|\mathcal{P}|=k$ blocks. Let $(l_1,\dots,l_k)\in\mathbb{N}^k$ denote the sizes of these blocks, $l_1+\cdots+l_k=n$, $l_i\geqslant 1$. There are $\binom{n}{l_1,\dots,l_k}$ possibilities to distribute n elements among k different blocks of specified size and order.[3] However, for a partition the order of the blocks is irrelevant. In a sum over the multiindex $(l_1,\dots,l_k)\in\mathbb{N}^k$ we thus have to divide by the number of permutations of $\{l_1,\dots,l_k\}$ to reduce it to a sum over all partitions. In total we get

$$\begin{aligned}(\boldsymbol{\mathcal{S}}\circ Z)^{(n)}(F^{\otimes n}) &\\ = \sum_{k=1}^{n}\frac{1}{k!}\boldsymbol{\mathcal{S}}^{(k)}\cdot\sum_{\substack{l_1+\cdots+l_k=n\\ l_i\geqslant 1}}\binom{n}{l_1,\dots,l_k}&Z^{(l_1)}(F^{\otimes l_1})\otimes\cdots\otimes Z^{(l_k)}(F^{\otimes l_k}),\end{aligned} \tag{VI.10}$$

which was the starting point in [FGBV05] for the derivation of

$$\begin{aligned}(\boldsymbol{\mathcal{S}}\circ Z)^{(n)}(F^{\otimes n}) &\\ = \sum_{k=1}^{n}\sum_{\lambda_1,\dots,\lambda_n}\frac{n!}{\lambda_1!\cdots\lambda_n!}\boldsymbol{\mathcal{S}}^{(k)}\cdot&\left(\frac{Z^{(1)}}{1!}\right)^{\otimes\lambda_1}\otimes\cdots\otimes\left(\frac{Z^{(n)}}{n!}\right)^{\otimes\lambda_n}(F^{\otimes n}).\end{aligned} \tag{VI.11}$$

[3] $\binom{n}{l_1,\dots,l_k} := \frac{n!}{l_1!\cdots l_k!}$ denotes the *multinomial coefficient*, see, e.g., [HHM08].

Here $\lambda_l \in \mathbb{N}_0$ denotes the number of blocks of size l. Equation (VI.11) is the version, which is usually cited as Faà di Bruno's formula in the literature, and often is the starting point for the introduction of the Faà di Bruno bialgebra. We will learn more about this bialgebra in Section VI.4.

VI.3.2. Minimal Subtraction. Regard an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix, e.g., $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$. Then Lemma VI.4 implies a recursion relation for the counterterms $Z^{(n)}_{\mu,\zeta}$ in the minimal subtraction renormalization scheme introduced in the previous chapter.

COROLLARY VI.5 (Recursion Relation for MS Counterterms). *In the minimal subtraction renormalization scheme (MS), a recursion relation for the counterterms $Z^{(n)}_{\mu,\zeta}$ is given by*

$$Z^{(n)}_{\mu,\zeta} = -\mathrm{pp} \sum_{\mathcal{P}\in \mathrm{Part}\{1,\ldots,n\}\setminus\{\mathcal{P}_1\}} \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}_{\mu,\zeta} \cdot \left(\bigotimes_{I\in\mathcal{P}} Z^{(|I|)}_{\mu,\zeta} \right). \tag{VI.12}$$

The counterterms are local, and all counterterms on the right hand side are of lower order than n, since the only partition in $\mathrm{Part}\{1,\ldots,n\}$ *with a block containing n elements is removed from the sum, $\mathcal{P}_1 = \{\{1,\ldots,n\}\}$.*

PROOF. Using the starting element condition [C2] we get from (VI.7),

$$\left(\boldsymbol{\mathcal{S}}_{\mu,\zeta} \circ Z_{\mu,\zeta}\right)^{(n)} = Z^{(n)}_{\mu,\zeta} + \sum_{\mathcal{P}\in \mathrm{Part}\{1,\ldots,n\}\setminus\{\mathcal{P}_1\}} \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}_{\mu,\zeta} \cdot \left(\bigotimes_{I\in\mathcal{P}} Z^{(|I|)}_{\mu,\zeta} \right).$$

By assumption $Z_{\mu,\zeta} \in \mathcal{R}$ is a renormalization group transformation which renders the limit $\zeta \to 0$ of the left hand side finite, cf. Equation (VI.6). Thus the counterterm $Z^{(n)}_{\mu,\zeta}$ has to subtract at least the principal part of the sum on the right hand side. In the minimal subtraction scheme, $Z^{(n)}_{\mu,\zeta}$ is fixed by the requirement to remove exactly the principal part, hence formula (VI.12). The locality of $Z^{(n)}_{\mu,\zeta}$ is implied directly by the functional calculus, cf. (VI.3). □

The expert reader[4] readily recognizes the similarity of (VI.12) to the recursive formula for the antipode in the Faà di Bruno bialgebra. However, observe that there are two products involved in the recursion for the counterterms. The tensor product $\otimes$ and the composition of linear maps " $\cdot$ ". What might be obvious for the expert, namely that this is a structure which cannot be described by a commutative Hopf algebra alone, will be derived "by foot" in Section VI.4. However, let us first give the derivation of a forest formula for the n-fold finitely regularized time-ordered product $\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta,\mathrm{ren}}$. The forest formula will solve the inductive construction of the renormalization group transformation $Z_{\mu,\zeta}$, which renders $\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}} = \boldsymbol{\mathcal{S}}_{\mu,\zeta} \circ Z_{\mu,\zeta}$ finite (in the sense of formal power series in $\hbar$) in the limit $\zeta \to 0$.

[4] I assume here that $\{\text{readers}\} \neq \emptyset$. If you have a proof, please tell me: kai.johannes.keller@desy.de

VI.3.3. Derivation of the forest formula. We begin with the definition of a forest as it was given by Zimmermann. We will then derive some relations to partitions which will motivate the definition of an Epstein-Glaser forest and make it possible to give a transparent proof of the forest formula.

Let us start with Zimmermann's definition, which was contained in [Zim69]. However, we incorporate directly his results from [Zim76] and will only consider subgraphs, which are given by a subset of the set of vertices of a graph Γ and all lines in Γ connecting them, cf. Definition V.5. Zimmermann called such subgraphs *full vertex parts*. Observe that this makes it possible to work directly with the set of vertices instead of the set of general graphs or full vertex parts. However, to keep the relation to the original definitions transparent, let $\Gamma \in \mathcal{G}$ be a graph.

A Γ-*forest* U is a set of subgraphs $\gamma \subseteq \Gamma$, such that any two elements $\gamma, \gamma' \in U$ are *non-overlapping*, i.e.

$$\text{(VI.13)} \qquad \text{either} \quad \gamma \subset \gamma' \quad \text{or} \quad \gamma' \subset \gamma \quad \text{or} \quad \gamma \cap \gamma' = \varnothing\,.$$

The empty set is referred to as the empty forest. The notation $\gamma \cap \gamma' = \varnothing$ means that $V(\gamma) \cap V(\gamma') = \varnothing$, and it follows from the definition of a subgraph (Definition V.5) that then also the sets of edges are disjoint, $E(\gamma) \cap E(\gamma') = \varnothing$. A graph $\gamma \in U$ is called *maximal* if there is no other graph in U containing it. A Γ-forest U is called *maximal*, if there is no other Γ-forest containing it. A forest is called *restricted*, if it contains only "divergent graphs", $\operatorname{div}(\gamma) \geqslant 0$, cf. (IV.7). Note that a (Feynman-) graph Γ has more than one maximal restricted forest, if and only if it has *overlapping divergences*, i.e., at least two divergent subgraphs $\gamma, \gamma' \subset \Gamma$ for which (VI.13) does not hold. Given a forest U of Γ and a subgraph $G \in U$, we define the set

$$U(G) := \{\gamma \in U : \gamma \subseteq G\}\,,$$

and note that $U(G)$ is a G-forest as well as a Γ-forest.

LEMMA VI.6 (Structure of Maximal Forests). *Let U be a maximal Γ-forest.*

(1) With any element $G \in U$, U contains also its vertex complement, $\Gamma \oslash G \in U$.
(2) For any graph $G \in U$ the set $U(G) = \{\gamma \in U : \gamma \subseteq G\}$ is a maximal G-forest.
(3) The forest $U \setminus \{\Gamma\}$ is the disjoint union of two maximal forests. If $G \in U \setminus \{\Gamma\}$ is a maximal element, then

$$U \setminus \{\Gamma\} = U(G) \dot\cup U(\Gamma \oslash G)\,.$$

PROOF. (1). Let $G \in U$, then for any element $\gamma \in U$, we have either $\gamma \subset G$ or $\gamma \subset \Gamma \oslash G$. Hence $U \cup \{\Gamma \oslash G\}$ is a forest, and by maximality of U: $\Gamma \oslash G \in U$.

(2). Let U' be a G-forest properly containing $U(G)$. Then there is a subgraph $\gamma' \subset G$ such that $\gamma' \notin U(G)$ does not overlap with any of the elements in $U(G)$. Since $G \in U$ it follows that $\gamma' \notin U$ is non-overlapping with any element in U. Thus $U \dot\cup \{\gamma'\}$ is a forest, in contradiction with the maximality of U.

(3) follows from (1) and (2). □

The combinatorial result, which leads to the forests formula for Epstein-Glaser renormalization, is that we can write (maximal) forests of the kind defined above in terms of (complete) sets of partitions of the vertex set.

LEMMA VI.7 (Partitions and Forests). *The set of partitions of n elements,* $\mathrm{Part}\{1,\dots,n\}$*, is a partially ordered set (poset). The partial order is defined by saying that $\mathcal{P}_1$ is finer than $\mathcal{P}_2$ (and $\mathcal{P}_2$ is coarser than $\mathcal{P}_1$),*

$$\mathcal{P}_1 \leqslant \mathcal{P}_2\,,$$

if for any block $I \in \mathcal{P}_1$ there is a block $J \in \mathcal{P}_2$ containing I. $(\mathrm{Part}\{1,\dots,n\},\leqslant)$ *is a complete partial order (cpo) with finest element $\{\{1\},\dots,\{n\}\}$ and coarsest element $\{\{1,\dots,n\}\}$.*

Let $\mathbb{P} \subset \mathrm{Part}\{1,\dots,n\}$ be a totally ordered subset. We call $\mathbb{P}$ maximal, if there is no totally ordered subset $\mathbb{P}' \subset \mathrm{Part}\{1,\dots,n\}$ containing $\mathbb{P}$. The union of any totally ordered subset $\mathbb{P}$ is a forest,

$$U(\mathbb{P}) := \bigcup_{\mathcal{P}'\in\mathbb{P}} \mathcal{P}'\,.$$

$U(\mathbb{P})$ is a maximal forest, if $\mathbb{P}$ is maximal.

PROOF. "$\leqslant$" is reflexive ($\mathcal{P} \leqslant \mathcal{P}$), antisymmetric ($\mathcal{P}_1 \leqslant \mathcal{P}_2 \wedge \mathcal{P}_2 \leqslant \mathcal{P}_1 \Rightarrow \mathcal{P}_1 = \mathcal{P}_2$), and transitive ($\mathcal{P}_1 \leqslant \mathcal{P}_2 \wedge \mathcal{P}_2 \leqslant \mathcal{P}_3 \Rightarrow \mathcal{P}_1 \leqslant \mathcal{P}_3$), hence a partial order. For $(\mathrm{Part}\{1,\dots,n\},\leqslant)$ to be a cpo, we have to show that any pair $\mathcal{P}_1,\mathcal{P}_2 \in \mathrm{Part}\{1,\dots,n\}$ has a least upper bound $\mathcal{P}_1 \sqcup \mathcal{P}_2$ and a greatest lower bound $\mathcal{P}_1 \sqcap \mathcal{P}_2$ in $\mathrm{Part}\{1,\dots,n\}$. The assertion is trivial, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are related by "$\leqslant$", hence let $\mathcal{P}_1$ and $\mathcal{P}_2$ not be related by "$\leqslant$". $\mathcal{P}_1 \sqcup \mathcal{P}_2$ is the partition where all overlapping blocks (cf. (VI.13)) are replaced by their union; filled up with the larger blocks of either $\mathcal{P}_1$ or $\mathcal{P}_2$. In $\mathcal{P}_1 \sqcap \mathcal{P}_2$ overlapping blocks are replaced by their intersection and filled up with the smaller sets of either $\mathcal{P}_1$ or $\mathcal{P}_2$, see Figure 2. Since $\mathcal{P}_1 \sqcup \mathcal{P}_2$ and $\mathcal{P}_1 \sqcap \mathcal{P}_2$ are partitions of $\{1,\dots,n\}$, we infer that $\mathrm{Part}\{1,\dots,n\}$ is a cpo.

Regard a totally ordered subset $\mathbb{P} \subset \mathrm{Part}\{1,\dots,n\}$ and let $\mathcal{P} \in \mathbb{P}$, then by definition $I \cap J = \varnothing$ for all $I,J \in \mathcal{P}$. Let $\mathcal{P},\mathcal{P}' \in \mathbb{P}$ be two different partitions, $\mathcal{P}' \neq \mathcal{P}$, then we have either $I \cap J = \varnothing$, $I \subset J$, or $J \subset I$ for any pair $(I,J) \in \mathcal{P} \times \mathcal{P}'$ since $\mathbb{P}$ is totally ordered. Thus I and J are non-overlapping, and $U(\mathbb{P})$ is a forest. Let $\mathbb{P}$ be maximal, then it contains $\{\{1,\dots,n\}\}$ and $\{\{1\},\dots,\{n\}\}$, and with any partition $\mathcal{P} \in \mathbb{P}$ it contains all partitions which can be constructed out of $\mathcal{P}$ by successively dividing any of its blocks into a pair of disjoint subsets. Let $I = I_1 \dot\cup I_2$ be an index set, then there is no partition $\mathcal{P}_I$ of I such that $\{I_1,I_2\} \leqslant \mathcal{P}_I \leqslant \{I\}$ and $\{I_1,I_2\} \neq \mathcal{P}_I \neq \{I\}$. Hence the maximal set $\mathbb{P}$ can be constructed out of $\{\{1,\dots,n\}\}$ by the successive division procedure described above. Conversely, let U be a maximal forest, by Lemma VI.6, U can be constructed in exactly the same way. □

We will have to regard in the sequel unions of totally ordered sets of partitions, which is non-trivial, since the set union of two totally ordered sets of partitions will

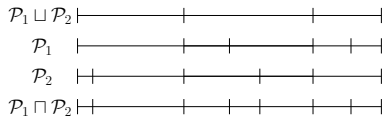


Figure 2: Least upper and greatest lower bound of a pair of partitions.

not be a totally ordered set of partitions, just as the union of forests is not a forest in general.

DEFINITION VI.8 (Position and Disjoint Union). Let $\mathbb{P} \subset \mathrm{Part}\{1,\dots,n\}$ be a totally ordered set, then we define the *position* of any element $\mathcal{P} \in \mathbb{P}$ by

$$\operatorname*{pos}_{\mathbb{P}}(\mathcal{P}) := \left|\left\{\mathcal{P}' \in \mathbb{P} : \mathcal{P} \leqslant \mathcal{P}'\right\}\right| .$$

It is easy to see that $\mathrm{pos}_{\mathbb{P}}(\mathcal{P}_c) = 1$, if $\mathcal{P}_c$ is the coarsest element of $\mathbb{P}$.

Let $\mathbb{P}(I)$ and $\mathbb{P}(J)$ be totally ordered subsets of Part$I$ and Part$J$, respectively. Any subset of

$$\left\{\mathcal{P} \dot\cup \mathcal{P}' : \mathcal{P} \in \mathbb{P}(I) \text{ and } \mathcal{P}' \in \mathbb{P}(J)\right\} ,$$

which is a totally ordered set of partitions of $I \dot\cup J$, we call a *disjoint union* of $\mathbb{P}(I)$ and $\mathbb{P}(J)$.

Observe that there are forests which do not correspond to a totally ordered subset of partitions, e.g., the empty forest or any forest containing just one proper subset of $\{1,\dots,n\}$. We now come to the definition of an Epstein-Glaser forest; a similar definition for forests was considered in [FGB05] to establish the relation to incidence Hopf algebras.

DEFINITION VI.9 (Epstein-Glaser Forest). Let $V = \{1,\dots,n\}$ be a (vertex) set. Then we call any totally ordered subset $\mathbb{F}$ of the set of partitions, $\mathbb{F} \subset \mathrm{Part}\{1,\dots,n\}$, containing the finest partition $\mathcal{P}_n := \{\{1\},\dots,\{n\}\}$ of V, an *Epstein-Glaser forest (EG forest)*, i.e., $\mathbb{F}$ has the form

$$\mathbb{F} = \{\cdots \geqslant \mathcal{P}_n\} , \quad \mathcal{P}_n = \{\{1\},\dots,\{n\}\} .$$

The EG forest containing only $\mathcal{P}_n$ we denote by $\mathbb{F}_n := \{\mathcal{P}_n\}$. If an EG forest contains the coarsest partition $\mathcal{P}_1 := \{\{1,\dots,n\}\}$ we call it a *full EG forest (full forest)* and write $\overline{\mathbb{F}}$. If an EG forest does not contain $\mathcal{P}_1$, we call it a *normal EG forest (normal forest)* and write $\underline{\mathbb{F}}$. For $n = 1$ there is just one forest, the one with one vertex, and we define this forest to be full. For $n > 1$ there is a one to one correspondence between full and normal forests, given by

$$\overline{\mathbb{F}} = \underline{\mathbb{F}} \cup \{\mathcal{P}_1\} .$$

An Epstein-Glaser forest $\mathbb{F}$ is called maximal, if $\mathbb{F}$ is maximal as a totally ordered set of partitions.

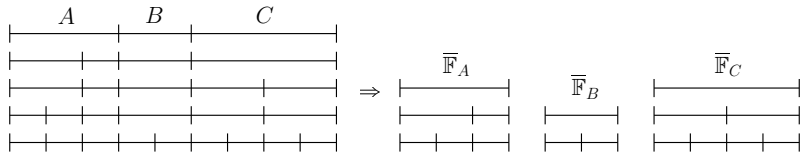


Figure 3: The depicted normal forest of $A \dot\cup B \dot\cup C$ decomposes into tree full forests $\overline{\mathbb{F}}_A$, $\overline{\mathbb{F}}_B$, and $\overline{\mathbb{F}}_C$. Conversely, given the three full forests, any composition of them, which preserves the order of partitions in each component separately, gives a different disjoint union, i.e. a different normal forest of $A \dot\cup B \dot\cup C$.

COROLLARY VI.10. *Any normal Epstein-Glaser forest $\underline{\mathbb{F}}$ is a disjoint union of at least two full Epstein-Glaser forests. If the coarsest partition in $\underline{\mathbb{F}}$ has k elements, then $\underline{\mathbb{F}}$ decomposes into k full forests $\overline{\mathbb{F}}_1, \dots, \overline{\mathbb{F}}_k$, $k \geqslant 2$, and we call $\underline{\mathbb{F}}$ k-fold connected. Conversely one can say that $\underline{\mathbb{F}}$ is a disjoint union of $\overline{\mathbb{F}}_1, \dots, \overline{\mathbb{F}}_k$ in the sense of Definition VI.8,*

$$\underline{\mathbb{F}} = \dot{\bigcup_{i \in \{1,\dots,k\}}} \overline{\mathbb{F}}_i \,. \tag{VI.14}$$

If I_i is the block of the coarsest partition in the full forest $\overline{\mathbb{F}}_i$, then $\overline{\mathbb{F}}_i = \underline{\mathbb{F}}(I_i)$. In this sense the decomposition (VI.14) of $\underline{\mathbb{F}}$ is unique. See also Figure 3. □

So far for the combinatorial part, we now have to define the analytic part, namely, the minimal subtractions in the blocks of a partition.

DEFINITION VI.11 (MS Operator). For any partition $\mathcal{P} \in \mathrm{Part}\{1, \dots, n\}$ define the minimal subtraction operator (MS operator)

$$-T^{\mathrm{MS}}_{\mathcal{P}} \boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta} := \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}_{\mu,\zeta} \cdot \left(\bigotimes_{I \in \mathcal{P}} R_{|I|} \left(\boldsymbol{\mathcal{S}}^{(|I|)}_{\mu,\zeta} \right) \right), \quad \text{where } R_k = \begin{cases} \mathrm{id} & \text{if } k = 1 \\ -\mathrm{pp} & \text{if } k > 1 \,, \end{cases}$$

cf. Definition V.6.

This defines the operator $-T^{\mathrm{MS}}_{\mathcal{P}}$ on the whole regularized time-ordered product $\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}$. Observe, however, that the above definition implies that we have chosen another regularization parameter in each block I of the partition; there is one operator $R_{|I|}$ for each block. To consider all partitions, which possibly contribute to the principal part, we will regard the n-fold regularized time-ordered product as being regularized in $\binom{n}{2}$ different regularization parameters, one for each pair of vertices. This is certainly possible regarding the fact that we have "regularized the lines" of each given diagram, i.e. the propagators. At the stage of only one MS operator $-T^{\mathrm{MS}}_{\mathcal{P}}$ this consideration is not very important, since the singularities in different blocks are independent, anyway. However, the fact that we can choose the regularization parameters freely for any pair of vertices becomes important as soon as we want to define products of the MS operators applied to the same regularized time-ordered product. Such products occur in the forest formula below, and we briefly discuss one example in order to clarify this point. We choose the

position space representation for convenience. Let $\mathcal{P}_1 \leqslant \mathcal{P}_2$ be different partitions in $\mathrm{Part}\{1,\dots,n\}$. Let $I_1 \subset I_2$ be a pair of blocks, $I_1 \in \mathcal{P}_1$, $I_2 \in \mathcal{P}_2$, and regard a special graph contribution, say $S_\Gamma^{\mu,\zeta}$, to $\boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)}$ for simplicity, $|V(\Gamma)| = n$. Let γ be the full vertex part (in Γ) to the vertex set I_1 and let G be the full vertex part of I_2, then $\gamma \subset G$. Regard the successive subtraction

$$s_{\Gamma\setminus G}^{\mu,\zeta}\, R_{I_2}\left(s_{G\setminus\gamma}^{\mu,\zeta}\, R_{I_1} s_\gamma^{\mu,\zeta}\right),$$

and insert the definition from the previous chapter, cf. (V.6),

$$= \sum_{n=-\infty}^{-1} \zeta^n \frac{1}{2\pi i}\oint_C d\xi\, \frac{1}{\xi^{n+1}} \sum_{k=-\infty}^{-1} \xi^k \frac{1}{2\pi i}\oint_{C'} d\xi'\, \frac{1}{\xi'^{k+1}} s_{\Gamma\setminus G}^{\mu,\xi} \cdot s_{G\setminus\gamma}^{\mu,\xi} \cdot s_\gamma^{\mu,\xi'},$$

where we had to introduce regularization parameters ξ and ξ' for the subgraphs in order to get independent subtractions in all subgraphs. With these remarks concerning the regularization, we have for the composition of MS operators corresponding to related partitions, $\mathcal{P} \geqslant \mathcal{P}'$,

$$T_{\mathcal{P}}^{\mathrm{MS}} T_{\mathcal{P}'}^{\mathrm{MS}} \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)} = \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(|\mathcal{P}|)} \cdot \left[\bigotimes_{I\in\mathcal{P}} R_{|I|}\left[\boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(|I'/I|)} \cdot \left(\bigotimes_{I'\subset I} R_{|I'|}\left(\boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(|I'|)}\right)\right)\right]\right],$$

where $|I'/I|$ denotes the cardinality of $\{I' \in \mathcal{P}' : I' \subset I\}$.

THEOREM VI.12 (Forest Formula for Epstein-Glaser Renormalization). *Let $T_{\mathcal{P}}^{\mathrm{MS}}$ be the minimal subtraction operator of Definition VI.11, and let the product of two MS operators corresponding to related partitions be defined as described above. Then*

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}}^{(n)} := \sum_{\mathbb{F}\subset \mathrm{Part}\{1,\dots,n\}} \left(\prod_{\mathcal{P}\in\mathbb{F}}^{\geqslant} -T_{\mathcal{P}}^{\mathrm{MS}}\right) \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)} \tag{VI.15}$$

gives a finite regularization of the n-fold regularized time-ordered product $\boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)} \equiv \boldsymbol{\mathcal{T}}_{\mu,\zeta}^{n}$. The sum is taken over all Epstein-Glaser forests. The product of the operators is taken in the order prescribed by "$\geqslant$", such that the coarsest partition in $\mathbb{F}$ stands to the very left.

PROOF. The forest formula implies an expression for the nth counterterm in the renormalization scheme of analytic regularization and minimal subtraction. We can split (VI.15) into a sum over full and normal forests,

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}}^{(n)} = \sum_{\underline{\mathbb{F}}} \left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T_{\mathcal{P}}^{\mathrm{MS}}\right) \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)} + \sum_{\overline{\mathbb{F}}} \left(\prod_{\mathcal{P}\in\overline{\mathbb{F}}}^{\geqslant} -T_{\mathcal{P}}^{\mathrm{MS}}\right) \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)}.$$

Observe that for $n = 1$ there are no normal forests, and the first sum is empty. Since any forest in the second sum on the right hand side contains the coarsest partition, $\mathcal{P}_1 = \{\{1,\dots,n\}\}$, we can factor out the corresponding MS operator and get from [C2],

$$\boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}}^{(n)} = \sum_{\underline{\mathbb{F}}} \left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T_{\mathcal{P}}^{\mathrm{MS}}\right) \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)} + R_n\left[\sum_{\underline{\mathbb{F}}} \left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T_{\mathcal{P}}^{\mathrm{MS}}\right) \boldsymbol{\mathcal{S}}_{\mu,\zeta}^{(n)}\right].$$

We show in the sequel, that

$$C^{(n)}_{\mu,\zeta} := R_n\left[\sum_{\underline{\mathbb{F}}}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\right] \equiv \sum_{\overline{\mathbb{F}}}\left(\prod_{\mathcal{P}\in\overline{\mathbb{F}}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}, \tag{VI.16}$$

is the local counterterm of regularized causal perturbation theory in the minimal subtraction scheme. That is $C^{(n)}_{\mu,\zeta}$ fulfills the recursion relation of Corollary VI.5. We proceed by induction.

For $n = 1$ we have by [C2] and the definition of R_1,

$$C^{(1)}_{\mu,\zeta} = R_1\boldsymbol{\mathcal{S}}^{(1)}_{\mu,\zeta} = \mathrm{id} = Z^{(1)}_{\mu,\zeta}\,.$$

Thus $C^{(1)}_{\mu,\zeta} = \mathrm{id} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ is local, and we can assume that $C^{(k)}_{\mu,\zeta} = Z^{(k)}_{\mu,\zeta}$ for all $k < n$. For the induction step we have to show that

$$R_n\left[\sum_{\underline{\mathbb{F}}}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\right] = -\mathrm{pp}\sum_{\mathcal{P}\in\mathrm{Part}\{1,\dots,n\}\setminus\{\mathcal{P}_1\}}\boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}_{\mu,\zeta}\cdot\left(\bigotimes_{I\in\mathcal{P}} C^{(|I|)}_{\mu,\zeta}\right).$$

for $n \geqslant 2$. By the definition of R_n this is the case, if

$$\sum_{\underline{\mathbb{F}}}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta} = \sum_{\mathcal{P}\in\mathrm{Part}\{1,\dots,n\}\setminus\{\mathcal{P}_1\}}\boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}_{\mu,\zeta}\cdot\left(\bigotimes_{I\in\mathcal{P}} C^{(|I|)}_{\mu,\zeta}\right). \tag{VI.17}$$

Regard the left hand side of this equation. Any normal forest is a disjoint union of at least two full forests (Corollary VI.10). Hence we can write the sum over all normal forests as

$$\mathrm{lhs} = \sum_{k=2}^{n}\sum_{\substack{\mathcal{P}'\in\mathrm{Part}\{1,\dots,n\}\\ |\mathcal{P}'|=k}}\sum_{\underline{\mathbb{F}}=\dot{\bigcup}_{I\in\mathcal{P}'}\underline{\mathbb{F}}(I)}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\,.$$

The product splits and we get

$$= \sum_{k=2}^{n}\sum_{\substack{\mathcal{P}'\in\mathrm{Part}\{1,\dots,n\}\\ |\mathcal{P}'|=k}}\sum_{\underline{\mathbb{F}}=\dot{\bigcup}_{I\in\mathcal{P}'}\underline{\mathbb{F}}(I)}\bigotimes_{I\in\mathcal{P}'}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}(I)}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\,. \tag{VI.18}$$

The sum over all normal, i.e., multiply connected forests $\underline{\mathbb{F}}$ can be performed by summing over all full forests $\underline{\mathbb{F}}(I)$ in its connected components, cf. Corollary VI.10. And we have to perform the sum in these components, in order to get a well-defined expression for (VI.18),

$$\mathrm{lhs} = \sum_{k=2}^{n}\sum_{\substack{\mathcal{P}'\in\mathrm{Part}\{1,\dots,n\}\\ |\mathcal{P}'|=k}}\bigotimes_{I\in\mathcal{P}'}\sum_{\underline{\mathbb{F}}(I)}\left(\prod_{\mathcal{P}\in\underline{\mathbb{F}}(I)}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}}\right)\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\,.$$

Inserting the inductive assumption, $C^{(k)}_{\mu,\zeta} = Z^{(k)}_{\mu,\zeta}\ \forall k < n$, gives the desired result (lhs = rhs). □

COROLLARY VI.13. *Let $\mathbb{F}_1,\dots,\mathbb{F}_c$ be the maximal forests of the vertex set $\{1,\dots,n\}$.*

Then we can write (VI.15) equivalently as

$$\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta,\mathrm{ren}} = \sum_{\substack{\emptyset\neq\{i_1,\dots,i_\nu\}\\ \subset\{1,\dots,c\}}} (-1)^{\nu+1} \left(\prod_{\mathcal{P}\in\mathbb{F}_{i_1}\cap\dots\cap\mathbb{F}_{i_\nu}}^{\geqslant} \left(1 - T^{\mathrm{MS}}_{\mathcal{P}}\right)\right) \boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta}\,. \tag{VI.19}$$

PROOF. This is merely another way of summing up the contributions to (VI.15). Multiplying out the factors $\left(1 - T^{\mathrm{MS}}_{\mathcal{P}}\right)$ gives the result, cf. [Zim69, Thm. 3.3]. □

COROLLARY VI.14 (EG Forest Formula in terms of graphs). *Let Γ be a graph with n vertices, $|V(\Gamma)| = n$. For any partition $\mathcal{P}$ of $V(\Gamma)$ the action of the MS operator on the level of graphs is given by*

$$-T^{\mathrm{MS}}_{\mathcal{P}} \left\langle s^{\mu,\zeta}_{\Gamma} \right\rangle := \left\langle s^{\mu,\zeta}_{\Gamma/\mathcal{P}} \right\rangle \left(\bigotimes_{I\in\mathcal{P}} R_{|I|} \left\langle s^{\mu,\zeta}_{\gamma_I} \right\rangle \right), \quad \textit{where } R_k = \begin{cases} \mathrm{id} & \textit{if } k = 1 \\ -\mathrm{pp} & \textit{if } k > 1\,, \end{cases}$$

where $\Gamma/\mathcal{P}$ is the graph with the blocks $I \in \mathcal{P}$ as vertices and as lines all lines in Γ which connect different blocks of $\mathcal{P}$. For each block $I \in \mathcal{P}$, the graph γ_I is the full vertex part of I. $\left\langle s^{\mu,\zeta}_{\Gamma} \right\rangle$ denotes the analytically regularized amplitude in any representation (momentum or position space). Then the limit

$$\left\langle s^{\mu}_{\Gamma,\mathrm{ren}} \right\rangle = \lim_{\zeta\to 0} \sum_{\mathbb{F}\subset\mathrm{Part}\{1,\dots,n\}} \left(\prod_{\mathcal{P}\in\mathbb{F}}^{\geqslant} -T^{\mathrm{MS}}_{\mathcal{P}} \right) \left\langle s^{\mu,\zeta}_{\Gamma} \right\rangle$$

is well-defined and gives a UV finite, i.e., renormalized amplitude.

PROOF. The MS operators are tensor products of the corresponding operators on sets (Definition V.6) and thus linear. Hence the corollary is a direct consequence of the discussion given in Section V.4 and the above Theorem VI.12. □

COROLLARY VI.15 (Prepared Amplitude). *Let Γ be a graph with n vertices, $n > 1$, i.e., $\left\langle S^{\mu,\zeta}_{\Gamma}, \boldsymbol{\delta}^{\alpha} \right\rangle$ with $\alpha \in \mathbb{N}^n$ and $|\alpha| = 2\,|E(\Gamma)|$ is a contribution to the n-fold, regularized time ordered product*

$$\boldsymbol{\mathcal{S}}^{(n)}_{\mu,\zeta} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}(\mathbb{M})[[\hbar]].$$

Then the prepared amplitude to Γ is given by,

$$S^{\mu,\zeta}_{\Gamma,\mathrm{prep}} := \sum_{\substack{\emptyset\neq\{i_1,\dots,i_\nu\}\\ \subset\{1,\dots,c\}}} (-1)^{\nu+1} \left(\prod_{\mathcal{P}\in\mathbb{F}_{i_1}\cap\dots\cap\mathbb{F}_{i_\nu}\setminus\{\mathcal{P}_1\}}^{\geqslant} \left(1 - T^{\mathrm{MS}}_{\mathcal{P}}\right)\right) S^{\mu,\zeta}_{\Gamma}\,. \tag{VI.20}$$ □

VI.4. More than Hopf Algebra

The investigation of the combinatorial structure of perturbative renormalization theory is a vivid field of research in mathematical physics and for the characterization of the underlying pattern Hopf algebras, and in particular the Faà di Bruno bialgebra introduced by Joni and Rota in 1982 [JR82], became more and more important in recent years [FGB05]. We will show in this last section, that there is a more intricate pattern underlying the combinatorial structure of

renormalization than is described by a Hopf algebra. However, the relation to the Hopf algebra of graphs originally encountered by Connes and Kreimer in BPHZ [Kre98, CK00, CK01] and later by Gracia-Bondía, Lazzarini, and Pinter in Epstein-Glaser renormalization [GBL00, Pin00b] will become transparent. The attractive feature of our derivation is that we can understand the emerging Hopf algebraic structure as a direct consequence of the main theorem of renormalization (Theorem VI.1). The elements of this Hopf algebra will be differential operators which give the time-ordered products, when acting on a $\boldsymbol{\mathcal{S}}$-matrix, and local counterterms when acting on an element of the Stückelberg-Petermann renormalization group. Hence, in a graphical representation they correspond to sums of graphs with the same set of vertices. The Hopf algebra structure for individual graphs is regained by linearity. The regularized Feynman rules and the renormalization map will appear naturally as soon as one specializes to an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix. This is a major difference to the Connes-Kreimer approach, where the Feynman rules had to be put by hand as characters into the commutative ring of Laurent series. We will show, however, that the recursion formula for minimally subtracted counterterms (VI.12), which was seen to be a direct consequence of the main theorem, cannot be described within the commutative Hopf algebra of Connes and Kreimer [CK00, CK01].

As a first step we will use the Faà di Bruno Hopf algebra introduced by Joni and Rota to derive the commutative, non-cocommutative Hopf algebra of graphs described briefly above. It will be clear from the given presentation that one needs an additional, non-commutative, composition product, to get back the recursion formula for the counterterms (Corollary VI.5). We will sketch in the last section how this composition product can be implemented into the Hopf algebra to describe algebraically the construction of counterterms in pAQFT. An interpretation of the maps in terms of graph operations will be given in the end.

VI.4.1. The Hopf Algebra. As shown by Joni and Rota, Faà di Bruno's formula for the chain rule gives rise to a natural bialgebra structure, which the authors called the Faà di Bruno bialgebra [JR82]. In modern presentations it is often introduced as a bialgebra of the coefficients in (VI.11), cf. [FGB05]. However, to keep the correspondence to causal perturbation theory transparent also in this last section of the present thesis we regard instead directly the partition version given in Lemma VI.4. Apart from that we follow essentially the steps of [FGB05] as far as the Hopf algebra structure is concerned.

Regard Faà di Bruno's formula (VI.7) in the termwise form of the main theorem of renormalization (Lemma VI.4). We denote the coefficients by

$$a_n(\boldsymbol{\mathcal{S}}) := \boldsymbol{\mathcal{S}}^{(n)}(0) \qquad \text{and} \qquad a_n(Z) := Z^{(n)}(0)$$

and get

$$a_n(\boldsymbol{\mathcal{S}} \circ Z) = \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} a_{|\mathcal{P}|}(\boldsymbol{\mathcal{S}}) \cdot \bigodot_{I \in \mathcal{P}} a_{|I|}(Z)\,. \tag{VI.21}$$

We want to make the symmetry of the functional derivative explicit here and replaced the tensor product in (VI.7) by the symmetrized tensor product

$$\bigodot_{i=1}^{k} A_i := \frac{1}{k!} \sum_{\sigma \in \mathrm{Perm}(k)} \bigotimes_{i=1}^{k} A_{\sigma(i)} \,,$$

where $\mathrm{Perm}(k)$ denotes the group of permutations of k elements. Observe that besides this commutative product ($\odot$), there is a second, non-commutative product in formula (VI.21). Namely, the composition of linear maps

$$\mathtt{C} : a_{|\mathcal{P}|}(\boldsymbol{\mathcal{S}}) \otimes \bigodot_{I \in \mathcal{P}} a_{|I|}(\mathrm{Z}) \mapsto a_{|\mathcal{P}|}(\boldsymbol{\mathcal{S}}) \cdot \bigodot_{I \in \mathcal{P}} a_{|I|}(\mathrm{Z}) \,, \tag{VI.22}$$

cf. (VI.8). This second product is absent if we regard the coefficients in (VI.21) as scalars. And we will sketch how to implement this additional non-commutative product into the Hopf algebra in the next section. Let us first regard the commutative part.

Regard the coefficients a_k in (VI.21) as differential operators

$$a_k : \boldsymbol{\mathcal{S}} \mapsto a_k(\boldsymbol{\mathcal{S}}) \equiv \boldsymbol{\mathcal{S}}^{(k)}(0) \quad \text{and} \quad a_k : \mathrm{Z} \mapsto a_k(\mathrm{Z}) \equiv \mathrm{Z}^{(k)}(0) \,. \tag{VI.23}$$

The operators will produce multi-linear maps on local functionals; with local image in the case they act on $\mathrm{Z} \in \mathcal{R}$,

$$a_k(\mathrm{Z}) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes k} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \,,$$

and with possibly non-local image if they act of $\boldsymbol{\mathcal{S}}$ fulfilling [C1]-[C4],

$$a_k(\boldsymbol{\mathcal{S}}) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes k} \to \mathcal{F}(\mathbb{M})[[\hbar]] \,.$$

Denote by $\mathfrak{H}$ the space of these coefficients a_k. Since the $\mathrm{Z}^{(k)}$ and $\boldsymbol{\mathcal{S}}^{(k)}$ (evaluation at zero understood) are linear maps on tensor products of local functionals, $\mathfrak{H}$ carries a natural $\mathbb{C}$-vector space structure induced by the $\mathbb{C}$-vector space structure on local functionals. As already remarked above, the symmetry of the functional derivative induces a commutative product on $\mathfrak{H}$,

$$\begin{array}{rccl} M_{\odot} : & \mathfrak{H} \otimes \mathfrak{H} & \to & \mathfrak{H} \\ & (a_k \otimes a_l) & \mapsto & a_k \odot a_l := \frac{1}{2}\left[a_k \otimes a_l + a_l \otimes a_k\right] , \end{array}$$

where we set

$$\left(a_k \odot a_l\right)(\mathrm{Z}) := a_k(\mathrm{Z}) \odot a_l(\mathrm{Z}) \,, \tag{VI.24}$$

and likewise for $\boldsymbol{\mathcal{S}}$. We regard $\mathfrak{H}$ as the free, commutative algebra generated by the a_k, and commit the usual abuse of notation by using the same symbol for the (symmetrized) tensor product of linear maps on the right hand side of (VI.24) and the commutative product of the algebra $(\mathfrak{H}, \odot)$ on the left. It will be clear from the context, where we mean which. $(\mathfrak{H}, \odot)$ is a unital algebra with unit

$$\mathbb{1} : \mathrm{Z} \mapsto \mathbb{1}(\mathrm{Z}) := \mathrm{id} \,, \qquad \mathbb{1} : \boldsymbol{\mathcal{S}} \mapsto \mathbb{1}(\boldsymbol{\mathcal{S}}) := \mathrm{id} \,,$$

where $\mathrm{id} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$ denotes the identity map on the space of

local functionals. We denote the corresponding unit map by

$$\begin{array}{rccc} e: & \mathbb{C} & \to & \mathfrak{H} \\ & \alpha & \mapsto & \alpha\,\mathbb{1}\,. \end{array}$$

We set $\mathfrak{H} \odot \mathbb{1} = \mathfrak{H}$. Joni and Rota interpreted Faà di Bruno's formula as a coproduct rule for the coefficients a_k , and we can do exactly the same thing here, by defining the coproduct, $\Delta : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$, as

$$(\Delta a_n) := \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} a_{|\mathcal{P}|} \otimes \left(\bigodot_{I \in \mathcal{P}} a_{|I|} \right). \tag{VI.25}$$

It is obvious from (VI.21) that this coproduct is induced by

$$(\Delta a_n)\,(\boldsymbol{\mathcal{S}} \otimes Z) = a_n(\boldsymbol{\mathcal{S}} \circ Z)\,,$$

and we break with the tradition of flipping the arguments of Δa_n here,[5] since, in contrast to the chain rule for functions or formal power series with scalar coefficients, we have a composition of linear maps with a prescribed order rather than a commutative product on the right hand side of Faà di Bruno's formula, cf. (VI.22). Consequently we have the linear part of the coproduct on the left hand side of $\otimes$. Equipping $\mathfrak{H}$ with the counit, defined on generators

$$\begin{array}{rccl} \bar{e}: & \mathfrak{H} & \to & \mathbb{C} \\ & a_n & \mapsto & \bar{e}(a_n) := \begin{cases} 1 & \text{if } a_n = \mathbb{1} \\ 0 & \text{else}\,, \end{cases} \end{array}$$

gives the usual commutative, non-cocommutative Faà di Bruno bialgebra, however, now interpreted in terms of the functional differential operators (VI.23).

It will be helpful for the construction of the antipode to first discuss the natural gradings on $\mathfrak{H}$.[6] As any tensor algebra, $\mathfrak{H}$ is graded by the number of factors,

$$\begin{array}{rccl} \deg^{\odot}: & \mathfrak{H} & \to & \mathbb{N} \\ & \bigodot_{i=1}^{k} a_{l_i} & \mapsto & \deg^{\odot}(a_{l_1} \odot \cdots \odot a_{l_k}) := k\,. \end{array}$$

That is, $\mathfrak{H}$ can be written as the direct sum

$$\mathfrak{H} = \bigoplus_{k=0}^{\infty} \mathfrak{H}^{\odot k} \quad , \quad \mathfrak{H}^{\odot k} = \left\{ a \in \mathfrak{H} : \deg^{\odot}(a) = k \right\}, \quad \mathfrak{H}^{\odot 0} = \mathbb{C}\,.$$

Subordinate to this tensor algebra grading is a naturally induced grading of the individual $a_l \in \mathfrak{H}^{\odot 1}$ given by the order of the derivative (minus one),

$$\deg^{v}(a_l) := l - 1\,,$$

[5] see, e.g. page 2 of [FGBV05].

[6] Two articles by Kastler were very helpful in learning about the relevance of grading and other Hopf related topics [Kas00, Kas04]. They are probably not the standard references to be cited at this point, however, they contain explicit proofs of the results from Hopf algebra theory needed here. See also more standard literature like [Swe69, Abe77]. Since we will not make any connection to more advanced structures in algebraic geometry, the given references will fully suffice for the discussion in this section.

and we have

$$\mathfrak{H}^{\odot 1} = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n^{\odot 1} \quad , \quad \mathfrak{H}_n^{\odot 1} = \left\{ a \in \mathfrak{H}^{\odot 1} : \deg^v(a) = n \right\} .$$

Observe that $\deg^v + 1$ corresponds to the order of derivative of $\boldsymbol{\mathcal{S}}$ (or Z) at zero. This determines the number of interaction functionals in the argument of the corresponding linear map

$$a_n(\boldsymbol{\mathcal{S}}) = \boldsymbol{\mathcal{S}}^{(n)}(0) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \to \mathcal{F}(\mathbb{M})[[\hbar]] \, , \quad n = \deg^v + 1 \, ,$$

and hence $\deg^v + 1$ is the number of vertices of the graphs contributing to the graph expansion of $\boldsymbol{\mathcal{S}}^{(n)}(0) \equiv \boldsymbol{\mathcal{T}}_n$, cf. Equation (IV.2).

This grading by vertex number can be extended to the tensor product $\mathfrak{H}^{\odot k}$ by setting

$$\deg^v(a_k \odot a_l) := \deg^v(a_k) + \deg^v(a_l) \, , \quad \text{i.e.,} \quad \deg^v(a_{l_1} \odot \cdots \odot a_{l_k}) = \sum_{i=1}^{k} l_i - k \, .$$

With this definition also $\deg^v$ is an algebra grading of the algebra $\mathfrak{H}$, and we have

$$\mathfrak{H} = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n \quad , \quad \mathfrak{H}_n = \{ a \in \mathfrak{H} : \deg^v(a) = n \} \ .$$

Furthermore the vertex grading $\deg^v$ is compatible with the coproduct (VI.25),

$$\deg^v(\Delta a_n) = |\mathcal{P}| - 1 + \sum_{I \in \mathcal{P}} (|I| - 1) = n - 1 \quad \forall \mathcal{P} \in \mathrm{Part}\{1, \ldots, n\} \, ,$$

and hence $(\mathfrak{H}, \odot, e, \Delta, \bar{e})$ is graded as a bialgebra. The starting element conditions [C2] and [RG2],

$$a_1(\boldsymbol{\mathcal{S}}) = \boldsymbol{\mathcal{S}}^{(1)}(0) = \mathrm{id} \quad \text{and} \quad a_1(Z) = Z^{(1)}(0) = \mathrm{id}$$

imply that $a_1 = \mathbb{1}$ and hence $\mathfrak{H}$ is $\mathbb{N}_0$-graded connected as a bialgebra, i.e., $\mathfrak{H}_0 = \mathbb{C}$ (we implicitly identify $\mathbb{1} = \mathbb{1} \odot \mathbb{1}$ here). It is a well-known fact of Hopf algebra theory that any $\mathbb{N}_0$-graded connected bialgebra possesses an antipode [Kas00, Prop. 2.7],

$$\mathcal{A} : \mathfrak{H} \to \mathfrak{H} \, ,$$

and thus $\mathfrak{H}$ is a Hopf algebra. We will now derive a recursion formula for this antipode. By definition, the antipode $\mathcal{A}$ of a Hopf algebra is the inverse of the identity with respect to the induced convolution product on the Hopf algebra automorphisms $\mathrm{Aut}(\mathfrak{H})$. The convolution product on $\mathrm{Aut}(\mathfrak{H})$ is induced by the product and coproduct on $\mathfrak{H}$, we denote it by

$$\phi \circledast \psi := M_{\odot} \circ (\phi \otimes \psi) \circ \Delta \, , \quad \phi, \psi \in \mathrm{Aut}(\mathfrak{H}) \, .$$

It is a standard computation to prove that

$$e \circ \bar{e} : \mathfrak{H} \to \mathfrak{H}$$

defines a unit in the algebra $(\mathrm{Aut}(\mathfrak{H}), \circledast)$. A similar computation will be done

below for the second product on the Hopf algebra, so we leave it out here. The antipode of an $\mathbb{N}_0$-graded connected bialgebra can then be constructed directly from its defining condition,

$$\mathrm{id}_{\mathfrak{H}} \circledast \mathcal{A} = e \circ \bar{e}\,.$$

We have

$$M_{\odot} \circ (\mathrm{id}_{\mathfrak{H}} \otimes \mathcal{A}) \circ \Delta(a_n) = e(\bar{e}(a_n))$$

$$\sum_{\mathcal{P}\in\mathrm{Part}\{1,\ldots,n\}} a_{|\mathcal{P}|} \otimes \left(\bigodot_{I\in\mathcal{P}} \mathcal{A}(a_{|I|}) \right) = \begin{cases} \mathbb{1} & \text{if } n = 1 \\ 0 & \text{else}\,, \end{cases}$$

since, in particular, $\mathcal{A}$ is an algebra homomorphism. From the case $n = 1$ we get $\mathcal{A}(a_1) = \mathbb{1}$, and since there is only one partition of $\{1,\ldots,n\}$ with one block, $\mathcal{P}_1 = \{\{1,\ldots,n\}\}$, we infer by connectedness of $\mathfrak{H}$, i.e., by $a_1 = \mathbb{1}$ that

$$\mathcal{A}(a_n) = -\sum_{\mathcal{P}\in\mathrm{Part}\{1,\ldots,n\}\setminus\{P_1\}} a_{|\mathcal{P}|} \odot \left(\bigodot_{I\in\mathcal{P}} \mathcal{A}(a_{|I|}) \right). \tag{VI.26}$$

Observe the similarity to the recursion for the counterterms in Corollary VI.5. However, observe also that the composition structure of (VI.12) is completely absent in (VI.26). This is no problem, if we regard the Feynman rules as characters of $(\mathfrak{H}, \odot)$ into a commutative ring of Laurent series with scalar coefficients, as it was done in [CK00, CK01]. We want to emphasize the relation of the Hopf algebra $(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \mathcal{A})$ to the Connes-Kreimer Hopf algebra of graphs. The elements of $\mathfrak{H}$ are differential operators whose order is determined by the vertex grading $\deg^v$. By implementing the graph expansion (IV.2), the elements $a_k \in \mathfrak{H}$ can be interpreted as sums over graphs with the same set of vertices. By linearity of the maps and the fact that only finitely many graphs contribute to the perturbative expansion at a given order of $\hbar$, [C3], we can break the Hopf algebra structure down to the level of graphs. However, since the structure for the algebraic construction of counterterms is not complete yet, we will give a more detailed account of this interpretation only at the end of the next section.

The fact that we found the Hopf algebra structure in the sums of graphs is in accordance with the results of Brouder and Frabetti, who found in different examples (including gauge theories) that Connes and Kreimer's Hopf algebra structure is preserved when one sums up the graph contributions at certain orders or perturbation theory [BF00b, BF01, Fra07], see also [vS07a]. Brouder and Frabetti, in collaboration with Krattenthaler and Menous, respectively, also observed the relation to the Faà di Bruno Hopf algebra [BFK06, BFM09], however, the relation to the main theorem of perturbative renormalization as proven in [DF04, DF07, BDF09] was, to the best of my knowledge, unobserved before.

We will now incorporate the non-commutative composition structure into the commutative Hopf algebra $(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \mathcal{A})$ constructed above.

VI.4.2. Algebraic Construction of Counterterms. In contrast to the Connes-Kreimer approach to renormalization, in our approach the Feynman rules are naturally induced as evaluation maps of the differential operators $a_n \in \mathfrak{H}$. The basic evaluation operator, which gives the n-fold time-ordered product, and thus corresponds to the Feynman rules is given by

$$\texttt{feyn} : a_n \mapsto a_n(\boldsymbol{\mathcal{S}}) = \boldsymbol{\mathcal{S}}^{(n)} \equiv \boldsymbol{\mathcal{T}}^n \,.$$

The image of `feyn` is a multi-linear map between spaces of (local) functionals. On linear maps there are two natural products, one is the (symmetrized) tensor product discussed above, and the other is the composition. Composition of linear maps is a non-commutative operation in the generic case, and as a consequence it is impossible to derive the action of the counterterms $Z^{(n)}$ on the time-ordered products $\boldsymbol{\mathcal{S}}^{(n)}$ described by Lemma VI.4 from the commutative Hopf algebra $(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \mathcal{A})$ alone. Regard once again the expression given in the lemma,

$$(\boldsymbol{\mathcal{S}} \circ Z)^{(n)}(0) = \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} \boldsymbol{\mathcal{S}}^{(|\mathcal{P}|)}(0) \cdot \left(\bigodot_{I \in \mathcal{P}} \left[Z^{(|I|)}(0) \right] \right),$$

and observe that we need both, the commutative product $\odot$ and the non-commutative composition " $\cdot$ ", as well as the coproduct Δ for defining the action of $\mathcal{R}$ on itself and on the set of $\boldsymbol{\mathcal{S}}$-matrices term by term in an algebraic fashion.

We want to incorporate the composition as an additional product in the commutative, non-cocommutative Hopf algebra constructed above,

$$(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \mathcal{A})$$

with generators $a_k \in \mathfrak{H}$, $k \in \mathbb{N}$. We define a map

$$\texttt{C} : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H} \,,$$

induced by the composition of linear maps (VI.22),

$$\texttt{C}(a_k \otimes \bigodot_{i=1}^{k} a_{l_i}) = a_k © \bigodot_{i=1}^{k} a_{l_i} \,, \qquad \text{with} \qquad \left(a_k © \bigodot_{i=1}^{k} a_{l_i} \right)(Z) := a_k(Z) \cdot \bigodot_{i=1}^{k} a_{l_i}(Z) \,.$$

Observe that the application on $\boldsymbol{\mathcal{S}}$ is ill-defined in the generic case, since the derivatives $\boldsymbol{\mathcal{S}}^{(n)}$ do not have local images for $n > 1$. The composition product © can be seen as (the dual of) the termwise group action of the Stückelberg-Petermann group on itself. We implement compatibility with the vertex grading by defining

$$\begin{aligned} \deg^v(a_k © \bigodot_{i=1}^{k} a_{l_i}) &:= \deg^v(a_k) + \deg^v(\bigodot_{i=1}^{k} a_{l_i}) \\ &= k - 1 + \sum_{i=1}^{k} (l_i - 1) = \sum_{i=1}^{k} l_i - 1, \end{aligned}$$

and with the Hopf algebra unit

$$\mathbb{1} \textcircled{c} a_k = a_k \textcircled{c} \mathbb{1} = a_k \,,$$

in accordance with [RG2]. The product is also compatible with the coproduct, in the sense that we can define another convolution on $\mathrm{Aut}(\mathfrak{H})$,

$$\phi \bullet_{\mathtt{C}} \psi := \mathtt{C} \circ (\phi \otimes \psi) \circ \Delta \,.$$

We check that $e \circ \bar{e}$ is a (both sided) identity for this convolution,

$$\begin{aligned}
(e \circ \bar{e} \bullet_{\mathtt{C}} \psi)\,(a_k) &= \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,k\}} e \circ \bar{e}\left(a_{|\mathcal{P}|}\right) \textcircled{c} \psi(\bigodot_{I \in \mathcal{P}} a_{|I|}) \\
&= \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,k\}} \delta_{|\mathcal{P}|,1}\, \mathbb{1} \textcircled{c} \psi(\bigodot_{I \in \mathcal{P}} a_{|I|}) \\
&= \psi(a_k)\,,
\end{aligned}$$

where $\delta_{|\mathcal{P}|,1}$ is the Kronecker-δ. Observe that $|\mathcal{P}| = 1$ implies $\mathcal{P} = \{\{1,\ldots,k\}\}$. Conversely we get

$$\begin{aligned}
(\phi \bullet_{\mathtt{C}} e \circ \bar{e})\,(a_k) &= \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,k\}} \phi\left(a_{|\mathcal{P}|}\right) \textcircled{c} \bigodot_{I \in \mathcal{P}} e \circ \bar{e}(a_{|I|}) \\
&= \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,k\}} \phi\left(a_{|\mathcal{P}|}\right) \textcircled{c} \bigodot_{I \in \mathcal{P}} \delta_{|I|,1} \\
&= \phi(a_k)\,.
\end{aligned}$$

Here the result is obtained since $|I| = 1\ \forall I \in \mathcal{P}$ implies $\mathcal{P} = \{\{1\},\ldots,\{k\}\}$. We define the Z-Feynman rules,

$$\mathtt{feyn}_Z(a_k) := a_k(Z) \equiv Z^{(k)}(0) : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes k} \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]\,.$$

These are algebra homomorphisms with respect to both algebra products,

$$\mathtt{feyn}_Z : (\mathfrak{H}, \odot) \to (\mathrm{Lin}, \odot) \qquad \text{and} \qquad \mathtt{feyn}_Z : (\mathfrak{H}, \textcircled{c}) \to (\mathrm{Lin}, \cdot)\,,$$

where we denoted by Lin the space of multi-linear maps between spaces of local functionals. We get at order n of causal perturbation theory the finite renormalizations, the changes of renormalization scheme, or the action of the Stückelberg-Petermann renormalization group on itself,

$$\left(\mathtt{feyn}_{Z_1} \bullet_{\mathtt{C}} \mathtt{feyn}_{Z_2}\right)(a_n) = \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} Z_1^{(|\mathcal{P}|)} \cdot \left(\bigodot_{I \in \mathcal{P}} Z_2^{(|I|)}\right).$$

Let us compute the right sided antipode of the convolution $\bullet_{\mathtt{C}}$,

$$\begin{aligned}
\mathrm{id}_{\mathfrak{H}} \bullet_{\mathtt{C}} \mathcal{A}_{\mathtt{C}}(a_n) &= e \circ \bar{e}(a_n) \\
\sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} a_{|\mathcal{P}|} \textcircled{c} \mathcal{A}_{\mathtt{C}}(\bigodot_{I \in \mathcal{P}} a_{|I|}) &= \delta_{n,1}
\end{aligned}$$

From $n = 1$ we get $\mathcal{A}_{\mathtt{C}}(a_1) = \mathbb{1}$. Furthermore we get from the connectedness of the

Hopf algebra, i.e., from $a_1 = \mathbb{1}$,

$$\mathcal{A}_{\mathtt{C}}(a_n) = -\sum_{\mathcal{P}\in\mathrm{Part}\{1,\ldots,n\}\setminus\{\mathcal{P}_1\}} a_{|\mathcal{P}|} \textcircled{c} \mathcal{A}_{\mathtt{C}}(\bigodot_{I\in\mathcal{P}} a_{|I|}),$$

where, as before, $\mathcal{P}_1 = \{\{1,\ldots,n\}\}$. Observe the similarity to the result of Connes-Kreimer in the commutative case, i.e., the recursion formula (VI.26) for the antipode $\mathcal{A}$. And observe the difference, the composition product $\textcircled{c}$. The augmented Hopf algebra

$$(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \textcircled{c}, \mathcal{A}_{\mathtt{C}})$$

constructed above can be interpreted as the algebraic dual of the Stückelberg-Petermann renormalization group. We want to discuss now the action of this algebraic dual on the (regularized) time-ordered products, and its relation to the original formulation of the Connes-Kreimer theory of renormalization.

Given the nice circumstance that we have a preferred renormalization prescription at all orders of perturbation theory as it is provided by any analytic regularization of the $\boldsymbol{\mathcal{S}}$-matrix combined with minimal subtraction. Then we define the regularized Feynman rules as

$$\mathtt{feyn}_{\mu,\zeta}(a_k) := a_k(\boldsymbol{\mathcal{S}}_{\mu,\zeta}) \equiv \boldsymbol{\mathcal{S}}^{(k)}_{\mu,\zeta} : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes k} \to \mathcal{F}(\mathbb{M})[[\hbar]]\,.$$

Since the derivatives of $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$ have non-local images in the generic case, we have that, in contrast to the Z-Feynman rules above, the regularized Feynman rules cannot be iterated. This is reflected in the algebraic setting by the fact that the regularized Feynman rules are algebra homomorphisms with respect to $\odot$, but not with respect to $\textcircled{c}$,

$$\mathtt{feyn}_{\mu,\zeta}(a_k \odot a_l) = \mathtt{feyn}_{\mu,\zeta}(a_k) \odot \mathtt{feyn}_{\mu,\zeta}(a_l)\,.$$

Following the idea of Kreimer [Kre99], we define

$$\mathcal{A}^{\mathtt{feyn}_{\mu,\zeta}}_{\mathtt{C},\mathrm{MS}} := R \circ \mathtt{feyn}_{\mu,\zeta} \circ \mathcal{A}_{\mathtt{C}}\,,$$

where R denotes minimal subtraction, i.e., the "renormalization map",

$$R(\boldsymbol{\mathcal{S}}^{(k)}_{\mu,\zeta}) = \begin{cases} \mathrm{id} & \text{for } k = 1 \\ \mathrm{pp}(\boldsymbol{\mathcal{S}}^{(k)}_{\mu,\zeta}) & \text{for } k > 1\,. \end{cases}$$

Assuming that R is only applied to prepared time-ordered products, we have that $R(\boldsymbol{\mathcal{S}}^{(k)}_{\mu,\zeta})$ is a multi-linear map from local functionals to local functionals, and then $R \circ \mathtt{feyn}_{\mu,\zeta}$ defines an algebra homomorphism of $(\mathfrak{H}, \odot, \textcircled{c})$ with respect to both products

$$R \circ \mathtt{feyn}_{\mu,\zeta}\left(a_k \textcircled{c} \bigodot_{i=1}^{k} a_{l_i}\right) = R(\boldsymbol{\mathcal{S}}^{(k)}_{\mu,\zeta}) \cdot \left(\bigodot_{i=1}^{k} R(\boldsymbol{\mathcal{S}}^{(l_i)}_{\mu,\zeta})\right).$$

That R is applied only to prepared time-ordered products is guaranteed by the recursive definition of the antipode. In particular R itself is a homomorphism of the symmetrized tensor product $\odot$, which makes the Rota-Baxter argument of

[Kre99] redundant in the presented framework (see Remark VI.16 below). We infer that $\mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn},\mu,\zeta}$ is an algebra homomorphism

$$\mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn},\mu,\zeta} : (\mathfrak{H}, \odot) \to (\mathrm{Lin}, \odot) \ .$$

We have that $\mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn}_{\mu,\zeta}}(a_1) = \mathrm{id}$, the identity on local functionals, and for $n > 1$,

$$\mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn}_{\mu,\zeta}}(a_n) = -R \sum_{\mathcal{P}\in\mathrm{Part}\{1,\ldots,n\}\setminus\{\mathcal{P}_1\}} a_{|\mathcal{P}|}(\boldsymbol{\mathcal{S}}_{\mu,\zeta}) \cdot \mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn}_{\mu,\zeta}}(\bigodot_{I\in\mathcal{P}} a_{|I|}) \, .$$

This is just the recursion formula we got for the counterterms (VI.12) by applying the minimal subtraction condition to the main theorem of renormalization in the form of Lemma VI.4. Observe that one loses the information on the product ⓒ if one regards the amplitudes, or regularized time-ordered products as elements in a commutative ring of Laurent series, only. However, similar as in Connes-Kreimer theory of renormalization, we can define the algebra homomorphism

$$\mathcal{A}_{\mathtt{C},\mathrm{ren}}^{\mathtt{feyn}_{\mu,\zeta}} = \mathtt{feyn}_{\mu,\zeta} \bullet_{\mathtt{C}} \mathcal{A}_{\mathtt{C},\mathrm{MS}}^{\mathtt{feyn}_{\mu,\zeta}} \, , \tag{VI.27}$$

which gives the finitely regularized n-fold time ordered product, when applied to a generator $a_n \in \mathfrak{H}$,

$$\mathcal{A}_{\mathtt{C},\mathrm{ren}}^{\mathtt{feyn}_{\mu,\zeta}}(a_n) = \boldsymbol{\mathcal{S}}_{\mu,\zeta,\mathrm{ren}}^{(n)}. \tag{VI.28}$$

However, although (VI.27) gives the correct result in the general case, i.e., for arbitrary local interactions, this is merely a compact notation for the successive subtraction of counterterms in the sense of BPH rather than a forest formula in the sense of Zimmermann [Zim69, Zim76] or Theorem VI.12. Solving the recursion would be equivalent to giving a closed formula for the character $\mathcal{A}_{\mathtt{C},\mathrm{ren}}^{\mathtt{feyn}_{\mu,\zeta}}$ in (VI.28). This was done in Section VI.3, although not in this abstract algebraic setting. After the remark on Rota-Baxter maps, we give a graphical interpretation of the maps constructed above.

REMARK VI.16 (Rota-Baxter Maps). For the reader, less familiar with the "Hopf algebra school" in renormalization theory, we probably have to remark here that a linear map fulfilling the algebraic relation

$$R(a)R(b) = R\left[R(a)b\right] + R\left[aR(b)\right] - R(ab) \, , \tag{VI.29}$$

is called a *Rota-Baxter map of weight one*. Examples of such maps are the projections in a *Birkhoff sum*, i.e., an algebra which splits into a direct sum of algebras, both closed under multiplication,

$$A = A^+ \oplus A^- .$$

The Laurent series with scalar coefficients are elements of a Birkhoff sum. It is straight forward to show the above claim that any linear projection in a Birkhoff sum, which projects to one of its components is Rota-Baxter of weight one. Denote by $a^+ + a^- \in A^+ \oplus A^-$ the elements of a Birkhoff sum, then by linearity and the

fact that the components are closed under multiplication we have

$$\begin{aligned}\left[\left(a^+ + a^-\right)\left(b^+ + b^-\right)\right]^- &= \left[a^- b^+ + a^+ b^-\right]^- + a^- b^- \\ &= \left[a^- b^+ + a^+ b^- + 2a^- b^-\right]^- - a^- b^- \\ &= \left[a^- b\right]^- + \left[ab^-\right]^- - a^- b^- .\end{aligned}$$

Following a suggestion by Brouder, Kreimer used this property for the projection to the principal part of a Laurent series in order to solve the "multiplicativity constraints", see [Kre99, Sec. 3]. Kreimer's "multiplicativity constraints" were originally formulated to have the map

$$\mathrm{pp} \circ \phi \circ \mathcal{A} : (\mathfrak{H}, \odot) \to \mathcal{L}^- ,$$

defined as a character of the commutative Hopf algebra to the ring of scalar Laurent series $\mathcal{L}$. $\phi : \mathfrak{H} \to \mathcal{L}$ denotes the character which induces Kreimer's (regularized) Feynman rules, $\mathcal{A}$ is the antipode of the commutative Hopf algebra, and $\mathrm{pp} : \mathcal{L}^+ \oplus \mathcal{L}^- \to \mathcal{L}^-$ is the projection to the principal part. Kreimer was able to show that the Rota-Baxter condition implies the multiplicativity of $R \circ \phi \circ \mathcal{A}$ (Prop. 2, loc. cit.). More advanced topics related to Rota-Baxter algebras in Physics and Mathematics partially induced by Kreimer's observation can be found, e.g., in [EFG07].

Observe, however, that in the framework advocated here all maps, and in particular R, are homomorphisms of the commutative algebra $(\mathfrak{H}, \odot)$, so that a Rota-Baxter argument is not necessary, since any algebra homomorphism, trivially, is a Rota-Baxter map of weight one.

VI.4.3. Graphs. The role of the additional composition product $\textcircled{c}$ and antipode $\mathcal{A}_{\mathsf{C}}$ in

$$(\mathfrak{H}, \odot, e, \Delta, \bar{e}, \mathcal{A}, \textcircled{c}, \mathcal{A}_{\mathsf{C}}) .$$

may become clearer if we break them down to the graph level. This graphwise interpretation is regained, if we insert for $\boldsymbol{\mathcal{S}}$ its perturbative expansion (IV.12), and regard the corresponding operations on the level of the graph contributions. This can be done since all maps involved are linear and since by condition [C3] all sums are finite at each fixed order of $\hbar$. Remember that we only regard full vertex parts as subgraphs.

In accordance with the structure of $(\mathfrak{H}, \odot, e)$, we regard the abstract algebra of graphs with disjoint union $\dot{\cup}$ as product and the empty set $\varnothing$ as unit. Let $\Gamma \in \mathcal{G}$ be a graph with n vertices, hence a contribution to $a_n(\boldsymbol{\mathcal{S}})$. Let $\mathrm{Part}^c V(\Gamma)$ be the set of all connected partitions of the vertex set of Γ. By connected partition we mean a partition $\mathcal{P}$ whose blocks $I \in \mathcal{P}$ give rise to connected full vertex parts $\gamma_I \subset \Gamma$. We can restrict to connected partitions, since the principal parts of the regularized amplitudes corresponding to disconnected graphs vanish. Denote by $\Gamma/\mathcal{P}$ the graph, which has the blocks $I \in \mathcal{P}$ as vertices and as lines the lines in Γ, which connect different blocks of $\mathcal{P}$. Let γ_I, $I \in \mathcal{P}$, be the full vertex part of the

block $I \in \mathcal{P}$. Then we can write the coproduct on the level of graphs as

$$\Delta\Gamma = \sum_{\mathcal{P}\in\mathrm{Part}^c V(\Gamma)} \Gamma/\mathcal{P} \otimes \dot{\bigcup_{I\in\mathcal{P}}} \gamma_I \,.$$

Applying the map $\mathrm{id}_{\mathfrak{H}} \otimes \mathcal{A}_{\mathtt{C}}$ gives

$$(\mathrm{id}_{\mathfrak{H}} \otimes \mathcal{A}_{\mathtt{C}}) \circ \Delta\Gamma = \sum_{\mathcal{P}\in\mathrm{Part}^c V(\Gamma)} \Gamma/\mathcal{P} \otimes \dot{\bigcup_{I\in\mathcal{P}}} \mathcal{A}_{\mathtt{C}}(\gamma_I) \,,$$

which corresponds to (recursively) computing the counterterms for the connected subgraphs γ_I, e.g., in DimReg+MS. Applying the composition product $\mathtt{C}$, inserts the counterterms $\mathcal{A}_{\mathtt{C}}(\gamma_I)$ at the vertices of $\Gamma/\mathcal{P}$ to give one contribution to the "renormalized graph",

$$\mathrm{id}_{\mathfrak{H}} \bullet_{\mathtt{C}} \mathcal{A}_{\mathtt{C}} = \sum_{\mathcal{P}\in\mathrm{Part}^c V(\Gamma)} \Gamma/\mathcal{P} \textcircled{c} \dot{\bigcup_{I\in\mathcal{P}}} \mathcal{A}_{\mathtt{C}}(\gamma_I) \,.$$

It is one of the results of the present thesis that this procedure, performed on the level of analytically regularized amplitudes in arbitrary representation (momentum or position space) leads to finitely regularized, i.e., UV convergent integrals, and local counterterms in all orders of perturbation theory.

Conclusion

The investigation undertaken for this thesis has shown that the methods of dimensional regularization and minimal subtraction can consistently be implemented into causal perturbation theory in the framework of perturbative Algebraic Quantum Field Theory (pAQFT). This enriches the framework by a renormalization technique, which has a preferred extension at all orders of causal perturbation theory, given the fact one disposes of an analytic regularization of the $\boldsymbol{\mathcal{S}}$-matrix. A concrete form of such a regularization was given in Minkowski position space in terms of the dimensionally regularized scattering matrix, $\boldsymbol{\mathcal{S}}_{\mu,\zeta}$. It was proven that the incorporation of an analytically regularized $\boldsymbol{\mathcal{S}}$-matrix makes it possible to solve the Epstein-Glaser induction at all orders in perturbation theory, and the result was given in terms of the Epstein-Glaser forest formula (Theorem VI.12). This result was derived directly from the main theorem of renormalization and was given in a form which is independent of the chosen representation. In particular the derived forest formula is valid in both, momentum and position space, whatever space its better suited for the concrete calculation.

Besides this forest formula, I gave a direct derivation of the Hopf algebra of Feynman graphs from the main theorem of perturbative renormalization. This Hopf algebra first occurred in the work of Kreimer and Connes-Kreimer in their analysis of BPHZ renormalization [Kre98, CK00, CK01] and was later found also in causal perturbation theory [GBL00, Pin00b]. In the present thesis the Hopf algebra of graphs was derived in a summed up form, i.e. the elements can be regarded as sums over all graphs with the same set of vertices. This is in accordance with the findings of [BF00b, BF01, Fra07, vS07a]. The reduced Hopf algebra of Pinter with only full vertex parts can be derived by linearity, however, the pure BPHZ subgraphs of the Connes-Kreimer approach do not emerge here. This is in accordance with the proof of Zimmermann that pure BPHZ graphs do not contribute to the renormalized amplitude [Zim76]. Although the Hopf algebra of graphs emerged as a commutative, non-cocommutative Hopf algebra, we could show that it is necessary to augment it with a non-commutative product stemming from the composition of linear maps in order to get the recursion relation for the pAQFT counterterms, which has been derived independently from the main theorem. The recursion relation for the counterterms is described algebraically as the antipode of the convolution induced by the coproduct and the additional non-commutative composition product. A main difference to the Connes-Kreimer

theory of renormalization is that in the algebraic setting described here the Feynman rules emerge naturally from the theory and are not assumed to give Laurent series with scalar coefficients. I see applications of these results in three main areas of current research in physics, mathematical physics, and mathematics.

First, physics. The forest formula was proven directly for the time-ordered products. However, we also gave a "graph form" of the formula, which could be relevant for concrete computations. I want to emphasize, that the combinatorial pattern underlying the Epstein-Glaser forest formula is much simpler than the one underlying Zimmermann's original version, which is still used in modern computations. Spurious subtractions do not occur in the EG forest formula. Although the spurious subtractions do not play a role in QED calculations, they do occur in Quantum Chromo Dynamics (QCD), since this theory has a four valent vertex. Besides this, there is a second simplification in the forest formula proven in this work. I showed that one can replace the Zimmermann forests of full vertex parts by totally ordered sets of partitions of the vertex set. This simplifies a lot the intricate combinatorics of Zimmermann's forest formula, and might make it possible to implement the advocated method in an algorithm. This, in turn, is certainly relevant for the computation of higher order contributions to the perturbative expansion in high energy physics phenomenology. Since the method was proven for any analytic regularization, also gauge symmetries should be preserved if one chooses a regularization which preserves these symmetries. This assertion is affirmed by the concrete computations of [FHS10], however, the case of gauge theories was not discussed in the present thesis.

Second, mathematical physics. The covariant formulation of perturbative Algebraic Quantum Field Theory makes the formalism applicable also in curved, globally hyperbolic spacetimes. Although the construction of the regularized $\mathcal{S}$-matrix was done in the present thesis for Minkowski space, the results of the last chapter, in particular the recursion relation for the counterterms (Corollary VI.5) and the Epstein-Glaser forest formula (Theorem VI.12) were derived in the more general, covariant framework. Thus they can be applied directly, given the fact that one disposes of an analytically regularized $\mathcal{S}$-matrix. Considering the convenient properties dimensional regularization has in flat spacetime when it comes to gauge theories, one may want to have a similar concept in curved spacetime. However, in the construction of the dimensionally regularized $\mathcal{S}$-matrix in Minkowski spacetime we made explicit use of translation invariance, and the choice of relative coordinates was made using the graph cohomology. As shown in [BF00a] the wave front set condition on local functionals can be understood as a microlocal remnant of translation invariance. However, one has to understand better the interplay of this microlocal condition with the graph cohomology in order to give a direct translation of the results.

Third, for mathematics. The analysis of algebraic structures is an active field of

research in pure mathematics which is of interest in its own right. The Hopf algebra found by Connes and Kreimer in perturbative renormalization theory affected this research on a profound basis, and the relation to the main theorem of perturbative renormalization and the framework of pAQFT which was established in this thesis could possibly be a new seed for research in this field. I showed that it is necessary to incorporate an additional composition structure into the Hopf algebra to have an interpretation for the antipode in terms of minimally subtracted counterterms in pAQFT, with the merit of having naturally emerging Feynman rules. There are many more intriguing questions about the connection of perturbative quantum field theory to pure mathematics. Questions about the role of multiple zeta values, graph polynomials, shuffle and stuffle products and the like in (algebraic) quantum field theory and causal perturbation theory. Such relations were established on the level of examples in the pioneering works of Connes and Kreimer [CK00, CK01] and Bloch, Esnault, and Kreimer [BEK06]. One might hope that the tools developed in this thesis contribute to further investigation of the suggested relations, and to a better understanding of the relation of the framework of perturbative algebraic quantum field theory to the more abstract algebraic setting of Connes and Marcolli [CM04a, CM04b].

APPENDIX A

Solutions of the Modified Bessel equation

In this Appendix we briefly review the solution theory of the (modified) Bessel equation. The interested reader may want to refer to [SS70] (e.g.) for a more detailed discussion of the topic. The modified Bessel equation[1]

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}\left(x^2+\nu^2\right)y = 0\,, \quad \mathrm{Re}(\nu) \geqslant 0 \tag{A.1}$$

is a second order ordinary differential equation with a *regular singular point* at the origin. That is, (at least one of) the coefficients $p(x)$ of $\frac{dy}{dx}$ and $q(x)$ of y are singular at $x = 0$, but $x\,p(x)$ and $x^2q(x)$ are regular in a neighborhood of zero. Let

$$x\,p(x) = \sum_{n=0}^{\infty} p_n x^n = 1 \qquad \text{and} \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n = -\left(x^2+\nu^2\right)$$

be the corresponding Taylor expansions. A differential equation with a regular singular point at 0 is solved with the ansatz

$$y = x^\alpha \sum_{s=0}^{\infty} c_s x^s\,, \quad c_0 \neq 0\,.$$

In order x^α one finds the *indicial equation*

$$\alpha\,(\alpha-1) + p_0\alpha + q_0 = 0\,,$$

whose roots are called the *exponents* of the differential equation. In the case of the (modified) Bessel equation (A.1), we evidently have

$$\alpha\,(\alpha-1) + \alpha - \nu^2 = (\alpha+\nu)\,(\alpha-\nu) = 0\,,$$

hence (A.1) has exponents $\alpha \in \{\pm\nu\}$. It is a straight forward calculation to see that, in the case $\nu \in \mathbb{C}\backslash\mathbb{N}_0$, we have $c_1 = 0$ and

$$c_s = \frac{c_{s-2}}{(s+\alpha+\nu)\,(s+\alpha-\nu)}\,, \quad \alpha \in \{\pm\nu\}\,, \quad \nu \in \mathbb{C}\backslash\mathbb{N}_0. \tag{A.2}$$

This leads to the linear independent set of solutions $\{I_{-\nu}, I_\nu\}$, where

$$I_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(\nu+s+1)}\left(\frac{x}{2}\right)^{\nu+2s}\,, \quad \nu \in \mathbb{C}\backslash\mathbb{N}_0. \tag{A.3}$$

[1]The modified Bessel equation (A.1) is related to Bessel's differential equation by the coordinate transform $x \mapsto ix$.

The functions I_ν and $I_{-\nu}$ are called modified Bessel function of first kind, they are related to the Bessel functions of first kind J_ν by

$$I_\nu = i^{-\nu} J_\nu(ix)\,.$$

Observe that in the case of integer order, $n \in \mathbb{N}_0$, $I_n = I_{-n}$, as can also be seen from the recursion relations (A.2). This is a general feature of (second order) ordinary differential equations with regular singular point. Problems occur, when their exponents differ by an integer value.

A linearly independent set of solutions of (A.1) for arbitrary order can be constructed in the following way. For non-integer order, $\nu \in \mathbb{C}\backslash\mathbb{N}_0$, we replace $I_{-\nu}$ by

$$K_\nu := \frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu} - I_\nu\right]\,, \quad \nu \in \mathbb{C}\backslash\mathbb{N}_0\,, \tag{A.4}$$

evidently giving a linearly independent set of solutions $\{I_\nu, K_\nu\}$ for non-integer order. The limit

$$K_n := \lim_{\nu\to n} K_\nu$$

exists for all integers $n \in \mathbb{N}_0$, and $\{I_n, K_n\}$ is a complete, linear independent set of solutions of (A.1). That $\{I_n, K_n\}$ is linearly independent, also in the case $n \in \mathbb{N}_0$ can be seen by the following argument. Just setting $\nu = n$ clearly results in the situation $\frac{0}{0}$, hence we apply l'Hôspital's rule to compute the limit

$$\lim_{\nu\to n} K_\nu = \frac{\pi}{2}\left[\frac{(\partial_\nu I_{-\nu}) - \partial_\nu I_\nu}{\pi\cos(\nu\pi)}\right]_{\nu=n} = \frac{(-1)^n}{2}\left[(\partial_\nu I_{-\nu}) - \partial_\nu I_\nu\right]_{\nu=n}$$

We have seen that $I_\nu(x) \sim x^\nu f_\nu(x^2)$, with entire analytic functions f_ν, hence the derivatives of I_ν with respect to ν will introduce logarithmic terms, $\partial_\nu\left(x^\nu\right) = \ln(x)\cdot x^\nu$, which do not cancel, since $f_\nu \neq f_{-\nu}$ for all $\nu \neq 0$.

The introduction of logarithmic terms in the limit $\nu \to n$ is not a feature of the special choice of K_ν but merely a consequence of the singularity structure of the (modified) Bessel differential equation at the origin (see e.g. [SS70, Sec. 1.6]).

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References

[Abe77] E. Abe. *Hopf algebras*. Cambridge UP, 2004 edition, 1977.

[AS70] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 1970.

[BF09] C. Bär and K. Fredenhagen, editors. *Quantum Field Theory on Curved Spacetimes - Concepts and Mathematical Foundations*, volume 786 of *Lecture Notes in Physics*, Berlin Heidelberg, 2009. Springer.

[BFF+78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization I+II. *Ann. Physics*, 111(1):61, 111, 1978.

[BRS75] C. Becchi, A. Rouet, and R. Stora. Renormalization of the abelian Higgs-Kibble model. *Communications in Mathematical Physics*, 42(2):127–162, June 1975.

[BBK09] C. Bergbauer, R. Brunetti, and D. Kreimer. Renormalization and resolution of singularities. 2009. arXiv:0908.0633.

[BEK06] S. Bloch, H. Esnault, and D. Kreimer. On Motives Associated to Graph Polynomials. *Commun. Math. Phys.*, 267:181–225, 2006.

[BW09] C. Bogner and S. Weinzierl. Periods and Feynman integrals. *J. Math. Phys.*, 50:042302,2009, 2009. arXiv:0711.4863 [hep-th].

[BW10] C. Bogner and S. Weinzierl. Feynman graph polynomials. 2010. arXiv:1002.3458 [hep-th].

[BS59] N. N. Bogoliubov and D. V. Shirkov. *Introduction to the Theory of Quantized Fields*. Interscience Publishers, 1959.

[BP57] N. N. Bogoliubow and O. S. Parasiuk. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. *Acta Mathematica*, 97(1-4):227–266, 1957.

[BG72a] C. G. Bollini and J. J. Giambiagi. Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter. *Nuovo Cimento B*, 12:20–25, 1972.

[BG96] C. G. Bollini and J. J. Giambiagi. Dimensional regularization in configuration space. *Phys. Rev. D*, 53(10):5761, May 1996.

[BM77a] P. Breitenlohner and D. Maison. Dimensional Renormalization and the Action Principle. *Commun. Math. Phys.*, 52:11–38, 1977.

[BM77b] P. Breitenlohner and D. Maison. Dimensionally Renormalized

Green's Functions for Theories with Massless Particles. 1. *Commun. Math. Phys.*, 52:39–54, 1977.

[BM77c] P. Breitenlohner and D. Maison. Dimensionally Renormalized Green's Functions for Theories with Massless Particles. 2. *Commun. Math. Phys.*, 52:55–75, 1977.

[BF00b] C. Brouder and A. Frabetti. Noncommutative renormalization for massless QED. 2000. arXiv:hep-th/0011161v2.

[BF01] C. Brouder and A. Frabetti. Renormalization of QED with planar binary trees. *The European Physical Journal C - Particles and Fields*, 19(4):715–741, March 2001.

[BD08] C. Brouder and M. Dütsch. Relating on-shell and off-shell formalism in perturbative quantum field theory. *J. Math. Phys.*, 49:052303, 2008. arXiv:0710.3040v2 [hep-th].

[BFK06] C. Brouder, A. Frabetti, and C. Krattenthaler. Non-commutative Hopf algebra of formal diffeomorphisms. *Advances in Mathematics*, 200(2):479–524, March 2006.

[BFM09] C. Brouder, A. Frabetti, and F. Menous. Combinatorial Hopf algebras from renormalization. 2009. arXiv:/abs/0909.3362.

[BDF09] R. Brunetti, M. Dütsch, and K. Fredenhagen. Perturbative Algebraic Quantum Field Theory and the Renormalization Groups. 2009. arXiv:0901.2038.

[BF00a] R. Brunetti and K. Fredenhagen. Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds. *Commun. Math. Phys.*, 208(3):623–661, 2000.

[BF04] R. Brunetti and K. Fredenhagen. Algebraic approach to Quantum Field Theory. 2004. arXiv:math-ph/0411072.

[BF07] R. Brunetti and K. Fredenhagen. Towards a Background Independent Formulation of Perturbative Quantum Gravity. In B. Fauser, J. Tolksdorf, and E. Zeidler, editors, *Quantum Gravity - Mathematical Models and Experimental Bounds*, pages 151–159. Birkhäuser, Basel, 2007. gr-qc/0603079v3.

[BFK96] R. Brunetti, K. Fredenhagen, and M. Köhler. The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes. *Commun. Math. Phys.*, 180(3):633–652, 1996.

[BFLR10] R. Brunetti, K. Fredenhagen, and P. Lauridsen Ribeiro. 2010. forthcoming paper.

[BFV03] R. Brunetti, K. Fredenhagen, and R. Verch. The Generally Covariant Locality Principle - A New Paradigm for Local Quantum Field Theory. *Commun. Math. Phys.*, 237:31–68, 2003.

[CK82] W. E. Caswell and A. D. Kennedy. A Simple Approach to Renormalization Theory. *Phys. Rev.*, D25:392, 1982.

[CK00] A. Connes and D. Kreimer. Renormalization in Quantum Field Theory and the Riemann-Hilbert Problem I: The Hopf Algebra

Structure of Graphs and the Main Theorem. *Commun. Math. Phys.*, 210(1):249–273, 2000.

[CK01] A. Connes and D. Kreimer. Renormalization in Quantum Field Theory and the Riemann-Hilbert Problem II: The β-Function, Diffeomorphisms and the Renormalization Group. *Commun. Math. Phys.*, 216(1):215–241, 2001.

[CM04a] A. Connes and M. Marcolli. From Physics to Number Theory via Noncommutative Geometry. Part I: Quantum Statistical Mechanics of Q-lattices. 2004. arXiv:math/0404128.

[CM04b] A. Connes and M. Marcolli. From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory. 2004. arXiv:hep-th/0411114.

[CM07] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives*. 2007. `http://www.alainconnes.org/downloads.html`.

[CK98] A. Connes and D. Kreimer. Hopf Algebras, Renormalization and Noncommutative Geometry. *Commun. Math. Phys.*, 199:203–242, 1998.

[Con78] J. B. Conway. *Functions of One Complex Variable*, volume 11 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 1978.

[DFP08] C. Dappiaggi, K. Fredenhagen, and N. Pinamonti. Stable cosmological models driven by a free quantum scalar field. *Phys. Rev. D*, 77:104015, 2008. arXiv:0801.2850.

[Dit90] J. Dito. Star-product approach to quantum field theory: The free scalar field. *Lett. Math. Phys.*, 20(2):125–134, 1990.

[DS67] N. Dunford and J. T. Schwartz. *Linear Operators, Part I: General Theory*. Pure and Applied Mathematics. Wiley Interscience Publishers, 1967.

[DF99] M. Dütsch and K. Fredenhagen. A Local (Perturbative) Construction of Observables in Gauge Theories: The Example of QED. *Commun. Math. Phys.*, 203(1):71–105, 1999.

[DF01a] M. Dütsch and K. Fredenhagen. Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion. *Commun. Math. Phys.*, 219(1):5–30, 2001.

[DF01b] M. Dütsch and K. Fredenhagen. Perturbative Algebraic Field Theory, and Deformation Quantization. In R. Longo, editor, *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, volume 30 of *Fields Institute Communications*, Providence, RI, 2001. AMS. arXiv:hep-th/0101079.

[DF03] M. Dütsch and K. Fredenhagen. The Master Ward Identity and

Generalized Schwinger-Dyson Equation in Classical Field Theory. *Commun. Math. Phys.*, 243(2):275–314, 2003.

[DF04] M. Dütsch and K. Fredenhagen. Causal Perturbation Theory in Terms of Retarded Products, and a Proof of the Action Ward Identity. *Rev. Math. Phys.*, 16(10):1291–1348, 2004.

[DF07] M. Dütsch and K. Fredenhagen. Action Ward Identity and the Stückelberg-Petermann renormalization group. *Prog. Math.*, 251:113–124, 2007. arXiv:hep-th/0501228.

[EFG07] K. Ebrahimi-Fard and L. Guo. Rota-Baxter Algebras in Renormalization of Perturbative Quantum Field Theory. *Fields Institute Communications*, 50:47–105, 2007. arXiv:hep-th/0604116.

[EG73] H. Epstein and V. Glaser. The Role of Locality in Perturbation Theory. *Ann. Inst. Henri Poincaré*, 19(3):211–295, 1973.

[Est98a] R. Estrada. Regularization of distributions. *International Journal of Mathematics and Mathematical Sciences*, 21(4):625–636, 1998.

[FdB55] F. Faà di Bruno. Sullo sviluppo delle Funzioni. *Annali di Scienze Matematiche e Fisiche*, 6:479–480, 1855.

[FHS10] S. Falk, R. Häußling, and F. Scheck. Renormalization in Quantum Field Theory: An Improved Rigorous Method. *J. Phys. A*, 43:035401, 2010. arXiv:0901.2252.

[FGB05] H. Figueroa and J. M. Gracia-Bondía. Combinatorial Hopf Algebras In Quantum Field Theory I. *Rev. Math. Phys.*, 17(8):881–976, 2005.

[FGBV05] H. Figueroa, J. M. Gracia-Bondía, and J. C. Varilly. Faà di Bruno Hopf algebras. 2005. arXiv:math/0508337.

[Fra07] A. Frabetti. Renormalization Hopf algebras and combinatorial groups. lecture notes for summer school *Geometric an Topological Methods for Quantum Field Theory*, Villa de Leyva, Colombia., 2007. `http://math.univ-lyon1.fr/~frabetti/papers/frabetti2007v1.pdf`.

[Fre09] K. Fredenhagen. The impact of the algebraic approach on perturbative quantum field theory. Talk at conference Algebraic Quantum Field Theory - the first 50 years, Göttingen, 29-31 July 2009.

[GML54] M. Gell-Mann and F. E. Low. Quantum electrodynamics at small distances. *Phys. Rev.*, 95:1300–1312, 1954.

[GB03] J. M. Gracia-Bondía. Improved Epstein-Glaser Renormalization in Coordinate Space I. Euclidean Framework. *Mathematical Physics, Analysis and Geometry*, 6(1):59–88, 2003. arXiv:hep-th/0202023.

[GBL03] J. M. Gracia-Bondía and S. Lazzarini. Improved Epstein-Glaser renormalization II. Lorentz invariant framework. *J. Math. Phys.*, 44(9):3863–3875, 2003. arXiv:hep-th/0212156.

[GBL00] J. M. Gracia-Bondía and S. Lazzarini. Connes-Kreimer-Epstein-Glaser Renormalization. 2000. arXiv:hep-th/0006106.

[Gri01b] D. R. Grigore. Scale invariance in the causal approach to renormalization theory. *Annalen der Physik*, 10:473–496, 2001.

[GY03b] J. L. Gross and J. Yellen, editors. *Handbook of Graph Theory*. Discrete Mathematics and its Applications. CRC Press, 2003.

[Haa58] R. Haag. Quantum field theories with composite particles and asymptotic conditions. *Phys. Rev.*, 112:669–673, 1958.

[HW57] D. Hall and A. S. Wightman. A Theorem on Invariant Analytic Functions with Applications to Relativistic Quantum Field Theory. *Matematisk-fysiske Meddelelse*, 31(5):1–41, 1957.

[Ham82] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N. S.)*, 7(1):65, 1982.

[HHM08] J. M. Harris, J. L. Hirst, and M. J. Mossinghoff. *Combinatorics and Graph Theory*. Undergraduate Texts in Mathematics. Springer, 2008.

[Hep66] K. Hepp. Proof of the Bogoliubov-Parasiuk Theorem on Renormalization. *Commun. Math. Phys.*, 2(4):301–326, 1966.

[HW03] S. Hollands and R. M. Wald. On the Renormalization Group in Curved Spacetime. *Commun. Math. Phys.*, 237(1-2):123–160, 2003.

[Hol04] S. Hollands. Algebraic approach to the 1/N expansion in quantum field theory. *Rev. Math. Phys.*, 16:509, 2004.

[HW01] S. Hollands and R. M. Wald. Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime. *Commun. Math. Phys.*, 223(2):289–326, 2001.

[HW02] S. Hollands and R. M. Wald. Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime. *Commun. Math. Phys.*, 231(2):309–345, December 2002.

[Hör03] L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Classics in Mathematics. Springer, 2003.

[Joh02] W. P. Johnson. The Curious History of Faà di Bruno's Formula. *The American Mathematical Monthly*, 109(3):217–234, 2002.

[JR82] S. Joni and G.-C. Rota. Coalgebras and Bialgebras in Combinatorics. In R. Morris, editor, *Umbral Calculus and Hopf Algebras*, volume 6 of *Contemporary Mathematics*. AMS, 1982.

[Kas00] D. Kastler. Connes-Moscovici-Kreimer Hopf Algebras. In R. Longo, editor, *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, volume 30 of *Fields Institute Communications*, 2000. arXiv:math-ph/0104017.

[Kas04] D. Kastler. On the exterior structure of graphs. *J. Math. Phys.*, 45(5):1777–1786, 2004.

[Kel09] K. J. Keller. Euclidean Epstein-Glaser Renormalization. *J. Math. Phys.*, 50(10):103503, 2009. arXiv:0902.4789.

[Kre98] D. Kreimer. On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.*, 2(2):303–334, 1998.

[Kre99] D. Kreimer. Chen's Iterated Integral represents the Operator Product Expansion. *Adv. Theor. Math. Phys.*, 3(3):627–670, 1999. hep-th/9901099v5.

[KM97] A. Kriegl and P. W. Michor. *The Convenient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs*. AMS, 1997.

[KTV96] A. N. Kuznetsov, F. V. Tkachov, and V. V. Vlasov. Techniques of Distributions in Perturbative Quantum Field Theory. 1996. arXiv:hep-th/9612037.

[LSZ55] H. Lehmann, K. Symanzik, and W. Zimmermann. Zur Formulierung quantisierter Feldtheorien. *Il Nuovo Cimento*, 1(1):205–225, 1955.

[LSZ57] H. Lehmann, K. Symanzik, and W. Zimmermann. On the Formulation of Quantized Field Theories - II. *Il Nuovo Cimento*, 6(2):319–333, 1957.

[Nee05] K.-H. Neeb. Monastir Summer School: Infinite-Dimensional Lie Groups. lecture notes, 2005. `http://www.math.uni-hamburg.de/home/wockel/data/monastir.pdf`.

[Pin00a] G. Pinter. *Epstein-Glaser Renormalization: Finite Renormalizations, the S-Matrix of ϕ^4-Theory and the Action Principle*. PhD thesis, Universität Hamburg, 2000. `http://unith.desy.de/research/aqft/doctoral_theses/`.

[Pin00b] G. Pinter. The Hopf Algebra Structure of Connes and Kreimer in Epstein-Glaser Renormalization. *Letters in Mathematical Physics*, 54(3):227–233, November 2000.

[Pin01] G. Pinter. Finite Renormalizations in the Epstein Glaser Framework and Renormalization of the S-Matrix of ϕ^4-Theory. *Annalen der Physik*, 10:333–363, 2001. arXiv:hep-th/9911063v3.

[PS82] G. Popineau and R. Stora. A Pedagogical Remark on the Main Theorem of Perturbative Renormalization Theory. unpublished notes, 1982.

[Pra99] D. Prange. Epstein-Glaser renormalization and differential renormalization. *J. Phys. A*, 32(11):2225–2238, 1999. hep-th/9710225.

[Pra00] D. Prange. *Energy Momentum Tensor and Operator Product Expansion in Local Causal Perturbation Theory*. PhD thesis, Universität Hamburg, 2000. `http://unith.desy.de/research/aqft/doctoral_theses/`.

[Rad96] M. J. Radzikowski. Micro-Local Approach to the Hadamard

Condition in Quantum Field Theory on Curved Space-Time. *Commun. Math. Phys.*, 179:529–553, 1996.

[Rue62] D. Ruelle. On the asymptotic condition in quantum field theory. *Helv. Phys. Acta*, 35:147ff, 1962.

[SS70] B. Spain and M. Smith. *Functions of Mathematical Physics*. Van Nostrand Reinhold Company, London, 1970.

[Ste71] O. Steinmann. *Perturbation Expansions in Axiomatic Field Theory*, volume 11 of *Lecture Notes in Physics*. Springer, Berlin and Heidelberg, 1971.

[Ste00] O. Steinmann. *Perturbative Quantum Electrodynamics and Axiomatic Field Theory*. Springer, 2000.

[SP53] E. C. G. Stückelberg and A. Petermann. La normalization des constantes dans la theorie des quanta. *Helv. Phys. Acta*, 26:499–520, 1953.

[Swe69] M. E. Sweedler. *Hopf Algebras*. W. A. Benjamin, Inc., 1969.

[tHV72] G. 't Hooft and M. Veltman. Regularization and renormalization of gauge fields. *Nuclear Physics B*, 44(1):189–213, July 1972.

[vS07a] W. D. van Suijlekom. Multiplicative renormalization and Hopf algebras. 2007. arXiv:0707.0555.

[Wal07] S. Waldmann. *Poisson-Geometrie und Deformationsquantisierung*. Springer, 2007.

[Wat22] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge UP, 1922.

[Wei06] S. Weinzierl. The Art of Computing Loop Integrals. In *Toronto 2005, Universality and Renormalization*, pages 345–395, 2006. arXiv:hep-ph/0604068.

[WW02] E. T. Whittaker and G. Watson. *A Course of Modern Analysis*. Cambridge UP, 1902.

[Zim69] W. Zimmermann. Convergence of Bogoliubov's Method of Renormalization in Momentum Space. *Commun. Math. Phys.*, 15(3):208–234, 1969.

[Zim70] W. Zimmermann. Local Operator Products and Renormalization in Quantum Field Theory. In *Lectures on Elementary Particles and Quantum Field Theory*, volume 1 of *Brandeis University Summer Institute in Theoretical Physics*, 1970.

[Zim76] W. Zimmermann. Remark on Equivalent Formulations for Bogoliubov's Method of Renormalization. In G. Velo and A. S. Wightman, editors, *Renormalization theory*, NATO advanced study institutes series C: Mathematical and physical sciences, Dordrecht and Boston, 1976. Reidel. Proceedings of Erice Conference, 1975.