Discrete Duality Symmetries in String- and M-theory

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Abstract

In this thesis, the structure of non-perturbative U-duality symmetry in string- and M-theory is investigated. Non-perturbative duality symmetries have dramatically changed string theory and led to the conjecture that all string theories are different limits of an eleven-dimensional theory called M-theory, whose exact shape is yet unclear. An investigation of its symmetries could give hints to a possible construction. U-duality is one of these symmetries, and a precise definition of U-duality and its effects in $d > 2$ is the main subject of this thesis. This is investigated in a toy model closely resembling low energy M-theory, and then in M-theory itsself. In the latter, the known classical $E_{7(+7)}$ duality symmetry is made manageable by embedding its algebra within $\mathfrak{e}_{8(+8)}$, and the definition of U-duality leads to study discrete subgroups of Lie groups acting on admissible lattices in basic representations. This yields generators for $E_{7(+7)}(\mathbb{Z})$ and higher dimensional U-dualities. Known subgroups of the symmetry are investigated, and different U-duality definitions are compared with the one found. U-duality in $d = 3$ is constructed by using different orders of compactification. Applications of U-duality in the context of stringy black holes and solitons are given. Elementary solitons in $d = 3$ are shown to be vortex solutions. Solutions admitting simultaneously null- and space-like Killing vectors are considered from a supersymmetric perspective. Finally, `Maple` codes to generate highest weight matrix representations of Lie groups used in this thesis are presented and discussed.

Zusammenfassung

In dieser Arbeit wird die Struktur nicht-perturbativer U-dualitätsgruppen in der String- und M-theory untersucht. Nichtstörungstheoretische Dualitätssymmetrien haben die Stringtheorie dramatisch verändert und zu dem Vorschlag geführt, daß alle Stringtheorien als unterschiedliche Grenzwerte einer elfdimensionalen, sogenannten M-theorie verstanden werden können, deren Struktur bisher noch unklar ist. Eine Untersuchung ihrer Symmetrien erlaubt Rückschlüsse darauf, wie diese Theorie zu konstruieren ist. U-dualität ist eine dieser Symmetrien, und eine präzise Definition der U-dualitätsgruppe und ihrer Effekte in $d > 2$ ist Hauptthema dieser Arbeit. Diese wird erst in einem Spielzeugmodell untersucht, das der M-theorie strukturell stark ähnelt, und dann in der M-theorie selbst. In letzterem Fall wird die Behandlung der klassischen $E_{7(+7)}$ Symmetrie in $d = 4$ durch Einbettung ihrer Algebra in $\mathfrak{e}_{8(+8)}$ stark vereinfacht, und die Definition von U-dualität führt zur Untersuchung diskreter Lie Gruppen auf sogenannten zulässigen Gittern in sogenannten einfachen Darstellungen. Dies ergibt Generatoren von $E_{7(+7)}(\mathbb{Z})$ und den höher dimensionalen U-dualitätsgruppen. Bekannte Untergruppen werden untersucht, und die hier benutzte Definition der U-dualitätsgruppe wird mit anderen Definitionen in der Literatur verglichen, und Übereinstimmung festgestellt. Die U-dualitätsgruppe in $d = 3$ wird durch verschiedene Kompaktifizierungsreihenfolgen konstruiert. Anwendungen von U-dualität im Zusammenhang mit Schwarzlochsolitonen in der Stringtheorie werden gegeben. Es wird gezeigt, daß elementaren Objekten in $d = 3$ Vortexlösungen entsprechen. Lösungen mit gleichzeitigem Null- und raumartigen Killingvektor werden von einer supersymmetrischen Perpektive beleuchtet. Abschließend werden `Maple` Programme vorgestellt, die in dieser Arbeit benutzt wurden, zur Erzeugung von Matrix-Höchstgewichtsdarstellungen von Liegruppen.

To Ruben and Ruth
To my parents

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Chapter 1

Introduction

Die Symmetrie . . . bildet auch den eigentlichen Kern jener Grundgleichung. [52]

What is $E_{8(+8)}(\mathbb{Z})$*? [1]*

One of the most influential and, at the same time, most challenging topics in modern physics is unification. After it was realized in the sixtieth and seventies of the last century that the electromagnetic force and the weak nuclear force may be understood in an electroweak theory with spontaneous symmetry breaking, a main focus of research was to find out if this is also true when incorporating the strong nuclear force, thus unifying these three forces in one interaction broken spontaneously. This led to several ”grand unified theories” (GUTs) that, on the other hand, all predict a proton decay not seen in nature. This problem could partially be removed by introducing supersymmetry, a symmetry between bosonic and fermionic particles, and experiments hint that the coupling constants indeed approach each other and might unify at an energy scale known as ”GUT” scale that is, however, far from being observable in experiment.

The force in nature most directly experienced in daily life, gravity, has proven to be much more stubborn and to withstand most approaches to incorporate it into a unified quantum theory. The main problem with ”conservative” approaches to quantum gravity or supergravity is that those theories are not renormalizable.

At current stage, string theories are promising candidates for quantum theories that incorporate gravity. The perturbative description of string theory admits consistently a spin two particle propagating on a gravitational background and coupling to other matter fields. However, the main problem with perturbative string theories is the variety of possible constructions. Perturbative superstring theories have been formulated in ten dimensions and need to be compactified to $d = 4$ to make contact with phenomenology. However, there is a wide and seemingly unconstrained variety how to do this, while none of the approaches has been able to reproduce the Standard Model of elementary particle physics exactly, though similar structures have been found.

However, string theories have considerably changed in recent years, and this is due to the study of non-perturbative duality symmetries in these theories. The five perturbative string theories are now understood to be related by a web of duality symmetries connecting them. Furthermore, new objects called D-branes have been introduced in the theory, and this has led to successfully address the question of microscopic black hole entropy in string theory. The theory has considerably been en-richened by the inclusion of non-perturbative effects.

Recent development has thus addressed the unification of string theories themselves. It has been argued that a theory exists containing all perturbative string theories in different regions of moduli space. This theory is called M-theory, but still, the M seems to reflect mystery. Though several attempts have been made to understand the structure of this eleven dimensional theory, none has been completely successful yet.

It seems the more important to have a clear picture of the symmetries M-theory admits in order to probe its structure. This has been a main motivation for the work presented in this thesis. M-theory, when compactified on the torus, is known to have a large self-duality group known as U-duality, corresponding to discrete subgroups of Lie groups of the exceptional series. A precise definition of its generators lacked in the literature, and this is the main focus of this thesis. The construction of U-duality will be investigated, and a set of generators will be given and proven to be correct. Applications of U-duality that are connected to black holes in string theory are also given.

In order to address these questions, the necessary background shall be briefly reviewed in this chapter. For more detailed descriptions, the reader is referred to the references and the textbook [81].

1.1 DSZ quantization condition

In this section an issue is discussed that will be used when defining U-duality in later chapters. It is related to a phenomenon in physics that, due to its absence in experiment, still seems mysterious: magnetic charge.

The striking fact that the Maxwell equations of electromagnetism admit a rotational symmetry between electric and magnetic sector, but this symmetry is broken by the absence of magnetic charge, inspired Dirac to introduce his famous monopole solution [30].

Consider a sphere around a monopole. Since the integrated flux on this sphere is nonzero if the magnetic charge is nonzero, a vector potential cannot be introduced globally. However, by introducing a vector potential that is defined everywhere but on a "Dirac-string", a nonzero magnetic charge g arises when calculating the total flux on a sphere surrounding the monopole, e.g. by introducing two copies of the gauge field for the northern and southern hemisphere of the sphere (Wu-Yang type argument, see e.g. [76]) that are related by a gauge transformation.

This may be given as follows: In $\mathbb{R}^3$, choose coordinates $\{i\} = \{x, y, z\}$ and let $r = \sqrt{x^2 + y^2 + z^2}$ as usual, and consider the Dirac-string to be along the negative z-axis. The corresponding "northern" hemisphere gauge potential A is given by

$$A_x^N = \frac{p}{2}\frac{-y}{r(r+z)}, \quad A_y^N = \frac{p}{2}\frac{x}{r(r+z)}, \quad A_z^N = 0$$

which yields

$$\vec{\nabla} \times \vec{A}^N = \vec{B} = \frac{p}{2}\frac{\vec{r}}{r^3}$$

except along the negative z-axis. The normalization is chosen for later convenience, as will become clear in a moment. On this "southern" hemisphere, introduce

$$A_x^S = \frac{p}{2}\frac{y}{r(r-z)}, \quad A_y^S = \frac{p}{2}\frac{-x}{r(r-z)}, \quad A_z^S = 0$$

which gives

$$\vec{\nabla} \times \vec{A}^S = \vec{B} = \frac{p}{2}\frac{\vec{r}}{r^3}$$

except along the positive z-axis. In spherical coordinates, one has

$$A_\phi^N = \frac{p}{2}\frac{(1-\cos\theta)}{r\sin\theta}, \quad A_\phi^S = -\frac{p}{2}\frac{(1+\cos\theta)}{r\sin\theta}$$

the other components zero. Both gauge fields are related by a gauge transformation

$$A_i^N - A_i^S = \frac{2}{r\sin\theta}\delta_{i\phi} = (\vec{\nabla}p\phi)_i,$$

so the gauge parameter is $\Lambda = p\phi$. The total flux is then, by cutting the sphere in northern and southern hemisphere U_N, U_S, respectively,

$$\begin{aligned}\Phi = \oint_S(\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \int_{U_N}(\vec{\nabla}\times\vec{A}^N)\cdot d\vec{S} + \int_{U_S}(\vec{\nabla}\times\vec{A}^S)\cdot d\vec{S}\\ &= \oint_{equator}(\vec{A}^N - \vec{A}^S)\cdot d\vec{s} = 2\pi p.\end{aligned}$$

Consider a point particle with electric charge e, moving in this field. In a quantized context, the wave function of the particle acquires a phase, induced by the minimal coupling of the electromagnetic field, of the form

$$\exp(-ie\Lambda)$$

(in "natural" units $\hbar = c = 1$) upon a gauge transformation. However, completing a circle around the equator, the wavefunction should be unique. This implies

$$ep = n, n \in \mathbb{Z},$$

the famous Dirac quantization condition [30]. Assuming the existence of a single monopole in the universe would therefore imply the quantization of electric charge seen in nature.

The same quantization condition may be obtained in a more heuristic way by considering the quantization of angular momentum. When incorporating an electromagnetic field, the purely mechanical angular momentum is not a conserved quantity, but needs to be augmented by the momentum of the electromagnetic field to be conserved. Taking the magnetic field as above, the quantization of this conserved angular momentum yields the same quantization condition.

These arguments have been studied for a situation with two particles with electric charge q_1 and q_2 *and* magnetic charge p_1 and p_2 in [84, 85, 99, 98] in a relativistic context. The quantization condition then has to be modified, and is given by

$$q_1p_2 - q_2p_1 = n, n \in \mathbb{Z}$$

called (referring to the above authors) the DSZ-quantization condition.

Introducing the symplectic form Ω with

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the quantization condition may be written in a more compact form by introducing charge vectors $Z = (p,q)^t$. It may then be written as

$$Z_1\Omega Z_2 = n, n \in \mathbb{Z}.$$

For theories admitting a number N of $U(1)$ gauge fields, one may define N-dimensional charge vectors $Z = (p,q)^t$ analogously. The DSZ quantization condition may then be written as above by using the $2N \times 2N$ symplectic form.

The above condition shall be applied in the following in supergravity theories that are low energy actions of string theories. In supergravity theories, magnetic monopoles reflecting the above structure, that is, with magnetic U(1) field admitting a Dirac string, have been found [50].

Like for the Reissner Nordström black hole, the Maxwell equations are those for the source-free case on curved space, since the singularity inducing nonzero flux is a singularity of space-time. If one assigns charges, e.g. in the asymptotically flat limit, to the gauge fields, these charges are restricted by the same semiclassical quantization condition. This has been a main tool to restrict global symmetries one encounters in supergravity to discrete subgroups that are symmetries of a quantized theory.

The Dirac quantization condition may be extended by the same Wu-Yang type argument to p-forms potentials in higher dimensions where the configuration is asymptotically flat (see e.g. [8] and references).

1.2 Electric-Magnetic Duality

Another important ingredient of this thesis is the notion of electric-magnetic duality.

For nonzero sources, the Maxwell equations are

$$\partial_\mu F^{\mu\nu} = j^\nu, \;\; \partial_\mu \star F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = k^\nu,$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength, ϵ is the Levi-Civita symbol with $\epsilon_{0123} = -1$ and j^ν, k^ν are the electric and magnetic currents. The Maxwell equations admit a symmetry

$$\begin{aligned} F^{\mu\nu} &\to \star F^{\mu\nu}, \quad \star F^{\mu\nu} \to -F^{\mu\nu}, \\ j^\mu &\to k^\mu, \quad k^\mu \to -j^\mu, \end{aligned} \tag{1.1}$$

where the minus sign takes account of the fact that $(\star)^2 = -1$ in Minkowski spacetime. This implies on the charge vector $Z = (p, q)^t$

$$\begin{pmatrix} p \\ q \end{pmatrix} = Z \to \Omega Z = \begin{pmatrix} -q \\ p \end{pmatrix}.$$

This is also a symmetry of the above DSZ quantization condition. This symmetry is called electric-magnetic duality.

The above quantization condition actually admits a larger symmetry that could be any group leaving the symplectic form invariant under conjugation. This is $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$ in this case, and the fact that the charge are quantized to integers breaks the group to a discrete $SL(2, \mathbb{Z})$ group.

Actually, for electromagnetic theory as $SO(3)$ gauge theory spantanously broken to $U(1)$ (see [51] for a review), this is a duality group that has been proposed. With Higgs field φ transforming in the adjoint of $SO(3)$, and including instanton effects by adding a "θ term" to the Lagrangian, the Lagrangian gets in the Higgs vacuum

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2\theta}{32\pi^2}F^{\mu\nu} \star F_{\mu\nu}.$$

The $SL(2, \mathbb{Z})$ symmetry is then spanned by two generators: a transformation corresponding to electric-magnetic duality for $\theta = 0$, and shifts of the θ vacuum. Defining

$$\tau = \frac{\theta}{2\pi} + \frac{g}{e}$$

where e, g are the elementary quantums of magnetic and electric charge, (in this theory the elementary quantum of magnetic charge is twice the one introduced above [51]), these two transformations correspond to

$$\tau \to -\tau^{-1}$$

and

$$\tau \to \tau + 1.$$

These transformation also induce a transformation on the charges. The electric charge is now related to the gauge field

$$\star H = 2\frac{\delta \mathcal{L}}{\delta F}$$

due to the presence of the θ term. One may define charges

$$Q = \oint_\Sigma \star F, \;\; p = \oint_\Sigma F,$$

and

$$q = \oint_\Sigma H$$

where Q is a "θ shifted" charge obeying

$$Q = q + \frac{\theta e}{2\pi} p.$$

If the above fractional linear transformation may be represented by a matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \;\; \tau \to \frac{a\tau + b}{c\tau + d}$$

then it may be compensated by relabelling the states by acting with the inverse group element on the vector $(p, q)^t$.

However, this conjecture of electric magnetic duality for the quantum theory [74] meets several obstacles: it maps a weakly coupled theory to a strongly coupled one and is therefore hard to test. Nonetheless, it seems very attractive to have a description of a theory with strong coupling by a weakly coupled one. However, it is known that no exact matching of states is possible, since the monopole is spherical symmetric corresponding to spin zero, and the gauge boson carries spin one. Furthermore, quantum corrections will affect the mass spectra.

This situation is improved in supersymmetric theories. In [86], a duality map was used to explicitly determine the vacuum structure in $N = 2$ supersymmetric Yang-Mills theory. With $N = 4$ supersymmetry, monopoles and gauge bosons correspond to the same multiplets, and masses are protected by non-renormalization theorems. This theory is therefore supposed to admit the above symmetry as self-duality, similar to the toroidally compactified string theories and their low energy supergravity counterparts, that have $N \geq 4$ supersymmetry. They shall be considered now.

The vacuum Maxwell equations admit a larger symmetry than electric-magnetic duality: they allow rotations of the electric and magnetic fields into each other, and since there are no charges, this symmetry is unrestricted. However, in the supergravity case, large symmetry groups exists known as classical duality symmetries (see table 1.1 below) that rotate equations of motion and Bianchi identities into each other. In this case, the fields carry charges due to the nontrivial gravitational background, and the symmetries, as mentioned above, get broken to discrete subgroups. All these groups are have been shown to be subgroups of $Sp(2N, \mathbb{Z})$ [41], and this is in accordance with the above DSZ condition.

1.3 Bogomolnyi Bound

A main tool in studying duality symmetries in string theory and supersymmetric field theory has been the Bogomolnyi bound. It gives a bound on the mass of supersymmetric states. States saturating this bound live in ”shortened” multiplets that can be used to test duality conjectures.

Since in this thesis the focus is on four dimensions, the four-dimensional supersymmetry algebra [46, 55] shall be considered in the following.

Consider the N-extended supersymmetry algebra in $d = 4$. It is given by

$$\{Q^{\alpha\, I}, Q^{\dagger\beta\, J}\} = (\gamma^0\gamma_M)^{\alpha\beta}P^M\delta^{IJ} + iU^{IJ}\gamma^{0\,\alpha\beta} + V^{IJ}(\gamma^5\gamma^0)^{\alpha\beta}$$

where the following notations are used: Use the gamma matrices in the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix},$$

where the σ^i are the Pauli matrices. The indices $\{\alpha, \beta\}$ are spinor indices, while the indices I, J are $SO(N)$ indices. The supersymmetry charges Q^I are Majorana spinors in $d = 4$, obeying

$$Q^I = i\gamma^2(Q^I)^*,$$

and the U^{IJ} and V^{IJ} are antisymmetric central charge matrices commuting with the rest of the algebra, that shall be specified further below.

Consider now the expectation value of a physical state

$$\langle\{Q^{\alpha\, I}, Q^{\dagger\beta\, J}\}\rangle = T^{(\alpha I),(\beta J)}.$$

The operator on the left hand side is manifestly positive definite. This can be used to derive a bound on the mass of physical states in supersymmetry, known as Bogomolnyi bound.

For this, consider the matrix T. It must be a a positive definite $4N \times 4N$ matrix. Going to the rest frame where $P^M = (M, 0)$, it is

$$T^{(\alpha I),(\beta J)} = \langle M\delta^{\alpha\beta}\delta^{IJ} + i(\gamma^0)^{\alpha\beta}U^{IJ} + (\gamma^5\gamma^0)^{\alpha\beta}V^{IJ}\rangle$$

Look at the eigenvalue problem of $\langle T^{(mI),(nJ)}\rangle$. This amounts to finding 32 component eigenspinors ϵ satisfying

$$((M - \lambda)\mathbf{1}_{4N})\,\epsilon = (i(\gamma^0)^{\alpha\beta}U^{IJ} + (\gamma^5\gamma^0)^{\alpha\beta}V^{IJ})\,\epsilon$$

To solve this, consider the square of the above equation. On the right hand side, in the Weyl representation of the gamma matrices, one has the $4N \times 4N$ matrix

$$\begin{pmatrix} -Z^\dagger Z & 0 & 0 & 0 \\ 0 & -Z^\dagger Z & 0 & 0 \\ 0 & 0 & -Z^\dagger Z & 0 \\ 0 & 0 & 0 & -Z^\dagger Z \end{pmatrix}$$

where

$$Z^{IJ} = \langle U^{IJ} - iV^{IJ}\rangle.$$

The matrix Z is complex antisymmetric and may therefore always be brought in skew diagonal form. The hermitian matrix $Z^\dagger Z$ thus can be diagonalized with real eigenvalues $(z_n)^2, n = 1 \ldots [N/2]$ where $[N/2]$ is the integer part of $N/2$. This implies that the matrix T has eigenvalues

$$\lambda_n = M \pm |z_n|.$$

However, since T is a positive definite matrix, the eigenvalues are bounded by

$$M \geq \max|z_n|.$$

This is the Bogomolnyi bound on the mass of a state in supersymmetric theories.

In the case of $N = 8$ supersymmetry that will be of interest here, Z can have four different eigenvalues. Suppose all are equal, and furthermore that the mass equals this eigenvalue.

Then T has 16 zero eigenvalues, and thus a 16 component supersymmetry exists annihilating the state. Translated into Majorana spinors, this means however the state preserves $N = 4$ or half of the total supersymmetry.

Correspondingly, if two eigenvalues are equal and the mass equal to them, the state preserves a quarter of the supersymmetry, and finally, if the mass is equals one eigenvalue, the state preserve one eighth of the supersymmetry.

The supersymmetries acting non-trivially span a multiplet of states, but this multiplet is smaller than the generic multiplet or is "shortened", since some super-charges annihilate the state. This shortening makes these states very attractive for tests of duality conjectures. For theories with $N \geq 4$, mass and charges are not renormalized. The shortening of multiplets then cannot be disturbed e.g. by changing the coupling constant, and one therefore may follow such a shortened multiplet from weak to strong coupling.

In perturbative string theory, a BPS state corresponds to a state whose mass equals its (electric) charge. For supergravity theories (e.g. corresponding to low energy expansions of string theories), the above global supersymmetry algebra is the asymptotic limit of a local supersymmetry algebra generated by spinors ϵ^I, and BPS solutions correspond to solitons of the theory. Usually, one is interested in purely bosonic solutions. In such a case, M may be chosen to correspond to the ADM mass of an asymptotically flat solution, and the charges are the asymptotic charges of the $U(1)$ fields of the theory, dressed by the asymptotic (constant) values of the scalars.

In the case of $N = 8$ supersymmetry, one may write the central charge matrix as [56]

$$Z_{IJ} = t^i_{IJ}(\bar{q}_i + i\bar{p}^i)$$

where the t^{IJ}_i are generators of the vector representation of $SO(8)$, and the where $(\bar{q}_i + i\bar{p}^i)$ are the dressed charges.

In setting all fermions to zero, the supersymmetry transformations of the bosonic fields of the theory vanish trivially. The Bogomolnyi bound may then be seen to be saturated if an exact local spinor ϵ^I exists such that the supersymmetry transformations of the fermions vanish in the whole space [55, 45]. This leads to equations that are called Killing spinor equations, since the supersymmetry transformation of the gravitino equals the supercovariant derivative of the spinor, where the spin connection of this derivative contains the usual gravitational spin connection, plus a term containing fermions and gauge fields. Such equations and the corresponding solutions that are solitons of string theory will be the subject of chapter 4.

1.4 Perturbative String Theories and Low Energy Approximation

With this background at hand, one is now set to discuss string theories and their dualities.

String theories are maps of a two-dimensional world-sheet with metric $g_{\alpha\beta}$ and coordinates σ, τ into a target space with coordinate X^M and metric G_{MN}. It shall be briefly described how string theories are related to space-time fields.

The $d = 2$ gravity of the world sheet may be decoupled by restricting the string to propagate in a specific dimension, which is $d = 26$ if only bosonic fields live on the world sheet, and $d = 10$ for the superstring with fermionic world sheet partners. Two different cases may be distinguished: the closed string where the spacetime embedding coordinates are periodic with respect to the world sheet coordinate σ,

$$X^M(\sigma + 2\pi) = X^M(\sigma)$$

and the case when the string is open and ends in spacetime with specific boundary conditions.

The closed superstring shall be taken as illustration here. The simplest action may be given by

$$S = -\frac{1}{8\pi}\int d\sigma d\tau \eta^{\alpha\beta}\left(\partial_\alpha X^M \partial_\beta X^N + 2i\bar{\Psi}^M \rho^\gamma \partial_\gamma \Psi^N\right)\eta_{MN}$$

where $\eta^{\alpha\beta}$ is the $d = 2$ Minkowski metric, η_{MN} the Minkowski metric in target space. Ψ_α is a real $d = 2$ Majorana spinor, and ρ^γ are the $d = 2$ gamma matrices.

Denoting spinor indices by $\{\pm\}$ and light cone coordinates by $\sigma^\pm = \tau \pm \sigma$, one may expand the fields Ψ_α, X^M into modes by

$$\begin{aligned}
\partial_\pm X^M &= \sum_{n\in\mathbb{Z}} \alpha^{\pm\,M}_{-n} e^{-in(\tau\pm\sigma)} \\
\Psi^M_\pm &= \sum_{(r+\phi)\in\mathbb{Z}} \beta^{\pm\,M}_{-r} e^{-in(\tau\pm\sigma)}.
\end{aligned}$$

ϕ takes account of the following: For the fermions, one may choose periodic (R=Ramond) or antiperiodic (NS=Neveu-Schwarz) boundary conditions, implying $\phi = 0$ or $\phi = 1/2$. The canonical quantization then uses commutation relations

$$\begin{aligned}
[\alpha^{\pm\,M}_n, \alpha^{\pm\,N}_n] &= n\delta_{m+n,0}\eta^{MN} \\
\{\beta^{\pm\,M}_r, \beta^{\pm\,N}_s\} &= \delta_{r+s,0}\eta^{MN}
\end{aligned}$$

to define states

$$\alpha^{+\,M_1}_{-n_1} \ldots \beta^{+\,M_k}_{-n_k} |p_+, a\rangle \alpha^{-\,M_1}_{-n_1} \ldots \beta^{-\,M_k}_{-n_k} |p_-, b\rangle.$$

where the p_+, p_- are momentum zero modes. These states carry representations of the spacetime Lorentz group and are identified with fields in spacetime. The parameters a, b indicate chirality of a fermionic ground state.

The tensoring of the right and left moving states leads to different sectors. Bosons correspond to either R-R or NS-NS states. One gets two theories in $d = 10$ with common NS-NS sector containing a spin 2 field identified with the graviton, an antisymmetric two-form potential and a dilaton. The theories have different RR sector: A one-form and a three-form potential or another scalar, two-form and a four-form potential, corresponding to type IIA and type IIB $N = 2$ supergravity in $d = 10$.

The bosonic string propagates in $d = 26$, but has a tachyon in the spectrum. However, it may be used (by compactification, see below) to construct a chiral theory in $d = 10$ by combining a right-moving bosonic and left-moving superstring to obtain a theory with field content corresponding to the above NS-NS sector, and additional gauge fields transforming in the adjoint of $E_8 \times E_8$ or $SO(32)$. This corresponds to $N = 1$ supergravity in $d = 10$.

Finally, one can also consider open strings, that lead to the spectrum of type I supergravity theory in $d = 10$ corresponding to a graviton, antisymmetric two-form, dilaton, and a gauge field transforming in the adjoint of $SO(32)$.

If the curvature of the space is small, the theory and dynamic can be analyzed perturbatively. If the characteristic length scale is long compared to the string, it is possible to ignore the internal structure of the string and use a low energy effective theory that corresponds to a usual point particle quantum field theory with specific cutoff. This theory is determined as follows. The massless spectrum of the string gives the field content. Weyl invariance of the world-sheet implies the equations of motion for these fields, and they are in lowest order identical to those of the above supergravity theories. One may then also check that the corresponding partition functions agree.

For many applications and discussions the low-energy approximation is used in the literature, especially, since the "traditional" string theories are defined only perturbatively. The low energy theory however admits solitons carrying charges that are not carried by the perturbative string. This issue will become important when discussing dualities. Especially solitons that correspond to exact solutions of string theory, that is, where the higher order corrections to the equations of low energy supergravity are known to vanish, are of interest, and such solutions will be discussed in chapter 4.

1.5 String Theory Dualities

1.5.1 T-duality

T-duality, or target space duality, is a perturbative symmetry of string theory connecting strings propagating on different target spaces (see [49] for a review).

Consider the purely closed bosonic closed string in the following notation

$$S = \frac{1}{4\pi\alpha'}\int d\sigma^2 \eta^{\mu\nu}\partial_\mu X^M \partial_\nu X^N \eta_{MN}$$

where σ, τ are coordinates on the two-dimensional world-sheet, and X^M are coordinates of the embedding target space. As mentioned, an obstacle to string theories is that they are confined to 26 for the bosonic and 10 dimensions for the superstring by the condition that the spectrum is free of ghosts. One is therefore interested in compactifications of string theory, that is, when the target space splits into a lower dimensional space and a decoupled "internal" space that is supposed to be invisible for the physical fields.

The most simple example is given in the above framework by assuming that one coordinate is compact. Say X^{25} is this coordinate. The periodicity condition on this coordinate is

$$X^{25}(\sigma + 2\pi) \to X^{25}(\sigma) + 2\pi R m, m \in \mathbb{Z}.$$

which reflect the fact that the string can wind m times along the compact direction. The mode expansion of this coordinate is then given by

$$\begin{aligned} X^{25} &= X^{25}_R + X^{25}_L \\ X^{25}_R(\tau - \sigma) &= \tfrac{1}{2}x^{25} + \tfrac{\alpha'}{2}p_R(\tau - \sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\tfrac{1}{n}\alpha^{-\,25}_n e^{-in(\tau-\sigma)} \\ X^{25}_L(\tau + \sigma) &= \tfrac{1}{2}x^{25} + \tfrac{\alpha'}{2}p_L(\tau + \sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\tfrac{1}{n}\alpha^{+\,25}_n e^{-in(\tau+\sigma)} \end{aligned}$$

where the momentum along the compact direction is quantized, since the direction is compact, by

$$p^{25} = \frac{n}{R}$$

and the left- and right-moving momentum zero-modes in the above expansion are defined by

$$p_R^{25} = \sqrt{\frac{1}{\alpha'}}\left(\frac{\sqrt{\alpha'}}{R}n - \frac{R}{\sqrt{\alpha'}}m\right)$$
$$p_L^{25} = \sqrt{\frac{1}{\alpha'}}\left(\frac{\sqrt{\alpha'}}{R}n + \frac{R}{\sqrt{\alpha'}}m\right).$$

The mass of the state is then given by evaluating the Hamiltonian constraint of the string: The canonical momentum is given by

$$P^M = \frac{1}{2\pi\alpha'}\partial_\tau X^M$$

(its zero mode equals the momentum zero modes of the coordinates) and the Hamiltonian is given by

$$H = \int d\sigma((\partial_+ X)^2 + (\partial_- X)^2)$$

and annihilates a physical state. Inserting the mode expansions of the coordinates leads to, with

$$m^2 = -\sum_{M=0...24} p_M p^M$$

where the p_M are the momentum zero modes of X^M, to

$$M^2 = \frac{2}{\alpha'}(N_R + N_L - 2) + \frac{1}{\alpha'}\left(n^2\frac{\alpha'}{R^2} + m^2\frac{R^2}{\alpha'}\right)$$

where N_R, N_L are the oscillator quantum numbers, obeying the constraint

$$N_R - N_L = mR$$

resulting the condition that rigid σ translations on the world-sheet should not change the physics for the closed string [81].

The mass is invariant under the map

$$\frac{R}{\sqrt{\alpha'}} \leftrightarrow \frac{\sqrt{\alpha'}}{R}, \quad n \leftrightarrow m.$$

which indicates that the string "looks" the same whether it is considered at small or at large radius of the internal circle. This symmetry is called target space or T-duality, and it is known that it is a symmetry of the string partition function to all orders.

The radius is often called modulus and is associated with the vacuum expectation value of a scalar field new in the string spectrum; it is generated by the internal creation operators along the compact direction.

So far, only the transformation of the momentum zero modes has been addressed. The other modes, however, transform as well. More generally, it can be seen that the OPEs of the world sheet conformal field theory do not change if the T-duality transformation is summarized, including the remaining modes, by

$$X_R \to -X_R, \;\; X_L \to X_L.$$

for the compact coordinate. This is important in the context of the superstring, where world-sheet fermions and their oscillators are included. The above transformation amounts to a space-time chirality in target space, interchanging type IIA and type IIB theory. Both theories are thus related by T-duality (see [2] for a recent discussion).

To compactify to lower dimensions, the easiest example is compactification on the torus. Consider again the closed bosonic string. The "internal" part of the action is given by

$$S = \frac{1}{4\pi}\int d\sigma^2 \sqrt{g} g^{\mu\nu}\partial_\mu X^i \partial_\nu X^j G_{ij} + \epsilon^{\mu\nu}\partial_\mu X^i \partial_\nu X^j B_{ij} - \tfrac{1}{2}\sqrt{g}\Phi R^{(2)}$$

where the constant matrices G_{ij}, B_{ij} describe the geometry of the torus and are interpreted as vacuum expectation values of the corresponding string fields. Defining a vielbein $G_{ij} = E_i{}^a E_j{}^a$, one has an orthonormal basis in target space.

The coordinates get identified as

$$X^i(\sigma + 2\pi) = X^i(\sigma) + 2\pi m^i$$

where the m^i are winding modes parallel to above, and the canonical momentum of the coordinate X^i is now

$$2\pi P_i = G_{ij}\partial_\tau X^j + B_{ij}\partial_\sigma X^j$$

and the zero mode gets again quantized due to the compactness of the coordinate X_i with integer momentum n_i.

The zero mode part of the mass of the string is then of the form, by calculating the Hamiltonian parallel to above,

$$\begin{aligned} M^2 &= \tfrac{1}{2}\left(n_i G^{-1\,ij} n_j + m^i (G - BG^{-1}B)_{ij} m^j + 2m^i B_{ik}(G^{-1})^{kj} n_j\right) \\ &= \tfrac{1}{2} Z^t M Z, \end{aligned} \tag{1.2}$$

where

$$Z = (m_i, n_j)^t, \quad M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$

The mass is invariant under the $O(d, d, \mathbb{Z})$ transformation acting simultaneously as

$$M \to gMg^t, \;\; Z \to gZ,$$

and the same is true for the partition function. The internal momenta are given by

$$p_R = (n^t + m^t(B - G))_i E_a{}^i, \quad p_L = (n^t - m^t(B - G))_i E_a{}^i,$$

they lie on an even self-dual lattice. In the low energy effective field theory, T-duality is represented by acting on the above moduli and the charges of the corresponding U(1) fields arising by action of the compact creation operators.

1.5.2 S-duality

In contrast to the above perturbative symmetry, non-perturbative symmetries have drawn much attention. Among those, duality symmetries relating theories at strong and weak coupling are of special interest, since they allow to address questions at strong coupling by means of weak coupling methods. As will be explained in chapter 4, the microscopic interpretation of black hole entropy relies on such a symmetry.

For the heterotic string theory compactified on the torus, strong-weak coupling duality was investigated in a series of papers (see e.g. [87, 89, 88]). The theory admits a scalar field which is a complex combination of an axion and the string coupling constant (originally called S in [37]), and an $SL(2,\mathbb{R})$ acts by fractional linear transformation on this field and the gauge fields of the theory, resembling an electromagnetic duality transformation. It was conjectured that this symmetry is an exact symmetry of the heterotic string theory. Since such a duality seems hard to "prove" due to the poor understanding of physics at strong coupling (and the fact that the heterotic string is defined perturbatively only), several recipes that "test" these dualities were developed. The low energy effective theory should admit such a strong-weak coupling duality as a symmetry of the equations of motion and/or the Lagrangian. Furthermore, as discussed above, states that saturate the Bogomolnyi bound live in shortened multiplets and are protected from quantum corrections if the theory has supersymmetry $N \geq 4$, and thus this spectrum, calculated with the tree-level Lagrangian, should show S-duality symmetry. Such tests have been performed successfully e.g. in [87].

This and similar tests have been applied to several different string theories. In contrast to the above case which is a self-duality of one string theory, different string theories on different manifolds were compared, and agreement was found in a astonishing variety of cases that led to the name "web of dualities" (see e.g. [90] for a review).

This thesis is mostly concerned with the case corresponding to toroidal compactifications of the superstring. As will become clear shortly, the superstring is special in some respects. Before addressing this point, it shall first be mentioned that the type IIB string was suggested to have a similar $SL(2,\mathbb{Z})$ duality as the heterotic string. The type IIB string has two scalar fields in the massless bosonic spectrum: the dilaton and a scalar field form the RR sector of the theory sometimes called an axion. Thus, a similar $SL(2,\mathbb{Z})$ acting on a complex combination of these fields as in the heterotic case has been suggested in [83], involving a strong-weak coupling duality. On the other hand, this symmetry mixes the three-form field strength from the NS-NS sector and the R-R sector of the theory, where the first carries charge of the fundamental string, while the second corresponds to the charge of a solitonic string.

1.5.3 M-theory Conjecture

It is well known that the type IIA supergravity, the low energy effective field theory, is a maximal supergravity and related to simple eleven-dimensional supergravity by Kaluza-Klein reduction. More specifically, the dilaton of type IIA theory, corresponding to the string coupling constant, is proportional to the radius of the circle eleven dimensional supergravity is compactified on. Using this, it was argued that, for strong coupling, the theory decompactifies to an eleven dimensional theory, which is now commonly called M-theory. It has eleven dimensional supergravity as low energy effective limit [93, 97].

Considering BPS states, type IIA theory should then contain states that carry charges with respect to the Kaluza Klein field along the compact direction. However, this field is in the R-R sector, and the perturbative type IIA string does not couple to it. It was noticed, however, that objects exist within a quantized theory that couple to these charges: they correspond to open strings that have Neumann boundary conditions in (p+1) dimensions and Dirichlet boundary conditions in the remaining directions, and are usually called Dp-branes [81]. The introduction of Dp-branes has been one of the most revolutionary developments in string theory, which is remarkable since they were known for six years before they caught attention in this context [27]. The D0-brane carrying charge with respect to the Kaluza-Klein field was shown to be exactly the state needed here. In the context of the M-theory conjecture, the above type IIB S-duality may be reinterpreted: It was found that, if type IIB theory is compactified on S^1, the S-duality is simply the modular group of the torus when including the eleventh direction. Compactifications of M-theory have been studied, and it is of special interest that the heterotic string arises by a compactification on an interval (orbifold) (see [90]).

What is M-theory? Eleven dimensional simple supergravity, though being the "maximal" supergravity filling the spin 2 multiplet, is known to be non-renormalizable and therefore suffers from ultraviolet divergencies. It is tempting to suspect that the supergravity is just a low energy approximation in the same way it is for string theories. Since the theory admits a membrane and a five-brane solution, M-theory could be thought of as theory of membranes and five-branes. A review of the quantized supermembrane theory has recently been given in [78], and the reader is referred to this review for a discussion of membrane theory. Recently, a theory of free D0-particles has been understood to give the same Lagrangian as the super-membrane in the infinite momentum frame. This theory is known as Matrix theory. For a discussion of successes and drawbacks in identifying it with M-theory the reader is referred to the extensive literature (see e.g. [54]).

While the structure of M-theory is not determined yet, it seems profitable to study its symmetries. This is a main subject of this thesis, and shall be addressed now by discussing U-duality.

1.5.4 U-duality

Consider M-theory/type II string theory compactifications on the torus.

In this theory, one expects the above type IIB S-duality to be present, as well as the target space symmetry arising from the compactification on the torus. However, both groups do not need to commute, a phenomenon first discovered in [89].

This has led to the conjecture [97] that a larger duality group is spanned by the non-commuting action of the type IIB S-duality and T-duality, forming a duality group $G(\mathbb{Z})$ by

$$G(\mathbb{Z}) = SL(2,\mathbb{Z}) \bowtie SO(d,d;\mathbb{Z}), \tag{1.3}$$

where $\bowtie$ refers to the non-commuting action of both groups.

For M-theory compactified on T^{d+1}, the modular group of the $d+1$-torus is $SL(d+1,\mathbb{Z})$, containing the above $SL(2,\mathbb{Z})$ as a subgroup, as stated above. Therefore, rather than (1.3), the definition

$$G(\mathbb{Z}) = SL(d+1,\mathbb{Z}) \bowtie SO(d,d;\mathbb{Z}) \tag{1.4}$$

may be adopted.

The original U-duality conjecture, however, was stated differently [56] and used the hidden symmetries of low energy supergravity ([59, 19], see also [21, 22]).

The low-energy effective field theory of type II theory on T^6 is $d = 4$ $N = 8$ maximal supergravity and has $E_{7(+7)}$ global symmetry [20] acting in the fundamental **56** representation. It is a electromagnetic duality transformation acting on the charges, and simultaneously on the scalar fields.

The Dirac-Schwinger-Zwanziger quantization condition in four dimensions discretizes electric and magnetic charges of the $U(1)$ gauge fields of the theory. The global $E_{7(+7)}$ symmetry is therefore broken to a discrete $E_{7(+7)}(\mathbb{Z})$ symmetry, inducing integer shifts on the charge lattice. It was conjectured in [56] that this group is the full non-perturbative duality symmetry group of type II string theory compactified on the torus to $d = 4$.

This has an interesting consequences. The global $E_{7(+7)}$ symmetry acts on the 70 scalars of the theory. They are therefore put on the same footing by this symmetry, and, specifically, the eleventh compact "M-theory" direction is treated on the same footing as the other compact directions, which makes this symmetry a true M-theory symmetry.

It was furthermore shown in [56] that elementary solitons exists for each type of charge in the **56**-plet. This is a necessary condition for the DSZ quantization condition to restrict all charges to integers. It will be shown that this is actually a purely group theoretical consequence of the

Dimension	Supergravity Duality Group G	String T-duality	Conjectured U-duality
$10A$	$SO(1,1)/\mathbb{Z}_2$	$\mathbf{1}$	$\mathbf{1}$
$10B$	$SL(2,\mathbb{R})$	$\mathbf{1}$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times O(1,1)$	$\mathbb{Z}_2$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$O(2,2;\mathbb{Z})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$O(5,5)$	$O(3,3;\mathbb{Z})$	$O(5,5;\mathbb{Z})$
6	$SL(5,\mathbb{R})$	$O(4,4;\mathbb{Z})$	$SL(5,\mathbb{Z})$
5	$E_{6(6)}$	$O(5,5;\mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}$	$O(6,6;\mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}$	$O(7,7;\mathbb{Z})$	$E_{8(8)}(\mathbb{Z})$
2	$E_{9(9)}$	$O(8,8;\mathbb{Z})$	$E_{9(9)}(\mathbb{Z})$
1	$E_{10(10)}$	$O(9,9;\mathbb{Z})$	$E_{10(10)}(\mathbb{Z})$

Table 1.1: U-duality groups

56 representation $E_{7(+7)}$ transforms under. The above solutions were derived by compactifying and/or wrapping higher dimensional solitonic brane solutions on the torus. The fact that the complete multiplet was found to be solitonic indicated that actually the complete type II string and/or M-theory may be understood as a soliton and is remarkable concerning the structure of M-theory.

The discrete group $E_{7(+7)}(\mathbb{Z})$ was specified in [56] by the following argument: on the charge lattice, choose normalizations such that all charges are integers. Then $E_{7(+7)}(\mathbb{Z})$ is the subgroup in $E_{7(+7)}$ of matrices whose entries are entirely integers, that is, that preserve the charge lattice. A direct construction of $E_{7(+7)}(\mathbb{Z})$ was not given, rather, the following argument was used: it is known that the most general duality group on the field configuration discussed is $Sp(2N,\mathbb{R})$ [41]. Thus, acting on the charge lattice, the $E_{7(+7)}(\mathbb{Z})$ discrete group is

$$E_{7(+7)}(\mathbb{Z}) = E_{7(+7)} \cap Sp(56,\mathbb{Z}) \tag{1.5}$$

where $Sp(56,\mathbb{Z})$ consists of matrices with entirely integer entries.

In this thesis, the intersection with the symplectic group is not used for construction, rather, the discrete group will be constructed by defining it directly by the condition that it preserves the charge lattice. It follows from the above construction that this is identical to (1.5).

For applications of U-duality it is instructive to have a set of generators of the duality group. The above definition as intersection with the discrete symplectic group is however unsuitable for this, since the corresponding embedding is rather complicated.

A main focus in this thesis is to determine a set of generators for the above discrete U-duality group. Once a charge lattice has been defined, the group will be shown to be uniquely determined by a remarkable fact: **56** is the unique minimal representation of $E_{7(+7)}$, and this will be used to determine the set of generators. The corresponding definition is then used to show that the definitions (1.3) and (1.4) agree with the one found here.

Higher dimensional dualities of M-theory on the torus are found by intersections with the discrete group $E_{7(+7)}(\mathbb{Z})$. They correspond to table 1.1 given in [56]. However, for $d \leq 4$, the situation is more complicated.

The classical continuous duality symmetry in three dimensions is $E_{8(+8)}$, explicitly used for construction in [67]. But the meaning of a duality group is not clear since only scalars remain in the theory and electric charge seems ill defined. It will be shown that elementary solitons in this theory, given by compactifying arrays of solitons in $d = 4$ carrying just one charge, are not asymptotically flat and correspond to vortex solutions.

In this thesis, an explicit construction of the U-duality group in three dimensions is given by extending a conjecture made by Hull and Townsend [56], parallel to a method applied to the heterotic string by Sen [89, 88]. It relies on the fact that eight different ways exist how to compactify first to four and then to three dimensions. This gives eight different four dimensional U-duality groups embedded into the $d = 3$ theory. It will be discussed how these groups merge together to give the $d = 3$ duality.

The compactification to $d = 3$ has several interesting features. First of all, the above method will give a quantization condition in $d = 3$ form a $d = 4$ one. One the other hand, reduced models are often useful to uncover the structure of a theory. The two last rows in table 1.1 are interesting: the procedure outlined above is suitable for lower dimensions as well, and it is an interesting question what a discrete subgroup of a affine or hyperbolic Kač-Moody group might be. This question is left for future work based on the results given here, hence the last two lines in table 1.1 still are somewhat of a mystery.

U-duality has several applications. It is a main tool in finding proper configurations that allow a microscopic interpretation of black hole entropy, as will be shown in chapter 4. For all these applications, an explicit knowledge of the way U-duality acts is useful and sometimes seems to have lacked in the literature. This was also a motivation to study the way these group acts on M-theory fields closely by using the results of [20].

It shall also be noted, though it will not be discussed in this thesis, that U-duality, rephrased in an algebraic language [33], has been used in the infinite momentum frame of Matrix theory, predicting BPS states for compactifications not yet accessible to Matrix theory and new "mysterious" states especially for compactifications to three dimensions. U-duality extends to a generalized electric-magnetic duality of the Super-YM theory. Upon inclusion of the light-like momentum on the M-theory circle, the group was even proposed to be extended by rank (see [79] for a review and references therein).

In [33], the Weyl group of U-duality was used and charge orbits discussed, while in [79] a set of generators within the algebraic formulation is proposed using the definition 1.4. These generators will be compared with the set of generators found in this thesis. U-duality in [79] was applied to study BPS spectra in diverse dimensions, to generate U-duality invariant mass formulae and to study R^4 corrections to type II theory as well as the implications on Matrix theory, and the reader is referred for these subjects to this extensive review.

1.6 Structure of the Thesis

This thesis is structured as follows. In chapter two, U-duality will be studied in a toy model that closely resembles the structure of M-theory and serves to introduce the main concepts. It will be shown how U-duality in $d = 4$ is defined, and how the $d = 3$ group may be constructed, in detail.

In chapter three, U-duality in M-theory will be discussed. U-duality in $d = 4$ will be identified by the discretization implied by the DSZ-quantization condition, and a proof is presented for a set of generators for arbitrary Lie groups acting on admissible lattices in basic representations. T- and S-duality subgroups are identified and discussed, and the definitions of U-duality outlined above are compared. U-duality is then constructed for $d = 3$ following the same procedure as in the toy model, but, as will be shown, remarkably with different result.

In chapter four, some applications of U-duality are shown in the context of black hole solitons and black hole entropy. Furthermore, a result on BPS-conditions and Null-Killing [42] is discussed from a strictly supersymmetric perspective.

Chapter five concludes this thesis.

In the appendix, Lie algebras are reviewed and `Maple` Procedures described that generate highest weight matrix representations and have been used for this thesis.

Chapter 2

Discrete Duality Symmetries in $d = 4$ and Lower Dimensions: The $G_{2(+2)}$ Toy Model

2.1 Motivation

It has been known for a long time that simple $d = 5$ supergravity closely resembles $d = 11$ supergravity [13, 19]. Apart from the gravitational fünfbein, it contains a five-dimensional vector field and a Dirac spinor Ψ_M, parallel to the elfbein, three-from potential and Rarita-Schwinger Majorana fermion of simple $d = 11$ supergravity. Its Lagrangian turns out to have exactly the same structure as the $N = 1$ supergravity in $d = 11$.

The theory does not contain a scalar field and may therefore, referring to a possible higher dimensional "origin", be called a no-moduli theory. Such theories are of interest for string phenomenology (see e.g. [29]). It is known that the model arises by a Calabi-Yau compactification of $d = 11$ simple supergravity [10, 34, 35], but an additional truncation is needed in the scalar multiplets. To obtain the model from a Calabi-Yau or orbifold compactification of string theory or M-theory without truncations seems difficult, since the dilaton needs to be "orbifolded away". However, asymmetric orbifold compactifications have been studied intensely in the literature, and it is tempting to associate this theory with a strong coupling limit of a theory in $d = 4$ with only the dilaton as modulus, the way M-theory is associated with type IIA string theory. However, such a construction has not been given yet, though the authors of [26] state that there seems to be "no fundamental reason" that it should not exist.

The theory has a solitonic string and particle solution of very similar structure than the M2 and M5 brane [47, 44]. In [72, 40, 39], this similarity has been studied and motivated investigations of the AdS/CFT conjecture ([66], see [3] for a recent review) in this theory. In [16], cosmological solutions of this model have been constructed, motivated by the manifestly U-duality covariant solutions in $d = 5$ from Calabi-Yau three-fold compactifications of M-theory [64].

In this thesis, the model will be used as a toy model for U-duality in M-theory. It will be shown how a symmetry resembling U-duality may be found, what its generators are and how the interplay of different compactifications yields the generators in lower dimensions. The important mechanisms will be introduced and discussed in detail in this model in order to simplify the discussion of M-theory in the next chapter.

In section 2.2, the five dimensional model is introduced, and in order to study the symmetry resembling U-duality, it is compactified to $d = 4$. The analogue of stringy U-duality is constructed and discussed in section 2.3. The construction of the U-duality group in $d = 3$ is then discussed in section 2.4.

2.2 M-theory Analogue: $d = 5$ Simple Supergravity

The arguments of the following sections and the definition of U-duality will be given on the purely bosonic side of the theory, as is usual in the literature. This is possible since the fermionic side, corresponding to the definition of supersymmetry, follows by using supersymmetry transformations.

The corresponding part of the Lagrangian of $d = 5$ simple supergravity [73, 72, 13, 19] will be used in the following form:

$$\mathcal{L} = -E^{(5)}(\mathcal{R}^{(5)} + \frac{1}{4}F_{MN}F^{MN}) - \frac{1}{12\sqrt{3}}\epsilon^{MNPQR}F_{MN}F_{PQ}A_R, \tag{2.1}$$

where $F_{MN} = 2\partial_{[M}A_{N]}$. $M, N, \dots$ are curved and $A, B, \dots$ flat five-dimensional indices.

The Lagrangian is invariant under reparametrizations and gauge transformations (up to a total derivative), with parameters χ^M, ξ, and local Lorentz transformations $\Lambda^A{}_B$, acting as

$$\begin{aligned} E_M^{(5)\,A} &\to E_M^{(5)\,A} + \partial_M\chi^N E_N^{(5)\,A} + \chi^N\partial_N E_M^{(5)\,A} + \Lambda^A{}_B E_M^{(5)\,B}, \\ A_M &\to A_M + \partial_M\chi^N A_N + \chi^N\partial_N A_M + \partial_M\xi. \end{aligned} \tag{2.2}$$

The analogue of U-duality will be defined by using $d = 4$ arguments, and hence, the compactification to $d = 4$ shall be addressed now. Note that, here as well as in the next chapter, it will be important to keep in touch with the $d = 5$ fields. This will be explicitly used when discussing $d = 3$ U-duality. Quite generally, many questions and phenomena in M-theory are addressed by looking at the $d = 11$ fields, and thus it is important, in the compactified theory, to know the corresponding relations and constraints.

Assuming that the 5th direction is compact and cannot be probed with energies below a certain cutoff set, a consistent low-energy approximation is to keep only the zeroth Fourier component of the fields with respect to the fifth coordinate. It shall therefore be assumed that the above Lagrangian is independent of the fifth coordinate. The Lagrangian may then be reduced to an effectively $d = 4$ Lagrangian. Consider first the Einstein-Hilbert action alone.

2.2.1 Reduction to $d = 4$: Gravity

The compactification shall be discussed generally as reduction step from $d + 1$ to d dimensions in this section. This will be useful for later sections. Consider the Einstein-Hilbert action in the form

$$\mathcal{S} = \int d^{d+1}x - \sqrt{-G}\mathcal{R}.$$

It is convenient to use a non-coordinate basis in tangent space by defining

$$\hat{E}_A = E_A{}^M\partial_M, \;\; g(\hat{E}_A, \hat{E}_B) = E_A{}^M E_B{}^N G_{MN} = \eta_{AB}$$

(see e.g. [96], [76]), where η_{AB} is the Minkowski metric with signature $(+ - - \dots -)$, and $E_A{}^M$ is the vielbein with inverse $E_M{}^A$ such that

$$G_{MN} = E_M{}^A E_N{}^B \eta_{AB}.$$

$A, B, \dots$ are flat indices, lowered and raised by the Minkowski metric, and $M, N, \dots$ curved indices. The Lie bracket of the non-coordinate basis is non-vanishing,

$$[\hat{E}_A, \hat{E}_B] = \Omega_{AB}{}^C\hat{E}_C,$$

where

$$\Omega_{AB}{}^{C} = 2\, E_A{}^M E_B{}^N \partial_{[N} E_{M]}^{C}$$

are the anholonomy coefficients, and $[AB]$ denotes antisymmetrization as usual,

$$c_{[AB]} = \tfrac{1}{2!}(c_{AB} - c_{BA}).$$

The connection coefficients in this basis are given by

$$\omega_{AMN} = (E_M^{\ C} \nabla_A E_N{}^D)\eta_{CD}.$$

Demanding that the connection has no torsion, one may express the Ricci scalar by the anholonomy coefficients only. Vanishing torsion implies

$$\Omega_{ABC} = \omega_{BCA} - \omega_{ACB},$$

and this yields the Ricci scalar

$$\mathcal{R} = \mathcal{R}^A{}_{BA}{}^B = 2\, \nabla_A \Omega_B^{A\ \ B} - \tfrac{1}{4}\, \Omega_{ABC}\Omega^{ABC} + \tfrac{1}{2}\, \Omega_{ABC}\Omega^{CAB} + \Omega_{CB}{}^B \Omega^C{}_A{}^A.$$

Thus the Einstein-Hilbert action becomes

$$\mathcal{S} = \int d^D x \;\; E\, \tfrac{1}{4}\left(\Omega_{ABC}\Omega^{ABC} - 2\, \Omega_{ABC}\Omega^{CAB} + 4\, \Omega_{CB}{}^B \Omega^C{}_A{}^A\right)$$

where $E = \det E_M^A$, neglecting the total derivative.

Assume now that the Einstein-Hilbert action does not depend on one specific coordinate, and call this coordinate z as curved and $\bar{z}$ as flat index here. The indices in d dimensions will denoted by $\mu, \nu, \ldots$ for curved and $\alpha, \beta, \ldots$ for flat indices. A convenient gauge choice for the vielbein, using local $SO(1, d)$ invariance, is then

$$E_M^{(d+1)\,A} = \begin{pmatrix} e^{y\phi} E_\mu^{(d)\,\alpha} & e^{x\phi} B_\mu \\ 0 & e^{x\phi} \end{pmatrix}. \tag{2.3}$$

where x, y are real numbers. Let $\Omega^{(d)}_{\alpha\beta\gamma}$ be the anholonomy coefficients with respect to $E_\mu^{(d)\,\alpha}$. In terms of the above vielbein, one has

$$\begin{aligned}
\Omega^{(d+1)}_{\alpha\beta\gamma} &= e^{-y\phi}\left(\Omega^{(d)}_{\alpha\beta\gamma} - 2y\, E^{(d)\,\mu}_{[\alpha} \partial_\mu \phi \eta_{\beta]\gamma}\right), \\
\Omega^{(d+1)}_{\alpha\beta\bar{z}} &= e^{(-2y+x)\phi} E_\alpha^{(d)\,\mu} E_\beta^{(d)\,\nu} B_{\mu\nu}, \;\; B_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}, \\
\Omega^{(d+1)}_{\alpha\bar{z}\bar{z}} &= x e^{-y\phi} E_\alpha^{(d)\,\mu} \partial_\mu \phi,
\end{aligned}$$

all remaining anholonomy coefficients equal zero. This yields, after integrating out the coordinate z and demanding $x = y(2 - d)$, the action

$$\mathcal{S} = \int d^{(d)}x\, E^{(d)} \;\left(-\mathcal{R}^{(d)} + y^2(d^2 - 3d + 2)\, \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{4} e^{2y(d-1)\phi} B_{\mu\nu} B^{\mu\nu}\right). \tag{2.4}$$

The field $B_{\mu\nu}$ is a $U(1)$ gauge field: Consider reparametrization invariance in $d+1$. The vielbein transforms as

$$E_M^{(d+1)\,A} \to E_M^{(d+1)\,A} + \partial_M \chi^N E_N^{(d+1)\,A} + \chi^N \partial_N E_M^{(d+1)\,A}.$$

This implies

$$E_\mu^{(d+1)\,A} E_A^{(d+1)\,z} = E_\mu^{(d+1)\,\bar{z}} E_{\bar{z}}^{(d+1)\,z} = B_\mu \to B_\mu + \partial_\mu \chi^z,$$

that is, the residual reparametrization invariance with respect to the coordinate z acts as $U(1)$ gauge transformation of the gauge field B_μ. This is actually the key point of the Kaluza-Klein mechanism: $d + 1$ gravity with one compact dimension yields gravity plus electromagnetism in d dimensions, plus a dilaton that may be set to zero.

2.2.2 Reduction to $d = 4$: $d = 5$ Simple Supergravity

Consider now the full theory. As will be seen, the reduction yields a theory that has a generalized electromagnetic $SL(2,\mathbb{R})$ symmetry mixing magnetic and electric sector. This symmetry will be discussed in the next section. Since it is not visible in the "naïve" reduction, it is often called a "hidden" symmetry, and one needs to define the fields of the theory in a proper way to uncover it.

For the reduction, the fünfbein shall be used as

$$E^{(5)A}_{\ M} = \begin{pmatrix} \rho^{-\frac{1}{2}} E^{(4)\bar{\alpha}}_{\bar{\mu}} & \rho B_{\bar{\mu}} \\ 0 & \rho \end{pmatrix},$$

$\bar{\mu}, \bar{\nu}, \ldots$ are curved and $\bar{\alpha}, \bar{\beta}, \ldots$ flat indices in four dimensions.

The metric gets

$$G^{(5)\bar{\rho}\bar{\kappa}} = \rho G^{(4)\bar{\rho}\bar{\kappa}}, \;\; G^{(5)4\bar{\kappa}} = -\rho B^{\bar{\kappa}},$$
$$G^{(5)44} = -\rho^{-2} + \rho B_{\bar{\kappa}} B^{\bar{\kappa}}, \;\; E^{(5)} = \rho^{-1} E^{(4)}.$$

Writing $A_M = \{A_{\bar{\mu}}, A_4\}$, the Lagrangian gets up to a total derivative, using (2.4),

$$\begin{aligned} \mathcal{L} \;=\; & -E^{(4)} R^{(4)} + \frac{3}{2} E^{(4)} \partial_{\bar{\mu}} \ln \rho \, \partial^{\bar{\mu}} \ln \rho + \frac{1}{2} E^{(4)} \rho^{-2} \, \partial_{\bar{\mu}} A_4 \, \partial^{\bar{\mu}} A_4 \\ & - \frac{1}{4} E^{(4)} \rho^3 \, B_{\bar{\mu}\bar{\nu}} B^{\bar{\mu}\bar{\nu}} - \frac{1}{4} E^{(4)} \rho \, F^{(4)}_{\bar{\mu}\bar{\nu}} F^{(4)\,\bar{\mu}\bar{\nu}} \\ & - \frac{1}{4\sqrt{3}} \epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \left(A_4 \, F^{(4)}_{\bar{\mu}\bar{\nu}} F^{(4)}_{\bar{\rho}\bar{\sigma}} - A_4^2 \, F^{(4)}_{\bar{\mu}\bar{\nu}} B_{\bar{\rho}\bar{\sigma}} + \frac{1}{3} A_4^3 \, B_{\bar{\mu}\bar{\nu}} B_{\bar{\rho}\bar{\sigma}} \right), \end{aligned} \tag{2.5}$$

where the field strength

$$B_{\bar{\mu}\bar{\nu}} = 2\partial_{[\bar{\mu}} B_{\bar{\nu}]}$$

is the Kaluza-Klein $U(1)$ gauge field strength as above. A second gauge field strength in the theory is

$$F'_{\bar{\mu}\bar{\nu}} = 2\partial_{[\bar{\mu}} A'_{\bar{\nu}]}, \;\; A'_{\bar{\mu}} = A_{\bar{\mu}} - B_{\bar{\mu}} A_4.$$

$A'_{\bar{\mu}}$ is invariant with respect to reparametrizations along the compact directions, and from (2.2) is a $U(1)$ gauge field. It is convenient to define

$$F^{(4)}_{\bar{\mu}\bar{\nu}} = F'_{\bar{\mu}\bar{\nu}} + B_{\bar{\mu}\bar{\nu}} A_4$$

to simplify the Lagrangian.

To put the Lagrangian in a from such that the mentioned "hidden" symmetry gets manifest, one may follow the recipe as used in [20],[72]. $A'_{\bar{\mu}}$ is dualized by adding a Lagrange multiplier

$$\mathcal{L}_{\text{Lag.mult.}} = \frac{1}{2} \epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \tilde{A}_{\bar{\sigma}} \partial_{\bar{\rho}} F'_{\bar{\mu}\bar{\nu}}.$$

Defining the new field strength $\tilde{A}_{\bar{\mu}\bar{\nu}} = 2\partial_{[\bar{\mu}} \tilde{A}_{\bar{\nu}]}$, this yields

$$\tilde{A}_{\bar{\mu}\bar{\nu}} = \rho \star F'_{\bar{\mu}\bar{\nu}} - \tfrac{2}{\sqrt{3}} A_4 F'_{\bar{\mu}\bar{\nu}} + \rho A_4 \star B_{\bar{\mu}\bar{\nu}} - \tfrac{1}{\sqrt{3}} (A_4)^2 B_{\bar{\mu}\bar{\nu}} \tag{2.6}$$

and the vector part of the Lagrangian may be rewritten, using the vector notation

$$\mathcal{G}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} \tilde{A}_{\bar{\mu}\bar{\nu}} \\ B_{\bar{\mu}\bar{\nu}} \end{pmatrix},$$

as

$$\mathcal{L}_V = -\tfrac{1}{4}E^{(4)}\mathcal{G}^t_{\bar{\mu}\bar{\nu}}N^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}\mathcal{G}_{\bar{\mu}\bar{\nu}}$$

with

$$N^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} = n\mathbf{1}^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} + m\star^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}$$

and

$$\begin{aligned} n &= (1+\tfrac{4}{3}\rho^{-2}(A_4)^2)^{-1}\begin{pmatrix} \rho^{-1} & -(\sqrt{3}\rho)^{-1}(A_4)^2 \\ -(\sqrt{3}\rho)^{-1}(A_4)^2 & \rho^3+\tfrac{4}{3}\rho(A_4)^2+\tfrac{1}{3}\rho^{-1}(A_4)^4 \end{pmatrix}, \\ m &= (1+\tfrac{4}{3}\rho^{-2}(A_4)^2)^{-1}\begin{pmatrix} -2(\rho^2\sqrt{3})^{-1}A_4 & -A_4-\tfrac{2}{3}\rho^{-2}(A_4)^3 \\ -A_4-\tfrac{2}{3}\rho^{-2}(A_4)^3 & \tfrac{2}{3\sqrt{3}}(A_4)^3+\tfrac{2}{9\sqrt{3}}\rho^{-2}(A_4)^5 \end{pmatrix}. \end{aligned}$$

This may be simplified considerably by combining the scalars of the theory, A_4 and ρ, in a field $\mathcal{V}^{(4)} \in SL(2,\mathbb{R})/SO(2)$ (which reflects two free parameters). A convenient representation of this field is given by using an Iwasawa decomposition for a group element $g \in SL(2,\mathbb{R})$ [53]: G admits a decomposition

$$g = khb$$

where k is an element of the maximal compact subgroup, h is an element of the Cartan subgroup, and b is an element of the subgroup with Lie algebra corresponding to all positive roots. Stripping k off, one may thus define

$$\mathcal{V}^{(4)} = P^{-1}\ \exp\left(-\frac{1}{2}\ln\rho H\right)\exp\left(-\frac{1}{\sqrt{3}}A_4E\right)\ P$$

where the Chevalley generators H, E, F of $\mathfrak{sl}(2,\mathbb{R})$ are in the **4** representation, given by

$$H = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix} \tag{2.7}$$

and P is a basis transformation to be specified.

The **4** representation is symplectic. One has

$$\Omega' X \Omega' = X^t, \quad X \in \mathfrak{sl}(2,\mathbb{R})$$

with symplectic from $\Omega' = P\Omega P^{-1}$. If

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (P^{-1})^t, \tag{2.8}$$

Ω is of the usual form

$$\Omega = \begin{pmatrix} & -\mathbf{1} \\ \mathbf{1} & \end{pmatrix}.$$

Coming back to the vector fields, one may define the dual magnetic field strength by

$$H^{\tilde{A}}_{\bar{\mu}\bar{\nu}} = -\frac{2}{E^{(4)}} \star \left(\frac{\delta\mathcal{L}}{\delta\tilde{A}^{\bar{\mu}\bar{\nu}}}\right), \ H^{B}_{\bar{\mu}\bar{\nu}} = -\frac{2}{E^{(4)}} \star \left(\frac{\delta\mathcal{L}}{\delta B^{\bar{\mu}\bar{\nu}}}\right), \ \mathcal{H}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} H^{\tilde{A}}_{\bar{\mu}\bar{\nu}} \\ H^{B}_{\bar{\mu}\bar{\nu}} \end{pmatrix}.$$

Explicitly, one has

$$H^{\tilde{A}}_{\bar{\mu}\bar{\nu}} = -F'_{\bar{\mu}\bar{\nu}}, \quad H^{B}_{\bar{\mu}\bar{\nu}} = (\rho^3 + \rho(A_4)^2) \star B_{\bar{\mu}\bar{\nu}} - \tfrac{2}{3\sqrt{3}}(A_4)^3 B_{\bar{\mu}\bar{\nu}} + \rho A_4 \star F'_{\bar{\mu}\bar{\nu}} - \tfrac{1}{\sqrt{3}}(A_4)^2 F'_{\bar{\mu}\bar{\nu}}). \tag{2.9}$$

It may then be verified that the vector fields obey a relation called a twisted self-duality in [21] by combining all electric and magnetic fields in one four-dimensional vector. The relations is then given by

$$\mathcal{F}_{\bar{\mu}\bar{\nu}} \equiv \begin{pmatrix} \mathcal{G}_{\bar{\mu}\bar{\nu}} \\ \mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix} = \Omega \mathcal{V}^{(4)t}\mathcal{V}^{(4)} \begin{pmatrix} \star\mathcal{G}_{\bar{\mu}\bar{\nu}} \\ \star\mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix}.$$

For this, one writes

$$\mathcal{V}^{t(4)}\mathcal{V}^{(4)} = \begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$$

and notices

$$d^{-1} = m, \ \ bd^{-1} = n.$$

The vector Lagrangian may therefore be written as

$$\mathcal{L}_V = E^{(4)} \left(\frac{1}{4}\mathcal{G}^T_{\bar{\mu}\bar{\nu}} \star \mathcal{H}^{\bar{\mu}\bar{\nu}}\right).$$

It may also be written as

$$\mathcal{L}_V = E^{(4)} \left(\frac{1}{4}\mathcal{G}^T_{\bar{\mu}\bar{\nu}} \star \mathcal{H}^{\bar{\mu}\bar{\nu}}\right) = E^{(4)} \left(\frac{1}{8}\mathcal{F}^T_{\bar{\mu}\bar{\nu}} L \star \mathcal{F}^{\bar{\mu}\bar{\nu}}\right) \tag{2.10}$$

with

$$L = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}.$$

For the scalar part of the Lagrangian, one may easily derive

$$(\partial_{\bar{\mu}}\mathcal{V}^{(4)})(\mathcal{V}^{(4)})^{-1} = (\partial_{\bar{\mu}}(-\tfrac{1}{2}\ln\rho))H - ((\sqrt{3}\rho)^{-1}\partial_{\bar{\mu}}A_4)E.$$

A field $P^{(4)}_{\mu}$ may then be defined by

$$\partial_{\mu}\mathcal{V}^{(4)}\mathcal{V}^{(4)-1} = Q^{(4)}_{\mu} + P^{(4)}_{\mu}, \ \ Q^{(4)}_{\mu} \in \mathfrak{so}(2), \ \ P^{(4)}_{\mu} \in \mathfrak{sl}(2) - \mathfrak{so}(2).$$

Since the maximal compact $\mathfrak{so}(2)$ subalgebra is generated by $\sqrt{2}^{-1}(E - F)$, one has

$$P^{(4)}_{\mu} = (\partial_{\bar{\mu}}(-\tfrac{1}{2}\ln\rho))H - ((\sqrt{6}\rho)^{-1}\partial_{\bar{\mu}}A_4)\sqrt{2}^{-1}(E + F),$$

and with tr $\frac{1}{2}(E+F)^2 = 10$ one arrives at the Lagrangian

$$\mathcal{L} = -E^{(4)}R^{(4)} + E^{(4)}\left(\frac{1}{4}\mathcal{G}^T_{\bar{\mu}\bar{\nu}} \star \mathcal{H}^{\bar{\mu}\bar{\nu}} + \frac{3}{10}\mathrm{Tr}(P^{(4)}_{\bar{\mu}}P^{(4)\bar{\mu}})\right). \tag{2.11}$$

2.3 U-duality in $d = 4$

Apart from reparametrization invariance and gauge symmetries, the Lagrangian (2.11) has further symmetries that will be investigated in this section. Consider first the scalar part of the Lagrangian. It is invariant under a transformation

$$\mathcal{V}^{(4)} \to h(x)\mathcal{V}^{(4)}\Lambda, \;\; \Lambda \in SL(2,\mathbb{R}), \;\; h(x) \in SO(2),$$

where the local $SO(2)$ transformation is parallel to the local Lorentz transformations in the vielbein formalism and is used to restore the $SL(2,\mathbb{R})/SO(2)$ gauge chosen in coset space.

The vector part of the Lagrangian is invariant with respect to the local symmetry, or, in other words, depends only on the "metric" $\mathcal{V}^{t(4)}\mathcal{V}^{(4)}$, but the global $SL(2,\mathbb{R})$ is not a symmetry of the Lagrangian. Rather, the form (2.10) of the Lagrangian indicates a group

$$G = O(2,2) \cap SL(2,\mathbb{R}),$$

which allows $SO(2)$. However, the full $SL(2,\mathbb{R})$ is a symmetry of the equations of motion, since it interchanges Bianchi identities

$$\partial_{\bar{\mu}}(E^{(4)} \star \tilde{A}^{\bar{\mu}\bar{\nu}}) = 0, \;\; \partial_{\bar{\mu}}(E^{(4)} \star B^{\bar{\mu}\bar{\nu}}) = 0$$

with equations of motion

$$\partial_{\bar{\mu}}(E^{(4)} \star H^{\tilde{A}\,\bar{\mu}\bar{\nu}}) = 0, \;\; \partial_{\bar{\mu}}(E^{(4)} \star H^{B\,\bar{\mu}\bar{\nu}}) = 0$$

by acting on the vector $\mathcal{F}^{\bar{\mu}\bar{\nu}}$ as

$$\mathcal{F}_{\hat{\mu}\hat{\nu}} \to \Lambda^{-1}\mathcal{F}_{\hat{\mu}\hat{\nu}}, \;\; \mathcal{V}^{(4)} \to \mathcal{V}^{(4)}\Lambda, \;\; \Lambda \in SL(2,\mathbb{R}).$$

This symmetry is usually called classical (hidden) duality symmetry. It should be kept in mind that the scalars of the theory in the field $\mathcal{V}^{(4)}$ transform simultaneously.

Following [56], one may define the analogue of U-duality in this theory. Define the four-dimensional charge vector

$$\mathcal{Z} = \begin{pmatrix} p \\ q \end{pmatrix}, \;\; p = \frac{1}{2\pi}\oint_{\Sigma} \mathcal{G}, \;\; q = \oint_{\Sigma} \mathcal{H}$$

with

$$p = \begin{pmatrix} p^{\tilde{A}} \\ p^{B} \end{pmatrix}, \;\; q = \begin{pmatrix} q_{\tilde{A}} \\ q_{B} \end{pmatrix},$$

where the position of the indices reflects the symplectic character of the representation introduced above.

The p charges are magnetic, the q charges Noether electric charges, and the charge vector $\mathcal{Z}$ transforms as

$$\mathcal{Z} \to \Lambda^{-1}\mathcal{Z}, \; \Lambda \in SL(2,\mathbb{R})$$

under the classical duality symmetry.

One may now impose the semiclassical DSZ quantization condition on dyons,

$$\mathcal{Z}^t\;\Omega\;\mathcal{Z}' = p^{\tilde{A}}q'_{\tilde{A}} - p'^{\tilde{A}}q_{\tilde{A}} + p^{B}q'_{B} - p'^{B}q_{B} = n, \;\; n \in \mathbb{Z}. \tag{2.12}$$

The situation is complicated in this model by the fact that the fields $\tilde{A}_{\bar{\mu}\bar{\nu}}$ and $B_{\bar{\mu}\bar{\nu}}$ are not treated on the same footing by the $SL(2,\mathbb{R})$ symmetry, but the $B_{\bar{\mu}\bar{\nu}}$ carries weight -3, while $\tilde{A}_{\bar{\mu}\bar{\nu}}$ carries weight +1. It is easy to see that e.g. the solution

$$\{p^{\tilde{A}}, p^B, q_{\tilde{A}}, q_B\} \in \mathbb{Z}$$

to (2.12) is incompatible with the $SL(2,\mathbb{R})$ symmetry generated by (2.7), since no matrix in this basis exists in $SL(2,\mathbb{R})$ that is nontrivial and entirely integral.

An easy way to see this is to consider the Birkhoff decomposition [75] of an element $g \in SL(2,\mathbb{R})$: g admits a triangular decomposition

$$g = g_-\, n\, g_+,$$

where $g_- = \exp(c_{-2}F)$, $g_+ = \exp(c_2 E)$, $c_{-2}, c_2 \in \mathbb{C}$, and n is given by

$$n = \prod_i n_2(t_i), \;\; t_i \in \mathbb{C}, \;\; n_2(t) = \exp(tE)\exp(-t^{-1}F)\exp(tE).$$

Writing out the Birkhoff decomposition in the above basis, one sees that no matrix exists that is entirely integral!

However, this is just a problem of finding the correct basis in $SL(2,\mathbb{R})$ representation space. To find a solution compatible with the $SL(2,\mathbb{R})$ symmetry, change the basis by

$$\tilde{\mathcal{F}} = U\mathcal{F}, \tilde{\mathcal{V}}^{(4)} = U\mathcal{V}^{(4)}U^{-1}, \text{etc.}$$

with

$$U = \begin{pmatrix} 1 & & & \\ & \sqrt{3} & & \\ & & 1 & \\ & & & \sqrt{3} \end{pmatrix}.$$

Then the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ gets

$$\tilde{E} = \begin{pmatrix} 0 & 3 & & \\ & 0 & 2 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \tilde{F} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & 3 & 0 \end{pmatrix}, \tilde{H} = H \tag{2.13}$$

which implies that, in the new basis, the action of the generators $E^n/n!$, $F^n/n!$ is entirely integral. This basis is called an admissible lattice, and such basis' will be studied in more detail when M-theory is considered.

The DSZ quantization condition now reads

$$\tilde{\mathcal{Z}}^T\, U^{-1}\Omega\, U^{-1}\, \tilde{\mathcal{Z}}' = \tilde{p}^{\tilde{A}}\tilde{q}'_{\tilde{A}} - \tilde{p}'^{\tilde{A}}\tilde{q}'_{\tilde{A}} + \frac{1}{3}(\tilde{p}^B\tilde{q}'_B - \tilde{p}'^B\tilde{q}'_B) = n, \;\; n \in \mathbb{Z}. \tag{2.14}$$

The maximal subgroup of $SL(2,\mathbb{R})$ preserving this discretization is $SL(2,\mathbb{Z})$, as shall be shown now, and this group will be used as analogue of U-duality in M-theory.

2.3.1 $d = 4$ U-duality and $SL(2, \mathbb{Z})$

Consider the Lie algebra $\mathfrak{sl}_2$ associated with $SL(2, \mathbb{R})$ with generators $\{E, F, H\}$ obeying

$$[E, F] = H,\ [H, E] = 2E \text{ and } [H, F] = -2F.$$

Lie algebra conventions and representations are summarized and explained in appendix A.1, and the reader is referred to this appendix to clarify notations and conventions.

Remember that $\mathfrak{sl}_2$ may be represented on vector spaces that allow a basis characterized by the eigenvalues of the Cartan subalgebra, that is, H in this case, called weights. The irreducible representations are uniquely characterized by a highest weight λ_0 with associated basis vector v_0, satisfying $Ev_0 = 0$ and $Hv_0 = \lambda_0 v_0$.

The representations may then be given by defining basis vectors $v_i = (1/i!)F^i v_0$. Then

$$\begin{aligned} Hv_i &= (\Lambda^0 - 2i)v_i, \\ Fv_i &= (i+1)v_{i+1}, \\ Ev_i &= (\Lambda^0 - i + 1)v_{i-1}, \qquad\qquad (i \geq 0). \end{aligned}$$

The representation for $\lambda_0 = 3$ is exactly the representation (2.13)!

In the above situation, the gauge fields and charges, respectively, thus correspond to the basis vectors in the representation space of highest weight 3. Normalizations of the basis vectors in representation space corresponds to charge normalizations. A way to make the $SL(2, \mathbb{R})$ symmetry compatible with the discretization (2.14) is now to choose a basis in which $SL(2, \mathbb{R})$ is broken to a nontrivial subgroup corresponding to matrices with integral entries. This subgroup is usually called $SL(2, \mathbb{Z})$, and shall be studied now.

The generators of the discrete group $SL(2, \mathbb{Z})$ are well-known from the literature. It is however instructive to recall a proof for the generators here, since this allows to introduce concepts needed in the next chapter, dealing with M-theory, in a simplified context.

In order to make contact with the literature, consider first the fundamental representation with highest weight 1.

Fundamental Representation

For the fundamental representation, choose basis vectors $\{v^+, v^-\}$ in V with $Hv^+ = v^+$, $Hv^- = -v^-$ and normalizations such that $v^+ = Ev^-$ and $v^- = Fv^+$. The representation matrices $\rho(\mathfrak{sl}_2)$ of the generators may then be given by

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The associated group $G = SL(2, \mathbb{C})$ is given by representation matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ab - cd = 1.$$

For the discrete group $G(\mathbb{Z}) = SL(2, \mathbb{Z})$, this implies $a, b, c, d \in \mathbb{Z}$.

Proposition. The discrete group $SL(2, \mathbb{Z})$ is generated by

$$T_+ = \exp(E),\ \ T_- = \exp(F).$$

More common is the equivalent set $T = \exp(E)$ and $S = \exp(-E)\exp(F)\exp(-E)$, where S represents the Weyl group modulo a phase. E.g. in the fundamental representation, it maps $v^+ \to v^-$ and $v^- \to -v^+$.

Proof. The simple proof given here uses the Birkhoff decomposition and the Euclidian algorithm. As shall be shown in the next chapter, for more complicated groups these are the essential tools as well.

Consider an element $g \in SL(2,\mathbb{Z})$ of the form

$$\rho(g) = \begin{pmatrix} (\rho(g))_{++} & (\rho(g))_{+-} \\ (\rho(g))_{-+} & (\rho(g))_{--} \end{pmatrix}$$

with all entries in $\mathbb{Z}$. g admits a triangular decomposition (Birkhoff decomposition)

$$g = g_- \, n \, g_+,$$

where, as above, $g_- = \exp(c_{-2}F)$, $g_+ = \exp(c_2 E)$, $c_{-2}, c_2 \in \mathbb{C}$, and n is given by

$$n = \prod_i n_2(t_i), \;\; t_i \in \mathbb{C}, \;\; n_2(t) = \exp(tE)\exp(-t^{-1}F)\exp(tE).$$

Remembering $v^+ = Ev^-$ and $v^- = Fv^+$, one has

$$n_2(t)v^+ = -t^{-1}v^-, \quad n_2(t)v^- = tv^+.$$

Any matrix $\rho(n)$, $n \in N$ is therefore either diagonal or off-diagonal. Furthermore, $(n_2(-1))^2$ acts as -**Id** on $\{v^+, v^-\}$. Note that $S = n_2(-1)$.

Suppose $(\rho(g))_{++} \neq 0$ (if $(\rho(g))_{++} = 0$, consider $\rho(Sg)$), then the Birkhoff decomposition implies that $\rho(n)$ is a diagonal matrix. Let $(\rho(n))_{++} = c$, then

$$\begin{aligned} (\rho(g))_{++} =& c \\ (\rho(g))_{-+} =& c\, c_{-2} \\ (\rho(g))_{+-} =& c\, c_2. \end{aligned}$$

This implies $c \in \mathbb{Z}$ and $c_{-2}, c_2 \in \mathbb{Q}$. Consider now $\rho((T_-)^m Sg)$ for $m \in \mathbb{Z}$ (note that $T_- = TST$, $(T_-)^{-1} = STSSS$). One has

$$\begin{aligned} (\rho((T_-)^m Sg))_{++} =& -(c\, c_{-2}) \\ (\rho((T_-)^m Sg))_{-+} =& +c - m \cdot (c\, c_{-2}). \end{aligned}$$

There always exists some integer m such that

$$|(\rho(g'))_{-+}| = |(\rho((T_-)^m Sg))_{-+}| < |(\rho(g))_{-+}|.$$

Thus, by repeating this operation, one arrives at a $g' \in G$ such that $(\rho(g'))_{-+} = 0$ (Euclidian algorithm). This, on the other hand, forces $(\rho(g'))_{++} = \pm 1$ by the condition on the determinant. If $(\rho(g'))_{++} = -1$, multiply with $(S)^2$ to have $(\rho(g'')) = \rho((S)^2 g')_{++} = 1$, if $(\rho(g'))_{++} = 1$, take $g'' = g'$. $(\rho(g''))_{+-} = c_2''$ then implies $c_2'' \in \mathbb{Z}$, and $g'' = \exp(c_2'' E) = (T_+)^{c_2''}$.

Thus, each $g \in SL(2,\mathbb{Z})$ is generated by $T_+ = \exp(E)$ and $T_- = \exp(F)$, or equivalently $T = T_+ = \exp(E)$ and $S = (T_+)^{-1} T_- (T_+)^{-1}$ □.

The Representation 4

Note that the definition of the generators and the resulting group $G(\mathbb{Z})$ turns out to be independent of the representation. However, the proof of the above proposition may easily be repeated for the representation with highest weight 3, and thus shall be given for completeness.

Consider the highest weight vector v_0 that satisfies $Ev_0 = 0$ and $Hv_0 = \lambda_0 v_0$ and has weight 3. As seen, the remaining basis vectors are $v_i = (1/i!)F^i v_0$ in the representation (2.13).

Denote the **4** representation by ρ. The first row and first column in the Birkhoff decomposition $g = g_- n g_+$, $g_- = \exp(c_{-2}F)$, $g_+ = \exp(c_2 E)$ are, parallel to above,

$$\begin{aligned}
\rho(g)_{00} &= c \in \mathbb{Z} \\
\rho(g)_{i0} &= \{cc_{-2}, cc_{-2}^2, cc_{-2}^3\} \in \mathbb{Z} \\
\rho(g)_{0i} &= \{3cc_2, 3cc_2^2, cc_2^3\} \in \mathbb{Z}, \qquad\qquad i \neq 0.
\end{aligned}$$

Say

$$cc_{-2} = n, \quad n \in \mathbb{Z}.$$

Consider again $\rho((T_-)^m Sg)$. One has

$$\begin{aligned}
\rho(g)_{30} &= \frac{1}{c^2} n^3 \\
\rho((T_-)^m Sg)_{30} &= \frac{1}{c^2}(c - mn)^3,
\end{aligned}$$

and one can always find an $m \in \mathbb{Z}$ such that

$$|c - mn| \leq |n|$$

and hence find a sequence of operations that yields a g' such that $\rho(g')_{30} = 0$, which implies, by the Birkhoff decomposition, $\rho(g')_{i0} = 0$ and $\rho(g')_{00} = \pm 1$. Since in the above representation again $S^2 = -\mathbf{Id}$, if $\rho(g')_{00} = -1$, multiply with S to have $\rho(g'') = \rho(Sg')_{00} = 1$, and consider $g'' = g'$ if $\rho(g')_{00} = 1$. Then, however, necessarily $g'' = \exp(c_2'' E) = (T_+)^{c_2''}$ $\square$.

2.3.2 $d = 4$ U-duality Group

The U-duality group is thus generated by the modular group generators $P^{-1}SP$ and $P^{-1}TP$, where

$$S = \exp(-\tilde{E}) \exp \tilde{F} \exp(-\tilde{E}) = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}, \quad T = \exp \tilde{E} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ & 1 & 2 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

This symmetry shall be used as analogue of U-duality in M-theory. Note two physical effects of this symmetry.

Take a look at the charges. If the $\{\tilde{p}^{\tilde{A}}, \tilde{q}_{\tilde{A}}\} = \{p^A, q_A\}$ are chosen to be integer, the $SL(2,\mathbb{Z})$ symmetry and DSZ condition yield that $\{\tilde{p}^{B}, \tilde{q}_B\}$ are in $3\mathbb{Z}$.

The $SL(2,\mathbb{Z})$ induces the familiar modular transformations $z \to z + 1$ and $z \to -1/z$ under T and S on the scalar $z \equiv -1/\sqrt{3}A_4 + i\rho$.

This may be understood from another perspective: consider the Lagrangian (2.5) truncated by demanding $B_{\mu\nu} \equiv 0$. In an asymptotic limit where $A_4, \rho \to$ const., z is exactly the τ parameter of electromagnetism plus theta term for the field $F'_{\mu\nu}$. However, the above $SL(2,\mathbb{Z})$ symmetry, since the representation is irreducible, will always mix all four types of charges and not preserve such a truncation.

2.4 U-duality in $d = 3$

Having defined the U-duality analogue in $d = 4$, it shall now be demonstrated how the analogue of U-duality in lower dimensions may be given.

This is done by explicitly following a conjecture made by Hull and Townsend [56], parallel to a method applied to the heterotic string by Sen [89, 88]. The focus is to find a set of generators for a discrete quantum symmetry. For this, is it assumed, as in [89], that the U-duality, discretized by the DSZ quantization in $d = 4$, is not broken by assuming that another coordinate is compact and is a symmetry of the quantized theory in $d = 3$ as well.

Consider the coordinates $\{0, 1, 2\}$ as three-dimensional coordinates, while the coordinates 3 and 4 are taken to be compact. Interpreting the coordinates $\{0, 1, 2, 3\}$ as four-dimensional, this results in a discrete 4-dimensional U-duality. The theory in $d = 3$ will be shown to have a classical $G_{2(+2)}$ symmetry. The corresponding Lie algebra $\mathfrak{g}_{2(+2)}$ is the maximally noncompact real form ("normal form") of $\mathfrak{g}_2$ corresponding to canonical normalization of the generators (see [48]). The $d = 4$ U-duality is a discrete subgroup in this symmetry in $d = 3$.

However, one may also interpret the coordinates $\{0, 1, 2, 4\}$ as four-dimensional, which gives another discrete U-duality, which acts *differently* on the fields of theory. Again, this symmetry is a discrete subgroup of the $G_{2(+2)}$ symmetry in $d = 3$, and the subgroup rotating these two symmetries into each other is the analogue of T-duality. Merging both symmetries together, one has the analogue of U-duality in $d = 3$. This philosophy is illustrated in figure 2.1. It is instructive

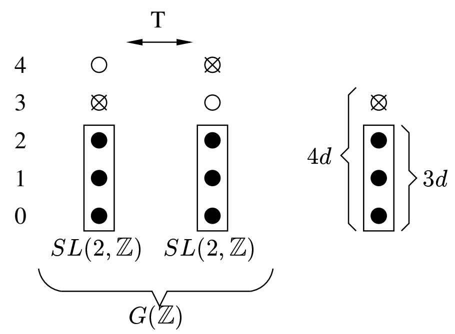


Figure 2.1: Construction of Three-Dimensional U-duality in the $G_{2(+2)}$ Toy Model

to study the compactification and symmetry groups by considering "elementary" solitons in $d = 4$ and the action of U-duality on their charges. One may then consider their counterparts in the $d = 3$ theory. It will be shown later how these solitons emerge and that the action of the $d = 4$ U-dualities exactly corresponds to the $d = 4$ action on the charge lattice.

2.4.1 $d = 3$ Theory

Starting from (2.11), consider first the situation that the coordinate 3 is compact. This may be used to further compactify the theory to $d = 3$. In doing so, the fields will be "maximally dualized" [21]: Since two-forms are dual to one-forms in $d = 3$, the situation where only scalars remain in the theory will be discussed here.

The compactification procedure is given in the following. Again, the results derived here are useful for the next chapter as well, and the discussion is therefore done in a more general fashion. Consider in $d = 4$ the Lagrangian

$$\mathcal{L} = -E^{(4)}\mathcal{R}^{(4)} + \tfrac{1}{4}\mathcal{G}^{t}_{\bar\mu\bar\nu} \star \mathcal{H}^{\bar\mu\bar\nu} + \tfrac{1}{c_{\mathbf{2n}}} E^{(4)}\mathrm{tr}(D(P^{(4)}_{\bar\mu})D(P^{(4)\,\bar\mu})) \tag{2.15}$$

where $\mathcal{G}_{\bar\mu\bar\nu} = 2\partial_{[\bar\mu}\mathcal{G}_{\bar\nu]}$ is a vector of $U(1)$ gauge field strengths, and the $\mathcal{H}^{\bar\mu\bar\nu}$ are dual magnetic field strength, determined by the condition

$$\mathcal{F}_{\bar\mu\bar\nu} \equiv \begin{pmatrix} \mathcal{G}_{\bar\mu\bar\nu} \\ \mathcal{H}_{\bar\mu\bar\nu} \end{pmatrix} = \Omega D(\mathcal{V}^{(4)})^t D(\mathcal{V}^{(4)}) \begin{pmatrix} \star\mathcal{G}_{\bar\mu\bar\nu} \\ \star\mathcal{H}_{\bar\mu\bar\nu} \end{pmatrix},$$

where

$$\Omega = \begin{pmatrix} & -\mathbf{1} \\ \mathbf{1} & \end{pmatrix}$$

is the symplectic form in $2n$ dimensions, and $D(\mathcal{V}^{(4)}) \in G^{(4)} \subset Sp(2n)$ is a field containing scalar matter, with

$$\partial_\mu D(\mathcal{V}^{(4)})D(\mathcal{V}^{(4)})^{-1} = D(Q^{(4)}_\mu) + D(P^{(4)}_\mu),\ \ Q^{(4)}_\mu \in \mathfrak{g}^{(4)},\ \ P^{(4)}_\mu \in \mathfrak{g}^{(4)} - \mathfrak{k}^{(4)}.$$

$\mathfrak{g}^{(4)}$ is the Lie algebra of $G^{(4)}$, and $\mathfrak{k}^{(4)}$ its maximal compact subalgebra.

$D(\mathcal{V}^{(4)})$ is in the $\mathbf{2n}$ representation of $G^{(4)}$, and $c_{\mathbf{2n}}$ is assumed to be the Dynkin (or second) index of the $\mathbf{2n}$ representation, so that one may write

$$\tfrac{1}{c_{\mathbf{2n}}} E^{(4)}\mathrm{tr}(D(P^{(4)}_{\bar\mu})D(P^{(4)\,\bar\mu})) = E^{(4)}\langle P^{(4)}_{\bar\mu}, P^{(4)\,\bar\mu}\rangle$$

where $\langle\cdot,\cdot\rangle$ is the abstract (representation independent) inner product of $\mathfrak{g}^{(4)}$.

Since the $\mathbf{2n}$ is symplectic, writing

$$D(\mathcal{V}^{(4)})^t D(\mathcal{V}^{(4)}) = \begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$$

the symplectic condition

$$(\mathcal{V}^{(4)})^{-1} = -\Omega(\mathcal{V}^{(4)})^t\Omega$$

implies $(\Omega D(\mathcal{V}^{(4)})^t D(\mathcal{V}^{(4)}))^2 = -\mathbf{Id}$, therefore

$$b^t d = db,\ \ ab^t = ba,\ \ -\mathbf{1} = (b^t)^2 - da.$$

Thus

$$\mathcal{H}^{\bar\mu\bar\nu} = d^{-1}\star\mathcal{G}_{\bar\mu\bar\nu} - bd^{-1}\mathcal{G}_{\bar\mu\bar\nu}.$$

Assume that the above Lagrangian does not depend on the coordinate denoted by z in the following, with flat counterpart $\bar z$. The vielbein is chosen to be

$$E^{(4)\,\bar\alpha}_{\bar\mu} = \begin{pmatrix} e^{\phi/2}E^{(d)\,\alpha}_\mu & e^{-\phi/2}\hat B_\mu \\ 0 & e^{-\phi/2} \end{pmatrix},$$

with signature $(+---)$, which implies

$$G^{(4)\rho\kappa} = e^{-\phi}G^{(3)\rho\kappa},\ \ G^{(4)z\kappa} = -e^{-\phi}\hat B^\kappa,$$
$$G^{(4)zz} = -e^{\phi} + e^{-\phi}\hat B_\kappa \hat B^\kappa,\ \ E^{(4)} = e^{\phi}E^{(3)}.$$

The gravitational Lagrangian is then treated as above, and it becomes, together with the scalar Lagrangian,

$$\mathcal{L}_{\text{gr+s}} = -E^{(3)}\mathcal{R}^{(3)} + \tfrac{1}{2}E^{(3)}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}e^{-2\phi}E^{(3)}\hat B_{\mu\nu}\hat B^{\mu\nu} + E^{(3)}\langle P^{(4)}_\mu, P^{(4)\,\mu}\rangle.$$

Consider the vector fields. The Levi-Civitta symbol is chosen to be $\epsilon^{1234} = 1,\ \ \epsilon^{\mu\nu\rho z} = \epsilon^{\mu\nu\rho}$. The vector fields get

$$\begin{aligned}\star\mathcal{G}_{\mu\nu} &= E^{(3)}\epsilon_{\mu\nu\rho}\left(e^{-\phi}\hat{B}_\kappa\mathcal{G}^{\rho\kappa} + e^{\phi}\mathcal{G}^{\rho}{}_z - e^{-\phi}\hat{B}_\kappa\hat{B}^\kappa\mathcal{G}^\rho{}_z + e^{-\phi}\hat{B}^\rho\hat{B}^\tau\mathcal{G}_{\tau z}\right)\\ \star\mathcal{G}_{\mu z} &= \tfrac{1}{2}E^{(3)}\epsilon_{\mu\rho\sigma}\left(-e^{-\phi}\mathcal{G}^{\rho\sigma} + e^{-\phi}\hat{B}_\sigma\mathcal{G}^\rho{}_z - e^{-\phi}\hat{B}_\rho\mathcal{G}^\sigma{}_z\right).\end{aligned}$$

Defining the $d = 3\ U(1)$ field strengths

$$\mathcal{G}'_{\mu\nu} = 2\partial_{[\mu}(\mathcal{G}_{\nu]} - \hat{B}_{\nu]}\mathcal{G}_z),\ \ \hat{B}_{\mu\nu} = 2\partial_{[\mu}B_{\nu]},$$

one has

$$\begin{aligned}\mathcal{H}_{\mu\nu} &= d^{-1}E^{(3)}\epsilon_{\mu\nu\rho}\left(e^{-\phi}\hat{B}_\tau\mathcal{G}'^{\rho\tau} + e^{-\phi}\hat{B}^{\rho\tau}\hat{B}_\tau\mathcal{G}_z - e^{\phi}\partial^\rho\mathcal{G}_z\right)\\ &\quad -bd^{-1}\left(\mathcal{G}'_{\mu\nu} + 2\partial_{[\mu}\mathcal{G}_z\hat{B}_{\nu]} + \hat{B}_{\mu\nu}\mathcal{G}_z\right)\\ \mathcal{H}_{\mu z} &= -\tfrac{1}{2}\epsilon_{\mu\rho\sigma}E^{(3)}e^{-\phi}d^{-1}\left(\mathcal{G}'^{\rho\sigma} + \hat{B}^{\rho\sigma}\mathcal{G}_z\right) - bd^{-1}\partial_\mu\mathcal{G}_z.\end{aligned}$$

Inserting this into the Lagrangian yields

$$\begin{aligned}\mathcal{L}_V &= \tfrac{1}{4}\mathcal{G}^t_{\bar{\mu}\bar{\nu}} \star \mathcal{H}^{\bar{\mu}\bar{\nu}}\\ &= -\tfrac{1}{4}E^{(3)}e^{-\phi}\mathcal{G}'^t_{\mu\nu}d^{-1}\mathcal{G}'^{\mu\nu} - \tfrac{1}{2}E^{(3)}e^{-\phi}\mathcal{G}'^t_{\mu\nu}d^{-1}\mathcal{G}_z\hat{B}^{\mu\nu}\\ &\quad -\tfrac{1}{4}E^{(3)}e^{-\phi}\hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\mathcal{G}^t_z d^{-1}\mathcal{G}_z\\ &\quad +\tfrac{1}{2}E^{(3)}e^{\phi}\partial_\mu\mathcal{G}^t_z d^{-1}\partial^\mu\mathcal{G}_z - \tfrac{1}{2}\epsilon^{\mu\nu\rho}(\mathcal{G}'^t_{\mu\nu}bd^{-1}\partial_\rho\mathcal{G}_z + \hat{B}_{\mu\nu}\mathcal{G}^t_z bd^{-1}\partial_\rho\mathcal{G}_z).\end{aligned}$$

In $d = 3$, the vector fields may now be dualized by adding the Lagrange multipliers

$$\mathcal{L}_{\text{Lag.mult.}} = \frac{1}{2}\epsilon^{\mu\nu\rho}(\bar{\eta}^t\partial_\rho\mathcal{G}'_{\mu\nu} + f\partial_\rho\hat{B}_{\mu\nu}).$$

Integrating out $\mathcal{G}'_{\mu\nu}$ yields

$$\partial_\mu\bar{\eta} = \mathcal{H}_{\mu z}$$

which together with the definition

$$\partial_\mu\eta = \mathcal{G}_{\mu z}$$

and

$$\mathcal{Y} = \begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}$$

is a key relation for $d = 3$ theory and the discussion of dualities. Integrating out the Kaluza-Klein field strength gives

$$\partial_\mu f = -\frac{1}{2}E^{(3)}\epsilon^{\mu\nu\rho}e^{-2\phi}\hat{B}_{\nu\rho} - \frac{1}{2}\mathcal{Y}^t\Omega\partial_\mu\mathcal{Y}.$$

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= -E^{(3)}\mathcal{R}^{(3)} + \tfrac{1}{2}E^{(3)}\partial_\mu\phi\partial^\mu\phi\\ &\quad +\tfrac{1}{2}E^{(3)}e^{2\phi}(\partial_\mu f + \tfrac{1}{2}\mathcal{Y}^t\Omega\partial_\mu\mathcal{Y})(\partial^\mu f + \tfrac{1}{2}\mathcal{Y}^t\Omega\partial^\mu\mathcal{Y})\\ &\quad +\tfrac{1}{2}E^{(3)}e^{\phi}\partial_\mu\mathcal{Y}^t D(\mathcal{V}^{(4)})^t D(\mathcal{V}^{(4)})\partial^\mu\mathcal{Y}\\ &\quad +E^{(3)}\langle P^{(4)}_\mu, P^{(4)\,\mu}\rangle \end{aligned}\tag{2.16}$$

and contains only gravity and scalars in $d = 3$.

Assume now that a group $G^{(3)}$ exists containing $G^{(4)}$ as subgroup, and the following holds (the reader should consult appendix A.1 for notations and conventions):

(i) If $\mathfrak{g}^{(4)}$ is of rank n, $\mathfrak{g}^{(3)}$ is of rank $n+1$, and $\mathfrak{g}^{(4)}$ corresponds to the Dynkin sub-diagram of $\mathfrak{g}^{(3)}$ given by erasing the vertex of the simple root α_{n+1} ($\mathfrak{g}^{(4)}$ is a direct subalgebra of $\mathfrak{g}^{(3)}$).

(ii) The space of roots $\alpha \in \Delta^{\mathfrak{g}^{(3)}}$ allows a $\mathbb{Z}$ grading with respect to the root coordinate r^{α}_{n+1}, such that, for the positive roots $\alpha \in \Delta^{\mathfrak{g}^{(3)}}_+$, one has sets $\Delta^{\mathfrak{g}^{(3)}}_+ = \{\Delta^{\mathfrak{g}^{(4)}}_+, \Phi, \alpha_+\}$ with

$$\begin{aligned}\Delta^{\mathfrak{g}^{(4)}}_+ &= \{\alpha \mid r^{\alpha}_{n+1} = 0\},\\ \Phi &= \{\beta \mid r^{\beta}_{n+1} = 1\},\\ \alpha_+ &= \{\gamma \mid r^{\gamma}_{n+1} = 2\}\end{aligned}$$

where Φ is the set of weights of $\mathfrak{g}^{(4)}$ of the representation carried by $\mathcal{V}^{(4)}$, and α_+ is the highest root of $\mathfrak{g}^{(3)}$.

The generators E_{β_i}, $\beta_i \in \Phi$ then admit an ordering

$$\begin{aligned}[E_{\beta_i}, E_{\beta_j}] &= \Omega_{ij} E_{\alpha_+},\\ [\mathfrak{g}^{(4)}, E_{\beta_i}] &= D(\mathfrak{g}^{(4)})_{ji} E_{\beta_j}\end{aligned}$$

where D is the $\mathbf{2n}$ representation such that $D(\mathcal{V}^{(4)})$ is written as $D(\mathcal{V}^{(4)}) = \exp(D(\mathfrak{g}^{(4)}))$. Two further formulas concerning the exponential mapping are useful:

$$\begin{aligned}\partial_\mu e^{t_i X_i} &= \left(\partial_\mu t_i X_i + \tfrac{1}{2!}[t_j X_j, \partial_\mu t_i X_i] + \tfrac{1}{3!}[t_k X_k, [t_j X_j, \partial_\mu t_i X_i]] + \dots\right) e^{t_l X_l}\\ e^B A e^{-B} &= A + [B, A] + \tfrac{1}{2}[B, [B, A]] + \dots, \qquad t_i \in \mathbb{C},\ X_i, A, B \in \mathfrak{g}.\end{aligned}$$

Consider generators E_{β_i} with $\beta_i \in \Phi$. It is assumed that canonical normalizations have been chosen, that is,

$$\mathrm{tr}(D(E_{\beta_i})D(E_{\beta_j})) = 0,\ \ \mathrm{tr}(D(E_{\beta_i})D(E_{-\beta_j})) = c_D$$

which implies, for $H_\alpha = [E_\alpha, E_{-\alpha}]$, the trace $\mathrm{tr}(D(H_{\beta_i})D(H_{\beta_i})) = 2c_D$, and c_D is the Dynkin index of the representation D.

One may now define a scalar field in $d = 3$, using the Iwasawa decomposition of $G^{(3)}$, as

$$\mathcal{V}^{(3)} = \mathcal{V}^{(4)} \exp(\tfrac{1}{2}\phi H_{\alpha_+}) \exp(\mathcal{Y}_i E_{\beta_i}) \exp(f E_{\alpha_+})$$

in a representation independent way. Using

$$\begin{aligned}\exp(\phi/2 H_{\alpha_+}) E_{\alpha_+} \exp(-\phi/2 H_{\alpha_+}) &= e^{\phi} E_{\alpha_+}\\ \exp(\phi/2 H_{\alpha_+}) E_{\beta_i} \exp(-\phi/2 H_{\alpha_+}) &= e^{\phi/2} E_{\beta_i}\\ \mathcal{V}^{(4)} E_{\beta_i} (\mathcal{V}^{(4d)})^{-1} = D(\mathcal{V}^{(4d)})_{ji} E_{\beta_j} &\end{aligned}$$

one may consider *any* faithful irreducible representation $\tilde{D}$ of $\mathfrak{g}^{(3)}$ and $G^{(3)}$ to get

$$\begin{aligned}\partial_\mu \tilde{D}(\mathcal{V}^{(3)})(\tilde{D}(\mathcal{V}^{(3)}))^{-1} &= \partial_\mu \tilde{D}(\mathcal{V}^{(4)})(\tilde{D}(\mathcal{V}^{(4)}))^{-1}\\ &\quad + \tfrac{1}{2}\partial_\mu \phi\, \tilde{D}(H_{\alpha_+}) + e^{\phi/2} \partial_\mu \mathcal{Y}_i D(\mathcal{V}^{(4)})_{ji}\, \tilde{D}(E_{\beta_j})\\ &\quad + e^{\phi}(\partial_\mu B + \tfrac{1}{2}\mathcal{Y}_i \Omega_{ij} \partial_\mu \mathcal{Y}_j)\, \tilde{D}(E_{\alpha_+}),\end{aligned}$$

and thus

$$\begin{aligned}\langle P^{(3)}_\mu, P^{(3)\,\mu}\rangle &= \langle P^{(4)}_\mu, P^{(4)\,\mu}\rangle + \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi\\ &\quad + \tfrac{1}{2} e^{\phi} \partial_\mu \mathcal{Y}^t D(\mathcal{V}^{(4)})^t D(\mathcal{V}^{(4)}) \partial^\mu \mathcal{Y}\\ &\quad + \tfrac{1}{2} e^{2\phi}(\partial_\mu B + \tfrac{1}{2}\mathcal{Y}^t \Omega \partial_\mu \mathcal{Y})(\partial^\mu B + \tfrac{1}{2}\mathcal{Y}^t \Omega \partial^\mu \mathcal{Y}).\end{aligned}$$

The Lagrangian (2.5) therefore takes the simple form

$$\mathcal{L} = -E^{(3)}\mathcal{R}^{(3)} + E^{(3)}\langle P^{(3)}_\mu, P^{(3)\,\mu}\rangle.$$

2.4.2 $\mathfrak{g}_{2(+2)}$

Thus, the Lagrangian may be written in the above simple form if a group exists obeying the conditions specified.

This group is $G_{2(+2)}$, and the algebra $\mathfrak{g}_{2(+2)}$ will be used in conventions that shall be explained in this subsection. $\mathfrak{g}_{2(+2)}$ is the maximally noncompact real form of $\mathfrak{g}_2$, it is a real Lie algebra corresponding to canonical normalizations of the generators (see [48]). The exact definitions of the generators as used here are given below.

$\mathfrak{g}_2$ is a rank two non-simply laced Lie algebra. Its roots are given by, as explained in appendix (A.1),

$$\begin{aligned} \boldsymbol{\epsilon}_{ij} &\equiv \mathbf{e}_i - \mathbf{e}_j \qquad (1 \leq i \neq j \leq 3), \\ \boldsymbol{\epsilon}_i &\equiv \pm(\mathbf{e}_i - \frac{1}{3}\sum_{l=1}^{3}\mathbf{e}_l) \qquad (1 \leq i \leq 3) \end{aligned} \tag{2.17}$$

displayed in figure 2.2. $\{\mathbf{e}_i \mid 1 \leq i \leq 3\}$ is a set of orthonormal vectors in $\mathbf{R^3}$, and the roots lie in a hyperplane is normal to $\sum_{i=1}^{3}\mathbf{e}_i$.

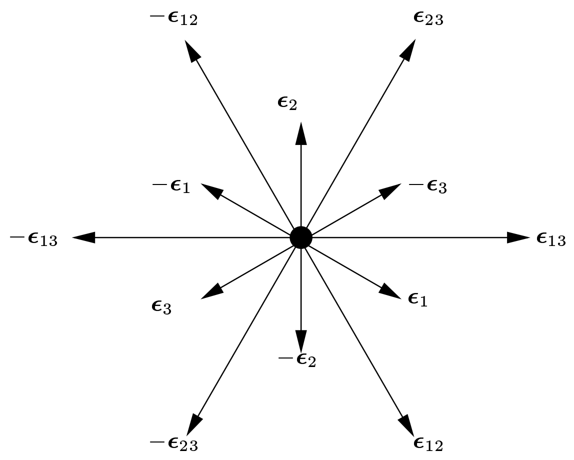


Figure 2.2: Roots of $\mathfrak{g}_2$

The corresponding positive and negative root generators of $\mathfrak{g}_2$ will be denoted by

$$\begin{aligned} E_{\mathbf{e}_{ij}} &= E^{i}{}_{j}, \\ E_{\mathbf{e}_i} &= E^{i}, \\ E_{-\mathbf{e}_i} &= E^{*}_{i}, \end{aligned}$$

and

$$\{H_i \equiv [E^{i}{}_{i+1}, E^{i+1}{}_{i}] \mid i = 1, 2\}$$

is a basis of the Cartan subalgebra. The positive roots generators are $E^{i}{}_{j}, 1 \leq i \leq j \leq 3$, $E^i, 1 \leq i \leq 2$ and E^*_3, and the generator E^1_2 corresponds to the long simple positive root, the generator E^2 to the short one.

There are several ways to find the commutation relations for the complete algebra, e.g. by using the bilinear asymmetry function of appendix A.1. However, since the model is studied as toy model for M-theory, the conventions shall be as parallel as possible to the conventions used in the next chapter for $\mathfrak{e}_{8(+8)}$, where a decomposition of exceptional Lie algebras with respect to their $\mathfrak{sl}_n$ direct subalgebras is used [38]. $\mathfrak{g}_{2(+2)}$ has a straightforward embedding into $\mathfrak{e}_{8(+8)}$ (see appendix A.1). This embedding is explicitly described in the next chapter, at this point, the result shall just be given. The commutation relations are

$$
\begin{aligned}
[\,H_i\,,\,H_j\,] &= 0,\\
[\,H_i\,,\,E^j{}_k\,] &= \delta^j_i E^i{}_k - \delta^j_{i+1} E^{i+1}{}_k - \delta^i_k E^j{}_i + \delta^{i+1}_k E^j{}_{i+1},\\
[\,H_i\,,\,E^j\,] &= \delta^j_i E^i - \delta^j_{i+1} E^{i+1},\\
[\,H_i\,,\,E^*_j\,] &= -(\delta^i_j E^*_i - \delta^{i+1}_j E^*_{i+1}),\\
[\,E^i{}_j\,,\,E^k{}_l\,] &= \delta^k_j E^i{}_l - \delta^i_l E^k{}_j,\\
[\,E^i{}_j\,,\,E^k\,] &= \delta^k_j E^i,\\
[\,E^i{}_j\,,\,E^*_k\,] &= -\delta^i_k E^*_j,\\
[\,E^i, E^j\,] &= -2\sum_k^3 \epsilon^{ijk} E^*_k,\\
[\,E^*_i, E^*_j\,] &= -2\sum_k^3 \epsilon_{ijk} E^k,\\
[\,E^i, E^*_j\,] &= 3E^i{}_j \quad \text{if } i \neq j,\\
[\,E^1, E^*_1\,] &= 2H_1 + H_2,\\
[\,E^2, E^*_2\,] &= -H_1 + H_2,\\
[\,E^3, E^*_3\,] &= -H_1 - 2H_2.
\end{aligned}
\tag{2.18}
$$

2.4.3 The $G_{2(+2)}$ Coset

$G_{2(+2)}$ has a maximal subgroup $SL(2) \times SL(2)$, where the two $SL(2)$ groups are generated by the short simple and the lowest root (see figure 2.3).

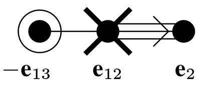


Figure 2.3: Decomposition of $G_{2(+2)} \supset SL(2) \times SL(2)$. The root surrounded by a circle is the lowest root added to the Dynkin diagram.

In the Iwasawa decomposition, the $SL(2)$ generated by the short simple root may be associated with $\mathcal{V}^{(4)}$, while the $SL(2)$ generated by the lowest root carries the $d = 3$ dilaton and dualized Kaluza-Klein gauge field. To see this, consider the vector

$$\mathcal{S}^t = (\frac{1}{\sqrt{3}} E^*_3, -E^1{}_2, \frac{1}{\sqrt{3}} E^1, E^2{}_3).$$

It is easy to see from 2.18 that this corresponds exactly to the vector $\{E_{\beta_i}\}$ of the last subsection by noting

$$[E^2, \mathcal{S}_i] = (P^{-1}EP)^t_{ij}\, \mathcal{S}_j, \quad [E^*_2, \mathcal{S}_i] = (P^{-1}FP)^t_{ij}\, \mathcal{S}_j,$$

and

$$[\mathcal{S}_i, \mathcal{S}_j] = \Omega_{ij} E^1{}_3$$

where the E, F are the $\mathfrak{sl}(2, \mathbb{R})$ generators given in (2.7), P is given in (2.8), and the $\mathcal{S}_i$ denote the components of the vector $\mathcal{S}$ as given above.

The Dynkin index of the **4** representation D of $\mathfrak{sl}(2)$ is 10, while the Dynkin index of the **14** representation $\tilde{D}$ of $\mathfrak{g}_2$ is 8. The generator E^2 corresponds to a short root of $\mathfrak{g}_2$ and obeys $\mathrm{tr}(\tilde{D}(E^2)\tilde{D}(E_2^*)) = 3c_{\mathbf{14}}$ in the normalizations chosen[1]. Thus, one arrives at the field

$$\begin{aligned} \mathcal{V}^{(3)} &= \exp\left(-\frac{1}{2}\ln\rho\,(H_2 - H_1)\right)\exp\left(-\frac{1}{\sqrt{3}}A_4\,E^2\right) \\ &\quad \exp\left(\frac{1}{2}\phi\,(H_1 + H_2)\right)\exp\left(\mathcal{Y}_i\,\mathcal{S}_i\right)\exp\left(f\,E^1{}_3\right) \\ &= \mathcal{V}^{(4)}\exp\left(\frac{1}{2}\phi\,(H_1 + H_2)\right)\exp\left(\mathcal{Y}_i\,\mathcal{S}_i\right)\exp\left(f\,E^1{}_3\right) \end{aligned}$$

with $\mathcal{V}^{(3)} \in G_{2(+2)}/SO(4)$, and as usual

$$\partial_\mu \mathcal{V}^{(3)} \mathcal{V}^{(3)-1} = Q_\mu^{(3)} + P_\mu^{(3)}, \;\; Q_\mu^{(3)} \in \mathfrak{so}(4), \;\; P_\mu^{(3)} \in \mathfrak{g}_{2(+2)} - \mathfrak{so}(4)$$

and the Lagrangian

$$\mathcal{L} = -E^{(3)}R^{(3)} + \frac{1}{8}E^{(3)}\mathrm{Tr}(P_\mu^{(3)}P^{(3)\mu}).$$

The four-dimensional vector $\mathcal{Y}$ is given by

$$\partial_\mu \bar{\eta} = \mathcal{H}_{\mu 3}, \;\; \partial_\mu \eta = \mathcal{G}_{\mu 3}, \;\; \mathcal{Y} = \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}. \tag{2.19}$$

Thus it corresponds to the $d = 4$ gauge fields. When discussing arrays of solitons in $d = 4$ that yields solitons in $d = 3$, it will be shown that it carries the $d = 4$ charges of these "fundamental" particles.

The field f is given by

$$\partial_\mu f = -\frac{1}{2}E^{(3)}\epsilon^{\mu\nu\rho}e^{-2\phi}\hat{B}_{\nu\rho} - \frac{1}{2}\mathcal{Y}^t\Omega\partial_\mu\mathcal{Y}. \tag{2.20}$$

and carries the dualized Kaluza-Klein field. Note that, considering the bosonic Lagrangian only, *no* representation of $G_{2(+2)}$ is fixed from the equations of motion in contrary to the situation in $d = 4$. However, the **14** decomposes, as already used, as

$$\begin{array}{rcl} G_{2(+2)} & \supset & SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \\ \mathbf{14} & \supset & (\mathbf{2},\mathbf{4}) + (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1}) \end{array}$$

and thus contains the familiar **4** representation from $d = 4$ of $SL(2)$ in its representation matrices, while e.g. the **7** decomposes as

$$\begin{array}{rcl} G_{2(+2)} & \supset & SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \\ \mathbf{7} & \supset & (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3}) \end{array}$$

[1]Note that the factor $\frac{3}{10} = 3(c_{\mathbf{4}})^{-1}$ in (2.11) indicates that, in a reduced model, the $d = 4$ $\mathfrak{sl}_2$ symmetry algebra should be embedded into a larger algebra such that it corresponds to a *short* root, and actually the factor 3 uniquely determines this algebra to be $\mathfrak{g}_2$!

where the first $SL(2)$ corresponds to the short simple root, and thus does not contain the **4** representation as a sub-representation. The **14** representation matrices look with respect to this decomposition like

$$\mathcal{V}'^{(4)} = \begin{pmatrix} \boxed{\mathbf{4}} & & & & & \\ & \boxed{\mathbf{4}} & & & & \\ & & \boxed{\mathbf{3}} & & & \\ & & & \boxed{\mathbf{1}} & & \\ & & & & \boxed{\mathbf{1}} & \\ & & & & & \boxed{\mathbf{1}} \end{pmatrix},$$

containing the familiar **4** block. However, it is again stressed that no explicit representation is needed for the theory in $d = 3$.

2.4.4 Identifying $d = 4$ U-duality in the $d = 3$ Theory

Consider

$$\Lambda \in SL(2,\mathbb{R}), \quad \Lambda = e^X, \; X = a\, E^2 + b\, E_2^* + c\, (h_2 - h_1), \; a, b, c \in \mathbb{R}$$

The $d = 4$ symmetry is then easily found:

$$\begin{aligned} \mathcal{V}^{(3)} \to \mathcal{V}^{(3)}\Lambda^{-1} &= \mathcal{V}'^{(4)}\Lambda^{-1} \exp\left(\frac{1}{2}\phi\,(H_1 + H_2)\right) \Lambda \exp\left(\mathcal{Y}_i\, \mathcal{S}_i\right) \Lambda^{-1} \exp\left(f\, E^1_{\ 3}\right) \\ &= \mathcal{V}'^{(4)}\Lambda^{-1} \exp\left(\frac{1}{2}\phi\,(H_1 + H_2)\right) \exp\left(\mathcal{Y}_i [\exp D(X)]^t_{ij}\, \mathcal{S}_j\right) \exp\left(f\, E^1_{\ 3}\right) \\ &= \mathcal{V}'^{(4)}\Lambda^{-1} \exp\left(\frac{1}{2}\phi\,(H_1 + H_2)\right) \exp\left([D(\Lambda)\mathcal{Y}]_i\, \mathcal{S}_i\right) \exp\left(f\, E^1_{\ 3}\right) \end{aligned} \tag{2.21}$$

therefore

$$\mathcal{V}'^{(4)} \to \mathcal{V}'^{(4)}\Lambda^{-1} \text{ and } \mathcal{Y} \to D(\Lambda)\mathcal{Y}$$

where D is the spin 3/2 representation (2.7).

This is exactly the four dimensional U-duality in the reduced model. The vector $\mathcal{Y}$ carries the $d = 4$ charges, as will be illustrated when considering solitons.

The symmetry was broken to a discrete subgroup by the DSZ condition, and thus the discrete $d = 4$ U-duality in $d = 3$ is generated by

$$S^2 = \exp(-E_2^*) \exp(E^2) \exp(-E_2^*), \; T^2 = \exp(E^2).$$

2.4.5 Connection to $d = 5$ Fields

In order to compare the different compactification in figure 2.1 and join the $d = 4$ U-duality groups together to a $d = 3$ U-duality group, a parametrization of the coset matrix will be given that simplifies this comparison. Define the fields

$$\begin{aligned} \varphi &= \eta^{\tilde{A}} + \frac{1}{\sqrt{3}} A_3 A_4 - \frac{1}{\sqrt{3}} B_3 A_4^2, \\ \Psi_1 &= f + \frac{1}{2} B_3 \Psi_2 - \frac{1}{4} B_3 A_4 \varphi + \frac{1}{6\sqrt{3}} A_3 A_4 (A_3 - B_3 A_4), \\ \Psi_2 &= \bar{\eta}_B - \frac{1}{2} A_4 \varphi - \frac{1}{3\sqrt{3}} A_4^2 (A_3 - B_3 A_4) \end{aligned}$$

polynomially connected to the fields used so far. This indicates that the factors added are BCH terms, and indeed, using (2.18), one arrives at the field

$$\begin{aligned}
\mathcal{V}^{(3)} &= \exp\left(\frac{1}{2}\left((\phi + \ln\rho)\, H_1 + (\phi - \ln\rho)\, H_2\right)\right) \\
&\quad \exp\left(-B_3\, E^1{}_2\right) \exp\left(\Psi_1\, E^1{}_3 + \Psi_2\, E^2{}_3\right) \\
&\quad \exp\left(\frac{1}{\sqrt{3}}\left(-A_3\, E^1 - A_4\, E^2 + \varphi\, E^*_3\right)\right) \\
&= \exp\left(-\ln(e_1{}^{\dot 1})\, H_1 - \ln(e_1{}^{\dot 1} e_2{}^{\dot 2})\, H_2\right) \exp\left(-e_1{}^{\dot 2} e_{\dot 2}{}^2\, E^1{}_2\right) \\
&\quad \exp\left(\Psi_i\, E^i{}_3\right) \\
&\quad \exp\left(\frac{1}{\sqrt{3}}\left(-A_{2+i}\, E^i + \varphi\, E^*_3\right)\right)
\end{aligned}$$

where the $d = 5$ ”internal” vielbein has been used, defined by

$$\begin{aligned}
E^{(5)A}_M &= \begin{pmatrix} e^{-1} E^{(3)\alpha}_\mu & B^i_\mu e^a_i \\ 0 & e^a_i \end{pmatrix} \\
&= \begin{pmatrix} e^{\phi/2}\rho^{-\frac{1}{2}} E^{(3)\alpha}_\mu & e^{-\phi/2}\rho^{-\frac{1}{2}}\hat{B}_\mu & \rho B_\mu \\ 0 & \rho^{-\frac{1}{2}} e^{-\phi/2} & \rho B_3 \\ & 0 & \rho \end{pmatrix}.
\end{aligned} \tag{2.22}$$

Explicitly, φ and Ψ_i obey, using (2.6), (2.9), (2.19), (2.20) and (2.22)

$$\begin{aligned}
\partial_\mu\varphi &= -e^2 E^{(3)}\epsilon_{\mu\nu\rho}(\partial^{[\nu}A^{\rho]} + B^{i[\nu}\partial^{\rho]}A_{(2+i)}) + \frac{1}{\sqrt{3}}\epsilon^{ij}A_{2+i}\partial_\mu A_{2+j}, \\
\partial_\mu\Psi_i &= -\frac{1}{2}e^2 E^{(3)}\epsilon_{\mu\nu\rho}B_i^{\nu\rho} - \frac{1}{2}(\varphi\partial_\mu A_{2+i} - \partial_\mu\varphi A_{2+i}) - \frac{1}{3\sqrt{3}}\epsilon^{jk}A_{2+i}A_{2+j}\partial_\mu A_{2+k}.
\end{aligned}$$

This exactly reproduces the result of [72] obtained by direct reduction to $d = 3$.

2.4.6 T-duality in $d = 3$

An analogon of T-duality may be identified within $G_{2(+2)}$ by investigating the diagonal subgroup H. The components of the diagonal of the internal vielbein $e_m^{\dot m}$ ($\dot m$ is flat index m) correspond to the radii of the compactification torus. Consider

$$\tilde{\mathcal{V}}^{(3)} = \exp\left(-\ln(e_1^{\dot 1})H_1 - \ln(e_1^{\dot 1}e_2^{\dot 2})H_2\right).$$

Since one has $H_1 = H_{\boldsymbol{\epsilon}_{12}}$ and $H_2 = H_{\boldsymbol{\epsilon}_{23}}$, one may write $H_1 = H_{\boldsymbol{\epsilon}_{13}} - H_{\boldsymbol{\epsilon}_{23}}$, and this yields

$$\tilde{\mathcal{V}}^{(3)} = \exp\left(-\ln(e_1^{\dot 1})H_{\boldsymbol{\epsilon}_{13}} - \ln(e_2^{\dot 2})H_{\boldsymbol{\epsilon}_{23}}\right). \tag{2.23}$$

A transformation exchanging the two radii is a Weyl reflection in root space exchanging $\mathbf{e}_1$ and $\mathbf{e}_2$. This is exactly given by $S^1{}_2$, defined as

$$S^1{}_2 = \exp(-E^2{}_1)\exp(E^1{}_2)\exp(-E^2{}_1)$$

and it may be easily checked that a conjugation with $S^1{}_2$ exactly exchanges $e_1^{\dot 1}$ with $e_2^{\dot 2}$ in (2.23). Since the ”T-duality” transforms the left hand side of figure 2.1 to the right hand side, one may expect this operator to be the transformation merging the $d = 4$ U-dualities together. This will be shown now.

2.4.7 Different Orders of Compactification

The compactification presented above corresponds to the left side of figure 2.1. It was done stepwise by considering

$$\mathcal{L}^{(5)}(E_M^{(5)A}, A_M) \to \mathcal{L}^{(4)}_{\#1}(E_M^{(5)A}, A_{\bar\mu}, A_4) \to \mathcal{L}^{(3)}(E_M^{(5)A}, A_\mu, A_3, A_4)$$

where $\{\bar\mu\} = \{0,1,2,3\}$. The resulting scalar coset matrix will be called $\mathcal{V}^{(3)}_{\#1}$. The $d=4$ U-duality is generated by

$$S^2 = \exp(-E_2^*)\exp(E^2)\exp(-E_2^*),\ T^2 = \exp(E^2).$$

Consider now the right hand side of figure 2.1. The compactification corresponds to the steps

$$\mathcal{L}^{(5)}(E_M^{(5)A}, A_M) \to \mathcal{L}^{(4)}_{\#2}(E_M^{(5)A}, A_{\bar\mu}, A_3) \to \mathcal{L}^{(3)}(E_M^{(5)A}, A_\mu, A_4, A_3)$$

where $\{\bar\mu\} = \{0,1,2,3\}$. This compactification may be performed strictly parallel to the above by putting $E_M^{(5)A}$ into a suitable gauge using local Lorentz symmetry in $d=5$. Starting from (2.22), consider the transformation

$$\Lambda_B^A = \begin{pmatrix} \delta_\beta^{\ \alpha} & & 0 \\ 0 & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$$

such that

$$E_M^{(5)B}\Lambda^B{}_A = \begin{pmatrix} e^{-1}E_\mu^{(3)\alpha} & B_\mu^i e_i'^{\,a} \\ 0 & e_i'^{\,a} \end{pmatrix}$$

with

$$e_i'^{\,a} = \begin{pmatrix} \rho' & 0 \\ \rho' B_3' & \rho'^{-\frac{1}{2}} e^{-\phi'/2} \end{pmatrix} = \begin{pmatrix} \rho^{-\frac{1}{2}} e^{-\phi/2} & \rho B_3 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which yields $\tan\theta = \rho^{3/2}e^{\phi/2}B_3$. Using this parameterization, the compactification is identical to above, when taking instead of $\{\bar\mu\} = \{0,1,2,3\}$ the coordinates $\{\bar\mu\} = \{0,1,2,4\}$ as $d=4$ coordinates. A subtlety arises from the Chern-Simons term in $d=4$ (see (2.5)) that changes sign ($\epsilon^{01234} = -\epsilon^{01243}$). Taking this into account, one notices that

$$\mathcal{L}^{(4)}_{\#1}(\tilde E_M^{(5)A}, \tilde A_{\bar\mu}, \tilde A) = \mathcal{L}^{(4)}_{\#2}(\tilde E_M^{(5)A}, -\tilde A_{\bar\mu}, -\tilde A)$$

for some $\tilde E_M^{(5)A}, \tilde A_{\bar\mu}, \tilde A$. Considering $\mathcal{V}^{(3)}_{\#1} = \mathcal{V}^{(3)}_{\#1}(e_i^a, A_3, A_4, \Psi_1, \Psi_2, \phi)$, this yields

$$\mathcal{V}^{(3)}_{\#2} = \mathcal{V}^{(3)}_{\#1}(e_i'^{\,a}, -A_4, -A_3, \Psi_2, \Psi_1, -\phi)$$

and thus

$$\begin{aligned} \mathcal{V}^{(3)}_{\#2} \;=\; & \exp\left(\frac{1}{2}\left((\phi' + \ln\rho')\,h_1 + (\phi' - \ln\rho')\,h_2\right)\right) \\ & \exp\left(-B_3'\,E^1{}_2\right)\exp\left(\Psi_2\,E^1{}_3 + \Psi_1\,E^2{}_3\right) \\ & \exp\left(\frac{1}{\sqrt{3}}\left(A_4\,E^1 + A_3\,E^2 - \varphi\,E_3^*\right)\right). \end{aligned}$$

The $d=4$ discrete U-duality is again generated by

$$S^2 = \exp(-E_2^*)\exp(E^2)\exp(-E_2^*),\ T^2 = \exp(E^2).$$

2.4.8 Joining $d = 4$ U-dualities in $d = 3$

The above two compactifications lead to the same theory in $d = 3$, and the corresponding coset matrices should therefore be related by a symmetry of the theory. Indeed, one finds

$$\mathcal{V}^{(3)}_{\#2} = (PS^1{}_2)^{-1} \ \exp\left(\theta(E^1{}_2 - E^2{}_1)\right) \ \mathcal{V}^{(3)}_{\#1} \ PS^1{}_2 \tag{2.24}$$

where θ is given by $\tan\theta = \rho^{3/2} e^{\phi/2} B_3$ as above. $S^1{}_2$ is indeed the T-duality introduced above.

Somewhat unexpected is the appearance of P in (2.24). P is a "parity" transformation given by

$$P = (-1)^{(h_1+h_2)} = (S^1{}_3)^2 = (S^2)^2$$

and is an element of $d = 4$ U-duality corresponding to a charge conjugation of the $d = 4$ charges, since in the representation (2.7), $(S^2)^2 = -\mathbf{Id}$ as may be easily checked.

Turning to the U-duality transformations, consider a transformation U on $\mathcal{V}^{(3)}_{\#1}$. One has

$$\begin{aligned}
\mathcal{V}'^{(3)}_{\#2} &= (PS^1{}_2)^{-1} \ \exp\left(\theta(E^1{}_2 - E^2{}_1)\right) \mathcal{V}'^{(3)}_{\#1} \ PS^1{}_2 \\
&= (PS^1{}_2)^{-1} \ \exp\left(\theta(E^1{}_2 - E^2{}_1)\right) \mathcal{V}^{(3)}_{\#1} \ PS^1{}_2 \ (PS^1{}_2)^{-1} \ U \ (PS^1{}_2) \\
&= \mathcal{V}^{(3)}_{\#2} \ (PS^1{}_2)^{-1} \ U \ (PS^1{}_2) \\
&= \mathcal{V}^{(3)}_{\#2} \ \tilde{U}
\end{aligned}$$

$\tilde{U}$ is thus a matrix generated by

$$S^1 = \exp(-E^*_1)\exp(E^1)\exp(-E^*_1), \; T^1 = \exp(E^1).$$

The joint U-duality group $U(\mathbb{Z})$ in three dimensions is therefore generated by

$$S^1, \; S^2, \; T^1, \; T^2.$$

2.4.9 $d = 3$ U-duality Group

The fact that one arrives at the above group $U(\mathbb{Z})$ is an interesting result. It will be shown in the next chapter that discrete subgroups of Lie groups acting on admissible lattices are generated by S, T generators for *all* Chevalley generators of the Lie algebra. This would mean that one has to take $S^i{}_j$, S^i, $T^i{}_j$, T^i for all positive roots of $\mathfrak{g}_{2(+2)}$.

But quite obviously $U(\mathbb{Z})$ is *not* $G_{2(+2)}(\mathbb{Z})$, but the former is strictly *smaller* than the latter. All generators of $U(\mathbb{Z})$ correspond to short roots of $\mathfrak{g}_{2(+2)}$. It is only possible to generate e.g. $(T^1{}_2)^3$ from them, but not $(T^1{}_2)$.

It will be shown that the above definition of $G_{2(+2)}(\mathbb{Z})$ relies on the action on admissible lattices in basic representations where all nontrivial weights or of same length. This is *not* the case for the **14** of $\mathfrak{g}_2$, but would be for the **7**. The definition of the group, generated by S's and T's for all roots, is however representation independent.

That the definition does not agree with the U-duality group found is obviously connected to the fact that $G_{2(+2)}$ is not simply laced. $d = 5$ simple supergravity has no moduli fields, and no string compactification described by this supergravity at low energies is known. Therefore one cannot determine which is the "correct" U-duality group until such a microscopic realization is found.

The disagreement will be further discussed at the end of the next chapter.

2.5 Summary

In this chapter, a toy model was discussed that resembles the structure of M-theory in the low energy limit.

It was shown how the theory is compactified to $d = 4$, exhibiting a ”hidden” classical $SL(2, \mathbb{R})$ duality symmetry mixing electric and magnetic sector.

The analogue of U-duality was derived from first principles by imposing the DSZ quantization condition. The resulting discrete group was shown to be standard $SL(2, \mathbb{Z})$ with two generators S, T, and its transformation on charges and scalars was discussed.

The U-duality group for further compactification to $d = 3$ was then derived. Only scalars remain in the theory after dualization, and the Lagrangian resulting from this compactification was shown to exhibit an enlarged symmetry, which is $G_{2(+2)}$. The discrete subgroup analogue of U-duality was derived by using different orders of compactification and merging the resulting $d = 4$ duality groups embedded into $d = 3$ together, and the analogue of T-duality was identified. Remarkably, the resulting discrete group was shown to be strictly smaller than the one expected from group theoretical analysis as discussed in the next chapter.

Chapter 3

U-duality in M-theory

U-duality in M-theory was introduced and motivated in the first chapter. In this chapter, generators of this symmetry will be given in dimensions $d > 2$. For this, parallel to the toy model in the preceding chapter, the construction of [56] will be applied. The treatment will therefore be semiclassical.

The discussion will lead to a discrete subgroup definition, and generators for these groups will be suggested and proven to span the whole group.

In the section 3.1, the low energy effective action of M-theory is introduced and the compactification to $d = 4$ is discussed. In section 3.2, the construction of U-duality is given, and physical subgroups are discussed. Section 3.3 the discusses the construction of U-duality in $d = 3$.

3.1 Maximal Supergravity

M-theory is supposed to have eleven-dimensional supergravity as low energy limit. Again, in this chapter, only the bosonic part of the theory will be considered, and the fermionic side will be implied to follow from supersymmetry transformations.

The bosonic part of the Lagrangian of eleven-dimensional supergravity is given by [20]

$$\mathcal{L} = -\frac{1}{4}E^{(11)}(R^{(11)} + \frac{1}{12}F_{MNPQ}F^{MNPQ}) + \frac{2}{12^4}\epsilon^{MNPQRSTUVWX}F_{MNPQ}F_{RSTU}A_{VWX},$$

where $F_{MNPQ} = 4\partial_{[M}A_{NPQ]}$, and antisymmetrization of n indices is weighted with $(n!)^{-1}$. Indices $M, N, \ldots$ now run from 0...10, the metric has signature $(+ - - \cdots -)$. Comparing this to (2.1), the similarity of the two Lagrangians is immediately apparent.

The Lagrangian has general reparametrization invariance, local Lorentz invariance and $d = 11$ gauge invariance generated with parameters χ^N, $\Lambda^A{}_B$ and ξ_{NP} as

$$\begin{aligned} E_M^{(11)\,A} &\to E_M^{(11)\,A} + \partial_M\chi^N E_N^{(11)\,A} + \chi^N\partial_N E_M^{(11)\,A} + \Lambda^A{}_B E_M^{(11)\,B} \\ A_{MNP} &\to A_{MNP} + 3\partial_{[M}\chi^N A_{NP]Q} + \chi^Q\partial_Q A_{MNP} + 3\partial_{[M}\xi_{NP]}. \end{aligned}$$

To define U-duality, as in [56], the DSZ quantization in $d = 4$ shall be used. Thus, the compactification on the torus to four dimensions shall be described now.

For this, it is assumed that the dimensions $\{5, \ldots 11\}$ cannot be probed with energies below a certain cutoff that shall be imposed. Thus, the theory has a low energy approximation where the Lagrangian does not depend on these coordinates. Consider, again, pure gravity first.

3.1.1 Reduction to $d = 4$: Gravity

Consider again the Einstein-Hilbert action. If the $d + n$ dimensional Einstein-Hilbert action does not depend on n coordinates corresponding to n commuting Killing vectors, the action may be reduced e.g. in a stepwise fashion as in section 2.2.1. The reduction may be performed in one step as well: Choosing a vielbein

$$E_M^{(d+n)\,A} = \begin{pmatrix} e^{y\phi} E_\mu^{(d)\,\alpha} & B_\mu^a \\ 0 & e_m{}^a \end{pmatrix}, \tag{3.1}$$

with det $e_m{}^a = e^{x\phi}$, $x, y \in \mathbb{R}$, the nonzero anholonomy coefficients are now

$$\begin{aligned} \Omega^{(d+n)}_{\alpha\beta\gamma} &= e^{-y\phi}\left(\Omega^{(d)}_{\alpha\beta\gamma} - 2y\ E^{(d)\,\mu}_{[\alpha}\partial_\mu\phi\eta_{\beta]\gamma}\right), \\ \Omega^{(d+n)}_{\alpha\beta a} &= e^{(-2y)\phi}E^{(d)\,\mu}_\alpha E^{(d)\,\nu}_\beta \eta_{ab} e_i{}^b B^i_{\mu\nu},\ \ B^i_{\mu\nu} = 2\partial_{[\mu}\left(B^a_{\nu]}e_a{}^i\right), \\ \Omega^{(d+n)}_{\alpha bc} &= -e^{-y\phi}e_b{}^i E^{(d)\,\mu}_\alpha \partial_\mu e_i{}^d \eta_{dc}. \end{aligned}$$

Integrating out the coordinates corresponding to the above isometries and demanding $x = y(2-d)$ yields

$$\mathcal{S} = \int d^{(d)}x\ E^{(d)}\ \left(-\mathcal{R}^{(d)} + y^2(d-2)\ \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} + \tfrac{1}{4}e^{-2y\phi}B^k_{\mu\nu}B_k^{\mu\nu}\right),$$

where $g_{mn} = e_m{}^a e_n{}^b(-\delta_{ab})$ is an "internal" metric with signature $(-\,-\cdots-)$, describing the geometry of internal space. Parallel to above, the fields $B^i_\mu = e_a{}^i B^a_\mu$ are n $U(1)$ gauge field, since the internal reparametrization invariance yields

$$B^i_\mu \to B^i_\mu + \partial_\mu \chi^i.$$

Though not used in this thesis, consider for a moment an Einstein-Hilbert action invariant under a set of non-commuting isometries with Killing vectors $\hat{e}_a$ obeying

$$\mathcal{L}_{\hat{e}_k} G_{MN} = 0,\ \ [\hat{e}_k, \hat{e}_l] = f_{kl}{}^{\ m}\hat{e}_m.$$

One has, with vielbein parallel to (3.1) and $\hat{e}_k = \partial_k$,

$$\partial_k G^{(d)}_{\mu\nu} = 0,\ \ \partial_k B^j_\mu = -f_{kl}{}^{\ j} B^l_\mu,\ \ \partial_k g_{mn} = f_{km}{}^j g_{jn} + f_{kn}{}^j g_{jm}.$$

Thus the vielbein components in $d+n$ dimensions depend on the coordinates k, however, if the isometry of the internal space is unimodular, the Einstein-Hilbert action gets independent of these coordinates, and one has an action of the form (see e.g. [18])

$$\mathcal{S} = \int d^{(d)}x E^{(d)}\left(-\mathcal{R}^{(d)} - e^{u\phi}\mathcal{R}^{(n)} + v\ \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}D_\mu g_{mn}D^\mu g^{mn} + \tfrac{1}{4}e^{w\phi}B^k_{\mu\nu}B_k^{\mu\nu} + \Lambda\right),$$

where $B^k_{\mu\nu} = 2\partial_{[\mu}B^l_{\nu]} - gf_{lm}{}^k B^l_\mu B^m_\nu$ is now a non-abelian field strength with gauge group corresponding to the above isometry group, D_μ is the covariant derivative with respect to this group, $\mathcal{R}^{(n)}(g_{mn})$ is the Ricci scalar for the "internal" space and describes a self-interaction of the scalar fields g_{mn}, and Λ is a cosmological constant.

3.1.2 Reduction to $d = 4$: Maximal Supergravity

The reduction to $d = 4$ was carried out in detail e.g. in [20] (see also [21] for a more recent treatment) and shall be briefly recalled. Choose the vielbein to be parametrized as

$$E_M^{(11)A} = \begin{pmatrix} \Delta^{-\frac{1}{4}}E^{(4)\bar{\alpha}}_{\bar{\mu}} & B^{(4)\bar{i}}_{\bar{\mu}}\rho_{\bar{i}}{}^{\bar{a}} \\ 0 & \rho_{\bar{i}}{}^{\bar{a}} \end{pmatrix}$$

with the internal vielbein $\rho_{\bar{i}}^{\bar{a}}$. $\bar{i}$, $\bar{a}$ are curved and flat internal indices respectively. Actually, it will be assumed later that the vielbein is triangular, reflecting the fact that the reduction could have been performed by repeating reduction steps $d+1$ to d.

In the following, it is important to keep in mind that internal coordinates are labeled as 2...8. This choice might seem odd at first sight, but will be useful when the symmetries of the theory are addressed, and later when the $d=3$ theory is discussed. However, the starting formulas when reducing to $d=4$ will look a little messy, but patience will pay off when the $E_{7(+7)}$ symmetry is addressed.

An internal metric may be chosen as usual to be $g_{\bar{m}\bar{n}} = \rho_{\bar{m}\ \bar{a}}\rho_{\bar{n}}^{\bar{a}}$ and has signature $--\cdots-$. Its determinant is $\sqrt{\Delta} = \det \rho_{\bar{m}}^{\bar{a}}$. $\bar{\mu}, \bar{\nu}, \ldots$ are curved and $\bar{\alpha}, \bar{\beta}, \ldots$ flat four dimensional indices, they run in $0\ldots 3$.

The Einstein-Hilbert action is then treated as above, and together with the vector field part, using

$$G^{(11)\bar{\rho}\bar{\kappa}} = \Delta^{1/2}G^{(4)\bar{\rho}\bar{\kappa}},\ \ G^{(11)\bar{k}\bar{\kappa}} = \Delta^{1/2}B^{(4)\bar{k}\ \bar{\kappa}},$$
$$G^{(5)\bar{k}\bar{l}} = g^{\bar{m}\bar{n}} + \Delta^{1/2}B_{\bar{\kappa}}^{(4)\bar{k}}B^{(4)\bar{l}\ \bar{\kappa}},\ \ E^{(11)} = \Delta^{-1/2}E^{(4)}$$

one has

$$\begin{aligned}
\mathcal{L} \ = \ & -\frac{1}{4}E^{(4)}R^{(4)} + \frac{1}{32}E^{(4)}\partial_{\bar{\mu}}\ln\Delta\ \partial^{\bar{\mu}}\ln\Delta - \frac{1}{16}E^{(4)}\partial_{\bar{\mu}}g_{\bar{m}\bar{n}}\ \partial^{\bar{\mu}}g^{\bar{m}\bar{n}} \\
& -\frac{1}{12}E^{(4)}\partial_{\bar{\mu}}A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}\ \partial^{\bar{\mu}}A^{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)} \\
& +\frac{1}{16}E^{(4)}\ \sqrt{\Delta}B_{\bar{\mu}\bar{\nu}}^{(4)\bar{i}}B_{\bar{i}}^{(4)\bar{\mu}\bar{\nu}} - \frac{1}{12}E^{(4)}\Delta\ F^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}}F^{(4)\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}} - \frac{1}{8}E^{(4)}\sqrt{\Delta}\ F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}F^{(4)\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} \\
& -\frac{2}{12^3}\epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}\epsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}\bar{o}}\Big(\ 4F^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}}\partial_{\bar{\rho}}A_{(\bar{j}+2)(\bar{k}+2)(\bar{l}+2)}A_{(\bar{m}+2)(\bar{n}+2)(\bar{o}+2)} \\
& \qquad -9F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}F^{(4)}_{\bar{\rho}\bar{\sigma}\ \bar{k}\bar{l}}A_{(\bar{m}+2)(\bar{n}+2)(\bar{o}+2)} \\
& \qquad +9F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}B^{(4)\bar{p}}_{\bar{\rho}\bar{\sigma}}A_{(\bar{p}+2)(\bar{k}+2)(\bar{l}+2)}A_{(\bar{m}+2)(\bar{n}+2)(\bar{o}+2)} \\
& \qquad -3B^{(4)\bar{p}}_{\bar{\mu}\bar{\nu}}B^{(4)\bar{q}}_{\bar{\rho}\bar{\sigma}}A_{(\bar{p}+2)(\bar{i}+2)(\bar{j}+2)}A_{(\bar{q}+2)(\bar{k}+2)(\bar{l}+2)}A_{(\bar{m}+2)(\bar{n}+2)(\bar{o}+2)}\Big).
\end{aligned}$$

The theory has the 7 $U(1)$ Kaluza-Klein field strengths $B_{\bar{\mu}\bar{\nu}}^{(4)\bar{k}} = 2\partial_{[\bar{\mu}}B_{\bar{\nu}]}^{(4)\bar{k}}$ whose gauge symmetry is generated by the internal diffeomorphisms. For the fields $F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}$ and $F^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}}$, the following definitions are used:

$$\begin{aligned}
F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} &= F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} + B^{(4)\bar{k}}_{\bar{\mu}\bar{\nu}}A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)} \\
F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} &= 2\partial_{[\bar{\mu}}A'_{\bar{\nu}]\ (\bar{i}+2)(\bar{j}+2)}, \\
A'_{\bar{\mu}\ (\bar{i}+2)(\bar{j}+2)} &= A_{\bar{\mu}\ (\bar{i}+2)(\bar{j}+2)} - B^{(4)\bar{k}}_{\bar{\mu}}A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}, \\
F^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}} &= F'_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}} + 3B^{(4)\bar{k}}_{[\bar{\mu}\bar{\nu}}A_{\bar{\rho}](\bar{i}+2)(\bar{k}+2)} \\
F'_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}} &= 3\partial_{[\bar{\mu}}A'_{\bar{\nu}\bar{\rho}]\ (\bar{i}+2)}, \\
A'_{\bar{\mu}\bar{\nu}\ (\bar{i}+2)} &= A_{\bar{\mu}\bar{\nu}\ (\bar{i}+2)} - 2B^{(4)\bar{j}}_{[\bar{\mu}}A_{\bar{\nu}](\bar{i}+2)(\bar{j}+2)} - B^{(4)\bar{j}}_{\bar{\mu}}B^{(4)\bar{k}}_{\bar{\nu}}A_{(\bar{j}+2)(\bar{k}+2)(\bar{i}+2)}.
\end{aligned} \tag{3.2}$$

They ensure that the 21 field strengths $F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}$ are $U(1)$ gauge fields with gauge symmetry generated by $\xi_{(\bar{k}+2)(\bar{l}+2)}$, while they are invariant under the symmetries generated by $\xi_{\bar{\mu}(\bar{l}+2)}$ and internal $d=11$ diffeomorphisms. The three-form field strengths $F'_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}}$ is invariant with respect to the internal diffeomorphisms and gauge symmetries $\xi_{(\bar{k}+2)(\bar{l}+2)}$, but admits a gauge symmetry generated by $\xi_{\bar{\mu}(\bar{l}+2)}$ as

$$A'_{\bar{\mu}\bar{\nu}\ (\bar{i}+2)} \rightarrow A'_{\bar{\mu}\bar{\nu}\ (\bar{i}+2)} + 2\partial_{[\bar{\mu}}\xi_{\bar{\nu}](\bar{i}+2)}.$$

The fields $F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}$ and $F'_{\bar{\mu}\bar{\nu}\bar{\rho}\ \bar{i}}$ are dualized by adding

$$\mathcal{L}_{\text{Lag.mult.}} = \frac{1}{12}\varphi^{(4)\bar{i}}\epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}\partial_{\bar{\mu}}F'_{\bar{\nu}\bar{\rho}\bar{\sigma}\ \bar{i}} + \frac{1}{4}\tilde{A}^{\bar{i}\bar{j}}_{\bar{\sigma}}\epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}\partial_{\bar{\rho}}F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}.$$

Integrating out $F'_{\bar{\nu}\bar{\rho}\bar{\sigma}\ \bar{i}}$ and $F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}$ yields

$$\begin{aligned}
\partial^{\bar{\rho}}\varphi^{(4)\ \bar{i}} &= -\tfrac{1}{3}(E^{(4)})^{-1}\epsilon^{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\rho}}\Delta F^{(4)\qquad \bar{i}}_{\bar{\mu}\bar{\nu}\bar{\rho}} + \tfrac{1}{18}\epsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}\bar{o}}\partial^{\bar{\rho}}A_{(2+\bar{i})(2+\bar{j})(2+\bar{k})}A_{(2+\bar{m})(2+\bar{n})(2+\bar{o})} &(3.3)\\
\tilde{A}^{\bar{i}\bar{j}}_{\bar{\mu}\bar{\nu}} &= E^{(4)}\sqrt{\Delta}\star F^{(4)\ \bar{i}\bar{j}}_{\bar{\mu}\bar{\nu}} - \varphi^{(4)\ [\bar{i}}B^{(4)\ \bar{j}]}_{\bar{\mu}\bar{\nu}}\\
&\quad +\tfrac{1}{6}\epsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}\bar{o}}\Big(F^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}A_{(\bar{m}+2)(\bar{n}+2)(\bar{o}+2)}\\
&\qquad -\tfrac{1}{2}B^{(4)\ \bar{p}}_{\bar{\mu}\bar{\nu}}A_{(2+\bar{p})(2+\bar{k})(2+\bar{l})}A_{(2+\bar{m})(2+\bar{n})(2+\bar{o})}\Big) &(3.4)
\end{aligned}$$

with $\tilde{A}^{\bar{i}\bar{j}}_{\bar{\mu}\bar{\nu}} = 2\partial_{[\bar{\mu}}\tilde{A}^{\bar{i}\bar{j}}_{\bar{\nu}]}$. Parallel to the preceding section, one may introduce the vector notation

$$\mathcal{G}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} \tilde{A}^{\bar{i}\bar{j}}_{\bar{\mu}\bar{\nu}} \\ \tilde{A}^{\bar{i}9}_{\bar{\mu}\bar{\nu}} \end{pmatrix},\ \mathcal{H}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} \\ H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{i}9} \end{pmatrix} \tag{3.5}$$

where

$$\tilde{A}^{\bar{i}9}_{\bar{\mu}\bar{\nu}} = -\tfrac{1}{2}B^{(4)\bar{i}}_{\bar{\mu}\bar{\nu}}, \tag{3.6}$$

and the vectors are thus 28-dimensional, and the dual magnetic fields $\mathcal{H}_{\bar{\mu}\bar{\nu}}$ obey

$$H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} = -\frac{4}{E^{(4)}}\ \star\left(\frac{\delta\mathcal{L}}{\delta\tilde{A}^{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}}\right),\ \ H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{i}9} = -\frac{4}{E^{(4)}}\ \star\left(\frac{\delta\mathcal{L}}{\delta\tilde{A}^{\bar{\mu}\bar{\nu}\ \bar{i}9}}\right).$$

One has

$$\begin{aligned}
H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} &= -F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}} &(3.7)\\
2H^{\tilde{A}}_{\bar{\mu}\bar{\nu}\ \bar{u}9} &= \sqrt{\Delta}\star B^{(4)}_{\bar{\mu}\bar{\nu}\ \bar{u}} - 2\tilde{A}^{\bar{i}\bar{j}}_{\bar{\mu}\bar{\nu}}A_{(2+\bar{i})(2+\bar{j})(2+\bar{u})}\\
&\quad -\tfrac{1}{9}\epsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}\bar{o}}A_{(2+\bar{m})(2+\bar{n})(2+\bar{o})}A_{(2+\bar{i})(2+\bar{j})(2+\bar{u})}A_{(2+\bar{k})(2+\bar{l})(2+\bar{q})}B^{(4)\ \bar{q}}_{\bar{\mu}\bar{\nu}}\\
&\quad +\tfrac{1}{6}\epsilon^{\bar{i}\bar{j}\bar{r}\bar{s}\bar{m}\bar{n}\bar{o}}A_{(2+\bar{m})(2+\bar{n})(2+\bar{o})}A_{(2+\bar{r})(2+\bar{s})(2+\bar{u})}F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}\\
&\quad +2\varphi^{(4)\ [\bar{i}}\delta^{\bar{j}]}_{\bar{u}}F'_{\bar{\mu}\bar{\nu}\ \bar{i}\bar{j}}. &(3.8)
\end{aligned}$$

Turn to the scalar sector. It is shown in [20] that the the 70 scalars of the theory, $A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}$, $\rho_{\bar{i}}^{\ \bar{a}}$ and $\varphi^{(4)\ \bar{i}}$, may be joint together in a field $\mathcal{V}^{(4)} \in E_{7(+7)}/SU(8)$, where the corresponding algebra $\mathfrak{e}_{7(+7)}$ is again the maximally noncompact form [48] of $\mathfrak{e}_7$. $\mathcal{V}^{(4)}$ is a representation matrix in the fundamental **56** representation. This representation is given in chapter VIII of [12] and presented here as in [20].

The representation space is spanned by two antisymmetric tensors $x^{\hat{i}\hat{j}}, y_{\hat{i}\hat{j}}$, where indices $\hat{i}, \hat{j}$ are chosen to run in $2\ldots 9$. On a vector $(x^{\hat{i}\hat{j}}|y_{\hat{i}\hat{j}})^t$, the algebra $\mathfrak{e}_{7(+7)}$ acts by the real matrices

$$\begin{aligned}\Lambda &= \begin{pmatrix} 2\Lambda^{[\hat{i}}{}_{[\hat{k}}\delta^{\hat{j}]}_{\hat{l}]} & \\ & 2\Lambda_{[\hat{i}}{}^{[\hat{k}}\delta^{\hat{l}]}_{\hat{j}]} \end{pmatrix}, \\ \Sigma &= \begin{pmatrix} & \Sigma^{*\hat{i}\hat{j}\hat{k}\hat{l}} \\ \Sigma_{\hat{i}\hat{j}\hat{k}\hat{l}} & \end{pmatrix}\end{aligned} \tag{3.9}$$

where $\Lambda^{\hat{i}}{}_{\hat{k}} = -\Lambda_{\hat{k}}^{\hat{i}}$, $\Lambda^{\hat{i}}{}_{\hat{i}} = 0$ obviously represents an $\mathfrak{sl}(8)$ subalgebra, $\Sigma_{\hat{i}\hat{j}\hat{k}\hat{l}}$ is totally antisymmetric, and $\Sigma^{*\hat{i}\hat{j}\hat{k}\hat{l}} = \frac{1}{24}\epsilon^{\hat{i}\hat{j}\hat{k}\hat{l}\hat{m}\hat{n}\hat{o}\hat{p}}\Sigma_{\hat{m}\hat{n}\hat{o}\hat{p}}$. The representation (3.9) is symplectic and preserves the symplectic from

$$\Omega = \begin{pmatrix} & -\mathbf{1} \\ \mathbf{1} & \end{pmatrix}$$

where $\mathbf{1} = \delta^{\hat{i}\hat{j},\hat{k}\hat{l}}$.

The field $\mathcal{V}^{(4)}$ was found to be

$$\mathcal{V}^{(4)} = \mathcal{V}_+^{(4)}\mathcal{V}_-^{(4)}$$

with

$$\mathcal{V}_+^{(4)} = \begin{pmatrix} v^{[\hat{a}}{}_{[\hat{i}}v^{\hat{b}]}{}_{\hat{j}]} & 0 \\ 0 & v_{[\hat{a}}{}^{[\hat{i}}v_{\hat{b}]}{}^{\hat{j}]} \end{pmatrix} \tag{3.10}$$

where $v^{\hat{a}}{}_{\hat{i}}$ is an $SL(8)$ vielbein, given by

$$v^{\hat{i}}{}_{\hat{a}} = \Delta^{-1/8}\begin{pmatrix} \rho_{\bar{i}}{}^{\bar{a}} & 0 \\ \varphi^{(4)\,\bar{j}}\rho_{\bar{j}}{}^{\bar{a}} & \sqrt{\Delta} \end{pmatrix} \tag{3.11}$$

and

$$\mathcal{V}_-^{(4)} = \exp\begin{pmatrix} 0 & A^{\star\hat{m}\hat{n}\hat{p}\hat{q}} \\ A_{\hat{m}\hat{n}\hat{p}\hat{q}} & 0 \end{pmatrix} \tag{3.12}$$

with the definition

$$A_{\hat{m}\hat{n}\hat{p}\hat{q}} = 4A_{[\bar{m}\bar{n}\bar{p}}\delta^9_{\hat{q}]}.$$

The coupling of the vector fields to the scalars is given by the twisted self-duality relation

$$\mathcal{F}_{\bar{\mu}\bar{\nu}} \equiv \begin{pmatrix} \mathcal{G}_{\bar{\mu}\bar{\nu}} \\ \mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix} = \Omega\mathcal{V}^{(4)T}\mathcal{V}^{(4)}\begin{pmatrix} \star\mathcal{G}_{\bar{\mu}\bar{\nu}} \\ \star\mathcal{H}_{\bar{\mu}\bar{\nu}} \end{pmatrix} \tag{3.13}$$

where, with respect to the representation (3.9), the 56-dimensional vector $\mathcal{F}_{\bar{\mu}\bar{\nu}}$ corresponds to the representation space vector $(x,y)^t$.

From $\mathcal{V}^{(4)}$, a field $P_\mu^{(4)}$ is defined as usual by

$$\partial_\mu\mathcal{V}^{(4)}\mathcal{V}^{(4)-1} = Q_\mu^{(4)} + P_\mu^{(4)},\ \ Q_\mu^{(4)} \in \mathfrak{su}(8),\ \ P_\mu^{(4)} \in \mathfrak{e}_{7(+7)} - \mathfrak{su}(8).$$

The Lagrangian is then

$$\mathcal{L} = -\frac{1}{4}E^{(4)}R^{(4)} + E^{(4)}\left(\frac{1}{8}\mathcal{G}^T_{\bar{\mu}\bar{\nu}}\star\mathcal{H}^{\bar{\mu}\bar{\nu}} + \frac{1}{48}\mathrm{Tr}(P^{(4)}_{\bar{\mu}}P^{(4)\bar{\mu}})\right). \tag{3.14}$$

In passing, note that the vector field part of the action may also be written as

$$\frac{1}{16}E^{(4)}\left(\mathcal{F}^{T}_{\bar\mu\bar\nu}\ \ L\ \star\mathcal{F}^{\bar\mu\bar\nu}\right)$$

with

$$L=\begin{pmatrix} & \mathbf{1}\\ \mathbf{1} & \end{pmatrix}.$$

3.1.3 $\mathfrak{e}_{7(+7)}$ and $\mathfrak{e}_{8(+8)}$

The fact that the above theory admits a hidden $E_{7(+7)}$ is remarkable and has fascinated many authors. How ever, the number of fields of this symmetry is large and the explicit form of fields rather complicated. However, for many applications within M-theory such as the identification of states with higher dimensional branes wrapped around the torus, the connection to the $d=11$ fields is needed, as given above. Several steps shall now be taken to simplify the situation.

In this thesis, the symmetries will rephrased in a more modern form by expressing the scalar field carrying the **56** of $E_{7(+7)}$ by the exponential mapping of the Lie algebra $\mathfrak{e}_{7(+7)}$.

For this, commutation relations and generators of $\mathfrak{e}_{7(+7)}$ need to be given. It will prove to be useful to consider $\mathfrak{e}_{7(+7)}$ as subalgebra of the maximally noncompact real form $\mathfrak{e}_{8(+8)}$ (already) at this stage. As will be shown, this gives a nice way to describe the **56** representation, discussed in the next section, and simplifies the discussion of the $d=3$ theory.

$\mathfrak{e}_{8(+8)}$ is given conveniently by using its decomposition with respect to its $\mathfrak{sl}_9$ subalgebra, parallel to [38].

Realization of $\mathfrak{e}_{8(+8)}$ and $\mathfrak{e}_{7(+7)}$

$\mathfrak{e}_8$ decomposes with respect to its maximal subalgebra $\mathfrak{sl}_9$ (see figure 3.1) as

$$\mathbf{248}=\mathbf{80}\oplus\mathbf{84}\oplus\overline{\mathbf{84}},$$

where **80** is the embedded adjoint of $\mathfrak{sl}_9$, and the **84** ($\overline{\mathbf{84}}$) are in the notation of appendix A.1 the [0 0 1 0 0 0 0 0] and [0 0 0 0 0 1 0 0] irreducible highest weight representations, corresponding to (3,0) ((0,3)) antisymmetric tensor representations. The generators of $\mathfrak{e}_8$ may therefore be equipped with an index structure reflecting this decomposition. Any element of $\mathfrak{e}_8$ may be specified by a triple:

$$[X_{i'}^{\ j'},\ v_{i'j'k'},\ v^{*i'j'k'}]\quad (X\text{ is traceless; }v,v^*\text{ are totally antisymmetric}, i',j',k'=1,\dots,9).$$

The Lie bracket of the two elements $[X_{(1)},v_{(1)},v^*_{(1)}]$ and $[X_{(2)},v_{(2)},v^*_{(2)}]$ is given again by a triple $[X_{(3)},v_{(3)},v^*_{(3)}]$ with [38, 73] (repeated indices are summed over)

$$\begin{aligned}
X_{(3)i'}^{\ \ \ j'} &= X_{(1)i'}^{\ \ \ l'}X_{(2)l'}^{\ \ \ j'}-X_{(2)i'}^{\ \ \ l'}X_{(1)l'}^{\ \ \ j'}\\
&\quad -\frac{1}{2!}\Big(v_{(1)i'p'q'}v^{*j'p'q'}_{(2)}-v_{(2)i'p'q'}v^{*j'p'q'}_{(1)}\\
&\qquad -\frac{1}{9}\left(v_{(1)p'q'r'}v^{*p'q'r'}_{(2)}-v_{(2)p'q'r'}v^{*p'q'r'}_{(1)}\right)\delta^{j'}_{i'}\Big)\\
v_{(3)i'j'k'} &= 3(X_{(1)[i'}^{\ \ \ \ l'}v_{(2)j'k']\,l'}-X_{(2)[i'}^{\ \ \ \ l'}v_{(1)j'k']\,l'})\\
&\quad -\frac{1}{3!}\epsilon_{i'j'k'l'm'n'p'q'r'}v^{*l'm'n'}_{(1)}v^{*p'q'r'}_{(2)}\\
v^{*i'j'k'}_{(3)} &= -3(X_{(1)l'}^{\ \ \ [i'}v^{*j'k']\,l'}_{(2)}-X_{(2)l'}^{\ \ \ [i'}v^{*j'k']\,l'}_{(1)})\\
&\quad +\frac{1}{3!}\epsilon^{i'j'k'l'm'n'p'q'r'}v_{(1)l'm'n'}v_{(2)p'q'r'},
\end{aligned}\tag{3.15}$$

where $\epsilon_{i'j'k'l'm'n'p'q'r'} = -\epsilon^{i'j'k'l'm'n'p'q'r'}$. The indices are raised and lowered by the metric diag$[-1, \dots, -1]$. If all the tensors $X_{i'}^{\ j'}$, $v_{i'j'k'}$ and $v^{*i'j'k'}$ are restricted to real numbers, then the relation (3.15) defines the maximally noncompact real form $\mathfrak{e}_{8(+8)}$.

In the above algebra, generators for the Cartan subalgebra H_i, H_{ijk} and generators $E^i_{\ j}$, E^{ijk}, E^*_{ijk} may be defined by introducing

$$
\begin{aligned}
H_i &= [\delta^i_{i'}\delta^{j'}_i - \delta^{i+1}_{i'}\delta^{j'}_{i+1}, 0, 0], \\
H_{ijk} &= [\delta^i_{i'}\delta^{j'}_i + \delta^j_{i'}\delta^{j'}_j + \delta^k_{i'}\delta^{j'}_k - \frac{1}{3}\delta^{j'}_{i'}, 0, 0], \\
E^i_{\ j} &= [\delta^i_{i'}\delta^{j'}_j, 0, 0], \\
E^{ijk} &= [0, 3!\delta^i_{[i'}\delta^j_{j'}\delta^k_{k']}, 0], \\
E^*_{ijk} &= [0, 0, -3!\delta^{[i'}_i\delta^{j'}_j\delta^{k']}_k].
\end{aligned}
\qquad (3.16)
$$

This definition implies for E^{ijk}

$$E^{ijk} = E^{jki} = E^{kij} = -E^{ikj} = -E^{jik} = -E^{kji},$$

and likewise for E^*_{ijk}.

The generators H_i, H_{ijk} may be expressed as $E^i_{\ i} - E^{i+1}_{\ \ i+1}$, $E^i_{\ i} + E^k_{\ k} + E^k_{\ k} - \frac{1}{3}\sum_{l=1}^9 E^l_{\ l}$, respectively. Writing

$$
\begin{aligned}
&E_{\boldsymbol{\epsilon}_{ij}} = E^i_{\ j} \quad \text{(total 72)}, \\
&E_{\boldsymbol{\epsilon}_{ijk}} = E^{ijk} \quad \text{(total 84)}, \\
&E_{-\boldsymbol{\epsilon}_{ijk}} = E^*_{ijk} \quad \text{(total 84)},
\end{aligned}
$$

the $E_{\boldsymbol{\epsilon}_{ij}}$, $E_{\boldsymbol{\epsilon}_{ijk}}$, $E_{-\boldsymbol{\epsilon}_{ijk}}$ are positive and negative root generators of $\mathfrak{e}_{8(+8)}$, and the $\boldsymbol{\epsilon}_{ij}$, $\pm\boldsymbol{\epsilon}_{ijk}$ are $\mathfrak{e}_{8(+8)}$ roots given by

$$
\begin{aligned}
\boldsymbol{\epsilon}_{ij} &\equiv \mathbf{e}_i - \mathbf{e}_j \qquad (1 \le i \ne j \le 9), \\
\pm\boldsymbol{\epsilon}_{ijk} &\equiv \pm(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k - \frac{1}{3}\sum_{l=1}^{9}\mathbf{e}_l) \qquad (1 \le i < j < k \le 9)
\end{aligned}
\qquad (3.17)
$$

in a hyperplane of $\mathbb{R}^9$ orthonormal to $\sum_{i=1}^9 \mathbf{e}_i$, where $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ is a orthonormal basis of $\mathbb{R}^9$ (see appendix A.1, especially (A.3) and (A.6)).

Figure 3.1: $\mathfrak{e}_8 \to \mathfrak{sl}_9$ decomposition. The root surrounded by a circle is the lowest root added to the Dynkin diagram. For the simple root convention see (3.17).

Take

$$H_i \equiv [E^i_{\ i+1}, E^{i+1}_{\ \ i}] \quad \text{(total 8)}$$

as the basis of the Cartan subalgebra $\mathfrak{h} = \{H_i \mid i = 1, ..., 8\}$ of $\mathfrak{e}_{8(+8)}$. The commutation relations of the $\mathfrak{e}_{8(+8)}$ generators are then from (3.15) given by (repeated indices are *not* summed over unless stated explicitly)

$$
\begin{aligned}
[\, H_i \,,\, H_j \,] &= 0, \\
[\, H_i \,,\, E^j{}_k \,] &= \delta^j_i E^i{}_k - \delta^j_{i+1} E^{i+1}{}_k - \delta^i_k E^j{}_i + \delta^{i+1}_k E^j{}_{i+1}, \\
[\, H_i \,,\, E^{jkl} \,] &= 3(\delta_i^{[l} E^{jk]i} - \delta_{i+1}^{[l} E^{jk]\, i+1}), \\
[\, H_i \,,\, E^*_{jkl} \,] &= -3(\delta^i_{[l} E^*_{jk]i} - \delta^{i+1}_{[l} E^*_{jk]\, i+1}), \\
[\, E^i{}_j \,,\, E^k{}_l \,] &= \delta^k_j E^i{}_l - \delta^i_l E^k{}_j, \\
[\, E^i{}_j \,,\, E^{klm} \,] &= 3\delta_j^{[m} E^{kl]i}, \\
[\, E^i{}_j \,,\, E^*_{klm} \,] &= -3\delta^i_{[m} E^*_{kl]j}, \\
[\, E^{ijk}, E^{lmn} \,] &= -\frac{1}{3!} \sum_{p,q,r}^{9} \epsilon^{ijklmnpqr} E^*_{pqr}, \\
[\, E^*_{ijk}, E^*_{lmn} \,] &= -\frac{1}{3!} \sum_{p,q,r}^{9} \epsilon_{ijklmnpqr} E^{pqr}, \\
[\, E^{ijk}, E^*_{lmn} \,] &= 6\delta^j_{[m} \delta^k_n E^i{}_{l]} \quad \text{if } i \neq l, m, n, \\
[\, E^{ijk}, E^*_{ijk} \,] &= -\frac{1}{3} \sum_{l=1}^{8} l h_l + \sum_{l=i}^{8} h_l + \sum_{l=j}^{8} h_l + \sum_{l=k}^{8} h_l \\
&\equiv h_{ijk}.
\end{aligned}
\tag{3.18}
$$

Within this algebra, the $\mathfrak{e}_{7(+7)}$ subalgebra may be identified by selecting the proper roots within $\mathfrak{e}_{8(+8)}$ root space. Using (A.6), it is generated by

$$
h_{1\bar{i}9},\ E^{1\bar{i}9},\ E^*_{1\bar{i}9},\ E^{\bar{i}}_{\bar{j}},\ E^{\bar{i}\bar{j}\bar{k}},\ E^*_{\bar{i}\bar{j}\bar{k}}. \tag{3.19}
$$

Before addressing the **56** representation of this algebra, it shall be shown, as promised, how $\mathfrak{g}_{2(+2)}$ is embedded.

Realization of $\mathfrak{g}_{2(+2)}$

Quite parallel to above, the Lie algebra $\mathfrak{g}_2$ is known to allow a $\mathbb{Z}_3$ grading and decompose into the adjoint and two fundamental representations of its maximal subalgebra $\mathfrak{sl}(3)$ (see figure 3.2):

$$
\mathbf{14} = \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}.
$$

This implies a realization parallel to above by traceless matrices $X_{i'}{}^{j'}$, $1 \leq i', j' \leq 3$ and two vectors $\tilde{v}_{i'}$ and $\tilde{v}^{*i'}$, $1 \leq i' \leq 3$ with similar commutation relations. As discussed in appendix A.1,

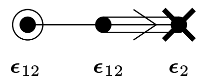
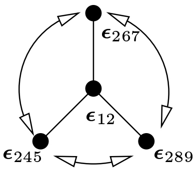


Figure 3.2: $\mathfrak{g}_{2(+2)} \to \mathfrak{sl}_3$ decomposition. The root surrounded by a circle is the lowest root added to the Dynkin diagram. For the root convention see (2.17).

$\mathfrak{g}_{2(+2)}$ can be embedded into $\mathfrak{e}_{8(+8)}$ by choosing any $\mathfrak{d}_4$ subalgebra. Consider the $\mathfrak{d}_4$ subalgebra displayed in figure 3.3 (for the $\mathfrak{e}_{8(+8)}$ roots remember (3.17)). The $\mathfrak{d}_4$ roots are



$\mathfrak{g}_{2(+2)} \leftrightarrow \mathfrak{d}_4$

Figure 3.3: Construction of $\mathfrak{g}_{2(+2)}$: $\mathfrak{d}_4$ Subalgebra of $\mathfrak{e}_{8(+8)}$

$$\Delta^{\mathfrak{d}_4} = \{\pm\boldsymbol{\epsilon}_{i45}, \pm\boldsymbol{\epsilon}_{i67}, \pm\boldsymbol{\epsilon}_{i89} (1 \le i \le 3); \pm\boldsymbol{\epsilon}_{ij} (1 \le i \le j \le 3)\}.$$

The corresponding $\mathfrak{g}_{2(+2)}$ generators are given by, following [61]

$$\begin{aligned} E^i &= E^{i45} + E^{i67} + E^{i89}, \\ E^*_i &= E^*_{i45} + E^*_{i67} + E^*_{i89} && (1 \le i \le 3); \\ E^i{}_j & && (1 \le i, j \le 3). \end{aligned}$$

It is then easy to shown that the commutator relations (2.18) follow from (3.18).

For the realization parallel to (3.15), consider two triples $[X_{(1)}, \tilde{v}_{(1)}, \tilde{v}^*_{(1)}]$, $[X_{(2)}, \tilde{v}_{(2)}, \tilde{v}^*_{(2)}]$. Their bracket gives rise to a new element of $\mathfrak{g}_{2(+2)}$ specified by $[X_{(3)}, \tilde{v}_{(3)}, \tilde{v}^*_{(3)}]$, and may be derived easily from 3.15 by putting $\tilde{v}_{i'} = v_{i'45} + v_{i'67} + v_{i'89}$, $\tilde{v}^{*i'} = v^{*i'45} + v^{*i'67} + v^{*i'89}$, $1 \le i' \le 3$. It is given by

$$\begin{aligned} X_{(3)}{}_{i'}^{j'} &= X_{(1)}{}_{i'}^{l'} X_{(2)}{}_{l'}^{j'} - X_{(2)}{}_{i'}^{l'} X_{(1)}{}_{l'}^{j'} \\ &\quad -3\Big(\tilde{v}_{(1)i'}\tilde{v}^{*j'}_{(2)} - \tilde{v}_{(2)i'}\tilde{v}^{*j'}_{(1)} \\ &\qquad -\frac{1}{3}\,(\tilde{v}_{(1)p'}\tilde{v}^{*p'}_{(2)} - \tilde{v}_{(2)p'}\tilde{v}^{*p'}_{(1)})\delta^{j'}_{i'}\Big) \\ \tilde{v}_{(3)i'} &= X_{(1)}{}_{i'}^{l'}\tilde{v}_{(2)l'} - X_{(2)}{}_{i'}^{l'}\tilde{v}_{(1)l'} \\ &\quad -2\epsilon_{i'j'k'}\tilde{v}^{*j'}_{(1)}\tilde{v}^{*k'}_{(2)} \\ \tilde{v}^{*i'}_{(3)} &= -(X_{(1)}{}_{l'}^{i'}\tilde{v}^{*l'}_{(2)} - X_{(2)}{}_{l'}^{i'}\tilde{v}^{*l'}_{(1)}) \\ &\quad +2\epsilon^{i'j'k'}\tilde{v}_{(1)j'}\tilde{v}_{(2)k'}, \end{aligned} \tag{3.20}$$

where $\epsilon_{i'j'k'} = -\epsilon^{i'j'k'}$, and indices are raised and lowered by the metric diag$[-1,-1,-1]$. Again, if all $X_{i'}^{j'}$, $\tilde{v}_{i'}$ and $\tilde{v}^{*i'}$ are restricted to real numbers, the relations (3.20) define the maximally noncompact real form denoted by $\mathfrak{g}_{2(+2)}$.

The generators H_i, E^i, E^*_i, $E^i{}_j$ may be simplified to be

$$\begin{aligned} H_i &= [\delta^i_{i'}\delta^{j'}_i - \delta^{i+1}_{i'}\delta^{j'}_{i+1}, 0, 0], \\ E^i{}_j &= [\delta^i_{i'}\delta^{j'}_j, 0, 0], \\ E^i &= [0, \delta^i_{i'}, 0], \\ E^*_i &= [0, 0, -\delta^{i'}_i] \end{aligned}$$

in this notation.

3.1.4 The 56 Representation of $\mathfrak{e}_{7(+7)}$

In [20], a main key while exhibiting this hidden $E_{7(+7)}$ structure was the fact that $\mathfrak{e}_{7(+7)}$ has the above **56** dimensional irreducible representation, consisting of two antisymmetric tensors $x^{\hat{i}\hat{j}}, y_{\hat{i}\hat{j}}$, that could be identified with the index structure carried by the electric and magnetic fields of the theory. The fact that this representation is the unique *minimal* irreducible representation of $\mathfrak{e}_{7(+7)}$, that is, all weights are mapped into each other by the Weyl group, and no zero weights exist, will be of interest when discussing U-duality.

For a reduction to $d = 3$ discussed later, the coset matrix $\mathcal{V}^4$ carrying the scalar matter fields in $d = 4$ will need to be expressed as exponential of a sum over Lie algebra generators. Furthermore, many calculations will become simple when a Cartan subalgebra and the corresponding positive and negative weight generators are identified within (3.9).

For this, the **56** representation of $\mathfrak{e}_{7(+7)}$ is constructed as representation of highest weight [1 0 0 0 0 0 0] and the corresponding representation matrices of the generators of $\mathfrak{e}_{7(+7)}$ are given, and a transformation of the representation space basis is found such that the algebra is put into the form (3.9). Normalizations and conventions follow appendix A.1 and the preceding section.

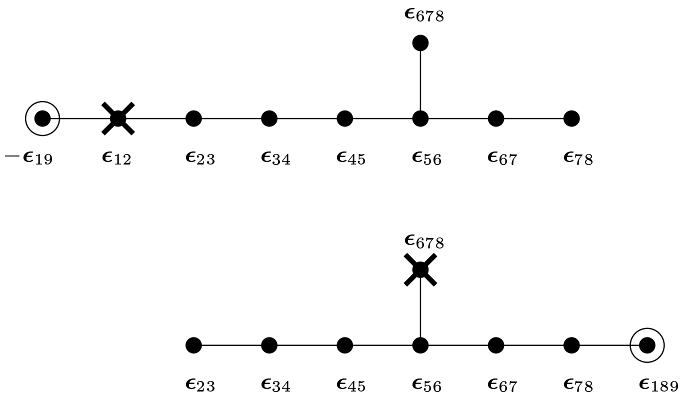


Figure 3.4: Decomposition of $E_{8(+8)} \supset SL(2) \times E_{7(+7)}$ and $E_{7(+7)} \supset SL(8)$

The weights of the **56** representation are given in table 3.1. Corresponding basis vectors need to be normalized in representation space.

This shall be done in the following way: The **56** is minimal, thus a simple root is subtracted or added to a weight at most once (see appendix A.1.2), and all subspaces corresponding to weights Λ are of dimension one. One may then choose, if α is a simple root and $\Lambda' = \Lambda - \alpha$, where Λ', Λ are weights,

$$v^{\Lambda} = \pm E_{\alpha} v^{\Lambda'}, \quad v^{\Lambda'} = \pm E_{-\alpha} v^{\Lambda}.$$

In this theory, gauge fields and charges again correspond to basis vectors of specific weight. Normalizations in the representation space thus corresponds to charge normalizations.

If for the basis $\{v^{\Lambda}\}$ the lattice $\{\mathbb{Z}v^{\Lambda}\}$ is considered, the above choice corresponds to an admissible lattice, that is, a lattice that is invariant with respect the $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra, see A.1.2. That the charges can be chosen to live on such a lattice will enormously simplify the definition of discrete U-duality.

The above normalizations shall now be specified further by considering a basis easily accessible: a basis within the adjoint representation of $\mathfrak{e}_{8(+8)}$.

The adjoint representation of $\mathfrak{e}_{8(+8)}$ decomposes with respect to its maximal $\mathfrak{e}_{7(+7)} + \mathfrak{sl}_2$

subalgebra (see figure 3.4) as

$$\mathbf{248} = (\mathbf{133},\mathbf{1}) \oplus (\mathbf{56},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3}) \tag{3.21}$$

The **133** corresponds to the $\mathfrak{e}_{7(+7)}$ subalgebra of figure 3.4 generated within $\mathfrak{e}_{8(+8)}$ by

$$H_{1\bar{\imath}9},\ E^{1\bar{\imath}9},\ E^*_{1\bar{\imath}9},\ E^{\bar{\imath}}_{\ \bar{\jmath}},\ E^{\bar{\imath}\bar{\jmath}\bar{k}},\ E^*_{\bar{\imath}\bar{\jmath}\bar{k}},$$

and one **56** of the above decomposition is carried by the positive root generators in the coset $\mathfrak{e}_{8(+8)} - \mathfrak{e}_{7(+7)}$, corresponding to the roots

$$\Phi = \{-\boldsymbol{\epsilon}_{\bar{\imath}\bar{\jmath}9}, \boldsymbol{\epsilon}_{1\bar{\imath}}, \boldsymbol{\epsilon}_{1\bar{\imath}\bar{\jmath}}, \boldsymbol{\epsilon}_{\bar{\imath}9}\}$$

where $2 \le \bar{\imath}, \bar{\jmath} \le 8$, while the other is carried by the corresponding negative root generators. The decomposition (3.21) is a $\mathbb{Z}$-graded decomposition with respect to the simple root $\boldsymbol{\epsilon}_{12}$ of $\mathfrak{e}_{8(+8)}$, that is, the roots $\alpha \in \mathfrak{e}_{7(+7)}$ have $r^{\alpha}_{\boldsymbol{\epsilon}_{12}} = 0$, the roots $\alpha \in \Phi$ satisfy $r^{\alpha}_{\boldsymbol{\epsilon}_{12}} = 1$ and $\boldsymbol{\epsilon}_{19}$ corresponding to the $\mathfrak{sl}_2$ factor in (3.21) has $r^{\boldsymbol{\epsilon}_{19}}_{\boldsymbol{\epsilon}_{12}} = 2$.

The roots may be associated with their $\mathfrak{e}_{7(+7)}$ weights given for $H \in \{H_i, H_{678}\}$ by

$$[H, E_{\Lambda_i}] = \lambda^{\Lambda_i}_H E_{\Lambda_i},\ \ \Lambda_i \in \Phi$$

as in table 3.1. Defining the representation matrices for the simple positive and negative root generators in $\mathfrak{e}_{7(+7)}$ by

$$[E_{\pm\alpha_k}, E_{\Lambda_i}] = D(E_{\pm\alpha_k})_{ji} E_{\Lambda_j},\quad \alpha_k \in \Pi^{\mathfrak{e}_{7(+7)}},\ \Lambda_i \in \Phi,$$

the corresponding entries may be read off from table 3.1 using (3.18). They are given in table 3.2.

To obtain all generators of $\mathfrak{e}_{7(+7)}$ in this representation, the procedures presented in appendix A.1 were used to generate the adjoint of $\mathfrak{e}_{8(+8)}$ as a whole, and the 56×56 submatrix corresponding to the adjoint action of the generator in the $\mathfrak{e}_{7(+7)}$ subalgebra on the generators with roots in Φ was extracted (the result may be verified by hand using table 3.1). Consider

$$\begin{aligned} X_{\mathfrak{e}_{7(+7)}} = & \left(h_{189} H_{189} + \sum_{i=2}^{7} h_{i(i+1)} H_i \right) + \\ & \left(\sum_{\bar{\imath}<\bar{\jmath}=2}^{8} e_{\bar{\imath}\bar{\jmath}} E^{\bar{\imath}}_{\ \bar{\jmath}} + \sum_{\bar{\imath}=2}^{8} e_{-1\bar{\imath}9} E^*_{1\bar{\imath}9} + \sum_{\bar{\imath}<\bar{\jmath}<\bar{k}=2}^{8} e_{\bar{\imath}\bar{\jmath}\bar{k}} E^{\bar{\imath}\bar{\jmath}\bar{k}} \right) + \\ & \left(\sum_{\bar{\imath}>\bar{\jmath}=2}^{8} f_{-\bar{\imath}\bar{\jmath}} E^{\bar{\imath}}_{\ \bar{\jmath}} + \sum_{\bar{\imath}=2}^{8} f_{1\bar{\imath}9} E^{1\bar{\imath}9} + \sum_{\bar{\imath}<\bar{\jmath}<\bar{k}=2}^{8} f_{-\bar{\imath}\bar{\jmath}\bar{k}} E^*_{\bar{\imath}\bar{\jmath}\bar{k}} \right) \end{aligned}$$

with $h_{189}, h_{i\,i+1}, e_{\bar{\imath}\bar{\jmath}}, e_{-1\bar{\imath}9}, e_{\bar{\imath}\bar{\jmath}\bar{k}}, f_{-\bar{\imath}\bar{\jmath}}, f_{1\bar{\imath}9}, f_{-\bar{\imath}\bar{\jmath}\bar{k}} \in \mathbb{R}$. $X_{\mathfrak{e}_{7(+7)}}$ is given in table 3.4.

To get $X_{\mathfrak{e}_{7(+7)}}$ in the form (3.9), the $\mathfrak{sl}_8$ subalgebra (figure (3.4)) in (3.9) and table 3.4 may be compared. For this, the representation (3.9) is conveniently generated on the computer. The corresponding `Maple code` is given in figure 3.3.

The resulting relations between the above parameters and the $\Lambda_{\hat{\imath}}^{\ \hat{\jmath}}$ and $\Sigma_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}}$ may be summed up by defining

$$\begin{aligned} \Lambda_{\mathfrak{e}_{8(+8)}} &= \sum_{\bar{\imath}=2}^{8} \left(\Lambda_{\bar{\imath}}^{\ \bar{\imath}} h_{1\bar{\imath}9} + \Lambda_{\bar{\imath}}^{\ 9} E^{1\bar{\imath}9} + \Lambda_{9}^{\ \bar{\imath}} E^*_{1\bar{\imath}9} \right) + \sum_{\bar{\imath},\bar{\jmath}=2}^{8} \Lambda_{\bar{\imath}}^{\ \bar{\jmath}} E^{\bar{\imath}}_{\ \bar{\jmath}}, \\ \Sigma_{\mathfrak{e}_{8(+8)}} &= \sum_{\bar{\imath},\bar{\jmath},\bar{k}=2}^{8} \frac{2}{3!} \left(\Sigma_{\bar{\imath}\bar{\jmath}\bar{k}9} E^{\bar{\imath}\bar{\jmath}\bar{k}} + \Sigma^{*\bar{\imath}\bar{\jmath}\bar{k}9} E^*_{\bar{\imath}\bar{\jmath}\bar{k}} \right). \end{aligned} \tag{3.22}$$

#	$\mathfrak{e}_{7(+7)}$ weight							level	$\mathfrak{e}_{8(+8)}$
	a_{23}	a_{34}	a_{45}	a_{56}	a_{67}	a_{78}	a_{678}		root
1	1	0	0	0	0	0	0	0	ϵ_{29}
2	-1	1	0	0	0	0	0	1	ϵ_{39}
3	0	-1	1	0	0	0	0	2	ϵ_{49}
4	0	0	-1	1	0	0	0	3	ϵ_{59}
5	0	0	0	-1	1	0	1	4	ϵ_{69}
6	0	0	0	0	-1	1	1	5	ϵ_{79}
7	0	0	0	0	1	0	-1	5	$-\epsilon_{789}$
8	0	0	0	0	0	-1	1	6	ϵ_{89}
9	0	0	0	1	-1	1	-1	6	$-\epsilon_{689}$
10	0	0	0	1	0	-1	-1	7	$-\epsilon_{679}$
11	0	0	1	-1	0	1	0	7	$-\epsilon_{589}$
12	0	0	1	-1	1	-1	0	8	$-\epsilon_{579}$
13	0	1	-1	0	0	1	0	8	$-\epsilon_{489}$
14	0	1	-1	0	1	-1	0	9	$-\epsilon_{479}$
15	0	0	1	0	-1	0	0	9	$-\epsilon_{569}$
16	1	-1	0	0	0	1	0	9	$-\epsilon_{389}$
17	1	-1	0	0	1	-1	0	10	$-\epsilon_{379}$
18	0	1	-1	1	-1	0	0	10	$-\epsilon_{469}$
19	-1	0	0	0	0	1	0	10	$-\epsilon_{289}$
20	-1	0	0	0	1	-1	0	11	$-\epsilon_{279}$
21	1	-1	0	1	-1	0	0	11	$-\epsilon_{369}$
22	0	1	0	-1	0	0	1	11	$-\epsilon_{459}$
23	-1	0	0	1	-1	0	0	12	$-\epsilon_{269}$
24	1	-1	1	-1	0	0	1	12	$-\epsilon_{359}$
25	0	1	0	0	0	0	-1	12	ϵ_{123}
26	-1	0	1	-1	0	0	1	13	$-\epsilon_{259}$
27	1	0	-1	0	0	0	1	13	$-\epsilon_{349}$
28	1	-1	1	0	0	0	-1	13	ϵ_{124}
29	-1	1	-1	0	0	0	1	14	$-\epsilon_{249}$
30	-1	0	1	0	0	0	-1	14	ϵ_{134}
31	1	0	-1	1	0	0	-1	14	ϵ_{125}
32	0	-1	0	0	0	0	1	15	$-\epsilon_{239}$
33	-1	1	-1	1	0	0	-1	15	ϵ_{135}
34	1	0	0	-1	1	0	0	15	ϵ_{126}
35	0	-1	0	1	0	0	-1	16	ϵ_{145}
36	-1	1	0	-1	1	0	0	16	ϵ_{136}
37	1	0	0	0	-1	1	0	16	ϵ_{127}
38	0	-1	1	-1	1	0	0	17	ϵ_{146}
39	-1	1	0	0	-1	1	0	17	ϵ_{137}
40	1	0	0	0	0	-1	0	17	ϵ_{128}
41	0	0	-1	0	1	0	0	18	ϵ_{156}
42	0	-1	1	0	-1	1	0	18	ϵ_{147}
43	-1	1	0	0	0	-1	0	18	ϵ_{138}
44	0	0	-1	1	-1	1	0	19	ϵ_{157}
45	0	-1	1	0	0	-1	0	19	ϵ_{148}
46	0	0	0	-1	0	1	1	20	ϵ_{167}
47	0	0	-1	1	0	-1	0	20	ϵ_{158}
48	0	0	0	-1	1	-1	1	21	ϵ_{168}
49	0	0	0	0	0	1	-1	21	ϵ_{18}
50	0	0	0	0	-1	0	1	22	ϵ_{178}
51	0	0	0	0	1	-1	-1	22	ϵ_{17}
52	0	0	0	1	-1	0	-1	23	ϵ_{16}
53	0	0	1	-1	0	0	0	24	ϵ_{15}
54	0	1	-1	0	0	0	0	25	ϵ_{14}
55	1	-1	0	0	0	0	0	26	ϵ_{13}
56	-1	0	0	0	0	0	0	27	ϵ_{12}

Table 3.1: The **56** Representation of $\mathfrak{e}_{7(+7)}$

i	j	$(E^2{}_3)_{ij}$	i	j	$(E^3{}_4)_{ij}$	i	j	$(E^4{}_5)_{ij}$	i	j	$(E^5{}_6)_{ij}$
1	2	1	2	3	1	3	4	1	4	5	1
16	19	-1	13	16	-1	11	13	-1	9	11	-1
17	20	-1	14	17	-1	12	14	-1	10	12	-1
21	23	-1	18	21	-1	15	18	-1	18	22	-1
24	26	-1	22	24	-1	24	27	-1	21	24	-1
27	29	-1	25	28	1	26	29	-1	23	26	-1
28	30	1	29	32	-1	28	31	1	31	34	1
31	33	1	33	35	1	30	33	1	33	36	1
34	36	1	36	38	1	38	41	1	35	38	1
37	39	1	39	42	1	42	44	1	44	46	1
40	43	1	43	45	1	45	47	1	47	48	1
55	56	-1	54	55	-1	53	54	-1	52	53	-1
i	j	$(E^6{}_7)_{ij}$	i	j	$(E^7{}_8)_{ij}$	i	j	$(E^{678})_{ij}$			
5	6	1	6	8	1	5	7	1			
7	9	-1	9	10	-1	6	9	-1			
12	15	-1	11	12	-1	8	10	1			
14	18	-1	13	14	-1	22	25	1			
17	21	-1	16	17	-1	24	28	-1			
20	23	-1	19	20	-1	26	30	1			
34	37	1	37	40	1	27	31	1			
36	39	1	39	43	1	29	33	-1			
38	42	1	42	45	1	32	35	1			
41	44	1	44	47	1	46	49	-1			
48	50	1	46	48	1	48	51	1			
51	52	-1	49	51	-1	50	52	-1			

Table 3.2: Matrices for the $\mathfrak{e}_{7(+7)}$ Simple Root Generators in the **56** Representation

The transformed representation space base is given by

$$\begin{aligned} \mathcal{S}^t &= \Big(- E^{*}_{\bar{i}\bar{j}9}, +E^{1}{}_{\bar{i}} \mid - E^{1\bar{i}\bar{j}}, -E^{\bar{i}}{}_{9} \Big), \\ \mathcal{X}^t &= \Big(x^{\bar{i}\bar{j}}, x^{\bar{i}\bar{9}} \mid y_{\bar{i}\bar{j}}, y_{\bar{i}\bar{9}} \Big). \end{aligned} \tag{3.23}$$

Then

$$[\Lambda_{\mathfrak{e}_{8(+8)}}, \mathcal{X}\cdot\mathcal{S}] = \mathcal{X}'\cdot\mathcal{S}, \quad [\Sigma_{\mathfrak{e}_{8(+8)}}, \mathcal{X}\cdot\mathcal{S}] = \mathcal{X}''\cdot\mathcal{S} \tag{3.24}$$

with

$$\mathcal{X}' = \Lambda\cdot\mathcal{X}, \quad \mathcal{X}'' = \Sigma\cdot\mathcal{X}$$

exactly reproduces the action of (3.9), and therefore, the adjoint action on $\mathcal{S}$ may be used to get the **56** representation $\rho_{\mathbf{56}}$ of $\mathfrak{e}_{7(+7)}$ as subalgebra of $\mathfrak{e}_{8(+8)}$.

```
# e7 in (Lambda, Sigma) representation
#
#
# Cremmer, Julia, NPB 159 p.200-204
#
#
# e7 parameters
S := array(1..8,1..8,1..8,1..8,antisymmetric):
L:=array(1..8,1..8):
Sst:= array(1..8, 1..8, 1..8, 1..8,antisymmetric):
eps:=array(1..8,1..8,1..8,1..8,1..8,1..8,1..8,1..8,
 antisymmetric):
eps[1,2,3,4,5,6,7,8]:=1:
for i1 from 1 to 8 do
 for j1 from 1 to 8 do
  for k1 from 1 to 8 do
   for l1 from 1 to 8 do
     Sst[i1,j1,k1,l1]:=1/24*sum(sum(sum(sum(
      eps[i1,j1,k1,l1,r1,s1,t1,u1]*S[r1,s1,t1,u1],
      r1=1..8),s1=1..8),t1=1..8),u1=1..8):
   od:
  od:
 od:
od:
# representation space
x:=array(1..8,1..8,antisymmetric):
y:=array(1..8,1..8,antisymmetric):
dx:=array(1..8,1..8,antisymmetric):
dy:=array(1..8,1..8,antisymmetric):
```

```
# transformations
for a from 1 to 8 do
 for b from 1 to 8 do
  dx[a,b]:= sum(-L[c,a]*x[c,b],c=1..8)+
   sum(-L[c,b]*x[a,c],c=1..8)+
   sum(sum(Sst[a,b,c,d]*y[c,d], c=1..8),d=1..8):
  dy[a,b]:= sum(L[a,c]*y[c,b],c=1..8)+
   sum(L[b,c]*y[a,c],c=1..8)+
   sum(sum(S[a,b,c,d]*x[c,d], c=1..8),d=1..8):
 od:
od:
e7:=matrix(56,56):
# representation matrix
In1:=0:
for i from 1 to 8 do
 for j from i+1 to 8 do
  In2:=0:
  In1:=In1+1:
  for k from 1 to 8 do
   for l from k+1 to 8 do
    In2:=In2+1:
     e7[In1,In2]:=coeff(dx[i,j],x[k,l]):
     e7[In1,In2+28]:=coeff(dx[i,j],y[k,l]):
     e7[28+In1,In2]:=coeff(dy[i,j],x[k,l]):
     e7[28+In1,28+In2]:=coeff(dy[i,j],y[k,l]):
   od:
  od:
 od:
od:
```

Table 3.3: `Maple` Code: **56** Representation of $\mathfrak{e}_{7(+7)}$ as given in [20]

Table 3.4: **56** of $\mathfrak{e}_{7(+7)}$: Representation Matrix

x_1	$h_{-189}+h_{23}$	x_{15}	$h_{45}-h_{67}$	x_{29}	$-h_{23}+h_{34}-h_{45}$	x_{43}	$-h_{-189}-h_{23}+h_{34}-h_{78}$
x_2	$h_{-189}-h_{23}+h_{34}$	x_{16}	$h_{-189}+h_{23}-h_{34}+h_{78}$	x_{30}	$-h_{23}+h_{45}$	x_{44}	$-h_{45}+h_{56}-h_{67}+h_{78}$
x_3	$h_{-189}-h_{34}+h_{45}$	x_{17}	$h_{23}-h_{34}+h_{67}-h_{78}$	x_{31}	$h_{23}-h_{45}+h_{56}$	x_{45}	$-h_{-189}-h_{34}+h_{45}-h_{78}$
x_4	$h_{-189}-h_{45}+h_{56}$	x_{18}	$h_{34}-h_{45}+h_{56}-h_{67}$	x_{32}	$-h_{34}$	x_{46}	$-h_{56}+h_{78}$
x_5	$h_{-189}-h_{56}+h_{67}$	x_{19}	$h_{-189}-h_{23}+h_{78}$	x_{33}	$-h_{23}+h_{34}-h_{45}+h_{56}$	x_{47}	$-h_{-189}-h_{45}+h_{56}-h_{78}$
x_6	$h_{-189}-h_{67}+h_{78}$	x_{20}	$-h_{23}+h_{67}-h_{78}$	x_{34}	$h_{23}-h_{56}+h_{67}$	x_{48}	$-h_{-189}-h_{56}+h_{67}-h_{78}$
x_7	$h_{-189}+h_{67}$	x_{21}	$h_{23}-h_{34}+h_{56}-h_{67}$	x_{35}	$-h_{34}+h_{56}$	x_{49}	h_{78}
x_8	$-h_{78}$	x_{22}	$h_{34}-h_{56}$	x_{36}	$-h_{23}+h_{34}-h_{56}+h_{67}$	x_{50}	$-h_{-189}-h_{67}$
x_9	$h_{-189}+h_{56}-h_{67}+h_{78}$	x_{23}	$-h_{23}+h_{56}-h_{67}$	x_{37}	$h_{23}-h_{67}+h_{78}$	x_{51}	$-h_{-189}+h_{67}-h_{78}$
x_{10}	$h_{56}-h_{78}$	x_{24}	$h_{23}-h_{34}+h_{45}-h_{56}$	x_{38}	$-h_{34}+h_{45}-h_{56}+h_{67}$	x_{52}	$-h_{-189}+h_{56}-h_{67}$
x_{11}	$h_{-189}+h_{45}-h_{56}+h_{78}$	x_{25}	h_{34}	x_{39}	$-h_{23}+h_{34}-h_{67}+h_{78}$	x_{53}	$-h_{-189}+h_{45}-h_{56}$
x_{12}	$h_{45}-h_{56}+h_{67}-h_{78}$	x_{26}	$-h_{23}+h_{45}-h_{56}$	x_{40}	$-h_{-189}+h_{23}-h_{78}$	x_{54}	$-h_{-189}+h_{34}-h_{45}$
x_{13}	$h_{-189}+h_{34}-h_{45}+h_{78}$	x_{27}	$h_{23}-h_{45}$	x_{41}	$-h_{45}+h_{67}$	x_{55}	$-h_{-189}+h_{23}-h_{34}$
x_{14}	$h_{34}-h_{45}+h_{67}-h_{78}$	x_{28}	$h_{23}-h_{34}+h_{45}$	x_{42}	$-h_{34}+h_{45}-h_{67}+h_{78}$	x_{56}	$-h_{-189}-h_{23}$

3.1.5 The $d = 4$ Scalar Sector Revisited

Assuming the internal vielbein $\rho_{\bar{m}}{}^{\bar{a}}$ to be of upper triangular shape, the relation (3.22) may now be used to reexpress the field $\mathcal{V}^{(4)}$ by the embedding the algebra $\mathfrak{e}_{7(+7)}$ into $\mathfrak{e}_{8(+8)}$. From the results of [20] (3.9), (3.10), (3.11), (3.12), one gets, using (3.18),(3.19), (3.22), (3.23)

$$\begin{aligned}\mathcal{V}^{(4)} \quad = \quad & \exp\left(-\sum_{\bar{m}=2}^{8} \ln\left(\rho_{\bar{m}}{}^{\dot{\bar{m}}}\right)\ h_{1\bar{m}9} + \frac{1}{8}\ln \Delta h_{1\bar{m}9}\right) \\ & \prod_{\bar{p}=0}^{5} \exp\left(-\sum_{\bar{q},\bar{r}=8-\bar{p}}^{8} \rho_{7-\bar{p}}{}^{\dot{\bar{q}}} (\rho^{(7-\bar{p})\ -1})_{\dot{\bar{q}}}{}^{\bar{r}}\ E^{7-\bar{p}}{}_{\bar{r}}\right) \\ & \exp\left(-\sum_{\bar{i}=2}^{8} \varphi^{(4)\ \bar{i}}\ E^{*}_{1\bar{i}9}\right) \\ & \exp\left(\frac{2}{3!}\sum_{\bar{i},\bar{j},\bar{k}=2}^{8} A_{(2+\bar{i})(2+\bar{j})(2+\bar{k})}\ E^{\bar{i}\bar{j}\bar{k}}\right),\end{aligned} \tag{3.25}$$

where the **56** representation is given by the adjoint action on the basis

$$\mathcal{S}^t = \Big(-E^{*}_{\bar{i}\bar{j}9}, +E^{1}{}_{\bar{i}}\ |\ -E^{1\bar{i}\bar{j}}, -E^{\bar{i}}{}_{9}\Big),$$

of the $\mathfrak{e}_{7(+7)}$ algebra generated by

$$h_{1\bar{i}9},\ E^{1\bar{i}9},\ E^{*}_{1\bar{i}9},\ E^{\bar{i}}{}_{\bar{j}},\ E^{\bar{i}\bar{j}\bar{k}},\ E^{*}_{\bar{i}\bar{j}\bar{k}}$$

as stated above. This relation will allow to directly and easily address the further compactification to $d = 3$. $\rho_{\bar{i}}^{(n)\ \bar{a}}$ is the submatrix of $\rho_{\bar{i}}{}^{\bar{a}}$ with columns and rows $(n+1)\ldots 8$.

3.2 U-Duality in $d = 4$

The scalar part of the action is invariant under

$$\mathcal{V}^{(4)} \to h(x)\mathcal{V}^{(4)}\Lambda,\ \ \Lambda \in E_{7(+7)},\ \ h(x) \in SU(8)$$

where the local $SU(8)$ is used to restore the parameterization of coset space.

Turn to the vector Lagrangian. A global symmetry has to act as

$$\mathcal{F}_{\hat{\mu}\hat{\nu}} \to \Lambda^{-1}\mathcal{F}_{\hat{\mu}\hat{\nu}},\ \ \mathcal{V}^{(4)} \to \mathcal{V}^{(4)}\Lambda,\ \ \Lambda \in G \subset E_{7(+7)}.$$

Not the full $E_{7(+7)}$ is a symmetry of the Lagrangian, since it would have to preserve L. It follows that only $\Lambda^{\hat{i}}{}_{\hat{k}}$ in (3.9) can be nonzero, the action is invariant under the $SL(8)$ subgroup in figure 3.4. However, the equations of motion of the theory show the full $E_{7(+7)}$ invariance, since the above symmetry rotates equations of motion and Bianchi identities. This is the classical (hidden) duality symmetry of the theory, and U-duality is a discrete subgroup of this symmetry.

To define U-duality as in [56], consider charges

$$\mathcal{Z} = \begin{pmatrix} p \\ q \end{pmatrix},\ \ p = \frac{1}{2\pi}\oint_{\Sigma}\mathcal{G},\ \ q = \oint_{\Sigma}\mathcal{H}.$$

It has been argued in [56] that all magnetic and electric charges exist. However, this is actually clear from the basis by noting that the **56** representation of $E_{7(+7)}$ is minimal. Hence, all weights of the representation and the corresponding basis vectors in the representation space V are rotated into each other by the Weyl group of $E_{7(+7)}$. If a solution with one nonzero charge exists, solutions with a single charge carried by all other gauge fields may be obtained. The discussion in [56] led to identify such states with higher dimensional branes compactified to $d = 4$, as mentioned in the introduction.

The DSZ condition now reads

$$\mathcal{Z}^t \, \Omega \, \mathcal{Z}' = n, \; n \in \mathbb{Z}.$$

Hence it breaks the continuous symmetry to a discrete subgroup $E_{7(+7)}(\mathbb{Z})$, demanding integer shifts on the lattice defined by the basis vectors (3.23). Thus, the matrix representation of $E_{7(+7)}(\mathbb{Z})$ consists of matrices with integer entries. As discussed in the first chapter, this group has been proposed to be a unified duality symmetry of type II string theory in [56], called U-duality for short, unifying strong-weak coupling dualities and target space dualities and putting all 70 moduli of the theory, including the string coupling constant, on the same footing. The subgroups corresponding to T- and S-duality are discussed below after introducing generators for the discrete group.

3.2.1 Discrete Subgroups of Lie Groups

The question what proper generators of the above group denoted by $E_{7(+7)}(\mathbb{Z})$ are shall be addressed now. This will be done in a general context by addressing discrete subgroups $G(\mathbb{Z})$ of the Lie groups G acting on admissible lattices, since this allows to cover a wider class of discrete groups, including $G_{2(+2)}(\mathbb{Z})$ and $E_{8(+8)}(\mathbb{Z})$. For the necessary Lie algebra conventions and notations, the reader is referred to appendix A.1.

Definition and Remarks

Let G be a complex simple Lie group, and $\mathfrak{g}$ be the Lie algebra of G. Consider a nontrivial representation ρ of G in an irreducible module V, and let Φ be the set of all weights in V. Demand that all nontrivial weights of V are transformed into each other by the Weyl group. Then all nontrivial weights are of multiplicity one, and the representation V is basic.

Actually, if one is interested in string dualities, it is enough to consider only this particular class of representations, since actually all the representations relevant to string duality symmetries are of this type. E.g., the **248** of E_8 is basic, while the **56** of E_7 is minimal.

Choose a basis $\{v_i\} \in V$ such that the lattice $V_{\mathbb{Z}} = \{\mathbb{Z}v_i\}$ forms an admissible lattice, that is, a lattice stable under the action of the $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra of $\mathfrak{g}$ (see appendix A.1.2).

Definition. The *discrete group* $G(\mathbb{Z})$ is defined by

$$G(\mathbb{Z}) = \{g \in G \mid \rho(g) \text{ stabilizes } V_{\mathbb{Z}}\}.$$

An admissible lattice fixes in which basis of V the entries of the elements of $G(\mathbb{Z})$ are restricted to integer values. For example,

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, \; ad - bc = 1 \right\}$$

stabilizes the admissible lattice

$$\mathbb{Z}\begin{pmatrix}1\\0\end{pmatrix}\oplus\mathbb{Z}\begin{pmatrix}0\\1\end{pmatrix}.$$

In other generic basis' of V, $SL(2,\mathbb{Z})$ is not represented as a group of matrices with integral entries. In $d=4$ duality in general, each $U(1)$ gauge field corresponds to different weight space and one can always normalize the charge lattice so that it may coincide with an admissible lattice. Therefore this definition of $G(\mathbb{Z})$ and the concept of discrete duality groups agree.

Discrete Subgroups in Basic Representations

Let G be a complex simple Lie group, and $\mathfrak{g}$ be the Lie algebra of G. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let Δ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, $\Delta_\pm$ the set of positive/negative roots and Π the set of simple roots.

Consider a nontrivial representation ρ of G in an irreducible module V of highest weight Λ^+, and let Φ be the set of all weights Λ in V, and demand that V is basic, that is, all nontrivial weights are transformed into each other be the Weyl group. All nontrivial weights are then of multiplicity one. Choose a basis $\{v^\Lambda, v_i^0\}$ and demand that the lattice $V_\mathbb{Z}=\{\mathbb{Z}v^\Lambda,\mathbb{Z}v_i^0\}$ is admissible. The following proposition shall be proven:
Proposition.

(i) If V is a basic module of $\mathfrak{g}$ and Φ does not contain zero weights, $G(\mathbb{Z})$ coincides with the group generated by

$$\{\exp E_\alpha|\alpha\in\Phi\}$$

denoted by $E(\mathbb{Z})$ in the following.

(ii) If V is a basic module of $\mathfrak{g}$ with zero weights in Φ,

$$G(\mathbb{Z})=E(\mathbb{Z})$$

if the same is true for the direct subgroups of type $SL(2), SL(3)$ and $Sp(4)$ in G.

Proof.

Basic representation without zero weights. All corresponding representations are fundamental representations [69, 57]. The proof is conducted via reduction by rank: It will be shown that, if $g\in G(\mathbb{Z})$, one has an element g' with $g=u^+g'u^-$, $u^+,u^-\in E(\mathbb{Z})$, obeying $g'\in G'(\mathbb{Z})$ with $G'\subset G$ and rank $G'<$ rank G. The representation of G decomposes again into representations of G' of same type, and thus, by repeating these operations, one may reduce any g to a $u^+g'u^-$ where $g'\in SL(2)$ in the fundamental representation, implying $G(\mathbb{Z})=E(\mathbb{Z})$.

Let α_1 be the simple root dual to the highest weight Λ_+ of V. Let Δ'_+ be the set of positive roots orthogonal to λ_1: $\Delta'_+=\{\alpha\in\Delta_+|(\lambda_1,\alpha)=0\}$.

An order may then be defined for the positive roots (denoted by $<$) such that $\alpha<\beta$ if $\alpha\in\Delta'_+$ and $\beta\in\Delta_+-\Delta'_+$. Similarly, for $-\alpha,-\beta\in\Delta_-^{\mathfrak{g}}$, we define $-\beta<-\alpha$ if $\alpha<\beta$. Within $\Delta'^{\mathfrak{g}}$ and $\Delta^{\mathfrak{g}}-\Delta'^{\mathfrak{g}}$, fix an order of the simple roots and order the remaining roots by their height with respect to the simple roots, that is, if height$(\alpha)<$ height(β), $\alpha<\beta$ and $-\beta<-\alpha$ for $\alpha,\beta\in\Delta'_+$ or $\alpha,\beta\in\Delta_+-\Delta'_+$.

It is known that any $g\in G$ can be written in the form (Birkhoff decomposition) [75]

$$g \;=\; \prod_{\alpha\in\Phi^+}\exp(c_{-\alpha}E_{-\alpha})\cdot n(g)\cdot\prod_{\alpha\in\Phi^+}\exp(c_\alpha E_\alpha) \tag{3.26}$$

with $n(g)$ being an element

$$n(g) = \prod_i n_{\beta_i}(t_i), \;\; t_i \in \mathbb{C}, \beta_i \in \Delta^{\mathfrak{g}}, \;\; n_\beta(t) = \exp(tE_\beta)\exp(-t^{-1}E_{-\beta})\exp(tE_\beta).$$

Let the multiple products in (3.26) be ordered with respect to the above order among the positive roots. Consider

$$\rho(g)v_{\Lambda^+} = \sum_{\Lambda\in\Phi} (\rho(g))_{\Lambda,\Lambda^+} v_\Lambda, \quad (\rho(g))_{\Lambda,\Lambda^+} \in \mathbb{Z}.$$

If $(\rho(g))_{\Lambda^+,\Lambda^+} = 0$, let $s_\alpha = n_\alpha(-1) = \exp(-E_\alpha)\exp(E_{-\alpha})\exp(-E_\alpha) \in E(\mathbb{Z})$, $\alpha \in \Delta^{\mathfrak{g}}$. Then $\rho(s_\alpha)$ sends v_Λ to a vector proportional to $v_{\sigma_\alpha(\Lambda)}$, where σ_α is the Weyl reflection with respect to α. Since any weight of ρ is transformed to the highest weight by the Weyl group, one can find some $s \in E(\mathbb{Z})$ (written as a product of s_α's) such that $\rho(s)v_\Lambda \propto v_{\Lambda^+}$ for any weight Λ. Thus, assume $(\rho(g))_{\Lambda^+,\Lambda^+} \neq 0$.

Since $\rho(E_\alpha)v_{\Lambda^+} = 0$ for any positive root α, and since $\rho(E_\alpha)_{\Lambda^+,\Lambda} \neq 0$ only if $\Lambda = \Lambda^+$ for any negative root α and any weight Λ, the assumption $(\rho(g))_{\Lambda^+,\Lambda^+} \neq 0$ implies that $n(g)$ in (3.26) stabilizes the highest weight Λ^+, i.e. $\rho(n(g))v_{\Lambda^+} = cv_{\Lambda^+}$ for some constant $c \in \mathbb{C}$.

The fact that the representation is basic without zero weights implies that $\langle\Lambda^+,\alpha\rangle \in \{0,\pm1\}$ for $\alpha \in \Delta$. Thus the first column and row of the matrix $\rho(g)$ are from (3.26)

$$\begin{aligned}(\rho(g))_{\Lambda,\Lambda^+} &= c\cdot\rho\left(\prod_{\alpha\in\Delta_+-\Delta'_+}(1+c_{-\alpha}E_{-\alpha})\right)_{\Lambda,\Lambda^+} \in \mathbb{Z},\\ (\rho(g))_{\Lambda^+\Lambda} &= c\cdot\rho\left(\prod_{\alpha\in\Delta_+-\Delta'_+}(1+c_\alpha E_\alpha)\right)_{\Lambda^+,\Lambda} \in \mathbb{Z}.\end{aligned}$$

Consider first the case when *G is simply laced*. For simply laced algebras, any $\alpha \in \Delta_+ - \Delta'_+$ cannot be decomposed into a sum of two other such roots (consider $\langle\Lambda^+,\alpha\rangle$). For $\Lambda^+ - \Lambda = \alpha \in \Delta_+ - \Delta'_+$, one therefore has

$$\begin{aligned}(\rho(g))_{\Lambda^+,\Lambda^+} &= c \in \mathbb{Z},\\ (\rho(g))_{\Lambda^+,\Lambda} &= \pm cc_\alpha \in \mathbb{Z},\\ (\rho(g))_{\Lambda,\Lambda^+} &= \pm cc_{-\alpha} \in \mathbb{Z}.\end{aligned} \tag{3.27}$$

If Λ is a weight but cannot be reached from the highest weight Λ^+ by a single Weyl reflection, then $(\rho(g))_{\Lambda,\Lambda^+}$ ($(\rho(g))_{\Lambda^+,\Lambda}$) is expressed as c times a polynomial of $c_{-\alpha}$ (c_α).

Suppose that $(\rho(g))_{\Lambda,\Lambda^+} \neq 0$, $\Lambda = \Lambda^+ - \alpha$ for the minimal $\alpha \in \Delta_+ - \Delta'_+$ with respect to this order. Parallel to $SL(2,\mathbb{Z})$ in the previous section, define $t_\alpha = \exp E_\alpha \in E(\mathbb{Z})$, and $s_\alpha = t_\alpha^{-1}t_{-\alpha}t_\alpha^{-1} \in E(\mathbb{Z})$. Then for $N \in \mathbb{Z}$ one has

$$\begin{aligned}\rho(t^n_{-\alpha}s_\alpha g)_{\Lambda^+,\Lambda^+} &= \mp(\rho(g))_{\Lambda,\Lambda^+}\\ \rho(t^N_{-\alpha}s_\alpha g)_{\Lambda,\Lambda^+} &= \pm(\rho(g))_{\Lambda^+,\Lambda^+} - N(\rho(g))_{\Lambda,\Lambda^+}.\end{aligned}$$

One may always find an integer N such that $|\rho(t^n_{-\alpha}s_\alpha g)_{\Lambda,\Lambda^+}| < |(\rho(g))_{\Lambda,\Lambda^+}|$. Therefore, by repeating this operation, one may arrive at an g' such that $(\rho(g))_{\Lambda,\Lambda^+}$ equals 0 (Euclidean algorithm).

Assume next that there exists a $\beta \in \Delta_+ - \Delta'_+$ such that $(\rho(g))_{\Lambda,\Lambda^+} = 0$ for all $\Lambda = \Lambda^+ - \alpha$, $\alpha < \beta$, and $(\rho(g))_{\Lambda^+-\beta,\Lambda^+} \neq 0$. Then applying a similar operation using s_β, $t_{-\beta}$, we have $\rho(u_- g)_{\Lambda^+-\beta,\Lambda^+} = 0$ with $u_- \in E(\mathbb{Z})$.

Since any $\alpha \in \Delta_+ - \Delta'_+$ cannot be decomposed into a sum of two other such roots, the order in $\Delta_+ - \Delta'_+$ ensures that the inductive assumption $(\rho(g))_{\Lambda,\Lambda^+} = 0$ for all $\Lambda = \Lambda^+ - \alpha$, $\alpha < \beta$ can never be violated by the operation for $\Lambda^+ - \beta$. Thus, one arrives at $\rho(u_- g)_{\Lambda^+ - \alpha,\Lambda^+} = 0$ for all $\alpha \in \Delta_+ - \Delta'_+$ for some element $u_- \in E(\mathbb{Z})$.

The Birkhoff decomposition of $u_- g$ may be written as

$$u_- g \;=\; \prod_{\alpha\in\Delta'_+} \exp(c'_{-\alpha}E_{-\alpha}) \cdot w(g) \cdot \prod_{\alpha\in\Delta_+} \exp(c'_\alpha E_\alpha). \tag{3.28}$$

Therefore, for any weight $\Lambda \neq \Lambda^+$, we have $\rho(u_- g)_{\Lambda,\Lambda^+} = 0$ and, by the condition that the determinant of the representation matrix equals one, $\rho(u_- g)_{\Lambda^+,\Lambda^+} = \pm 1$. Take $g' = u_- g$, and if $\rho(u_- g)_{\Lambda^+,\Lambda^+} = -1$, consider $g' = s_\alpha^2 u_- g$ for $\alpha \in \Delta_+ - \Delta'_+$ to have $\rho(g')_{\Lambda^+,\Lambda^+} = 1$.

One then arrives at

$$\begin{aligned} \rho(g')_{\Lambda^+\Lambda^+} &= 1, \\ \rho(g')_{\Lambda^+\Lambda} &= c'_\alpha \in \mathbb{Z}, \\ \rho(g')_{\Lambda\Lambda^+} &= 0, \end{aligned}$$

where $\alpha \in \Delta_+ - \Delta'_+$. Thus $u_+ = \prod_{\alpha\in\Delta_+ - \Delta'_+} \exp(-c'_\alpha E_\alpha)$ (where the product is arranged in reverse order with respect to the one in (3.28)) belongs to $E(\mathbb{Z})$, and

$$\begin{aligned} \rho(u_- g u_+)_{\Lambda^+,\Lambda^+} &= 1 \\ \rho(u_- g u_+)_{\Lambda^+,\Lambda} &= 0, \\ \rho(u_- g u_+)_{\Lambda,\Lambda^+} &= 0, \end{aligned}$$

for all the weights $\Lambda \neq \Lambda^+$. Consider a Birkhoff decomposition for $u_- g u_+$. The elements n contained in this decomposition that correspond to roots $\alpha \in \Delta - \Delta'$ are restricted to be diagonal by the fact that they stabilize the highest weight. On the other hand, diagonal elements are in H, but $H = H' \exp(cH_{\alpha_1})$, where H' corresponds to the Cartan subgroup of the group G' with algebra $\mathfrak{g}'$ and roots Δ'. But $\rho(u_- g u_+)_{\Lambda^+,\Lambda^+} = 1$ implies $c = 0$. Therefore, $u_- g u_+$ belongs to the subgroup $G' \subset G$ with Lie algebra $\mathfrak{g}'$ whose simple roots are $\Pi^{\mathfrak{g}'} = \Pi^{\mathfrak{g}} - \alpha_1$, and the representation ρ of G is again decomposed into a direct sum of representations of G' of same type. The reduction chain therefore ends with fundamental representations of $SL(2)$, proving the proposition.

As illustration, the corresponding chain for the **56** representation of $E_{7(+7)}$ is given in figure 3.5.

If G is not simply laced, the same arguments apply, except for the modification in the consistent order of operations. Some roots in $\Delta_+ - \Delta'_+$ can be written as a sum of two roots (sum of three roots for G_2) in $\Delta_+ - \Delta'_+$ if G is not simply laced, and (3.27) gets additional terms of c times polynomials in the corresponding c_α's ($c_{-\alpha}$'s). The reduction is performed as above, but by changing the order of operations, since the order induced by ordering the roots with respect to their height fails to ensure consistency. Such orders however always exists. If α decomposes into two roots, that is, $\alpha, \beta, \gamma \in \Delta_+ - \Delta'_+$, $\alpha = \beta + \gamma$ and $\beta < \gamma$, a consistent order is given by setting the $\{\Lambda^+ - \beta, \Lambda^+\}$ entry, then $\{\Lambda^+ - \alpha, \Lambda^+\}$ and $\{\Lambda^+ - \gamma, \Lambda^+\}$ entry to zero.

For G_2, the consistent order is illustrated in appendix 3.2.1. Since its (unique) basic representation involves a zero weight, representation with such weights will be discussed first.

Basic representations with zero weights. The only basic representations of this type are those where the nontrivial weights in Φ correspond to the short roots of $\mathfrak{g}$ [69]. They are fundamental representation, with the exception of the adjoint of $SL(n)$. Suppose first that *the representation is fundamental*.

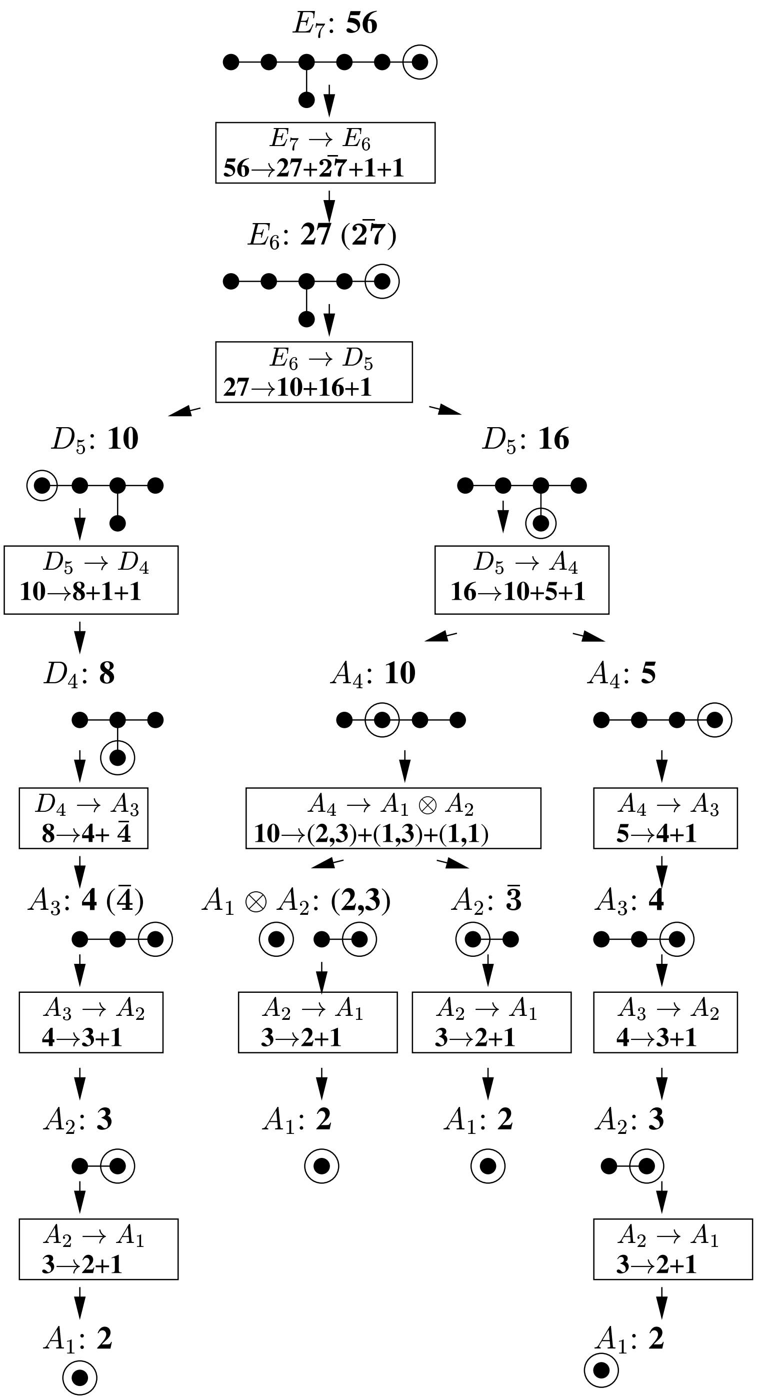


Figure 3.5: $E_{7(+7)}(\mathbb{Z})$: Reduction Chain for the **56** Representation

For a proof of the above proposition, a reduction by rank parallel to the one described above holds, but modifications for the entries $(\rho(g))_{0i,\Lambda^+}$ and $(\rho(g))_{-\Lambda^+,\Lambda^+}$ are needed. V is a subspace of the adjoint module, and the representation matrices for $X = \exp(Y)$ may be given by the action

$$\exp(\mathrm{ad}Y)(v^\Lambda) = \rho(X)_{\Lambda',\Lambda} v^{\Lambda'}, \quad X \in \mathfrak{g}.$$

$V_\mathbb{Z}$ is admissible, and without loss of generality one may set $v^{\Lambda^+} = E_{\Lambda^+}$, since the lattice is defined only modulo a multiplication of the highest weight vector with a scalar. Furthermore, the H_{α_i} with $\alpha_i \in \Pi \cap \Phi$ are lattice vectors in $V_\mathbb{Z}$[1] . Again, consider the Birkhoff decomposition (3.26) and assume with the same reasoning as above that $(\rho(g))_{\Lambda^+,\Lambda^+} \neq 0$ (if $(\rho(g))_{0i,\Lambda^+} \neq 0$ is the only nonzero entry, multiply with a suitable $\exp(E_\alpha) \in E(\mathbb{Z})$ and proceed as above) and therefore $\rho(n(g))v_{\Lambda^+} = c\, v_{\Lambda^+}$, $c \in \mathbb{C}$. The first column and row are then given by

$$\begin{aligned}
(\rho(g))_{\Lambda,\Lambda^+} &= c \cdot \rho \left(\prod_{\alpha\in\Delta_+ - \Delta'_+} (1 + c_{-\alpha} E_{-\alpha} + \tfrac{1}{2}(c_{-\alpha})^2 (E_{-\alpha})^2) \right)_{\Lambda,\Lambda^+} \in \mathbb{Z}, \\
(\rho(g))_{\Lambda^+,\Lambda} &= c \cdot \rho \left(\prod_{\alpha\in\Delta_+ - \Delta'_+} (1 + c_{\alpha} E_{\alpha} + \tfrac{1}{2}(c_{\alpha})^2 (E_{\alpha})^2) \right)_{\Lambda^+,\Lambda} \in \mathbb{Z}.
\end{aligned}$$

For $\Lambda^+ - \Lambda = \alpha \in \Delta_+ - \Delta'_+$,

$$\begin{aligned}
(\rho(g))_{\Lambda^+,\Lambda^+} &= c \in \mathbb{Z}, \\
(\rho(g))_{\Lambda^+,\Lambda} &= \pm c c_\alpha + \cdots \in \mathbb{Z}, \\
(\rho(g))_{\Lambda,\Lambda^+} &= \pm c c_{-\alpha} + \cdots \in \mathbb{Z}.
\end{aligned}$$

parallel to above, where the $\ldots$ are nonzero only for non-simply laced groups, where an $\alpha \in \Delta_+ - \Delta'_+$ may decompose into roots $\beta_i \in \Delta_+ - \Delta'_+$, and contain terms of type

$$c \prod_i c_{\beta_i} \quad \left(c \prod_i c_{-\beta_i} \right), \quad \alpha = \sum_i \beta_i, \quad i \leq 3, \quad \alpha, \beta_i \in \Delta_+ - \Delta'_+.$$

parallel to above. If Λ is a weight but cannot be reached from the highest weight Λ^+ by a single Weyl reflection, then $(\rho(g))_{\Lambda\Lambda^+}$ $((\rho(g))_{\Lambda^+\Lambda})$ is expressed as c times a polynomial in the corresponding $c_{-\alpha}$ (c_α). Furthermore, one has

$$\begin{aligned}
(\rho(g))_{\Lambda^+,0i} &= \tilde{m}_i\, c c_{\Lambda^+} + \cdots \in \mathbb{Z}, \\
(\rho(g))_{0i,\Lambda^+} &= m_i\, c c_{-\Lambda^+} + \cdots \in \mathbb{Z}, \\
(\rho(g))_{\Lambda^+,-\Lambda^+} &= \mp c (c_{\Lambda^+})^2 + \cdots \in \mathbb{Z}, \\
(\rho(g))_{-\Lambda^+,\Lambda^+} &= \mp c (c_{-\Lambda^+})^2 + \cdots \in \mathbb{Z},
\end{aligned} \tag{3.29}$$

where the dots contain terms as above. Since $[E_{-\Lambda^+}, E_{\Lambda^+}] = -H_{\Lambda^+}$ one has $\tilde{m}_i, m_i \in \mathbb{Z}$.

The reduction is then carried out exactly as in the case when Φ does not contain zero weights. Choosing a suitable order of operations, one arrives at a g' of the form

[1] It is known [57], if a highest weight vector v has been fixed, that any module V of a Lie algebra $\mathfrak{g}$ contains a minimal and a maximal admissible lattice. The minimal lattice is given by $\mathcal{U}_\mathbb{Z} v$ (see appendix A.1.2), containing the H_{α_i} in the above case, and any admissible lattice in V contains this lattice. The maximal lattice is dual to the minimal lattice of the dual representation space V^*. Any admissible lattice is contained in this maximal lattice. For example, for the adjoint module of $\mathfrak{sl}_2$, the basis of the minimal admissible lattice is given by $\{E_2, H_2, E_{-2}\}$, while the basis of the maximal admissible lattice is $\{E_2, 1/2H_2, E_{-2}\}$.

$$\begin{aligned}(\rho(g'))_{\Lambda^+,\Lambda^+} &= +c \in \mathbb{Z},\\ (\rho(g'))_{0i,\Lambda^+} &= -m_i\, c\, c_{-\Lambda^+} \in \mathbb{Z},\\ (\rho(g'))_{-\Lambda^+,\Lambda^+} &= \mp c\,(c_{-\Lambda^+})^2 \in \mathbb{Z}.\end{aligned}$$

Consider $(t_{-\Lambda^+})^N s_{\Lambda^+} g'$, $N \in \mathbb{N}$. This yields

$$\begin{aligned}(\rho((t_{-\Lambda^+})^N s_{\Lambda^+} g'))_{\Lambda^+,\Lambda^+} &= +c(c_{-\Lambda^+})^2 \in \mathbb{Z},\\ (\rho((t_{-\Lambda^+})^N s_{\Lambda^+} g'))_{0i,\Lambda^+} &= -m_i(c\, c_{-2} - Nc\,(c_{-\Lambda^+})^2) \in \mathbb{Z},\\ (\rho((t_{-\Lambda^+})^N s_{\Lambda^+} g'))_{-\Lambda^+,\Lambda^+} &= \mp(c + 2Nc\, c_{-\Lambda^+} + N^2 c\,(c_{-\Lambda^+})^2) \in \mathbb{Z}.\end{aligned}$$

Using $c_{-2} = n/m \cdot 1/c,\ n \in \mathbb{Z}, m \in \{m_i\} \in \mathbb{Z}$, one gets

$$\begin{aligned}c\,(c_{-\Lambda^+})^2 &= \frac{1}{m^2 c} n^2\\ (c + 2Nc\, c_{-\Lambda^+} + N^2 c\,(c_{-\Lambda^+})^2) &= \frac{1}{m^2 c}(mc + Nn)^2\end{aligned}$$

and one can always find an integer N such that

$$|mc + Nn| < |n|,$$

and repeat this operation until a $u_- g'$, $u_- \in E(\mathbb{Z})$ is reached with $(\rho(u_- g'))_{-\Lambda^+,\Lambda^+} = 0$, implying $(\rho(u_- g'))_{0i,\Lambda^+} = 0$ (Euclidian algorithm).

Thus, reduction by rank holds also for basic representations including zero weights. However, the endpoints of the corresponding chains need further study. If the endpoint corresponds to basic representations with zero weights of $Sp(4)$, $SL(3)$ or $SL(2)$, further study is needed.

If the reduction chain ends at a group of rank two in a representation whose highest weight is the sum of the two simple roots (**5** of $Sp(4)$ or **8** of $SL(3)$), then no further reduction to rank one is possible. Consider $g = u_- g' u_+$, $u_\pm \in E(\mathbb{Z})$ with $g' \in Sp(4)$ ($g' \in Sl(3)$), and $\rho(g')$ acts in the **5** (**8**) plus singlets on $V_\mathbb{Z}$. One may then apply $(t_{\pm\alpha})^N s_\alpha$ to reach $g' = u_- n u_+$, $u_\pm \in E(\mathbb{Z})$, where n is an element of G' with $\rho(n)_{\tilde{\Lambda}^+,\tilde{\Lambda}^+} = \pm 1$, $\tilde{\Lambda}^+$ being the highest weight of the **5** (**8**) representation. But still such element may exist that are not in $E(\mathbb{Z})$. If the reduction chain ends with $Sl(2)$ acting in the adjoint representation plus singlets, e.g. $n_2(I)n_2(-1)$ is a matrix with integral entries, but not in $E(\mathbb{Z})$.

However, if the above representations are accompanied by further basic representations in $V_\mathbb{Z}$ that do not involve zero weights, the latter fix the maximal set of integral representation matrices on $V_\mathbb{Z}$. In these cases, one therefore still has $G(\mathbb{Z}) = E(\mathbb{Z})$. This is the case for the adjoint representations of $E_{8(+8)}$ (and $E_{7(+7)}$), where the endpoints of the reduction chain corresponds to $SL(2)$, but adjoint representation are accompanied by fundamental ones. The corresponding chains are given in the figures 3.6, 3.7.

Finally, consider *the adjoint of* $SL(n)$. Here, one can again achieve by the same reasoning that $(\rho(g'))_{\Lambda^+,\Lambda} = 0$ and $(\rho(g'))_{\Lambda,\Lambda^+} = 0$ for $\Lambda \neq \Lambda^+$, and $(\rho(g'))_{\Lambda^+,\Lambda^+} = 1$ where $g' = u_- g u_+$, and $u_-, u_+ \in E(\mathbb{Z})$. One may however still have elements of G/G' in the normalizer contained in the Birkhoff decomposition of g' of the form $\exp c(H_{\alpha_1} - H_{\alpha_{n-1}})$, where α_1, α_{n-1} correspond to the endpoints of the Dynkin diagram of $\mathfrak{sl}_n$. However, this elements acts as identity times a factor on each sub-lattice of $G' = Sl(n-2)$ (the next reduction step) in $V_\mathbb{Z}$, and therefore, the reduction may be carried out as before, and $c = 0$.

This completes the proof of the proposition. □

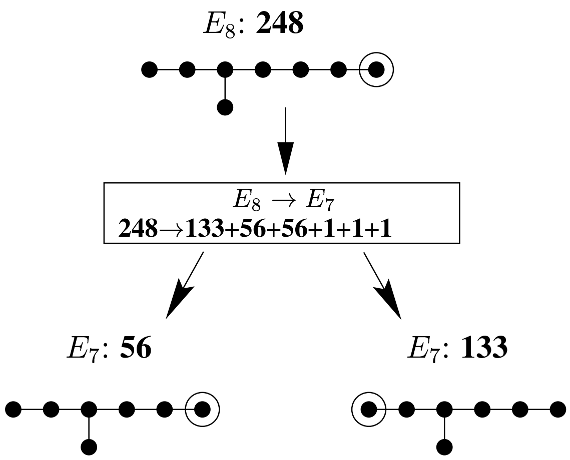


Figure 3.6: $E_{8(+8)}(\mathbb{Z})$: Reduction Chain for the Adjoint Representation

Illustration: $G_2(\mathbb{Z})$

Consider the algebra $\mathfrak{g}_2$ with conventions as in 2.4.2. Consider the (unique) basic representation **7** with highest weight $\Lambda^+ = [0, 1]$ in Dynkin components. Choose as admissible lattice the sub-lattice of the adjoint representation with weights corresponding to the short roots (weights 3,4,6,8,10,11,12 in figure A.9, roots numbering as in this figure). The corresponding representation matrices are sub-matrices of the one given in figure A.10. With the above Λ^+, one has $\Delta' = \{\alpha_1\}$.

A general group element g admits a Birkhoff decomposition (3.26). Suppose $\rho(g)_{\Lambda^+,\Lambda^+} \neq 0$, else, if $\rho(g)_{\Lambda,\Lambda^+} \neq 0$, multiply with the corresponding s_α for nontrivial Λ or with $s_{\alpha_3} t_{\alpha_2}$ for the zero weight to get $\rho(u_- g)_{\Lambda^+,\Lambda^+} \neq 0$, $u_- \in E(\mathbb{Z})$.

The first column $\rho(g)_{\Lambda,\Lambda^+}$ in the **7** representation is then given by

$$\begin{pmatrix} c \\ c\,c_2 \\ -c\,c_3 \\ c\,c_4 - c\,c_2\,c_3 \\ -c\,c_5 + 2\,c\,c_2\,c_4 - c\,(c_2)^2\,c_3 \\ c\,c_6 - 2c\,c_3\,c_4 \\ c\,c_2\,c_6 - c\,c_3\,c_5 + c\,(c_4)^2 - 2\,c\,c_2\,c_3\,c_4 \end{pmatrix}$$

Note that $\alpha_4 = \alpha_3 + \alpha_2$, $\alpha_5 = \alpha_4 + \alpha_3 + \alpha_2 = 2\alpha_2 + \alpha_3$ and $\alpha_6 = \alpha_3 + \alpha_4$, as the above Birkhoff decomposition indicates.

This implies operations $((t_{-\alpha})^N s_\alpha)$ in the order α_2, α_5, α_4, α_6, α_3 to set all $\rho(g)_{\Lambda,\Lambda^+}, \Lambda \neq \Lambda^+$ consistently to zero, that is, without violating a zero of a previous operation.

Consider the resulting group element given by $u'_- g$. Its Birkhoff decomposition implies $(\rho(u'_- g))_{\Lambda^+,\Lambda^+} = \pm 1$. If $(\rho(u'_- g))_{\Lambda^+,\Lambda^+} = -1$, consider e.g. $(\rho((s_{\alpha_2})^2 u'_- g))_{\Lambda^+,\Lambda^+} = 1$. The Birkhoff decomposition for the first row then implies $g = u_- g' u_+$ with $g' \in G' = SL(2)$ with Lie algebra corresponding to the root α_1. The **7** decomposes into two fundamental representations plus three singlets with respect to G', as can be read off from the weights in figure A.9. Thus, $G_2(\mathbb{Z}) = E(\mathbb{Z})$.

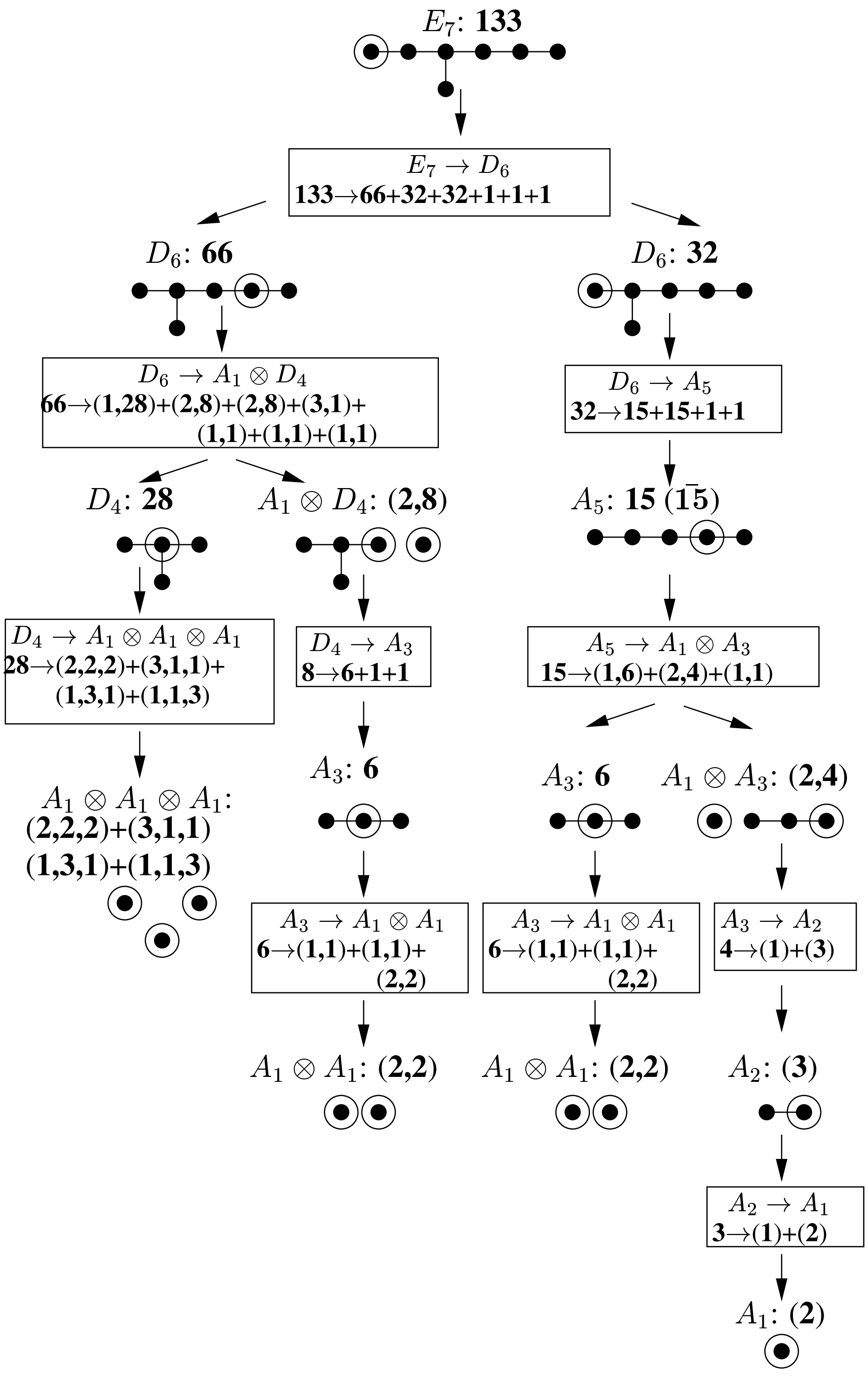


Figure 3.7: $E_{7(+7)}(\mathbb{Z})$: Reduction Chain for the Adjoint Representation

Groups over $\mathbb{Z}$

In this section, an abstract definition of discrete subgroups of Lie groups [62, 68, 69] using group schemes over $\mathbb{Z}$ due to Chevalley [17] is reviewed. The resulting groups are analogous to the analytic definitions given above and might, by their connection to more powerful algebraic definitions, proof useful for further study of dualities symmetries and their associated groups, especially in lower dimensions.

Consider a connected semi-simple algebraic group G of automorphisms on a vector space V over $\mathbb{C}$. Let $\mathfrak{g}$ be the corresponding Lie algebra and consider the universal enveloping algebra $\mathcal{U}$ as introduced in appendix A.1.2. $\mathcal{U}$ is a Hopf algebra over $\mathbb{C}$ [62] with diagonal map $d : dx = x \otimes 1 + 1 \otimes x$, co-unit $\epsilon : \epsilon(x) = 0$ and antipode $s : s(x) = -x$ for $x \in \mathfrak{g}$, and the $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ is a Hopf algebra over $\mathbb{Z}$.

A Hopf algebra $\mathbb{Z}[G]$ over the group G may then be as follows [62]: Consider an admissible lattice $V_\mathbb{Z}$ in V and fix a basis $\{v_i\}$ with dual basis $\{w_i\}$ in V^*. Define then $f_{ij}(u) = \langle uv_i, w_j\rangle$ for all $u \in \mathcal{U}_\mathbb{Z}$. The f_{ij} then generate a Hopf algebra $\mathbb{Z}[G]$ over $\mathbb{Z}$, and the set of ring homomorphisms of $\mathbb{Z}[G]$ into $\mathbb{Z}$

$$G_\mathbb{Z} = \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$$

has a group structure, induced by the Hopf algebra structure of $\mathbb{Z}[G]$. $G_\mathbb{Z}$ is called the *group over* $\mathbb{Z}$ *associated with* G.

A subgroup X of G induces the corresponding subgroup $X_\mathbb{Z}$ of $G_\mathbb{Z}$. For each root $\alpha \in \Delta^\mathfrak{g}$, define a subgroup $N^\alpha = \exp(\mathbb{C}E_\alpha)$. The elements in the union of $N^\alpha_\mathbb{Z}$ are called *fundamental unipotents*, and the subgroup of $G_\mathbb{Z}$ generated by all $N^\alpha_\mathbb{Z}$ will be denoted by E.

In [69], it is proven that $G_\mathbb{Z}$ and E agree if the same holds for the subgroups $SL(2)$, $SL(3)$, $Sp(4)$ by considering basic representations and reducing the problem with respect to the rank.

3.2.2 $d = 4$ U-duality Group

The above considerations directly apply to the maximally noncompact real forms, since the corresponding Lie algebras are in the above normalization. Thus, one may now identify U-duality with the group generated by

$$\begin{aligned} T^{\bar i}{}_{\bar j} &= \exp(E^{\bar i}{}_{\bar j}),\ \ \bar i < \bar j \quad &;\quad T^{\bar i}{}_{\bar j} &= \exp(E^{\bar i}{}_{\bar j}),\ \ \bar i > \bar j, \\ T^{1\bar i 9} &= \exp(E^{1\bar i 9}) \quad &;\quad T^*_{1\bar i 9} &= \exp(E^*_{1\bar i 9}), \\ T^{\bar i\bar j\bar k} &= \exp(E^{\bar i\bar j\bar k}) \quad &;\quad T^*_{\bar i\bar j\bar k} &= \exp(E^*_{\bar i\bar j\bar k}) \end{aligned} \tag{3.30}$$

or alternatively

$$\begin{aligned} T^{\bar i}{}_{\bar j} &= \exp(E^{\bar i}{}_{\bar j}), \quad &;\quad S^{\bar i}{}_{\bar j} &= \exp(-E^{\bar j}{}_{\bar i})\exp(E^{\bar i}{}_{\bar j})\exp(-E^{\bar j}{}_{\bar i})\ \ \bar i < \bar j, \\ T^{1\bar i 9} &= \exp(E^{1\bar i 9}) \quad &;\quad S^{1\bar i 9} &= \exp(-E^*_{1\bar i 9})\exp(E^{1\bar i 9})\exp(-E^*_{1\bar i 9}), \\ T^{\bar i\bar j\bar k} &= \exp(E^{\bar i\bar j\bar k}) \quad &;\quad S^{\bar i\bar j\bar k} &= \exp(-E^*_{\bar i\bar j\bar k})\exp(E^{\bar i\bar j\bar k})\exp(-E^*_{\bar i\bar j\bar k}) \end{aligned} \tag{3.31}$$

in the representation $\mathbf{56}$, where the S generators are known to carry a representation of the Weyl group modulo $\mathbb{Z}_2$ (see e.g. [63]).

Two things should be noted: on the basis $\mathcal{S}$, the actions of the S generators are Weyl reflections and may therefore be easily given. The action of the T generators is significantly simplified by the fact that the **56** representation is minimal. This implies that the square of all Chevalley generators vanishes on this basis, and this the action is again easily read off from the basis $\mathcal{S}$.

This discrete group was constructed by compactifying the theory to $d = 4$ and using a suitable quantization condition. For compactifications to higher dimensions, the symmetry groups of the theory are direct subgroups (see chapter one), and the corresponding U-duality groups are given by their embeddings into the discrete subgroup constructed here.

This is simplified enormously by noting that the discrete subgroup found is actually representation independent. Hence the notion of the above generators is representation independent, and the $E_{6(+6)}(\mathbb{Z})$ etc. U-duality generators follow directly from truncating the Dynkin diagram and identifying the corresponding subsets of generators.

Note that, apart from this fact, the symmetry groups in higher dimensions all act in minimal representations, which can be read off explicitly from the basis (3.23). Actually, in this sense, all U-dualities follow from the adjoint representation of $E_{8(+8)}$, and this may be taken as indication that $E_{8(+8)}$ is a fundamental underlying structure of M-theory.

The explicit shape of this U-duality transformations shall be used in the next chapter when discussing solitons. Here, some well known subgroups shall be investigated: T- and S-duality, along the lines of [87, 89].

3.2.3 T-duality

T-duality shall be discussed in two ways: first, it shall be identified by considering the subset of fields called NS-NS sector. Then, it shall be discussed by considering the moduli of the internal torus, parallel to the preceding chapter.

The known superstring theories in ten dimensions have a common low-energy sector whose spectrum is the same as the NS-NS sector of type II theories. The corresponding low-energy fields will be called NS-NS fields. In $d = 4$, the fundamental string can carry electric charge with respect to the U(1) fields in the NS-NS sector ([77], see also chapter one). T-duality is identified with the subgroup of $E_{7(+7)}$ that stabilizes the NS-NS charge lattice.

The NS-NS sector in $d = 10$ consists of the metric, the dilaton and an antisymmetric two-form. In order to identify the corresponding $d = 4$ fields, take the direction 10 as compact eleventh spatial dimensions, and $\{0, \dots, 9\}$ as space time coordinates the string propagates in. Considering (3.1.2), the corresponding $d = 4$ fields are

$$E_{\bar{\mu}}^{(4)\bar{\alpha}},\ \rho_8{}^{\dot{8}},\ \rho_{\tilde{i}}{}^{\tilde{a}},\ B_{\mu}^{(4)\tilde{i}},$$

where indices with tilde run in $2\dots 7$. The $d = 10$ antisymmetric two-form corresponds to the components

$$A_{\bar{\mu}\bar{\nu}8},\ A_{\bar{\mu}\tilde{i}8},\ A_{\tilde{i}\tilde{j}8}$$

of the $d = 11$ three-form potential. The $d = 4$ U(1) field strengths in the NS-NS sector are then easily identified using (3.6,3.7,3.2) to be

$$-H_{\bar{\mu}\bar{\nu}\ \tilde{i}8},\ \ -\tilde{A}^{\tilde{i}9}_{\bar{\mu}\bar{\nu}},\ \ \tilde{i} \in \{2, \dots 7\}.$$

where it is assumed that the remaining (RR) fields not in the NS-NS sector vanish.

With (3.23),(3.5),(3.13) this corresponds to the representation space basis

$$(E^{1\tilde{i}8},\ \ -E^{1}_{\ \tilde{i}}). \tag{3.32}$$

With (3.18) one may verify that the subgroup stabilizing this basis is the obvious $O(6, 6)$ subgroup generated as indicated in figure 3.8. With the above representation basis one may verify that it acts in the fundamental representation and preserves the metric

$$\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \tag{3.33}$$

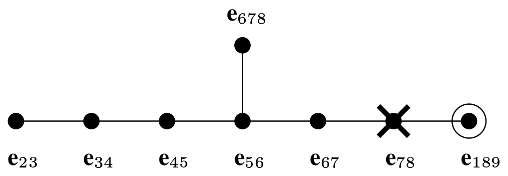


Figure 3.8: Decomposition of $E_{7(+7)} \supset O(6,6) \times SL(2)$

where $\mathbf{1}$ is the six dimensional unit matrix, as expected. The dual fields

$$-H_{\bar\mu\bar\nu\,\tilde i 9},\ \ -\tilde A^{\tilde i 8}_{\bar\mu\bar\nu},\ \ \tilde i \in \{2,\dots 7\}$$

correspond to the basis

$$(E^*_{\tilde i 89},\ \ E^{\tilde i}{}_9)$$

and transform in the transpose inverse representation of the above fundamental representation of $O(6,6)$, again as expected.

Consider now the full theory. The above split of the basis explicitly reflects the decomposition

$$\begin{array}{rcl} E_{7(+7)} & \supset & O(6,6) \times SL(2,\mathbb{R}) \\ \mathbf{56} & \supset & (\mathbf{12},\mathbf{2}) + (\mathbf{32},\mathbf{1}) \end{array}$$

as mentioned in [56]. The $U(1)$ field strength in the R-R sector transform in the $\mathbf{32}$ spinor representation of $O(6,6)$, which is unexpected, since this means that T-duality mixes electric and magnetic fields in the R-R sector.

Note that one Kaluza-Klein field (corresponding to the compactification $d = 11$ to $d = 10$) is in the RR sector and is therefore neglected when determining the basis (3.32). One might speculate if, including it into the discussion and thereby changing indices $\tilde i \to \bar i$ in (3.32), a larger subgroup exists stabilizing this basis, which e.g. could be $O(7,7)$. However, this is not the case, since $SL(2) \otimes O(6,6)$ is a maximal subgroup and does not allow an extension by rank, and e.g. no $O(7,7)$ subgroup exists in $E_{7(+7)}$, which, from this side, justifies that the eleventh direction plays a different rôle than the other directions (see chapter one).

However, the eleventh direction does not need to be identified with the coordinate 10, but, of course, any compact coordinate can be chosen as eleventh direction. This yields different bases

$$(E^{1\bar i n},\ \ -E^{1}{}_{\bar i}),\ \ \ \bar i \neq n$$

leading to different $O(6,6)$ subgroups, connected to the above by the Weyl reflection exchanging $\mathbf{e}_n$ with $\mathbf{e}_8$, generated by $S^n{}_8$.

Turn back to the above $O(6,6)$ subgroup. It shall now be investigated from another perspective: by looking at the action on the moduli. The action of $O(6,6)$ on the Cartan subalgebra part of (3.25) can be seen by looking at the submatrix of (3.25) corresponding to the basis (3.32). This yields a matrix with the $\rho_m^{\dot m}\Delta^{-\frac{1}{8}}$ on the diagonal. The $\rho_m^{\dot m}$ correspond to the radii of the compactification torus transforming under $O(6,6)$, while $\Delta^{1/2}$ is the volume of the 7-torus including the eleventh direction. The appearance of the factor $\Delta^{-1/8}$ indicates that $\rho_8^{\dot 8}$, corresponding to the radius of the eleventh compact direction, transforms as well under T-duality and is not decoupled, as was pointed out e.g. in [79]. This transformation therefore mixes strong weak coupling duality of the type IIB string [83, 4], that corresponds to modular transformations involving $\rho_8^{\dot 8}$, with a transformation of the moduli corresponding to the compactification torus of type II string theory.

Note that the $O(6,6)$ in our formulation is not a symmetry of the action, but only of the equations of motion.

3.2.4 S-duality

Turn to the commuting $SL(2,\mathbb{R})$ factor in figure 3.8, which is a symmetry of the action (3.14)[2].

It was suggested in [56] to interpret this factor like in the heterotic case [87] as a $d = 4$ S-duality, not to be confused with the $d = 10$ S-duality of the type IIB string.

Interpreting the above $SL(2,\mathbb{R})$ factor as S-duality, a $\mathbb{Z}_2$ symmetry exchanging electric and magnetic sector is expected to be present. The natural candidate is S^{189}. Using (3.18), it is interesting to note that S^{189} transforms the magnetic into the electric sector and vice versa, but, considering the NS-NS-fields, takes the Kaluza-Klein sector into the 3-form field sector (and vice versa) as well.

In the basis we have chosen, S^{189} has to be accompanied by an $O(6,6)$ transformation in order to transform the magnetic to the electric field strength of a specific vector field. This corresponds to the above preserved metric (3.33), which is equal to

$$\prod_{\tilde{i}<\tilde{j}} S^{\tilde{i}}{}_{\tilde{j}} \prod_{\tilde{i}<\tilde{j}} S^{\tilde{i}\tilde{j}8}.$$

3.2.5 Definitions of U-duality

Finally, the definition found for the U-duality group $E_{7(+7)}(\mathbb{Z})$ in (3.30), (3.31) shall be compared with the definitions (1.3) and (1.4).

Since E_7 is simply laced, the generators in (3.30) corresponding to simple roots generate all other generators and therefore the whole $E_{7(+7)}(\mathbb{Z})$. On the other hand, the same is true for the simply laced groups $O(6,6,\mathbb{Z})$ and $SL(7,\mathbb{Z})$ as subgroups of $E_{7(+7)}(\mathbb{Z})$. Both groups correspond to subdiagrams of the Dynkin diagram of E_7 and act in minimal representations. The subgroup $O(6,6)$ is indicated in figure 3.8, the subgroup $SL(7)$ simply corresponds to erasing the root $\mathbf{e}_{678}$ for the E_7 Dynkin diagram. Joining their generators together, we get the whole set of generators of $E_{7(+7)}(\mathbb{Z})$. The two definitions are therefore equivalent.

In the algebraic approach reviewed in [79], Weyl and Borel generators were used to define the discrete group. These actually correspond to the S and T generators in (3.31). For the Weyl group, the identification

$$S^{i}{}_{j} = \hat{S}_{ij}, \quad S^{ijk} = \hat{T}_{ijk}$$

holds, where hatted indices are the Weyl generators of [79]. The S^i_j correspond to the exchange of two radii, while the S^{ijk} correspond to a simultaneous inversion of three radii. Only the S^{ij8} are elements of the T-duality group $O(6,6)$, corresponding, as pointed out above, to a simultaneous inversion of two radii and the radius corresponding to the eleventh dimension connected to type IIB S-duality. The Borel generators are identified correspondingly.

3.3 U-Duality in $d = 3$

The above construction has given generators for U-duality in dimensions $d > 3$. However, an analogous construction in $d = 3$ seems difficult, since a proper quantization condition is unclear. Therefore, the procedure demonstrated for the toy model in the last chapter will be now applied to the full theory, assuming the the assumption of an additional compact coordinate does not break

[2]This is analogous to the manifestly $SL(2,\mathbb{R})$ invariant action from the dual $N = 1$ $d = 10$ supergravity theory in [87] for the heterotic string. The connection is obvious when all R-R fields are set to zero.

the symmetry quantized by the DSZ condition in $d = 4$. Note that this has an attractive side effect: starting from a quantization condition in $d = 4$, one constructs a symmetry in $d = 3$ that shall hold in the proper quantum theory as well.

The procedure is illustrated in figure 3.9. By compactifying M-theory on the torus, we can choose eight different ways how to compactify first to four dimensions. This results in eight $E_{7(+7)}(\mathbb{Z})$ acting *differently* on M-theory fields. By compactifying the theory further to three dimensions, these groups are merged together to form the three dimensional duality group.

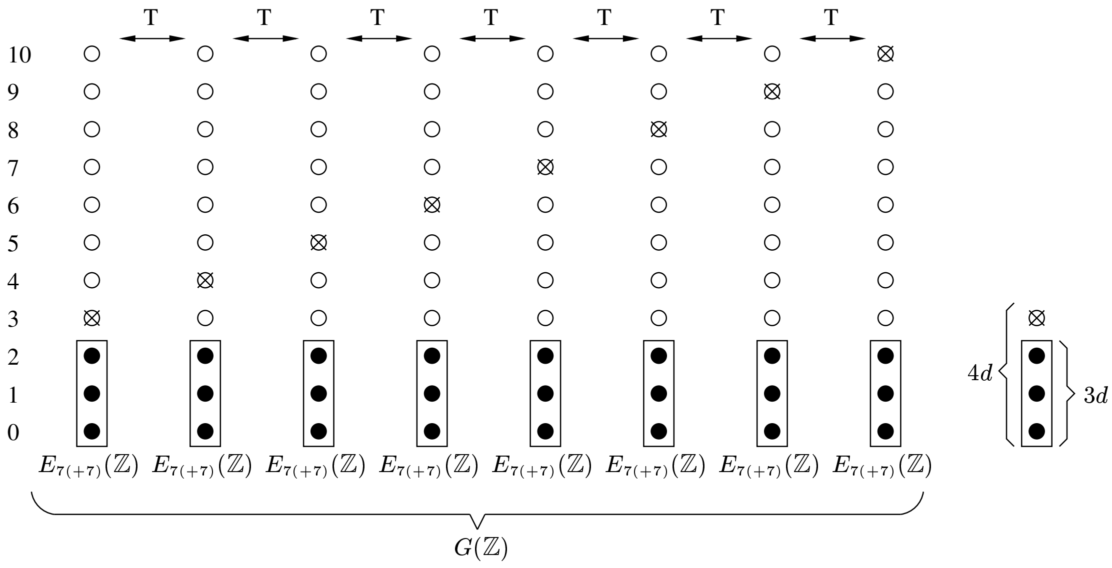


Figure 3.9: Construction of Three-Dimensional U-duality

3.3.1 $d = 3$ Theory: The $E_{8(+8)}$ Coset

The compactification to $d = 3$ is strictly parallel to the toy model: Choose the vielbein to be

$$E^{(4)\bar{\alpha}}_{\bar{\mu}} = \begin{pmatrix} e^{\phi/2}E^{(3)\alpha}_{\mu} & e^{-\phi/2}\hat{B}_{\mu} \\ 0 & e^{-\phi/2} \end{pmatrix},$$

where μ, α now run from $0 \ldots 2$. Using the results of section (2.4.1), the Lagrangian may be written as

$$\mathcal{L} = -\tfrac{1}{4}E^{(3)}R^{(3)} + \frac{1}{c}E^{(3)}\mathrm{Tr}(P^{(3)}_{\mu}P^{(3)\mu}) \tag{3.34}$$

provided a proper group exists obeying the conditions stated there. This group is (of course) $E_{8(+8)}$, and the basis

$$\mathcal{S}^t = \Big(- E^{*}_{\bar{i}\bar{j}9}, +E^{1}_{\;\bar{i}} \mid - E^{1\bar{i}\bar{j}}, -E^{\bar{i}}_{\;9}\Big).$$

has already been introduced above and is discussed. (3.22), (3.24) show that the representation generated by the adjoint action on $\mathcal{S}$ is indeed the **56** representation in the form needed, and

$$[\mathcal{S}, \mathcal{S}] = \Omega E^{1}_{9}$$

follows directly from (3.18).

Choosing proper normalizations, the Lagrangian is given by (3.34) with $c = 240$, and the scalar field $\mathcal{V}^{(3)}$ is

$$\mathcal{V}^{(3)} = \mathcal{V}^{(4)} \exp\Big(\frac{1}{2}\phi \sum_{i=1}^{8} h_i\Big) \exp\Big(\mathcal{Y}\cdot\mathcal{S}\Big) \exp\Big(f\, E^1_{\ 9}\Big).$$

with

$$\partial_\mu \mathcal{V}^{(3)}\mathcal{V}^{(3)-1} = Q^{(3)}_\mu + P^{(3)}_\mu,\ \ Q^{(3)}_\mu \in \mathfrak{e}_{8(+8)},\ \ P^{(3)}_\mu \in \mathfrak{e}_{8(+8)} - \mathfrak{so}(16)$$

The scalar fields are

$$\begin{aligned} &\partial_\mu\eta = \mathcal{G}_{\mu 3},\ \partial_\mu\bar\eta = \mathcal{H}_{\mu 3},\ \mathcal{Y} = \begin{pmatrix}\eta\\ \bar\eta\end{pmatrix}, \\ &\partial_\mu f = -\tfrac{1}{2}E^{(3)}\epsilon^{\mu\nu\rho}e^{-2\phi}\hat B_{\nu\rho} - \mathcal{Y}^t\Omega\partial_\mu\mathcal{Y} \quad . \end{aligned}$$

The Lagrangian admits local $SO(16)$ and global $E_{8(+8)}$ symmetry. Note that, again, the bosonic Lagrangian does not single out a specific representation of $E_{8(+8)}$. However, the **248** adjoint representation of $E_{8(+8)}$ is the smallest (and unique basic) representation of $E_{8(+8)}$, and has a nice additional property: as already seen, it decomposes as

$$\begin{array}{rcl} E_{8(+8)} & \supset & SL(2,\mathbb{R})\times E_{7(+7)} \\ \mathbf{248} & \supset & (\mathbf{2},\mathbf{56}) + (\mathbf{1},\mathbf{133}) + (\mathbf{3},\mathbf{1}). \end{array}$$

$\mathcal{V}'^{(4)}$ therefore has block structure

$$\mathcal{V}'^{(4)} = \begin{pmatrix} \boxed{\mathbf{56}} & & & & & \\ & \boxed{\overline{\mathbf{56}}} & & & & \\ & & \boxed{\mathbf{133}} & & & \\ & & & \boxed{\mathbf{1}} & & \\ & & & & \boxed{\mathbf{1}} & \\ & & & & & \boxed{\mathbf{1}} \end{pmatrix}.$$

and contains the familiar $\mathbf{56}$ blocks as representation matrix.

3.3.2 Identifying $d = 4$ U-Duality in the $d = 3$ Theory

Parallel to (2.21), an element

$$\Lambda \in E_{7(+7)},\quad \Lambda = e^X,\ X \in \mathfrak{e}_{7(+7)}$$

acts within $\mathcal{V}^{(3)}$ as

$$\mathcal{V}^{(4)} \to \mathcal{V}^{(4)}\Lambda^{-1} \text{ and } \mathcal{Y} \to \mathbf{D_{56}}(\Lambda)\mathcal{Y}$$

which is the $d = 4$ U-duality embedded into the $d = 3$ theory. As already mentioned, in the next chapter it will be shown that the vector $\mathcal{Y}$ carries the $d = 4$ charges by considering solitons.

The generators of the discrete group are identical to (3.30) resp. (3.31) in the **248** adjoint representation of $E_{8(+8)}$.

3.3.3 Connection to $d = 11$ fields

Parallel to the toy model, a suitable parametrization of the scalar coset matrix will prove useful when addressing different orders of compactifications.

Define the fields

$$
\begin{aligned}
\varphi^{1\bar{i}} &= \varphi^{(4)\bar{i}}, \\
\varphi^{\bar{i}\bar{j}} &= 2\eta^{\bar{i}\bar{j}} + 4\varphi^{(4)[\bar{j}}\eta^{\bar{i}]9} + \frac{1}{6}\epsilon^{\bar{i}\bar{j}\bar{l}\bar{m}\bar{n}\bar{p}\bar{q}} A_{(\bar{l}+2)(\bar{m}+2)(\bar{n}+2)}\bar{\eta}_{\bar{p}\bar{q}}, \\
\Psi_1 &= f - \eta^{\bar{i}9}(-2\bar{\eta}_{\bar{i}9} + 2\bar{\eta}_{\bar{i}\bar{j}}\varphi^{\bar{j}} - 2\eta^{\bar{j}\bar{k}} A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}) \\
&\quad -\frac{1}{36}\epsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{p}\bar{q}\bar{r}}(\bar{\eta}_{\bar{i}\bar{j}} - 2\eta^{\bar{m}9} A_{(\bar{i}+2)(\bar{j}+2)(\bar{m}+2)}) A_{(\bar{p}+2)(\bar{q}+2)(\bar{r}+2)}\bar{\eta}_{\bar{k}\bar{l}} \\
\Psi_{\bar{i}} &= -\bar{\eta}_{\bar{i}9} + \bar{\eta}_{\bar{i}\bar{j}}\varphi^{\bar{j}} - A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)}\eta^{\bar{j}\bar{k}} \\
&\quad -\frac{1}{36}\epsilon^{\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}\bar{p}\bar{q}} A_{(\bar{i}+2)(\bar{j}+2)(\bar{k}+2)} A_{(\bar{l}+2)(\bar{m}+2)(\bar{n}+2)}\bar{\eta}_{\bar{p}\bar{q}}.
\end{aligned}
$$

Again, the polynomial transformation indicates that the additional terms might be understood as BCH terms. Of course, this has been done on purpose, since one finds, using (3.18)

$$
\begin{aligned}
\mathcal{V}^{(3)} &= \exp\left(\sum_{m=1}^{8} \ln\left(-\prod_{n=1}^{m} e_n^{\ \dot{n}}\right)\ h_m\right) \prod_{p=0}^{6} \exp\left(-\sum_{q,r=8-p}^{8} e_{7-p}{}^{\dot{q}}(e^{(7-p)\ -1})_{\dot{q}}{}^{r}\ E^{7-p}{}_{r}\right) \\
&\quad \exp\left(\sum_{i=1}^{8} \Psi_i\ E^{i}{}_{9}\right) \\
&\quad \exp\left(\frac{2}{3!}\sum_{i,j,k=1}^{8} A_{(2+i)(2+j)(2+k)}\ E^{ijk} - \frac{1}{2!}\sum_{i,j=1}^{8} \varphi^{ij}\ E^{*}_{ij9}\right)
\end{aligned}
\tag{3.35}
$$

where summations have been spelled out, and the eleven-dimensional vielbein

$$
\begin{aligned}
E^{(11)A}_{M} &= \begin{pmatrix} e^{-1}E^{(3)\alpha}_{\mu} & B^{(3)i}_{\mu}e^{a}_{i} \\ 0 & e^{a}_{i} \end{pmatrix} \\
&= \begin{pmatrix} e^{\phi/2}\Delta^{-\frac{1}{4}}E^{(3)\alpha}_{\mu} & e^{-\phi/2}\Delta^{-\frac{1}{4}}\hat{B}_{\mu} & B^{(4)\bar{i}}_{\mu}\rho_{\bar{i}}{}^{\bar{a}} \\ 0 & \Delta^{-\frac{1}{4}}e^{-\phi/2} & B^{(4)\bar{i}}_{3}\rho_{\bar{i}}{}^{\bar{a}} \\ & 0 & \rho_{\bar{i}}{}^{\bar{a}} \end{pmatrix}
\end{aligned}
$$

has been used. The φ^{ij}, Ψ_i obey, using (3.2),(3.4),(3.8)

$$
\begin{aligned}
\partial_\mu\varphi^{ij} &= -2e^2E^{(3)}\epsilon_{\mu\nu\rho}(\partial^{[\nu}A^{\rho](i+2)(j+2)} + B^{(3)k[\nu}\partial^{\rho]}A_{(k+2)(i+2)(j+2)}) \\
&\quad +\frac{1}{18}\epsilon^{ijklmnpq}\partial_\mu A_{(2+k)(2+l)(2+m)}A_{(2+n)(2+p)(2+q)} \\
\partial_\mu\Psi_i &= -\frac{1}{2}e^2E^{(3)}\epsilon_{\mu\nu\rho}B_i^{\nu\rho} - \frac{1}{2}(\varphi^{kl}\partial_\mu A_{(2+k)(2+l)(2+i)} - \partial_\mu\varphi^{kl}A_{(2+k)(2+l)(2+i)}) \\
&\quad -\frac{1}{54}\epsilon^{jklmnpqr}A_{(2+i)(2+j)(2+k)}\partial_\mu A_{(2+l)(2+m)(2+n)}A_{(2+p)(2+q)(2+r)}
\end{aligned}
$$

where $i, j, k, \dots = 1 \dots 8$. This is exactly the result of [71] found by direct reduction to $d = 3$.

3.3.4 Different Orders of Compactification

Consider the different compactifications indicated in figure 3.9. Like in the toy model, eight different ways exists how to identify a set of $d = 4$ coordinates with fixed $d = 3$ coordinates, leading to eight different Lagrangians. Note that this construction puts indeed *all* directions on the same footing, that is, no specific eleventh "M-theory" direction is singled out, and therefore goes beyond the string oriented conjecture of [56]. The compactification is then straightforward, and the corresponding eight coset matrices may be read off directly from (3.35), if the vielbein is put into a proper shape by a local Lorentz transformation, that is,

$$
\begin{pmatrix}
e^{\phi/2}\Delta^{-\frac{1}{4}}E_\mu^{(3)\alpha} & e^{-\phi/2}\Delta^{-\frac{1}{4}}\hat{B}_\mu & B_\mu^{(4)\bar{i}}\rho_{\bar{i}}{}^{\bar{a}} & & & & & \\
 & \Delta^{-\frac{1}{4}}e^{-\phi/2} & B_3^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{2}} & B_3^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{3}} & & & \dots & B_3^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{8}} \\
 & 0 & \rho_2{}^{\dot{2}} & & & & \dots & \rho_2{}^{\dot{8}} \\
 & 0 & 0 & \rho_3{}^{\dot{3}} & & & \dots & \rho_3{}^{\dot{8}} \\
\mathbf{0} & & & & & & \dots & \\
 & 0 & \dots & & 0 & \rho_n{}^{\dot{n}} & \dots & \rho_n{}^{\dot{8}} \\
 & & & & & & \dots & \\
 & 0 & 0 & & & \dots & 0 & \rho_8{}^{\dot{8}}
\end{pmatrix}
$$

$$
\to \begin{pmatrix}
e^{\tilde{\phi}/2}\tilde{\Delta}^{-\frac{1}{4}}E_\mu^{(3)\alpha} & e^{-\tilde{\phi}/2}\tilde{\Delta}^{-\frac{1}{4}}\tilde{\hat{B}}_\mu & \tilde{B}_\mu^{(4)\bar{i}}\tilde{\rho}_{\bar{i}}{}^{\bar{a}} & & & & & \\
 & 0 & \dots & & 0 & \tilde{\rho}_1{}^{\dot{n}} & \dots & \tilde{\rho}_1{}^{\dot{8}} \\
 & 0 & \tilde{\rho}_2{}^{\dot{2}} & & & & \dots & \tilde{\rho}_2{}^{\dot{8}} \\
 & 0 & 0 & \tilde{\rho}_3{}^{\dot{3}} & & & \dots & \tilde{\rho}_3{}^{\dot{8}} \\
\mathbf{0} & & & & & & \dots & \\
 & \tilde{\Delta}^{-\frac{1}{4}}e^{-\tilde{\phi}/2} & \tilde{B}_{2+n}^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{2}} & \tilde{B}_{2+n}^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{3}} & & & \dots & \tilde{B}_{2+n}^{(4)\bar{i}}\rho_{\bar{i}}{}^{\dot{8}} \\
 & & & & & & \dots & \\
 & 0 & 0 & & & \dots & 0 & \tilde{\rho}_8{}^{\dot{8}}
\end{pmatrix}. \tag{3.36}
$$

Taking the sign change in the Chern-Simons term into account, the scalar coset matrix for the compactification obtained by taking $\{0, 1, 2, (2+n)\}$ as four dimensional coordinates is

$$
\mathcal{V}_{\#n}^{(3)} = \mathcal{V}_{\#1}^{(3)}(\tilde{e}_i{}^a, \tilde{\Psi}_i, \tilde{A}_{(i+2),(j+2),(k+2)}, \tilde{\varphi}^{ij})
$$

$$
\begin{aligned}
\tilde{A}_{(i+2),(j+2),(k+2)} &= -A_{(i+2),(j+2),(k+2)}, \;\; i,j,k \neq 1,n, \\
\tilde{A}_{3,(j+2),(k+2)} &= -A_{(n+2),(j+2),(k+2)}, \;\; j,k \neq 1,n, \\
\tilde{A}_{(n+2),(j+2),(k+2)} &= -A_{3,(j+2),(k+2)}, \;\; j,k \neq 1,n, \\
\tilde{A}_{3,(n+2),(k+2)} &= -A_{(n+2),3,(k+2)}, \;\; k \neq 1,n
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\varphi}^{ij} &= -\varphi^{ij}, \;\; i,j \neq 1,n, \\
\tilde{\varphi}^{1j} &= -\varphi^{nj}, \;\; j \neq 1,n, \\
\tilde{\varphi}^{nj} &= -\varphi^{1j}, \;\; j \neq 1,n, \\
\tilde{\varphi}^{1n} &= -\varphi^{n1},
\end{aligned}
$$

finally

$$
\begin{aligned}
\tilde{\Psi}_i &= \Psi_i, \;\; i \neq 1,n, \\
\tilde{\Psi}_1 &= \Psi_n, \\
\tilde{\Psi}_n &= \Psi_1.
\end{aligned}
$$

In each compactification, the U-duality group is generated by (3.30) resp. (3.31).

3.3.5 Joining U-dualities in $d = 3$

As in the toy model, the coset matrices may be easily seen to be connected by, using (3.18),

$$\mathcal{V}^{(3)}_{\#n} = (P_n S^1{}_n)^{-1}\, h_n\ \mathcal{V}^{(3)}_{\#1}\ P_n S^1{}_n.$$

h_n is the natural lift to $E_{8(+8)}$ of the local transformation (3.36), $S^1{}_n$ generates the Weyl reflection exchanging $\mathbf{e}_1$ with $\mathbf{e}_n$ and is actually the additional $d = 3$ T-duality in the first compactification as in the toy model, and P_n is again a "parity" transformation that obeys

$$\begin{aligned} P_n &= (-1)^{h_{678}+h_6+\sum\limits_{m=n}^{7} h_m} = (S^{678})^2 (S^6{}_7)^2 (S^n{}_8)^2 \quad 2 \le n \le 7, \\ P_8 &= (-1)^{h_{678}+h_6} = (S^{678})^2 (S^6{}_7)^2. \end{aligned}$$

P_n is an $E_{7(+7)}(\mathbb{Z})$ transformation corresponding to a charge conjugation in $d = 4$, but leaves the fields $\tilde{A}^{n\bar{j}}_{\bar{\mu}\bar{\nu}}$, $B^{(4)\bar{j}}_{\bar{\mu}\bar{\nu}}$, $\bar{j} \neq n$ unchanged. As we have seen, this exactly corresponds to a specific set of NS-NS fields.

The $d = 3$ U-duality is therefore given by joining all

$$\Lambda_n = P_n S^1{}_n\ \Lambda\ (P_n S^1{}_n)^{-1}$$

where Λ is an element of $E_{7(+7)}$ spanned by (3.19), and in the discrete case by (3.30) resp. (3.31).

3.3.6 $d = 3$ U-duality group

It is easily seen from appendix A.1 that the intersection of two different U-dualities is exactly $E_{6(+6)}$ as expected. Joining all Λ_n gives, in this case, the whole of $E_{8(+8)}$. Thus, the $d = 3$ discrete U-duality is generated by the set of generators obtained by exponentiating the Chevalley generators for all roots.

Since the **248** is the unique basic representation of $E_{8(+8)}$ and the adjoint itself corresponds to the minimal admissible lattice, this coincides with $E_{8(+8)}(\mathbb{Z})$ as defined above.

This completes the discussion of generators of U-duality in $d = 3$.

3.4 $G_{2(+2)}$ in $E_{8(+8)}$

It has been shown that the algebra $\mathfrak{g}_{2(+2)}$ is embedded into $\mathfrak{e}_{8(+8)}$ by choosing any $\mathfrak{d}_4$ subalgebra in $\mathfrak{e}_{8(+8)}$. This is parallel to the truncation of the physical theory

$$\begin{gathered} ds^{(11)\,2} = ds^{(5)\,2} + ds^{(E6)\,2} \\ A^{(11)} = -\tfrac{1}{\sqrt{3}} A^{(5)} \wedge J,\ J = \tfrac{1}{2}(dx^5 \wedge dx^6 + dx^7 \wedge dx^8 + dx^9 \wedge dx^{10}) \end{gathered} \tag{3.37}$$

in [80], where $E6$ is the flat six dimensional Euclidean space, J is its Kähler form and $A^{(11)}$, $A^{(5)}$ is the eleven dimensional three form and the five dimensional one-form potential.

In the toy model, the discrete U-duality construction does not yield $G_{2(+2)}(\mathbb{Z})$, but a smaller group. However, this may be seen as a consequence of the embedding of $\mathfrak{g}_{2(+2)}$ into $\mathfrak{e}_{8(+8)}$, involving generator sums at the level of the algebra, indicating a non-simply laced algebra.

3.5 Summary

In this chapter, the structure of U-duality in M-theory has been investigated, and generators have been given in all dimensions $d > 2$.

The low energy effective theory of M-theory compactified on the torus to $d = 4$ has been reviewed, and rephrased by giving the fundamental **56** representation of $\mathfrak{e}_{7(+7)}$ embedded into the adjoint of $\mathfrak{e}_{8(+8)}$. By using the exponential mapping, the assignment of the 70 scalars of the theory to $\mathfrak{e}_{7(+7)}$ generators was given.

U-duality was defined following [56] from first principles by imposing the DSZ quantization condition. The resulting discrete subgroup $E_{7(+7)}(\mathbb{Z})$ was analyzed by investigating discrete Lie groups stabilizing admissible lattices in basic representations, using two simple ingredients: the Birkhoff decomposition and the Euclidian algorithm. This gave also a definition of $E_{8(+8)}(\mathbb{Z})$, to be compared with the $d = 3$ U-duality, and $G_{2(+2)}(\mathbb{Z})$, where disagreement with toy model U-duality in $d = 3$ is found as already stated.

T- and S-duality subgroups have been identified in $d = 4$, and their role and action on scalars (or moduli, respectively) and charges has been seen in some points not to correspond to the picture in the literature, and clarified.

The definition of discrete U-duality as found in this thesis has been compared to the definitions present in the literature, and it was found that they agree.

U-duality in $d = 3$ has been constructed by applying the same method as in the toy model and extending the original conjectures. The merged U-duality in $d = 3$ has been found to agree with the above definition of $E_{8(+8)}(\mathbb{Z})$. The disagreement of $G_{2(+2)}(\mathbb{Z})$ and toy model U-duality has been argued to be caused by the embedding of $\mathfrak{g}_{2(+2)}$ into $\mathfrak{e}_{8(+8)}$ leading to a non-simply laced algebra.

Chapter 4

String Theory Solitons

4.1 Stringy Black Holes

4.1.1 Motivation

In this section, applications of U-duality on solitonic solutions of M-theory are discussed briefly. A general review on solitons in string theory shall not be given, the reader is referred to extensive literature (for a recent review, see [92]). Rather, some examples shall be given and put into the framework developed in the preceding chapters.

Solitons in theories that correspond to the low-energy effective action on string theories have been of interest for several reasons. On the one hand, they may be viewed as solitonic "counterparts" of strings [56]. The charges carried by these solitons are in the **56** multiplet introduced above. However [81], it is known that the fundamental string does couple only to the charges in the NS-NS sector introduced above, while the R-R fields appear in the vertex operator only through their field strengths. Thus, since U-duality demands objects that carry these charges, this has led to extend the former picture of string theory considerably by open strings ending on boundary surfaces, now known as D-branes. A detailed description of these objects may be found in [81].

Macroscopic solitons of string theory that have a regular horizon are interesting in the context of a microscopic interpretation of black hole entropy ([65], for a recent account, see [11] and references). Since the Schwarzschild radius of a black hole is proportional to its mass, for sufficiently large mass the Schwarzschild radius of a black hole will be larger than the string scale. A string of this mass will then lie inside the black hole, and may thus be interpreted as black hole itself. However, this gives a possibility to define black hole entropy by statistical means. One may associate it with the degeneracy of string states carrying the quantum numbers associated with the black hole. Attempts to calculate the black hole entropy for the general non-charged Schwarzschild solutions have been made, but are problematic since the string state masses can get renormalized. However, for a recent calculation where a agreement was found see [28].

However, the situation is much simplified in the case where no quantum corrections are expected, that is, for black holes saturating the Bogomolnyi bound and in theories admitting enough supersymmetry. In this case, the general procedure is: fix the set of charges and find the string and/or D-brane configurations that correspond to BPS states (mass=charge). One may then calculate their degeneracy at weak coupling where the microscopic description of the states is reliable. BPS states, since they break a fraction of supersymmetry, live in shortened multiplets, and the degeneracy is not expected to change when the string coupling is changed. Therefore, one may then increase the string coupling to sufficiently large value such that the state becomes a black hole. The entropy of this black hole should then be given by the degeneracy calculated at weak coupling. To verify the correctness of the result, one may then look for a string soliton corresponding to black hole carrying the same charges and mass, and calculate the Bekenstein-Hawking entropy, and agreement was found for various cases. For $N = 2$ supersymmetric theories, it was found that higher derivative terms need to be incorporated into the low energy effective action to reach agreement with the microscopic calculation (see [11] and references).

Some application of U-duality within the context of black holes in string theory shall be given now.

4.1.2 Black holes and U-duality

Those solitonic solutions that saturate the Bogomolnyi bound and carry just one charge in the **56** dimensional charge vector will be called elementary. They preserve half of the supersymmetry, corresponding to the fact that just one non-zero charge exists and hence all eigenvalues of the central charge matrix are equal. The Killing spinor equation will be solved explicitly in section 4.3.1.

Such a solution was found in [43]. An easy embedding into $d = 4$ maximal supergravity is given by setting $A_{MNP} = 0$, which corresponds to the above solitons charged under one Kaluza-Klein gauge field.

The solution is then given by one harmonic function H. The metric is of the form

$$ds^2 = H^{-1/2}dt^2 - H^{1/2}(dr^2 + r^2 d\Omega).$$

where standard spherical coordinates $\{r, \theta, \phi\}$ have been chosen. Select and fix one specific $\bar{k} \in \{2...8\}$. Set the remaining diagonal components of the internal metric to one (notations are as in the preceding chapter), all off-diagonal components and the internal three-form potential to zero. Furthermore, set all gauge fields to zero except one, which is the Kaluza-Klein gauge field corresponding to the direction $\bar{k}$. One then has

$$B_{tr}^{(4)\bar{k}} = -H^{-2}\partial_r H, \quad g_{\bar{k}\bar{k}} = H,$$

all other fields zero, with

$$H = 1 + \frac{Q}{|\vec{r}|}.$$

As shown in the next section, this solution corresponds to a boosted $d = 5$ Schwarzschild solution and preserves half of the supersymmetry. It has a nasty property: in the canonical Einstein metric, it is a naked singularity, whose gravitational force is ”dressed” by the anti-gravitating effect of the dilaton. The interpretation as soliton in this frame seems difficult, however, in the Weyl rescaled ”string-frame” it has an ordinary interpretation as soliton interpolating between two vacua, which may be seen to be Minkowski spacetime in the asymptotically flat limit at spatial infinity and the Robinson-Bertotti vacuum down an infinite Einstein-Rosen throat [31]. These solitons with charges in the NS-NS sector are exact classical solutions to type II string theory [56], not only of the low energy sector.

Since the solution saturates the Bogomolnyi bound, gravitational and electromagnetic forces cancel each other, and is is not surprising that one also has a multi-black hole solution

$$H = 1 + \sum_n \frac{Q_n}{|\vec{r} - \vec{r_n}|}$$

reflecting the fact that there are no forces between the constituents, allowing a superposition despite the nonlinearity of the Einstein equations. Such a multi-black hole will be used in the next section.

The solution has a dual magnetic solution given by

$$B^{(4)\bar{k}}_{\theta\phi} = r^2 \sin\theta \partial_r H, \quad g_{\bar{k}\bar{k}} = H^{-1},$$

with the harmonic form

$$H = 1 + \frac{P}{|\vec{r}|}$$

for the single soliton with magnetic charge P, and

$$H = 1 + \sum_n \frac{P_n}{|\vec{r} - \vec{r_n}|}$$

for the multi-magnetically charged black hole.

Remember from the preceding chapter that the gauge fields are multiplied a basis vector

$$\mathcal{S}^t = \Big(- E^*_{\bar{i}\bar{j}9}, +E^1_{\ \bar{i}} \mid - E^{1\bar{i}\bar{j}}, -E^{\bar{i}}_{\ 9} \Big)$$

that spans the **56** representation of $E_{7(+7)}$. Since

$$\boldsymbol{\epsilon}_{\bar{k}9} = \boldsymbol{\epsilon}_{1\bar{k}} - \boldsymbol{\epsilon}_{189} + \boldsymbol{\epsilon}_{\bar{k}\bar{j}8} - \boldsymbol{\epsilon}_{\bar{j}\bar{k}}$$

both solutions are transformed into each other by

$$S^{189} S^{\bar{k}\bar{j}8} S^{\bar{j}}_{\ \ \bar{k}}$$

reflecting a mixed T- and S-duality transformation, as defined in the preceding chapter.

Solutions charged with respect to the other gauge fields of the theory may now be obtained by the action of $E_{7(+7)}$ (respectively, $E_{7(+7)}(\mathbb{Z})$). They correspond to wrapped membrane and five-brane solutions of M-theory. While the S generators take single charge solutions to single charge solutions, the T generators will yield solitons with multiple charge.

Note that the solitons may carry electric or magnetic charges, but there are no dyonic states in this Kaluza-Klein multiplet. Looking the above representation space basis, one sees that no corresponding T generators exist. This is not surprising, since the dyonic solutions are known to have a different conformal structure, while U-duality does not act on the space-time metric.

Dyonic solutions were considered in [43] and shown to preserve a quarter of supersymmetry in [25], corresponding to two eigenvalues of the central charge matrix being equal to the mass.

They are given by choosing two nonzero gauge fields in the Kaluza-Klein sector and two nontrivial diagonal metric components, while the other diagonal metric components are one, and all other fields except the metric zero. Choose and fix two numbers $\bar{k}, \bar{j} \in \{2..8\}$. Then, the solution is given by

$$B^{(4)\bar{k}}_{tr} = -H_1^{-2} \partial_r H_1, \quad g_{\bar{k}\bar{k}} = H_1$$

$$B^{(4)\bar{j}}_{\theta\phi} = r^2 \sin\theta \partial_r H_2, \quad g_{\bar{j}\bar{j}} = H_2^{-1},$$

with the harmonic functions

$$H_1 = 1 + \frac{Q}{|\vec{r}|}, H_2 = 1 + \frac{P}{|\vec{r}|},$$

and the spacetime is given by

$$ds^2 = H_1^{-1/2} H_2^{-1/2} dt^2 - H_1^{1/2} H_2^{1/2} (dr^2 + r^2 d\Omega) \tag{4.1}$$

which reflects the "harmonic function rule" for intersecting solutions of $d = 11$ supergravity reduced to $d = 4$ [95].

We may now give another solution by transforming e.g. the vector fields $B^{(4)\,\bar{k}}$ to $F'_{\bar{i}\bar{k}} = -H^{\tilde{A}}_{\bar{i}\bar{k}}$ and $B^{(4)\,\bar{j}}$ to $F'_{\bar{i}\bar{j}} = -H^{\tilde{A}}_{\bar{i}\bar{j}}$ simultaneously for a fixed $\bar{\imath}$, such that $F'_{\bar{i}\bar{k}}$ carries charge $Q/2$ and $F'_{\bar{i}\bar{j}}$ carries charge $P/2$.

Looking again at the basis $\mathcal{S}$, this corresponds to the following $E_{7(+7)}(\mathbb{Z})$ transformation:

$$S^{189} S^{\bar{j}\bar{k}8} S^{\bar{k}}{}_{\bar{j}} S^{1\bar{\imath}9}$$

The transformation of the scalars under this $E_{7(+7)}(\mathbb{Z})$ transformation is given by the relations

$$\begin{aligned}
g'_{\bar{i}\bar{i}} g'_{\bar{j}\bar{j}} \Delta'^{-1/2} &= g^{-1}_{\bar{j}\bar{j}} \Delta^{-1/2} \\
g'_{\bar{i}\bar{i}} g'_{\bar{k}\bar{k}} \Delta'^{-1/2} &= g^{-1}_{\bar{k}\bar{k}} \Delta^{-1/2} \\
g'^{-1}_{\bar{j}\bar{j}} \Delta'^{-1/2} &= g_{\bar{i}\bar{i}} g_{\bar{j}\bar{j}} \Delta^{-1/2} \\
g'^{-1}_{\bar{k}\bar{k}} \Delta'^{-1/2} &= g_{\bar{i}\bar{i}} g_{\bar{k}\bar{k}} \Delta^{-1/2} \\
g'_{\bar{r}\bar{r}} g'_{\bar{l}\bar{l}} \Delta'^{-1/2} &= g'_{\bar{r}\bar{r}} g'_{\bar{l}\bar{l}} \Delta^{+1/2} \qquad\qquad (\bar{r}, \bar{l} \neq \bar{k}, \bar{\imath}, \bar{\jmath})
\end{aligned}$$

which is identical to the relations given in [24] as discrete $E_{7(+7)}$ transformation between the "3-form" solution and the KK dyonic solution. The latter was calculated from scratch and compared with the KK dyonic solution in [24]. Note that the explicit knowledge of the action of $E_{7(+7)}$ on the fields coming from the dimensional reduction renders this calculation obsolete!

Other solutions corresponding to BPS solitons that preserve 1/4 and 1/8 supersymmetry have been identified with multi-particle bound states of the "elementary" solitons charged with respect to different gauge fields [32]. Generically these correspond to intersecting p-brane configurations in higher dimensions [94]. The most general 1/8 BPS soliton is known to depend on five parameters, four charges and one relative phase.

This solution was given in [23] in the NS-NS sector of the heterotic string and shown to be an exact solution of the string theory to all orders. It shall be embedded into the notation of this thesis here. The solution carries five charges that are associated with the gauge fields of the theory as follows: Pick two internal dimensions, say 9 and 10. One then has electric and magnetic charges

$$\begin{aligned}
B^{(4)\,9}_{\bar{\mu}} &: (Q_1, 0) \qquad & A'_{\bar{\mu}\,911} &: \frac{1}{2}(Q_2, 0) \\
B^{(4)\,10}_{\bar{\mu}} &: (q, P_1) \qquad & A'_{\bar{\mu}\,1011} &: \frac{1}{2}(-q, P_2)
\end{aligned}$$

For completeness, the explicit form of the fields shall be given here: Define 5 harmonic functions

$$\begin{aligned}
F^{-1} = 1 + \frac{Q_2}{r}, &\quad K = 1 + \frac{Q_1}{r}, \\
f = 1 + \frac{P_2}{r}, &\quad F^{-1} = 1 + \frac{P_1}{r}, \\
A = \frac{q}{r} & \frac{r + \frac{1}{2}(P_1 + P_2)}{r + P_1}
\end{aligned} \tag{4.2}$$

and

$$D = FKfk - A^2F^2.$$

The gauge fields are then

$$\begin{aligned} B_t^{(4)\,9} &= FfkD^{-1} \\ B_\phi^{(4)\,9} &= 0, \\ A'_{t\;9,11} &= \frac{1}{2}F^2kfkD^{-1}, \\ A'_{\phi\;9,11} &= 0 \\ B_t^{(4)\,10} &= -AF^2D^{-1}, \\ B_\phi^{(4)\,9} &= P_1(1-\cos\theta), \\ A'_{t\;10,11} &= AF^2fkD^{-1}, \\ A'_{\phi\;10,11} &= P_2(1-\cos\theta) \end{aligned}$$

and finally the metric and moduli are

$$\begin{aligned} ds^2 &= \lambda(r)dt^2 - \lambda^{-1}(dr^2 + r^2 d\Omega), \\ \lambda &= FkD^{-1/2}, \\ (\rho_{11,11})^{1/2}\Delta^{-1/2} &= FfD^{-1/2}, \\ (\rho_{11,11})^{1/2}\rho_{9,9} &= FK, \\ (\rho_{11,11})^{1/2}\rho_{10,10} &= fk, \\ (\rho_{11,11})^{1/2}\rho_{9,10} &= AF, \\ A_{9,10,11} &= \tfrac{1}{2}AF, \\ \partial_r\varphi^{(4)\,11} &= AF^{-2}k^{-1}\partial_r(fk). \end{aligned}$$

That this solution corresponds to the "most general" solution can be seen as follows: Consider the asymptotic values $\mathcal{V}_\infty^{(4)}$ of the scalars. In the above solution, it is actually the identity matrix. The combination $\mathcal{V}_\infty^{(4)}\mathcal{Z}$, where $\mathcal{Z}$ is the charge vector, is then invariant with respect to $E_{7(+7)}$, but transforms with respect to $SU(8)$. The subgroup of $SU(8)$ that stabilizes the above charge configuration is $SO(4)\otimes SO(4)$, and thus the symmetry $SU(8)/(SO(4)\otimes SO(4))$, acting on the solution, generates new charge configurations with 51 new charge parameters. The general black hole is then given by acting with an $E_{7(+7)}$ transformation on the scalars, which leaves $\mathcal{V}_\infty^{(4)}\mathcal{Z}$ invariant, but changes the asymptotic values of the scalars and dresses the charges.

The generating solution was recently discussed in another framework in [5] which corresponds to a truncation to $N=2$ supergravity within $N=8$ supergravity. However, both solution should be related by a field redefinition and $E_{7(+7)}$ transformation. This is work in progress that should not prove to be too difficult. Since the solution has a regular horizon and Bekenstein-Hawking black hole entropy

$$S = 2\pi\sqrt{Q_1Q_2P_1P_2 - \tfrac{1}{4}q^2(P_1+P_2)^2}.$$

In [5], the entropy was derived by using the central charge matrix and the quartic invariant of $E_{7(+7)}$ directly. It is a most interesting task to perform a state counting for this solution and compare the result to the above entropy, since this would indeed give "the most general" counting.

A useful tool in this is again U-duality. Since it transforms the charges, it may be used to find different charge configurations that simplify the calculation. In [6] a transformation to a purely RR charged soliton is considered, and the corresponding relation to the charges of a system of D3-branes intersecting at nontrivial angles such that 1/8 supersymmetry is preserved is considered, as well as a D0-D4 brane system. The microscopic state counting has, for the general solution, not been performed yet. However it has been done for the solution with $q = 0$ where agreement was found (see references in [6]). This is a stimulating subject for further study.

4.2 Elementary solitons in $d = 3$

In this section, the elementary solitons are embedded into the $d = 3$ theory.

Consider the elementary multi-BPS solitons given above where copies of a single BPS soliton of the four dimensional theory are put along the 3-direction with distance $2\pi R$ among two of them. The force equals zero condition allows this, though configurations of this type have also been studied for Schwarzschild solutions. Usually, this reduction is called direct or vertical reduction.

This corresponds to the harmonic form

$$H = 1 + \sum_n \frac{Z}{\sqrt{x_1^2 + x_2^2 + (x_3 + 2\pi R n)^2}}$$

where $Z = Q, P$ for the electric resp. magnetic case.

The above sum is logarithmically divergent, but the divergence can be regularized by adjusting an additive constant. One can add e.g. a regulator of the form

$$-\frac{Z}{2\pi R n}.$$

The derivatives of H with respect to the Cartesian coordinates shall be studied. It has been shown in [89] that the dependence of H on x_3 falls off exponentially if the compactification radius is small. To study the derivatives with respect to x_1, x_2, the summation is approximated by an integration [65], which finally yields

$$H \propto \frac{Z}{2\pi R} \ln \rho + C$$

where we introduced polar coordinates $\rho = \sqrt{x_1^2 + x_2^2}$ and θ[1]. The result depends logarithmically on ρ and therefore corresponds to the naïve solution of the $d = 3$ reduced field equations (harmonic function in $d = 3$). The $d = 3$ fields read

$$ds^2 = H^{-1} dt^2 - dx_i^2, \quad e^{\phi} = H^{-1/2},$$

plus

$$g_{\bar{k}\bar{k}} = H, \quad \bar{\eta}_{\bar{k}9} = \frac{Q}{4\pi R}\,\theta$$

in the electric and

$$g_{\bar{k}\bar{k}} = H^{-1}, \quad \eta^{\bar{k}9} = \frac{P}{4\pi R}\,\theta$$

[1] See also [7].

in the magnetic case, where $\eta, \bar{\eta}$ as in the preceding chapter. This corresponds to the vortex solutions studied in [89]. Note that this solution is not asymptotically flat, therefore an interpretation as soliton seems difficult [79].

Under $\theta \to \theta + 2\pi$, one encounters the discrete $d = 4$ charges by the shifts $\bar{\eta}_{\bar{k}9} \to \bar{\eta}_{\bar{k}9} + Q/2R$ and $\eta^{\bar{k}9} \to \eta^{\bar{k}9} + P/2R$. Full rotations translate to discrete $E_{7(+7)}$ transformations parallel to [89]. Note that the charges are "dressed" by the compactification radius.

Note that the electric and magnetic solutions are embedded in a perfectly equal way. The spectrum significantly unifies in $d = 3$, since only particle and string-like configurations survive.

4.3 Null-Killing Vectors and Killing Spinors

In this section, a result by Gebert and Mizoguchi [42] shall be discussed.

4.3.1 Example

Consider the metric

$$ds^{(5)\,2} = \left(1 - \frac{r_0}{r}\right) dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\Omega - dy^2$$

in five dimensions, corresponding to a Schwarzschild black hole smeared along the fifth spacial direction ($M_{\text{Schwarzschild}} \times \mathbb{R}$). Apply an infinite boost along the fifth direction as

$$\begin{pmatrix} t \\ y \end{pmatrix} \to \begin{pmatrix} \text{ch}\beta & -\text{sh}\beta \\ -\text{sh}\beta & \text{ch}\beta \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix}.$$

In the limit $\beta \to \infty$, $r_0 \to 0$ and $r_0 e^{2\beta} \to 4q$, the metric takes the form

$$ds^{(5)\,2} = \left(1 - \frac{q}{r}\right) dt^2 - 2\frac{q}{r} dt dy - \left(1 + \frac{q}{r}\right) dy^2 - dr^2 - r^2 d\Omega. \tag{4.3}$$

The parameter q has a physical interpretation: it may be viewed as electric charge when compactifying the fifth dimension. To see this, one may write the metric as

$$ds^{(5)\,2} = \rho^{-1} G^{(4)}_{\mu\nu} dx^\mu dx^\nu - \rho^2 (dy + B_\mu dx^\mu)^2$$

corresponding to the standard Kaluza-Klein reduction with fünfbein

$$E^{(5)A}_{\;M} = \begin{pmatrix} \rho^{-\frac{1}{2}} E^{(4)\alpha}_{\;\mu} & \rho B_\mu \\ 0 & \rho \end{pmatrix}, \tag{4.4}$$

which yields an Einstein-Maxwell system plus dilaton in $d = 4$. Explicitly, one gets

$$\rho = \left(1 + \frac{q}{r}\right)^{\frac{1}{2}}, \quad B_\mu = \delta_{\mu t} \frac{q}{r} \left(1 + \frac{q}{r}\right)^{-1},$$
$$ds^{(4)} = \left(1 + \frac{q}{r}\right)^{-\frac{1}{2}} dt^2 + \left(1 + \frac{q}{r}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega)$$

which is a static, asymptotically flat solution, with electric charge

$$Q = \frac{1}{4} \lim_{r\to\infty} r^2 \partial_r B_t = \frac{q}{4}$$

and ADM mass, by evaluating the Komar integral for the Killing vector ∂_t,

$$M = -\frac{1}{2} \lim_{r\to\infty} r^2 \partial_r g_{rr} = \frac{q}{4} = Q.$$

The above metric therefore saturates the Bogomolnyi bound $M = Q$. In this limit, the spacetime structure corresponds to a naked singularity, which is "dressed" by the anti-gravitating effect of the scalar field.

The infinite boost has yielded a configuration that saturates the Bogomolnyi bound upon reduction. On the other hand, it has given a configuration with covariantly constant Killing vector in $d = 5$. To see this, write the above metric (4.3) as

$$ds^2 = -dudv - \frac{2q}{r}du^2 + r^2 d\Omega$$

where light-cone coordinates

$$u = \sqrt{2}^{-1}(y-t), \quad v = \sqrt{2}^{-1}(y+t)$$

are used. This is a special case of a pp- or Brinkmann wave [9], admitting a covariantly constant null-Killing vector

$$\chi^M \partial_M = \partial_u, \quad D_M \chi_N = 0.$$

Therefore, the authors of [42] investigated if the existence of a null-Killing vector together with a another space-like Killing vector *necessarily* implies the saturation of the Bogomolnyi bound upon reduction along space-like direction. This was done by considering the geometrical properties of such solutions to the Einstein equations.

Here, the question if a null-Killing vector together with a (commuting) space-like Killing vector imply saturation of the Bogomolnyi bound shall be viewed from another perspective.

Consider again the above example. One may easily see that the $d = 5$ solution, in the context supergravity, preserves a fraction of local supersymmetry, as is well known in the literature. This, on the other hand, automatically implies the saturation of the Bogomolnyi bound in the reduced theory [25]!

To see this, consider a supersymmetric embedding of the above situation, that is, e.g. simple $d = 5$ supergravity. As already mentioned, $d = 5$ simple supergravity contains, apart from pure gravity, a 1-form potential A_M and a Dirac spinor Ψ_M. Truncating the theory to pure gravity by setting $A_M = 0$, $\Psi_M = 0$, the supersymmetry transformations are [13]

$$\begin{aligned} \delta E^{(5)A}_{M} &= 0 \\ \delta A_M &= 0 \\ \delta \Psi_M &= D_M \epsilon = \partial_M \epsilon + \frac{1}{4}\omega_{MAB}\gamma^{AB}\epsilon \end{aligned}$$

with the $d = 5$ gamma matrices

$$\gamma_M = \{\gamma_i, \gamma_0, i\gamma_5\},$$

where γ_i, γ_0 are the $d = 4$ gamma matrices, and γ_5 as usual, and γ^{AB} is the antisymmetrized product of γ^A and γ^B. Thus, a nontrivial supersymmetry transformation is given by finding a Dirac spinor ϵ that is regular and covariantly constant in the whole space, that is,

$$\delta \Psi_M = D_M \epsilon = 0. \tag{4.5}$$

The integrability condition of this equation is

$$[D_M, D_N]\epsilon = R_{MN}{}^{OP}\gamma_{OP}\epsilon = 0$$

where R_{MNOP} is the Riemann tensor, which upon contraction with γ^P by using the first Bianchi identity yields

$$R_{MN}\gamma^N\epsilon = 0,$$

and by multiplication with $R_{ML}\gamma^L$, where M is not summed over

$$R_{MN}R_M{}^N\epsilon = 0$$

which was called a "Ricci-null" condition in [36]. Thus, if the gravitational equations of motion (Ricci tensor equal to zero) is satisfied, the equation is integrable, however, finding a Killing spinor does not always imply that the equations of motion are satisfied, since there might be cases when the Ricci tensor is "null" but not zero.

Here, a solution is easily found: Split the curved indices into $M = \{i, u, v\}$ and the flat into $A = \{a, -, +\}$. Choosing the vielbein

$$E^{(5)A}_M = \begin{pmatrix} \delta_i{}^a & 0 & 0 \\ 0 & 1 & H \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} -\delta_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

where $H = q/r$, the only nonzero connection coefficient is

$$\omega_{ua-} = \partial_a H$$

and thus a solution is given by a constant Dirac spinor ϵ_0 obeying

$$\gamma_+\epsilon_0 = 0.$$

Thus, the above solution is a solution to $d = 5$ supergravity that preserves half of the supersymmetry. When performing a Kaluza-Klein reduction along the Killing direction y, this is also a solution of $d = 4$ supergravity. The reduction does not break supersymmetry, and thus the $d = 4$ solution is also supersymmetric, and furthermore, as seen above, asymptotically flat and static. For such configurations, the Bogomolnyi bound holds between mass and charges [25]. It is saturated, if and only if (4.5) holds, that is, a regular Killing spinor exists globally, which is the case.

Thus, the saturation of the Bogomolnyi bound in $d = 4$ may be understood as a consequence of the fact that the $d = 5$ solution admits an exact Killing spinor (or is a "spin-manifold"). This is a simple condition, and it should be interesting to compare it to the perspective of [42].

4.3.2 Null Killing Vectors and the BPS Condition

General metrics admitting a null-Killing vector were investigated closely in [60]. A corresponding vielbein was the starting point of the investigation of [42], augmented by the further assumption that another space-like Killing vector exists, commuting with the null-Killing vector. The saturation of the BPS bound was then derived by using a coordinate transformation to obtain a space-like and time-like Killing vector from the above and reducing along the space-like Killing vector. Choosing a Weyl rescaling of the metric such that the $d = 4$ gravitational Lagrangian is Einstein-Hilbert, the Komar integral was then (by using the equations of motion) evaluated for the time-like Killing vector and shown to give the mass equal to the purely electric charge, assuming that the spacetime is asymptotically Minkowskian.

As in the example, this result would follow automatically if the space considered in [42] admits a Killing spinor. If the theory is reduced along a space-like direction and is asymptotically flat, the Bogomolnyi is saturated if an exact Killing spinor exists tending to a constant at infinity.

It shall be investigated here if this is the case by considering a general vielbein obeying the above properties.

Consider a metric admitting a null-Killing-vector $\chi^M \partial_M = \partial_u$ and a space-like Killing vector $\xi^M \partial_M = \partial_v$ in $d+1$ dimensions, where u and v are light-cone coordinates as above. Split the curved coordinates into $M = \{\mu, u, v\}$ and the flat coordinates into $A = \{\alpha, -, +\}$. A general vielbein in this configuration is given by [60]

$$E_M^{(d+1)A} = \begin{pmatrix} E_\mu^{(d-1)\alpha} & w_m & SC_m \\ 0 & w_u & SC_u \\ 0 & 0 & S \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} -\delta_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In [42], a coordinate transformation was used to obtain a time-like Killing vector ∂_t and a space-like Killing vector ∂_y from ∂_u, ∂_v by

$$u = p\,t + r\,y, \quad v = q\,t + s\,y.$$

The vielbein may then be identified, using a local Lorentz boost, with Kaluza-Klein reduction vielbein (4.4). This results in the $d = 4$ U(1) gauge potential

$$B_m = \sqrt{2}^{-1}(u_m + SC_m)\rho^{-1}, \quad B_t = \frac{p}{r} - (ps - rq)\rho^{-2} S w_u$$

and dilaton

$$\rho = r[2Sw_u(C_u + s/r)]^{1/2}.$$

This configuration was argued to be purely electric in [42]. Here, this shall be used from the start to simplify the calculation. Assume therefore

$$B_m = 0, \quad SC_m = -u_m$$

This will significantly simplify the above vielbein. The null-Killing vector obeys a normality condition [60] that implies

$$Sw_M = W\partial_M u$$

where u shall be identified with the above light-cone coordinate. Then the vielbein may be put into the simple form

$$E_M^{(d+1)A} = \begin{pmatrix} E_\mu^{(d-1)\alpha} & 0 & 0 \\ 0 & x & xy \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} E_\mu^{(d-1)\alpha} & 0 \\ 0 & e_i^{\ a} \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} -\delta_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

where $i = \{u, v\}$ and $a = \{-, +\}$.

A necessary condition for the Killing spinor equation

$$D_M \epsilon = 0$$

to be integrable is that the Ricci tensor obeys the condition that R_M^N is null. Since it is assumed that the above vielbein obeys the equations of motion, that is, the space is Ricci flat, the Killing spinor equation is integrable. It remains however to check whether a Killing spinor exists admitting the correct properties for a saturation of the Bogomolnyi bound in $d = 4$. For this, it has to be regular on the whole space and to approach a constant on a sphere at infinity.

To investigate the Killing spinor equations, one calculates the non-vanishing components of the spin connection with the above vielbein. They are

$$\begin{aligned}\omega^{(d+1)}_{\alpha\beta\gamma} &= \omega^{(d-1)}_{\alpha\beta\gamma}\\ \omega^{(d+1)}_{a\gamma b} &= e_{(b}{}^{i}\partial_\alpha e_{i\ c)}\end{aligned}$$

where (ab) means symmetrized, $[ab]$ antisymmetrized indices. Let $\{\Gamma_M\}$ be a representation of the gamma matrices in $d+1$. The Killing spinor equations then reduce to

$$\begin{aligned}&(\partial_u + \frac{1}{4}\omega_{uAB}\Gamma^{AB})\epsilon = \partial_u\epsilon + \frac{1}{4}(\Gamma^\alpha(\partial_\alpha(xy))\Gamma^- + \Gamma^\alpha(\partial_\alpha x)\Gamma^+)\epsilon = 0\\ &(\partial_v + \frac{1}{4}\omega_{vAB}\Gamma^{AB})\epsilon = \partial_v\epsilon + \frac{1}{4}(\Gamma^\alpha(\partial_\alpha x)\Gamma^+)\epsilon = 0\\ &(\partial_\mu + \frac{1}{4}\omega^{(d-1)}_{\mu ab}\Gamma^{ab})\epsilon = 0\end{aligned}$$

The Killing spinor should be asymptotically constant in the reduced $d = 4$ theory in order to saturate the Bogomolnyi bound. Therefore, one has to demand

$$\partial_u\epsilon = \partial_v\epsilon = 0.$$

Thus, the first two equations become

$$\begin{aligned}(\Gamma^\alpha\partial_\alpha(xy)\Gamma^- + \Gamma^\alpha(\partial_\alpha x)\Gamma^+)\epsilon &= 0\\ \Gamma^\alpha(\partial_\alpha x)\Gamma^+\epsilon &= 0\end{aligned}$$

These equations have a nontrivial solution only for $\partial_\mu x = 0$, that is, x is constant.

One may now investigate if this is a consequence of the equations of motion and the above assumptions. The equations of motion show that x is a harmonic function on the $d-1$ dimensional Riemannian space with Weyl-rescaled vielbein $x^{-2}E^{(d-1)\alpha}_\mu$. However, only if the function x has compact support, it is actually constant! But one may not assume this to be the generic case, which seems to make it impossible to find a proper solution!

If x is constant, this however implies that the above solution is actually a Brinkmann wave, with covariantly constant null-Killing vector [60] (the covariant derivative of χ^M vanishes for constant x). Such waves and their supersymmetry have been studied closely in the literature (see e.g. [36] and references for a recent treatment). In the above situation, the Killing spinor equations reduce by demanding

$$\gamma_+\epsilon = 0 \tag{4.6}$$

to

$$(\partial_\mu + \frac{1}{4}\omega^{(d-1)}_{\mu ab}\Gamma^{ab})\epsilon = 0$$

Thus (at least) half of the supersymmetry is broken[2]. The integrability condition of this equation is Ricci-flatness for the connection $\omega^{(d-1)}_{\mu ab}$, which, using the constancy of x, is again the $d+1$ dimensional field equation $R_{\mu\nu} = 0$.

Coming back to a five-dimensional example as in the preceding section, for $d-1 = 3$ Ricci flatness implies flatness, and the spinor is not further constrained. For higher dimensions, the supersymmetry might be further broken on the transverse space.

[2] Note that, if $d+1$ is even, the condition (4.6) may be seen to be consistent with the spinor decomposition when reducing to d dimensions [25]

Finally, it shall be commented on the speculations that these solitons might be suitable to probe the structure of the hyperbolic E_{10}. E_{10} is supposed to arise upon a null reduction to one-dimensional supergravity as a symmetry. If the above solitons, however, do *by construction* not depend on the light cone coordinates as assumed here, a direct reduction to $d = 1$ by forming periodic arrays in transverse space will always yield a trivial, that is constant solution in $d = 1$, and it seems hard to imagine a nontrivial interpretation of this scenario.

4.4 Summary

In this chapter, soliton solutions of M-theory and connected applications of U-duality have been discussed.

Soliton solutions that correspond to black holes with regular horizon allow a microscopic interpretation of black hole entropy by using duality symmetries and counting of D-brane and string states. These corresponding concepts have been reviewed and explained.

Elementary solitons carrying just one nonzero charge have been introduced, as well as dyonic solutions, and applications of U-duality on these solutions have been given. The most general spherical symmetric black hole has been embedded into the conventions of this thesis and reviewed, its relevance to the microscopic entropy discussion and the importance of U-duality in this context has been shown.

Elementary solitons in $d = 3$ have been constructed by forming periodic arrays in $d = 4$ and direct reduction to $d = 3$. They have been shown to carry the $d = 4$ charges, dressed by the compactification radius, and discrete U-duality transformations correspond to travelling around the solutions on full circles.

The question if the existence of a null-Killing vector implies the saturation of the Bogomolnyi bound has been addressed from a supersymmetric point of view. It has been shown that, in general, both assumptions are not equivalent in this view. The solutions have, however, been argued not to be suitable to probe a hyperbolic symmetry in M-theory.

Chapter 5

Conclusions and Outlook

The main results of this thesis shall briefly be summarized.

M-theory admits a class of large non-perturbative duality symmetries, called U-dualities, putting all the moduli of the theory on the same footing. These symmetries may be constructed from first principles in the low energy effective approximation by imposing the DSZ quantization condition. This breaks the classical duality symmetries present in these theories to discrete subgroups. For M-theory compactifications on the torus, these groups were investigated in this thesis.

In a toy model resembling the low energy effective limit of M-theory, it was shown that a hidden classical duality symmetry may be ”uncovered” by using a proper set of fields. The DSZ condition was shown to break the classical group to a discrete group, which turned out to be $SL(2,\mathbb{Z})$, acting in the **4** representation, with standard generators T, S. The $d = 4$ discrete groups acts as electric-magnetic duality with theta term in the asymptotic limit, but always mixes all four types of charges.

Upon further compactification to $d = 3$, different views of the compact theory by choosing different orders of compactification lead to different $d = 4$ U-duality groups embedded into the $d = 3$ theory. These groups merge together to form the $d = 3$ U-duality. It turned out that the discrete group given by this construction is strictly smaller than the one expected from the action of admissible lattices in basic representations. This indicates that, even in the most simple case of toroidal compactification, the discrete groups need to addressed with care, and the same should be kept in mind when addressing truncations of M-theory, as the toy model corresponds to such a truncation.

For the low energy limit of M-theory, compactification on the torus leads to a $d = 4$ theory that exhibits a classical $E_{7(+7)}$ duality. This theory can be made more manageable by embedding the $\mathfrak{e}_{7(+7)}$ algebra into $\mathfrak{e}_{8(+8)}$ and defining the fundamental **56** carried by the vector fields and the scalars of the theory by the adjoint action of the $\mathfrak{e}_{7(+7)}$ subalgebra within $\mathfrak{e}_{8(+8)}$.

The discrete U-duality is given by imposing the DSZ quantization condition. The representations space may then be chosen to be an admissible lattice. Lie algebras acting on those lattices in basic representations are generated by the analogies of the $SL(2,\mathbb{Z})$ S, T generators for all roots, and thus, a set of generators for the discrete U-duality group $E_{7(+7)}(\mathbb{Z})$ is found. Subgroups identified with T- and S-duality involve subtleties and mix non-trivially, e.g., a true electric-magnetic duality for one specific gauge field is always a combination of S- and T-duality, and the string coupling constant related to the $d = 11$ compactification radius always transforms non-trivially under a T-duality.

Using the definition of generators of $E_{7(+7)}(\mathbb{Z})$, the different definitions of U-duality given in the literature are seen to agree.

U-duality in $d = 3$ is given by merging the discrete $d = 4$ duality groups embedded into the

$d = 3$ theory together. For M-theory, this gives the expected result: The U-duality group in $d = 3$ agrees with the definition of $E_{8(+8)}(\mathbb{Z})$ by the action on admissible lattices.

Solitons in the theory play an important role as counterpart of strings, and especially for a microscopic interpretation of black hole entropy if they correspond to black holes with nonzero horizon. Elementary solitons correspond to naked singularities dressed by an anti-gravitating dilaton, and fill the **56**-plet by the fact that the representation is minimal. Explicit knowledge of U-duality proves useful when calculating multiplets of dyonic black holes. The most general spherically symmetric black hole has five nonzero charges, and all other charged black holes of same topology are generated from it. However, no state counting procedure has been found yet for this black hole, and U-duality is a useful tool to look for configuration that might allow such a counting.

The $d = 3$ analogies of the elementary $d = 4$ solitons are shown to be vortex solutions, and discrete duality transformations correspond to traveling around them on a full circle.

Finally, purely gravitational solutions that admit a null-Killing vector and a space-like Killing vector and are asymptotically flat when reducing along a space-like Killing direction were shown in the literature to saturate the Bogomol'nyi bound. This would follow immediately if these solution can generally be shown to be supersymmetric, however, for the most general case, this seems not to be possible. The solutions seem not to be suitable to address the questions of hyperbolic symmetries in M-theory, since they are trivial in the corresponding reduction.

Several future directions of research are suggested by the results of this thesis.

The methods to study U-duality are quite general and also apply to compactifications of M-theory that admit less supersymmetry, and it is interesting to study corresponding U-duality groups from a physical and mathematical point of view in this context.

U-duality has several other applications for which a precise knowledge of the discrete group proves useful. Apart from the search for a proper regime that allows a state counting of the most general spherical symmetric BPS black hole, it may e.g. be applied to determine counter-terms and non-perturbative effects in string theory, a subject not reviewed in this thesis. A precise definition of U-duality as given here might prove useful in this context as well.

The method used to construct $d = 3$ U-duality can be extended to lower dimensions. This would then yield a discrete subgroup of an affine or hyperbolic group, and would, which is even more attractive, make contact to the quantized approaches in these dimensions. This might give new insights into the structure of M-theory, and it should be interesting to consider this "stringy" symmetry and compare it to e.g. the known quantized symmetries in $d = 2$.

Finally, it is interesting to connect the discrete group construction found here to the supergroups discussed in [58] and references.

Appendix A

Lie Algebras and `Maple` Procedures

In this appendix, definitions and conventions for Lie algebras are introduced and procedures are presented that generate representation matrices for highest weight representations of simple Lie algebras. The weights of the representation are demanded to be nondegenerate, except for adjoint and basic representations that include zero weights with degeneracy equal to the number of simple roots among the weights. A procedure to study tensor products is discussed as well.

For the work presented in this thesis, these procedures were used to generate and investigate representations of $\mathfrak{e}_{6(+6)}$, $\mathfrak{e}_{7(+7)}$, $\mathfrak{e}_{8(+8)}$ as well as $\mathfrak{g}_{2(+2)}$, for discrete subgroup calculus and for the calculation of coset spaces. For further study, the procedures were extended to general Lie algebras.

As platform `Maple V Release 5` was used. A description of its syntax and procedures may be found in [14, 15]. However, the procedures were programmed to be as simple as possible and may be migrated to any other platform or programming languages like `FORTRAN`, `C` or `LISP` with minor changes in the syntax and by using standard procedures e.g. for matrix inversion.

A.1 Lie Algebras: Definitions and Conventions

For the necessary mathematical background in Lie algebras used in this appendix see [61, 57, 48]. A collection of tables and useful recipes can be found in [70, 91]. The necessary definitions and conventions shall be briefly recalled in this section.

A.1.1 Lie Algebras

Some general definitions shall be stated first.

A *Lie algebra* may be defined as a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ (taken to be $\mathbb{C}$ or $\mathbb{R}$ in the following) with bilinear bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : [x, y] \to z$ satisfying

$$\begin{aligned} &[x, x] = 0, && x \in \mathfrak{g}, \\ &[x, [y, z]] + [z, [z, x]] + [z, [x, y]] = 0 \text{ (Jacobi identity)}, && x, y, z \in \mathfrak{g}. \end{aligned}$$

A subspace $\mathfrak{g}' \subset \mathfrak{g}$ closed under the bracket is called a (Lie) *subalgebra* of $\mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{g}'] \subseteq \mathfrak{g}'$, it is said to be *invariant*. Any algebra that contains an invariant subalgebra is called *reducible*, while an irreducible Lie algebra is called *simple*. Only those algebras will be of interest here.

An important class of subalgebras may be given by using two further definitions: For a subalgebra $\mathfrak{g}'$, one may define the *normalizer* $\mathfrak{n}$ in $\mathfrak{g}$ by $\mathfrak{n} = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}'] \subset \mathfrak{g}'\}$. Furthermore, the *adjoint* of an element $y \in \mathfrak{g}$ is defined by the map ad : $x \in \mathfrak{g}, y \mapsto [x, y]$, and an algebra $\mathfrak{g}$ is said to be *nilpotent* if (ad $\mathfrak{g})^n$=0 for some $n \in \mathbb{N}$. A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is then defined to be a subalgebra that is nilpotent and equals its normalizer in $\mathfrak{g}$. For simple Lie algebras over $\mathbb{C}$, Cartan subalgebras are abelian, and the action of a fixed Cartan subalgebra may be simultaneously diagonalized. This plays a central role in defining proper generators of a Lie algebra. A general set of generators shall now be presented and discussed.

Define an $l \times l$ *(generalized) Cartan matrix* $A^{\mathfrak{g}}$ by demanding that $A^{\mathfrak{g}}_{ii} = 2$, $A^{\mathfrak{g}}_{ij} \in -\mathbb{N}$ for $i \neq j$ and $A^{\mathfrak{g}}_{ij} = 0 \Rightarrow A^{\mathfrak{g}}_{ji} = 0$. It is assumed that $A^{\mathfrak{g}}$ is symmetrizable, that is, a nondegenerate diagonal matrix D exists such that $DA^{\mathfrak{g}}$ is symmetric, and that it may not be put into block-diagonal form by the exchange of rows and columns (ensuring irreducibility).

To each such Cartan matrix $A^{\mathfrak{g}}$ a *Kač-Moody algebra* may be associated generated by the sets $\{E_i, F_i \mid 1 \leq i \leq l\}$ and $\{H_i \mid 1 \leq i \leq l\}$, and the generators satisfy the following relations:

$$\begin{aligned}
&[H_i, H_j] = 0, && 1 \leq i, j \leq l,\\
&[E_i, F_i] = H_i,\ [E_i, F_j] = 0 && 1 \leq i \neq j \leq l,\\
&[H_i, E_j] = A^{\mathfrak{g}}_{ji} E_j,\ [H_i, F_j] = -A^{\mathfrak{g}}_{ji} E_j, && 1 \leq i \neq j \leq l,\\
&(\text{ad } E_i)^{-A^{\mathfrak{g}}_{ji}+1}(E_j) = 0, && i \neq j,\\
&(\text{ad } E_i)^{-A^{\mathfrak{g}}_{ji}+1}(E_j) = 0, && i \neq j.
\end{aligned} \tag{A.1}$$

The $\{H_i \mid 1 \leq i \leq l\} \in \mathfrak{h}$ obviously form a Cartan subalgebra.

The last two relations are called *Serre relations*. If the matrix $DA^{\mathfrak{g}}$ is indefinite or positive semi-definite, the algebra is infinite dimensional. If $DA^{\mathfrak{g}}$ is positive definite, the above Lie algebra is finite dimensional. This will be assumed in the following.

A realization of $A^{\mathfrak{g}}_{ij}$ may then be given on an n-dimensional real Euclidian space with scalar product $\cdot$ by defining sets of vectors $\Pi^{\mathfrak{g}} = \{\alpha_1, \dots, \alpha_l\}$ for which

$$A^{\mathfrak{g}}_{ij} = \langle \alpha_i, \alpha_j \rangle = 2\,\frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j},\ \ \alpha_i, \alpha_j \in \Pi^{\mathfrak{g}}, i, j = 1 \dots l.$$

The sets $\Pi^{\mathfrak{g}}$ are called sets of *simple roots*. In view of (A.1), the E_i, F_i will be denoted $E_{\alpha_i}, E_{-\alpha_i}$, and the H_i by H_{α_i} in the following.

A multiple commutator $X = (\text{ad } E_{\alpha_{b_1}})^{a_1} \dots (\text{ad } E_{\alpha_{b_n}})^{a_n} E_{\alpha_{b_{n+1}}}$, if nontrivial, satisfies $[H_{\alpha_i}, X] = \langle \alpha, \alpha_i \rangle X$, where $\alpha = \sum_i a_i \alpha_{b_i} + \alpha_{b_{n+1}}$, $b_i \in \{1, \dots, l\}$, $a_i \in \mathbb{N}$, and is therefore associated with the *root* α. The *set of all roots* is called $\Delta^{\mathfrak{g}}$, and from (A.1) it may be written as $\Delta^{\mathfrak{g}} = \Delta^{\mathfrak{g}}_+ \cup \Delta^{\mathfrak{g}}_-$, where

$$\begin{aligned}
\Delta^{\mathfrak{g}}_+ &= \{\alpha \mid \alpha = \sum_i r_i^{\alpha} \alpha_i, r_i^{\alpha} \in \mathbb{N}_0\},\\
\Delta^{\mathfrak{g}}_- &= \{\alpha \mid \alpha = \sum_i r_i^{\alpha} \alpha_i, r_i^{\alpha} \in -\mathbb{N}_0\}
\end{aligned}$$

are called the sets of *positive and negative roots*. The $\{r_i^{\alpha}\}$ will be called *root coordinates*.

For the roots $\alpha \in \Delta^{\mathfrak{g}}$ the following properties may be shown from (A.1):

$$\begin{aligned}
0 &\notin \Delta^{\mathfrak{g}}\\
\alpha \in \Delta^{\mathfrak{g}} &\ \Rightarrow\ n\alpha \notin \Delta^{\mathfrak{g}}, n \in \mathbb{Z}, |n| \neq 1,\\
\alpha \in \Delta^{\mathfrak{g}} &\ \Rightarrow\ -\alpha \in \Delta^{\mathfrak{g}},\\
\alpha, \beta \in \Delta^{\mathfrak{g}} &\ \Rightarrow\ \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in \Delta^{\mathfrak{g}},\\
\alpha, \beta \in \Delta^{\mathfrak{g}} &\ \Rightarrow\ \langle \beta, \alpha \rangle \in \mathbb{Z}.
\end{aligned} \tag{A.2}$$

The σ_α are called *Weyl reflections* and form the *Weyl group* $\mathcal{W}$. It may be shown that, if $\alpha \in \Delta^{\mathfrak{g}}$, $\sigma \in \mathcal{W}$ exists such that $\sigma(\alpha) \in \Pi^{\mathfrak{g}}$.

Recall that, investigating (A.2), it may be shown that the finite dimensional simple Lie algebras split into *simply laced* Lie algebras, where all roots are of the same length, and *non-simply laced* Lie algebras with exactly two different lengths. The simply laced algebras fall into three classes, denoted by $\mathfrak{a}_l$, $\mathfrak{d}_l$ and $\mathfrak{e}_l$, while the non-simply laced algebras are classified as $\mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{f}_4$ and $\mathfrak{g}_2$. Their Cartan matrices may be conveniently given by *Dynkin diagrams*.

Dynkin diagrams are defined to have l vertices, with vertices i and j connected by $|A^{\mathfrak{g}}_{ij}|$ lines if $|A^{\mathfrak{g}}_{ij}| \geq |A^{\mathfrak{g}}_{ji}|$, and an arrow pointing towards vertex j if $|A^{\mathfrak{g}}_{ij}| > 1$, that is, pointing towards the shorter root. The Dynkin diagrams for simple finite dimensional Lie algebras as used here are listed in figure A.1.

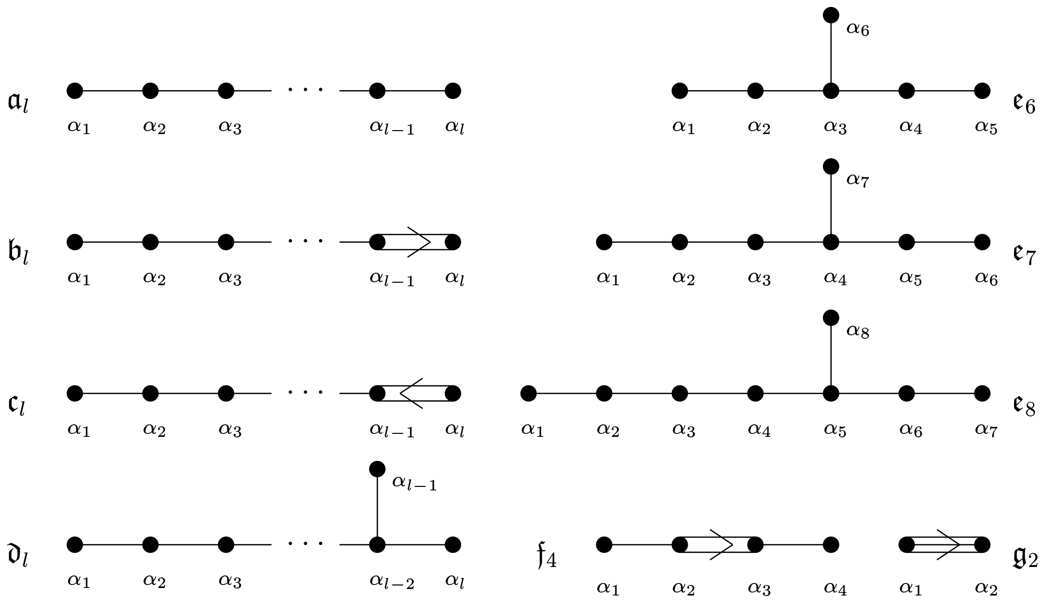


Figure A.1: Lie Algebra Conventions

The root systems shall now be described in more detail in order to present generators and relations for the full algebras. Consider an Euclidian space with standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_l$ and scalar product $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. For the simply laced Lie algebras $\mathfrak{a}_l, \mathfrak{d}_l, \mathfrak{e}_l$ the simple roots may be represented as

$$\begin{aligned}
\Pi^{\mathfrak{a}_l} &= \{\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \ldots, \alpha_l = \mathbf{e}_l - \mathbf{e}_{l+1}\},\\
\Pi^{\mathfrak{d}_l} &= \{\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \alpha_{l-1} = \mathbf{e}_{l-1} - \mathbf{e}_l, \alpha_l = \mathbf{e}_{l-1} + \mathbf{e}_l\},\\
\Pi^{\mathfrak{e}_l} &= \{\alpha_1 = \mathbf{e}_{9-l} - \mathbf{e}_{10-l}, \ldots, \alpha_{l-1} = \mathbf{e}_7 - \mathbf{e}_8, \alpha_l = -\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8\}
\end{aligned} \tag{A.3}$$

satisfying $\alpha_i \cdot \alpha_i = 2$ for all simple roots. They span the remaining roots by

$$\Delta^{\mathfrak{g}} = \{\alpha \mid \alpha = \sum_i r_i^\alpha \alpha_i, r_i^\alpha \in \mathbb{N}_0 \vee r_i^\alpha \in -\mathbb{N}_0, \alpha \cdot \alpha = 2\}.$$

The roots for the non-simply laced algebras $\mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{f}_4, \mathfrak{g}_2$ are connected to those of the simply laced algebras by second and third order automorphisms $\bar{\mu}$ on specific root spaces [61]. Consider

$$\begin{aligned}
\Delta^{\mathfrak{d}_{l+1}}: &\quad \bar{\mu}(\alpha_i) = \alpha_i, \;\; 1 \leq i \leq l-1, \;\; \bar{\mu}(\alpha_l) = \alpha_{l+1}, \bar{\mu}(\alpha_{l+1}) = \alpha_l;\\
\Delta^{\mathfrak{a}_{2l-1}}: &\quad \bar{\mu}(\alpha_{2l-k}) = \alpha_k, \;\; k = 1 \ldots 2l-1;\\
\Delta^{\mathfrak{e}_6}: &\quad \bar{\mu}(\alpha_1) = \alpha_5, \bar{\mu}(\alpha_2) = \alpha_4, \bar{\mu}(\alpha_3) = \alpha_3, \bar{\mu}(\alpha_4) = \alpha_2, \bar{\mu}(\alpha_5) = \alpha_1, \bar{\mu}(\alpha_6) = \alpha_6;\\
\Delta^{\mathfrak{d}_4}: &\quad \bar{\mu}(\alpha_1) = \alpha_3, \bar{\mu}(\alpha_3) = \alpha_4, \bar{\mu}(\alpha_4) = \alpha_1, \bar{\mu}(\alpha_2) = \alpha_2.
\end{aligned} \tag{A.4}$$

and define

$$\Delta_l^{\mathfrak{g}} = \{\alpha = \alpha' \in \Delta^{\mathfrak{g}'} \mid \bar{\mu}(\alpha') = \alpha'\},$$
$$\Delta_s^{\mathfrak{g}} = \{\alpha = r^{-1}(\bar{\mu}(\alpha') + \cdots + \bar{\mu}^r(\alpha')) \mid \alpha' \in \Delta^{\mathfrak{g}'}, \bar{\mu}(\alpha') \neq \alpha'\} \tag{A.5}$$

for $(\mathfrak{g}, \mathfrak{g}', r) = \{(\mathfrak{b}_l, \mathfrak{d}_{l+1}, 2), (\mathfrak{c}_l, \mathfrak{a}_{2l-1}, 2), (\mathfrak{f}_4, \mathfrak{e}_6, 2), (\mathfrak{g}_2, \mathfrak{d}_4, 3)\}$. This gives the *long and short roots* for the non-simply laced algebras, their union gives the set of all roots for these algebras. For the simple roots one obtains

$$\begin{aligned}
\Pi^{\mathfrak{b}_l} &= \{\alpha_1 = \mathbf{v}_1 - \mathbf{v}_2, \ldots, \alpha_{l-1} = \mathbf{v}_{l-1} - \mathbf{v}_l, \alpha_l = \mathbf{v}_l\},\\
\Pi^{\mathfrak{c}_l} &= \{\alpha_1 = \tfrac{1}{\sqrt{2}}\mathbf{v}_1 - \mathbf{v}_2, \ldots, \alpha_{l-1} = \tfrac{1}{\sqrt{2}}(\mathbf{v}_{l-1} - \mathbf{v}_l), \alpha_l = \sqrt{2}\mathbf{v}_l\},\\
\Pi^{\mathfrak{f}_4} &= \{\alpha_1 = \mathbf{v}_2 - \mathbf{v}_3, \alpha_2 = \mathbf{v}_3 - \mathbf{v}_4, \alpha_3 = \mathbf{v}_4, \alpha_4 = \tfrac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4)\},\\
\Pi^{\mathfrak{g}_2} &= \{\alpha_1 = \mathbf{v}_1 - \mathbf{v}_2, \alpha_2 = \tfrac{1}{3}(-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3)\}
\end{aligned}$$

where the $\mathbf{v}_i$ form again a basis of $\mathbb{R}^n$ and $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$[1]. The root spaces (A.5) are spanned by

$$\Delta^{\mathfrak{g}} = \{\alpha \mid \alpha = \sum_i r_i^\alpha \alpha_i, r_i^\alpha \in \pm\mathbb{N}_0,\ \ \alpha\cdot\alpha = 1 \text{ or } 2, \mathfrak{g} \in \{\mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{f}_4\}\},$$
$$\Delta^{\mathfrak{g}_2} = \{\alpha \mid \alpha = \sum_i r_i^\alpha \alpha_i, r_i^\alpha \in \pm\mathbb{N}_0,\ \ \alpha\cdot\alpha = \tfrac{2}{3} \text{ or } 2\}.$$

Finally, the sets of all roots for the simple Lie algebras following from the construction introduced above shall be given.

$$\begin{aligned}
\Delta^{\mathfrak{a}_l} &= \{\pm(\mathbf{e}_i - \mathbf{e}_j)(1 \le i < j \le l+1)\},\\
\Delta^{\mathfrak{b}_l} &= \{\pm\mathbf{v}_i \pm \mathbf{v}_j(1 \le i < j \le l), \pm\mathbf{v}_i(1 \le i \le l)\},\\
\Delta^{\mathfrak{c}_l} &= \{\tfrac{1}{\sqrt{2}}(\pm\mathbf{v}_i \pm \mathbf{v}_j)(1 \le i < j \le l), \pm\sqrt{2}\mathbf{v}_i(1 \le i \le l)\},\\
\Delta^{\mathfrak{d}_l} &= \{\pm\mathbf{e}_i \pm \mathbf{e}_j\ (1 \le i < j \le l+1)\},\\
\Delta^{\mathfrak{e}_8} &= \{\pm(\mathbf{e}_i - \mathbf{e}_j)(1 \le i < j \le 9),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)(1 \le i < j < k \le 8),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_9)(1 \le i < j \le 8)\},\\
\Delta^{\mathfrak{e}_7} &= \{\pm(\mathbf{e}_i - \mathbf{e}_j)(2 \le i < j \le 8),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)(2 \le i < j < k \le 8),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_9)(2 \le i \le 8)\},\\
\Delta^{\mathfrak{e}_6} &= \{\pm(\mathbf{e}_i - \mathbf{e}_j)(3 \le i < j \le 8),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k)(3 \le i < j < k \le 8),\\
&\qquad \pm(-\tfrac{1}{3}(\mathbf{e}_1 + \cdots + \mathbf{e}_9) + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_9)\},\\
\Delta^{\mathfrak{f}_4} &= \{\pm\mathbf{v}_i(1 \le i \le 4), \pm\mathbf{v}_i \pm \mathbf{v}_j(1 \le i < j \le 4), \tfrac{1}{2}(\pm\mathbf{v}_1 \pm \mathbf{v}_2 \pm \mathbf{v}_3 \pm \mathbf{v}_4)\},\\
\Delta^{\mathfrak{g}_2} &= \{\pm(\mathbf{v}_i - \mathbf{v}_j)(1 \le i < j \le 3), \pm(-\tfrac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + \mathbf{v}_i)(1 \le i \le 3)\}.
\end{aligned} \tag{A.6}$$

[1] The above construction gives specific orthonormal base vectors $\{\mathbf{v}_i\}$ for $\mathfrak{g} \in \{\mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{f}_4, \mathfrak{g}_2\}$. Explicitly,
$\Pi^{\mathfrak{b}_l}: \mathbf{v}_i = \mathbf{e}_i;$
$\Pi^{\mathfrak{c}_l}: \mathbf{v}_i = \frac{1}{\sqrt{2}}(\mathbf{e}_i - \mathbf{e}_{2l+1-i});$
$\Pi^{\mathfrak{f}_4}: \mathbf{v}_1 = -\frac{1}{3}\sum_{k=1}^{9}\mathbf{e}_k + \frac{1}{2}\sum_{k=4}^{7}\mathbf{e}_k + \mathbf{e}_3, \mathbf{v}_2 = -\frac{1}{3}\sum_{k=1}^{9}\mathbf{e}_k + \frac{1}{2}\sum_{k=4}^{7}\mathbf{e}_k + \mathbf{e}_8,$
$\mathbf{v}_3 = \frac{1}{2}(\mathbf{e}_4 + \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7), \mathbf{v}_4 = \frac{1}{2}(\mathbf{e}_4 - \mathbf{e}_5 + \mathbf{e}_6 - \mathbf{e}_7);$
$\Pi^{\mathfrak{g}_2}: \mathbf{v}_1 = \mathbf{e}_2, \mathbf{v}_2 = \mathbf{e}_3, \mathbf{v}_3 = -\mathbf{e}_1.$

With these conventions, the simple Lie algebras $\mathfrak{g}$ of figure A.1 with simple roots $\alpha_i \in \Pi^{\mathfrak{g}}$ spanning root spaces $\{\alpha \mid \alpha \in \Delta^{\mathfrak{g}}\}$ may be given completely. They are

$$\mathfrak{g} = \sum_i h_{\alpha_i} H_{\alpha_i} + \sum_\alpha e_\alpha E_\alpha, \quad h_{\alpha_i}, e_\alpha \in \mathbb{C},$$

$$\begin{aligned}
&[H_{\alpha_i}, H_{\alpha_j}] = 0, && \\
&[H_{\alpha_i}, E_\alpha] = \langle \alpha, \alpha_i \rangle E_\alpha && \alpha \in \Delta^{\mathfrak{g}}, \\
&[E_\alpha, E_{-\alpha}] = H_\alpha && \alpha \in \Delta^{\mathfrak{g}}, \\
&[E_\alpha, E_\beta] = 0 && \alpha, \beta \in \Delta^{\mathfrak{g}}, \alpha + \beta \notin \Delta^{\mathfrak{g}} \cup \{0\}, \\
&[E_\alpha, E_\beta] = \epsilon(\alpha, \beta) E_{\alpha+\beta} && \alpha, \beta, \alpha + \beta \in \Delta^{\mathfrak{g}}, \mathfrak{g} \in \{\mathfrak{a}_l, \mathfrak{d}_l, \mathfrak{e}_l\}, \\
&[E_\alpha, E_\beta] = (p+1)\epsilon(\alpha', \beta') E_{\alpha+\beta} && \alpha, \beta, \alpha + \beta \in \Delta^{\mathfrak{g}}, \mathfrak{g} \in \{\mathfrak{c}_l, \mathfrak{b}_l, \mathfrak{f}_4, \mathfrak{g}_2\}, \\
& && p \in \mathbb{N}, \alpha - p\beta \in \Delta^{\mathfrak{g}}, \alpha - (p+1)\beta \notin \Delta^{\mathfrak{g}}
\end{aligned} \tag{A.7}$$

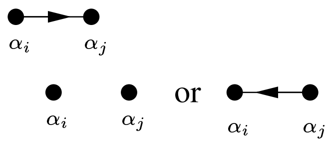
where $\epsilon(\alpha, \beta)$ is a bimultiplicative asymmetry function [61] obeying

$$\begin{aligned}
\epsilon(\alpha + \tilde{\alpha}, \beta) &= \epsilon(\alpha, \beta)\epsilon(\tilde{\alpha}, \beta) \\
\epsilon(\alpha, \beta + \tilde{\beta}) &= \epsilon(\alpha, \beta)\epsilon(\alpha, \tilde{\beta}), \ \ \alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \Delta^{\mathfrak{g}}.
\end{aligned}$$

It may be given by choosing an orientation for the Dynkin diagram of $\mathfrak{g}$ as in figure A.2.

$\epsilon(\alpha_i, \alpha_j) = -1$ if $i = j$ or if

$\epsilon(\alpha_i, \alpha_j) = +1$ otherwise, i.e.,



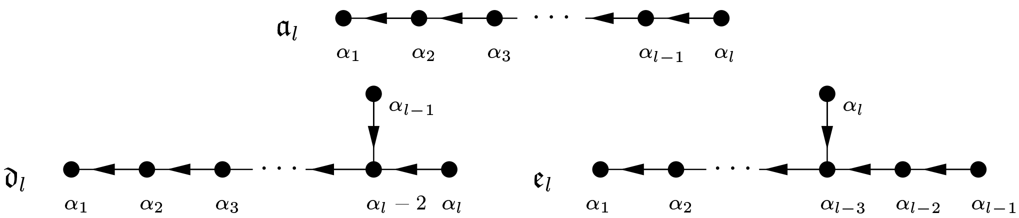


Figure A.2: Bimultiplicative Asymmetry Function

For the algebras $\mathfrak{g} \in \{\mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{f}_4, \mathfrak{g}_2\}$, the primed roots refer to the embedding (A.5)[2], and orientations are used invariant under the automorphisms (A.4), displayed in figure A.3. The automorphisms (A.4), called *diagram automorphisms*, are indicated by arrows.

A.1.2 Representations

For a Lie algebra $\mathfrak{g}$, consider an n-dimensional vector space V, endowed with the operation $\mathfrak{g} \times V \to V : (X, v) \mapsto Xv$ that satisfies

$$\begin{aligned}
(aX + bY)v &= a(Xv) + b(Yv), \\
X(av + bw) &= a(Xv) + b(Xw), \\
[X, Y]v &= XYv - YXv, \qquad (X, Y \in \mathfrak{g}; v, w \in V; a, b \in \mathbb{F}).
\end{aligned} \tag{A.8}$$

[2]The corresponding Lie algebra generators are embedded by $E_\alpha = E'_{\alpha'}, \ \alpha \in \Delta^{\mathfrak{g}}_l$, $E_\alpha = E'_{\bar{\mu}(\alpha')} + \cdots + E'_{\bar{\mu}^r(\alpha')}, \ \alpha \in \Delta^{\mathfrak{g}}_s$.

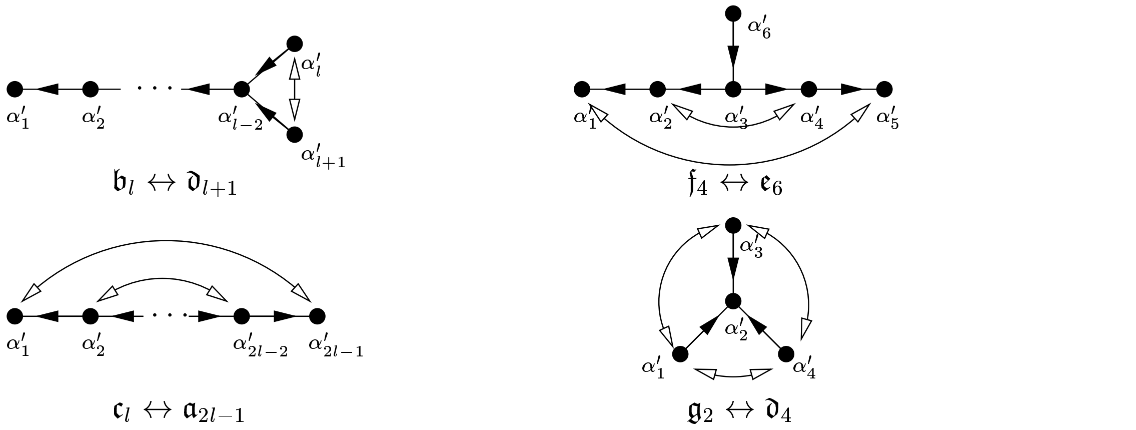


Figure A.3: Bimultiplicative Asymmetry Function for Non-Simply Laced Algebras

Such a vector space is called a $\mathfrak{g}$-*module* and defines a representation $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$. An *irreducible* $\mathfrak{g}$-*module* contains only itself and 0 as submodules.

The Cartan subalgebra $\mathfrak{h}$ acts diagonally on V, and V may be decomposed into a direct sum over V_Λ where $\Lambda = \{\lambda_i^\Lambda\}$ and $V_\Lambda = \{v \in V \mid H_{\alpha_i} v = \lambda_i^\Lambda v\}$. The Λ are called *weights* and V_Λ the *weight space* of Λ.

For a finite dimensional irreducible module V a unique weight Λ^0 and vector v_0 exist such that $E_\alpha v_0 = 0$ for all $\alpha \in \Delta_+^{\mathfrak{g}}$. This weight is called the *highest weight* of V, and v_0 the *highest weight vector*. Conversely, for each such highest weight vector an irreducible $\mathfrak{g}$-module exists.

The representation theory of $\mathfrak{g} = \mathfrak{a}_1$ with generators $E_\alpha = E$, $E_{-\alpha} = F$, $H_\alpha = H$ shall be recalled briefly. Choose a maximal weight Λ^0 and consider a maximal vector $v_0 \in V^{\Lambda^0}$. Define $v_i = (1/i!)F^i v_0$. Then

$$\begin{aligned} Hv_i &= (\Lambda^0 - 2i)v_i, \\ Fv_i &= (i+1)v_{i+1}, \\ Ev_i &= (\Lambda^0 - i + 1)v_{i-1}, \qquad (i \geq 0) \end{aligned} \tag{A.9}$$

as may be easily shown by induction. Furthermore, if m is the smallest integer such that $v_m \neq 0, v_{m+1} = 0$, the last equation shows $m = \Lambda^0$ for $i = m+1$, that is, Λ^0 is a nonnegative integer.

In the general case of an algebra $\mathfrak{g}$ and a finite dimensional $\mathfrak{g}$-module V, define for each simple root $\alpha_i \in \Pi^{\mathfrak{g}}$ the corresponding algebra $(\mathfrak{a}_1)_i$ consisting of E_{α_i}, $E_{-\alpha_i}$ and H_{α_i}. Then a highest weight of $\mathfrak{g}$ is also a highest weight of the $(\mathfrak{a}_1)_i$, showing that in the general case for $\Lambda^0 = \{\lambda_i^{\Lambda^0}\}$ the $\lambda_i^{\Lambda^0} \in \mathbb{N}$, and it may be seen that V is the sum of finite dimensional $(\mathfrak{a}_1)_i$-modules.

A $\mathfrak{g}$-module with highest weight Λ^0 is called *fundamental* if $\sum_i \lambda_i^{\Lambda^0} = 1$. All irreducible representations of $\mathfrak{g}$ may be generated by tensor products from the fundamental ones. One such fundamental irreducible representation exists that generates all others, and has $\lambda_k^{\Lambda^0} = 1$, where α_k corresponds to an endpoint of the Dynkin diagram of $\mathfrak{g}$.

A $\mathfrak{g}$-module with highest weight Λ^0 will be called *basic* if all nontrivial weights are transformed into each other by the Weyl group. Then all weights are of multiplicity one, and the multiplicity of the weight 0 equals the number of simple roots among the weights of V. The only basic representations that contain zero weights are those whose set of weights equals the the short roots of $\mathfrak{g}$ [57, 69].

A $\mathfrak{g}$-module with highest weight Λ^0 will be called *minimal* if no other highest weight $\tilde{\Lambda}^0$ of $\mathfrak{g}$ exists such that $\Lambda^0 - \tilde{\Lambda}^0 = \sum_i b_i \beta_i$, where $b_i \in \mathbb{N}$ and $\beta_i \in \Delta_+^{\mathfrak{g}}$. Then $\langle \Lambda^0, \alpha \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Delta^{\mathfrak{g}}$, and the Weyl-orbit of the highest weight is saturated (thus, basic representations containing no zero weights are minimal).

For the procedures to follow, the space V shall be specified more explicitly by describing possible basis' of V and considering the action of the generators $E_{\alpha_i}, E_{-\alpha_i}$ on them.

For this, consider all roots $\{\beta_1, ..., \beta_d\} \in \Delta_+^{\mathfrak{g}}$ and fix a specific ordering[3]. Define $A = (a_1, \ldots, a_d)$, $B = (b_1, \ldots, b_l)$ and $C = (c_1, \ldots, c_d)$, where $a_i, b_i, c_i \in \mathbb{N}$ and consider

$$\begin{aligned} F_A =& \frac{(E_{-\beta_1})^{a_1}}{a_1!} \ldots \frac{(E_{-\beta_d})^{a_d}}{a_d!}, \\ H_B =& \begin{pmatrix} H_{\alpha_1} \\ b_1 \end{pmatrix} \ldots \begin{pmatrix} H_{\alpha_l} \\ b_l \end{pmatrix}, \\ E_C =& \frac{(E_{\beta_1})^{c_1}}{c_1!} \ldots \frac{(E_{\beta_d})^{c_d}}{c_d!} \end{aligned}$$

where

$$\begin{pmatrix} H_{\alpha_i} \\ b_i \end{pmatrix} = \frac{H_{\alpha_i}(H_{\alpha_i} - 1) \ldots (H_{\alpha_i} - b_i + 1)}{b_i!}.$$

The elements $F_A H_B E_C$ form an $\mathbb{F}$ basis of $\mathcal{U}(\mathfrak{g})$, the *universal enveloping algebra* of $\mathfrak{g}$. Denote by $\mathcal{U}_{\mathbb{Z}}^-, \mathcal{U}_{\mathbb{Z}}^0$ and $\mathcal{U}_{\mathbb{Z}}^+$ the $\mathbb{Z}$ spans of F_A, H_B and E_C, and by $\mathcal{U}_{\mathbb{Z}}$ the $\mathbb{Z}$ span of $F_A H_B E_C$.

The $\mathbb{Z}$-span of a specific basis in V shall be called a *lattice* of V. A lattice M that is invariant under the action of $\mathcal{U}_{\mathbb{Z}}$ is called *admissible*. Obviously, this implies the action of all $E_\alpha^n/n!$ in V to be integral.

For an admissible lattice M, consider a basis vector $\tilde{v}_0 \in V^\Lambda$, and let $\Lambda = \{\lambda_i^\Lambda\}$ be a highest weight for $(\mathfrak{a}_1)_j$ for some $1 \leq j \leq l$. Consider another basis vector of M denoted by $\tilde{v}_{\lambda_j^\Lambda} \in V^{\Lambda - \lambda_j^\Lambda \alpha_j}$. By considering

$$\frac{E_{\alpha_j}^{\lambda_j^\Lambda}}{\lambda_j^\Lambda!} \frac{F_{\alpha_j}^{\lambda_j^\Lambda}}{\lambda_j^\Lambda!} \tilde{v}_0$$

it may be shown that necessarily

$$\tilde{v}_{\lambda_j^\Lambda} = \pm \frac{F_{\alpha_j}^{\lambda_j^\Lambda}}{\lambda_j^\Lambda!} \tilde{v}_0.$$

This fact will be of interest in section 3.2.1. For the procedures to follow, the basis shall be specified further. Consider now the $\mathfrak{g}$-module V with highest weight Λ^0. The *minimal admissible lattice* of V may then be defined by $M = \mathcal{U}_{\mathbb{Z}}^- v_0$. For each $V^\Lambda \subset V$ one may choose basis vectors v_k^Λ by the intersection $M \cap V^\Lambda$ (M spans V over $\mathbb{F}$). Consider such a basis $\{v_k^\Lambda\}$ in V. The action of the $(\mathfrak{a}_1)_j$ on such a basis may then be given quite analogously to (A.9), that is, if Λ is a weight of V, λ_j^Λ a highest weight of an $(\mathfrak{a}_1)_j$-module and $\tilde{v}_{\Lambda - k\alpha_j} \in V^{\Lambda - k\alpha_j}$ a basis vector of M,

$$\begin{aligned} H_{\alpha_j} \tilde{v}_{\Lambda - k\alpha_j} =& (\lambda_j^\Lambda - 2k) \tilde{v}_{\Lambda - k\alpha_j}, \\ F_{\alpha_j} \tilde{v}_{\Lambda - k\alpha_j} =& (k+1) \tilde{v}_{\Lambda - (k+1)\alpha_j}, \\ E_{\alpha_j} \tilde{v}_{\Lambda - k\alpha_j} =& (\lambda_j^\Lambda - k + 1) \tilde{v}_{\Lambda - (k-1)\alpha_j}, \end{aligned}$$

where $\tilde{v}_{\Lambda - (k+1)\alpha_j} \in V^{\Lambda - (k+1)\alpha_j}$ and $\tilde{v}_{\Lambda - (k-1)\alpha_j} \in V^{\Lambda - (k-1)\alpha_j}$ are again basis vectors of M. Note that the spaces $V^{\Lambda - n\alpha_j}$, $1 \leq n \leq \lambda_j^\Lambda$ need not be one-dimensional (in this case the weight $\Lambda - n\alpha_j$ is called *degenerate*).

[3] E.g., fix an ordering of the simple roots and order the remaining roots by their *height*, that is, by the number of simple roots they decompose into. For roots of the same height, the ordering of simple roots may be used again.

For the following procedures, roots α will either be given by their root coordinates $\{r_i^\alpha\}$, or in *Dynkin components* $\{a_i^\alpha\}$ given by

$$a_i^\alpha = \langle \alpha, \alpha_i \rangle = r_j^\alpha A_{ji}^{\mathfrak{g}}.$$

Weights Λ will be given by $\{\lambda_i^\Lambda\}$ as defined above, corresponding to Dynkin components.

A.2 `Maple` Procedures

The Maple procedures used to generate highest weight matrix representations are grouped into several subroutines.

A.2.1 Lie Algebra Data

The procedure `initialize` generates the necessary Lie algebra data to be used in subsequent procedures. It is called with a specific group (1 for $\mathfrak{a}_l$, 2 for $\mathfrak{b}_l$, 3 for $\mathfrak{c}_l$, 4 for $\mathfrak{d}_l$, 5 for $\mathfrak{e}_l$, 6 for $\mathfrak{f}_4$, 7 for $\mathfrak{g}_2$) and Rank l and generates the Cartan matrix with conventions as in figure A.1, as well as the simple roots and the highest root in Dynkin components. The corresponding `Maple` code is given in table A.1.

A.2.2 Normalizations

The procedure `Kacnorm`, when called with two roots α, β of a specified simply laced algebra $\mathfrak{g}$, returns the asymmetry function $\epsilon(\alpha, \beta)$. Let $\{r_i^\alpha\}$, $\{r_i^\beta\}$ be the corresponding root coordinates. Using bimultiplicativity, one has

$$\epsilon(\alpha, \beta) = \prod_{i,j} (\epsilon(\alpha_i, \alpha_j))^{r_i^\alpha r_j^\beta}$$

with the definitions of figure A.2.

For non-simply laced algebras $\mathfrak{g}$, for $\alpha, \beta \in \Delta^{\mathfrak{g}}$ corresponding $\alpha', \beta' \in \Delta^{\mathfrak{g}'}$ of the embeddings (A.5) need to be determined. Defining

$$\begin{aligned}
\Pi^{\mathfrak{d}_{l+1}}: &\quad \hat{\alpha}_i = \alpha_i, \ \ i = 1 \ldots l; \\
\Pi^{\mathfrak{a}_{2l-1}}: &\quad \hat{\alpha}_i = \alpha_i, \ \ i = 1 \ldots l; \\
\Pi^{\mathfrak{e}_6}: &\quad \hat{\alpha}_1 = \alpha_6, \hat{\alpha}_2 = \alpha_3, \hat{\alpha}_3 = \alpha_2, \hat{\alpha}_4 = \alpha_1; \\
\Pi^{\mathfrak{d}_4}: &\quad \hat{\alpha}_1 = \alpha_2, \hat{\alpha}_2 = \alpha_1;
\end{aligned}$$

they are calculated as follows: Let $\alpha \in \Delta^{\mathfrak{g}}$ and $\{r_i^\alpha\}$ be the corresponding root coordinates, $\hat{\alpha} \in \Delta^{\mathfrak{g}'}$, $\hat{\alpha}_i \in \Pi^{\mathfrak{g}'}$, $(\mathfrak{g}, \mathfrak{g}') = \{(\mathfrak{b}_l, \mathfrak{d}_{l+1}), (\mathfrak{c}_l, \mathfrak{a}_{2l-1}), (\mathfrak{f}_4, \mathfrak{e}_6)\}$, then one may use

$$\hat{\alpha} = \sum_i z(r_i^\alpha)\hat{\alpha}_i + z_{\bar{\mu}}(r_i^\alpha)\bar{\mu}(\hat{\alpha}_i),$$

with $\bar{\mu}$ given in (A.4) and

r_i^α	$z(r_i^\alpha)$	$z_{\bar{\mu}}(r_i^\alpha)$
1	1	0
2	1	1
3	2	1
4	2	2

It may be easily seen that this yields a root $\hat{\alpha} \in \Delta^{\mathfrak{g}'}$. The procedure then checks for $\alpha' \in \{\hat{\alpha}, \bar{\mu}(\hat{\alpha})\}$ and $\beta' \in \{\hat{\beta}, \bar{\mu}(\hat{\beta})\}$ if $\alpha' + \beta' \in \Delta^{\mathfrak{g}'}$. If so, $\epsilon(\alpha', \beta')$ is calculated with orientations as in figure A.3, and returned.

For $\alpha \in \Delta^{\mathfrak{g}_2}$, $\hat{\alpha} \in \Delta^{\mathfrak{d}_4}$, $\hat{\alpha}_i \in \Pi^{\mathfrak{d}_4}$,

$$\hat{\alpha} = \sum_i z(r_i^\alpha)\hat{\alpha}_i + z_{\bar{\mu}}(r_i^\alpha)\bar{\mu}(\hat{\alpha}_i) + z_{\bar{\mu}^2}(r_i^\alpha)\bar{\mu}^2(\hat{\alpha}_i)$$

with $\bar{\mu}$ as in (A.4) and

r_i^α	$z(r_i^\alpha)$	$z_{\bar{\mu}}(r_i^\alpha)$	$z_{\bar{\mu}^2}(r_i^\alpha)$
1	1	0	0
2	1	1	0
3	1	1	1

For $\alpha' \in \{\hat{\alpha}, \bar{\mu}(\hat{\alpha}), \bar{\mu}^2(\hat{\alpha})\}$ and $\beta' \in \{\hat{\beta}, \bar{\mu}(\hat{\beta}), \bar{\mu}^2(\hat{\beta})\}$ it is checked if $\alpha' + \beta' \in \Delta^{\mathfrak{d}_4}$ and, if so, $\epsilon(\alpha', \beta')$ is calculated using orientations as in figure A.3, and returned. The corresponding `Maple` code is given in table A.2.

Apart from the realization (A.7), for the exceptional series and $\mathfrak{g}_2$ the realization using a decomposition with respect to $\mathfrak{sl}_n$ subalgebras following Freudenthal [38] is used in this thesis, as described in sections 2.4.2, 3.1.3. This amounts to calculating different normalizations $\epsilon(\alpha, \beta)$ (where $\epsilon(\alpha, \beta)$ is not necessarily bimultiplicative) in (A.7) to obtain the commutators (2.18) and (3.18). The corresponding calculations have therefore been implemented as well.

For this, define (cf. (A.3))

$$\begin{aligned}
\boldsymbol{\epsilon}_{ij} &\equiv \mathbf{e}_i - \mathbf{e}_j \qquad (1 \le i \ne j \le 9), \\
\pm\boldsymbol{\epsilon}_i &\equiv \pm(\mathbf{e}_i - \frac{1}{3}\sum_{l=1}^{3}\mathbf{e}_l) \qquad (1 \le i \le 3), \\
\pm\boldsymbol{\epsilon}_{ijk} &\equiv \pm(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k - \frac{1}{3}\sum_{l=1}^{9}\mathbf{e}_l) \qquad (1 \le i < j < k \le 9),
\end{aligned}$$

Given a root in Dynkin components, the procedure `freuden` extracts the indices i, j, k of $\boldsymbol{\epsilon}_i$, $\boldsymbol{\epsilon}_{ij}$ and $\boldsymbol{\epsilon}_{ijk}$. The procedure `freudeneps` is then used to calculate $\epsilon(\alpha, \beta)$ when generating representation matrices. For this, as will become clear in subsequent subsections, it is sufficient to consider $\alpha \in \Pi^{\mathfrak{g}}$ and $\beta \in \Delta_-^{\mathfrak{g}}$. The procedure is called with indices for α, β and $\alpha + \beta$ and contains the necessary information to calculate and return $\epsilon(\alpha, \beta)$. The corresponding `Maple` code is given in table A.3.

A.2.3 Matrix Calculus

Since most representation matrices, especially in higher dimensional representations, have only a very limited number of nonzero entries, procedures were used for matrix calculus that benefit from this fact. Computation times and the usage of memory could be shortened considerably by this.

Matrices are stored in arrays that list only the nonzero entries and their position within the matrix, as well as the total number of nonzero entries. The basic operations (scalar multiplication `scalmult`, matrix multiplication `mult`, matrix sum `addi`) are then straightforward, as well as the procedure for commutators `commu`. In order to generate matrices in their "ordinary" form to make them accessible for more complicated operations within `Maple`, the procedure `matri` may be used.

All `Maple` codes are listed in table A.4.

A.2.4 Generation of Representation Matrices

Representation matrices of the algebra $\mathfrak{g}$ on the $\mathfrak{g}$-module V with highest weight Λ^0 are determined by the procedure `highweightrep` called with a specific group, rank and highest weight vector.

If degenerate weights exist in V, the calculation of the action of the generators on V involves in general a deeper study of its basis. In the case of no degenerate weights, using the conventions of section A.1, the action can be simply read off from the Dynkin components of the weights of V. In this thesis, only modules without degenerate weights are of interest, and therefore the procedure is restricted to assume that no degenerate weights exist, with two exceptions: Adjoint representations that have a weight 0 with degeneracy equal to the rank of the algebra, and general basic representations that have a zero weight of degeneracy equal to the number of short roots in $\Pi^{\mathfrak{g}}$.

For the following, consider V with highest weight Λ_0, and call the set of all weights of V Φ^{Λ_0}. The procedure works in two steps.

In the first step, the weight lattice Φ^{Λ_0} is constructed, and the entries in the generator matrices are produced. The procedure starts by checking if Λ_0 corresponds to the highest root or yields a basic representation. Consider first the case that this is not the case. The weight Λ_0 is checked for nonzero $\lambda_k^{\Lambda_0}$, $1 \leq k \leq l$. If the $\lambda_k^{\Lambda_0} > 0$ for specific k, the corresponding chain of weights $\Lambda_i = \Lambda_0 - i\alpha_k, 1 \leq i \leq \lambda_k^{\Lambda_0}$ is produced, and the Λ_i are added to a set $\tilde{\Phi}^{\Lambda_0}$. The entries in the representation matrices for E_{α_k}, $E_{-\alpha_k}$ are then made corresponding to

$$E_{\pm\alpha_k} v_i = D(E_{\pm\alpha_k})_{ji} v_j$$

where $v_i \in V^{\Lambda_i}$ and the base vector v_i and the entries $D(E_{\pm\alpha})_{ji}$ follow the convention introduced in section A.1.2. This is then done subsequently for all the following weights Λ_i, and if $\lambda_k^{\Lambda_i} > 0$, $1 \leq k \leq l$, new weights and the corresponding entries in the representation matrices are created. Since it is assumed that no degenerate weights exist, when creating a new weight, it is checked if the weight already has been added to $\tilde{\Phi}^{\Lambda_0}$, and if so, the entries of the representation matrices are produced correspondingly. This finally yields $\tilde{\Phi}^{\Lambda_0} = \Phi^{\Lambda_0}$ and representation matrices for all simple roots of $\mathfrak{g}$. Note that the matrices have entries in $\mathbb{N}_0$ in this construction.

In the case of adjoint representations, the procedure works as above, but additional normalizations are added. If the weight Λ_i corresponds to the root α and the simple root α_k is added or subtracted, the representation matrix entry for $E_{\pm\alpha_k}$ is produced as above and multiplied with $\epsilon(\pm\alpha_k, \alpha)$, such that the representation matrices correspond to

$$[E_{\pm\alpha_k}, E_{\Lambda_i}] = D(E_{\pm\alpha_k})_{ji} E_{\Lambda_j}$$

where algebra conventions (A.7) are used for all algebras except $\mathfrak{e}_l$, $l \geq 6$ and $\mathfrak{g}_2$. For the latter, conventions corresponding to sections 2.4.2, 3.1.3 are applied as default, but conventions (A.7) may be used as well. If a zero weight is reached, the procedure adds all zero weights at once and then generates matrix entries corresponding to (A.7) and 2.4.2, refe8freud, respectively. The representation matrices in this case have entries in $\mathbb{Z}$.

The weights of basic representations are, as already mentioned in section A.1.2, simply subsets of the weights of the adjoint representations. This is used for all basic representations with degenerate zero weight. The procedure creates the representation matrices as above, yielding matrices with entries in $\mathbb{Z}$.

In the second step, matrices for the remaining generators are produced by multiple commutators of the simple root generator matrices, with normalizations adjusted to (A.7) and sections 2.4.2, 3.1.3 respectively. This is done again subsequently by generating the negative weights (corresponding to negative roots) of the adjoint representation. If the simple root α_k is subtracted from the weight Λ corresponding to the root $-\alpha$, $[E_{-\alpha_k}, E_{-\alpha}]$ and $[E_{\alpha_k}, E_{\alpha}]$ is calculated and normalized to give $E_{-\alpha-\alpha_k}$ and $E_{+\alpha+\alpha_k}$, where the normalization is again simply given from the Dynkin indices of Λ and by calculating $\epsilon(\alpha_k, \alpha)(= -\epsilon(-\alpha_k, -\alpha))$ corresponding to (A.7) and sections 2.4.2, 3.1.3, respectively.

The corresponding `Maple` code is given in table A.5.

A.2.5 Lie Algebra Check

In the process of developing the above algorithms, a procedure was also created to check if the generated representation matrices are correct and give a faithful representation of the algebra.

The procedure `testCartan` is called with the representation matrices for the positive and negative root generators, along with their corresponding roots in Dynkin components, for a specific group and rank. It then checks all relations (A.1) for these matrices, and the commutator $[E_\alpha, E_{-\alpha}] = H_\alpha$ for all positive roots α. The corresponding `Maple` code is given in table A.6.

A.2.6 Tensor Products

For a given finite dimensional simple Lie algebra $\mathfrak{g}$, all irreducible representations can in principle be generated from the ones generated by the above procedures by tensor products. Therefore, a simple procedure `tens` was generated. Let $X \in \mathfrak{g}$, and consider two $\mathfrak{g}$-modules V^1 and V^2 of dimension d_1 and d_2 with basis $\{v_a^1\}$ and $\{v_i^2\}$. Let the corresponding representation matrices be given by X_{ab}^1, X_{ij}^2 and X_{IJ} on the tensor product module $V^1 \otimes V^2$, where $I = ai$ runs from $1 \ldots d_1 \cdot d_2$. One has

$$X_{IJ}(v^1 \otimes v^2)_{JK} = X_{ai,bj}(v_b^1 \otimes v_j^2) = (X_{ab}^1 \delta_{ij} + \delta_{ab} X_{ij}^2)(v_b^1 \otimes v_j^2).$$

The procedure `tens` is called with the matrices X_{ab}^1 and X_{ij}^2 and dimensions d_1,d_2. It returns X_{IJ}. The corresponding `Maple` code is given in table A.7.

Lie Algebra Data

```
initialize:=proc(grou,Ran,Cart,highest,Dimen,car)
local k,k1,k2,Temp,zaes;
with(linalg):
#
# Cartan matrices, highest root, Dimension
#
for k from 1 to 7 do
 Cart[k]:=array(1..Ran,1..Ran,sparse):
od:
highest:=array(1..7):
for k from 1 to 7 do
 highest[k]:=array(1..1,1..Ran,sparse):
od:
#
# A series (=sl(Ran+1))
#
Dimen[1]:=(Ran+1)^2-1:
Cart[1][1,1]:=2:
Cart[1][1,2]:=-1:
Cart[1][Ran,Ran-1]:=-1:
Cart[1][Ran,Ran]:=2:
for k from 1 to Ran-2 do
 Cart[1][1+k,k]:=-1:
 Cart[1][1+k,k+1]:=2:
 Cart[1][1+k,k+2]:=-1:
od:
for k1 from 1 to Ran do
car[1][k1]:=submatrix(Cart[1],k1..k1,1..Ran):od:
for k from 1 to Ran do
 highest[1]:=
  evalm(highest[1]+submatrix(Cart[1],k..k,1..Ran)):
od:
#
# B series (=so(2Ran+1))
#
if Ran>1 then
 Dimen[2]:=(2*Ran+1)*Ran:
 if Ran=2 then
  Cart[2]:=matrix([[2,-2],[-1,2]]):
 fi:
 if Ran>2 then
  Cart[2][1,1]:=2:
  Cart[2][1,2]:=-1:
  Cart[2][Ran-1,Ran-2]:=-1:
  Cart[2][Ran-1,Ran-1]:=2:
  Cart[2][Ran-1,Ran]:=-2:
  Cart[2][Ran,Ran-1]:=-1:
  Cart[2][Ran,Ran]:=2:
  for k from 1 to Ran-3 do
   Cart[2][1+k,k]:=-1:
   Cart[2][1+k,k+1]:=2:
   Cart[2][1+k,k+2]:=-1:
  od:
 fi:
 for k from 1 to Ran do
  car[2][k]:=submatrix(Cart[2],k..k,1..Ran):
 od:
 highest[2]:=car[2][1]:
 for k from 2 to Ran do
  highest[2]:=evalm(highest[2]+2*car[2][k]):
 od:
fi:
#
# C series (=sp(2Ran))
#
if Ran>1 then
 Dimen[3]:=(2*Ran+1)*Ran:
 Cart[3]:=transpose(Cart[2]):
 for k from 1 to Ran do
  car[3][k]:=submatrix(Cart[3],k..k,1..Ran):
 od:
 highest[3]:=car[3][Ran]:
 for k from 1 to Ran-1 do
  highest[3]:=evalm(highest[3]+2*car[3][k]):
 od:
fi:
#
# D series (=so(2Ran))
#
if Ran>3 then
 Dimen[4]:=(2*Ran-1)*Ran:
 Cart[4][1,1]:=2:Cart[4][1,2]:=-1:
 Cart[4][Ran-2,Ran-3]:=-1:Cart[4][Ran-2,Ran-2]:=2:
 Cart[4][Ran-2,Ran-1]:=-1:Cart[4][Ran-2,Ran]:=-1:
 Cart[4][Ran-1,Ran-2]:=-1:Cart[4][Ran-1,Ran-1]:=2:
 Cart[4][Ran,Ran-2]:=-1:Cart[4][Ran,Ran]:=2:
 for k from 1 to Ran-3 do
  Cart[4][1+k,k]:=-1:Cart[4][1+k,k+1]:=2:
  Cart[4][1+k,k+2]:=-1:
 od:
 for k from 1 to Ran do
  car[4][k]:=submatrix(Cart[4],k..k,1..Ran):
 od:
 highest[4]:=car[4][Ran]+car[4][Ran-1]+car[4][1]:
 for k from 2 to Ran-2 do
  highest[4]:=evalm(highest[4]+2*car[4][k]):
 od:
fi:
#
# E series
#
if Ran>5 and Ran<9 then
 Temp:=matrix([
 [2,-1,0,0,0,0,0,0],[-1,2,-1,0,0,0,0,0],
 [0,-1,2,-1,0,0,0,0],[0,0,-1,2,-1,0,0,0],
 [0,0,0,-1,2,-1,0,-1],[0,0,0,0,-1,2,-1,0],
 [0,0,0,0,0,-1,2,0], [0,0,0,0,-1,0,0,2]]):
 Cart[5]:=submatrix(Temp, 8-Ran+1..8, 8-Ran+1..8):
 for k from 1 to Ran do
  car[5][k]:=submatrix(Cart[5],k..k,1..Ran):
 od:
 if Ran=6 then
  highest[5]:=matrix(1,6,[0,0,0,0,0,1]):
  Dimen[5]:=78:
 fi:
 if Ran=7 then
  highest[5]:=matrix(1,7,[0,0,0,0,0,1,0]):
  Dimen[5]:=133:
 fi:
 if Ran=8 then
  highest[5]:=matrix(1,8,[1,0,0,0,0,0,0,0]):
  Dimen[5]:=248:
 fi:
fi:
#
# F4
#
if Ran=4 then
 Cart[6]:=matrix([
  [2,-1,0,0],[-1,2,-2,0],[0,-1,2,-1],[0,0,-1,2]]):
 for k from 1 to Ran do
  car[6][k]:=submatrix(Cart[6],k..k,1..Ran):
 od:
 highest[6]:=matrix(1,4,[1,0,0,0]):
 Dimen[6]:=52:
fi:
#
# G2
#
if Ran=2 then
 Cart[7]:=matrix([[2,-3],[-1,2]]):
 for k from 1 to Ran do
  car[7][k]:=submatrix(Cart[7],k..k,1..Ran):
 od:
 highest[7]:=matrix(1,2,[1,0]):
 Dimen[7]:=14:
fi:
#
# output
#
print(`Cartan matrix`, Cart[grou],`highest root`,
highest[grou],`Dimension`, Dimen[grou]);
end:
```

Table A.1: `Maple` Code: Lie Algebra Data

Kač's Normalization

```
Kacnorm:=proc(vec1,vec2,Ran,grou,Cart)
local eps,roco,k1,k2,Rann,si,sim,rocon,test,
 rococh1,rococh2,Test;
eps:=1:
roco[1]:=evalm(vec1&*inverse(Cart[grou])):
roco[2]:=evalm(vec2&*inverse(Cart[grou])):
# B series
if grou=2 and Ran>1 then
 # Cartan matrix of
 # mapped simply laced algebra (D_(Ran+1))
 Rann:=Ran+1:
 Test:=array(1..Rann,1..Rann,sparse):
 Test[1,1]:=2:Test[1,2]:=-1:
 if Rann>3 then
  Test[Rann-2,Rann-3]:=-1:Test[Rann-2,Rann-2]:=2:
  Test[Rann-2,Rann-1]:=-1:Test[Rann-2,Rann]:=-1:
  Test[Rann-1,Rann-2]:=-1:Test[Rann-1,Rann-1]:=2:
  Test[Rann,Rann-2]:=-1:Test[Rann,Rann]:=2:
  for k1 from 1 to Rann-3 do
   Test[1+k,k]:=-1:Test[1+k,k+1]:=2:Test[1+k,k+2]:=-1:
  od:
 fi:
 if Rann=3 then
  Test:=matrix([[2,-1,-1],[-1,2,0],[-1,0,2]]):
 fi:
 # calculate corresponding roots
 # of simply laced algebra
 for k1 from 1 to Ran-1 do
  si[k1]:=array(1..1,1..Rann,sparse):
  sim[k1]:=array(1..1,1..Rann,sparse):
  si[k1][1,k1]:=1:  sim[k1][1,k1]:=1:
 od:
 si[Ran]:=array(1..1,1..Rann,sparse):
 sim[Ran]:=array(1..1,1..Rann,sparse):
 si[Ran][1,Ran+1]:=1:
 sim[Ran][1,Ran]:=1:
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   rocon[k1,k2]:=array(1..1,1..Rann,sparse):
  od:
 od:
 for k2 from 1 to 2 do
  for k1 from 1 to Ran-1 do
   rocon[k2,1]:=
    evalm(rocon[k2,1]+roco[k2][1,k1]*si[k1]):
   rocon[k2,2]:=
    evalm(rocon[k2,2]+roco[k2][1,k1]*sim[k1]):
  od:
 od:
 for k2 from 1 to 2 do
  if roco[k2][1,Ran]=1 then
   rocon[k2,1]:=evalm(rocon[k2,1]+si[Ran]):
   rocon[k2,2]:=evalm(rocon[k2,2]+sim[Ran]):
  fi:
  if roco[k2][1,Ran]=2 then
   rocon[k2,1]:=evalm(rocon[k2,1]+si[Ran]+sim[Ran]):
   rocon[k2,2]:=evalm(rocon[k2,2]+sim[Ran]+si[Ran]):
  fi:
 od:
 # check and select
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   test:=evalm((rocon[1,k1]+rocon[2,k2])&*
     Test&*transpose(rocon[1,k1]+rocon[2,k2])):
   if test[1,1]=2 then
    rococh1:=rocon[1,k1]:rococh2:=rocon[2,k2]:
   fi:
  od:
 od:
 # calculate epsilon
 for k1 from 1 to Rann do
  for k2 from 1 to Rann do
   if k1>k2 and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1=k2 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
  od:
 od:
fi:
```

```
# F4
if grou=6 and Ran=4 then
 # Cartan matrix
 # of mapped simply laced algebra (E_6)
 Rann:=6:
 Test:=matrix([
  [2,-1,0,0,0,0],[-1,2,-1,0,0,0],
  [0,-1,2,-1,0,-1],[0,0,-1,2,-1,0],
  [0,0,0,-1,2,0], [0,0,-1,0,0,2]]):
 # calculate corresponding roots
 # of simply laced algebra
 for k1 from 1 to 4 do
  si[k1]:=array(1..1,1..Rann,sparse):
  sim[k1]:=array(1..1,1..Rann,sparse):
 od:
 si[1][1,6]:=1:sim[1][1,6]:=1:
 si[2][1,3]:=1:sim[2][1,3]:=1:
 si[3][1,2]:=1:sim[3][1,4]:=1:
 si[4][1,1]:=1:sim[4][1,5]:=1:
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   rocon[k1,k2]:=array(1..1,1..Rann,sparse):
  od:
 od:
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   rocon[k2,1]:=
    evalm(rocon[k2,1]+roco[k2][1,k1]*si[k1]):
   rocon[k2,2]:=
    evalm(rocon[k2,2]+roco[k2][1,k1]*sim[k1]):
  od:
 od:
 for k1 from 1 to 2 do
  for k2 from 3 to 4 do
   if roco[k1][1,k2]=1 then
    rocon[k1,1]:=evalm(rocon[k1,1]+si[k2]):
    rocon[k1,2]:=evalm(rocon[k1,2]+sim[k2]):
   fi:
   if roco[k1][1,k2]=2 then
    rocon[k1,1]:=evalm(rocon[k1,1]+si[k2]+sim[k2]):
    rocon[k1,2]:=evalm(rocon[k1,2]+si[k2]+sim[k2]):
   fi:
   if roco[k1][1,k2]=3 then
    rocon[k1,1]:=evalm(rocon[k1,1]+2*si[k2]+sim[k2]):
    rocon[k1,2]:=evalm(rocon[k1,2]+si[k2]+2*sim[k2]):
   fi:
   if roco[k1][1,k2]=4 then
    rocon[k1,1]:=
     evalm(rocon[k1,1]+2*si[k2]+2*sim[k2]):
    rocon[k1,2]:=
     evalm(rocon[k1,2]+2*si[k2]+2*sim[k2]):
   fi:
  od:
 od:
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   test:=evalm((rocon[1,k1]+rocon[2,k2])&*
    Test&*transpose(rocon[1,k1]+rocon[2,k2])):
   if test[1,1]=2 then
    rococh1:=rocon[1,k1]:rococh2:=rocon[2,k2]:
   fi:
  od:
 od:
 for k1 from 1 to Rann do
  for k2 from 1 to Rann do
   if k1>k2 and (k1<4 and k2<4)
    and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1<k2 and ((k1>2 and k1<>6)
    and (k2>2 and k2<>6)) and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1=6 and k2=3 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1=k2 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
  od:
 od:
fi:
```

Table A.2: Maple Code: Normalizations for Adjoint Representations (Kač)

```
# simply laced
if (grou=1 or grou=4 or grou=5) or Ran=1 then
 for k1 from 1 to Ran do
  for k2 from 1 to Ran do
   if k1>k2 and Cart[grou][k1,k2]<>0 then
    eps:=eps*(-1)^(roco[1][1,k1]*roco[2][1,k2]):
   fi:
   if k1=k2 then
    eps:=eps*(-1)^(roco[1][1,k1]*roco[2][1,k2]):
   fi:
  od:
 od:
fi:
# C series
if grou=3 and Ran>1 then
 # Cartan matrix of
 # mapped simply laced algebra (A_(2*Ran-1))
 Rann:=2*Ran-1:
 Test:=array(1..Rann,1..Rann,sparse):
 Test[1,1]:=2:Test[1,2]:=-1:
 Test[Rann,Rann-1]:=-1:Test[Rann,Rann]:=2:
 for k1 from 1 to Rann-2 do
  Test[1+k1,k1]:=-1:Test[1+k1,1k+1]:=2:
  Test[1+k1,k1+2]:=-1:
 od:
 # calculate corresponding roots of
 # simply laced algebra
 for k1 from 1 to Ran do
  si[k1]:=array(1..1,1..Rann,sparse):
  si[k1][1,k1]:=1:
  sim[k1]:=array(1..1,1..Rann,sparse):
  sim[k1][1,2*Ran-k1]:=1:
 od:
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   rocon[k1,k2]:=array(1..1,1..Rann,sparse):
  od:
 od:
 for k1 from 1 to Ran do
  for k2 from 1 to 2 do
   if roco[k2][1,k1]=1 then
    rocon[k2,1]:=evalm(rocon[k2,1]+si[k1]):
    rocon[k2,2]:=evalm(rocon[k2,2]+sim[k1]):
   fi:
   if roco[k2][1,k1]=2 then
    rocon[k2,1]:=evalm(rocon[k2,1]+si[k1]+sim[k1]):
    rocon[k2,2]:=evalm(rocon[k2,2]+sim[k1]+si[k1]):
   fi:
  od:
 od:
 # check and select
 for k1 from 1 to 2 do
  for k2 from 1 to 2 do
   test:=evalm((rocon[1,k1]+rocon[2,k2])&*
    Test&*transpose(rocon[1,k1]+rocon[2,k2])):
   if test[1,1]=2 then
    rococh1:=rocon[1,k1]:rococh2:=rocon[2,k2]:
   fi:
  od:
 od:
 # calculate epsilon
 for k1 from 1 to Rann do
  for k2 from 1 to Rann do
   if k1>k2 and k1<Ran+1
    and k2<Ran+1 and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1<k2 and k1>Ran-1
    and k2>Ran-1 and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
  od:
 od:
fi:
```

```
# G2
if grou=7 and Ran=2 then
 # Cartan matrix
 # of corresponding simply laced algebra (D_4)
 Rann:=4:
 Test:=matrix([
 [2,-1,0,0], [-1,2,-1,-1], [0,-1,2,0], [0,-1,0,2]]):
 # calculate corresponding roots
 # of simply laced algebra
 for k1 from 1 to 2 do
  for k2 from 1 to 3 do
   si[k1,k2]:=array(1..1,1..Rann,sparse):
  od:
 od:
 si[1,1][1,2]:=1:si[1,2][1,2]:=1:si[1,3][1,2]:=1:
 si[2,1][1,1]:=1:si[2,2][1,3]:=1:si[2,3][1,4]:=1:
 for k1 from 1 to 2 do
  for k2 from 1 to 3 do
   rocon[k1,k2]:=array(1..1,1..Rann,sparse):
  od:
 od:
 for k1 from 1 to 3 do
  rocon[1,k1]:=
   evalm(rocon[1,k1]+roco[1][1,1]*si[1,1]):
  rocon[2,k1]:=
   evalm(rocon[2,k1]+roco[2][1,1]*si[1,1]):
 od:
 for k1 from 1 to 2 do
  if roco[k1][1,2]=1 then
   rocon[k1,1]:=evalm(rocon[k1,1]+si[2,1]):
   rocon[k1,2]:=evalm(rocon[k1,2]+si[2,2]):
   rocon[k1,3]:=evalm(rocon[k1,3]+si[2,3]):
  fi:
  if roco[k1][1,2]=2 then
   rocon[k1,1]:=evalm(rocon[k1,1]+si[2,1]+si[2,2]):
   rocon[k1,2]:=evalm(rocon[k1,2]+si[2,2]+si[2,3]):
   rocon[k1,3]:=evalm(rocon[k1,3]+si[2,3]+si[2,1]):
  fi:
  if roco[k1][1,2]=3 then
   for k2 from 1 to 3 do
    rocon[k1,k2]:=
     evalm(rocon[k1,k2]+si[2,1]+si[2,2]+si[2,3]):
   od:
  fi:
 od:
 for k1 from 1 to 3 do
  for k2 from 1 to 3 do
   test:=evalm((rocon[1,k1]+rocon[2,k2])&*
    Test&*transpose(rocon[1,k1]+rocon[2,k2])):
   if test[1,1]=2 then
    rococh1:=rocon[1,k1]:  rococh2:=rocon[2,k2]:
   fi:
  od:
 od:
 for k1 from 1 to Rann do
  for k2 from 1 to Rann do
   if k1=2 and k2<>2 and Test[k1,k2]<>0 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
   if k1=k2 then
    eps:=eps*(-1)^(rococh1[1,k1]*rococh2[1,k2]):
   fi:
  od:
 od:
fi:
eps:
end:
```

Table A.2: `Maple` Code: Normalizations for Adjoint Representations (Kač)

Freudenthal's Normalization

```
freuden:=proc(Ran,grou,vec,inde)
local Temp,Cart,ro,rootco,epsis,c7;
inde:=array(1..5):
if grou=5 then
 Temp:=matrix([
  [2,-1,0,0,0,0,0,0],[-1,2,-1,0,0,0,0,0],
  [0,-1,2,-1,0,0,0,0],[0,0,-1,2,-1,0,0,0],
  [0,0,0,-1,2,-1,0,-1],[0,0,0,0,-1,2,-1,0],
  [0,0,0,0,0,-1,2,0], [0,0,0,0,-1,0,0,2]]):
 Cart:=submatrix(Temp, 8-Ran+1..8, 8-Ran+1..8):
 ro[7]:=ep[7]-ep[8]:
 ro[8]:=-1/3*(ep[1]+ep[2]+ep[3]+ep[4]+ep[5]+ep[6]+
  ep[7]+ep[8]+ep[9])+ep[6]+ep[7]+ep[8]:
 ro[6]:=ep[6]-ep[7]:
 ro[5]:=ep[5]-ep[6]:
 ro[4]:=ep[4]-ep[5]:
 ro[3]:=ep[3]-ep[4]:
 ro[2]:=ep[2]-ep[3]:
 ro[1]:=ep[1]-ep[2]:
 rootco:=evalm(vec&*inverse(Cart)):
 epsis:=sum(rootco[1,c6]*ro[c6+8-Ran],c6=1..Ran):
fi:
if grou=7 then
 Temp:=matrix([[2,-3],[-1,2]]):
 ro[1]:=ep[1]-ep[2]:
 ro[2]:=ep[2]-1/3*(ep[1]+ep[2]+ep[3]):
 rootco:=evalm(vec&*inverse(Temp)):
 epsis:=sum(rootco[1,c6]*ro[c6],c6=1..Ran):
fi:
inde[4]:=0:
for c7 from 1 to 9 do
 if (coeff(epsis,ep[c7])=1) then
  inde[4]:=inde[4]+1:  inde[inde[4]]:=c7:  inde[5]:=0:
 fi:
od:
for c7 from 1 to 9 do
 if (coeff(epsis,ep[c7])=-1) then
  inde[4]:=inde[4]+1:  inde[inde[4]]:=c7:  inde[5]:=0:
 fi:
od:
for c7 from 1 to 9 do
 if (coeff(epsis,ep[c7])=2/3) then
  inde[4]:=inde[4]+1:  inde[inde[4]]:=c7:  inde[5]:=+1:
 fi:
od:
for c7 from 1 to 9 do
 if (coeff(epsis,ep[c7])=-2/3) then
  inde[4]:=inde[4]+1:  inde[inde[4]]:=c7:  inde[5]:=-1:
 fi:
od:
end:
```

```
freudeneps:=proc(inde1,inde2,inde3,eps)
local epsilon;
epsilon:=array(1..9,1..9,1..9,1..9,1..9,1..9,
 1..9,1..9,1..9,antisymmetric):
epsilon[1,2,3,4,5,6,7,8,9]:=1:
eps:=1:
if inde1[4]=2 then
 if inde2[4]=2 then
  if inde1[1]=inde2[2] then eps:=-1:fi:
 fi:
 if inde2[4]=3 and inde2[5]=-1 then
  eps:=-1:
 fi:
 if inde2[4]=1 and inde2[5]=-1 then
  eps:=-1:
 fi:
fi:
if inde1[4]=3 then
 if inde2[4]=3 and inde2[5]=-1 then
  eps:=+epsilon[6,7,8,inde2[1],inde2[2],inde2[3],
   inde3[1],inde3[2],inde3[3]]:
 fi:
 if inde2[4]=3 and inde2[5]=+1 then
  if inde2[1]=6 and inde2[2]=7 then eps:=-1:fi:
  if inde2[2]=6 and inde2[3]=7 then eps:=-1:fi:
  if inde2[1]=7 and inde2[2]=8 then eps:=-1:fi:
  if inde2[2]=7 and inde2[3]=8 then eps:=-1:fi:
 fi:
 if inde2[4]=2 then
  if inde2[1]=7 then eps:=-1:fi:
 fi:
fi:
if inde1[4]=1 then
 if inde2[4]=1 and inde2[5]=-1 then
  eps:=+epsilon[2,inde2[1],inde3[1],4,5,6,7,8,9]:
 fi:
 if inde2[4]=1 and inde2[5]=+1 then
  eps:=-1:
 fi:
fi:
end:
```

Table A.3: Maple Code: Normalizations for Adjoint Representations (Freudenthal)

Conventions

Matrices are stored in arrays:
matrix $A \longleftrightarrow$ array a
a[1,4]: total number of nonzero entries in A
a[k,1]:i, a[k,2]:j, a[k,3]: A_{ij}, k=1...a[1,4]

Scalar Multiplication

```
scalmult:=proc(co1,c)
local z1,final;
#
# co1:  matrix, c:  scalar
#
# final:  c*co1
#
if c=0 then final[1,4]:=0:fi:
if c<>0 then
 for z1 from 1 to co1[1,4] do
  final[z1,1]:=co1[z1,1]:
  final[z1,2]:=co1[z1,2]:
  final[z1,3]:=c*co1[z1,3]:
 od:
 final[1,4]:=co1[1,4]:
fi:
final:
end:
```

Matrix Multiplication

```
mult:=proc(co1,co2)
local z1,z2,k1,k2,k3,k4,co3,indew,final;
#
# co1,co2:  matrices
#
#
# final:  co1*co2
#
z1:=0:
for k1 from 1 to co1[1,4] do
 for k2 from 1 to co2[1,4] do
  # check for nonzero entry
  if co1[k1,2]=co2[k2,1] then
   # if already exists, add
   indew:=0:
   for k3 from 1 to z1 do
    if co1[k1,1]=co3[k3,1]
     and co2[k2,2]=co3[k3,2] then
     indew:=1:
     co3[k3,3]:=co3[k3,3]+co1[k1,3]*co2[k2,3]:
    fi:
   od:
   # if does not exist, new
   if indew=0 then
    z1:=z1+1:
    co3[z1,1]:=co1[k1,1]:
    co3[z1,2]:=co2[k2,2]:
    co3[z1,3]:=co1[k1,3]*co2[k2,3]:
   fi:
  fi:
 od:
od:
# keep only nonzero entries
z2:=0:
for k3 from 1 to z1 do
 if co3[k3,3]<>0 then
  z2:=z2+1:
  for k4 from 1 to 3 do
   final[z2,k4]:=co3[k3,k4]:
  od:
 fi:
od:
final[1,4]:=z2:
final:
end:
```

Matrix Sum

```
addi:=proc(co1,co2)
local z1,z2,z3,z4,indew,co3,final;
#
# co1,co2:  matrices
#
# final:  co1+co2
#
# if one matrix is zero, result equals the other
if co1[1,4]=0 then final:=co2:fi:
if co2[1,4]=0 then final:=co1:fi:
# if both are not zero
if co1[1,4]<>0 and co2[1,4]<>0 then
 # add first matrix to result
 for z1 from 1 to co1[1,4] do
  for z2 from 1 to 3 do
   co3[z1,z2]:=co1[z1,z2]:
  od:
 od:
 z3:=co1[1,4]:
 # add second matrix
 for z1 from 1 to co2[1,4] do
  # if entry exists already, add
  indew:=0:
  for z2 from 1 to co1[1,4] do
   if co3[z2,1]=co2[z1,1] and co3[z2,2]=co2[z1,2] then
    indew:=1:
    co3[z2,3]:=co3[z2,3]+co2[z1,3]:
   fi:
  od:
  # if entry does not exists, new
  if indew=0 then
   z3:=z3+1:
   for z2 from 1 to 3 do
    co3[z3,z2]:=co2[z1,z2]:
   od:
  fi:
 od:
 # keep only nonzero entries
 z2:=0:
 for z1 from 1 to z3 do
  if co3[z1,3]<>0 then
   z2:=z2+1:
   for z4 from 1 to 3 do final[z2,z4]:=co3[z1,z4]:od:
  fi:
 od:
 final[1,4]:=z2:
fi:
final:
end:
```

Commutator

```
commu:=proc(co1,co2)
#
# co1,c2:  matrices
#
# final:  [co1,co2]
#
local final;
final:=
 addi(mult(co1,co2),scalmult(mult(co2,co1),-1)):
final:
end:
```

Matrices from Arrays

```
matri:=proc(co1,d)
#
# co1:  array, d:  dimension of the representation
#
# ma:  matrix
#
local k,ma;
ma:=array(1..d,1..d,sparse):
for k from 1 to co1[1,4] do
 ma[co1[k,1],co1[k,2]]:=co1[k,3]:
od:
ma:
end:
```

Table A.4: `Maple` Code: Matrix Calculus

```
Generation of Representation
Matrices

highweightrep:=proc(grou,Ran,vec,rep,Repdimen,pco,mco)
local a1,a2,z0,z1,z2,z3,z5,k1,k2,k3,adj,adjf,chain,
 test,dummy,indi1,indi2,indi3,indi4,indi5,indi6,
 indi7,indi8,eps,roco,indes,inde;
#
# grou:  group, Ran:  rank, vec:  highest weight
# rep:  weights/roots, Repdimen:  Dim.  of the rep.
# pco/mco:  pos./neg.  root generators (entries)
#
initialize(grou,Ran,'Cart','highest','Dimen','car'):
# test if adjoint
adj:=1:
for k1 from 1 to Ran do
 if vec[1,k1]<>highest[grou][1,k1] then
  adj:=0:
 fi:
od:
# test if basic
if (grou=6 and sum(vec[1,ka],ka=1..4)=1
 and vec[1,4]=1) or (grou=3 and
 sum(vec[1,ka],ka=1..Ran)=1 and vec[1,2]=1) then
 adjf:=1:adj:=1:
else
 adjf:=0:
fi:
# initial values:
rep:=array(1..2):
rep[1][1]:=vec:
pco:=array(1..Dimen[grou]):
mco:=array(1..Dimen[grou]):
z1:=1:
z5:=array(1..Ran,sparse):
chain[1]:=array(1..1000,1..Ran,sparse):
chain[2]:=array(1..1000,1..Ran,sparse):
indi1:=0:
# FIRST RUN: produce simple root generators
# SECOND RUN: produce remaining generators
for k2 from 1 to 2 do
 # initial values for the second run
 if k2=2 then
  adj:=1:
  indi1:=0:
  z1:=Ran:
  for k1 from 1 to z1 do
   rep[2][k1]:=evalm(-car[grou][k1]):
  od:
 fi:
 # Freudenthal's indices:  initial values
 if (grou=5 and Ran>5 and Ran<9) or grou=7 then
  indes:=array(1..2):  inde:=array(1..2):
  for k1 from 1 to Ran do
   freuden(Ran,grou,-car[grou][k1],indes[k2][k1]):
  od:
  freuden(Ran,grou,rep[k2][1],inde[k2][1]):
  if k2=2 then
   for k1 from 1 to Ran do
    freuden(Ran,grou,rep[k2][k1],inde[k2][k1]):
   od:
  fi:
 fi:
 # begin
 for z0 from 1 to 2000 while indi1=0 do
  # run as long as new weights are produced
  indi1:=1:
  # if there's more weights to look at, run further
  if z1>z0 then indi1:=0:fi:
  # check for positive entries
  for z2 from 1 to Ran do
   # if positive entry exists and
   # weight is not in a chain already
   # cared for (see below) proceed
   if rep[k2][z0][1,z2]>0 and chain[k2][z0,z2]=0 then
    # new chain starts if entry is larger than 1
    if rep[k2][z0][1,z2]>1 then
     chain[k2][z0,z2]:=rep[k2][z0][1,z2]:
    fi:
    # produce chain/ new weight
    test:=evalm(rep[k2][z0]):
    dummy:=z0:
    for k3 from 1 to rep[k2][z0][1,z2] do
     indi1:=0:
     test:=evalm(test-car[grou][z2]):
     # check if zero
     indi2:=0:
     for k1 from 1 to Ran do
      if test[1,k1]<>0 then indi2:=1:fi:
     od:
     # check if the new weight already exists
     indi3:=0:
     indi4:=0:
     for z3 from 1 to z1 do
      indi5:=0:
      for k1 from 1 to Ran do
       if rep[k2][z3][1,k1]<>test[1,k1] then
        indi5:=1:
       fi:
      od:
      if indi5=0 then
       # YES, EXISTS
       indi3:=1:
       # check if entry in generatormatrix
       # has been made, if not, do a new one
       indi6:=0:
       for k1 from 1 to z5[z2] do
        if k2=1 and pco[z2][k1,1]=z0
         and pco[z2][k1,2]=z3 then
         indi6:=1:
        fi:
       od:
       # if no entry yet and not zero weight
       # in adjoint rep or basic rep
       if indi6=0 and not(indi2=0 and adj=1)
        and not(indi2=0 and adjf=1) then
        if k2=1 then
         # Kacnorm
         eps:=1:
         indi8:=0:
         for k1 from 1 to Ran do
          if rep[k2][dummy][1,k1]<>0 then indi8:=1:fi:
         od:
         if adj=1 and indi8=1 then
          roco:=
           evalm(rep[k2][dummy]&*inverse(Cart[grou])):
          indi7:=0:
          for k1 from 1 to Ran do
           if roco[1,k1]<0 then indi7:=1:fi:
          od:
          if indi7=1 then
           eps:=-Kacnorm(
            car[grou][z2],-rep[k2][dummy],Ran,grou,Cart):
          fi:
          if indi7=0 then
           eps:=Kacnorm(
            car[grou][z2],rep[k2][z3],Ran,grou,Cart):
          fi:
         fi:
         # Freudenthal
         if ((grou=5 and Ran>5 and Ran<9) or grou=7)
          and adj=1 then
          eps:=1:
          freuden(Ran,grou,rep[k2][z3],inde[k2][z3]):
          freudeneps(indes[k2][z2],
           inde[k2][dummy],inde[k2][z3],'eps'):
         fi:
         # entries
         z5[z2]:=z5[z2]+1:
         pco[z2][z5[z2],1]:=dummy:
         pco[z2][z5[z2],2]:=z3:
         pco[z2][z5[z2],3]:=
          eps*(rep[k2][z0][1,z2]+1-k3):
         mco[z2][z5[z2],1]:=z3:
         mco[z2][z5[z2],2]:=dummy:
         mco[z2][z5[z2],3]:=eps*k3:
        fi:
        dummy:=z3:
       fi:
       # if zero weight in adjoint
       if indi2=0 and adj=1 and indi4=0 and k2=1 then
        a1:=Ran:a2:=0:
        if grou=6 and adjf=1 then a1:=2:a2:=2:fi:
        if grou=3 and adjf=1 then a1:=Ran-1:a2:=0:fi:
        for k1 from 1 to a1 do
         # Check CSA
         if Cart[grou][z2,k1+a2]=2 then
          z5[z2]:=z5[z2]+1:
          pco[z2][z5[z2],1]:=z0:
          pco[z2][z5[z2],2]:=z3+k1-1:
          pco[z2][z5[z2],3]:=-2:
          mco[z2][z5[z2],1]:=z3+k1-1:
          mco[z2][z5[z2],2]:=z0:
          mco[z2][z5[z2],3]:=-1:
          dummy:=z3+k1-1:
         fi:
         if Cart[grou][z2,k1+a2]<>2
          and Cart[grou][z2,k1+a2]<>0 then
          z5[z2]:=z5[z2]+1:
          pco[z2][z5[z2],1]:=z0:
          pco[z2][z5[z2],2]:=z3+k1-1:
```

Table A.5: Maple Code: Generation of Representation Matrices

```
      pco[z2][z5[z2],3]:=-Cart[grou][z2,k1+a2]:
      mco[z2][z5[z2],1]:=z1+1:
      mco[z2][z5[z2],2]:=z3+k1-1:
      mco[z2][z5[z2],3]:=Cart[grou][z2,k1+a2]:
     fi:
    od:
    # set indicator to cope with degeneracy
    # (do the above only once for zero root)
    indi4:=1:
   fi:
  fi:
 od:
 # NO, DOES NOT EXIST YET
 # if zero weight in basic/adjoint rep:
 # create all zero weights
 if (indi2=0 and indi3=0 and adj=1 and k2=1) then
  indi3:=1:
  a1:=Ran:a2:=0:
  if grou=6 and adjf=1 then a1:=2:a2:=2:fi:
  if grou=3 and adjf=1 then a1:=Ran-1:a2:=0:fi:
  for k1 from 1 to a1 do
   rep[k2][z1+k1]:=evalm(test):
   if Cart[grou][z2,k1+a2]=2 then
    z5[z2]:=z5[z2]+1:
    pco[z2][z5[z2],1]:=z0:
    pco[z2][z5[z2],2]:=z1+k1:
    pco[z2][z5[z2],3]:=-2:
    mco[z2][z5[z2],1]:=z1+k1:
    mco[z2][z5[z2],2]:=z0:
    mco[z2][z5[z2],3]:=-1:
    dummy:=z1+k1:
   fi:
   if Cart[grou][z2,k1+a2]<>2
    and Cart[grou][z2,k1+a2]<>0 then
    z5[z2]:=z5[z2]+1:
    pco[z2][z5[z2],1]:=z0:
    pco[z2][z5[z2],2]:=z1+k1:
    pco[z2][z5[z2],3]:=-Cart[grou][z2,k1+a2]:
    mco[z2][z5[z2],1]:=z1+a1+1:
    mco[z2][z5[z2],2]:=z1+k1:
    mco[z2][z5[z2],3]:=Cart[grou][z2,k1+a2]:
   fi:
  od:
  z1:=z1+a1:
 fi:
 # if nontrivial weight
 if indi3=0 then
  indi1:=0:
  z1:=z1+1:
  rep[k2][z1]:=evalm(test):
  # check if member of a chain
  if rep[k2][z0][1,z2]>1 then
   chain[k2][z1,z2]:=rep[k2][z0][1,z2]-k3:
  fi:
  # Kacnorm
  eps:=1:
  indi8:=0:
  for k1 from 1 to Ran do
   if rep[k2][dummy][1,k1]<>0 then
    indi8:=1:
   fi:
  od:
  if adj=1 and indi8=1 then
   roco:=evalm(rep[k2][dummy]&*
    inverse(Cart[grou])):
   indi7:=0:
   for k1 from 1 to Ran do
    if roco[1,k1]<0 then indi7:=1:fi:
   od:
   if indi7=1 then
    eps:=-Kacnorm(
     car[grou][z2],-rep[k2][dummy],Ran,grou,Cart):
   fi:
   if indi7=0 then
    eps:=Kacnorm(
       car[grou][z2],rep[k2][z1],Ran,grou,Cart):
     fi:
    fi:
    # Freudenthal
    if ((grou=5 and Ran>5 and Ran<9) or grou=7)
     and adj=1 then
     eps:=1:
     freuden(Ran,grou,rep[k2][z1],inde[k2][z1]):
     freudeneps(indes[k2][z2],
      inde[k2][dummy],inde[k2][z1],'eps'):
    fi:
    # entries
    if k2=1 then
     z5[z2]:=z5[z2]+1:
     pco[z2][z5[z2],1]:=dummy:
     pco[z2][z5[z2],2]:=z1:
     pco[z2][z5[z2],3]:=
      eps*(rep[k2][z0][1,z2]+1-k3):
     mco[z2][z5[z2],1]:=z1:
     mco[z2][z5[z2],2]:=dummy:
     mco[z2][z5[z2],3]:=eps*k3:
    fi:
    # second run:  produce generator
    if k2=2 then
     pco[z1]:=
      scalmult(commu(pco[z2],pco[dummy]),-1/k3*eps):
     mco[z1]:=
      scalmult(commu(mco[z2],mco[dummy]),1/k3*eps):
    fi:
    dummy:=z1:
   fi:
  od:
 fi:
od:
od:
for k1 from 1 to Ran do
 pco[k1][1,4]:=z5[k1]:mco[k1][1,4]:=z5[k1]:
od:
if k2=1 then
 print(`Weights:`);
 for k1 from 1 to z1 do
  print(k1, rep[k2][k1]);
 od;
 print(`Number of entries for
  simple root generator k`);
 for k1 from 1 to Ran do print(k1,z5[k1]); od;
 Repdimen:=z1:
fi:
if k2=2 then
 print(`positive root generators:`);
 for k1 from 1 to z1 do
  rep[k2][k1]:=evalm(rep[k2][k1]*(-1)):
  if grou<>5 and grou<>7 then
   print(`generator`,k1,`rootcoordinates`,
   evalm(rep[k2][k1]&*inverse(Cart[grou])));
  fi;
  if (grou=5 and Ran>5 and Ran<9) or grou=7 then
   if inde[2][k1][4]=2 then
    test:=seq(inde[2][k1][3-k3],k3=1..2):
   else
    test:=seq(inde[2][k1][k3],k3=1..inde[2][k1][4]):
   fi:
    print(`generator`,k1,`rootcoordinates`,
     evalm(rep[k2][k1]&*inverse(Cart[grou])),
     `-`,`epsilon`,test);
   else
    print(`generator`,k1,`rootcoordinates`,
     evalm(rep[k2][k1]&*inverse(Cart[grou])),
     `epsilon`,test);
   fi;
  fi;
 od:
fi:
od:
end:
```

Table A.5: `Maple` Code: Generation of Representation Matrices

Lie Algebra Check

```
testCartan:=proc(pco,mco,rep,grou,Ran)
local Le,k1,k2,k3,rest,roco,h,test;
initialize(grou,Ran,'Cart','highest','Dimen','car'):
# relative lengths of roots
for k1 from 1 to Ran do
 for k2 from 1 to Ran do
  for k3 from 1 to 7 do
   Le[k3][k1,k2]:=1:
  od:
 od:
od:
for k1 from 1 to Ran-1 do
 Le[2][k1,Ran]:=2:
 Le[2][Ran,k1]:=1/2:
 Le[3][k1,Ran]:=1/2:
 Le[3][Ran,k1]:=2:
od:
Le[6][1,2]:=1:Le[6][2,1]:=1:
Le[6][1,3]:=2:Le[6][3,1]:=1/2:
Le[6][1,4]:=2:Le[6][4,1]:=1/2:
Le[6][2,3]:=2:Le[6][3,2]:=1/2:
Le[6][2,4]:=2:Le[6][4,2]:=1/2:
Le[6][3,4]:=1:Le[6][4,3]:=1:
Le[7][1,2]:=3:Le[7][2,1]:=1/3:
# Calculate CSA
for k1 from 1 to Ran do
 h[k1]:=commu(pco[k1],mco[k1]):
od:
# Test Cartan Matrix
print(`Test [H_alpha_i, E_alpha_j]=A_ji E_alpha_j`);
for k1 from 1 to Ran do
 for k2 from 1 to Ran do
  rest:=addi(commu(h[k1],pco[k2]),
   scalmult(pco[k2],-Cart[grou][k2,k1])):
  if rest[1,4]=0 then print(k1,k2,ok); fi;
  if rest[1,4]<>0 then print(k1,k2,ups); fi;
 od;
od;
print(`Test [H_alpha_i, E_-alpha_j]=A_ji E_-alpha_j`);
for k1 from 1 to Ran do
 for k2 from 1 to Ran do
  rest:=addi(commu(h[k1],mco[k2]),
   scalmult(mco[k2],Cart[grou][k2,k1])):
  if rest[1,4]=0 then print(k1,k2,ok); fi;
  if rest[1,4]<>0 then print(k1,k2,ups); fi;
 od;
od;
# Test normalizations
# [E_alpha,F_alpha]=H_alpha for positive roots
# Calculate H_alpha
print(`Test [E_alpha,F_alpha]=H_alpha
 for positive roots`);
for k1 from Ran+1 to (Dimen[grou]-Ran)/2 do
 roco:=evalm(rep[2][k1]&*inverse(Cart[grou])):
 h[k1]:=scalmult(h[1],roco[1,1]*
  (sum((roco[1,e1])^2*Le[grou][e1,1],e1=1..Ran)+
  sum(sum(roco[1,e1]*roco[1,e2]*Cart[grou][e2,e1]*
  Le[grou][e1,1],e2=e1+1..Ran),
  e1=1..Ran))^(-1)):
 for k2 from 2 to Ran do
  h[k1]:=addi(scalmult(h[k2],roco[1,k2]*
   (sum((roco[1,e1])^2*Le[grou][e1,k2],e1=1..Ran)+
   sum(sum(roco[1,e1]*roco[1,e2]*Cart[grou][e2,e1]*
   Le[grou][e1,k2],e2=e1+1..Ran),
   e1=1..Ran))^(-1)),h[k1]):
 od:
od:
# Check with [E_alpha,F_alpha]
for k1 from Ran+1 to (Dimen[grou]-Ran)/2 do
 rest:=addi(h[k1],commu(mco[k1],pco[k1])):
 if rest[1,4]=0 then print(k1, `ok`):fi:
 if rest[1,4]<>0 then
  print(k1, `ups`,rest[1,4]):
 fi:
od:
# Check [E_alpha_i,F_alpha_j] for simple roots
print(`Check [E_alpha_i,F_alpha_j]=0 for simple roots,
 i not equal to j`):
for k1 from 1 to Ran do
 for k2 from 1 to Ran do
  if k1<>k2 then
   rest:=commu(pco[k1],mco[k1]):
   if rest[1,4]=0 then print(k1,k2,`ok`):fi:
   if rest[1,4]<>0 then print(k1,k2,`ok`):fi:
  fi:
 od:
od:
# Check Serre relations
print(`Test Serre relations
 (adj[E_alpha_i]^(-A_ji+1))(E_alpha_j)=0`);
for k1 from 1 to Ran do
 for k2 from k1+1 to Ran do
  test:=commu(pco[k2],pco[k1]):
  for k3 from 1 to -Cart[grou][k1,k2] do
   test:=commu(pco[k2],test):
  od:
  if test[1,4]=0 then print(k2,k1,ok):fi:
  if test[1,4]<>0 then print(k2,k1,ups):fi:
 od:
od:
for k2 from 1 to Ran do
 for k1 from k2+1 to Ran do
  test:=commu(pco[k2],pco[k1]):
  for k3 from 1 to -Cart[grou][k1,k2] do
   test:=commu(pco[k2],test):
  od:
  if test[1,4]=0 then print(k2,k1,ok):fi:
  if test[1,4]<>0 then print(k2,k1,ups):fi:
 od:
od:
end:
```

Table A.6: Maple Code: Lie Algebra Check

Tensor Products

```
tens:=proc(ma1,d1,ma2,d2)
local z1,z2,z3,z4,tens1,tens2,tens3;
z3:=0:
for z1 from 1 to ma1[1,4] do
 for z2 from 1 to d2 do
  z3:=z3+1:
  tens1[z3,1]:=z2+(ma1[z1,1]-1)*d2:
  tens1[z3,2]:=z2+(ma1[z1,2]-1)*d2:
  tens1[z3,3]:=ma1[z1,3]:
 od:
od:
tens1[1,4]:=z3:
z4:=0:
for z1 from 1 to ma2[1,4] do
 for z2 from 1 to d1 do
  z4:=z4+1:
  tens2[z4,1]:=ma2[z1,1]+(z2-1)*d2:
  tens2[z4,2]:=ma2[z1,2]+(z2-1)*d2:
  tens2[z4,3]:=ma2[z1,3]:
 od:
od:
tens2[1,4]:=z4:
tens3:=addi(tens1,tens2):
tens3:
end:
```

Table A.7: Maple Code: Tensor Products

A.3 Examples

In this section, some simple examples to show how the above procedures can be applied are presented.

A.3.1 $\mathfrak{a}_1$: 4

The first example is given in table A.8. The **4** of $\mathfrak{sl}_2$ is generated by the procedure `highweightrep`. The representation matrices, known from the main part of this thesis, are then printed.

```
> highweightrep(1,1,([[3]]),ro,'d',pco,mco);
```

Cartan matrix, [2], highest root, [2], Dimension, 3
Weights:
1, [3]
2, [1]
3, [-1]
4, [-3]
Number of entries for simple root generator k
1, 3
positive root generators:
generator, 1, rootcoordinates, [1]

```
> print(matri(pco[1],d),matri(mco[1],d),
    matri(commu(pco[1],mco[1]),d));
```

$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Table A.8: Example:**4** of $\mathfrak{a}_1$

A.3.2 $\mathfrak{g}_2$: 14

A more complicated example is the **14** adjoint representation of $\mathfrak{g}_2$, involving $\mathfrak{a}_1$ modules of dimension 4. It is generated by calling `highweightrep`. The normalization has been chosen to correspond to section 2.4.2, refe8freud.

The correctness of the representation matrices is tested by calling `testCartan`. The corresponding output is given in table A.9.

The whole algebra is then given in one matrix by

$$D(\mathfrak{g}_2) = \sum_i H_i D(H_{\alpha_i}) + \sum_i P_i D(E_{\beta_i}) + \sum_i M_i D(E_{-\beta_i}), \quad \alpha_i \in \Pi^{\mathfrak{g}_2}, \beta_i \in \Delta_+^{\mathfrak{g}_2}$$

where the roots β_i correspond to the list given by `highweightrep`. The $D(H_{\alpha_i})$, $D(E_{\beta_i})$, $D(E_{\beta_i})$ denote the representation matrices, and the parameters H_i, P_i, M_i can be chosen to be in $\mathbb{C}$ or $\mathbb{R}$ depending on the real form needed (the latter choice corresponds to the normal form $\mathfrak{g}_{2(+2)}$). The result is given in table A.10.

```
> highweightrep(7,2,([[1,0]]),ro,'d',pco,mco);
```

Cartan matrix, $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ *, highest root, [1 0], Dimension, 14*

Weights:
1, [1 0]
2, [-1 3]
3, [0 1]
4, [1 -1]
5, [2 -3]
6, [-1 2]
7, [0 0]
8, [0 0]
9, [-2 3]
10, [1 -2]
11, [-1 1]
12, [0 -1]
13, [1 -3]
14, [-1 0]
Number of entries for simple root generator k
1, 7
2, 9
positive root generators:
generator, 1, rootcoordinates, [1 0], epsilon, 1, 2
generator, 2, rootcoordinates, [0 1], epsilon, 2
generator, 3, rootcoordinates, [1 1], epsilon, 1
generator, 4, rootcoordinates, [1 2], -, epsilon, 3
generator, 5, rootcoordinates, [1 3], epsilon, 2, 3
generator, 6, rootcoordinates, [2 3], epsilon, 1, 3

```
> testCartan(pco,mco,ro,7,2);
```

Cartan matrix, $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ *, highest root, [1 0], Dimension, 14*

Test [h_alpha_i, e_alpha_j]=A_ji e_alpha_j
1, 1, ok
1, 2, ok
2, 1, ok
2, 2, ok
Test [h_alpha_i, f_alpha_j]=-A_ji f_alpha_j
1, 1, ok
1, 2, ok
2, 1, ok
2, 2, ok
Test [e_alpha,f_alpha]=h_alpha for all roots
3, ok
4, ok
5, ok
6, ok
Check [e_alpha_i,f_alpha_j]=0 for simple roots, i not equal to j
1, 2, ok
2, 1, ok
Test Serre relations (adj[e_alpha_i]^(-A_ji+1))(e_alpha_j)=0
2, 1, ok
1, 2, ok

Table A.9: Example:**14** of $\mathfrak{g}_2$

```
> G:=scalmult(pco[1],P[1])
> for k from 2 to 6 do G:=(addi(scalmult(pco[k],P[k]),G)):od:
> for k from 1 to 6 do G:=(addi(scalmult(mco[k],M[k]),G)):od:
> for k from 1 to 2 do
> G:=(addi(scalmult(commu(pco[k],mco[k]),H[k]),G)):od:
> print(matri(G,d));
```

$$\begin{pmatrix}
H_1 & P_1 & 3P_3 & -3P_4 & -P_5 & 0 & -P_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_1 & 3H_2-H_1 & 3P_2 & 0 & 0 & -3P_4 & P_5 & -3P_5 & -P_6 & 0 & 0 & 0 & 0 & 0 \\
M_3 & M_2 & H_2 & 2P_2 & 0 & -2P_3 & 0 & -P_4 & 0 & -P_5 & -P_6 & 0 & 0 & 0 \\
-M_4 & 0 & 2M_2 & -H_2+H_1 & -P_2 & P_1 & -P_3 & P_3 & 0 & -2P_4 & 0 & P_6 & 0 & 0 \\
-M_5 & 0 & 0 & -3M_2 & -3H_2+2H_1 & 0 & -2P_1 & 3P_1 & 0 & 3P_3 & 0 & 0 & P_6 & 0 \\
0 & -M_4 & -2M_3 & M_1 & 0 & 2H_2-H_1 & P_2 & -2P_2 & -P_3 & 0 & 2P_4 & P_5 & 0 & 0 \\
-2M_6 & -M_5 & -3M_4 & -3M_3 & -M_1 & 0 & 0 & 0 & P_1 & 0 & 3P_3 & 3P_4 & P_5 & 2P_6 \\
-M_6 & -M_5 & -2M_4 & -M_3 & 0 & -M_2 & 0 & 0 & 0 & P_2 & P_3 & 2P_4 & P_5 & P_6 \\
0 & -M_6 & 0 & 0 & 0 & -3M_3 & 2M_1 & -3M_1 & 3H_2-2H_1 & 0 & 3P_2 & 0 & 0 & P_5 \\
0 & 0 & -M_5 & -2M_4 & M_3 & 0 & -M_2 & 2M_2 & 0 & -2H_2+H_1 & -P_1 & 2P_3 & P_4 & 0 \\
0 & 0 & -M_6 & 0 & 0 & 2M_4 & M_3 & -M_3 & M_2 & -M_1 & H_2-H_1 & -2P_2 & 0 & P_4 \\
0 & 0 & 0 & M_6 & 0 & M_5 & 0 & M_4 & 0 & 2M_3 & -2M_2 & -H_2 & -P_2 & -P_3 \\
0 & 0 & 0 & 0 & M_6 & 0 & -M_5 & 3M_5 & 0 & 3M_4 & 0 & -3M_2 & -3H_2+H_1 & -P_1 \\
0 & 0 & 0 & 0 & 0 & 0 & M_6 & 0 & M_5 & 0 & 3M_4 & -3M_3 & -M_1 & -H_1
\end{pmatrix}$$

Table A.10: Example:**14** of $\mathfrak{g}_2$ (Representation Matrix)

A.3.3 $\mathfrak{f}_4$: 26

A third example is the **26** of $\mathfrak{f}_4$, the (only) basic representation of this algebra. Again, `highweightrep` is called (see table A.11) and the algebra is given as a whole in one representation matrix with conventions as above (table A.12).

```
> highweightrep(6,4,([[0,0,0,1]]),
    ro,'d',pco,mco);
```

Cartan matrix, $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$,

highest root, [1 0 0 0], Dimension, 52

Weights:
1, [0 0 0 1]
2, [0 0 1 -1]
3, [0 1 -1 0]
4, [1 -1 1 0]
5, [-1 0 1 0]
6, [1 0 -1 1]
7, [-1 1 -1 1]
8, [1 0 0 -1]
9, [0 -1 1 1]
10, [-1 1 0 -1]
11, [0 0 -1 2]
12, [0 -1 2 -1]
13, [0 0 0 0]
14, [0 0 0 0]
15, [0 0 1 -2]
16, [0 1 -2 1]
17, [0 1 -1 -1]
18, [1 -1 0 1]
19, [1 -1 1 -1]
20, [-1 0 0 1]
21, [-1 0 1 -1]
22, [1 0 -1 0]
23, [-1 1 -1 0]
24, [0 -1 1 0]
25, [0 0 -1 1]
26, [0 0 0 -1]
Number of entries for simple root generator k
1, 6
2, 6
3, 11
4, 11
positive root generators:
generator, 1, rootcoordinates, [1 0 0 0]
generator, 2, rootcoordinates, [0 1 0 0]
generator, 3, rootcoordinates, [0 0 1 0]
generator, 4, rootcoordinates, [0 0 0 1]
generator, 5, rootcoordinates, [1 1 0 0]
generator, 6, rootcoordinates, [0 1 1 0]
generator, 7, rootcoordinates, [0 1 2 0]
generator, 8, rootcoordinates, [0 0 1 1]
generator, 9, rootcoordinates, [1 1 1 0]
generator, 10, rootcoordinates, [1 1 2 0]
generator, 11, rootcoordinates, [0 1 1 1]
generator, 12, rootcoordinates, [0 1 2 1]
generator, 13, rootcoordinates, [0 1 2 2]
generator, 14, rootcoordinates, [1 1 1 1]
generator, 15, rootcoordinates, [1 2 2 0]
generator, 16, rootcoordinates, [1 1 2 1]
generator, 17, rootcoordinates, [1 1 2 2]
generator, 18, rootcoordinates, [1 2 2 1]
generator, 19, rootcoordinates, [1 2 2 2]
generator, 20, rootcoordinates, [1 2 3 1]
generator, 21, rootcoordinates, [1 2 3 2]
generator, 22, rootcoordinates, [1 2 4 2]
generator, 23, rootcoordinates, [1 3 4 2]
generator, 24, rootcoordinates, [2 3 4 2]

Table A.11: Example:**26** of $\mathfrak{f}_4$

```
> G:=scalmult(pco[1],P[1]):
> for k from 2 to 24 do G:=(addi(scalmult(pco[k],P[k]),G)):od:
> for k from 1 to 24 do G:=(addi(scalmult(mco[k],M[k]),G)):od:
> for k from 1 to 4 do
> G:=(addi(scalmult(commu(pco[k],mco[k]),H[k]),G)):od:
> print(matri(G,d));
```

$$\begin{pmatrix}
H_4 & P_4 & P_8 & -P_{11} & P_{14} & P_{12} & -P_{16} & -P_{13} & P_{18} & P_{17} & P_{20} & -P_{19} & -2P_{21} & P_{21} & 0 & P_{22} & 0 & -P_{23} & 0 & P_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\
M_4 & -H_4+H_3 & P_3 & -P_6 & P_9 & P_7 & -P_{10} & -P_{12} & P_{15} & P_{16} & 0 & -P_{18} & -P_{20} & -P_{20} & P_{21} & 0 & -P_{22} & 0 & P_{23} & 0 & -P_{24} & 0 & 0 & 0 & 0 & 0 \\
M_8 & M_3 & -H_3+H_2 & -P_2 & P_5 & P_6 & -P_9 & -P_{11} & 0 & P_{14} & -P_{15} & 0 & P_{18} & -2P_{18} & P_{19} & -P_{20} & -P_{21} & 0 & 0 & 0 & 0 & -P_{23} & P_{24} & 0 & 0 & 0 \\
-M_{11} & -M_6 & -M_2 & H_3-H_2+H_1 & -P_1 & P_3 & 0 & -P_8 & -P_9 & 0 & -P_{10} & P_{14} & P_{16} & -2P_{16} & P_{17} & 0 & 0 & P_{20} & P_{21} & 0 & 0 & P_{22} & 0 & -P_{24} & 0 & 0 \\
M_{14} & M_9 & M_5 & -M_1 & H_3-H_1 & 0 & P_3 & 0 & -P_6 & -P_8 & -P_7 & P_{11} & P_{12} & -2P_{12} & P_{13} & 0 & 0 & 0 & 0 & -P_{20} & -P_{21} & 0 & -P_{22} & P_{23} & 0 & 0 \\
M_{12} & M_7 & M_6 & M_3 & 0 & H_4-H_3+H_1 & -P_1 & P_4 & -P_5 & 0 & -P_9 & 0 & -P_{14} & -P_{14} & 0 & P_{16} & P_{17} & P_{18} & P_{19} & 0 & 0 & P_{21} & 0 & 0 & P_{24} & 0 \\
-M_{16} & -M_{10} & -M_9 & 0 & M_3 & -M_1 & H_4-H_3+H_2-H_1 & 0 & -P_2 & P_4 & -P_6 & 0 & -P_{11} & -P_{11} & 0 & P_{12} & P_{13} & 0 & 0 & -P_{18} & -P_{19} & 0 & -P_{21} & 0 & -P_{23} & 0 \\
-M_{13} & -M_{12} & -M_{11} & -M_8 & 0 & M_4 & 0 & -H_4+H_1 & 0 & -P_1 & 0 & -P_5 & -2P_9 & P_9 & -P_{14} & P_{10} & P_{16} & P_{15} & P_{18} & 0 & 0 & P_{20} & 0 & 0 & 0 & -P_{24} \\
M_{18} & M_{15} & 0 & -M_9 & -M_6 & -M_5 & -M_2 & 0 & H_4+H_3-H_2 & 0 & -P_3 & P_4 & -P_8 & -P_8 & 0 & 0 & 0 & -P_{12} & -P_{13} & -P_{16} & -P_{17} & 0 & 0 & P_{21} & P_{22} & 0 \\
M_{17} & M_{16} & M_{14} & 0 & -M_8 & 0 & M_4 & -M_1 & 0 & -H_4+H_2-H_1 & 0 & -P_2 & -2P_6 & P_6 & -P_{11} & P_7 & P_{12} & 0 & 0 & -P_{15} & -P_{18} & 0 & -P_{20} & 0 & 0 & P_{23} \\
M_{20} & 0 & -M_{15} & -M_{10} & -M_7 & -M_9 & -M_6 & 0 & -M_3 & 0 & 2H_4-H_3 & 0 & P_4 & -2P_4 & 0 & P_8 & 0 & P_{11} & 0 & P_{14} & 0 & -P_{13} & -P_{17} & -P_{19} & -P_{21} & 0 \\
-M_{19} & -M_{18} & 0 & M_{14} & M_{11} & 0 & 0 & -M_5 & M_4 & -M_2 & 0 & -H_4+2H_3-H_2 & -2P_3 & P_3 & -P_8 & 0 & 0 & -P_7 & -P_{12} & -P_{10} & -P_{16} & 0 & 0 & P_{20} & 0 & -P_{22} \\
-M_{21} & -M_{20} & 0 & 0 & 0 & -M_{14} & -M_{11} & -M_9 & -M_8 & -M_6 & 0 & -M_3 & 0 & 0 & 0 & P_3 & P_8 & P_6 & P_{11} & P_9 & P_{14} & 0 & 0 & 0 & P_{20} & P_{21} \\
0 & -M_{20} & -M_{18} & -M_{16} & -M_{12} & -M_{14} & -M_{11} & 0 & -M_8 & 0 & -M_4 & 0 & 0 & 0 & P_4 & 0 & P_8 & 0 & P_{11} & 0 & P_{14} & P_{12} & P_{16} & P_{18} & P_{20} & 0 \\
0 & M_{21} & M_{19} & M_{17} & M_{13} & 0 & 0 & -M_{14} & 0 & -M_{11} & 0 & -M_8 & -M_4 & 2M_4 & -2H_4+H_3 & 0 & P_3 & 0 & P_6 & 0 & P_9 & P_7 & P_{10} & P_{15} & 0 & -P_{20} \\
M_{22} & 0 & -M_{20} & 0 & 0 & M_{16} & M_{12} & M_{10} & 0 & M_7 & M_8 & 0 & 2M_3 & -M_3 & 0 & H_4-2H_3+H_2 & -P_4 & P_2 & 0 & P_5 & 0 & -P_{11} & -P_{14} & 0 & P_{18} & P_{19} \\
0 & -M_{22} & -M_{21} & 0 & 0 & M_{17} & M_{13} & M_{16} & 0 & M_{12} & 0 & 0 & M_8 & M_8 & M_3 & -M_4 & -H_4-H_3+H_2 & 0 & P_2 & 0 & P_5 & P_6 & P_9 & 0 & -P_{15} & -P_{18} \\
-M_{23} & 0 & 0 & M_{20} & 0 & M_{18} & 0 & M_{15} & -M_{12} & 0 & M_{11} & -M_7 & 2M_6 & -M_6 & 0 & M_2 & 0 & H_4-H_2+H_1 & -P_4 & P_1 & 0 & P_8 & 0 & -P_{14} & -P_{16} & -P_{17} \\
0 & M_{23} & 0 & M_{21} & 0 & M_{19} & 0 & M_{18} & -M_{13} & 0 & 0 & -M_{12} & M_{11} & M_{11} & M_6 & 0 & M_2 & -M_4 & -H_4+H_3-H_2+H_1 & 0 & P_1 & -P_3 & 0 & P_9 & P_{10} & P_{16} \\
M_{24} & 0 & 0 & 0 & -M_{20} & 0 & -M_{18} & 0 & -M_{16} & -M_{15} & M_{14} & -M_{10} & 2M_9 & -M_9 & 0 & M_5 & 0 & M_1 & 0 & H_4-H_1 & -P_4 & 0 & P_8 & P_{11} & P_{12} & P_{13} \\
0 & -M_{24} & 0 & 0 & -M_{21} & 0 & -M_{19} & 0 & -M_{17} & -M_{18} & 0 & -M_{16} & M_{14} & M_{14} & M_9 & 0 & M_5 & 0 & M_1 & -M_4 & -H_4+H_3-H_1 & 0 & -P_3 & -P_6 & -P_7 & -P_{12} \\
0 & 0 & -M_{23} & M_{22} & 0 & M_{21} & 0 & M_{20} & 0 & 0 & -M_{13} & 0 & -M_{12} & 2M_{12} & M_7 & -M_{11} & M_6 & M_8 & -M_3 & 0 & 0 & -H_3+H_1 & P_1 & -P_5 & -P_9 & -P_{14} \\
0 & 0 & M_{24} & 0 & -M_{22} & 0 & -M_{21} & 0 & 0 & -M_{20} & -M_{17} & 0 & -M_{16} & 2M_{16} & M_{10} & -M_{14} & M_9 & 0 & 0 & M_8 & -M_3 & M_1 & -H_3+H_2-H_1 & P_2 & P_6 & P_{11} \\
0 & 0 & 0 & -M_{24} & M_{23} & 0 & 0 & 0 & M_{21} & 0 & -M_{19} & M_{20} & -M_{18} & 2M_{18} & M_{15} & 0 & 0 & -M_{14} & M_9 & M_{11} & -M_6 & -M_5 & M_2 & H_3-H_2 & -P_3 & -P_8 \\
0 & 0 & 0 & 0 & 0 & M_{24} & -M_{23} & 0 & M_{22} & 0 & -M_{21} & 0 & M_{20} & M_{20} & 0 & M_{18} & -M_{15} & -M_{16} & M_{10} & M_{12} & -M_7 & -M_9 & M_6 & -M_3 & H_4-H_3 & -P_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -M_{24} & 0 & M_{23} & 0 & -M_{22} & 2M_{21} & -M_{21} & -M_{20} & M_{19} & -M_{18} & -M_{17} & M_{16} & M_{13} & -M_{12} & -M_{14} & M_{11} & -M_8 & -M_4 & -H_4
\end{pmatrix}$$

Table A.12: Example:**26** of $\mathfrak{f}_4$ (Representation matrix)

$$D(\mathfrak{f}_4) = \sum_i H_i D(H_{\alpha_i}) + \sum_i P_i D(E_{\beta_i}) + \sum_i M_i D(E_{-\beta_i}), \quad \alpha_i \in \Pi^{\mathfrak{f}_4}, \beta_i \in \Delta_+^{\mathfrak{f}_4}, \;\; H_i, P_i, M_i \in \mathbb{C}$$

A.3.4 Tensor Products

Finally, a simple example shall be given for the `tens` procedure introduced above. For $\mathfrak{sl}_2$, consider the tensor product $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$. The corresponding output is given in table A.13.

The base vectors for the two zero weights of the tensor product module **1 +3** need to be rotated in order to see the singlet directly. For general tensor products, to separate the irreducible blocks, the representation space basis will need to be transformed correspondingly. If the aim is to find a specific irreducible module by constructing it in a tensor product, finding the transformation to get block diagonal structure of the representation matrix might be nontrivial, and in general it might be easier to consider closely the basis of this module and to construct it directly.

```
> highweightrep(1,1,([[1]]),ro,'d',pco,mco):
> G1:=scalmult(pco[1],P):
> G1:=(addi(scalmult(mco[1],M),G1)):
> G1:=(addi(scalmult(commu(pco[1],mco[1]),H),G1)):
> print(matri(G1,d));
```

$$\begin{pmatrix} H & P \\ M & -H \end{pmatrix}$$

```
> x:=matri(tens(G1,2,G1,2),4):print(x);
```

$$\begin{pmatrix} H & P & P & 0 \\ M & 0 & 0 & P \\ M & 0 & 0 & P \\ 0 & M & M & -2H \end{pmatrix}$$

```
> y:=array(1..4,1..4,[[1,0,0,0],[0,1,0,-1],[0,1,0,1],[0,0,1,0]]):
> evalm(inverse(y)&*x&*y);
```

$$\begin{pmatrix} 2H & 2P & 0 & 0 \\ M & 0 & P & 0 \\ 0 & 2M & -2H & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Table A.13: Example:$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$ for $\mathfrak{a}_1$

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