Quantum Effects of Majorana Neutrinos in Precision Observables

Dissertation

zur Erlangung des Doktorgrades

des Fachbereichs Physik

der Universität Hamburg

vorgelegt von

Kai-Peer Oliver Diener

aus Hechingen

Hamburg

2000

Gutachter der Dissertation:	Prof. Dr. B.A. Kniehl Prof. Dr. G. Kramer
Gutachter der Disputation:	Prof. Dr. B.A. Kniehl Prof. Dr. J. Bartels
Datum der Disputation:	18.12.2000
Dekan des Fachbereichs und Vorsitzender des Promotionsausschusses:	Prof. Dr. F.-W. Büßer

Zusammenfassung

Die vorliegende Arbeit untersucht, inwieweit sich eine Erweiterung des Standardmodell–Leptonsektors um beliebig mischende Majorana–Neutrinos auf theoretische Vorhersagen für sehr genau gemessene elektroschwache Observablen auswirkt.
Eine die entsprechenden leptonischen Wechselwirkungen beschreibende Lagrangedichte wird explizit konstruiert, und die Kopplungen der Neutrinos an die Eichbosonen und Skalare des Standardmodells werden abgeleitet.
Um die Abhängigkeit der Formfaktoren vom Eichparameter untersuchen zu können, werden generische Ergebnisse für den Zerfall eines Eichbosons in ein Fermionpaar auf dem Ein–Schleifen–Niveau angegeben.
Bisher verwendete Counterterme für die Renormierung fermionischer Mischungsmatrizen im On–Shell–Schema werden vorgestellt und kritisiert. Wir geben eine Renormierungsbedingung an, aus der Counterterme für die Standardmodell–Quark–Mischungsmatrix sowie die leptonischen Mischungsmatrizen des besprochenen Modells bestimmt werden können.
Unter Verwendung unserer Renormierungsvorschrift für leptonische Mischungsmatrizen geben wir anschließend analytische Ausdrücke für die elektroschwache Pseudoobservable Δr in einem Modell mit erweitertem Leptonsektor an.
Im letzten Teil der Arbeit diskutieren wir unsere numerischen Ergebnisse für die relative Abweichung der (leptonischen) Zerfallsbreiten des W–Bosons und des Myons in diesem Modell gegenüber dem Standardmodell.

Abstract

In this work we investigate to what extent an extension of the standard model lepton sector accomodating Majorana neutrinos with arbitrary mixing changes the theoretical predictions for precisely measured electroweak observables.
A Lagrangian describing the respective leptonic interactions is explicitly constructed and the couplings of the neutrinos to the gauge bosons and scalars are derived.
To investigate the gauge parameter dependence of the form factors generic results for the decay of a gauge boson into a fermion pair at one–loop level are given.
The counterterms of fermionic mixing matrices in the on–shell scheme as found in the literature so far are presented and criticized. We give a renormalization condition which allows the derivation of counterterms for the standard model quark mixing matrix as well as the mixing matrices of the aforementioned extended lepton sector models.
Making use of our renormalization prescription for the leptonic mixing matrices we then present analytic results for the electroweak pseudo–observable Δr in such a framework.
In the last part of this work we discuss our numerical results for the relative deviation of the (leptonic) decay widths of the W–boson and the muon in this model compared to the standard model.

Contents

1 Introduction and Outline

For more than twenty years the standard model of particle interactions (SM) has been most successful in describing the phenomena of elementary particles at high energies. All matter fields predicted by the theory on the basis of self–consistency arguments have by now been experimentally observed.
Nevertheless strong theoretical arguments like the failure of unification of all interactions suggest that the SM, instead of fundamental should rather be considered an effective low–energy theory.
Although the SM has been subjected to a quite sizeable number of high–precision tests at large collider facilities, the only indication of non–conformance of the SM with experimental reality stems from a field, known for its notorious experimental difficulties and large uncertainties: neutrino physics.
The Super–Kamiokande experiment [1], which measures neutrino fluxes in an underground water Cerenkov detector, found evidence in atmospheric neutrino flux ratios that neutrinos not only should be massive but also that neutrino gauge interactions are not diagonal. Both phenomena raise doubts about the SM leptonic sector and motivate the search for theoretical frameworks which correctly describe the observed data. A fairly general class of such frameworks is constructed by adding an arbitrary number of right–handed gauge singlet neutrinos to the SM, allowing for the usual Dirac mass term connecting left– and right–handed fields as well as a Majorana mass term for the right–handed fields. The Dirac mass term can simply be introduced in strict analogy to the other SM fermions, the Majorana mass can be treated as an effective parameter or can alternatively be generated from the vacuum expectation value (VEV) of an extra gauge singlet neutral Higgs field, giving rise to spontaneous breaking of lepton number. Lepton family number is in principle broken by the Dirac and the Majorana mass terms. In this way one obtains the classical see–saw neutrino mass matrix with a 'texture' zero instead of a left–handed Majorana mass term. Thus, one is not forced to introduce new Higgs fields to ensure gauge invariance of a left–handed Majorana term and through the see–saw mechanism has a natural explanation for the small masses of the observed neutrino states.

The aim of this work is to investigate if, by comparison of high–accuracy electroweak observables with precision calculations at next–to–leading order (NLO) in such a framework, one is able to obtain constraints on neutrino masses and mixing angles complementary to those obtained from the 'classical' neutrino experiments.

At first we will describe in detail how the interaction Lagrangian of such a scenario is constructed. We explicitly derive all those leptonic interactions which differ from their SM analogues.
In the next section we shall give very generic formulae for the decay of a gauge boson into a pair of fermions at NLO in arbitrary R_ξ gauges. We analyze the structure of the proper vertex corrections and counterterms with special regard to their gauge parameter dependence, as this will be of great importance in the following section about the renormalization of fermionic mixing matrices in general and leptonic mixing in our scenario in particular.

At the beginning of this section we review two counterterms for the Cabbibo–Kobayashi–Maskawa matrix in the on–shell scheme suggested in the literature. We generalize one of these approaches in such a way that the resulting counterterm can be derived from a renormalization condition instead of being postulated ad hoc. In the following we show that a previously published counterterm for leptonic mixing matrices leads to gauge–parameter–dependent transition amplitudes and is thus unphysical. Using the results of the previous section, we apply our renormalization condition to the case of leptonic mixing and find that by construction it leads to gauge–parameter–independent and finite results.
Subsequently, we calculate the electroweak pseudo–observable Δr at one–loop level in our framework, making use of our renormalization method.
Finally, we give numerical values for the relative deviation of the W and muon one–loop decay widths in our extended lepton sector scenario from their SM analogues.

2 Extending the Leptonic Sector of the Standard Model

2.1 Charge Conjugation and Majorana Particles

Starting from the Dirac equation for electrons interacting with an electromagnetic field

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0\,, \tag{1}$$

we seek a transformation $\psi \to \psi^c$, such that

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi^c = 0 \tag{2}$$

be fulfilled with the requirement that the transformation be local and that the charge conjugation operator be its own inverse up to a phase.
In order to construct ψ^c, we hermitian conjugate and then transpose equation (1) to receive

$$[\gamma^{\mu T}(-i\partial_\mu - eA_\mu) - m]\bar{\psi}^T = 0\ .$$

Assuming the existence of a matrix C, with

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu\ , \tag{3}$$

we may define

$$\psi^c = \eta_c C\bar{\psi}^T\ ,$$

with a phase factor η_c, which obviously solves equation (2).
We prove the existence of C by giving it explicitly in the Dirac representation:

$$C = i\gamma^2\gamma^0 = -i\sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}\ ,$$

which enables us to conclude:

$$C^T \ = \ -C \quad \text{and} \quad C^\dagger C \ = \ 1\ .$$

These relations between C, C^T and $C^\dagger$ remain valid in any other possible representation.
The transformation $\psi \to \psi^c$ is called charge conjugation. Invariance up to a phase under this transformation i.e.

$$\psi^c = \eta_\psi\psi \tag{4}$$

characterizes so–called Majorana fields.
If we want to consider the charge conjugation of a quantized field ψ we define an associated operator $\mathcal{C}$ which acts on ψ like:

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \eta_c C\bar{\psi}^T.$$

Commutation properties of the chirality projectors

$$R = \frac{1}{2}(1+\gamma_5) \quad \text{and} \quad L = \frac{1}{2}(1-\gamma_5)$$

with charge conjugation are easily determined:

$$\begin{aligned}(\psi_R)^c &= C\overline{\psi_R}^T = C[(R\,\psi)^\dagger\gamma^0]^T = C[\psi^\dagger R\,\gamma^0]^T \\ &= C\gamma^0 R\,(\psi^\dagger)^T = C\,L\,\overline{\psi}^T = L\,C\overline{\psi}^T = L\,\psi^c \\ &= (\psi^c)_L\ . \end{aligned} \tag{5}$$

In the future we shall simply write ψ_L^c for $(\psi_L)^c$, $\overline{\psi}^c$ shall denote the adjoint of the charge conjugate spinor:

$$\overline{\psi}^c = \overline{(\psi^c)} = (C\overline{\psi}^T)^\dagger\gamma^0 = \psi^T\gamma^0 C^\dagger\gamma^0 = -\psi^T\gamma^0 C\gamma^0 = \psi^T C\ . \tag{6}$$

We shall now prove two useful identities for Dirac spinors, that we will need later to simplify neutrino interaction terms in the Lagrangian:

$$\begin{aligned}\overline{\psi}^c_R\chi_R &= (\overline{\psi}^c_R\chi_R)^T = (\psi^T C R\,\chi)^T = -\chi^T R^T C^T\psi \\ &= -\chi^T R\,(-C)\psi = \chi^T C R\,\psi \\ &= \overline{\chi}^c_R\psi_R\ , \end{aligned} \tag{7}$$

$$\begin{aligned}\overline{\psi}^c_L\chi^c_R &= \psi_L^T C C\overline{\chi}_R^T = -\psi_L^T C C^{-1}\overline{\chi}_R^T \\ &= -\psi_L^T\overline{\chi}_R^T = (\overline{\chi}_R\psi_L)^T \\ &= \overline{\chi}_R\psi_L\ . \end{aligned} \tag{8}$$

Note that fermionic fields are treated as Grassman variables.

2.2 Neutrino Mass Terms

As mentioned above, our goal is a generalization of the SM leptonic sector to account for neutrino masses and mixing between neutrino flavors as indicated by neutrino experiments. The most straightforward approach would be to add three right–handed neutrino flavors to the SM field content, thus restoring the symmetry between the neutrinos on one side and the other fermionic fields on the other side, and allowing for a Dirac–type mass term:

$$(m_D\,\overline{\nu}_L\,\nu_R + \text{h.c.})\ . \tag{9}$$

This is, however, not the most general type of mass term that is possible for neutral fermions. From equations (5) and (6) it can easily be understood, that there are two more ways of constructing neutrino mass terms, namely:

$$\frac{1}{2}\,(m_L\,\overline{\nu}^c_L\,\nu_L + \text{h.c.}) \tag{10}$$

and

$$\frac{1}{2}\,(m_R\,\overline{\nu}^c_R\,\nu_R + \text{h.c.})\ . \tag{11}$$

The advantages of this particular choice of prefactors will become apparent when we write the combined mass term in a more compact way.

We want to point out that writing down a mass term as in equations (10) and (11) one has implicitly assumed, that ν and ν^c represent the same field, as it does not make sense to construct a mass term connecting two fields carrying different quantum numbers. Such mass terms are called Majorana mass terms and can only be constructed from neutral fields. Accommodating all three terms in compact form and adopting the usual sign convention for the mass term we can write:

$$\mathcal{L}_m = -\frac{1}{2} \begin{pmatrix} \overline{\nu}_L & \overline{\nu}^c_R \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix} + \text{h.c.} \tag{12}$$

This is the most general structure we can write down for one right– and one left–handed Majorana field and it is (up to sign convention) exactly equal to the sum of the three terms in equations (9) – (11). To verify this one needs to apply equation (8) to the Dirac terms. Obviously this only works for our choice of prefactors in equations (9) – (11).

As we would like to describe particle interactions in the framework of a gauge theory, we have to ensure gauge invariance of all the mass terms introduced so far.
A Dirac mass as in equation (9) can be introduced in exact analogy to the SM mass terms for the up–type quarks using the usual SM Higgs doublet $\Phi = (\phi^+ \; \phi^0)^T$:

$$-g^{\text{Dirac}}_{ij} \; (\bar{\nu}_L \; \bar{l}_L)_i \; i\tau_2 \, \Phi^* \, \nu_{Rj} \; , \tag{13}$$

where we allowed interfamily mixing through the coupling matrices g^{Dirac}_{ij}.
We now turn to the right–handed neutrino states and introduce a mass term as in equation (11). As there are only gauge singlets involved, in principle no recourse to spontaneous symmetry breaking is necessary to keep the Lagrangian gauge–invariant. The mass parameter m_R could be introduced 'by hand' without affecting gauge invariance. Because it is desirable to treat all mass terms on the same grounds and to avoid an ad hoc parameter m_R in the Lagrangian, we generate m_R by spontaneous symmetry breaking of a complex scalar gauge singlet ϕ_R introduced for just this purpose:

$$-\frac{1}{2} g^{\text{R}}_{ij} \, \overline{\nu}^c_{Ri} \nu_{Rj} \; \phi_R . \tag{14}$$

As opposed to the Majorana mass for the right–handed neutrinos where we introduced ϕ_R for reasons of consistency and esthetics, there is a clear need for a new Higgs field if we want to endow the left–handed neutrinos with a Majorana mass term in analogy to the other mass terms of the model. They are no singlets under gauge transformations, and mass terms for the left–handed matter fields must clearly involve the whole doublet containing also the isopartners. To render such a mass term gauge–invariant, we are forced to introduce a triplet Higgs field:

$$h \; = \; \begin{pmatrix} h^+ & h^{++} \\ h^0 & -h^+ \end{pmatrix} \tag{15}$$

with hypercharge $Y = 2$. Given the multiplet h as above we are able to construct the following gauge–invariant mass term for the left–handed neutrinos:

$$-\frac{1}{2} g^{\text{L}}_{ij} \; (\overline{\nu}^c \, \overline{l}^c_f)_{Li} \; i\tau_2 \; h \; (\nu \; l_f)_{Lj} . \tag{16}$$

The elements of the mass matrix of equation (12) are the products of the couplings g_{ij} with the VEVs of the corresponding Higgs fields.

The most general theoretical scenario for massive neutrinos will contain three different types of mass terms and three different Higgs fields. Following reference [2] we call this most general case the 1–2–3 scenario because we use the VEVs of a singlet, doublet and a triplet Higgs field to generate the different mass terms. We observe that the singlet and triplet Higgs fields carry non–vanishing lepton number $l = -2$.
In the remainder of this work we shall confine ourselves to the 1–2 case, i.e. we shall not consider a Majorana mass for the left–handed neutrinos. As will be shown shortly this leads to the standard see–saw mechanism, explaining the lightness of the observed three neutrino flavors. Also we will not investigate the phenomenology of the degrees of freedom of the field ϕ_R of equation (14) after spontaneous symmetry breaking. Instead m_R will be treated as an effective parameter. It can be viewed as a remnant of the spontaneous breaking of lepton number at a much higher scale than that of electroweak symmetry breaking. At the electroweak scale the remaining degrees of freedom are assumed to be decoupled.

Now that we have discussed the different ways of endowing the neutrino degrees of freedom with mass we want to consider the scenario which we shall actually investigate in the phenomenological part of this work. We want to generalize the mass term (12) to several generations in compliance with the SM gauge group.
Up to this point we have been rather careless in distinguishing between right–handed and left–handed versus isosinglet and isodoublet neutrinos. For Majorana states this distinction is more involved than for the charged fermionic states of the SM. This is due to the fact that charge conjugation and chirality projection do not commute as we have seen in equation (5). The situation becomes even more confusing once we go back and forth between the flavor and mass eigenstates of the fields. We shall therefore construct our mass term in its final form from neutrino states which are manifestly invariant under charge conjugation but which are not eigenstates of the chirality projection operators.
We start from a number n_G of gauge doublet neutrino states ν_{L_i} and a number n_R of gauge singlet neutrino states ν_{R_i}. We call the number of doublet neutrinos n_G because in order to avoid anomalies it must agree with the number of matter field generations of our scenario, which will usually be set to three. From the ν_{L_i} and ν_{R_i} we construct the charge–conjugation–symmetric state vectors σ and ω:

$$\sigma = \begin{pmatrix} \nu_{L_1} + \nu_{L_1}^c \\ \vdots \\ \nu_{L_{n_G}} + \nu_{L_{n_G}}^c \end{pmatrix}, \quad \omega = \begin{pmatrix} \nu_{R_1} + \nu_{R_1}^c \\ \vdots \\ \nu_{R_{n_R}} + \nu_{R_{n_R}}^c \end{pmatrix}. \tag{17}$$

Every entry in the vectors thus defined contains only two degrees of freedom. Of course they can still be represented as Dirac spinors, and we will do so to stick to the standard notation of the SM and its Feynman rules, but it should be noted that the four entries of such an object cannot be independent.
It is also worth noticing that although ω contains only gauge singlet states, its left–handed projection does not vanish, just as the corresponding right–handed projection

of the entries of σ:

$$R\,\sigma_i = \nu^c_{L_i} \neq 0, \qquad L\,\omega_i = \nu^c_{R_i} \neq 0\ .$$

Written in terms of the σ and ω state vectors, the final form of the mass term in the 1–2 scenario is

$$\mathcal{L}_m = -\frac{1}{2}\left(\begin{array}{cc} \overline{\sigma} & \overline{\omega} \end{array}\right)_L \left(\begin{array}{cc} 0 & m_D \\ m_D^T & m_R \end{array}\right) \left(\begin{array}{c} \sigma \\ \omega \end{array}\right)_R + \text{h.c.} \tag{18}$$

As we have mentioned above, we want to assume m_L to vanish in our phenomenological studies. The symmetry in m_D, i.e. the appearance of m_D and m_D^T in the mass matrix is due to equation (8). Relation (7) forces m_R to be symmetric. We conclude that

$$m_\nu = \left(\begin{array}{cc} 0 & m_D \\ m_D^T & m_R \end{array}\right) = m_\nu^T. \tag{19}$$

The dimension of m_ν is in general $(n_G + n_R) \times (n_G + n_R)$ while m_D is $n_R \times n_G$ and m_R is $n_R \times n_R$.

2.3 The See–Saw Mechanism

Having fixed the structure of the mass matrix, we might wonder about its phenomenological consequences. We want to present the physical mass spectrum of the neutrino sector induced by our specific choice for m_ν in a qualitative manner. A detailed analysis strongly depends on the specific values of all parameters and shall not be our goal at this point. For this reason we think it is well justified to study the simplest conceivable case with $n_G = n_R = 1$.
For the two eigenvalues of m_ν we obtain

$$m_1 \quad = \quad \frac{1}{2}m_R - \frac{1}{2}\sqrt{m_R^2 + 4m_D^2} \quad \approx \quad -\frac{m_D^2}{m_R}$$

and

$$m_2 \quad = \quad \frac{1}{2}m_R + \frac{1}{2}\sqrt{m_R^2 + 4m_D^2} \quad \approx \quad M + \frac{m_D^2}{m_R}\ .$$

We have expanded the mass eigenvalues in $\frac{m_D}{m_R}$ because we want to assume $m_R \gg m_D$. This is well motivated as m_D and m_R have two physically different origins: m_D is generated through the SM Higgs mechanism and can due to the unitarity bound be expected to be of the order of the scale of electroweak symmetry breaking while m_R might well stem from some symmetry broken at a scale which is several orders of magnitude higher.
This in turn implies that we obtain one very light and one very heavy mass eigenstate. Because this feature is phenomenologically desirable and also theoretically satisfying as it follows only from the structure of m_ν without any further assumptions, the procedure of mass generation with matrices of our type is well known under the name 'see–saw mechanism'. It was first proposed by Gell–Mann, Ramond, Slansky and Yanagida [3]. The relation between mass and flavor eigenstates is governed by the matrix U that transforms m_ν to a positive diagonal matrix m_n. Note that the unphysical negative

sign of m_1 can easily be absorbed into a phase factor multiplying the neutrino fields. For the simple example under consideration here this transformation is readily calculated as

$$m_n = U^T\, m_\nu\, U \ , \tag{20}$$

where

$$U = \begin{pmatrix} -i\sqrt{\frac{m_2}{-m_1+m_2}} & \sqrt{\frac{-m_1}{-m_1+m_2}} \\ i\sqrt{\frac{-m_1}{-m_1+m_2}} & \sqrt{\frac{m_2}{-m_1+m_2}} \end{pmatrix} \tag{21}$$

with positive entries $\{-m_1, m_2\}$ in m_n.

In appendix A we show that any symmetric complex matrix can be diagonalized by a transformation of the type of equation (20). This implies that also in the more complicated cases of arbitrary numbers of isosinglet and isodoublet neutrinos, the neutrino mass matrix can be brought to a positive diagonal form by only one unitary transformation and its transposed. This property holds regardless of the specific type of model (1–2 versus 1–2–3) under consideration because the matrix m_L in the 1–2–3 scenario satisfies the same symmetry relation as m_R in the present case so that the neutrino mass matrix is always symmetric. This fact clearly distinguishes the neutrinos from all other SM matter fields. In case of the quarks and charged leptons the mass matrices in the weak basis have no symmetry properties so that their diagonalization must be performed by two different unitary matrices U_R and U_L transforming the right and left–handed matter fields independently.

2.4 The Interaction Lagrangian

As we can see from equation (20), the transition from flavor states $\nu = (\sigma\,\omega)^T$, where σ denotes the 'left–handed' states participating in the weak interaction and ω denotes the 'sterile' states, to mass eigenstates n is a unitary transformation:

$$\nu_R = U n_R \quad \text{and} \quad \nu_L = U^\star n_L \ . \tag{22}$$

In order to construct the interactions of the neutrinos with the charged gauge bosons and scalars we need to introduce the matrix U_l that appears in the diagonalization of the charged–lepton mass matrix, transforming left–handed flavor (l_f) to mass eigenstates (l). At this point we remark that the dimensions of U an U_l are not equal: while U_l is $n_G \times n_G$, U is $(n_G + n_R) \times (n_G + n_R)$. A product of these matrices is thus a priori undefined for simple dimensional reasons.
With these definitions at hand we write down the interaction of the neutrinos with the charged gauge bosons:

$$\mathcal{L}^W = \frac{e}{s_{\rm w}\sqrt{2}} W^\mu\, \bar{l}_f \gamma_\mu L \sigma \ + \ \text{h.c.} = \frac{e}{s_{\rm w}\sqrt{2}} W^\mu\, \bar{l}_f \gamma_\mu L\, A \nu_L \ + \ \text{h.c.}\ , \tag{23}$$

where

$$A = (\mathbf{1}_{n_G\times n_G}\ \mathbf{0}_{n_G\times n_R}) \tag{24}$$

is a matrix we need to introduce to project ν onto σ. In terms of mass eigenstates, equation (23) reads

$$\mathcal{L}^W = \frac{e}{s_\mathrm{w}\sqrt{2}} W^\mu \bar{l}_m U_l \gamma_\mu L A U^\star n + \text{h.c.} \tag{25}$$

From this expression we can directly read off the definition of the leptonic equivalent of the SM CKM matrix, which we shall call B:

$$B = U_l A U^\star. \tag{26}$$

It should be noted, that B is a rectangular matrix of dimension $n_G \times (n_G + n_R)$. Using B to express the charged current interaction, it reads in its final form:

$$\mathcal{L}^W = \frac{e}{s_\mathrm{w}\sqrt{2}} W^\mu \bar{l}_m B \gamma_\mu L n + \text{h.c.} \tag{27}$$

Unlike other, more 'exotic' extensions of the leptonic sector, like e.g. the E_6-inspired model of reference [4], the charged current interactions of our scenario are still pure $v - a$.

Next we want to consider the neutral weak current interactions. Its explicit form in the mass state basis follows just like in the charged current case:

$$\mathcal{L}^Z = \frac{e}{2 s_\mathrm{w} c_\mathrm{w}} Z^\mu \bar{\sigma} \gamma_\mu L \sigma = \frac{e}{2 s_\mathrm{w} c_\mathrm{w}} Z^\mu \bar{n} U^T A^T \gamma_\mu L A U^\star n \, . \tag{28}$$

We define a matrix C describing neutral weak current mixing of the neutrinos:

$$C = U^T A^T A U^\star. \tag{29}$$

The matrix C is hermitian by construction as required for the Z interaction term. From the hermiticity of C we can conclude symmetry and antisymmetry of the real and imaginary part of C, respectively:

$$\operatorname{Re} C + i \operatorname{Im} C = C = C^\dagger = \operatorname{Re} C^T - i \operatorname{Im} C^T \, . \tag{30}$$

In appendix B we show, that the vector and axial–vector currents of two Majorana particles are antisymmetric and symmetric, respectively, under particle exchange:

$$\overline{\chi}^c \gamma_\mu \psi^c = -\overline{\psi} \gamma_\mu \chi \quad \text{and} \quad \overline{\chi}^c \gamma_5 \gamma_\mu \psi^c = \overline{\psi} \gamma_5 \gamma_\mu \chi \, . \tag{31}$$

Taking into account the symmetry properties of the currents and the mixing matrix, we find the following equality:

$$\bar{n} C \gamma_\mu (1 - \gamma_5) n = \bar{n} \gamma_\mu (i \operatorname{Im} C - \operatorname{Re} C \gamma_5) n \, . \tag{32}$$

After insertion of relation (32) into expression (28), the Z interaction of Majorana neutrinos can be written as:

$$\mathcal{L}^Z = \frac{e}{4 s_\mathrm{w} c_\mathrm{w}} Z^\mu \bar{n} \gamma_\mu (i \operatorname{Im} C - \operatorname{Re} C \gamma_5) n \, . \tag{33}$$

The couplings of charged leptons to the Z–boson remain unaltered.

We turn to the derivation of the leptonic couplings to the SM scalars. Again we only need to work out the couplings involving neutrinos. The SM Higgs boson appearing in equation (13) can be parameterized as $\Phi = (\phi^+ \; \frac{1}{\sqrt{2}}(v + H + i\chi))^T$ in terms of the charged scalar ϕ, the pseudoscalar χ, the physical Higgs H and the vacuum expectation value v. For the interaction of the doublet Φ with the leptonic fields we have:

$$\mathcal{L}^{\text{Yukawa}} = -\left(\bar{L}\, g_l\, \Phi\, l_{fR} \;+\; \bar{L}\, g_\nu^{\text{Dirac}}\, i\tau_2 \Phi^\star \omega_R\right) \;+\; \text{h.c.}\,, \tag{34}$$

with the leptonic doublet $L = (\sigma_L,\; l_{fL})$ and Yukawa couplings g_f. It should be kept in mind that g_ν^{Dirac} is proportional only to the Dirac mass term of the neutrinos but not to the neutrino mass matrix as a whole.
For the leptonic interaction of the charged scalar ϕ we infer from equation (34):

$$\begin{aligned}
\mathcal{L}^\phi &= -\bar{\sigma}_L\, g_l\, l_{fR}\, \phi^+ \;+\; \bar{l}_{fL}\, g_\nu^{\text{Dirac}}\, \omega_R\, \phi^- \;+\text{h.c.} \\
&= \frac{e}{\sqrt{2} s_{\text{w}} m_W} \left(\bar{l}_L\, U_l\, A\, U^\star\, m_n\, U^\dagger\, U\, n_R\, \phi^- - \bar{n}_L\, U^T\, A^T\, U_l^\dagger\, m_l\, l_R\, \phi^+ \;+\; \text{h.c.}\right) \\
&= \frac{e}{\sqrt{2} s_{\text{w}} m_W} \left(\bar{l}_L\, B\, m_n\, n_R\, \phi^- - \bar{n}_L\, B^\dagger\, m_l\, l_R\, \phi^+ \;+\; \text{h.c.}\right) \\
&= \frac{e}{\sqrt{2} s_{\text{w}} m_W} \left[\bar{n}\, B^\dagger\, (m_n\, L - m_l\, R)\, l\, \phi^+ \;+\; \bar{l}\, B\, (m_n\, R - m_l\, L)\, n\, \phi^-\right] .
\end{aligned} \tag{35}$$

By the same principle we write the interaction Lagrangian of the neutral scalar and pseudoscalar H and χ in a compact form:

$$\begin{aligned}
\mathcal{L}^H &= -\frac{1}{\sqrt{2}} \left(\bar{\sigma}_L\, g_\nu^{\text{Dirac}}\, \omega_R \;+\; \text{h.c.}\right)\; H \\
&= -\frac{e}{2 s_{\text{w}} m_W} \left(\bar{n}_L\, U^T\, A^T\, A\, U^\star\, m_n\, U^\dagger\, U\, n_R \;+\; \text{h.c.}\right)\; H \\
&= -\frac{e}{2 s_{\text{w}} m_W} \left(\bar{n}\, C\, m_n\, R\, n \;+\; \bar{n}\, m_n\, C\, L\, n\right)\; H \\
&= -\frac{e}{4 s_{\text{w}} m_W} \bar{n} \left[i\, \text{Im}\,(C m_n - m_n C) \gamma_5 \;+\; \text{Re}\,(C m_n + m_n C)\right] n\; H
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\mathcal{L}^\chi &= \frac{i}{\sqrt{2}} \left(\bar{\sigma}_L\, g_\nu^{\text{Dirac}}\, \omega_R\right) \chi + \;\text{h.c.} \\
&= \frac{ie}{2 s_{\text{w}} m_W} \left(\bar{n}\, C\, m_n\, R\, n \;-\; \bar{n}\, m_n\, C\, L\, n\right)\; \chi \\
&= \frac{ie}{4 s_{\text{w}} m_W} \bar{n} \left[i\, \text{Im}\,(C m_n - m_n C) \;+\; \text{Re}\,(C m_n + m_n C)\, \gamma_5\right] n\; \chi\;,
\end{aligned} \tag{37}$$

where we have made use of the symmetry properties (30) of C and relation (8) for scalar products of Majorana fields.
Except for a deviation in the definition of the weak mixing angle our results for the interaction Lagrangian of the modified lepton sector agree with reference [5].

2.5 Transition Amplitudes with Majorana Neutrinos

In the last section we have shown how the introduction of right–handed neutrinos and a right–handed Majorana mass term induced certain modifications of the neutrino interactions with gauge bosons and scalars.
In principle the calculation of transition amplitudes with self–conjugate neutrinos does not differ from calculations with Dirac neutrinos. The only difference lies in the combinatorial factors that arise when loop diagrams with more than one neutral current interaction connected by internal neutrino lines are encountered.
We demonstrate this explicitly by considering the Majorana neutrino contribution to the Z self–energy at one–loop level.
For the T–matrix element $\mathcal{T}$ corresponding to this amplitude we have:

$$\begin{aligned}(2\pi)^4\delta(p-p')i\mathcal{T} &= \int \mathrm{d}x\mathrm{d}y\epsilon^{*\nu}(p',\lambda')\epsilon^{\mu}(p,\lambda)e^{ip'y}e^{-ipx}i\Gamma^{ij}_{\nu,\alpha\beta}i\Gamma^{ij}_{\mu,\gamma\delta}\\ &\quad\times\langle 0|T\{\bar{n}_{i\alpha}(y)n_{j\beta}(y)\bar{n}_{k\gamma}(x)n_{l\delta}(x)\}|0\rangle\ , \end{aligned} \tag{38}$$

where we have defined

$$\Gamma^{ij}_{\nu,\alpha\beta} = \frac{g}{4c_{\mathrm{w}}}\gamma_{\nu,\alpha\beta}[i\mathrm{Im}\,C_{ij} - \gamma_5\mathrm{Re}\,C_{ij}]\ ,$$

where μ is a Lorentz index, α,β are spinor indices, and $T\{\ldots\}$ denotes the time ordering operator.
We want to consider the time–ordered product of the n_i. If the n_i were Dirac particles, application of Wick's theorem would yield

$$\langle 0|T\{\bar{n}_{i\alpha}(y)n_{j\beta}(y)\bar{n}_{k\gamma}(x)n_{l\delta}(x)\}|0\rangle = -\langle 0|n_{j\beta}(y)\bar{n}_{k\gamma}(x)|0\rangle\langle 0|n_{l\delta}(x)\bar{n}_{i\alpha}(y)|0\rangle\ . \tag{39}$$

Using the Majorana property:

$$n_{i\,\alpha} = C_{\alpha\beta}\bar{n}_{i\beta}\ ,$$

we can also contract n_i with n_j and $\bar{n}_i$ with $\bar{n}_j$. We conclude that equation (39) is not complete. Instead it should read:

$$\begin{aligned}&\langle 0|T\{\bar{n}_{i\alpha}(y)n_{j\beta}(y)\bar{n}_{k\gamma}(x)n_{l\delta}(x)\}|0\rangle =\\ &\quad -\langle 0|n_{j\beta}(y)\bar{n}_{k\gamma}(x)|0\rangle\ \langle 0|n_{l\delta}(x)\bar{n}_{i\alpha}(y)|0\rangle\\ &\quad -\langle 0|C^{\dagger}_{\alpha\epsilon}n_{i\epsilon}(y)\bar{n}_{k\gamma}(x)|0\rangle\ \langle 0|n_{j\beta}(y)C_{\delta\chi}\bar{n}_{l\chi}(x)|0\rangle\ .\end{aligned} \tag{40}$$

Insertion of the propagator

$$\langle 0|n_{j\beta}(y)\bar{n}_{k\gamma}(x)|0\rangle = \delta_{jk}\,iS_j(y-x)_{\beta\gamma}$$

simplifies equation (38) to

$$\begin{aligned}&-\int \mathrm{d}x\mathrm{d}y\epsilon^{*\nu}(p',\lambda')\epsilon^{\mu}(p,\lambda)e^{ip'y}e^{-ipx}\\ &\times\mathrm{Sp}\left\{i\Gamma^{ik}_{\nu}iS_k(y-x)i\Gamma^{ki}_{\mu}iS_i(x-y) + CiS^T_j(y-x)i\Gamma^{ij\,T}_{\nu}C^{\dagger}iS_i(y-x)i\Gamma^{ij}_{\mu}\right\}\\ =\ &-\int \mathrm{d}x\mathrm{d}y\epsilon^{*\nu}(p',\lambda')\epsilon^{\mu}(p,\lambda)e^{ip'y}e^{-ipx}\end{aligned}$$

$$
\begin{aligned}
& \times \mathrm{Sp}\left\{ i\Gamma_\nu^{ij} iS_j(y-x) i\Gamma_\mu^{ji} iS_i(x-y) + i\Gamma_\nu^{ij\,c} iS_j(y-x) i\Gamma_\mu^{ji} iS_i(x-y) \right\} \\
= & -\int \mathrm{d}x\mathrm{d}y \epsilon^{*\nu}(p',\lambda')\epsilon^\mu(p,\lambda) e^{ip'y} e^{-ipx} \\
& \times \mathrm{Sp}\left\{ i\left(\Gamma_\nu^{ij} + \Gamma_\nu^{ij\,c}\right) iS_j(y-x) i\Gamma_\mu^{ji} iS_i(x-y) \right\} \ , \qquad (41)
\end{aligned}
$$

where we have used the definition $i\Gamma_\nu^{ij\,c} = C i \Gamma_\nu^{ij\,T} C^\dagger$ in the first equation of relations (41). Evaluating the sum of the coupling matrices Γ and using the hermiticity of C we obtain:

$$
\begin{aligned}
\left(\Gamma_\nu^{ij} + \Gamma_\nu^{ij\,c}\right) &= \frac{g}{4c_{\mathrm{w}}} \gamma_\nu \left[i\mathrm{Im}\,(C - C^T)_{ij} - \gamma_5 \mathrm{Re}\,(C + C^T)_{ij} \right] \\
&= 2\,\frac{g}{4c_{\mathrm{w}}} \gamma_\nu \left[i\mathrm{Im}\, C_{ij} - \gamma_5 \mathrm{Re}\, C_{ij} \right] \\
&= 2\Gamma_\nu^{ij} \ . \qquad (42)
\end{aligned}
$$

Up to a factor 2 this agrees with the usual result we would have expected for Dirac fermions:

$$
i\mathcal{T} = -2 \int \frac{\mathrm{d}k}{(2\pi)^4} \mathrm{Sp}\left\{ \Gamma_\nu^{ij} \frac{\not{k} + m_j}{k^2 - m_j^2} \Gamma_\mu^{ji} \frac{\not{k} + \not{p} + m_i}{(k+p)^2 - m_i^2} \right\} \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p',\lambda') \ . \qquad (43)
$$

In general, this combinatorial factor can be easily obtained by inspecting the diagram under consideration and counting the non–standard contractions.
To summarize in a brief notation, we find that except for the usual contraction

$$
\underbrace{n_a \bar{n}_b}
$$

the Majorana property of the neutrinos enables us to also perform contractions of the type

$$
\underbrace{\bar{n}_a \bar{n}_b} \quad \text{and} \quad \underbrace{n_a n_b} \ .
$$

For the time–ordered product we therefore obtain:

$$
\begin{aligned}
T\{N(\bar{n}_i(x) n_j(x)) N(\bar{n}_k(y) n_l(y))\} = & -\delta_{il} \underbrace{n_l(y)\bar{n}_i(x)} \delta_{jk} \underbrace{n_j(x)\bar{n}_k(y)} \\
& -\delta_{ik} \underbrace{\bar{n}_i(x)\bar{n}_k(y)} \delta_{jl} \underbrace{n_j(x) n_l(y)} \ . \qquad (44)
\end{aligned}
$$

In equation (42) we saw that the two terms on the right hand side of equation (44) are equal and we thus obtain the usual result for Dirac fermions except for an extra factor of 2. This factor accounts for the different possible ways of contracting fields at x with fields at y analogous to the combinatorial factors appearing in theories with self–coupled neutral scalars.

2.6 Properties of the Matrices B and C

There are some interesting properties of the matrices B and C defined through equations (26) and (29) that we want to collect and prove at this point.
In order to guarantee hermiticity of the interaction Lagrangian, we expect the matrices B and C to have certain symmetry properties. These are readily obtained from their

definitions.

$$
\begin{aligned}
BB^\dagger &= U_l A U^\star U^T A^T U_l^\dagger = \mathbf{1}_{n_G \times n_G}\,, \\
CC &= U^T A^T A U^\star U^T A^T A U^\star = U^T A^T A A^T A U^\star = U^T A^T A U^\star = C\,, \\
BC &= U_l A U^\star U^T A^T A U^\star = U_l A A^T A U^\star = U_l A U^\star = B\,, \\
B^\dagger B &= U^T A^T U_l^\dagger U_l A U^\star = U^T A^T A U^\star = C\,,
\end{aligned}
\tag{45}
$$

where we have used the fact that $A^T A$ is a projection operator with the same dimensionality as C and AA^T is the $n_G \times n_G$ identity matrix. Obviously the mixing matrices are not independent, but instead C can easily be obtained once we know B. Physically this means that neutral and charged current mixing is not independent, but if the elements of B are fixed also C is uniquely defined. The opposite is of course not true, as B contains contributions from U_l, occurring in the diagonalization of the charged–lepton mass matrix which are not present in C.

Apart from these quasi–unitarity and hermiticity relations, the structure of the mass matrix gives rise to some non–trivial identities involving the mixing matrices and the neutrino spectrum:

$$
\begin{aligned}
C m_n C^T &= U^T A^T A U^\star m_n U^\dagger A^T A U = U^T A^T A m_\nu A^T A U = 0\,, \\
B m_n C^T &= U_l A U^\star m_n U^\dagger A^T A U = U_l A m_\nu A^T A U = 0\,, \\
B m_n B^T &= U_l A U^\star m_n U^\dagger A^T U_l^T = U_l A m_\nu A^T U_l^T = 0\,,
\end{aligned}
\tag{46}
$$

with m_n and m_ν denoting the neutrino mass matrix in the mass and flavor basis, respectively.
Note that the second set of equations is a characteristic feature of the 1–2 scenario. Although we are primarily interested in this framework most of our results can easily be generalized to the 1–2–3 case.

3 The $W-$ and $Z-$Boson Decay Amplitudes

3.1 Generic Formulae for the Decay of a Gauge Boson into Two Fermions

In this section we give a very general result for the decay amplitude of a spin–one boson into a fermion pair up to the one–loop order.

Let us begin by writing the interaction Lagrangian of scalars and gauge bosons coupled to a fermionic current in our standard notation which we shall use throughout the following chapters:

$$\begin{aligned}\mathcal{L}^V &= V^\mu \, \bar{f}_i \, \gamma_\mu \, (g^+_{Vij} R + g^-_{Vij} L) \, f_j \, , \\ \mathcal{L}^S &= S \, \bar{f}_i \, (g^+_{Sij} R + g^-_{Sij} L) \, f_j \, , \end{aligned} \tag{1}$$

where the indices i, j do not necessarily refer to fermions of equal flavor or family quantum number. The couplings $g^\sigma_{\ldots}$ of lepton pairs to scalars and vector bosons are thus defined through equations (27), (33) and (35) through (37) of section 2.4. Because it is often much easier to read, we will interchangeably use $g^{+/-}_{\ldots}$ for $g^{R/L}_{\ldots}$. An analogous, self–explanatory notation is used for the couplings between gauge bosons and scalars where we adopted the convention of reference [6].
The tree level decay amplitude for a gauge boson can practically be read off from the interaction Lagrangian:

$$i\mathcal{M}_{Born} = i \, \bar{u}(m_{f_i}, k_i) \, \not{\epsilon} \, (g^+_{Vij} R + g^-_{Vij} L) \, v(m_{f_j}, k_j) \, . \tag{2}$$

For the investigation of gauge independence of our decay amplitudes we need to consider the full gauge boson propagators, which renders the one–loop virtual corrections to equation (2) fairly complicated. Because we investigate two different decay processes, but quoting explicit analytic results for both of them would imply a great amount of redundancy, we decided to give the higher–order corrections in the most generic representation possible. In this way, we can concentrate on the six generic diagrams shown in figure 1. We have decomposed each diagram into standard matrix elements and form factors that are expressed in terms of purely generic masses and couplings. Introducing suitable abbreviations for the occurring so–called Passarino–Veltman tensor coefficient functions [7], we were even able to quote this general result in arbitrary R_ξ gauge, which to our knowledge is not found in the literature.
In order to obtain the physical amplitude for a specific decay, we then insert all virtual field configurations contributing to the transition under consideration. Because we explicitly gave all relevant couplings in section 2, we shall not cite total transition amplitudes for the processes we want to investigate. With the given building blocks at hand the calculation of the transition amplitude is merely a matter of insertion and a suitable computer code for numerical evaluation.

We decompose the jth amplitude (from left to right) of figure 1 in the following way by projecting onto final states of definite polarization:

$$\delta\mathcal{M}^\sigma_{virt}(j) = \sum_{i=1,2} \mathcal{M}^\sigma_i \, F^\sigma_i(j) \, , \tag{3}$$

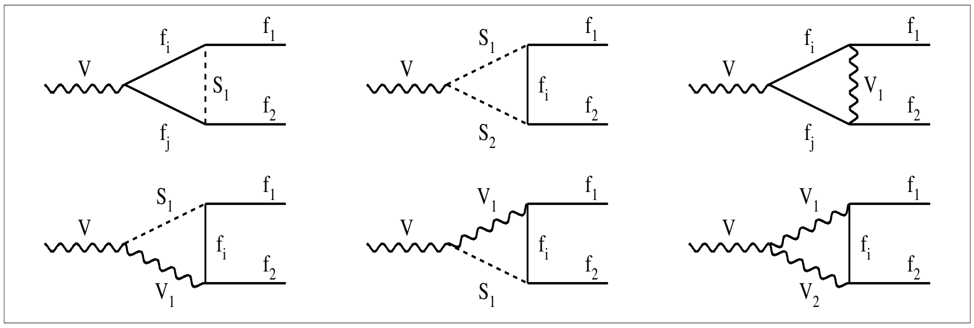

Figure 1: Generic one–loop contributions to fermionic decay of a gauge boson. V, S and F represent gauge bosons, scalars and fermions, respectively.

with $\sigma = L, R$ as introduced in section 2.1 and matrix elements $\mathcal{M}_i^\sigma$ defined as:

$$\begin{aligned}\mathcal{M}_1^\sigma &= \bar{u}(m_1, k_1)\, \sigma\, v(m_2, k_2)\, (\varepsilon k_1)\,, \\ \mathcal{M}_2^\sigma &= \bar{u}(m_1, k_1)\, \not{\varepsilon}\, \sigma\, v(m_2, k_2)\,.\end{aligned}$$

Although we would naively expect $3 \times 2 \times 2 = 12$ independent matrix elements accounting for the three possible polarizations of the gauge boson and the two chiralities for each fermion, four are actually sufficient due to conservation of angular momentum.

The form factors $F_i^\sigma(j)$ in R_ξ gauge are obtained only after tedious computations which we carried out using the *Mathematica*–based computer algebra packages FeynArts [8] and FeynCalc [9]. We checked our formulae by application to the decay of a W–boson into a quark pair and found excellent numerical agreement with other authors [10]. Results for individual form factors were even verified in explicit calculation by hand. To render the expressions more compact, we introduced a shorthand notation for the aforementioned Passarino–Veltman tensor coefficient functions, which occur after representing the tensorial one–loop integrals in a natural basis of external four–momenta and metric tensors. The coefficient functions and some special cases thereof are introduced in appendix C before the decomposition algorithm in part C.3 and our abbreviated notation in section C.4. Due to their excessive complexity, the expressions for the $F_i^\sigma(j)$ were also relegated to the appendix (C.5).

3.2 The Decay $W \to l_1\nu_1$

In this section we will present the decay amplitude of a negatively charged W–boson into a lepton pair l_1, ν_1.
We start by quoting the tree level decay amplitude

$$\mathcal{M}_{Born} = \frac{e}{\sqrt{2}s_{\mathrm{w}}} B_{l_1\nu_1} \mathcal{M}_2^- = g^-_{Wl_1\nu_1} \mathcal{M}_2^- \,, \tag{4}$$

and observe that, in contrast to the SM case, the tree level expression contains a non–trivial CKM–type mixing matrix.
Before we discuss the consequences of this fact, we turn to the virtual corrections. To illustrate the 'assembly' of the virtual triangle corrections, we describe all field configurations contributing to each of the generic diagrams defined in the previous section.
There are three possible insertions into the first diagram: $f_i, f_j, S_i \to e, \nu_i, H \,\mathrm{or}\, \chi$ and $f_i, f_j, S_i \to \nu_1, l_1, \phi$. Because this last insertion displays a genuinely new feature of our model, namely the possibility of fermion number violation, this insertion is illustrated in figure 2. For the second generic diagram, we can set $S_1, S_2, f_i \to \phi, H \,\mathrm{or}\, \chi, \nu_i$ and

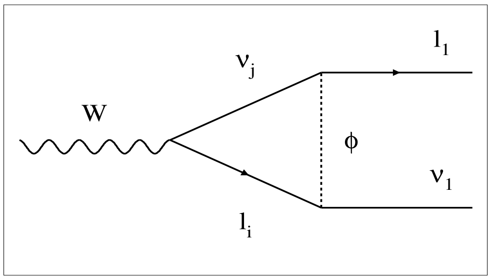

Figure 2: Fermion number violating contribution to $W \to l_1\nu_1$.

$S_1, S_2, f_i \to H \,\mathrm{or}\, \chi, \phi, e$.
The two possible substitutions in the third diagram are analogous to those of first with Z and W instead of χ and ϕ.
The possible virtual field configurations of diagram four are either $V_1, S_1, f_i \to Z, \phi, \nu_i$ or $V_1, S_1, f_i \to W, H, l_1$.
In diagram five we can replace $V_1, S_1, f_i \to \gamma \,\mathrm{or}\, Z, \phi, l_1$ or $V_1, S_1, f_i \to W, H, \nu_i$.
The possible insertions into the last generic diagram are analogous to those of diagram five with H and ϕ replaced by Z and W.
In total the virtual corrections consist of seventeen vertex correction diagrams, which can easily be obtained from the $F_i^\sigma(j)$ of section 3.1 by inserting the proper couplings and masses as described above. We find two fermion (and lepton) number violating contributions, the diagram depicted in figure 2 and the same diagram with a W instead of the charged scalar.

Let us now turn to the counterterm amplitude. Given the interaction Lagrangian, the W decay one–loop counterterm amplitude is readily calculated as

$$\delta\mathcal{M}_{ct} = \mathcal{M}_2^- \left(\delta g^-_{Wl_1\nu_1} + \frac{1}{2} g^-_{Wl_1\nu_1} \delta Z_W + \frac{1}{2} g^-_{Wl_1\nu_k} \delta Z^L_{\nu_k\nu_1} + \frac{1}{2} \delta Z^{L\dagger}_{l_1 l_k} g^-_{Wl_k\nu_1} \right) .$$

The counterterms in R_ξ–gauge according to reference [6] can be calculated from the self–energies given in appendix D.

In our rather formal notation the mixing matrix renormalization is hidden in the $\delta g^-_{Wl_1\nu_1}$ term:

$$\delta g^-_{Wl_1\nu_1} = \delta \left(\frac{e}{\sqrt{2} s_{\mathrm{w}}} B_{l_1\nu_1} \right) . \tag{5}$$

Following mostly the notation of reference [6] and writing

$$e^0 = e + \delta e \quad \text{and} \quad s^0_{\mathrm{w}} = s_{\mathrm{w}} + \delta s_{\mathrm{w}} \quad \text{and} \quad B^0_{l\nu} = B_{l\nu} + \delta B_{l\nu} ,$$

we conclude

$$\delta g^-_{Wl_1\nu_1} = \frac{e}{\sqrt{2} s_{\mathrm{w}}} B_{l_1\nu_1} \left(\frac{\delta e}{e} - \frac{\delta s_{\mathrm{w}}}{s_{\mathrm{w}}} + \frac{\delta B_{l_1\nu_1}}{B_{l_1\nu_1}} \right) .$$

Clearly the $\delta B_{l\nu}$ term is a new feature of our scenario and its definition or rather the defining renormalization condition cannot be found in reference [6] where a standard leptonic sector is assumed. At this point we have introduced a counterterm $\delta B_{l\nu}$ for formal reasons: The mixing matrix $B_{l\nu}$ is a collection of parameters appearing in the Lagrangian of our model in the mass eigenstate basis. The parameters contained in $B_{l\nu}$ can in principle be measured and should therefore undergo renormalization just as every other physical parameter of our model. For now we have not specified any scheme defining it, nor have we adopted any prescription existing in the literature and $\delta B_{l\nu}$ is merely a symbol here. The definition and structure of $\delta B_{l\nu}$ will be the subject of a detailed discussion in the next section.

Before we turn to the Z–boson decay, some remarks about the virtual corrections $\delta\mathcal{M}_{virt}$ are in order. We define

$$F_i^\sigma = \sum_{j=1,6} \sum_{\text{insertions}} F_i^\sigma(j) ,$$

where the sum over insertions symbolizes all possible virtual field configurations for a given generic amplitude as described above, such that

$$\delta\mathcal{M}^\sigma_{virt} = \sum_i \mathcal{M}_i^\sigma F_i^\sigma .$$

With $(\ldots)_{UV}$ denoting the ultra–violet–divergent part of the expression in parenthesis we have verified in an explicit calculation that

$$\begin{aligned} \left(F_1^+\right)_{UV} &= \frac{\partial}{\partial\xi_W} F_1^+ = \frac{\partial}{\partial\xi_Z} F_1^+ = 0 , \\ \left(F_1^-\right)_{UV} &= \frac{\partial}{\partial\xi_W} F_1^- = \frac{\partial}{\partial\xi_Z} F_1^- = 0 , \\ \left(F_2^+\right)_{UV} &= \frac{\partial}{\partial\xi_W} F_2^+ = \frac{\partial}{\partial\xi_Z} F_2^+ = 0 . \end{aligned} \tag{6}$$

This holds for the obvious reason that the formfactors of equations (6) have no counterterms which could cancel the ϵ or ξ dependence. As the total, renormalized amplitude has to be independent of the gauge parameters and finite, the statements of equations (6) are necessarily true in a renormalizable model.
Relations equivalent to equations (6) for F_2^- will turn out to be of great importance for the determination of a suitable counterterm for the mixing matrix and shall be investigated in the next section where we address the question of mixing renormalization.

3.3 The Decay $Z \to \nu_1\nu_2$

Given the notation of equation (1), the tree level decay amplitude of a Z–boson into a pair of neutrinos ν_1, ν_2 reads

$$\mathcal{M}_{Born} = g^-_{Z\nu_1\nu_2}\mathcal{M}_2^- + g^+_{Z\nu_1\nu_2}\mathcal{M}_2^+ . \tag{7}$$

Except for the fact that in the present case there is an additional $v+a$ contribution to the Born decay amplitude, there is no difference to the case of W decay in the previous section.
In order to calculate the virtual corrections, we identify all possible field configurations in the diagrams of figure 1 that contribute to the decay process.
There are four possible insertions into the first diagram: $f_i, f_j, S_1 \to \nu_i, \nu_j, H/\chi$ or l_i, l_i, ϕ with charge flow in both directions.
In the next diagram we can substitute $S_1, S_2, f_i \to H, \chi, \nu_i$ or $S_1, S_2, f_i \to \phi, \phi, l_i$ as well as the first substitution with H and χ interchanged and the second with opposite charge flow.
The insertions into the third diagram are the same as those into the first with Z instead of H and W instead of ϕ.
There are three possible virtual field configurations in each case contributing from diagrams four and five, namely $S_1, V_1, f_i \to H, Z, \nu_i$ or ϕ, W, l_i and its charge conjugate.
For the last diagram the two possible insertions are just $V_1, V_2, f_i \to W, W, l_i$ with charge flow in both directions.
Altogether there are nineteen possible field insertions into the generic diagrams of figure 1 constituting the virtual corrections to equation (7).

The counterterm amplitude for this process reads

$$\delta\mathcal{M}_{ct} = \sum_{\sigma=R/L} \mathcal{M}_2^\sigma \left(\delta g^\sigma_{Z\nu_1\nu_2} + \frac{1}{2} g^\sigma_{Z\nu_1\nu_2}\delta Z_Z + \frac{1}{2} g^\sigma_{Z\nu_1\nu_k}\delta Z^\sigma_{\nu_k\nu_2} + \frac{1}{2}\delta Z^{\sigma\dagger}_{\nu_1\nu_k} g^\sigma_{Z\nu_k\nu_2} \right) . \tag{8}$$

As for $\delta g^\sigma_{Wl\nu}$ before, $\delta g^\sigma_{Z\nu_1\nu_2}$ not only contains the counterterms of the electromagnetic coupling and the weak mixing angle, but also a contribution from δC stemming from the non–trivial neutral current mixing of the neutrinos:

$$\delta g^{-(+)}_{Z\nu_1\nu_2} = g^{-(+)}_{Z\nu_1\nu_2} \left(\frac{\delta e}{e} - \frac{\delta s_\mathrm{w}}{s_\mathrm{w}}(1 - \frac{s_\mathrm{w}^2}{c_\mathrm{w}^2}) + \frac{\delta C^{(\star)}_{\nu_1\nu_2}}{C^{(\star)}_{\nu_1\nu_2}} \right) . \tag{9}$$

Like δB in the previous section, δC is just a symbol for a counterterm we have not yet determined. A proper definition of those counterterms with the desired properties will be given in the following section.

4 Renormalization of Fermionic Mixing Matrices

In a theory where infinities arise in the ultra–violet or large–momentum regime, we need to introduce some cutoff or regularization so that we can identify and suitably subtract them. In practical calculations this is most conveniently done in dimensional regularization. Details on renormalization theory in general and dimensional regularization can be found in reference [11].

Once we know how to regularize the occurring infinities, we have to specify how to subtract the infinities in physical amplitudes. We define counterterms, containing the same divergences that plague our Feynman graphs. The finite part of the counterterms however, are a priori undetermined, and the objective of the renormalization scheme is to resolve the ambiguity related to the definition of the counterterm with its arbitrary finite part.

4.1 On–Shell Renormalization Scheme

For our calculations we have decided to use the so–called on–shell scheme as described e.g. in reference [6]. The SM can be parameterized in terms of different maximal sets of independent parameters. The input parameters of the on–shell scheme are the electromagnetic coupling e, the masses of the fermion fields m_{f_i} and the masses of the bosons m_W, m_Z and m_H.

The electromagnetic coupling is renormalized by the requirement that e be the effective one–loop coupling of a photon to an electron in the limit of vanishing momentum transfer (Thomson limit) and on–shell external fields. The renormalization of the masses is achieved by imposing that the physical mass be equal to the real part of the pole of the renormalized one–loop propagator. The non–diagonal entries of the fermionic field renormalization matrices are obtained by imposing diagonality on the renormalized one–particle–irreducible two–point functions if the external lines are on the mass shell. The diagonal elements are fixed such that the renormalized fields are properly normalized, i.e. the residues of the renormalized propagators are equal to one.

If the wave function renormalization (WFR) matrices have non–diagonal entries, we have tacitly assumed that there is mixing between fermionic fields of different flavor. Consequently there are mixing parameters in the Lagrangian which we have not mentioned in our brief description of the on–shell renormalization scheme, and which, like all other parameters of the Lagrangian have to be included in the renormalization procedure.

The aim of this chapter is to present a suitable renormalization prescription for the leptonic mixing matrices B and C introduced in section 2.4.

The first procedure [12] suggested in the literature to renormalize the SM CKM matrix in the on–shell scheme has only recently been discovered to violate gauge invariance [13]. We found that a generalization of reference [12] to the case of leptonic mixing matrices involving Majorana neutrinos [14] suffers from the same deficit. Because the renormalization of our matrices B and C and the quark mixing matrix are closely related we shall present in this section not only our solution for the problem of leptonic mix-

ing but also quote some 'historical' developments in the field that led the way to our renormalization technique.
Our main criticism concerning the procedures of references [12, 13, 15] is that they instead of deriving them from renormalization conditions postulate ad hoc counterterms for the CKM matrix elements. Consequently it is a priori not clear if the counterterms thus obtained are consistent with the symmetries of the theory, a problem in fact responsible for the failure of the method suggested in reference [12]. In particular none of the aforementioned procedures could be generalized to the case of leptonic mixing in the context of our model — a fact which we attribute to the same conceptual weakness.

The remaining sections of this chapter are organized as follows: We start by reviewing and criticizing the existing prescriptions for the renormalization of the SM CKM matrix in the on–shell scheme. In the next step we propose a generalization of one of the suggested methods in such a way that a counterterm can be *derived* from a renormalization condition in compliance with the theory's symmetries instead of being postulated ad hoc. We continue by showing that a naive translation of the procedures of references [12, 13, 15] to the leptonic case does not yield satisfactory results. The last section is devoted to our solution to the renormalization problem based on our aforementioned renormalization condition.

4.2 Renormalization of SM Quark Mixing

The first prescription for the renormalization of the CKM matrix in the on–shell scheme was given by Denner and Sack in reference [12]. The authors observed that in the one–loop W decay amplitude without counterterm for the mixing matrix, the finiteness of the total amplitude was spoiled by the antihermitian part of the WFR of the final state quarks, i.e.

$$\Big(F_2^- + g^-_{Wdu}(\frac{\delta e}{e} - \frac{\delta s_{\mathrm{w}}}{s_{\mathrm{w}}} + \frac{1}{2}\delta Z_W) \\ + \frac{1}{2} g^-_{Wdu_k}\delta Z^L_{u_k u} + \frac{1}{2}\delta Z^{L\dagger}_{dd_k} g^-_{Wd_k u}\Big)_{UV} \neq 0 \tag{1}$$

but

$$\Big(F_2^- + g^-_{Wdu}(\frac{\delta e}{e} - \frac{\delta s_{\mathrm{w}}}{s_{\mathrm{w}}} + \frac{1}{2}\delta Z_W) \\ + \frac{1}{4} g^-_{Wdu_k}(\delta Z^L_{u_k u} + \delta Z^{L\dagger}_{u_k u}) + \frac{1}{4}(\delta Z^{L\dagger}_{dd_k} + \delta Z^L_{dd_k}) g^-_{Wd_k u}\Big)_{UV} = 0\,, \tag{2}$$

where we used the same notation for the counterterms as in section 3.2 with the quark mixing matrix V instead of the leptonic mixing matrix B.
It seemed therefore natural to define the counterterm for the CKM matrix δV as

$$\delta V^{DS}_{du} = -\frac{1}{4}(\delta Z^{L\dagger}_{dd_k} - \delta Z^L_{dd_k})V_{d_k u} - V_{du_k}\frac{1}{4}(\delta Z^L_{u_k u} - \delta Z^{L\dagger}_{u_k u})\,, \tag{3}$$

leading according to equation (2) to a finite total one–loop transition amplitude. We note that only the off–diagonal elements of δZ^L enter equation (3) and that by construction the bare and the renormalized mixing matrix are unitary.

As shown by explicit calculation in reference [13], the expression in parentheses in equation (1) does not depend on the gauge parameters:

$$\begin{aligned}\frac{\partial}{\partial\xi_{W(Z)}}\Big(&F_2^- + g^-_{Wdu}\big(\frac{\delta e}{e} - \frac{\delta s_{\rm w}}{s_{\rm w}} + \frac{1}{2}\delta Z_W\big)\\ &+\frac{1}{2}g^-_{Wdu_k}\delta Z^L_{u_k u} + \frac{1}{2}\delta Z^{L\dagger}_{dd_k} g^-_{Wd_k u}\Big) = 0\,.\end{aligned} \tag{4}$$

The dependence of the decay amplitude on the photon gauge parameter does not affect our discussion and will therefore not be studied. Because the total one–loop decay amplitude must be independent of the gauge parameter, this in turn implies that a physically sensible counterterm for the CKM mixing matrix must not depend on the gauge parameters either, i.e.

$$\frac{\partial\delta V}{\partial\xi_{W(Z)}} = 0\,. \tag{5}$$

However, the counterterm δV^{DS} of reference [12] as defined in equation (3) does *not* satisfy this constraint:

$$\frac{\partial}{\partial\xi_W}\left(-\frac{1}{4}(\delta Z^{L\dagger}_{dd_k} - \delta Z^L_{dd_k})V_{d_k u} - V_{du_k}\frac{1}{4}(\delta Z^L_{u_k u} - \delta Z^{L\dagger}_{u_k u})\right) \neq 0\,. \tag{6}$$

Clearly, we have to reject this renormalization prescription as unphysical because transition amplitudes related to physical observables must never depend on the specific values of the gauge parameters.

This had been observed by Gambino, Grassi and Madricardo (GGM) in reference [13] and led them to a variation of the scheme of Denner and Sack [12] described above. Instead of constructing the desired counterterm δV of the off–diagonal parts of the usual WFR δZ as in equation (3), a pseudo wave function renormalization $\delta\widetilde{Z}$ is defined which, if substituted in equation (3), renders δV independent of the gauge parameters.

The off–diagonal entries δZ_{ij} of the WFR matrices of the quark fields are defined through the conditions

$$\widetilde{\rm Re}\ \Sigma^{\rm ren}_{ij}(\not q)\, u_j(q)\Big|_{q^2=m_j^2} = 0 \quad \text{and} \quad \widetilde{\rm Re}\ \bar u_i(q)\, \Sigma^{\rm ren}_{ij}(\not q)\Big|_{q^2=m_i^2} = 0 \tag{7}$$

imposed on the renormalized one–particle–irreducible two–point functions $\Sigma^{\rm ren}$. The $\widetilde{\rm Re}$ symbolizes that only the dispersive parts of the loop integrals are considered but imaginary parts of couplings should not be suppressed, so that the result need not necessarily be real. The renormalized two–point functions depend on the unrenormalized two–point functions as well as the WFR and mass counterterms.

$$\begin{aligned}\Sigma^{\rm ren}_{ij}(\not q) \;=\;& \left(\Sigma^L_{ij} + \frac{1}{2}(\delta Z^L_{ij} + \delta Z^{L\dagger}_{ij})\right)\not q L + \left(\Sigma^R_{ij} + \frac{1}{2}(\delta Z^R_{ij} + \delta Z^{R\dagger}_{ij})\right)\not q R\\ &+\left(\Sigma^D_{ij} - \frac{1}{2}(m_i\delta Z^L_{ij} + \delta Z^{R\dagger}_{ij}m_j) - \delta_{ij}\delta m_i\right)L\\ &+\left(\Sigma^{D\dagger}_{ij} - \frac{1}{2}(m_i\delta Z^R_{ij} + \delta Z^{L\dagger}_{ij}m_j) - \delta_{ij}\delta m_i\right)R\end{aligned} \tag{8}$$

is the most general form in compliance with hermiticity. It should be observed that our definition of Σ^D_{ij} differs slightly from that of references [13] and [6], where Σ^D_{ij} and $\Sigma^{L/R}_{ij}$ have equal dimensions.

In order to obtain $\delta\widetilde{Z}$, GGM imposed the following conditions on the renormalized two–point functions of equation (8) with the untilded quantities substituted by tilded ones:

$$\widetilde{\Sigma}^{\mathrm{ren}}_{ij}(\not{q})\Big|_{\not{q}=0} = 0 \quad \text{and} \quad \frac{\partial}{\partial\not{q}}\widetilde{\Sigma}^{\mathrm{ren}}_{ij}(\not{q})L\Big|_{\not{q}=0} = 0\,. \tag{9}$$

Only the tilded analogues of δZ^L are derived from these relations, and as in equation (3) only the left–handed pseudo WFR is employed in the mixing renormalization while the right–handed WFR $\delta\widetilde{Z}^R$ is set to zero.

At next–to–leading order this seems natural as mixing is a genuinely weak effect affecting just the left–handed fermion fields and the off–diagonal right–handed WFR are finite anyway.

A straightforward calculation yields

$$\begin{aligned}
\delta\widetilde{Z}^{L,A}_{ij} &= \frac{1}{m_i^2-m_j^2}\left((m_i^2+m_j^2)\Sigma^L_{ij}(0)+m_i\Sigma^D_{ij}(0)+m_j\Sigma^{D\dagger}_{ij}(0)\right)\,, \\
\delta\widetilde{Z}^{L,H}_{ij} &= -\Sigma^L_{ij}(0)
\end{aligned} \tag{10}$$

for the pseudo WFR decomposed into hermitian and antihermitian–hermitian parts. Note that our result specializes to that of reference [13] in the SM case where Σ^D_{ij} is hermitian if the convention of that reference is adopted instead of equation (8). The counterterm of reference [13] then reads:

$$\delta V^{GGM}_{du} = \frac{1}{2}\delta\widetilde{Z}^{L,A}_{dd_k}V_{d_k u} - V_{du_k}\frac{1}{2}\delta\widetilde{Z}^{L,A}_{u_k u}\,. \tag{11}$$

It should be observed that $\widetilde{\Sigma}^{\mathrm{ren}}$ has no physical significance whatsoever. It only suits the purpose of defining $\delta\widetilde{Z}_{ij}$ through equations (9). Substituting the $\delta\widetilde{Z}_{ij}$ in equation (3) in fact renders the mixing matrix counterterm $\delta\widetilde{V}^{GGM}$ independent of the gauge parameters without destroying the desired pole structure:

$$\frac{\partial\delta\widetilde{V}^{GGM}_{du}}{\partial\xi_{W(Z)}} = 0\,. \tag{12}$$

This scheme thus defines a valid counterterm for the CKM mixing matrix leading to finite, gauge–invariant transition amplitudes.

At this point some remarks about the CKM renormalization according to reference [13] are in order.

First we observe that condition (9) is to a large extent arbitrary. The subtraction point $\not{q}=0$ is not suggested by any underlying principle but instead seems to be an ad hoc choice. Indeed the important feature is that unlike in the usual on–shell scheme there is a common subtraction point for all flavors. Why this point should be zero is not clear.

In contrast, we have verified that subtraction in the limit $\not{q} \to \infty$ is also possible. This procedure, however, requires careful definition [16] of the limiting process because the naive limit of the self–energies diverges with q^2. As we will refer to this procedure once again in a later section we shall sketch it briefly:
Knowing its ultra–violet behavior, Weinberg's theorem [17] tells us that $\Sigma(s)$ diverges at most logarithmically in s. Our goal is then to find Σ_∞, the subtracted finite function in the limit $s \to \infty$. We define

$$\widehat{\Sigma}(s) = -\int_1^\infty \frac{\mathrm{d}\alpha}{\alpha}\left\{\alpha s\Sigma'(\alpha s) - \lim_{s\to\infty} s\Sigma'(s)\right\} + \ln(s)\lim_{s\to\infty} s\Sigma'(s)\,, \tag{13}$$

which coincides with Σ in the divergent terms in s. Then our subtracted function is

$$\Sigma_\infty = \Sigma(s) - \widehat{\Sigma}(s)\,. \tag{14}$$

Given the formulae of appendix D.2, one readily verifies that the definition

$$\delta\widetilde{Z}^{L,A}_{\infty ij} = \frac{1}{m_i^2 - m_j^2}\left((m_i^2 + m_j^2)\Sigma^L_{\infty ij} + m_i\Sigma^D_{\infty ij} + m_j\Sigma^{D\dagger}_{\infty ij}\right) \tag{15}$$

like the $\not{q} = 0$ prescription leads to a gauge–parameter–independent counterterm:

$$\frac{\partial\delta V_{\infty du}}{\partial\xi_{W(Z)}} = \frac{\partial}{\partial\xi_{W(Z)}}\left(\frac{1}{2}\delta\widetilde{Z}^{L,A}_{\infty dd_k}V_{d_k u} - V_{du_k}\frac{1}{2}\delta\widetilde{Z}^{L,A}_{\infty u_k u}\right) = 0\,. \tag{16}$$

Clearly the infinite–momentum subtraction method presents a serious alternative which cannot be ruled out and justifies doubts about the uniqueness and motivation of the $\not{q} = 0$ method.
The authors of reference [13] argue that their scheme has several desirable features as among other things it does not violate certain Ward–Takahashi and Slavnov–Taylor identities as opposed to the scheme of Denner and Sack. It was not possible, however, to prove the method's applicability beyond NLO. Especially the role of $\delta\widetilde{Z}^R_{ij}$ which has been set to zero and which at higher orders will contain off–diagonal divergent terms remains equivocal and is likely to create problems at higher orders.
Although the authors explicitly state that their scheme only applies to Dirac fermions, they suggest it might also be employed in the renormalization of the mixing matrices of our scenario. We will discuss this issue at a later point in this chapter.

An alternative, more intuitive counterterm which also leads to gauge–invariant results has recently been suggested by Barroso, Brücher and Santos (BBS) [15]. The authors observed that for the Born matrix element's form factor of the one–loop decay amplitude without quark WFR, the CKM matrix element factorizes in the ultra–violet divergent part as well as in the derivative with respect to the gauge parameters:

$$\begin{aligned}&\left(F_2^- + \frac{e}{\sqrt{2}s_\mathrm{w}}V_{du}\left(\frac{\delta e}{e} - \frac{\delta s_\mathrm{w}}{s_\mathrm{w}} + \frac{1}{2}\delta Z_W\right)\right)_{UV}\\ =\;& V_{du}\left(F_{2_1}^- + \frac{e}{\sqrt{2}s_\mathrm{w}}\left(\frac{\delta e_1}{e} - \frac{\delta s_{\mathrm{w}_1}}{s_\mathrm{w}} + \frac{1}{2}\delta Z_{W_1}\right)\right)_{UV}\end{aligned} \tag{17}$$

and

$$\begin{aligned} &\frac{\partial}{\partial \xi_{W(Z)}} \left(F_2^- + \frac{e}{\sqrt{2} s_{\rm w}} V_{du} \big(\frac{\delta e}{e} - \frac{\delta s_{\rm w}}{s_{\rm w}} + \frac{1}{2} \delta Z_W \big) \right) \\ &= V_{du} \frac{\partial}{\partial \xi_{W(Z)}} \left(F_{2_1}^- + \frac{e}{\sqrt{2} s_{\rm w}} \big(\frac{\delta e_{\mathbf{1}}}{e} - \frac{\delta s_{\rm w_1}}{s_{\rm w}} + \frac{1}{2} \delta Z_{W_1} \big) \right) , \end{aligned} \tag{18}$$

where the subscript $\ldots_{\mathbf{1}}$ indicates that the quantity is calculated imposing diagonality of the mixing matrix with respect to the flavors under consideration. From these equations and the fact that the transition amplitude without mixing is finite and gauge–parameter–independent we readily conclude that

$$F_2^- + \frac{e}{\sqrt{2} s_{\rm w}} V_{du} \big(\frac{\delta e}{e} - \frac{\delta s_{\rm w}}{s_{\rm w}} + \frac{1}{2} \delta Z_W + \frac{1}{2} \delta Z_{dd_1}^{L\dagger} + \frac{1}{2} \delta Z_{uu_1}^{L} \big) \tag{19}$$

is independent of the gauge parameters and finite just as the decay amplitude without mixing. This suggests the following definition for the CKM matrix counterterm δV_{du}:

$$\delta V_{du}^{BBS} = V_{du} \left(\frac{1}{2} \delta Z_{uu_1}^L + \frac{1}{2} \delta Z_{dd_1}^{L\dagger} \right) - \frac{1}{2} \delta Z_{dd_k}^{L\dagger} V_{d_k u} - V_{du_k} \frac{1}{2} \delta Z_{u_k u}^L , \tag{20}$$

which by construction leads to gauge–invariant finite transition amplitudes.

4.3 Generalizing the BBS Approach

The procedure described above was inspired by two observations. Firstly, the mixing matrix element factorizes in the ultra–violet–divergent part and in the relevant derivatives of the counterterms δe, $\delta s_{\rm w}$ and δZ_W and in F_2^- itself because each diagram contributing to the virtual correction contains one and only one charged current vertex. Secondly, there is the obvious fact that the usual on–shell result for trivial mixing is finite and gauge–independent. However, the constructive approach described above lacks an underlying principle expressible in a renormalization condition. We shall therefore give a more general condition for the counterterm that unlike the procedure of reference [15] and the resulting expression (20) is applicable also at higher orders in perturbation theory.

Up to finite contributions from δe, $\delta s_{\rm w}$ and δZ_W we would have obtained the same counterterm, had we simply imposed the renormalization condition that the mixing matrix element factorize in the renormalized transition amplitude. The counterterm $\delta \widetilde{V}$ thus obtained would differ from δV^{BBS} as in equation (20) only by finite mixing–effect contributions from δe, $\delta s_{\rm w}$ and δZ_W:

$$\delta \widetilde{V} - \delta V^{BBS} = V \left(- \frac{\delta e}{e} + \frac{\delta e_{\mathbf{1}}}{e} + \frac{\delta s_{\rm w}}{s_{\rm w}} - \frac{\delta s_{\rm w_1}}{s_{\rm w}} - \frac{1}{2} \delta Z_W + \frac{1}{2} \delta Z_{W_1} \right) . \tag{21}$$

If δe, $\delta s_{\rm w}$ and δZ_W were independent of the mixing we would have obtained δV as in equation (20).

We stress that the procedure presented by Barroso, Brücher and Santos in comparison with the approach of Gambino, Grassi and Madricardo is in no way favorable. Both

counterterms are completely ad hoc and not derived from a renormalization condition. As presented by the authors there is therefore no stringent generalization to higher orders.
This is different for our counterterm $\delta\widetilde{V}$: although seemingly more complicated it is derived from a very simple principle and the underlying condition is not constrained to a specific order in perturbation theory.
From this point of view the method of BBS is a simplified version of this concept at the one–loop order.
At higher orders $\delta\widetilde{V}$ will clearly become increasingly complicated, possibly containing three–point functions, but on the one hand it certainly presents at least a valid starting point in the search for an all–order prescription and on the other hand it turned out to be a fruitful concept to generalize to the case of neutrino mixing in our scenario, which will be the topic of the next section.

4.4 Generalizing the CKM Renormalization to the B and C Matrices

A first attempt to renormalize the mixing matrices of the 1–2 scenario was made by Kniehl and Pilaftsis [14]. The authors generalized the concept of Denner and Sack [12] to models with Majorana neutrinos.
Before we begin the discussion of a suitable mixing–matrix counterterm, we point out a general difference between $f_j \to f_i$ transition amplitudes for Dirac fermions and those for Majorana fermions. In case of Dirac fermions the most general form in compliance with hermiticity is

$$\Sigma(\not{q}) = \not{q} L \Sigma^L(q^2) + \not{q} R \Sigma^R(q^2) + L\Sigma^D(q^2) + R\Sigma^{D\dagger}(q^2)\,, \tag{22}$$

with $\Sigma^{L(R)} = \Sigma^{L(R)\dagger}$.
For Majorana fermions the corresponding expression reads

$$\Sigma(\not{q}) = \not{q} L \Sigma^L(q^2) + \not{q} R \Sigma^{L\star}(q^2) + L\Sigma^M(q^2) + R\Sigma^{M\star}(q^2)\,, \tag{23}$$

where we have used the self–conjugacy of the neutrino fields to imply that left– and right–handed field components and chirality projections of the transition amplitudes are not independent.
Following the lines of reference [12] the counterterm suggested by Kniehl and Pilaftsis for the matrix B is defined as:

$$\delta B^{KP}_{l\nu} = -\frac{1}{4}(\delta Z^{L\dagger}_{ll_k} - \delta Z^L_{ll_k})B_{l_k\nu} - B_{l\nu_k}\frac{1}{4}(\delta Z^L_{\nu_k\nu} - \delta Z^{L\dagger}_{\nu_k\nu})\,. \tag{24}$$

In analogy, the counterterm for C is given as:

$$\delta C^{KP}_{\nu_i\nu_j} = -\frac{1}{4}(\delta Z^{L\dagger}_{\nu_i\nu_k} - \delta Z^L_{\nu_i\nu_k})C_{\nu_k\nu_j} - C_{\nu_i\nu_k}\frac{1}{4}(\delta Z^L_{\nu_k\nu_j} - \delta Z^{L\dagger}_{\nu_k\nu_j})\,. \tag{25}$$

It is easy to verify that by virtue of this ansatz the bare and renormalized mixing matrices B and C satisfy equations (45) of section 2.6. As in the SM case of section 4.2, δB^{KP} and δC^{KP} also correctly subtract the poles of the virtual corrections to tree level amplitudes with charged and neutral current leptonic mixing.

Just as for the decay into a quark pair a relation analogous to equation (4) holds for the leptonic decay amplitudes of the $W-$ and $Z-$bosons in our scenario. This clearly implies that the mixing counterterms must not be gauge–parameter–dependent, i.e.

$$\frac{\partial \delta B}{\partial \xi_{W(Z)}} = 0 \quad \text{and} \quad \frac{\partial \delta C}{\partial \xi_{W(Z)}} = 0\,. \tag{26}$$

In a simple calculation we found, however, that the definitions of equations (24) and (25) do not satisfy this constraint.

$$\begin{aligned} \frac{\partial}{\partial \xi_{W(Z)}} \left(-\frac{1}{4}(\delta Z^{L\dagger}_{ll_k} - \delta Z^{L}_{ll_k}) B_{l_k\nu} - B_{l\nu_k}\frac{1}{4}(\delta Z^{L}_{\nu_k\nu} - \delta Z^{L\dagger}_{\nu_k\nu}) \right) &\neq 0\,, \\ \frac{\partial}{\partial \xi_{W(Z)}} \left(-\frac{1}{4}(\delta Z^{L\dagger}_{\nu_i\nu_k} - \delta Z^{L}_{\nu_i\nu_k}) C_{\nu_k\nu_j} - C_{\nu_i\nu_k}\frac{1}{4}(\delta Z^{L}_{\nu_k\nu_j} - \delta Z^{L\dagger}_{\nu_k\nu_j}) \right) &\neq 0\,. \end{aligned} \tag{27}$$

Thus, we show that the generalization of the procedure of Denner and Sack to Majorana fermions suffers from the same deficit as the original prescription for the CKM matrix. Given the analysis of section 4.2 this is not a surprising result, but nevertheless it should be noted for reasons of completeness and because it represented up to this point the only attempt to renormalize the mixing matrices occurring in models with Majorana neutrinos.

From the discussion of section 4.2 it follows that the natural strategy to pursue is to adopt the subtraction procedure of Gambino, Grassi and Madricardo, analogous to the development in the CKM case. We have applied this ansatz and found it to fail in the leptonic case:
The renormalized two–point function for self–conjugate fermions differs slightly from expression (8), instead it reads:

$$\begin{aligned} \Sigma^{\text{ren}}_{ij}(\not{q}) \;=\; & \left(\Sigma^{L}_{ij} + \frac{1}{2}(\delta Z^{L}_{ij} + \delta Z^{L\dagger}_{ij})\right)\not{q}L + \left(\Sigma^{L\star}_{ij} + \frac{1}{2}(\delta Z^{L\star}_{ij} + \delta Z^{LT}_{ij})\right)\not{q}R \\ & + \left(\Sigma^{M}_{ij} - \frac{1}{2}(m_i\delta Z^{L}_{ij} + \delta Z^{LT}_{ij} m_j) - \delta_{ij}\delta m_i\right) L \\ & + \left(\Sigma^{M\star}_{ij} - \frac{1}{2}(m_i\delta Z^{L\star}_{ij} + \delta Z^{L\dagger}_{ij} m_j) - \delta_{ij}\delta m_i\right) R\,. \end{aligned} \tag{28}$$

Application of conditions (9) to the tilded version of this equation yields the following hermitian and antihermitian pseudo WFR for Majorana neutrinos:

$$\begin{aligned} \delta \widetilde{Z}^{L,A}_{ij} &= \frac{1}{m_i^2 - m_j^2}\left((m_i^2 + m_j^2)\Sigma^{L}_{ij}(0) + 2m_i m_j \Sigma^{L\star}_{ij} + 2m_i\Sigma^{M}_{ij}(0) + 2m_j\Sigma^{M\star}_{ij}(0)\right)\,, \\ \delta \widetilde{Z}^{L,H}_{ij} &= -\Sigma^{L}_{ij}(0)\,. \end{aligned} \tag{29}$$

The charged–lepton contributions to δB^{GGM} are treated like the quark contributions in the CKM case. Then the counterterms for the leptonic mixing matrices corresponding to δV^{GGM} can be written as:

$$\begin{aligned} \delta B^{GGM}_{l\nu} &= \frac{1}{2}\delta Z^{L,A}_{ll_k} B_{l_k\nu} - B_{l\nu_k}\frac{1}{2}\delta Z^{L,A}_{\nu_k\nu}\,, \\ \delta C^{GGM}_{\nu_i\nu_j} &= \frac{1}{2}\delta Z^{L,A}_{\nu_i\nu_k} C_{\nu_k\nu_j} - C_{\nu_i\nu_k}\frac{1}{2}\delta Z^{L,A}_{\nu_k\nu_j}\,. \end{aligned} \tag{30}$$

It should be noted that the superscript GGM does not indicate that these counterterms were suggested in reference [13], but rather that they were obtained through condition (9).
Again, the crucial question is whether the counterterms (30) depend on the gauge parameters.

The surprising result that can be readily checked with the generic formulae of appendix D.2 is that δB^{GGM} and δC^{GGM} *do* indeed depend on ξ_W and ξ_Z.

As mentioned in section 4.2, the subtraction point $q^2 = 0$ is to some extent arbitrary. We have therefore investigated if there exists any other real positive subtraction point leading to gauge–parameter–independent counterterms. Regardless of what scale we choose, expression (29) and its charged–lepton counterpart only differ in the momentum squared at which the self–energies are evaluated. The pole structure, which is independent of q^2 is always correct as long as the coefficients of $\Sigma^{L(\star)}$ and $\Sigma^{M(\star)}$ remain unchanged. In fact the desired pole–structure fixes them uniquely. Finding a scale that renders the counterterm gauge–parameter–independent is therefore equivalent to solving

$$\frac{\partial}{\partial \xi_{W(Z)}} \left(\delta Z^{L,A}_{ll_k}(q^2) B_{l_k\nu} - B_{l\nu_k} \delta Z^{L,A}_{\nu_k\nu}(q^2) \right) = 0 \tag{31}$$

for q^2.
This equation is not analytically solvable, hence we investigated the ξ dependence of the expression on which the derivative acts numerically. Starting from any physical configuration of the leptonic sector, i.e. satisfying relations (45) and (46) of section 2.6, we calculated the sum of absolute differences of the counterterm evaluated at largely different values of ξ_W and ξ_Z. Plotting this quantity we found no zeroes for values of q^2 between 0 and the Planck scale.
We have also investigated analytically the limit $q^2 \to \infty$, which in the quark case leads to a gauge–parameter–independent result. We used the method described in section 4.2 and found gauge parameter dependence of the counterterm also in this limit.
It is therefore justified to claim, that there is no real positive value for q^2 in equation (29) resulting in a gauge–parameter–independent counterterm for the B and C mixing matrices of our scenario.

In the following we want to give two arguments for the failure of this method, even for an arbitrary subtraction point.
First we once again refer to the different effect of the left–handed chirality projection for Dirac and Majorana fermions and to the fact that the R and L operators do not project the Majorana fields on their gauge singlet and doublet components. Whereas in the CKM case the right–handed parts of the WFR have been explicitly excluded in the construction of δV, the authors of reference [14] employed δZ^L in their construction, containing gauge singlet and doublet contributions.
Secondly, we remind the reader of the relations (46) between the neutrino mass spectrum and the mixing matrices derived in section 2.6. These equations reflect the texture–zero of the neutrino mass matrix in the weak basis, which we imposed to

ensure gauge invariance in the absence of a triplet Higgs field needed to construct a gauge singlet left–handed Majorana mass term. Clearly they should hold not only for the bare masses and mixing matrices, but also for their renormalized counterparts. We mentioned that pseudo–unitarity and hermiticity of the bare and renormalized mixing matrices are ensured by construction. The mass–mixing angle relations (46) of section 2.6, however, are not automatically fulfilled at higher orders, with δB as in equations (30), we obtain for example

$$\delta B m B^T + B \delta m B^T + B m \delta B^T \neq 0\,. \tag{32}$$

Thus, gauge symmetry in the neutrino sector is radiatively broken unless all relations between mixing matrices and neutrino spectrum hold by construction for the bare and renormalized quantities.

4.5 Gauge–Parameter–Independent Renormalization of B and C

In this section we construct a gauge–parameter–independent counterterm δB.
We propose a 'hybrid' technique using the neutrino mass counterterms to renormalize the intrinsic mixing related to the mass–mixing angle relations (46) of section 2.6 and employing our renormalization condition of section 4.3 to renormalize the remaining degrees of freedom.
Because C can be deduced from B, its counterterm is fixed once we know δB, and we shall not worry about δC for now.

We will exemplify the procedure step by step in a concrete scenario and then give a review of our method without reference to a specific physical situation.
As stated in relations (46) the neutrino masses and their mixing angles are not independent. To be explicit, we shall for now assume that we have one right–handed neutrino per generation in addition to the usual SM particle content and only two generations. We want to introduce a standard form of the matrix U of section 2.3 depending only on the neutrino masses such that the mass–mixing angle relations (46) are automatically satisfied. We call this the 'canonical' form of U without inter–generational mixing, representing a possible parameterization of the intrinsic neutrino mixing induced by the structure of the mass matrix in the 1–2 model. It assumes mixing only between the right and left–handed degrees of freedom within one generation. As a matter of fact, the canonical form of U is a trivial generalization of equation (21) in the same section. For the simplest non–trivial case of two generations one obtains:

$$U^{\mathcal{C}} = \begin{pmatrix} -i\sqrt{\frac{m_2}{m_1+m_2}} & 0 & \sqrt{\frac{m_1}{m_1+m_2}} & 0 \\ 0 & -i\sqrt{\frac{m_4}{m_3+m_4}} & 0 & \sqrt{\frac{m_3}{m_3+m_4}} \\ i\sqrt{\frac{m_1}{m_1+m_2}} & 0 & \sqrt{\frac{m_2}{m_1+m_2}} & 0 \\ 0 & i\sqrt{\frac{m_3}{m_3+m_4}} & 0 & \sqrt{\frac{m_4}{m_3+m_4}} \end{pmatrix}, \tag{33}$$

with positive neutrino masses m_i. According to equation (20) of section 2.3, the mass matrix in the weak basis would then read

$$\begin{pmatrix} 0 & 0 & \frac{-m_1+m_3}{m_1+m_3}\sqrt{m_1 m_3} & 0 \\ 0 & 0 & 0 & \frac{-m_2+m_4}{m_2+m_4}\sqrt{m_2 m_4} \\ \frac{-m_1+m_3}{m_1+m_3}\sqrt{m_1 m_3} & 0 & -m_1+m_3 & 0 \\ 0 & \frac{-m_2+m_4}{m_2+m_4}\sqrt{m_2 m_4} & 0 & -m_2+m_4 \end{pmatrix}, \quad (34)$$

with the desired zero block matrix in the upper left, which shows that our canonical form is in agreement with the mass–mixing angle relations.
Given U as above we readily determine the canonical form $B^{\mathcal{C}}$ of the matrix B to be

$$B^{\mathcal{C}} = \begin{pmatrix} i\sqrt{\frac{m_3}{m_1+m_3}} & 0 & \sqrt{\frac{m_1}{m_1+m_3}} & 0 \\ 0 & i\sqrt{\frac{m_4}{m_2+m_4}} & 0 & \sqrt{\frac{m_2}{m_2+m_4}} \end{pmatrix}. \quad (35)$$

If the charged current mixing matrix is of this form, there are no new parameters other than the neutrino masses in our scenario.
For reasons of self–consistency, the 1–2 scenario in this form, i.e. without mixing angles introduced by hand, must therefore be renormalizable without introducing any new counterterms but instead with a $\delta B^{\mathcal{C}}$ constructed exclusively of neutrino mass counterterms. This is an important observation reflecting the fact that the mass–mixing angle relations, given the neutrino spectrum, dictate some intrinsic mixing among the neutrinos.
A natural conjecture for a counterterm to $B^{\mathcal{C}}_{1\nu_1}$,

$$\delta B^{\mathcal{C}}_{1\nu_1} = \delta\left(i\sqrt{\frac{m_3}{m_1+m_3}}\right) = \frac{i}{2}\left(\frac{\delta m_3}{m_3} - \frac{\delta m_1}{m_1}\right)\sqrt{\frac{m_1^2 m_3}{(m_1+m_3)^3}} \quad (36)$$

indeed renders the corresponding decay amplitude gauge–invariant and finite. Gauge invariance is guaranteed by construction as the mass counterterms are independent of the gauge parameters after inclusion of the tadpole contributions.
It is straightforward to show that $\delta B^{\mathcal{C}}$ defined by variation of the functional dependence of $B^{\mathcal{C}}$ on the m_i like in equation (36) makes the left hand side of relation (32) vanish. Thus, for a given neutrino spectrum we have achieved that

$$F_2^{\mathcal{C}-} + \frac{e}{\sqrt{2}s_{\mathrm{w}}}\left(B^{\mathcal{C}}_{l\nu}\left(\frac{\delta e^{\mathcal{C}}}{e} - \frac{\delta s^{\mathcal{C}}_{\mathrm{w}}}{s_{\mathrm{w}}} + \frac{1}{2}\delta Z^{\mathcal{C}}_W\right) + \frac{1}{2}\delta Z^{\mathcal{C}\dagger}_{ll_k} B^{\mathcal{C}}_{l_k\nu} + B^{\mathcal{C}}_{l\nu_k}\frac{1}{2}\delta Z^{\mathcal{C}}_{\nu_k\nu} + \delta B^{\mathcal{C}}_{l\nu}\right) \quad (37)$$

is gauge–parameter–independent and finite if evaluated with $B = B^{\mathcal{C}}$ and $\delta B^{\mathcal{C}}$ as explained above. Assuming our canonical form of B for given neutrino mass values, we have found a natural way of renormalizing the intrinsic neutrino mixing.

In the next step we shall generalize this procedure to the case where additional, 'extrinsic' mixing is introduced to tune the matrix B to the phenomenologically observed values and especially to allow for interfamily mixing.
We want to introduce interfamily mixing by allowing for an arbitrary unitary matrix U_l (see the definition of B in section 2.4) such that

$$B = U_l B^{\mathcal{C}}. \quad (38)$$

We retain our illustrative two–generation scenario and choose e.g.

$$U_l = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{39}$$

such that B attains the form

$$\begin{pmatrix} i\cos\alpha\sqrt{\frac{m_3}{m_1+m_3}} & i\sin\alpha\sqrt{\frac{m_4}{m_2+m_4}} & \cos\alpha\sqrt{\frac{m_1}{m_1+m_3}} & \sin\alpha\sqrt{\frac{m_2}{m_2+m_4}} \\ -i\sin\alpha\sqrt{\frac{m_3}{m_1+m_3}} & i\cos\alpha\sqrt{\frac{m_4}{m_2+m_4}} & -\sin\alpha\sqrt{\frac{m_1}{m_1+m_3}} & \cos\alpha\sqrt{\frac{m_2}{m_2+m_4}} \end{pmatrix}. \tag{40}$$

If we now calculate the one–loop decay amplitude of a W–boson into an electron and the first neutrino mass eigenstate using B as above, clearly the analogue of equation (37) with B substituted instead of $B^{\mathcal{C}}$ will not be finite. This is not surprising as we have not yet renormalized the angle α. We shall do this by imposing the renormalization condition introduced in section 4.3. This means that we want to require that the mixing angle introduced in addition to the intrinsic mixing factorize in the renormalized decay amplitude. In our simple example this means, we demand

$$\delta\mathcal{M}^{\mathrm{ren}}_{e\nu_1} = \cos\alpha\,\delta\mathcal{M}^{\mathcal{C}\mathrm{ren}}_{e\nu_1}. \tag{41}$$

This condition clearly guarantees gauge parameter independence as well as finiteness of the renormalized amplitude.

For practical purposes in one–loop calculations this approach might seem unnecessarily formal. The counterterm defined through equation (41) contains finite, gauge–parameter–independent contributions from δe, δs_{w}, δZ_W and even from three–point functions which make the calculation of the counterterm relatively difficult.
We will therefore give a concrete counterterm suitable for our one–loop problem and parameterizations similar to the one explained above with an arbitrary number of generations:

$$\delta B_{l\nu} = \frac{B_{l\nu}}{B^{\mathcal{C}}_{l\nu}}\left(\frac{1}{2}B^{\mathcal{C}}_{l\nu_k}\delta Z^{L\mathcal{C}}_{\nu_k\nu} + \frac{1}{2}\delta Z^{L\mathcal{C}\dagger}_{ll_k}B^{\mathcal{C}}_{l_k\nu} + \delta B^{\mathcal{C}}_{l\nu}\right) - \frac{1}{2}B_{l\nu_k}\delta Z^{L}_{\nu_k\nu} - \frac{1}{2}B_{l\nu_k}\delta Z^{L}_{\nu_k\nu}. \tag{42}$$

This is the analogue of δV^{BBS} of section 4.2 as opposed to $\delta\widetilde{V}$ corresponding to the counterterm we would have obtained from relation (41).

It should be understood that $B^{\mathcal{C}}_{l\nu}$ and all functions in parentheses of equation (42) are evaluated assuming that the flavors in the isodoublets are arranged such that l and ν couple if 'unit' mixing is assumed. Thus, $B^{\mathcal{C}}_{l\nu}$ in expression (42) *never vanishes* and the counterterm is always well–defined. What seems awkward at first glance actually reflects a deep principle. In the presence of mixing, the choice of flavors which are arranged into common isodoublets is completely random.

We have decomposed our renormalization procedure into two parts and explained it in case of a simple two–generation scenario. For clarity we will recapitulate the general principle without reference to a concrete scenario.

In the most general case we start from a matrix B parameterized in terms of the neutrino masses m_i, mixing angles α_i and phases ϕ_i satisfying relations (45) and (46) of section 2.6 for any values which the α_i and ϕ_i can attain. Note that the number of angles and phases strongly depends on the number of generations and right–handed neutrinos and that the α_i and ϕ_i reflect how under–determined the mixing matrix B is after imposing pseudo–unitarity and the mass–mixing angle relations. A 'canonical' form, which in most cases will not be unique, is then defined through

$$B^{\mathcal{C}}(m_i) = B(m_i, \alpha_i = 0, \phi_i = 0)\,.$$

A counterterm $\delta B^{\mathcal{C}}$ for $B^{\mathcal{C}}(m_i)$ is constructed in terms of the neutrino mass counterterms δm_i by variation of the functional dependence of $B^{\mathcal{C}}_{l\nu}$ on the m_i. This counterterm will properly renormalize transition amplitudes calculated with $B^{\mathcal{C}}$ and render them gauge–parameter–independent and finite. This is guaranteed for reasons of self–consistency of our model, as the intrinsic mixing must be renormalizable without introducing further counterterms.

Allowing for non–trivial values of the parameters α_i and ϕ_i is then comparable to departing from a diagonal CKM matrix in the quark sector of the SM and will analogously induce new divergences not subtracted by $\delta B^{\mathcal{C}}$ in the transition amplitudes. They can be renormalized by imposing the renormalization condition suggested in section 4.3:

$$\delta\mathcal{M}^{\mathrm{ren}}_{l\nu}(B) = \frac{B_{l\nu}}{B^{\mathcal{C}}_{l\nu}}\delta\mathcal{M}^{\mathrm{ren}}_{l\nu}(B^{\mathcal{C}})\,. \tag{43}$$

This procedure allows the renormalization of arbitrary 1–2 (and of course 1–2–3) scenarios, leading by construction to gauge–parameter–independent, finite results. Following this formal prescription might, however, lead to complicated expressions for the desired counterterms. Given a concrete scenario it might therefore be desirable to omit finite, gauge–parameter–independent contributions to obtain a simplified counterterm in a compact analytical form as for example in equation (42).

5 Muon Decay and Δr

In this section we want to present our results for the muon decay amplitude up to the one–loop order in our scenario with Majorana neutrinos. We recall the SM result first calculated by Sirlin [18] which we will denote Δr^{SM} in contrast to Δr, the result in our scenario. The muonic lifetime is one of the most precisely–measured electroweak observables and as a purely leptonic decay it should be very sensitive to alterations of the SM leptonic sector.

5.1 Born Amplitude and Lowest–Order Decay Rate

As our model allows for lepton flavor violation, there is more than one possible final state into which the muon can decay. To be as general as possible, we give the decay amplitude $\mu \to e\nu_i\nu_j$. We want to assume that ν_i and ν_j can be any of the three lightest neutrinos, i.e. the three lowest mass eigenstates are light compared to m_μ. Neglecting the Yukawa couplings of external fermions, two diagrams contribute to the Born amplitude. Due to their Majorana nature, each of the final–state neutrinos can be emitted either at the muon or electron charged current vertex. The relative sign between the crossed graphs reflects Fermi statistics.

$$\begin{aligned} i\mathcal{M}_{Born} &= \frac{ie}{\sqrt{2}s_{\mathrm{w}}}\, B^\dagger_{\nu_i\mu}\, \bar{\nu_i}(k_2)\gamma^\mu L\mu(p_1)\, \frac{-ig_{\mu\nu}}{(p_1-k_2)^2-m_W^2} \\ &\quad \times\frac{ie}{\sqrt{2}s_{\mathrm{w}}}\, B_{e\nu_j}\, \bar{e}(k_1)\gamma^\nu L\nu_j(k_3) \;-\; (\nu_i, k_2 \leftrightarrow \nu_j, k_3)\,. \end{aligned}$$

Neglecting higher orders of $\frac{m_\mu}{m_W}$ we obtain:

$$\begin{aligned} i\mathcal{M}_{Born} &= \frac{-ie^2 B^\dagger_{\nu_i\mu}B_{e\nu_j}}{2s_{\mathrm{w}}^2 m_W^2}\,(\bar{\nu_i}\gamma^\mu L\mu)\,(\bar{e}\gamma_\mu L\nu_j) \;-\; (\nu_i \leftrightarrow \nu_j) \\ &= i\mathcal{M}(i,j) - i\mathcal{M}(j,i)\,. \end{aligned} \tag{1}$$

Given $\mathcal{M}_{Born}$ we readily calculate the ν_i, ν_j contribution to the lowest–order decay rate $\Gamma_B(i,j)$:

$$\begin{aligned} \Gamma_B(i,j) &= (1-\frac{\delta_{ij}}{2})\,\frac{1}{2}\,\frac{1}{2m_\mu}\left(\prod_{l=1}^{3}\frac{d^3k_l}{(2\pi)^3 2k_l^0}\right)(2\pi)^4\delta^4(p_1-\sum_{l=1}^{3}k_l) \\ &\quad \times\sum_{spins}\left(\mathcal{M}(i,j)-\mathcal{M}(j,i)\right)\left(\mathcal{M}^\star(i,j)-\mathcal{M}^\star(j,i)\right)\,, \end{aligned} \tag{2}$$

where the first factor accounts for an identical–particle final state, the second for initial spin average, and the overall normalization and phase space measure is adopted from reference [19]. It should be observed that $\Gamma_B(i,j)$ is the same as $\Gamma_B(j,i)$ and Γ_B^{total} is therefore obtained as

$$\Gamma_B^{total} = \sum_{\substack{i,j=1 \\ i\le j}}^{3} \Gamma_B(i,j)\,. \tag{3}$$

In summing over the fermionic spins we find that the interference terms of equation (2) vanish i.e. $\mathcal{M}(i,j) \perp \mathcal{M}(j,i)$. Neglecting external momenta, we can rewrite equation (3) as

$$\Gamma_B^{total} = \sum_{i,j=1}^{3} \frac{1}{2}\frac{1}{2m_\mu} \left(\prod_{l=1}^{3} \frac{d^3 k_l}{(2\pi)^3 2k_l^0} \right) (2\pi)^4 \delta^4(p_1 - \sum_{l=1}^{3} k_l)\, |\mathcal{M}(i,j)|^2\,, \tag{4}$$

in exact analogy to the case of mixing Dirac neutrinos, where the sum a priori extends over all values for i and j, as ν_i and ν_j can always be distinguished by their fermion number. The same argument applies at the one–loop level, where we will use it to reduce the number of diagrams by a factor of one half.

Neglecting terms of $\mathcal{O}(\frac{m_e^2}{m_\mu^2})$ in the phase space integration, the lowest order width reads:

$$\Gamma_B^{total} = \frac{\alpha^2}{384\pi}\, m_\mu^5\, \frac{1}{(s_{\rm w}^2 m_W^2)^2} \sum_{i,j=1}^{3} |B_{\nu_i\mu}|^2\, |B_{e\nu_j}|^2\,. \tag{5}$$

Comparing with the well–known Fermi model result

$$\Gamma_{Fermi} = \frac{G_\mu^2}{192\pi^3}\, m_\mu^5\,, \tag{6}$$

we find that the usual relation between G_μ and the SM parameters α, m_W and $s_{\rm w}$ changes to

$$G_\mu = \frac{\alpha\pi}{\sqrt{2} s_{\rm w}^2\, m_W^2} \sum_{i,j=1}^{3} |B_{\nu_i\mu}|\, |B_{e\nu_j}|\,, \tag{7}$$

i.e. in comparison to the SM result we find an extra factor of $\sum_{i,j=1}^{3} |B_{\nu_i\mu}|\,|B_{e\nu_j}|$ just as we might have expected intuitively due to leptonic mixing. Clearly, equation (7) is a tree level relation which is modified by the inclusion of radiative corrections in equation (5) and (6).
The Fermi model is an effective theory valid up to energies well below the electroweak scale where the four–fermion interaction can be resolved into two charged current vertices and the W–boson propagator. It is non–renormalizable and apart from purely electromagnetic contributions no higher order effects can be taken into account. Including the electromagnetic corrections, i.e. bremsstrahlung, photon exchange between initial and final state charged leptons and QED wave function renormalization (see figure 3), the Born amplitude is multiplied by the well known QED factor Δq [20]:

$$\delta\mathcal{M}_{QED} = \mathcal{M}_{Born}\Delta q = \mathcal{M}_{Born}\frac{\alpha}{4\pi}\left(\frac{25}{4} - \pi^2\right)\,. \tag{8}$$

The electromagnetic corrections comprised in Δq have actually been included in the definition of G_μ.
Instead of the naive lowest–order relation in the Fermi model, the equation defining G_μ in terms of the measured width is:

$$\Gamma_\mu^{exp} = \frac{G_\mu^2}{192\pi^3}\, m_\mu^5 (1 + 2\Delta q)\,. \tag{9}$$

Except for higher–order QED corrections, which have by now been calculated to $\mathcal{O}(\alpha^2)$ [21], equation (9) cannot be improved any further in the Fermi model.

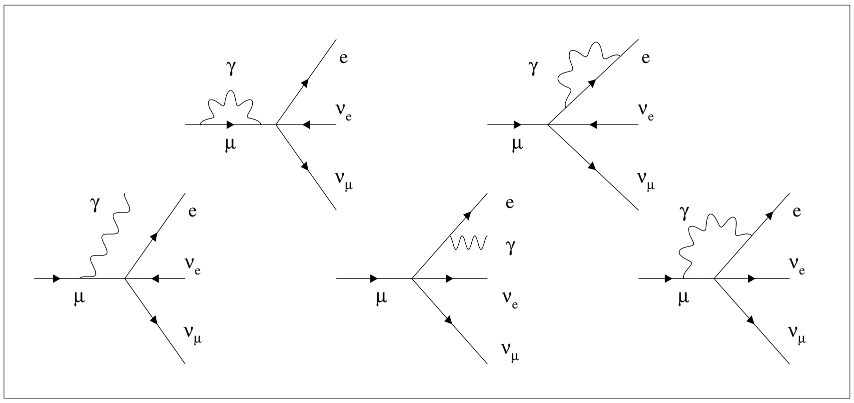

Figure 3: QED correction in the Fermi model, contributions to Δq

5.2 Δr^{SM}

Although the Fermi model does not permit the determination of non–electromagnetic higher–order corrections to equation (9), we can improve relation (7) by including virtual contributions in the SM calculation. These contributions were first calculated in a pioneering work by Sirlin [18] who coined the term Δr^{SM} for the genuinely weak corrections of the SM to the muon decay amplitude:

$$\delta\mathcal{M}_{weak} = \mathcal{M}_{Born}\Delta r^{SM} \,. \tag{10}$$

For completeness we quote the original result Sirlin obtained in the SM:

$$\begin{aligned}\Delta r^{SM} \;\; = \;\; & \frac{\mathrm{Re}\,\Sigma_{WW}(m_W^2) - \Sigma_{WW}(0)}{m_W^2} - 2\frac{\delta e}{e} + \frac{c_{\mathrm{w}}^2}{s_{\mathrm{w}}^2}\mathrm{Re}\left(\frac{\Sigma_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Sigma_{WW}(m_W^2)}{m_W^2}\right) \\ & + \frac{\alpha}{4\pi s_{\mathrm{w}}^2}\left(4(\Delta - \log\frac{m_Z^2}{\mu^2}) + \frac{1}{s_{\mathrm{w}}^2}\log c_{\mathrm{w}}^2(\frac{7}{2} - 6s_{\mathrm{w}}^2) + 6\right) \,,\end{aligned} \tag{11}$$

where $\Delta = \frac{2}{4-D} - \gamma_E + \log 4\pi$ in D space–time dimensions.
Because Δr^{SM} comprises only the genuinely weak corrections we need not recalculate the infrared–divergent contributions from the photonic diagrams. Instead, we can rely on the well–known result of reference [20]. In order to separate the QED contributions which have been accounted for in Δq the photon propagator has been split in the following way:

$$\frac{1}{k^2} = \frac{1}{k^2 - m_W^2} + \frac{m_W^2}{m_W^2 - k^2}\frac{1}{k^2} \,. \tag{12}$$

This amounts to a separation into a photon propagator with a regulator mass m_W and a second term with a Feynman cutoff at $\Lambda = m_W$. Following Sirlin, we shall call the first term on the right side of equation (12) $\gamma_>$, the second $\gamma_<$. Obviously IR singularities enter the amplitude exclusively through $\gamma_<$. In the SM calculation of the

muon decay amplitude at NLO, photonic corrections enter through the wave function renormalization, the vertex and the box diagrams.
In the limit of vanishing momentum transfer the SM box diagram with W and photon exchange and the full photon propagator is equivalent to the last diagram of figure 3, regularized with the aforementioned Feynman cutoff. The $\gamma_<$ and the $\gamma_>$ part of this diagram can hence be omitted completely in Δr^{SM}.
The $\gamma_<$ contribution to the wave function renormalization in the SM is identical to the result of the first and second diagrams in figure 3 regulated with the same cutoff. Thus in Δr^{SM} we only consider the $\gamma_>$ part of the photon propagator in the renormalization of the wave functions.
Proper vertex diagrams with photon exchange arise only due to the resolution of the contact interaction into two distinct vertices. They are not present in the Fermi theory and are included in Δr^{SM} with the full photon propagator.
Apart from the photonic contributions which have to be treated with some care, equation (11) is just the sum of all SM diagrams and counterterms. Of course diagrams with Yukawa couplings to external fermions are strongly suppressed and need not be considered.

In the following we want to generalize the expression for Δr^{SM} of equation (11) to include the effects of massive neutrinos and mixing in our scenario.

5.3 Counterterm Contribution to Δr

Because all electromagnetic couplings remain the same, it is clear that in our scenario the result for Δq remains the same as in the SM. The separation of QED from genuinely weak effects is accomplished in exactly the same way for Δr as for Δr^{SM}.
In our calculation of Δr we will use the same approximations that led to equation (11), i.e. we will neglect external masses and momenta relative to the electroweak scale.
From the interaction Lagrangian of our model which deviates from the SM Lagrangian only in the neutrino interactions that were explicitly calculated in section 2, we easily derive the counterterms for the one–loop decay amplitude $\mu \to e\nu_i\nu_j$ in the on–shell scheme. The structure of the counterterm amplitude, schematically shown in figure 4, is only a little more complicated than the Born amplitude structure. Nevertheless a new feature, characteristic for our model can be observed: the decay can proceed through a neutral current process with an intermediate Z–boson as shown in the last diagram in figure 4.
For the charged and neutral current counterterms, $\delta\mathcal{M}_{ct}^{CC}$ and $\delta\mathcal{M}_{ct}^{NC}$ we obtain the following results:

$$\begin{aligned}
\delta\mathcal{M}_{ct}^{CC} \;\; = \;\; & \frac{-e^2}{2s_{\rm w}^2 m_W^2}\,(\bar{\nu}_i\gamma^\mu L\mu)(\bar{e}\gamma_\mu L\nu_j)\left\{2B^\dagger_{\nu_i\mu}B_{e\nu_j}\left(\delta Z_e - \frac{\delta s_{\rm w}}{s_{\rm w}} - \frac{1}{2}\frac{\delta m_W^2}{m_W^2}\right)\right. \\
& + B_{e\nu_j}\left(\frac{1}{2}\delta Z^{L\dagger}_{\nu_i k}B^\dagger_{k\mu} + \frac{1}{2}B^\dagger_{\nu_i l}\delta Z^L_{l\mu} + \delta B^\dagger_{\nu_i\mu}\right) \\
& \left. + B^\dagger_{\nu_i\mu}\left(\frac{1}{2}\delta Z^{L\dagger}_{el}B_{l\nu_j} + \frac{1}{2}B_{ek}\delta Z^L_{k\nu_j} + \delta B_{e\nu_j}\right)\right\}\,, \qquad (13)
\end{aligned}$$

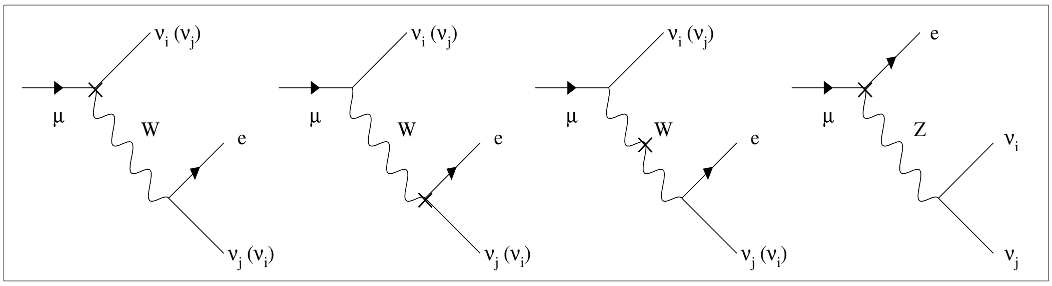

Figure 4: Counterterm diagrams for muon decay, labels in parentheses symbolize crossed graphs not given explicitly.

where summation over double indices is understood and the crossed graphs, with a relative sign and the interchange $(\nu_i \leftrightarrow \nu_j)$ have been omitted.
For the last diagram of figure 4, the neutral current counterterm contribution, we obtain:

$$\begin{aligned}\delta\mathcal{M}_{ct}^{NC} \;=\; & \frac{-e^2}{4s_\mathrm{w}^2 c_\mathrm{w}^2 m_Z^2}\, \Big\{ (\bar{e}\gamma^\mu R\mu)(\bar{\nu_j}\gamma_\mu L\nu_i)\, C_{\nu_j\nu_i}\, s_\mathrm{w}^2\, \left(\delta Z_{e\mu}^R + \delta Z_{e\mu}^{R\dagger}\right) \\ & + (\bar{e}\gamma^\mu R\mu)(\bar{\nu_j}\gamma_\mu R\nu_i)\,(-C_{\nu_i\nu_j}^\dagger)\, s_\mathrm{w}^2\, \left(\delta Z_{e\mu}^R + \delta Z_{e\mu}^{R\dagger}\right) \\ & + (\bar{e}\gamma^\mu L\mu)(\bar{\nu_j}\gamma_\mu L\nu_i)\, C_{\nu_j\nu_i}\,(s_\mathrm{w}^2 - \frac{1}{2})\, \left(\delta Z_{e\mu}^L + \delta Z_{e\mu}^{L\dagger}\right) \\ & + (\bar{e}\gamma^\mu L\mu)(\bar{\nu_j}\gamma_\mu R\nu_i)\,(-C_{\nu_i\nu_j}^\dagger)\,(s_\mathrm{w}^2 - \frac{1}{2})\, \left(\delta Z_{e\mu}^L + \delta Z_{e\mu}^{L\dagger}\right) \Big\}\,. \qquad (14)\end{aligned}$$

For the NC amplitude there exists no crossed contribution that has to be explicitly subtracted. Instead, as we shall see at the end of this section, Fermi statistics enters through the Feynman rules where it is implicitly 'built in'.

Like in the SM we want to write the virtual and counterterm contributions at NLO as a factor multiplying the Born matrix element. In equation (14), however, we encounter matrix elements not present in the Born decay amplitude. In the following we shall employ Fierz transformations (see e.g. reference. [19]) and the relations derived in appendix B to map neutral current on charged current matrix elements. Using the latter we see that:

$$\bar{\nu_j}\,\gamma_\mu\, R(L)\,\nu_i \;=\; \bar{\nu_i}\,\gamma_\mu\, L(R)\,\nu_j\,, \qquad (15)$$

where we have assumed that the particle names represent merely spinors, not Dirac operators. Keeping this in mind, the Fierz transformation yields:

$$(\bar{e}\gamma^\mu L\mu)(\bar{\nu_j}\gamma_\mu L\nu_i) = -(\bar{e}\gamma^\mu L\nu_i)(\bar{\nu_j}\gamma_\mu L\mu)\,, \qquad (16)$$

otherwise no factor (-1) were present on the right side of this identity.

Using equations (15) and (16), we can map the matrix elements in the last two rows of expression (14) on the Born matrix elements:

$$
\begin{aligned}
(\bar{e}\gamma^{\mu}L\mu)(\bar{\nu_j}\gamma_{\mu}L\nu_i) &= -(\bar{e}\gamma^{\mu}L\nu_i)(\bar{\nu_j}\gamma_{\mu}L\mu)\,, \\
(\bar{e}\gamma^{\mu}L\mu)(\bar{\nu_j}\gamma_{\mu}R\nu_i) &= -(\bar{e}\gamma^{\mu}L\nu_j)(\bar{\nu_i}\gamma_{\mu}L\mu)\,,
\end{aligned}
\tag{17}
$$

the matrix elements in the first two rows of (14) have finite form factors and are orthogonal to the Born matrix elements.
Relations (17) with the hermiticity of C show that the Fermi antisymmetry of the CC amplitude consistently carries over to the NC amplitude at NLO: under the interchange $(\nu_i \leftrightarrow \nu_j)$ the form factors in the first and last two rows of equation (14) are identical up to a factor (-1).

5.4 Virtual Corrections

Similarly to the counterterms, the virtual corrections can be separated into charged and neutral current contributions. Again, virtual contributions proportional to the matrix elements appearing in equation (14) can be mapped on factors multiplying the Born matrix elements, allowing us to give an expression for Δr analogous to the SM case yet considerably more complicated.
The calculation was performed in an arbitrary gauge to ensure explicitly that our final result obeys gauge invariance. After verifying that all gauge–parameter–dependent terms vanish in our final result, we decided to choose Feynman gauge in the following presentation.
As in the case of W decay, we generated the amplitude with the *MATHEMATICA* package *FeynArts* [8] and performed the loop integration in dimensional regularization with *FeynCalc* [9]. The occurring Passarino–Veltman tensor coefficient functions were then substituted with their limits for vanishing external momenta. We obtained explicit results for the zero–external–momenta limit, and an explanation of the technique we employed to obtain them is given in appendix C.6. The result thus obtained was expanded in $\frac{m_\mu}{m_W}$, and only leading–order terms were considered. In this approximation only the matrix elements appearing in the Born amplitudes have non–vanishing form factors, just as in the SM.
The virtual corrections fall into five different classes: charged current self–energy, vertex and box corrections ($\delta\mathcal{M}_{\Sigma}^{CC}$, $\delta\mathcal{M}_{\triangle}^{CC}$ and $\delta\mathcal{M}_{\square}^{CC}$) and neutral current vertex and box corrections ($\delta\mathcal{M}_{\triangle}^{NC}$ and $\delta\mathcal{M}_{\square}^{NC}$). We give the explicit results for the CC and NC combined box and triangle graphs in Feynman gauge. The W self–energy will be added later when we quote the combined result of virtual and counterterm contributions to Δr. The results are:

$$
\begin{aligned}
\delta\mathcal{M}_{\triangle+\square}^{CC} \;=\; & \frac{e^4(\bar{\nu_i}\gamma^{\mu}L\mu)(\bar{e}\gamma_{\mu}L\nu_j)}{256\pi^2 s_{\mathrm{w}}^2 c_{\mathrm{w}}^2 m_W^2 m_Z^2}\left\{\frac{c_{\mathrm{w}}^2 m_k^2 m_Z^2 \log(\frac{m_k^2}{m_W^2})}{2 s_{\mathrm{w}}^2 m_W^2 (m_H^2 - m_k^2)(m_k^2 - m_l^2)(m_k^2 - m_W^2)}\right. \\
& \times\frac{1}{(m_k^2 - m_Z^2)}\Big(B_{e\nu_l}B^{\dagger}_{\nu_k\mu}C_{\nu_i\nu_k}C_{\nu_l\nu_j}m_k^2(m_l^2(2m_k^2 - m_H^2 - m_Z^2)(m_k^2 - 4m_W^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{c_{\rm w}^2} m_W^2 (m_k^2 - m_H^2)(m_W^2 - m_l^2)) \; + \; (m_l \leftrightarrow m_k) \\
&+2(B_{e\nu_j} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} + B_{e\nu_k} B^\dagger_{\nu_i\mu} C_{\nu_k\nu_j})(m_k^2 - m_l^2) \Big((m_k^4 + 12 m_W^2 (m_k^2 - m_W^2)) \\
&\times m_H^2 + m_k^2 (m_Z^2 (m_k^2 - 4m_W^2) - 2(m_k^4 + 2m_W^2(2m_k^2 - 3m_W^2)) \\
&+\frac{2}{c_{\rm w}^2} m_W^2 (m_k^2 - m_H^2)(m_k^2 - 3m_W^2))\Big)\Big) \\
&-\frac{c_{\rm w}^2 (m_H^2 - 4m_W^2) m_Z^2 m_H^2 m_k^2 \log(\frac{m_H^2}{m_W^2})}{2 s_{\rm w}^2 m_W^2 (m_H^2 - m_k^2)(m_H^2 - m_l^2)(m_H^2 - m_W^2)} \Big(m_l^2 B_{e\nu_l} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} C_{\nu_l\nu_j} \\
&-2(m_H^2 - m_l^2)(B_{e\nu_j} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} + B_{e\nu_k} B^\dagger_{\nu_i\mu} C_{\nu_k\nu_j})\Big) - 2m_Z^2 \log(\frac{\mu^2}{m_W^2}) \\
&\times \left(\frac{c_{\rm w}^2}{s_{\rm w}^2}(B_{e\nu_k} B^\dagger_{\nu_i\mu} C_{\nu_k\nu_j} + B_{e\nu_j} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k})\frac{m_k^2}{m_W^2} - 2B_{e\nu_j} B^\dagger_{\nu_i\mu}(10 - 11\frac{1}{s_{\rm w}^2})\right) \\
&+\frac{m_Z^2 \log(\frac{m_Z^2}{m_W^2})}{2 s_{\rm w}^2 m_W^2 (m_W^2 - m_Z^2)(m_Z^2 - m_k^2)} \Big(-4 B_{e\nu_j} B^\dagger_{\nu_i\mu} m_W^2 (m_k^2 - m_Z^2)(c_{\rm w}^2 - s_{\rm w}^2) \\
&((c_{\rm w}^2 - s_{\rm w}^2)(m_W^2 - 6m_Z^2) - 6m_Z^2)) - (B_{e\nu_j} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} + B_{e\nu_k} B^\dagger_{\nu_i\mu} C_{\nu_k\nu_j}) m_Z^2 \\
&\times(m_k^2(m_Z^2 - 4m_W^2) + (c_{\rm w}^2 - s_{\rm w}^2)(8m_W^2(m_Z^2 - \frac{3}{2}m_W^2) + m_k^2(4m_W^2 + m_Z^2)) \\
&+12 m_W^2 m_Z^2) + B_{e\nu_l} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} C_{\nu_l\nu_j} \frac{m_Z^2}{m_Z^2 - m_l^2}(c_{\rm w}^2 m_k^2 m_l^2 (m_Z^2 - 4m_W^2) \\
&+4m_W^2(m_W^2 m_Z^2 - m_k^2 m_l^2))\Big) + 2B_{e\nu_j} B^\dagger_{\nu_i\mu} m_Z^2 \; \frac{1}{s_{\rm w}^2}(22 s_{\rm w}^2 - 21) \\
&-3\frac{c_{\rm w}^2}{s_{\rm w}^2}(B_{e\nu_j} B^\dagger_{\nu_k\mu} C_{\nu_i\nu_k} + B_{e\nu_k} B^\dagger_{\nu_i\mu} C_{\nu_k\nu_j})\frac{m_Z^2}{m_W^2} m_k^2 \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta\mathcal{M}^{NC}_{\triangle+\Box} \;=\; & \frac{-e^4 B^\dagger_{\nu_k\mu}(\bar{e}\gamma^\mu L\nu_j)(\bar{\nu}_i\gamma_\mu L\mu)}{256\pi^2 s_{\rm w}^2 c_{\rm w}^2 m_W^2 m_Z^2} \left\{ \frac{m_k^3 \log(\frac{m_k^2}{m_W^2})}{s_{\rm w}^2 m_W^2 (m_k^2 - m_W^2)^2} \Big(B_{e\nu_l} C_{\nu_i\nu_j} \frac{(m_k^2 - m_W^2)}{(m_k^2 - m_l^2)} \right. \\
&\times m_W^2 (C_{\nu_k\nu_l} m_l (m_k^2 - 4m_W^2) + 2C_{\nu_l\nu_k} m_k (m_l^2 - m_W^2)) + (m_k \leftrightarrow m_l) \\
&+\frac{B_{e\nu_k} m_k}{(m_k^2 - m_h^2)}(C_{\nu_i\nu_j}(m_h^2 - m_k^2) m_W^2 (s_{\rm w}^2 (m_k^2 - 8m_W^2) - c_{\rm w}^2 (m_k^2 + 12 m_W^2)) \\
&-c_{\rm w}^2 B_{h\nu_j} B^\dagger_{\nu_i h} m_Z^2 (16 m_W^4 + m_h^2 (m_k^2 - 8m_W^2)))\Big) \\
&-\frac{c_{\rm w}^2 B_{e\nu_k} B_{h\nu_j} B^\dagger_{\nu_i h} m_Z^2 m_h^4 \log(\frac{m_h^2}{m_W^2})}{s_{\rm w}^2 m_W^2 (m_h^2 - m_k^2)(m_h^2 - m_W^2)^2}(m_k^2 (m_h^2 - 8m_W^2) + 16 m_W^4) \\
&-\frac{C_{\nu_i\nu_j}}{s_{\rm w}^2} \log(\frac{\mu^2}{m_W^2})(B_{e\nu_l} C_{\nu_k\nu_l} m_k m_l + B_{e\nu_k}(4m_W^2 + (c_{\rm w}^2 - s_{\rm w}^2)(m_k^2 + 6m_W^2))) \\
&-\frac{1}{2 s_{\rm w}^2} B_{e\nu_l} C_{\nu_i\nu_j}(C_{\nu_k\nu_l} m_k m_l + 2C_{\nu_l\nu_k} m_W^2) + \frac{B_{e\nu_k}}{2(m_W^2 - m_k^2)} \Big(C_{\nu_i\nu_j}(m_k^2
\end{aligned}
$$

$$(17m_W^2 - 3m_k^2) + \frac{c_{\rm w}^2}{s_{\rm w}^2}(3m_k^4 + 19m_k^2 m_W^2 + 4m_W^4)) + \frac{2c_{\rm w}^2 B_{h\nu_j} B^\dagger_{\nu_i h} m_Z^2}{s_{\rm w}^2(m_h^2 - m_W^2)}$$
$$(16m_W^4 - 7m_h^2 m_k^2)\Big)\Big\}\ ,$$

where we have already applied equations (17) to write the NC corrections as a form factor to the Born matrix element and where $m_{k/l}$ denote internal neutrino or charged lepton masses as can be deduced from the couplings.

Combining the counterterm and virtual contributions, we obtain the following expression for Δr in our massive neutrino scenario:

$$\begin{aligned}
\Delta r(i,j) &= 2(\frac{\delta e}{e} - \frac{\delta s_{\rm w}}{s_{\rm w}}) + \frac{\Sigma^W(0) - \mathrm{Re}\,\Sigma^W(m_W^2)}{m_W^2} \\
&+ \frac{1}{B^\dagger_{\nu_i\mu}}\left(\frac{1}{2}\delta Z^{L\dagger}_{\nu_i k} B^\dagger_{k\mu} + \frac{1}{2} B^\dagger_{\nu_i l}\delta Z^L_{l\mu} + \delta B^\dagger_{\nu_i\mu}\right) \\
&+ \frac{1}{B_{e\nu_j}}\left(\frac{1}{2}\delta Z^{L\dagger}_{el} B_{l\nu_j} + \frac{1}{2} B_{ek}\delta Z^L_{k\nu_j} + \delta B_{e\nu_j}\right) \\
&+ \frac{s_{\rm w}}{c_{\rm w}}\frac{C_{\nu_i\nu_j}}{B^\dagger_{\nu_i\mu}B_{e\nu_j}}\frac{s_{\rm w}^2 - \frac{1}{2}}{s_{\rm w}c_{\rm w}}\frac{1}{2}\left(\delta Z^L_{e\mu} + \delta Z^{L\dagger}_{e\mu}\right) \\
&+ \frac{\delta\mathcal{M}^{CC}_{\Delta+\Box} + \delta\mathcal{M}^{NC}_{\Delta+\Box}}{\mathcal{M}(i,j)}\,.
\end{aligned} \tag{18}$$

One should keep in mind that the expressions for $\delta\mathcal{M}^{CC}$, $\delta\mathcal{M}^{NC}$ and Δr only comprise the virtual and counterterm contributions to one of the Born amplitudes, namely $\mathcal{M}(i,j)$ and that their crossed counterparts have been omitted for the sake of brevity. The calculation of the total one–loop–corrected muon decay width proceeds just as in section 5.1. For the one–loop corrected muon decay width in our scenario, we obtain:

$$\Gamma^{total}_{NLO} = \Gamma^{total}_B(1 + 2\Delta q + 2\sum_{i,j=1}^{3} \mathrm{Re}\,\Delta r(i,j)) \tag{19}$$

$$= \frac{G_\mu^2}{192\pi^3} m_\mu^5 (1 + 2\sum_{i,j=1}^{3} \mathrm{Re}\,\Delta r(i,j))\,. \tag{20}$$

6 Numerical Results and Conclusions

6.1 Values of Numerical Input Parameters

In this section we want to give explicit values for all input parameters which we employed for our numerical studies. We have chosen all values in compliance with the most recent publication of the Particle Data Group [22].
For the fine structure constant and the boson masses we used:

$$\alpha = 1/137.036\,, \qquad m_W = 80.419\,\mathrm{GeV}\,,$$
$$m_Z = 91.1882\,\mathrm{GeV}\,, \qquad m_H = 150\,\mathrm{GeV}\,.$$

The quark and charged–lepton masses were chosen as:

$$\begin{array}{lll} m_e = 0.5109989\,\mathrm{MeV}\,, & m_\mu = 105.658357\,\mathrm{MeV}\,, & m_\tau = 1.77703\,\mathrm{GeV}\,, \\ m_u = 3\,\mathrm{MeV}\,, & m_c = 1.5\,\mathrm{GeV}\,, & m_t = 174.3\,\mathrm{GeV}\,, \\ m_d = 6\,\mathrm{MeV}\,, & m_s = 120\,\mathrm{MeV}\,, & m_b = 4.7\,\mathrm{GeV}\,. \end{array}$$

In all of this section we have assumed a scenario with three light neutrinos denoted n_i and three heavy neutrinos N_i. In agreement with the aforementioned reference we have obeyed the following mass bounds for those states :

$$m_{n_1} < 3\,\mathrm{eV}\,, \quad m_{n_2} < 0.19\,\mathrm{MeV}\,, \quad m_{n_3} < 18.2\,\mathrm{MeV}\,, \quad m_{N_i} > 73.5\,\mathrm{GeV}\,. \tag{1}$$

These bounds have to be considered very conservative and actually the three lowest–lying neutrino mass states are widely believed to be much lighter.
The Super–Kamiokande experiment [1] which presently provides the best evidence for non–vanishing neutrino mass favors a value for the difference of the squares of the mixing neutrino state masses Δm^2 of $10^{-3}\ldots 0.1\,\mathrm{eV}^2$. Solar neutrino experiments even suggest values of $\Delta m^2 \leq 10^{-5}\,\mathrm{eV}^2$. Unless we assume some coincidental degeneracy of the light neutrino state masses these values for Δm^2 point to mass values far below the bounds given in relations (1).
Typically neither the Born amplitudes of the decay processes we consider nor their electroweak virtual corrections are as sensitive to Δm^2 and the mixing angles as the famous formula describing neutrino oscillations in vacuum. In our numerical study we will therefore not pay special attention to a particular region of parameter space. Instead it is our goal to see for what values of the input parameters the theoretical predictions for the observables under investigation show deviations from their SM values at all. We thus perform a relatively unbiased scan.
The parameterization of the leptonic mixing matrices B and C is a simple generalization of the form given in equations (38) and (40) of section 4.5 to three generations. We have the freedom to choose an arbitrary 3×3 unitary matrix U_l multiplying the intrinsic (or canonical) neutrino mixing matrix $B^{\mathcal{C}}$ from the left to yield B. We have not included any CP–violating phases in our parameterization of U_l, and thus we end up with three angles θ_{12}, θ_{23} and θ_{13} which we must fix in our analysis. We parameterize U_l adopting the 'standard' parameterization [23] for the CKM matrix:

$$U_l = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}, \tag{2}$$

where c_{ij} stands for $\cos\theta_{ij}$ and s_{ij} for the corresponding sine.
In the quark sector we have assumed unit mixing.

Given the analytic results of the previous sections one can study several physical observables in the framework of our extended lepton sector model. Although the leptonic partial decay widths of the $W-$ and $Z-$bosons cannot be considered precision observables, we have evaluated these partial widths to verify in the simplest non–trivial case that our renormalization prescription of section 4 indeed renders physical amplitudes independent of the gauge parameters. We found independence of the gauge parameters in both cases as well as in the muon decay amplitude. Besides this very strong test we compared our results with other authors and established agreement in the SM limit.
In the following we will give numerical results for $W-$boson and muon (partial) decay widths.

6.2 $W-$Boson Decay Results

Inserting all possible internal field configurations in agreement with section 2, into the formulae of section 3 one readily obtains the W decay amplitude into a lepton pair in our scenario. The hadronic decay amplitude is the same as in the SM. Apart from the comparison with reference [10] where we found excellent numerical agreement for the hadronic width, we also reproduced the measured SM value for the leptonic partial width. The infrared singularities were regulated by introducing a photon mass λ. We verified the results for the electromagnetic real radiation of reference [6] and found good infrared stability for variations of λ over five orders of magnitude. Cancellation of ultraviolet singularities was established analytically and successfully tested in our numerical evaluation using the FF–package [24].
The physical quantity $\delta\Gamma_W$ which we graphically depict in this section is defined as

$$\delta\Gamma_W = \frac{\Gamma^{1-2}_{W\to ln} - \Gamma^{SM}_{W\to l\nu}}{\Gamma^{SM}_{W\to l\nu}}\,, \tag{3}$$

i.e. the relative deviation of the leptonic width in the 1–2 Majorana neutrino scenario from the SM result at NLO. We sum over all charged lepton–light neutrino final states and assume that the heavy states N_i are not produced. This quantity is essentially free of large logarithmic contributions of the light quarks to the running of the electromagnetic coupling and the corresponding uncertainties. We need no special treatment of the light–quark contributions to $\Delta\alpha$ as they cancel in the denominator of definition (3).
The experimental value of the (averaged) leptonic branching ratio of the $W-$boson is 10.56(14)% [22], which implies a relative experimental error of about 1.3%. With the hadronic decay channels being unaffected by our extension of the SM leptonic sector and under the approximate assumption that the SM value agrees with the experimental data, we expect values of $\delta\Gamma_W$ to be physically relevant if they are of the order of 1%, given the current experimental accuracy.
Figure 5 shows $\delta\Gamma_W$ as a function of the sine of the mixing angle θ_{12}. Here and in the following all mixing angles are set to zero unless indicated otherwise. The masses of the light–neutrino states are set to their maximal values as given in relations (1) while the three heavy–neutrino states are assumed to be degenerate at 75, 250, 500

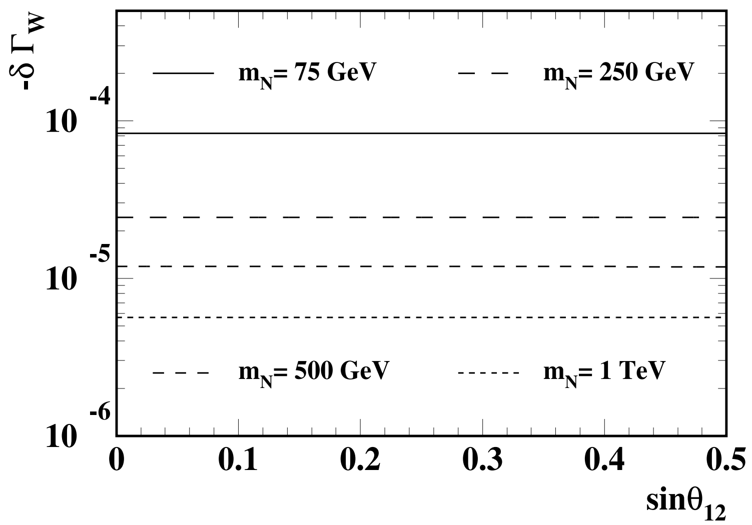

Figure 5: $\delta\Gamma_W$ over $\sin\theta_{12}$ with m_{n_i} at their upper bound and m_{N_i} degenerate between 75 and 1000 GeV.

and 1000 GeV, respectively. Throughout this range of m_N values we observe that the partial width is smaller in the 1–2 scenario than in the SM and that the relative deviation in the leptonic width is of the order of 10^{-6} to 10^{-4}. In this case as well as for $\delta\Gamma_W$ as a function of $\sin\theta_{23}$ or $\sin\theta_{13}$ which is not shown here, mixing effects are not observable. The deviation from the SM values is largest if the heavy neutrino masses approach their lower bound, but even then the effect is far below the current experimental sensitivity.

In figure 6 we plot $\delta\Gamma_W$ over $\sin\theta_{23}$ with different values of m_{n_2} for different curves. The masses of the n_1 and n_3 states are maximal, the masses of the heavy–neutrino states are set to 250 GeV and m_{n_2} is varied between 1 eV and 190 keV. The sign of $\delta\Gamma_W$ is still negative but unlike in figure 5 we can see slight mixing effects because the ordinate is linear and its range is much smaller than in the previous figure. $\delta\Gamma_W$ increases with the mixing angle and the mass of the n_2 state. The curves for the three lowest m_{n_2} values almost coincide on the chosen scale while $\delta\Gamma_W$ is maximal for m_{n_2} at its experimental upper bound. For $\sin\theta_{12}$ on the abscissa the angle dependence almost vanishes, for $\sin\theta_{13}$ it is about as weak as in the present case.

Figure 7 depicts the dependence of $\delta\Gamma_W$ on the common mass m_N of the heavy neutral lepton states. It shows four almost coinciding curves: one for vanishing mixing, and three where one mixing angle is maximal and the others are set to zero. The light–neutrino masses are assumed to be maximal in agreement with relations (1).

The important feature we want to point out here is the decoupling of the heavy states which is already manifest in the few–TeV range.

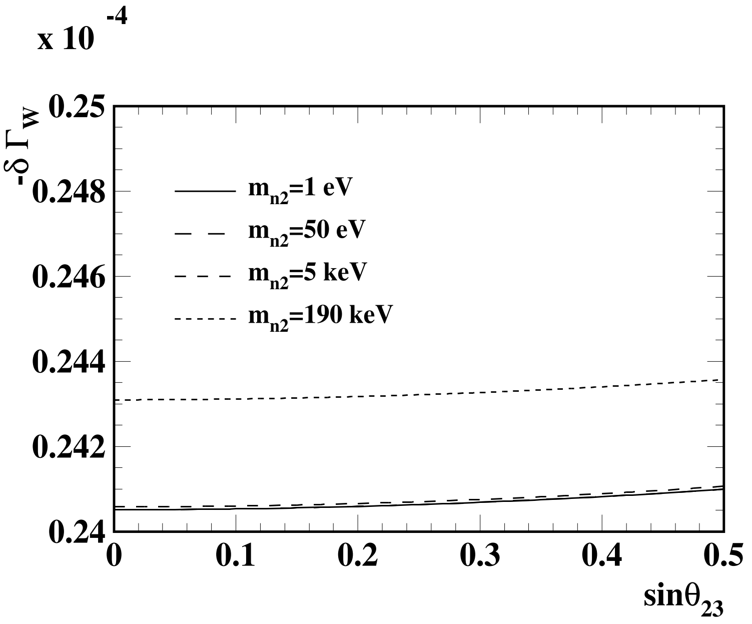

Figure 6: $\delta\Gamma_W$ over $\sin\theta_{23}$ with m_{n_1} and m_{n_3} at their upper bounds, m_{N_i} degenerate at 250 GeV and m_{n_2} between 1 eV and 190 keV.

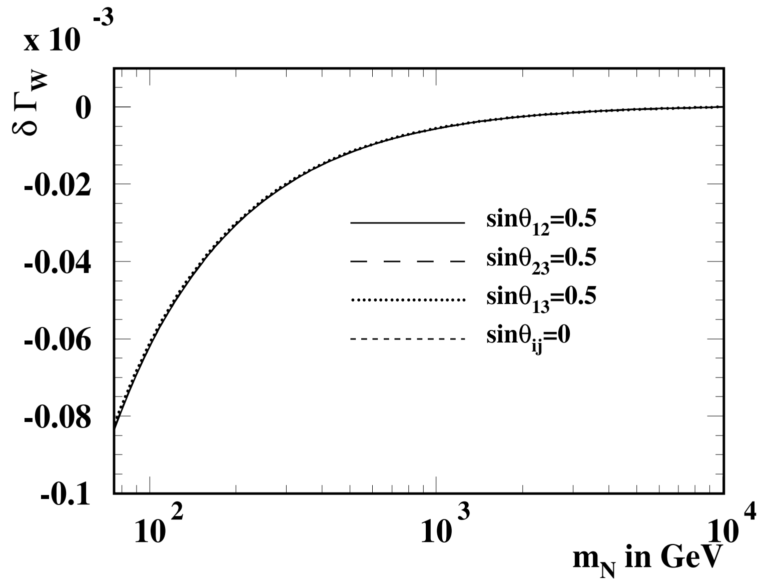

Figure 7: $\delta\Gamma_W$ over the common mass of the heavy neutrino states m_N for different values of the mixing angles with the m_{n_i} at their upper bounds.

Besides ultraviolet and infrared stability and gauge–parameter–independence, decoupling of the heavy states for large m_N values is a good check, specific to our model, reflecting the see–saw mechanism which the scenario is based on. With increasing masses of the heavy states their non–gauge singlet admixture decreases, decoupling them from the gauge interactions. It should be noted, although unobservable in this plot, that the curves do not converge to zero but instead to some very small but non–zero value. This is due to the fact that in our parameterization in the limit $m_N \to \infty$ we obtain three light but not massless, 'left–handed' neutrino states and the physical situation thus slightly differs from the SM. If instead the Majorana mass parameter of the neutrino mass matrix in the weak basis would be increased to very large values, the curves would asymptotically approach zero. In our parameterization this amounts to increasing the common value m_N and simultaneously decreasing the m_{n_i} values.
One also observes in figure (7) that mixing effects have practically no impact on $\delta\Gamma_W$. This is due to the fact that the leptonic W partial decay width involves a sum (or an average) over final–state flavors and this process is therefore quite flavor– and mixing–insensitive.
What is dominant instead, is the apparent pseudo–unitarity violation due to the fact that decays with final–state heavy neutrinos are kinematically not accessible. This also explains the sign of $\delta\Gamma_W$.

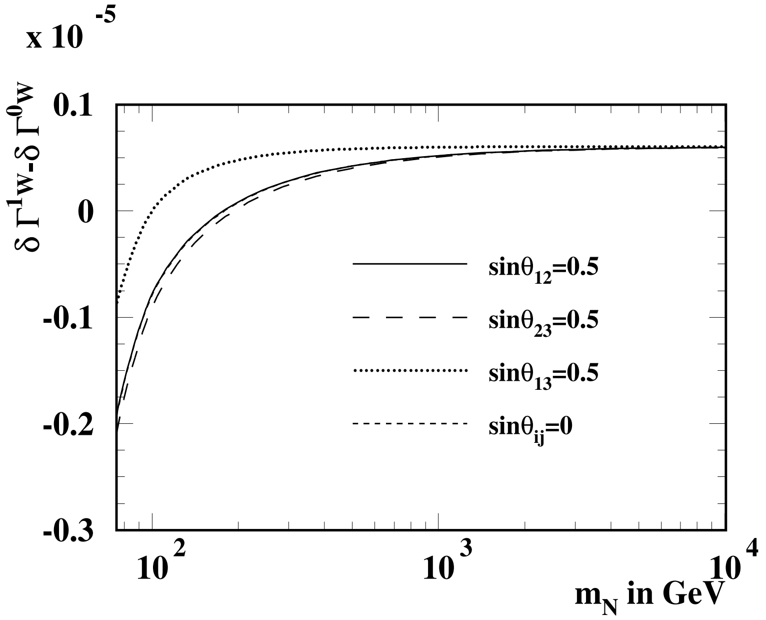

Figure 8: The difference $\delta\Gamma_W - \delta\Gamma_W^{\mathrm{tree}}$ between the one–loop and tree level relative correction of the extended lepton sector model to the SM result. All parameters are chosen as in figure 7.

To see if the effects observed so far really arise from quantum contributions of the neutrinos or if they are primarily due to the structure of the couplings and thus manifest

already at tree level we subtract from $\delta\Gamma_W$ up to one–loop level, as depicted in all plots so far, the corresponding lowest order result. Figure 8 shows this difference which can be interpreted as the purely quantum–induced contributions for the same set of parameters we have used in figure 7. We observe that the quantum contributions show a behavior very similar to the one observed for $\delta\Gamma_W$ in the previous plot, however, on a scale about one order of magnitude lower. It is striking that the quantum corrections change their sign at very low values of the common heavy–neutrino mass of about 100 GeV and then asymptotically approch a value of approximately 6×10^{-7}. The dependence on the mixing angles is generally feeble except for the slightly faster asymptotic convergence of the $\sin\theta_{13}$–curve.
We conclude that the quantum corrections have no significant influence on our prediction for $\delta\Gamma_W$.

Given the present experimental accuracy we have not found observable effects in any of the investigated regions of parameter space. In particular we find that the dependence of $\delta\Gamma_W$ on the neutrino masses and mixing angles is dominated by the tree level contributions and that genuine quantum effects are suppressed by about one order of magnitude.

6.3 Muon Decay Results

In the previous section we have found values for the quantity $\delta\Gamma_W$ of the order 10^{-4} which is about two orders of magnitude beyond current experimental means.
The muon lifetime is one of the most precisely measured electroweak observables with a present experimental value of $2.19703(4) \times 10^{-6}$ s [22] and an experimental error of only about 18 parts per million (ppm). Thus effects of the same approximate order of magnitude as in the previous process could well be physically relevant in this case.
For our numerical analysis we define a quantity $\delta\Gamma_\mu$ in analogy to equation (3):

$$\delta\Gamma_\mu = \frac{\Gamma^{1-2}_{\mu\to enn} - \Gamma^{SM}_{\mu\to e\bar{\nu}_e\nu_\mu}}{\Gamma^{SM}_{\mu\to e\bar{\nu}_e\nu_\mu}} \,. \tag{4}$$

As we have explained in the previous section, this quantity is free of uncertainties associated with the light–quark contributions to the running of the electromagnetic coupling. In fact any scheme dependence not directly related to the modified neutral lepton sector is canceled.
Our result for the muon decay amplitude has undergone rigorous testing on the analytic as well as the numerical level. As mentioned before we found analytic agreement with the original formulae by Sirlin [18] in the SM limit. Furthermore we have verified ultraviolet finiteness and gauge parameter independence of our formulae. Finally we could reproduce published [25] numerical values for Δr in the SM.
For figures 9 and 10 we have chosen the same input parameters as for figure 5 of the last section, i.e. the light neutrino masses at their upper bound and the masses of the heavy states degenerate between 75 and 1000 GeV. We plot $\delta\Gamma_\mu$ over $\sin\theta_{12}$ and $\sin\theta_{23}$ respectively.
We observe several features that could already be seen in previous figures. Again we find generally negative values for $\delta\Gamma$ with a magnitude of at most 10^{-4}. As before $\delta\Gamma_\mu$

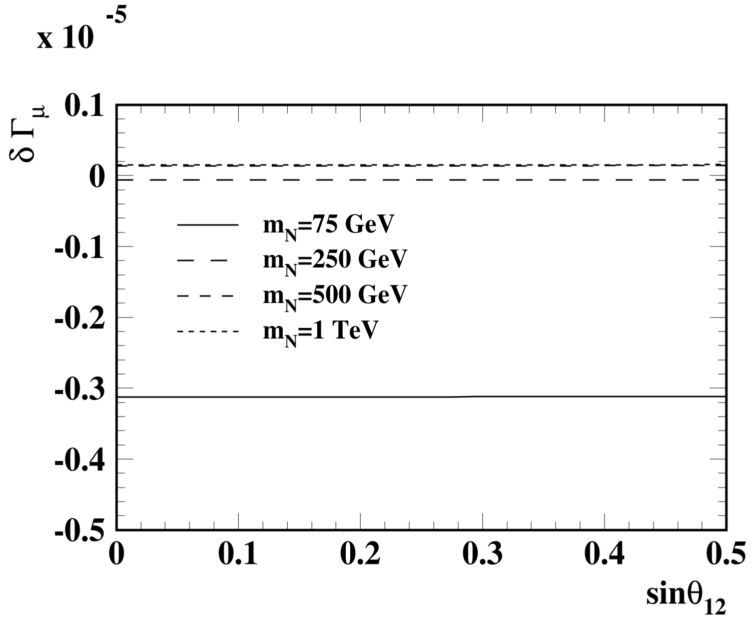

Figure 9: $\delta\Gamma_\mu$ over $\sin\theta_{12}$ with m_{n_i} at their upper bound and m_{N_i} degenerate between 75 and 1000 GeV.

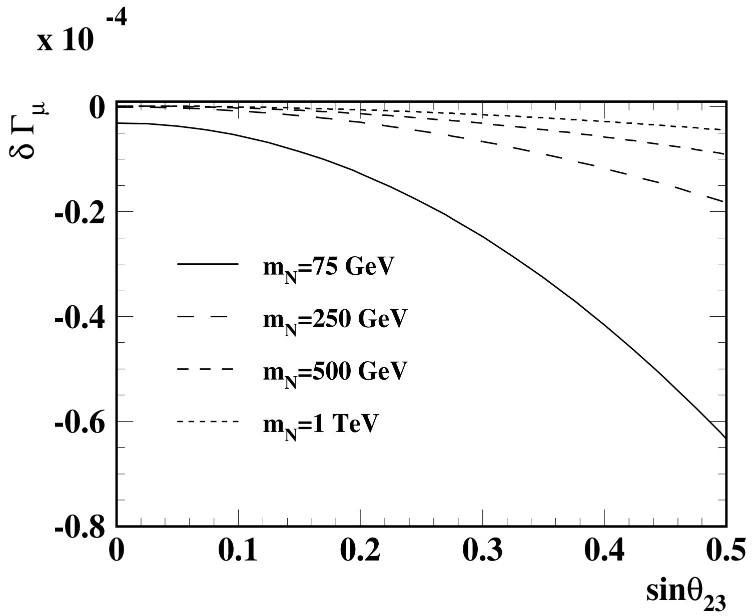

Figure 10: $\delta\Gamma_\mu$ over $\sin\theta_{23}$ with m_{n_i} at their upper bound and m_{N_i} degenerate between 75 and 1000 GeV.

rises in magnitude for decreasing values of m_N because the couplings of the heavy–neutrino states to the gauge fields become stronger. Figure 9 shows almost coinciding curves for a heavy–neutrino mass of 500 and 1000 GeV with positive values of $\delta\Gamma_\mu$. In this range of heavy–neutrino masses, decoupling apparently subdues pseudo–unitarity violation to such an extent that positive virtual effects prevail.

However, there is an important difference between this and the previously considered process. While the W–boson decay was fairly insensitive to the values of the mixing angles, we find a strong dependence of $\delta\Gamma_\mu$ on $\sin\theta_{23}$. In fact figures 9 and 10 show a surprisingly different behavior of the muonic width under variation of $\sin\theta_{12}$ versus $\sin\theta_{23}$. Our results show no visible functional dependence on θ_{12} while for $m_N = 75$ GeV it increases by a factor of 20 for vanishing versus maximal θ_{23} mixing.

We mentioned before that the relative insensitivity of $\delta\Gamma_W$ to the mixing angles was due to the summation over flavors and thus the flavor symmetry of this process. In muon decay there is only a partial flavor symmetry, namely between the first and second generation. Thus mixture of these generations has no appreciable effect on $\delta\Gamma_\mu$. Allowing however an admixture of the third generation to the second as in figure 10 disturbs the flavor symmetry and leads to significant mixing effects. As we would expect from this argument, the functional dependence of $\delta\Gamma_\mu$ on $\sin\theta_{13}$ is equally pronounced as that on $\sin\theta_{23}$. Indeed the corresponding graph with $\delta\Gamma_\mu$ over $\sin\theta_{13}$ differs only very little from figure 10 and is therefore omitted.

Alternatively one can understand the θ_{12}–induced mixing as having complementary effects at the two charged current vertices leading to an overall compensation.

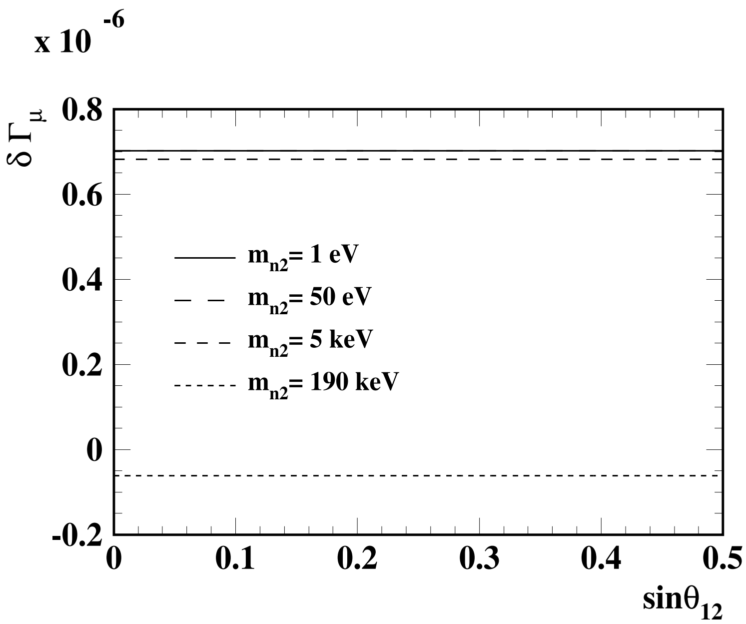

Figure 11: $\delta\Gamma_\mu$ over $\sin\theta_{12}$ with m_{n_1} and m_{n_3} at their upper bound, m_{N_i} equal to 250 GeV and m_{n_2} between 1 eV and 190 keV.

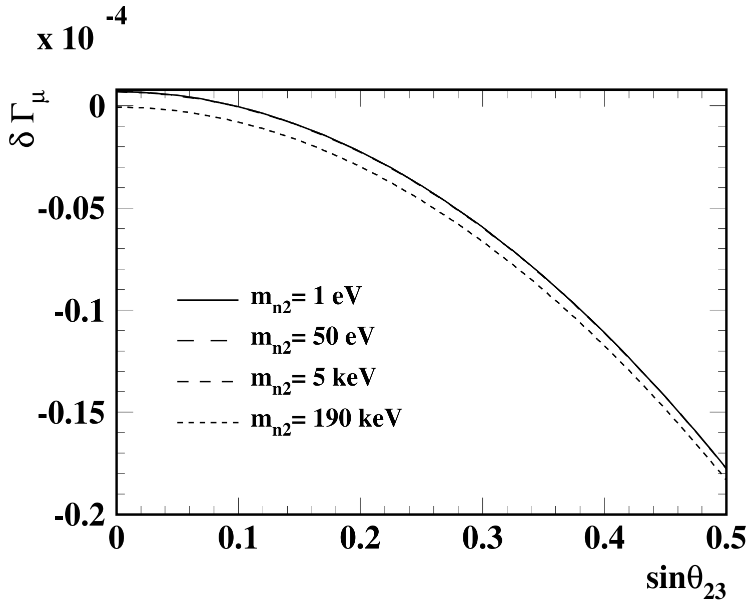

Figure 12: $\delta\Gamma_\mu$ over $\sin\theta_{23}$ with m_{n_1} and m_{n_3} at their upper bound, m_{N_i} equal to 250 GeV and m_{n_2} between 1 eV and 190 keV.

The parameter settings for figures 11 and 12 are identical to those of figure 6 of the last section, i.e. we have assumed the heavy–neutrino states to have degenerate masses of 250 GeV and the masses of the first and third light–neutrino states to be maximal in agreement with their experimental bounds. The value of m_{n_2} is varied between 1 eV and 190 keV. For arbitrary values of θ_{12} and small values of θ_{23} we find negligible deviation from the SM predictions of both signs. For larger values of $\sin\theta_{23}$ there is an increase in $\delta\Gamma_\mu$ to values comparable to those of figure 6. The sign ambiguity has the same interpretation as in figure 9, except that decoupling in this case is controlled by the smallness of m_{n_2}. For increasing values of the light–neutrino mass given a fixed m_N, the gauge couplings of the heavy states increase, leading to a stronger violation of pseudo–unitarity, as the heavy neutrinos cannot be produced in the final state. As stated earlier, this shifts $\delta\Gamma_\mu$ to negative values. In both figures the curves for the three low n_2 masses more or less coincide while only the $m_{n_2} = 190$ keV curve is appreciably separated, just as in figure 6.

Figure 13 is the counterpart of the last illustration in the previous section. We plot $\delta\Gamma_\mu$ over the common heavy–neutrino mass m_N with the light state masses at their upper bound. The curves for no mixing and maximal mixing between the first and second generation coincide as well as those for maximal θ_{23} and maximal θ_{13}. As we pointed out above, mixing with the third generation leads to appreciable effects, while due to the flavor symmetry between the first two generations in this decay process the $\sin\theta_{12} = 0.5$ curve cannot be distinguished from the graph with no mixing at all. All four curves clearly show the expected decoupling behavior in agreement with figure 7.

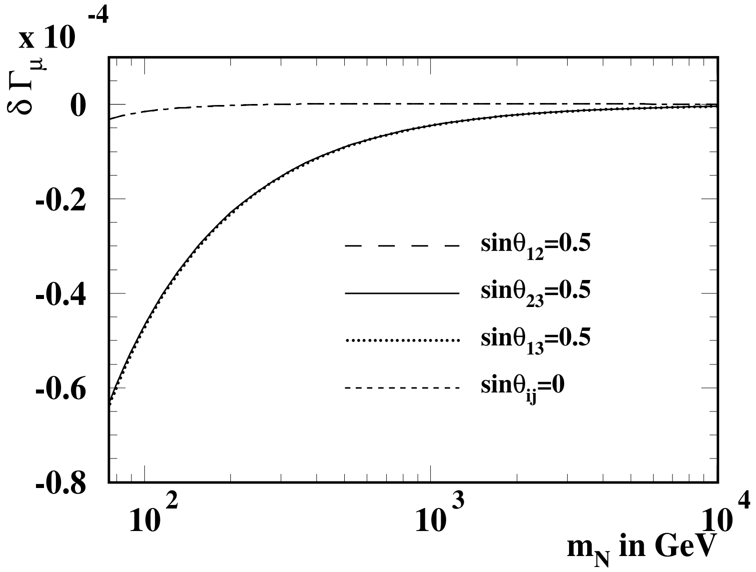

Figure 13: $\delta\Gamma_\mu$ over the common mass of the heavy neutrino states m_N for different values of the mixing angles with the m_{n_i} at their upper bounds.

The effects are of comparable magnitude in figure 7 and figure 13 which implies in the latter case that they might well be physically significant.

In figure 14 we investigate if the discrepancy of the theoretical prediction for $\delta\Gamma_\mu$ in our extended lepton sector model compared to the SM is due to virtual corrections or rather a consequence of the particular choice of couplings and masses dominated by the Born level result. As in figure 8 we plot the difference between the full one–loop result for $\delta\Gamma_\mu$ and the corresponding lowest–order quantity. For low values of m_N there is a steep rise in the quantum corrections due to the increasing mass splitting within the leptonic isodoublet. Towards larger values of the common heavy–neutrino mass this is suppressed by the decoupling of the N_i states. These competing effects lead to a pronounced maximum between 100 and 200 GeV. The impact of the isodoublet mass splitting is strongest in the W–boson propagator contributions which explains the absence of a noticeable maximum in the corresponding figure 8 for $\delta\Gamma_W$ in section 6.2. As in figure 13 we find that the solid and dotted and the two dashed curves, respectively, coincide. In contrast to the previous figure, however, we observe that the solid and the dotted curves show larger effects than the dashed graphs, especially in the region from 100 to 200 GeV where we observe relatively large positive quantum corrections.
Although in these cases the virtual corrections are relatively important we conclude again that they do not significantly affect the predictions for $\delta\Gamma_\mu$, which itself is very small.

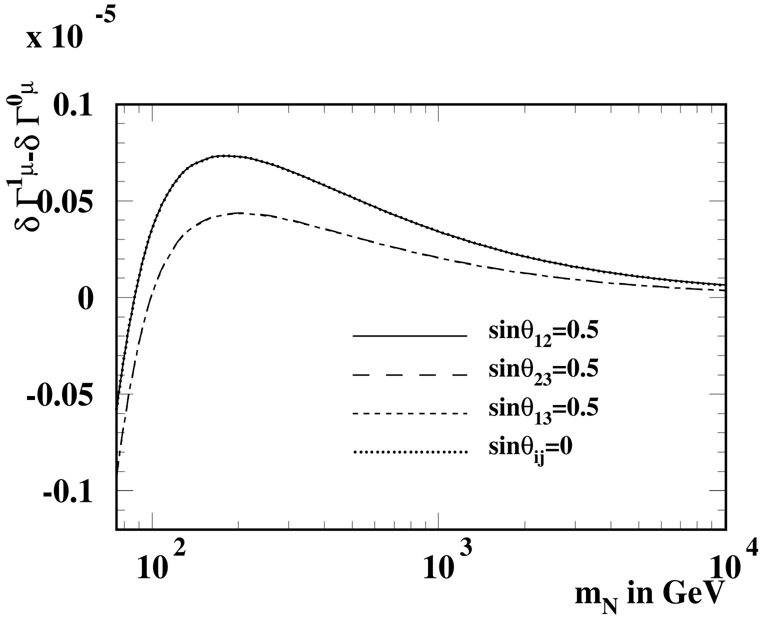

Figure 14: The difference $\delta\Gamma_\mu - \delta\Gamma_\mu^{\rm tree}$ between the one–loop and tree level relative correction of the extended lepton sector model to the SM result. All parameters are chosen as in figure 13.

In general our numerical studies present a consistent picture of the effects of our extended leptonic sector scenario up to NLO. In particular they prove in practice that our scheme indeed presents a suitable method to renormalize the leptonic mixing matrices B and C. The different flavor structures of the processes under investigation have interesting consequences for the potential observability of non–trivial mixing. It turns out that careful choice of the observable with respect to the experimental accuracy and its flavor properties could allow constraints on specific mixing angles as well as mass parameters of a see–saw Majorana neutrino scenario. Given a precision observable as the muon lifetime, the effects we see in figures 10, 12 and 13 could be physically relevant as they can be larger than the experimental uncertainty. Clearly we are aware of the fact that a value in excess of 18 ppm for $\delta\Gamma_\mu$ does not directly imply that we have found 'observable' effects which can be used to constrain our scenario's parameter space. This would either require a more detailed study involving the physical observable τ_μ^{1-2} that can directly be confronted with experimental data or careful consideration of all the uncertainties associated with the SM value of Γ_μ which entail uncertainties in $\delta\Gamma_\mu$.
We would also like to point out that in this concrete analysis we have only considered one of several possible parameterizations of the 1–2 model and that we have by far not scanned all of its parameter space.
Nevertheless our results indicate that a more detailed study, possibly considering other parameterizations and more of the mixing angle and mass value ranges, is worthwhile and will most likely produce relevant constraints on the considered model. In the present investigation the inclusion of quantum corrections did not have significant influence on our predictions.

6.4 Conclusions

In this section we briefly review contents and results of all the foregoing sections.
Following our introduction we explain the derivation of the interaction Lagrangian of a simple model which extends the SM's leptonic sector to account for neutrino masses and mixing. We focused on a see–saw scenario with only a right–handed Majorana mass term which does not require the introduction of additional Higgs fields to preserve gauge invariance. In our theoretical framework, we found relations between the mixing matrices and the neutrino spectrum.
We discussed different renormalization prescriptions for the SM CKM matrix in the on–shell scheme. An alternative method was suggested based on the renormalization condition that the mixing matrix element factorize in the renormalized amplitude. It was argued that this ansatz, unlike the ones suggested in the literature, by construction is not limited to first–order corrections because the underlying renormalization condition can be imposed at arbitrary orders.
We showed that the only renormalization prescription suggested in the literature for our Majorana neutrino scenario leads to gauge–parameter–dependent results. We argued that the failure of this procedure, among other things, stems from the violation of the mass–mixing–angle relations at higher than leading order, thus allowing for radiative breaking of gauge invariance. It was shown that in the trivial case where the intrinsic mixing is entirely parameterized in terms of neutrino masses and no angles are introduced, a mixing counterterm can be constructed in terms of mass counterterms leading to finite gauge–parameter–independent results. A 'hybrid' renormalization procedure was presented for the non–trivial case where the intrinsic mixing is renormalized in terms of neutrino mass counterterms and the additional degrees of freedom not determined by the mass–mixing–angle relations are renormalized with the aforementioned renormalization condition. This procedure by construction leads to gauge–invariant, finite transition amplitudes and is not limited to first–order corrections.
We gave generic formulae for gauge boson decays into a fermion pair at one loop in arbitrary R_ξ gauges. We applied our renormalization procedure to W– and Z–boson decays and found finite and gauge–parameter–independent transition amplitudes. We quoted analytic results for Δr in our see–saw scenario in Feynman gauge and reproduced published results in the SM limit. In contrast to the SM case, we also found neutral current contributions.
Finally we numerically investigated the relative deviation of the W–boson and muon decay width in our scenario as opposed to the SM. We numerically verified gauge parameter independence and ultraviolet and infrared stability of the decay amplitudes in our renormalization scheme with non–trivial mixing. We observed relative deviations of the order 10^{-6} to 10^{-4} and argued that in case of muon decay these could be physically significant and in a systematic study could lead to significant constraints on masses and mixing angles in a whole class of see–saw scenarios analogous to the one we considered.
In our parameterization the inclusion of quantum corrections did not have significant impact on our predictions. In the investigated regions of parameter space their contributions were at least one order of magnitude below the current experimental sensitivity. If, however, in future experiments the accuracy of electroweak precision observables

will be further improved as this was the case over the last decade, quantum corrections through a modified SM leptonic sector will inevitably play a more and more important role.

A Diagonalization of a Complex Symmetric Matrix

Given a complex, symmetric matrix M we claim, there exists a matrix U, such that

$$\tilde{M} = U^T M U$$

is positive diagonal.
It is well–known [19] that M can be made positive diagonal by a biunitary transformation with unitary matrices U_L and U_R:

$$\begin{aligned}\tilde{M} &= U_L^\dagger M U_R \\ &= U_R^T M U_L^* \, .\end{aligned}$$

This enables us to write:

$$\begin{aligned}\tilde{M}^2 &= U_L^\dagger M U_R U_R^\dagger M^* U_L = U_L^\dagger M M^* U_L \\ &= U_R^T M U_L^* U_L^T M^* U_R^* = U_R^T M M^* U_R^* \, .\end{aligned}$$

The matrix MM^* is hermitian and the unitary matrices which diagonalize MM^* are unique up to a diagonal phase matrix J. We therefore have:

$$U_R = U_L^* J$$

and

$$\tilde{M} = U_L^\dagger M U_L^* J \, .$$

Because $\tilde{M}$ and J are diagonal

$$\tilde{M} = J^{\frac{1}{2}} \tilde{M} J^{-\frac{1}{2}} = J^{\frac{1}{2}} U_L^\dagger M U_L^* J^{\frac{1}{2}} \, .$$

Defining $U = U_L^* J^{\frac{1}{2}}$, we obtain:

$$U U^\dagger = J^{\frac{1}{2}*} U_L^T U_L^* J^{\frac{1}{2}} = J^{\frac{1}{2}*} J^{\frac{1}{2}} = \mathbf{1} \, ,$$

i.e. U is unitary and we get

$$\tilde{M} = U^T M U \, ,$$

q.e.d.

B Useful Identities Involving Fermionic Currents

$$\begin{aligned}\overline{\psi}^c_R \chi_R &= (\overline{\psi}^c_R \chi_R)^T = (\psi^T C R \chi)^T = -\chi^T R^T C^T \psi \\ &= -\chi^T R\, (-C)\, \psi = \chi^T C R\, \psi \\ &= \overline{\chi}^c_R \psi_R \, ,\end{aligned}$$

$$\begin{aligned}\overline{\psi}^c_L \chi^c_R &= \psi_L^T C C \overline{\chi}_R^T = -\psi_L^T C C^{-1} \overline{\chi}_R^T \\ &= -\psi_L^T \overline{\chi}_R^T = (\overline{\chi}_R \psi_L)^T\end{aligned}$$

$$
\begin{aligned}
&= \overline{\chi}_R \psi_L \; , \\
\overline{\chi^c}\gamma_\mu\psi^c &= \chi^T C\gamma_\mu C\overline{\psi}^c = -\chi^T C\gamma_\mu C^{-1}\overline{\psi}^c = \chi^T\gamma_\mu^T\overline{\psi}^T \\
&= -(\overline{\psi}\gamma_\mu\chi)^T = -\overline{\psi}\gamma_\mu\chi \; , \\
\overline{\chi^c}\gamma_5\gamma_\mu\psi^c &= \chi^T C\gamma_5\gamma_\mu C\overline{\psi}^c = -\chi^T C\gamma_5\gamma_\mu C^{-1}\overline{\psi}^c = -\chi^T(\gamma_5\gamma_\mu)^T\overline{\psi}^T \\
&= (\overline{\psi}\gamma_5\gamma_\mu\chi)^T = \overline{\psi}\gamma_5\gamma_\mu\chi \; .
\end{aligned}
$$

C Standard 1-Loop Integrals

C.1 Definition of the Standard Integrals

We define a standard notation for the occurring loop integrals in dimensional regularization with D space–time dimensions:

$$
A_0(m_i) = \frac{\mu^{4-D}}{i\pi^2}\int \frac{d^Dq}{(2\pi)^{D-4}}\frac{1}{(q^2-m_i^2+i\epsilon)} \; ,
$$

$$
\begin{aligned}
&B_0, B_\mu, B_{\mu\nu}(k_1, m_i, m_j) = \\
&\quad \frac{\mu^{4-D}}{i\pi^2}\int \frac{d^Dq}{(2\pi)^{D-4}}\frac{1, q_\mu, q_\mu q_\nu}{(q^2-m_i^2+i\epsilon)((q+k_1)^2-m_j^2+i\epsilon)} \; ,
\end{aligned}
$$

$$
\begin{aligned}
&C_0, C_\mu, C_{\mu\nu}, C_{\mu\nu\rho}(k_1, k_2, m_i, m_j, m_k) = \\
&\quad \frac{\mu^{4-D}}{i\pi^2}\int \frac{d^Dq}{(2\pi)^{D-4}}\frac{1, q_\mu, q_\mu q_\nu, q_\mu q_\nu q_\rho}{(q^2-m_i^2+i\epsilon)((q+k_1)^2-m_j^2+i\epsilon)((q+k_2)^2-m_k^2+i\epsilon)} \; ,
\end{aligned}
$$

$$
\begin{aligned}
&D_0, D_\mu, D_{\mu\nu}, D_{\mu\nu\rho}, D_{\mu\nu\rho\sigma}(k_1, k_2, k_3, m_i, m_j, m_k, m_l) = \\
&\quad \int \frac{d^4q}{i\pi^2}\frac{1, q_\mu, q_\mu q_\nu, q_\mu q_\nu q_\rho, q_\mu q_\nu q_\rho q_\sigma}{(q^2-m_i^2+i\epsilon)((q+k_1)^2-m_j^2+i\epsilon)((q+k_2)^2-m_k^2+i\epsilon)((q+k_3)^2-m_l^2)} \; .
\end{aligned}
$$

C.2 Coefficient Functions of Standard Tensor Integrals

In order to separate the the tensorial structure from the argument–dependent scalar part of the standard tensor integrals, we define the following coefficient functions which can be calculated using a reduction algorithm given in section C.3:

$$
\begin{aligned}
B_\mu &= k_{1\mu}B_1 \; , \\
B_{\mu\nu} &= g_{\mu\nu}B_{00} + k_{1\mu}k_{1\nu}B_{11} \; , \\
C_\mu &= k_{1\mu}C_1 + k_{2\mu}C_2 \; , \\
C_{\mu\nu} &= g_{\mu\nu}C_{00} + k_{1\mu}k_{1\nu}C_{11} + k_{2\mu}k_{2\nu}C_{22} \\
&\quad +(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu})C_{12} \; ,
\end{aligned}
$$

$$
\begin{aligned}
C_{\mu\nu\rho} &= (g_{\mu\nu}k_{1\rho} + g_{\nu\rho}k_{1\mu} + g_{\mu\rho}k_{1\nu})C_{001} \\
&+(g_{\mu\nu}k_{2\rho} + g_{\nu\rho}k_{2\mu} + g_{\mu\rho}k_{2\nu})C_{002} \\
&+k_{1\mu}k_{1\nu}k_{1\rho}C_{111} + k_{2\mu}k_{2\nu}k_{2\rho}C_{222} \\
&+(k_{1\mu}k_{1\nu}k_{2\rho} + k_{1\mu}k_{2\nu}k_{1\rho} + k_{2\mu}k_{1\nu}k_{1\rho})C_{112} \\
&+(k_{2\mu}k_{2\nu}k_{1\rho} + k_{2\mu}k_{1\nu}k_{2\rho} + k_{1\mu}k_{2\nu}k_{2\rho})C_{122}\ , \\
D_{\mu} &= k_{1\mu}D_1 + k_{2\mu}D_2 + k_{3\mu}D_3\ , \\
D_{\mu\nu} &= g_{\mu\nu}D_{00} + k_{1\mu}k_{1\nu}D_{11} + k_{2\mu}k_{2\nu}D_{22} + k_{3\mu}k_{3\nu}D_{33} \\
&+(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu})D_{12} + (k_{2\mu}k_{3\nu} + k_{3\mu}k_{2\nu})D_{23} + (k_{1\mu}k_{3\nu} + k_{3\mu}k_{1\nu})D_{13} \\
D_{\mu\nu\rho} &= (g_{\mu\nu}k_{1\rho} + cycl.)D_{001} + (g_{\mu\nu}k_{2\rho} + cycl.)D_{002} + (g_{\mu\nu}k_{3\rho} + cycl.)D_{003} \\
&+(k_{1\mu}k_{1\nu}k_{2\rho} + cycl.)D_{112} + (k_{1\mu}k_{1\nu}k_{3\rho} + cycl.)D_{113} + (k_{2\mu}k_{2\nu}k_{3\rho} + cycl.)D_{223} \\
&+(k_{2\mu}k_{2\nu}k_{1\rho} + cycl.)D_{122} + (k_{3\mu}k_{3\nu}k_{1\rho} + cycl.)D_3^{133} + (k_{3\mu}k_{3\nu}k_{2\rho} + cycl.)D_{233} \\
&+(k_{1\mu}k_{2\nu}k_{3\rho} + perm.)D_{123}\ , \\
D_{\mu\nu\rho\sigma} &= g_{\mu\nu}g_{\rho\sigma}D_{0000} \\
&+ \sum_{i,j=1}^{3} (g_{\mu\nu}k_{i\rho}k_{j\sigma} + g_{\mu\rho}k_{i\nu}k_{j\sigma} + g_{\mu\sigma}k_{i\rho}k_{j\nu} \\
&+g_{\nu\rho}k_{i\mu}k_{j\sigma} + g_{\nu\sigma}k_{i\mu}k_{j\rho} + g_{\rho\sigma}k_{i\mu}k_{j\nu})D_{00ij} \\
&+ \sum_{i,j,k,l=1}^{3} k_{i\mu}k_{j\nu}k_{k\rho}k_{l\sigma}D_{ijkl}\ .
\end{aligned}
$$

C.3 Reduction of the Tensor Integrals

The coefficient functions introduced in section C.2 can be expressed solely with the help of the scalar loop integrals. In the following we will show how this reduction can be accomplished and give a brief example.
The integrand has the general structure:

$$I_n^{\mu_1\dots\mu_m} = \frac{q^{\mu_1}\dots q^{\mu_m}}{N_1\dots N_n}, \tag{5}$$

where q is the integration variable. The denominator can be written as

$$N_i = (q + p_{i-1})^2 - m_i^2.$$

Here p_0 is the zero momentum vector, the p_i with $i \geq 1$ are the external momenta and m_i the corresponding masses.
Contraction of the ith external momentum p_i with q yields

$$
\begin{aligned}
q \cdot p_i &= \frac{1}{2}(q + p_i)^2 - \frac{1}{2}q^2 - \frac{1}{2}p_i^2 \\
&= \underbrace{\frac{1}{2}(q + p_i)^2 - \frac{1}{2}m_i^2}_{=N_{i+1}} \underbrace{- \frac{1}{2}q^2 + \frac{1}{2}m_1^2}_{=-N_1} \underbrace{- \frac{1}{2}p_i^2 + \frac{1}{2}m_i^2 - \frac{1}{2}m_1^2}_{=:-f_i} \\
&= N_{i+1} - N_1 - f_i.
\end{aligned} \tag{6}
$$

Using relation(6) in contracting an external momentum with $I_n^{\mu_1 \cdots \mu_m}$ gives

$$\begin{aligned} p_{i,\mu_1} I_n^{\mu_1 \cdots \mu_m} &= \frac{p_{i,\mu_1} q^{\mu_1} ... q^{\mu_m}}{N_1 ... N_n} \\ &= \frac{(N_{i+1} - N_1 - f_i) q^{\mu_2} ... q^{\mu_m}}{N_1 ... N_n} \\ &= \frac{1}{2} \frac{q^{\mu_2} ... q^{\mu_m}}{N_1 ... N_n} - \frac{1}{2} \frac{q^{\mu_2} ... q^{\mu_m}}{N_2 ... N_n} - \frac{1}{2} f_i \frac{q^{\mu_2} ... q^{\mu_m}}{N_1 ... N_n}. \end{aligned}$$

In case $m > 2$ contraction with $g_{\mu\nu}$ leads to an additional, similar relation.
We have managed to write the integral over $I_n^{\mu_1 \cdots \mu_m}$ as a sum of three integrals each reduced by one power in the integration variable. To bring the term in the middle into the canonical form (5) we just need to shift $q \to q - p_1$.
As a simple example we perform the reduction of B_μ:

$$\begin{aligned} k_1^\mu B_\mu &= \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D q}{(2\pi)^{D-4}} \frac{k_1^\mu q_\mu}{(q^2 - m_i^2)\left((q + k_1)^2 - m_j^2\right)} \\ &= \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D q}{(2\pi)^{D-4}} \left(\frac{\frac{1}{2}(q + k_1)^2 - \frac{1}{2} m_j^2}{(q^2 - m_i^2)\left((q + k_1)^2 - m_j^2\right)} \right. \\ &\quad \left. - \frac{\frac{1}{2} q^2 - \frac{1}{2} m_i^2}{(q^2 - m_i^2)\left((q + k_1)^2 - m_j^2\right)} - \frac{\frac{1}{2} k_1^2 - \frac{1}{2} m_j - \frac{1}{2} m_i^2}{(q^2 - m_i^2)\left((q + k_1)^2 - m_j^2\right)} \right) \\ &= \frac{1}{2} A_0(m_i) - \frac{1}{2} A_0(m_j) - \frac{1}{2} f_1 B_0(k_1, m_i, m_j) \\ &= k_1^2 B_1, \end{aligned}$$

where the last line follows trivially from the definition in section C.2. Therefore B_1 reads

$$B_1 = \frac{1}{k^2} \cdot \left(\frac{1}{2} A_0(m_i) - \frac{1}{2} A_0(m_j) - \frac{1}{2} f_1 B_0(k_1, m_i, m_j) \right).$$

C.4 Abbreviations of Coefficient Functions

In section 3 we employed the following abbreviations to quote our generic analytic result in a more compact form:

$$\begin{aligned} B[1] &= B_0(0, m_{f_i}^2, m_{V_1}^2) \quad = B[3] \text{ if } V_1 \to V_2\,, \\ B[2] &= B_0(0, m_{f_i}^2, \xi_{V_1} m_{V_1}^2) \quad = B[4] \text{ if } V_1 \to V_2\,, \\ B[5] &= B_0(0, m_{V_2}^2, m_{V_2}^2)\,, \\ B[6] &= B_0(0, \xi_{V_2} m_{V_2}^2, \xi_{V_2} m_{V_2}^2)\,, \\ B[7] &= B_0(m_{f_2}^2, m_{f_j}^2, m_{V_1}^2) \quad = B[10] \text{ if } f_j \to f_i\,, \\ B[8] &= B_0(m_{f_2}^2, m_{f_j}^2, \xi_{S_1} m_{S_1}^2) \quad = B[9] \text{ if } S \to V\,, \\ B[11] &= B_0(m_{f_2}^2, m_{f_i}^2, \xi_{V_1} m_{V_1}^2)\,. \end{aligned}$$

All C and $\tilde{C}$ functions depend on the same momenta squared:

$$C_{...}[j] = C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, a, b, c)\,,$$

where the ellipses stand for the indices introduced above and where the value of j depends on the form factor under consideration and represents certain values for a, b, c. For the $\tilde{C}$ functions the number in brackets corresponds to a, b, c in the following way:

$$\begin{aligned}
\tilde{C}_{...}[1] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, m_{f_i}^2, m_{V_1}, m_{f_j}^2)\,,\\
\tilde{C}_{...}[2] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, m_{f_i}^2, \xi_{V_1} m_{V_1}, m_{f_j}^2)\,,
\end{aligned}$$

in all $F(i)$, except for $F(2)$, and V_1 replaced by S_1 in $F(2)$.
The $C_{...}[i]$ in $F(6)$ are defined as:

$$\begin{aligned}
C_{...}[1] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, m_{V_1}^2, m_{f_i}^2, m_{V_2}^2)\,,\\
C_{...}[2] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, m_{V_1}^2, m_{f_i}^2, \xi_{V_2} m_{V_2}^2)\,,\\
C_{...}[3] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, \xi_{V_1} m_{V_1}^2, m_{f_i}^2, m_{V_2}^2)\,,\\
C_{...}[4] &= C_{...}(m_{f_1}^2, m_{f_2}^2, m_V^2, \xi_{V_1} m_{V_1}^2, m_{f_i}^2, \xi_{V_2} m_{V_2}^2)\,.
\end{aligned}$$

In $F(5)$ one should replace V_2 by S_1, in $F(4)$ V_1 should be substituted by S_1 and V_2 by V_1.

C.5 The Result for the Form Factors $F_i^\sigma(j)$

$$\begin{aligned}
F_1^\sigma(1) &= \frac{1}{16\pi^2}(g^\sigma_{S_1 f_2 f_j} m_{f_2}(g^{\tilde{\sigma}}_{S_1 f_i f_1} g^\sigma_{V f_j f_i} m_{f_1}\tilde{C}_1[2] - g^\sigma_{S_1 f_i f_1}(-(g^{\tilde{\sigma}}_{V f_j f_i} m_{f_j}\tilde{C}_2[2])\\
&+g^\sigma_{V f_j f_i} m_{f_i}(\tilde{C}_2[2] + \tilde{C}[2]))) + g^{\tilde{\sigma}}_{S_1 f_2 f_j}(g^\sigma_{S_1 f_i f_1} g^{\tilde{\sigma}}_{V f_j f_i}(B[8] - 2\tilde{C}_{00}[2]\\
&+m_{f_1}^2\tilde{C}_1[2] + m_{V_1}^2\tilde{C}_2[2]) + g^\sigma_{S_1 f_i f_1} m_{f_i}(g^{\tilde{\sigma}}_{V f_j f_i} m_{f_i} - g^\sigma_{V f_j f_i} m_{f_j})\tilde{C}[2]\\
&+g^{\tilde{\sigma}}_{S_1 f_i f_1} m_{f_1}(g^\sigma_{V f_j f_i} m_{f_j}(\tilde{C}_1[2] + \tilde{C}_2[2]) - g^{\tilde{\sigma}}_{V f_j f_i} m_{f_i}(\tilde{C}_1[2] + \tilde{C}_2[2] + \tilde{C}[2]))))\,,
\end{aligned}$$

$$\begin{aligned}
F_2^\sigma(1) &= \frac{-1}{8\pi^2}(g^{\tilde{\sigma}}_{S_1 f_2 f_j} g^\sigma_{S_1 f_i f_1} g^{\tilde{\sigma}}_{V f_j f_i} m_{f_2}\tilde{C}_{12}[2] + g^\sigma_{S_1 f_2 f_j}(-(g^{\tilde{\sigma}}_{S_1 f_i f_1} g^\sigma_{V f_j f_i} m_{f_1}(\tilde{C}_{11}[2]\\
&+\tilde{C}_{12}[2] + \tilde{C}_1[2])) + g^\sigma_{S_1 f_i f_1}(-(g^{\tilde{\sigma}}_{V f_j f_i} m_{f_j}\tilde{C}_2[2]) + g^\sigma_{V f_j f_i} m_{f_i}(\tilde{C}_1[2] + \tilde{C}_2[2]\\
&+\tilde{C}[2]))))\,,
\end{aligned}$$

$$F_1^\sigma(2) = \frac{1}{8\pi^2} g^{\tilde{\sigma}}_{S_2 f_2 f_i} g^\sigma_{S_1 f_i f_1} g_{V S_1 S_2} C_{00}[4]\,,$$

$$\begin{aligned}
F_2^\sigma(2) &= \frac{-1}{8\pi^2} g_{V S_1 S_2}(-(g^{\tilde{\sigma}}_{S_2 f_2 f_i} g^\sigma_{S_1 f_i f_1} m_{f_2} C_{12}[4]) + g^\sigma_{S_2 f_2 f_i}(g^\sigma_{S_1 f_i f_1} m_{f_i} C_1[4]\\
&+g^{\tilde{\sigma}}_{S_1 f_i f_1} m_{f_1}(C_1[4] + C_{11}[4] + C_{12}[4])))\,,
\end{aligned}$$

$$F_1^\sigma(3) = \frac{1}{32 m_{V_1}^2\pi^2}(-(g^\sigma_{V_1 f_2 f_j}(-(g^\sigma_{V_1 f_i f_1}(-(g^\sigma_{V f_j f_i}(-((2m_{f_1}^2 + m_{f_2}^2 - 2m_{f_i}^2)(B[7] - B[9]))$$

$$
\begin{aligned}
&+m_{f_j}^2(-B[1]+B[2]+B[7]-B[9])+2m_{f_i}^2(-m_{f_2}^2+m_{f_i}^2+m_V^2)\tilde{C}[1]\\
&m_{V_1}^2(2+B[1]-3B[7]-\xi_{V_1}(B[2]+B[9])+8\tilde{C}_{00}[1]+4m_{f_2}^2\tilde{C}_1[1]\\
&-4m_V^2(\tilde{C}_1[1]+\tilde{C}_2[1])-4m_{f_i}^2\tilde{C}[1])+2(m_{f_1}^4(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2])\\
&+m_{f_1}^2(2\tilde{C}_{00}[1,2]-m_{f_2}^2(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2]+\tilde{C}_1[1,2])+m_{f_i}^2(2\tilde{C}_1[1,2]+\\
&\tilde{C}_2[1,2])+m_V^2(\tilde{C}_{11}[1,2]+3\tilde{C}_{12}[1,2]+\tilde{C}_1[1,2]+2\tilde{C}_{22}[1,2]+\tilde{C}_2[1,2]))\\
&-m_{f_i}^2(-(m_V^2(\tilde{C}_1[1,2]+2\tilde{C}_2[1,2]))+m_{f_2}^2(\tilde{C}_1[1,2]+\tilde{C}_2[1,2]-\tilde{C}[2])\\
&+(m_{f_i}^2+m_V^2)\tilde{C}[2]))))+2g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_i}m_{f_j}(B[7]-B[9]-2\tilde{C}_{00}[1,2]\\
&+(-m_{f_2}^2+m_V^2)\tilde{C}_2[1,2]-2m_{V_1}^2\tilde{C}[1]+m_{f_i}^2\tilde{C}[1,2]+m_{f_1}^2(2\tilde{C}_1[1,2]\\
&+\tilde{C}_2[1,2]+\tilde{C}[1,2]))))+2g^{\tilde{\sigma}}_{V_1f_if_1}m_{f_1}(-(g^{\sigma}_{Vf_jf_i}m_{f_i}(2\tilde{C}_{00}[1,2]+m_{f_1}^2(\tilde{C}_{11}[1,2]\\
&+\tilde{C}_{12}[1,2]+\tilde{C}_1[1,2])+m_{f_i}^2(\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2])-m_{f_2}^2(\tilde{C}_{11}[1,2]\\
&+\tilde{C}_{12}[1,2]+2\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2])+m_V^2(\tilde{C}_{11}[1,2]+3\tilde{C}_{12}[1,2]\\
&+2\tilde{C}_1[1,2]+2\tilde{C}_{22}[1,2]+3\tilde{C}_2[1,2]+\tilde{C}[1,2])))+g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_j}(B[7]-B[9]\\
&-2\tilde{C}_{00}[1,2]+m_{f_1}^2\tilde{C}_1[1,2]+(-m_{f_2}^2+m_V^2)\tilde{C}_2[1,2]+m_{f_i}^2(\tilde{C}_1[1,2]+\tilde{C}_2[1,2]\\
&+2\tilde{C}[1,2])))))+\frac{1}{m_{f_2}}(g^{\tilde{\sigma}}_{V_1f_2f_j}(g^{\tilde{\sigma}}_{V_1f_if_1}g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_1}(m_{f_j}^2(B[1]-B[2]-B[7]\\
&+B[9])+m_{V_1}^2(-B[1]+\xi_{V_1}B[2]+B[7]-\xi_{V_1}B[9]-4m_{f_2}^2\tilde{C}_1[1])m_{f_2}^2(3B[7]\\
&-3B[9]+2(-4\tilde{C}_{00}[1,2]+(m_{f_2}^2-m_V^2)\tilde{C}_{12}[1,2]+m_{f_1}^2(-\tilde{C}_{12}[1,2]+\tilde{C}_1[1,2])\\
&-2m_V^2\tilde{C}_{22}[1,2]+m_{f_i}^2(\tilde{C}_1[1,2]+2\tilde{C}[1,2]))))+g^{\sigma}_{V_1f_if_1}(g^{\sigma}_{Vf_jf_i}m_{f_j}(m_{f_j}^2(B[1]\\
&-B[2]-B[7]+B[9])+m_{V_1}^2(-B[1]+\xi_{V_1}B[2]+B[7]-\xi_{V_1}B[9])+m_{f_2}^2(B[7]\\
&-B[9]+2(m_{f_1}^2-m_{f_i}^2)\tilde{C}_2[1,2]))-g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_i}(m_{f_j}^2(B[1]-B[2]-B[7]+B[9])\\
&+m_{V_1}^2(-B[1]+\xi_{V_1}B[2]+B[7]-\xi_{V_1}B[9])+m_{f_2}^2(3B[7]-3B[9]\\
&+2(-4\tilde{C}_{00}[1,2]+(m_{f_2}^2-m_V^2)\tilde{C}_{12}[1,2]-2m_V^2\tilde{C}_{22}[1,2]+m_{f_i}^2(-\tilde{C}_2[1,2]\\
&+\tilde{C}[1,2])+m_{f_1}^2(-\tilde{C}_{12}[1,2]+2\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2])))))))\,,
\end{aligned}
$$

$$
\begin{aligned}
F_2^{\sigma}(3) \;=\; & \frac{-1}{16m_{f_2}^2m_{V_1}^2\pi^2}(g^{\tilde{\sigma}}_{V_1f_2f_j}(g^{\tilde{\sigma}}_{V_1f_if_1}m_{f_1}(g^{\tilde{\sigma}}_{Vf_jf_i}(m_{f_j}^2(-B[1]+B[2]+B[7]-B[9])\\
&+m_{V_1}^2(B[1]-B[7]+\xi_{V_1}(-B[2]+B[9])-4m_{f_2}^2(\tilde{C}_{11}[1]+\tilde{C}_{12}[1]))+m_{f_2}^2\\
&\times(-B[7]+B[9]+2(2\tilde{C}_{00}[1,2]+m_{f_1}^2\tilde{C}_{12}[1,2]+m_{f_i}^2\tilde{C}_2[1,2]+m_V^2(\tilde{C}_{12}[1,2]\\
&+2\tilde{C}_{22}[1,2]+\tilde{C}_2[1,2]))))-2g^{\sigma}_{Vf_jf_i}m_{f_2}^2m_{f_i}m_{f_j}(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2]+2\tilde{C}_1[1,2]\\
&+\tilde{C}_2[1,2]+\tilde{C}[1,2]))-g^{\sigma}_{V_1f_if_1}(g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_i}(m_{f_j}^2(-B[1]+B[2]+B[7]-B[9])\\
&+m_{V_1}^2(B[1]-\xi_{V_1}B[2]-B[7]+\xi_{V_1}B[9]+4m_{f_2}^2\tilde{C}_1[1])+m_{f_2}^2(-B[7]+B[9]\\
&+2(2\tilde{C}_{00}[1,2]+m_{f_1}^2\tilde{C}_{12}[1,2]+m_{f_i}^2\tilde{C}_2[1,2]+m_V^2(\tilde{C}_{12}[1,2]+2\tilde{C}_{22}[1,2]\\
&+\tilde{C}_2[1,2]))))-2g^{\sigma}_{Vf_jf_i}m_{f_2}^2m_{f_j}(-2m_{V_1}^2\tilde{C}_1[1]+m_{f_1}^2(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2]\\
&+\tilde{C}_1[1,2])+m_{f_i}^2(\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2]))))+2g^{\sigma}_{V_1f_2f_j}m_{f_2}^3(g^{\tilde{\sigma}}_{V_1f_if_1}m_{f_1}\\
&\times(-(g^{\tilde{\sigma}}_{Vf_jf_i}m_{f_j}(\tilde{C}_{12}[1,2]+\tilde{C}_2[1,2]))+g^{\sigma}_{Vf_jf_i}m_{f_i}(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2]
\end{aligned}
$$

$$+2\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2])) - g^{\sigma}_{V_1 f_i f_1}(-(g^{\tilde{\sigma}}_{V f_j f_i} m_{f_i} m_{f_j}(\tilde{C}_{12}[1,2]+\tilde{C}_2[1,2]))$$
$$+g^{\sigma}_{V f_j f_i}(-2m^2_{V_1}(\tilde{C}_{12}[1]+\tilde{C}_1[1]) + m^2_{f_1}(\tilde{C}_{11}[1,2]+\tilde{C}_{12}[1,2]+\tilde{C}_1[1,2])$$
$$+m^2_{f_i}(\tilde{C}_1[1,2]+\tilde{C}_2[1,2]+\tilde{C}[1,2]))))) \,,$$

$$F_1^\sigma(4) = \frac{1}{16m^2_{V_1}\pi^2} g_{S_1VV_1}(g^{\tilde{\sigma}}_{V_1 f_2 f_i} g^{\sigma}_{S_1 f_i f_1} m_{f_2}(C_{00}[3]-C_{00}[4]+m^2_{V_1}C_2[3])$$
$$+g^{\sigma}_{V_1 f_2 f_i}(g^{\sigma}_{S_1 f_i f_1} m_{f_i}(m^2_{V_1}C[3]-C_{00}[3]+C_{00}[4]) + g^{\tilde{\sigma}}_{S_1 f_i f_1} m_{f_1} m^2_{V_1}(C[3]$$
$$+C_1[3]+C_2[3]))) \,,$$

$$F_2^\sigma(4) = \frac{-1}{32m^2_{f_2}m^2_{V_1}\pi^2} g_{S_1VV_1}(2g^{\sigma}_{V_1 f_2 f_i} m^3_{f_2}(g^{\sigma}_{S_1 f_i f_1} m_{f_i}(C_1[3]-C_1[4]+C_{12}[3]-C_{12}[4])$$
$$+g^{\tilde{\sigma}}_{S_1 f_i f_1} m_{f_1}(C_1[3]-C_1[4]+C_{11}[3]-C_{11}[4]+C_{12}[3]-C_{12}[4])) + g^{\tilde{\sigma}}_{V_1 f_2 f_i}$$
$$\times(2g^{\tilde{\sigma}}_{S_1 f_i f_1} m_{f_1} m^2_{f_2} m_{f_i}(-C_1[3]+C_1[4]-C_{11}[3]+C_{11}[4]-C_{12}[3]+C_{12}[4])$$
$$+g^{\sigma}_{S_1 f_i f_1}(m^2_{f_i}(B[3]-B[4]-B[10]+B[11]) + m^2_{f_2}(B[10]-B[11]$$
$$+2(-2C_{00}[3]+2C_{00}[4]+m^2_{S_1}\xi_{S_1}(-C_1[3]+C_1[4]) + m^2_{V_1}(-C_{12}[3]+C_{12}[4])$$
$$+m^2_{f_1}(-C_1[3]+C_1[4]-2C_{11}[3]+2C_{11}[4]-C_{12}[3]+C_{12}[4]))) + m^2_{V_1}(-B[3]$$
$$+\xi_{V_1}B[4]+B[10]-\xi_{V_1}B[11]-4m^2_{f_2}C_2[3])))) \,,$$

$$F_1^\sigma(5) = \frac{-1}{16m^2_{V_1}\pi^2} g_{S_1VV_1}(g^{\tilde{\sigma}}_{S_2 f_2 f_i}(g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} - g^{\sigma}_{V_1 f_i f_1} m_{f_i})(m^2_{V_1}C[2]-C_{00}[2]+C_{00}[4])$$
$$+g^{\tilde{\sigma}}_{S_2 f_2 f_i} g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} m^2_{V_1} C_1[2] + (g^{\tilde{\sigma}}_{S_2 f_2 f_i} g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} + g^{\sigma}_{S_2 f_2 f_i} g^{\sigma}_{V_1 f_i f_1} m_{f_2})$$
$$\times m^2_{V_1} C_2[2]) \,,$$

$$F_2^\sigma(5) = \frac{-1}{16m^2_{V_1}\pi^2} g_{S_1VV_1}(-(g^{\tilde{\sigma}}_{S_2 f_2 f_i} m_{f_2}(g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} - g^{\sigma}_{V_1 f_i f_1} m_{f_i})(C_{12}[2]-C_{12}[4]))$$
$$+g^{\sigma}_{S_2 f_2 f_i}(g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} m_{f_i}(-C_{11}[2]+C_{11}[4]-C_{12}[2]+C_{12}[4]) + g^{\sigma}_{V_1 f_i f_1}$$
$$\times(-2C_{00}[2]+2C_{00}[4]+m^2_{f_1}(-C_{11}[2]+C_{11}[4]) + (m^2_{f_2}-m^2_{V_1})(C_{12}[2]$$
$$-C_{12}[4]) + m^2_{V_1}(2C[2]+C_1[2]+\xi_{V_1}C_1[4]+2C_2[2])))) \,,$$

$$F_1^\sigma(6) = \frac{-1}{192m^2_{V_1}m^2_{V_2}\pi^2} g_{VV_1V_2}(\frac{1}{m_{f_2}}(6g^{\tilde{\sigma}}_{V_1 f_2 f_i}(-2g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1} m^2_{f_2}(-(m^2_{V_1}C_{00}[1,4])$$
$$-m^4_{V_1}(C[1,2]+C_1[1,2]+C_2[1,2]) + m^2_{V_1}(C_{00}[1,2]+m^2_{V_2}(-3C[1]-3C_1[1]$$
$$+C_2[1]-\xi_{V_1}C_2[3]) + m^2_{V_1}(C[1,2]+C_1[1,2]+C_2[1,2])) + m^2_{V_2}(3C_{00}[1,3]$$
$$+(m^2_{f_1}-m^2_{f_2}+m^2_{V_1})C_{12}[1,3]+2m^2_{V_1}C_{22}[1,3])) + g^{\sigma}_{V_1 f_i f_1} m_{f_i}(m^2_{f_i} m^2_{V_1}$$
$$\times(-B[3]+B[4]+B[10]-B[11]) - 2m^2_{f_2} m^4_{V_1}(C[1,2]+C_2[1,2]) + m^2_{V_1}$$
$$\times(m^2_{V_2}(B[3]-B[10]+\xi_{V_2}(-B[4]+B[11]) + 2m^2_{f_2}(-3C[1]+C_2[1]$$
$$-\xi_{V_1}C_2[3])) + m^2_{f_2}(-B[10]+B[11]+2(C_{00}[1,2]+m^2_{V_1}(C[1,2]+C_2[1,2]))))$$
$$+2m^2_{f_2}(-(m^2_{V_1}C_{00}[1,4]) + m^2_{V_2}(3C_{00}[1,3]+(m^2_{f_1}-m^2_{f_2}+m^2_{V_1})C_{12}[1,3]$$

$$
\begin{aligned}
&+2m_{V_1}^2C_{22}[1,3]))))) + g^{\sigma}_{V_1f_2f_i}(12g^{\tilde{\sigma}}_{V_1f_if_1}m_{f_1}m_{f_i}(m_{V_2}^2(3C_{00}[1,3]\\
&+(m_{f_1}^2-m_{f_2}^2)(C_{11}[1,3]+C_{12}[1,3]))+m_{V_1}^2(C_{00}[1,2]+m_{V_2}^2(-3C[1]+C_1[1]\\
&+C_2[1]-\xi_{V_1}(C_1[3]+C_2[3])))-m_{V_1}^4(C[1,2]+C_1[1,2]+C_2[1,2])\\
&+m_{V_1}^2(-C_{00}[1,4]+m_{V_1}^2(C[1,2]+C_1[1,2]+C_2[1,2])+m_{V_2}^2(C_{11}[1,3]\\
&+3C_{12}[1,3]+2C_{22}[1,3])))+\frac{1}{m_{f_2}^2}(g^{\sigma}_{V_1f_if_1}(m_{V_1}^2(m_{V_2}^4(-B[3]+B[10]+\xi_{V_2}^2(B[4]\\
&-B[11]))+m_{f_i}^4(-B[3]+B[4]+B[10]-B[11])+2m_{f_i}^2m_{V_2}^2(B[3]-\xi_{V_2}B[4]\\
&-B[10]+\xi_{V_2}B[11]))+12m_{f_2}^6m_{V_2}^2C_{12}[1,3]+m_{f_2}^4(m_{V_1}^2(-5B[10]+5B[11]\\
&+12(C_{002}[1,2]+m_{V_2}^2(2C[1]+C_1[1]+\xi_{V_1}(C_1[3]+C_2[3]))))\\
&-12(m_{V_1}^2C_{002}[1,4]+m_{V_2}^2(2C_{00}[1,3]-C_{002}[1,3]+m_{f_1}^2(C_{11}[1,3]+C_{12}[1,3])\\
&+2m_{V_1}^2(C_{12}[1,3]+C_{22}[1,3])))))+m_{f_2}^2(12m_{V_1}^6C[1,2]+12m_{V_1}^4(B[10]-B[11]\\
&-m_{V_2}^2(C[1]+\xi_{V_1}^2C[3])-C_{00}[1,2]+m_{V_1}^2(-C[1,2]+C_2[1,2])+m_{f_1}^2(C[1,2]\\
&+2C_1[1,2]+C_2[1,2]))+m_{V_1}^2(m_{f_i}^2(5B[3]-5B[4]+4B[10]-4B[11])\\
&+m_{V_2}^2(-5B[3]+\xi_{V_2}(5B[4]-2(4+5B[6]+2B[11]))-2(-5B[5]+2(-5\\
&+(2+3\xi_{V_1})B[10]+15C_{00}[1]-3\xi_{V_1}C_{00}[3]-3m_{f_1}^2(C[1]+C_1[1]-2\xi_{V_1}C_1[3]\\
&-C_2[1])+3m_{V_1}^2(2C[1]+C_1[1]+\xi_{V_1}C_1[3]-C_2[1]+3\xi_{V_1}C_2[3]))))\\
&-12(m_{f_1}^2(C_{00}[1,2]+2C_{001}[1,2]+C_{002}[1,2])+m_{V_1}^4C_2[1,2]+m_{V_1}^2(B[10]\\
&-B[11]+\xi_{V_1}C_{00}[3]-\xi_{V_1}C_{00}[4]-C_{00}[1,2]+C_{002}[1,2]+m_{f_1}^2(C[1,2]\\
&+2C_1[1,2]+C_2[1,2]))))+12(m_{f_1}^4m_{V_2}^2C_{11}[1,3]+m_{f_1}^2(-(m_{V_2}^2(C_{00}[1,3]\\
&+2C_{001}[1,3]+C_{002}[1,3]))+m_{V_1}^2(C_{00}[1,4]+2C_{001}[1,4]+C_{002}[1,4]\\
&+m_{V_2}^2(C_{11}[1,3]+3C_{12}[1,3])))+m_{V_1}^2(m_{V_1}^2C_{002}[1,4]+m_{V_2}^2(2C_{00}[1,3]\\
&-C_{002}[1,3]+m_{V_1}^2(C_{12}[1,3]+2C_{22}[1,3]))))))))),
\end{aligned}
$$

$$
\begin{aligned}
F_2^{\sigma}(6) \;=\; &\frac{1}{96m_{f_2}^3m_{V_1}^2m_{V_2}^2\pi^2}g_{VV_1V_2}(g^{\sigma}_{V_1f_2f_i}(6g^{\tilde{\sigma}}_{V_1f_if_1}m_{f_1}m_{f_2}^4m_{f_i}(m_{V_1}^2C_{11}[1,2]\\
&+m_{V_2}^2C_{11}[1,3]-m_{V_1}^2C_{11}[1,4])+g^{\sigma}_{V_1f_if_1}(m_{f_2}^2m_{V_1}^2(4m_{f_i}^2(-B[3]+B[4]\\
&+B[10]-B[11])+m_{V_2}^2(-1+\xi_{V_2}+4B[3]-4\xi_{V_2}B[4]-2B[5]+2\xi_{V_2}B[6]\\
&-2B[10]+2\xi_{V_2}B[11]))+2m_{V_1}^2(m_{f_i}^4(B[3]-B[4]-B[10]+B[11])\\
&+2m_{f_i}^2m_{V_2}^2(-B[3]+\xi_{V_2}B[4]+B[10]-\xi_{V_2}B[11])+m_{V_2}^4(B[3]-B[10]\\
&+\xi_{V_2}^2(-B[4]+B[11])))-6m_{f_2}^6(m_{V_1}^2(C_{12}[1,2]+C_{122}[1,2])+m_{V_2}^2(C_{12}[1,3]\\
&+C_{122}[1,3])-m_{V_1}^2(C_{12}[1,4]+C_{122}[1,4]))-2m_{f_2}^4(3(-(m_{V_2}^2(2C_{00}[1,3]\\
&+2C_{002}[1,3]+m_{V_1}^2(C_{12}[1,3]+C_{122}[1,3])+m_{f_1}^2(C_{11}[1,3]+2C_{112}[1,3]\\
&+C_{12}[1,3]+C_{122}[1,3])))+m_{V_1}^2(2C_{00}[1,4]+2C_{002}[1,4]+m_{V_1}^2(C_{12}[1,4]\\
&+C_{122}[1,4])+m_{f_1}^2(C_{11}[1,4]+2C_{112}[1,4]+C_{12}[1,4]+C_{122}[1,4])))\\
&+3m_{V_1}^4(2C[1,2]+C_1[1,2]-C_{12}[1,2]+2C_2[1,2])+m_{V_1}^2(B[10]-B[11]\\
&-3(2C_{00}[1,2]+2C_{002}[1,2]+m_{f_1}^2(C_{11}[1,2]+2C_{112}[1,2]+C_{12}[1,2]
\end{aligned}
$$

$$
\begin{aligned}
&+C_{122}[1,2]) - m_{V_2}^2(2C[1] + C_1[1] - 5C_{12}[1] + \xi_{V_1}(C_1[3] + C_{12}[3]) - 2C_2[1]) \\
&+m_{V_1}^2(2C[1,2] + C_1[1,2] + \xi_{V_1}(C_1[3] - C_1[4] + C_{12}[3] - C_{12}[4]) + C_{122}[1,2] \\
&+2C_2[1,2])))))) + 3g^{\tilde{\sigma}}_{V_1 f_2 f_i} m_{f_2}(-(g^{\tilde{\sigma}}_{V_1 f_i f_1} m_{f_1}(m_{f_i}^2 m_{V_1}^2(B[3] - B[4] - B[10] \\
&+B[11]) + 2m_{f_2}^2 m_{V_1}^4(C_{11}[1,2] + C_{12}[1,2]) + m_{V_1}^2(m_{f_2}^2(B[10] - B[11] \\
&+2(4C_{001}[1,2] + 2C_{002}[1,2] + m_{V_1}^2(-C_{11}[1,2] + C_{112}[1,2] + \xi_{V_1}(C_{11}[3] \\
&-C_{11}[4] + C_{12}[3] - C_{12}[4]) - C_{12}[1,2] + C_{122}[1,2]) - m_{f_2}^2(C_{112}[1,2] \\
&+C_{12}[1,2] + C_{122}[1,2]) + m_{f_1}^2(C_{11}[1,2] + 2C_{111}[1,2] + 3C_{112}[1,2] + C_{12}[1,2] \\
&+C_{122}[1,2]))) + m_{V_2}^2(-B[3] + B[10] + \xi_{V_2}(B[4] - B[11]) + 2m_{f_2}^2(4C[1] \\
&+8C_1[1] + 5(C_{11}[1] + C_{12}[1]) - \xi_{V_1}(C_{11}[3] + C_{12}[3] - 2C_2[3])))) \\
&+2m_{f_2}^2(m_{V_1}^2(-4C_{001}[1,4] - 2C_{002}[1,4] - m_{V_1}^2(C_{112}[1,4] + C_{122}[1,4]) \\
&+m_{f_2}^2(C_{112}[1,4] + C_{12}[1,4] + C_{122}[1,4]) - m_{f_1}^2(C_{11}[1,4] + 2C_{111}[1,4] \\
&+3C_{112}[1,4] + C_{12}[1,4] + C_{122}[1,4])) + m_{V_2}^2(-4C_{00}[1,3] + 4C_{001}[1,3] \\
&+2C_{002}[1,3] - m_{f_2}^2(C_{112}[1,3] - C_{12}[1,3] + C_{122}[1,3]) + m_{f_1}^2(C_{11}[1,3] \\
&+2C_{111}[1,3] + 3C_{112}[1,3] - C_{12}[1,3] + C_{122}[1,3]) + m_{V_1}^2(C_{112}[1,3] \\
&-2C_{12}[1,3] + C_{122}[1,3] - 4C_{22}[1,3]))))) + g^{\sigma}_{V_1 f_i f_1} m_{f_i}(m_{f_i}^2 m_{V_1}^2(B[3] - B[4] \\
&-B[10] + B[11]) + 2m_{f_2}^2 m_{V_1}^4(2C[1,2] + C_1[1,2] + 2C_2[1,2]) + m_{V_1}^2(m_{V_2}^2 \\
&\times(-B[3] + B[10] + \xi_{V_2}(B[4] - B[11]) + 2m_{f_2}^2(6C[1] + 5C_1[1] + \xi_{V_1}C_1[3] \\
&-2C_2[1] + 2\xi_{V_1}C_2[3])) + m_{f_2}^2(B[10] - B[11] - 2(2C_{00}[1,2] + m_{f_1}^2 C_{11}[1,2] \\
&+m_{V_1}^2(2C[1,2] + \xi_{V_1}C_1[3] - \xi_{V_1}C_1[4] + C_1[1,2] + C_{12}[1,2] + 2C_2[1,2])))) \\
&+2m_{f_2}^2(m_{V_1}^2(2C_{00}[1,4] + m_{f_1}^2 C_{11}[1,4] + m_{V_1}^2 C_{12}[1,4]) - m_{V_2}^2(6C_{00}[1,3] \\
&-2m_{f_2}^2 C_{12}[1,3] + m_{f_1}^2(C_{11}[1,3] + 2C_{12}[1,3]) + m_{V_1}^2(3C_{12}[1,3] \\
&+4C_{22}[1,3])))))) \,,
\end{aligned}
$$

where $\tilde{\sigma}$ just denotes $-\sigma$.

C.6 Coefficient Functions for Vanishing External Momenta

In case the external momenta and their scalar products can be neglected in comparison to the masses of the virtual particles in the loop, the coefficient functions acquire a particularly simple form. We briefly present our method of deriving the zero–external–momentum limit of the coefficient functions through a generating functional in the most difficult case of the 4-point functions:

$$
\begin{aligned}
F_4(z; k_1, k_2, k_3, m_i, m_j, m_k, m_l) &:= \frac{\mu^{4-D}}{i\pi^2}\int \frac{d^D q}{(2\pi)^{D-4}} \\
&\times \frac{e^{iqz}}{(q^2 - m_i^2 + i\epsilon)((q+k_1)^2 - m_j^2 + i\epsilon)((q+k_2)^2 - m_k^2 + i\epsilon)((q+k_3)^2 - m_l^2 + i\epsilon)} \,.
\end{aligned}
$$

Therefore we have

$$
D_\mu(\ldots) \quad = \quad (-i)\,\frac{\partial}{\partial z^\mu} F_4(z;\ldots)\bigg|_{z=0} \,,
$$

$$D_{\mu\nu}(\ldots) = (-i)^2 \frac{\partial^2}{\partial z^\mu \partial z^\nu} F_4(z;\ldots)\bigg|_{z=0},$$

and the corresponding higher derivatives $D_{\mu\nu\rho}, D_{\mu\nu\rho\sigma}$.
Now $F_4(z;\ldots)$ can be written as

$$\begin{aligned}
F_4(z;\ldots) &= i\int_0^\infty \frac{d^4q}{i\pi^2}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4 \\
&\quad\times e^{i\left(qz+\alpha_1(q^2-m_i^2)+\alpha_2((q+k_1)^2-m_j^2)+\alpha_3((q+k_2)^2-m_k^2)+\alpha_4((q+k_3)^2-m_l^2)+i\epsilon(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right)} \\
&= -i\int_0^\infty \frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2} \\
&\quad\times e^{i\left(-\frac{(\frac{z}{2}+\alpha_2k_1+\alpha_3k_2+\alpha_4k_3)^2}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}-\alpha_1m_i^2+\alpha_2(k_1^2-m_j^2)+\alpha_3(k_2^2-m_k^2)+\alpha_4(k_3^2-m_l^2)+i\epsilon(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right)} \\
&= -i\int_0^\infty \mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\int_0^\infty \mathrm{d}\lambda\,\lambda \\
&\quad\times e^{i\lambda\left(-(\frac{z}{2}+\alpha_2k_1+\alpha_3k_2+\alpha_4k_3)^2-\alpha_1m_i^2+\alpha_2(k_1^2-m_j^2)+\alpha_3(k_2^2-m_k^2)+\alpha_4(k_3^2-m_l^2)+i\epsilon\right)}\,.
\end{aligned}$$

Differentiating with respect to z^μ and integrating at $z=0$ yields:

$$D_i(0,0,0,m_i,m_j,m_k,m_l) = \int_0^\infty \mathrm{d}x\mathrm{d}y\mathrm{d}z\frac{f_i}{(-m_i^2+x(m_i^2-m_j^2)+y(m_i^2-m_k^2)+z(m_i^2-m_l^2)+i\epsilon)^2}\,, \quad (7)$$

where $f_0=1$, $f_1=-x$, $f_2=-y$, $f_3=-z$, $f_{11}=x^2$, $f_{22}=y^2$, $f_{33}=z^2$, $f_{12}=xy$, $f_{23}=yz$, $f_{13}=xz$.
The analogous procedure (without the necessity to introduce α_4) yields the corresponding three–point coefficient functions in the zero–external–momentum limit. With $a=m_i^2$, $b=m_j^2$, $c=m_k^2$ and $d=m_l^2$ for the internal masses we obtain for example:

$$\begin{aligned}
C_0 &= \frac{a\,(-b+c)\,\log(\frac{a}{b})+(-a+b)\,c\,\log(\frac{c}{b})}{(a-b)\,(a-c)\,(b-c)}\,, \\
C_1 &= \frac{1}{2}\left(\frac{a^2\,\log(\frac{a}{b})}{(a-b)^2\,(a-c)}+\frac{\frac{b\,(-b+c)}{a-b}+\frac{c^2\,\log(\frac{c}{b})}{-a+c}}{(b-c)^2}\right)\,, \\
C_{12} &= \frac{1}{6\,(a-b)^2\,(a-c)^2\,(b-c)^3}\left(-\left(a^3\,(b-c)^3\,\log(\frac{a}{b})\right)\right. \\
&\quad +(a-b)\,\left((b-c)\,\left(a^2\,b^2-2\,a\,b^2\,c+a^2\,c^2-a\,b\,c^2+b^2\,c^2-a\,c^3+b\,c^3\right)\right. \\
&\quad \left.\left.+(a-b)\,c^2\,(3\,a\,b-a\,c-2\,b\,c)\log(\frac{c}{b})\right)\right)\,, \\
C_{00} &= \frac{3+2\,Diverg}{8}-\frac{a^2\,\log(a)}{4\,(a-b)\,(a-c)}+\frac{b^2\,\log(b)}{4\,(a-b)\,(b-c)}+\frac{c^2\,\log(c)}{4\,(a-c)\,(-b+c)}\,, \\
D_0 &= -\frac{a\,\log(\frac{a}{b})}{(a-b)\,(a-c)\,(a-d)}+\frac{c\,\log(\frac{c}{b})}{(a-c)\,(-b+c)\,(c-d)} \\
&\quad +\frac{d\,\log(\frac{d}{b})}{(a-d)\,(-b+d)\,(-c+d)}\,,
\end{aligned}$$

$$
\begin{aligned}
D_1 &= \frac{-b}{2\,(a-b)\,(b-c)\,(b-d)} + \frac{a^2\,\log(\frac{a}{b})}{2\,(a-b)^2\,(a-c)\,(a-d)} \\
&\quad -\frac{c^2\,\log(\frac{c}{b})}{2\,(a-c)\,(b-c)^2\,(c-d)} - \frac{d^2\,\log(\frac{d}{b})}{2\,(a-d)\,(b-d)^2\,(-c+d)}\,, \\
D_{11} &= \frac{1}{6}\left(\frac{b\,(a\,(b^2+5\,c\,d-3\,b\,(c+d))+b\,(b^2-3\,c\,d+b\,(c+d)))}{(a-b)^2\,(b-c)^2\,(b-d)^2}\right. \\
&\quad -\frac{2\,a^3\,\log(\frac{a}{b})}{(a-b)^3\,(a-c)\,(a-d)} + \frac{2\,c^3\,\log(\frac{c}{b})}{(a-c)\,(-b+c)^3\,(c-d)} \\
&\quad \left.+\frac{2\,d^3\,\log(\frac{d}{b})}{(a-d)\,(-b+d)^3\,(-c+d)}\right)\,, \\
D_{12} &= \frac{1}{6}\left(\frac{a\,(b\,c^2+b^2\,(c-d)-c^2\,d)+b\,c\,(c\,d+b\,(-2\,c+d))}{(a-b)\,(a-c)\,(b-c)^2\,(b-d)\,(c-d)}\right. \\
&\quad +\frac{b^2\,(b^3+b^2\,c+3\,a\,c\,d-b\,(2\,c\,d+a\,(2\,c+d)))\,\log(\frac{b}{c})}{(a-b)^2\,(b-c)^3\,(b-d)^2} \\
&\quad \left.+\frac{d^3\,\log(\frac{d}{c})}{(a-d)\,(b-d)^2\,(c-d)^2} - \frac{a^3\,\log(\frac{a}{c})}{(a-b)^2\,(a-c)^2\,(a-d)}\right)\,.
\end{aligned}
$$

D The Self–Energies

D.1 Bosonic Self–Energies

We give results for the gauge boson self–energies in arbitrary gauge. For simplicity and transparency we list contributions through fermions and bosons separately.

First we quote in the most generic way the fermionic contributions to the transverse part of the gauge boson self–energies $\Sigma_T^{V_1V_2}$ with $V_1V_2 = \gamma\gamma,\, \gamma Z,\, ZZ,\, WW$. Note that knowledge of the longitudinal parts is not necessary for our purposes.
In D space–time dimensions the transverse part can in its most general form be written as [4]:

$$
\begin{aligned}
\Pi_T^{V_1V_2}(p^2) &= \frac{1}{D-1}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\sum_{(f_i,f_j)}\frac{\alpha}{4\pi}N_c^f N_S^f\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \mathrm{d}^D q\,\mathrm{Tr}\left[\gamma_\nu\left(g^+_{ijV_2}\mathrm{P}_+\right.\right. \\
&\quad \left.\left.+g^-_{ijV_2}\mathrm{P}_-\right)\frac{1}{\not{q}+\not{p}-m_j+i\varepsilon}\gamma_\mu\left(g^+_{jiV_1}\mathrm{P}_+ + g^-_{jiV_1}\mathrm{P}_-\right)\frac{1}{\not{q}-m_i+i\varepsilon}\right] \\
&= \sum_{(f_i,f_j)}\pi\alpha N_c^f N_S^f\left\{\left(g^+_{jiV_1}g^+_{ijV_2}+g^-_{jiV_1}g^-_{ijV_2}\right)\right. \\
&\quad \times\left[\Pi_V(p^2,m_i,m_j)+\Pi_V(p^2,m_i,-m_j)\right] \\
&\quad \left.+\left(g^+_{jiV_1}g^-_{ijV_2}+g^-_{jiV_1}g^+_{ijV_2}\right)\left[\Pi_V(p^2,m_i,m_j)-\Pi_V(p^2,m_i,-m_j)\right]\right\}, \qquad (8)
\end{aligned}
$$

where N_c^f is the color factor, N_S^f is a Majorana symmetry factor with value 1 for Dirac-and $\frac{1}{2}$ for Majorana fermions and the $g^{\pm}_{ijV}$ correspond to the couplings as defined in section 3.1. Π_V is defined as [4]

$$\begin{aligned}\Pi_V(q^2,m_1,m_2) \;=\; & \frac{1}{12\pi^2}\left\{\left[q^2-\frac{m_1^2+m_2^2}{2}+3m_1m_2-\frac{\left(m_1^2-m_2^2\right)^2}{2q^2}\right]B_0(q,m_1,m_2)\right.\\ & +m_1^2\left(-1+\frac{m_1^2-m_2^2}{2q^2}\right)B_0(0,m_1,m_1)+m_2^2\left(-1+\frac{m_2^2-m_1^2}{2q^2}\right)\\ & \left.\times B_0(0,m_2,m_2)-\frac{q^2}{3}+\frac{\left(m_1^2-m_2^2\right)^2}{2q^2}\right\}. \qquad (9)\end{aligned}$$

Now we give the bosonic part of the ZZ, WW, $\gamma\gamma$ and γZ self–energies with the W– and Z–boson gauge parameters ξ_W and ξ_Z. As the unbroken electromagnetic $U(1)$ gauge is of no interest in our case, we have set $\xi_\gamma = 1$.

$$\begin{aligned}\Sigma^{ZZ}(s) \;=\; & \frac{e^2}{576\,c_w^6\,M_W^4\,\pi^2\,s\,s_w^2}\Big(-3\,c_w^2\,M_H^2\,M_W^4\,\Big(-M_W^2+c_w^2\,\Big(M_H^2-2\,s\Big)\Big)\\ & \times B_0(0,M_H^2,M_H^2)-3\,M_W^6\,\Big(M_W^2+c_w^2\,\Big(-M_H^2+s\Big)\Big)\,B_0(0,{M_Z}^2,{M_Z}^2)\\ & +3\,M_W^4\,\Big(M_W^4-2\,c_w^2\,M_W^2\,\Big(M_H^2-5\,s\Big)+c_w^4\,\Big(M_H^2-s\Big)^2\Big)\,B_0(s,M_H^2,{M_Z}^2)\\ & +3\,c_w^8\,s\,\Big(-48\,M_W^6-68\,M_W^4\,s+16\,M_W^2\,s^2+s^3\Big)\,B_0(s,M_W^2,M_W^2)\\ & +3\,c_w^4\,s\,\Big(c_w^4\,\Big(M_W^4-s^2\Big)+2\,c_w^2\,M_W^4\,s_w^2+M_W^4\,s_w^4\Big)\,B_0(s,M_W^2\,\xi_W,M_W^2\,\xi_W)\\ & \times\Big(-s+4\,M_W^2\,\xi_W\Big)+6\,c_w^4\,M_W^4\,B_0(0,M_W^2\,\xi_W,M_W^2\,\xi_W)\,\xi_W\\ & \times\Big(-{c_w}^4\,\Big(M_W^4+5\,M_W^2\,s-9\,s^2\Big)-4\,c_w^2\,M_W^2\,s\,s_w^2+{M_W}^2\,\Big(M_W^2+3\,s\Big)\,s_w^4\\ & +\Big({c_w}^4\Big(M_W^2-s\Big)^2-{M_W}^4\,s_w^4\Big)\,\xi_W\Big)+6\,c_w^4\,M_W^4\,B_0(0,M_W^2,M_W^2)\\ & \Big({c_w}^4\,\Big(M_W^4+23\,M_W^2\,s-9\,s^2\Big)-{M_W}^2\,\Big(M_W^2+s\Big)\,s_w^4\,\xi_W\\ & +\Big(-{c_w}^4\Big(M_W^2-s\Big)^2+{M_W}^4\,s_w^4\Big)\Big)-6\,c_w^4\,\Big(c_w^4\Big(M_W^2-s\Big)^2-M_W^4\,s_w^4\Big)\\ & \times B_0(s,M_W^2,M_W^2\,\xi_W)\,\Big({M_W}^4+10\,M_W^2\,s+s^2-2\,M_W^2\,\Big(M_W^2+s\Big)\,\xi_W\\ & +M_W^4\,{\xi_W}^2\Big)+9\,c_w^2\,M_W^6\,s\,B_0(0,{M_Z}^2\,\xi_Z,{M_Z}^2\,\xi_Z)\,\xi_Z+\\ & M_W^4\,\Big(-3\,c_w^4\,M_H^4+6\,c_w^2\,M_H^2\,M_W^2-3\,M_W^4+6\,c_w^8\,M_W^4-9\,c_w^2\,M_W^2\,s+\\ & 6\,c_w^8\,M_W^2\,s+2\,c_w^4\,s^2-36\,c_w^8\,s^2-4\,c_w^6\,s^2\,s_w^2-2\,c_w^4\,\Big(3\,M_W^4+9\,M_W^2\,s-s^2\Big)\\ & s_w^4+6\,c_w^4\,\xi_W\,\Big({c_w}^4\,\Big(-2\,M_W^4+M_W^2\,s+4\,s^2\Big)+{M_W}^2\,\Big(2\,M_W^2+3\,s\Big)\,s_w^4\\ & +\Big({c_w}^4\Big(M_W^2-s\Big)^2-{M_W}^4\,s_w^4\Big)\,\xi_W\Big)+9\,c_w^2\,M_W^2\,s\,\xi_Z\Big)\Big)\,,\end{aligned}$$

$$\Sigma^{WW}(s) \;=\; \frac{e^2}{1152\,c_w^4\,M_W^4\,\pi^2\,s\,s_w^2}\Big\{6\,M_H^2\,M_W^4\,\Big(-M_H^2+M_W^2+2\,s\Big)\,B_0(0,M_H^2,M_H^2)\,c_w^4$$

$$
\begin{aligned}
&+6\, M_W^4 \left(M_H^4 + M_W^4 + 10\, M_W^2\, s + s^2 - 2\, M_H^2 \left(M_W^2 + s\right)\right) B_0(s, M_H^2, M_W^2) \\
&\times c_{\rm w}^4 + 6\, B_0(s, M_W^2, M_Z^2) \left(M_W^4 - 2\, M_W^2 \left(M_W^2 + s\right) c_{\rm w}^2 + \left(M_W^2 - s\right)^2 c_{\rm w}^4\right) \\
&\times \left(M_W^4 + \left(M_W^4 + s^2\right) c_{\rm w}^4 + 5\, M_W^2\, c_{\rm w}^2 \left(2\, M_W^2 + 3\, s + s\, (c_{\rm w}^2 - s_{\rm w}^2)\right)\right) \\
&+\frac{3\, c_{\rm w}^2}{8} \left(M_W^2 - s\right) B_0(s, M_Z^2, M_W^2\, \xi_W) \left(-3\, M_W^2 + s + \left(M_W^2 + s\right) (c_{\rm w}^2 - s_{\rm w}^2)\right) \\
&\times \left(8\, M_W^4 + 40\, M_W^2\, s + 3\, s^2 + 4\, s \left(10\, M_W^2 + s\right) (c_{\rm w}^2 - s_{\rm w}^2) + 8\, M_W^4\, c_{\rm w}^4\, {\xi_W}^2\right. \\
&\left.-8\, M_W^2\, c_{\rm w}^2 \left(2\, M_W^2 + s + s\, (c_{\rm w}^2 - s_{\rm w}^2)\right) \xi_W + s^2\, (c_{\rm w}^4 - 6\, c_{\rm w}^2\, s_{\rm w}^2 + s_{\rm w}^4)\right) \\
&-3\, M_W^4\, B_0(0, M_W^2\, \xi_W, M_W^2\, \xi_W)\, c_{\rm w}^4\, \xi_W \left(3\, M_W^4 + 4\, M_W^2\, s - 19\, s^2\right. \\
&\left.+ \left(s^2 - M_W^4\right) (c_{\rm w}^2 - s_{\rm w}^2) - 4 \left(M_W^4 - M_W^2\, s\right) \xi_W + 2 \left(M_W^4 - s^2\right) c_{\rm w}^2\, \xi_Z\right) \\
&+\frac{3}{4}\, M_W^4\, B_0(0, M_W^2, M_W^2)\, c_{\rm w}^2 \left(4\, M_H^2\, M_W^2 + 3\, M_W^4 + 90\, M_W^2\, s - 37\, s^2\right. \\
&+4 \left(M_H^2\, M_W^2 - M_W^4 + 22\, M_W^2\, s - 9\, s^2\right) (c_{\rm w}^2 - s_{\rm w}^2) + \left(M_W^2 - s\right)^2 \\
&\left.\times (c_{\rm w}^4 - 6 c_{\rm w}^2 s_{\rm w}^2 + s_{\rm w}^4 - 8 c_{\rm w}^4 \xi_Z)\right) - 6 \left(M_W^2 - s\right)^2 B_0(s, M_W^2, M_Z^2 \xi_Z) \\
&\times c_{\rm w}^4 \left(\left(M_W^4 + 10\, M_W^2\, s + s^2\right) c_{\rm w}^4 - 2\, M_W^2 \left(M_W^2 + s\right) c_{\rm w}^2\, \xi_Z + M_W^4\, {\xi_Z}^2\right) \\
&-6 \left(M_W^4 - s^2\right) B_0(s, M_W^2\, \xi_W, M_Z^2\, \xi_Z)\, c_{\rm w}^4 \left(c_{\rm w}^4 \left(s - M_W^2\, \xi_W\right)^2\right. \\
&\left.-2\, M_W^2\, c_{\rm w}^2 \left(s + M_W^2\, \xi_W\right) \xi_Z + M_W^4\, {\xi_Z}^2\right) + 12\, M_W^2 \left(M_W^2 + s\right) \\
&\times \left(5\, M_W^4 - 22\, M_W^2\, s + 5\, s^2\right) B_0(s, 0, M_W^2)\, c_{\rm w}^4\, s_{\rm w}^2 - 12\, M_W^2 \left(M_W^2 - s\right) \\
&\times B_0(s, 0, M_W^2\, \xi_W)\, c_{\rm w}^4 \left(-\left(s \left(6\, M_W^2 + 5\, s\right)\right) - 2\, M_W^2\, s\, \xi_W + M_W^4\, {\xi_W}^2\right) s_{\rm w}^2 \\
&+6\, M_W^4\, B_0(0, M_Z^2, M_Z^2) \left(-9\, M_W^4\, c_{\rm w}^2 + M_W^2 \left(9\, M_W^2 + 14\, s\right) c_{\rm w}^4\right. \\
&+ \left(M_W^4 + 9\, M_W^2\, s - 9\, s^2\right) c_{\rm w}^6 + \frac{c_{\rm w}^4}{2} \left(M_W^2 - s\right) \left(s - 3\, M_W^2 + \left(M_W^2 + s\right)\right. \\
&\left.\left.\times (c_{\rm w}^2 - s_{\rm w}^2)\right) \xi_W - M_W^2 \left(M_W^2 + s\, c_{\rm w}^2\right) s_{\rm w}^4\right) + M_W^4 \left(-48\, M_W^4\, c_{\rm w}^2\right. \\
&+ \left(-6\, M_H^4 + 12\, M_H^2\, M_W^2 + 102\, M_W^4 + 15\, M_W^2\, s + 8\, s^2\right) c_{\rm w}^4 \\
&- \left(48\, M_W^4 + 21\, M_W^2\, s + 76\, s^2\right) c_{\rm w}^6 + 6\, c_{\rm w}^4 \left(2 \left(M_W^4 - M_W^2\, s\right) {\xi_W}^2\right. \\
&+\xi_Z \left(6\, M_W^2\, s + \left(-2\, M_W^4 - 5\, M_W^2\, s + 4\, s^2\right) c_{\rm w}^2 + 2 \left(M_W^4 - M_W^2\, s\right) \xi_Z\right) \\
&\left.+ \xi_W \left(4 M_W^2 s - 3 M_W^4 + 5 s^2 + \left(M_W^4 - s^2\right) (c_{\rm w}^2 - s_{\rm w}^2 - 2 c_{\rm w}^2 \xi_Z)\right)\right) \\
&\left.- \left(60 M_W^4 + 51 M_W^2 s + 52 s^2\right) c_{\rm w}^4 s_{\rm w}^2 - 6 M_W^2 \left(M_W^2 + 3 s c_{\rm w}^2 - 3 s c_{\rm w}^2 \xi_Z\right) s_{\rm w}^4\right) \\
&-6\, M_W^4\, B_0(0, M_Z^2\, \xi_Z, M_Z^2\, \xi_Z)\, c_{\rm w}^2\, \xi_Z \left(\left(M_W^4 + 9\, M_W^2\, s - 9\, s^2\right) c_{\rm w}^4 - 2\, M_W^2\, s\right. \\
&\left.\left.-2\, M_W^2\, s\, c_{\rm w}^2 + \left(M_W^4 - s^2\right) c_{\rm w}^4\, \xi_W + M_W^2 \left(2 \left(s - M_W^2\right) c_{\rm w}^2\, \xi_Z - s\, s_{\rm w}^4\right)\right)\right\} ,
\end{aligned}
$$

$$\begin{aligned}
\Sigma^{\gamma\gamma} &= \frac{e^2}{192\,\pi^2}\Big(14\,M_W^2\;(\xi_W-1)+9\,M_W^2\;(\xi_W-1)\;\xi_W+\Big(M_W^2\;(34-9\,\xi_W)-8\,s\Big)\\
&\times B_0(0,M_W^2,M_W^2)-(-1+\xi_W)\;\Big(-2\,s\;(5+\xi_W)+M_W^2\;(10+13\,\xi_W)\Big)\\
&\times B_0(0,M_W^2,M_W^2\,\xi_W)+\xi_W\;\Big(8\,s+M_W^2\;(14+9\,\xi_W)\Big)\;B_0(0,M_W^2\,\xi_W,M_W^2\,\xi_W)\\
&-\frac{1}{M_W^4}\Big(4\,M_W^2-s\Big)\;\Big(12\,M_W^4+20\,M_W^2\,s+s^2\Big)\;B_0(s,M_W^2,M_W^2)\\
&+\frac{2}{M_W^4}\;\Big(2\,M_W^2-s\Big)\;\Big(s^2-2\,M_W^2\,s\;(-5+\xi_W)+M_W^4\,(-1+\xi_W)^2\Big)\\
&\times B_0(s,M_W^2,M_W^2\,\xi_W)+\frac{s^2\;(s-4\,M_W^2\,\xi_W)\;B_0(s,M_W^2\,\xi_W,M_W^2\,\xi_W)}{M_W^4}\Big)\;,
\end{aligned}$$

$$\begin{aligned}
\Sigma^{\gamma Z} &= \frac{e^2}{576 c_{\rm w}\pi^2 s_{\rm w}}\Big(4s_{\rm w}^2\,\Big(s+6\,M_W^2\;(\xi_W-1)\Big)+c_{\rm w}^2\;\Big(4\,s+9\,M_W^2\;\Big(2+\xi_W-3\,\xi_W^2\Big)\Big)\\
&+3\;\Big(-4\,M_W^2\,s_{\rm w}^2+c_{\rm w}^2\;\Big(8\,s+M_W^2\;(-38+9\,\xi_W)\Big)\Big)\;B_0(0,M_W^2,M_W^2)\\
&-\frac{3}{s}\;(-1+\xi_W)\;\Big(2\,M_W^2\,s_{\rm w}^2\;\Big(s+M_W^2\;(-1+\xi_W)\Big)+c_{\rm w}^2\;\Big(2\,M_W^4\;(-1+\xi_W)\\
&+2\,s^2\;(5+\xi_W)-M_W^2\,s\;(8+13\,\xi_W)\Big)\Big)\;B_0(0,M_W^2,M_W^2\,\xi_W)\\
&-3\,\xi_W\;\Big(-12\,M_W^2\,s_{\rm w}^2+c_{\rm w}^2\;\Big(8\,s+M_W^2\;(2+9\,\xi_W)\Big)\Big)\;B_0(0,M_W^2\,\xi_W,M_W^2\,\xi_W)\\
&+\frac{3\,c_{\rm w}^2}{M_W^4}\;\Big(4\,M_W^2-s\Big)\;\Big(12\,M_W^4+20\,M_W^2\,s+s^2\Big)\;B_0(s,M_W^2,M_W^2)\\
&+\frac{6}{M_W^4\,s}\Big(c_{\rm w}^2\;\Big(M_W^2-s\Big)^2+M_W^4 s_{\rm w}^2\Big)\;\Big(s^2-2M_W^2 s\,(\xi_W-5)+M_W^4(\xi_W-1)^2\Big)\\
&\times B_0(s,M_W^2,M_W^2\,\xi_W)-\frac{3\,c_{\rm w}^2\,s^2}{M_W^4}\;\Big(s-4\,M_W^2\,\xi_W\Big)\;B_0(s,M_W^2\,\xi_W,M_W^2\,\xi_W)\Big)\;.
\end{aligned}$$

D.2 Fermionic Self–Energies

We give the fermionic self–energies in the most generic form and separate scalar from spin–one and tadpole contributions. The contribution through a scalar S in arbitrary gauge reads:

$$\begin{aligned}
\Sigma_{ij}^S(\not p) &= \frac{e^2}{16\pi^2}\sum_k\Big(-\not p\,B_1(p^2,m_k^2,M_S^2\,\xi_S)\,(g_{kj}^+\,g_{ik}^-\,R+g_{kj}^-\,g_{ik}^+\,L)\\
&+m_k\,B_0(p^2,m_k^2,M_S^2\,\xi_S)\,(g_{kj}^+\,g_{ik}^+\,R+g_{kj}^-\,g_{ik}^-\,L)\Big)\;,
\end{aligned}$$

where $\{S,\,M_S,\,\xi_S\}$ stand for the physical Higgs $\{H,\,M_H,\,\xi_H=1\}$ and the charged $\{\phi^\pm,\,M_W,\,\xi_W\}$ and neutral $\{\chi,\,M_Z,\,\xi_Z\}$ unphysical Higgs ghosts.
The analogous term due to a spin–one boson V is

$$
\begin{aligned}
\Sigma^V_{ij}(\not p) \; = \; & \frac{-e^2}{16\pi^2} \sum_k \Big\{ M_V^2 \, (1 + 2\, B_1(p^2, M_V^2, m_k^2)) \not p \, (g^+_{kj}\, g^+_{ik}\, R + g^-_{kj}\, g^-_{ik}\, L) \\
& + 2\, M_V^2 \, (2\, B_1(p^2, M_V^2, m_k^2) - 1)\, m_k \, (g^+_{kj}\, g^-_{ik}\, R + g^-_{kj}\, g^+_{ik}\, L) \\
& + \Big(4 (B_{00}(p^2, M_V^2\, \xi_V, m_k^2) - B_{00}(p^2, M_V^2, m_k^2)) \\
& + s\, (B_{11}(p^2, M_V^2\, \xi_V, m_k^2) - B_{11}(p^2, M_V^2, m_k^2)) - \frac{1}{2}\, M_V^2\, (\xi_V - 1) \Big) \\
& \times (m_k\, (g^+_{kj}\, g^-_{ik}\, R + g^-_{kj}\, g^+_{ik}\, L) - \not p\, (g^+_{kj}\, g^+_{ik}\, R + g^-_{kj}\, g^-_{ik}\, L)) \\
& + (B_1(p^2, M_V^2\, \xi_V, m_k^2) - B_1(p^2, M_V^2, m_k^2))\, (m_k^2 - s) \not p \, (g^+_{kj}\, g^+_{ik}\, R + g^-_{kj}\, g^-_{ik}\, L) \Big\} \, .
\end{aligned}
$$

The tadpole contributions are given as

$$
\begin{aligned}
\Sigma^{tad.}_{ij} \; = \; & \frac{e^2}{16\, \pi^2\, M_H^2\, s_{\rm w}\, M_W} \left(g^-_{ijH}\, L + g^+_{ijH}\, R \right) \bigg(\frac{3}{4}\, M_H^4 \, \left(1 + B_0(0, M_H^2, M_H^2) \right) \\
& + M_W^4 \, \left(1 + 3\, B_0(0, M_W^2, M_W^2) \right) + \frac{1}{2}\, M_Z^4 \, \left(1 + 3\, B_0(0, M_Z^2, M_Z^2) \right) \\
& + \frac{1}{2}\, M_H^2\, M_W^2 \, \left(1 + B_0(0, M_W^2\, \xi_W, M_W^2\, \xi_W) \right) \xi_W \\
& + \frac{1}{4}\, M_H^2\, M_Z^2 \, \left(1 + B_0(0, M_Z^2\, \xi_Z, M_Z^2\, \xi_Z) \right) \xi_Z \\
& + 2 \sum_f N_c^f\, N_S^f\, M_W\, s_{\rm w}\, (g^-_{ffH} + g^+_{ffH}) \left(1 + B_0(0, m_f^2, m_f^2) \right) m_f^3 \bigg) \, ,
\end{aligned}
$$

with $g^{\pm}_{ffS}$, N_c^f and N_S^f as defined in section D.1.

References

[1] T. Kajita for the Super–Kamiokande Collaboration, *Nucl. Phys. Proc. Suppl.* **77** (1999), 123

[2] J. Schechter and J.W.F. Valle, *Phys. Rev.* **D22**, 2227 (1980)

[3] M. Gell-Mann, P. Ramond, R. Slansky, *Supergravity*, North Holland (1979) und T. Yanagida, *Proc. Workshop on Unified Theories and baryon number conservation in the universe*, KEK (1979)

[4] K.-P.O. Diener, B.A. Kniehl, A. Pilaftsis, *Phys. Rev.* **D57**, 2771 (1998)

[5] A. Pilaftsis, *Z. Phys.* **C55** (1992), 275

[6] A. Denner, *Fortschr. Phys.* **41** (1993), 307

[7] G. Passarino and M. Veltman, *Nucl. Phys.* **B160** (1979), 151

[8] J. Küblbeck, M. Böhm and A. Denner, *Comp. Phys. Comm.* **60** (1990), 165
T. Hahn, *Acta Phys. Polon.* **B30** (1999), 3469

[9] R. Mertig, M. Böhm and A. Denner, *Comp. Phys. Comm.* **64** (1991), 345

[10] B. A. Kniehl, F. Madricardo, M. Steinhauser, *Phys. Rev.* **D62**, 073010 (2000)

[11] J. C. Collins, *Renormalization*, Cambridge University Press (1984)

[12] A. Denner, T. Sack, *Nucl. Phys.* **B347** (1990), 203

[13] P. Gambino, P. Grassi, F. Madricardo, *Phys. Lett.* **B454**, 98 (1999)

[14] B. Kniehl, A. Pilaftsis, *Nucl. Phys.* **B424** (1994), 18

[15] A. Barroso, L. Brücher, R. Santos, *Phys. Rev.* **D62**, 096003 (2000)

[16] P. A. Grassi, priv. comm.

[17] S. Weinberg, *Phys. Rev.* **118** (1960), 838

[18] A. Sirlin, *Phys. Rev.* **D22**, 971 (1980)

[19] C. Itzykson, J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill (1980)

[20] R. E. Behrends, R. J. Finkelstein, A. Sirlin, *Phys. Rev.* **101** (1956), 866
S. M. Berman, *Phys. Rev.* **112** (1958), 267
T. Kinoshita and A. Sirlin, *Phys. Rev.* **113** (1959), 1652

[21] T. van Ritbergen and R. Stuart, *Phys. Lett.* **B437**, 201 (1998)
T. van Ritbergen and R. Stuart, *Phys. Rev. Lett.* **82**, 3394 (1999)
T. Seidensticker and M. Steinhauser, *Phys. Lett.* **B467**, 271 (1999)

[22] Particle Data Group: C. Caso et al., *Eur. Phys. Jour.* **C15**, 1 (2000)

[23] L.-L. Chau and W.-Y. Keung, *Phys. Rev. Lett.* **53**, 1802 (1984)
H. Harari and M. Leurer, *Phys. Lett.* **B181**, 123 (1986)
H. Fritzsch and J. Plankl, *Phys. Rev.* **D35**, 1732 (1987)
F.J. Botella and L.-L. Chao, *Phys. Lett.* **B168**, 97 (1986)

[24] G.J. van Oldenborgh, *Comp. Phys. Comm.* **66** (1991), 1
T. Hahn, *Acta Phys. Polon.* **B30** (1999), 3469

[25] W. F. L. Hollik, *Fortschr. Phys.* **38** (1990), 165

Danksagung

Ich danke Herrn Prof. Dr. Bernd Kniehl für die interessante Themenstellung, seine Diskussionsbereitschaft und die Begutachtung der vorliegenden Arbeit. Desweiteren gilt mein Dank Herrn Prof. Dr. Gustav Kramer für seine Korrekturvorschläge und Fragen.
Bei allen (ehemaligen) Mitarbeitern des MPI für Physik in München, insbesondere Matthias Steinhauser, Pietro Antonio Grassi, Tobias Hurth und Alexander Lenz bedanke ich mich für viele interessante Diskussionen und die stets angenehme Atmosphäre.
Ebenso bin ich meinen hamburger Kollegen, speziell Michael Spira, wiederum Matthias Steinhauser und Michael Klasen für ihre Diskussionsbereitschaft verbunden. Dasselbe gilt für meine Kollegen Lennart Zwirner und Armando Barrientos.
Mein Dank gilt schließlich meinen Eltern, die das Entstehen dieser Arbeit ermöglicht haben sowie meiner Schwester Kathrin für ihren Rückhalt und ihre Ermutigung.