Low-energy Supergravities from Heterotic Compactification on Reduced Structure Backgrounds

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Abstract

In this thesis, the compactification of heterotic supergravity on six-dimensional manifolds with SU(2) and SU(3) structure is studied. For the SU(2)-structure backgrounds, the spectrum and the bosonic action of the effective theory in four dimensions are obtained. The results are gauged versions of the ungauged $\mathscr{N} = 2$ supergravity obtained after compactification on $\mathrm{K3} \times T^2$. The gauge algebra and the Killing prepotentials are also computed. For the SU(3)-structure backgrounds, the couplings of the resulting $\mathscr{N} = 1$ supergravity are computed by reducing terms in the heterotic supergravity action involving fermionic fields, and are further checked by computing the supersymmetry variations of the fermions.

Zussamenfassung

In dieser Dissertation wird die Kompaktifizierung der heterotischen Supergravitation auf sechsdimensionalen Mannigfaltigkeiten mit SU(2)- und SU(3)-Struktur untersucht. Für die SU(2)-Struktur-Hintergründe erhalten wir das Spektrum und die bosonische Wirkung der effektiven Theorie. Die Ergebnisse sind geeichte Versionen von der ungeeichten $\mathscr{N} = 2$-Supergravitation, die man aus der Kompaktifizierung auf $\mathrm{K3} \times T^2$ erhält. Die Eichalgebra und die Killing-Präpotentiale werden auch berechnet. Für die SU(3)-Struktur-Hintergründe werden die Kopplungen der resultierenden $\mathscr{N} = 1$-Supergravitation aus der Reduktion fermionischer Terme in der Wirkung der heterotischen Supergravitation berechnet. Diese werden durch die Berechnung der Supersymmetrie-Variationen der Fermionen verifiziert.

A mi madre y a Félix.

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Contents

1 Introduction **7**

1.1 Fluxes and torsion . . . 10

1.2 Outline of the thesis . . . 11

1.3 Brief comment on notation . . . 12

2 Heterotic compactification on $\mathrm{K3} \times T^2$ **15**

2.1 Heterotic supergravity . . . 15

2.2 The product manifold $\mathrm{K3} \times T^2$ . . . 17

2.2.1 Geometric moduli of $\mathrm{K3} \times T^2$ . . . 19

2.3 Compactification on $\mathrm{K3} \times T^2$ . . . 23

2.3.1 Effective theory . . . 25

3 Heterotic on SU(2)**-structure backgrounds** **37**

3.1 Manifolds with SU(2) structure . . . 37

3.1.1 Geometric moduli space of SU(2)-structures . . . 40

3.1.2 Ansatz for the SU(2)-structure backgrounds . . . 43

3.2 Compactification on K3 fibration over a torus . . . 46

3.2.1 Effective theory . . . 48

3.3 Compactification on 'K3 fibration over twisted torus' . . . 52

3.3.1 Effective theory . . . 54

3.3.2 General case . . . 60

3.4 Gauge algebra and Killing prepotentials . . . 61

4 Heterotic on SU(3)-structure backgrounds: fermionic approach **65**

4.1 Manifolds with SU(3) structure 65

4.2 Compactification on SU(3)-structure backgrounds 67

4.2.1 Bosonic sector 68

4.2.2 Fermionic spectrum and kinetic terms 71

4.2.3 Gravitino mass term and F-terms 74

4.2.4 D-terms 79

4.3 Supersymmetry transformations 80

5 Conclusions **85**

A $\mathscr{N} = 2$ **and** $\mathscr{N} = 1$ **supergravity theories in four dimensions** **87**

B Line element in the space of $\mathcal{Y}_4$ **metrics** **91**

C Almost product structures **95**

Chapter 1

Introduction

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ

Reducing all diverse, perceptible phenomena to one unique, fundamental principle has been a paradigm of Western reasoning –an obsession even– since at least the time of the Greeks. From the atomistic ideas of Leucippus and Democritus in around 400 BC to the Standard Model of Particle Physics in the 20th century, the principle known as Occam's razor or *lex parsimoniae* is perfectly recognizable as a most successful 'prejudice' underlying the quest for an explanation of the physical world.

The Standard Model of Particle Physics is undoubtedly a major achievement in attaining that paradigm. It describes the strong interaction of quarks and the electroweak interaction of leptons and quarks according to the single scheme of Yang-Mills or non-Abelian gauge theory. Its predictions have been tested to astonishing accuracy in numerous experiments up to the TeV scale. However, in spite of its tremendous success there are also reasons why the Standard Model can not be the ultimate story. First of all, it comes with an uncomfortably large number of free parameters in the form of Yukawa couplings, mixing angles, parameters of the Higgs potential and vacuum expectation value of the Higgs field. Not to mention the existence of hierarchy and naturalness problems concerning the values of these parameters! In an attempt to solve some of these problems, supersymmetry has been invoked. It is a symmetry relating bosons and fermions and it roughly doubles the matter content of the Standard Model in its minimal version [1]. Nevertheless, simply adding superpartners to the particle content of the Standard Model still leaves us with the plethora of free parameters.

But more importantly, gravity is absent in the Standard Model. The gravitational force is successfully described on large scales by the General Theory of Relativity, but when quantum effects are expected to play a role, as happens for example when trying to understand black holes or the Big Bang itself, amending this classical theory is inevitable [2]. Reconciling though the assumption made by this theory of a dynamical but otherwise smooth spacetime with the uncertainty principle of Quantum Mechanics at sub-Planckian scales has proven a very difficult puzzle. More technically stated: in quantizing gravity it is very difficult to avoid divergences, and a renormalizable quantum field theory based on Einsten's gravity theory seems to be very difficult to write down.*

String Theory [3–5] claims that the physical world is composed not of point-like particles but of tiny vibrating strings. Being unidimensional in space, these strings possess a richer structure than zero-dimensional objects, and one can assume that all the elementary particles known as such so far (leptons, quarks and bosons mediating interactions) are but this unique type of string vibrating in different fashions. In other words, particle flavor is traded for vibration mode of a single object: the string. Certainly an idea of which the aforementioned Democritus would have been delighted to get to know.

Although historically its first motivation was a different one, String Theory has proven to be able to deal in a very clean, often miraculous way with the puzzle of quantum gravity. With a minimum of assumptions, it manages to give us gauge interactions and gravity in a very natural, unified way and almost for free. There are indeed spin-2 massless excitations of the closed string that one can not fail to identify with the graviton. The nonlocal nature of string interactions (they interact by joining and splitting) intuitively explains how the short-distance singularities of point-particle interactions can be avoided. Moreover, String Theory natively incorporates supersymmetry, since the only known consistent string theories happen to be supersymmetric. The bosonic massless excitations for each of these theories consist universally of the graviton, a scalar named dilaton and the Neveu-Schwarz two-form, and additionally there are some number of antisymmetric tensors or p-form fields depending on the theory. The dynamics of these fields is described by a supergravity theory, i.e. a field theory with local supersymmetry. Another expression of its 'economy' is that String Theory contains only one free parameter, namely the string length or its inverse, the string tension, since the string coupling constant is fixed by the vacuum expectation value of the dilaton field.

However, our understanding of String Theory is far from being complete. The five consistent superstring theories (Type I, Type IIA, Type IIB and Heterotic with gauge group SO(32) or $E_8 \times E_8$) are known to be interrelated by dualities, some of which are nonperturbative in nature (see Figure 1.1). Since a nonperturbative control over String Theory is missing, those dualities remain conjectural, though they have proven to apply on the few situations where calculations are under control. Nowadays it is believed that all five superstring theories are like tips of a big iceberg called M-theory [6]. The latter is largely unknown, though at low energies it becomes eleven-dimensional supergravity coupled to a three-form field.

It turns out that the superstring moves consistently on a flat spacetime background only in ten dimensions and not in the four of the observable world. One possible way out of this dilemma is to take the six dimensions in excess to be curled up into a com-

*This would change if the hope for finiteness of $\mathscr{N} = 8$ supergravity in four dimensions proves right.

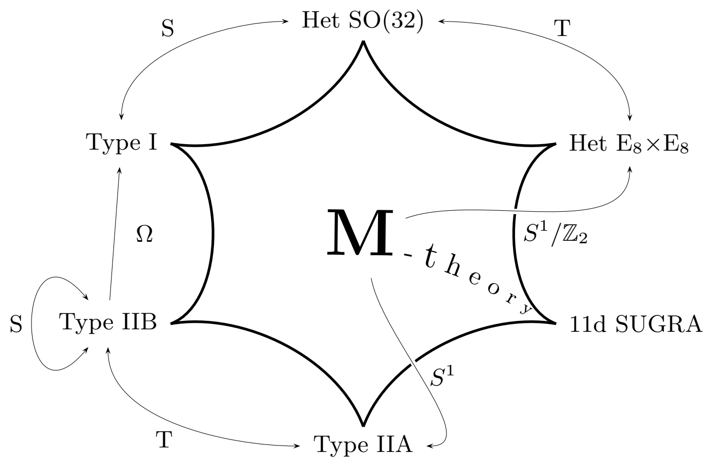

Figure 1.1: Web of superstring theories. Here Ω is an orientifold projection, T is perturbative T-duality, S is nonperturbative strong/weak duality and S^1 ($S^1/\mathbb{Z}_2$) denotes compactification on S^1 ($S^1/\mathbb{Z}_2$).

pact six-dimensional space $\mathcal{Y}$ *a là* Kaluza-Klein [7, 8]. In other words, the theory must be compactified to four dimensions. Demanding that at least some of the supersymmetry of the superstring survives the dimensional reduction imposes constraints on $\mathcal{Y}$. Considering backgrounds where all the p-form fields are set to zero, and requiring that the vacuum be supersymmetric, leads to the existence on $\mathcal{Y}$ of a global spinor that is covariantly constant with respect to the Levi-Civita connection. This preserves the minimal amount of supersymmetry in four-dimensions and is equivalent to choosing $\mathcal{Y}$ as a manifold with SU(3) holonomy or Calabi-Yau space [9–11]. The compactification ansatz then consists of an expansion of the ten-dimensional fields in harmonic forms on $\mathcal{Y}$.

The compactification of the low energy supergravity corresponding to, say, the heterotic string on a Calabi-Yau leads to an $\mathscr{N} = 1$ supergravity in four dimensions [9, 12]. If even more supersymmetry is to be preserved, one must increase the number of global covariantly constant spinors on $\mathcal{Y}$. The existence of two of these spinors imposes stronger restrictions on $\mathcal{Y}$. In this case, the manifold must have SU(2) holonomy and can only be the product manifold K3 $\times T^2$ [13]. As a result, an $\mathscr{N} = 2$ supergravity in four dimensions is obtained (see [14, 15] and references therein).

Calabi-Yau compactifications come though with a serious drawback: the moduli problem. The parameters or moduli defining, for example, the size and shape of the Calabi-Yau may vary from point to point in four-dimensional spacetime. They therefore appear as scalar fields in the effective four-dimensional theory. But there is no potential in the effective action for these fields and their values remain undetermined. More importantly, a flat potential corresponds to massless scalars or moduli, and these typically have a measurable effect on the gravitational force. No such effect has been detected. A flat potential also spoils the predictive power of the theory. Something must therefore be done to generate a potential that stabilizes these fields at some specific values. There are several mechanisms that can be implemented to give masses to the moduli. Some of them are nonperturbative, either at string level like instanton corrections or at four-dimensional level like gaugino condensation, but here a different sort of mechanism is analyzed.

1.1 Fluxes and torsion

One possibility that has been devised to overcome the moduli problem is to turn on p-form fluxes on the internal manifold [16–19]. In all string theories, there is at least one of these p-forms, namely the Neveu-Schwarz two-form. In Type II theories there are additionally the Ramond-Ramond p-forms, with odd p for IIA and even p for IIB. In the heterotic string one has a one-form or gauge field in the adjoint of either SO(32) or $E_8 \times E_8$. The idea is intuitively simple, and it is to consider a background where the field strengths for these p-form fields take a nonzero value giving a net contribution when integrated over $(p+1)$-cycles of the internal manifold. Of course, there are restrictions on the possible values of these fluxes. In particular, fluxes contribute a positive energy that must be compensated by introducing negative-tension sources like orientifold planes in Type II theories. Enough freedom is nevertheless left as to render the approach very fruitful. The energy contained in the flux certainly depends on the size and shape of the internal manifold and therefore the fluxes can in principle stabilize the moduli by generating a potential.

The conditions for a supersymmetric vacuum are modified in the presence of backgrounds fluxes. In this case, the global spinors on the internal manifold must be covariantly constant with respect to a connection that has a nonvanishing torsion [20]. Physically, the torsion is the backreaction of the geometry to the presence of fluxes. This leads to the idea of relaxing the special holonomy condition on $\mathcal{Y}$ and demanding no more than the existence of global nowhere-vanishing spinors. The existence of these spinors imposes restrictions on the possible manifolds $\mathcal{Y}$. It can be shown that it implies a reduction of the structure group of the manifold. For a generic six-dimensional manifold, the bundle of all possible orthonormal frames has structure group SO(6), since these are the transformations that preserve orthonormality. If a subbundle thereof can be constructed having the same fiber, but with transition functions taking values on a subgroup $G \subset \mathrm{SO}(6)$, one is in presence of reduced-structure or G-structure manifolds [21–23]. Special-holonomy manifolds or Calabi-Yau spaces are then a particular case of reduced-structure manifolds where the torsion happens to vanish. Compactifications of the heterotic string on these so-called generalized Calabi-Yau spaces have been studied, for example, in [24–31].

As already mentioned, the compactification ansatz for the case of special holonomy or Calabi-Yau spaces consists of an expansion in the harmonic forms of the manifold. The G-structure manifolds are also characterized by a finite set of forms that lead to light modes after a dimensional reduction [32]. These forms are in general not harmonic, and in fact their exterior differentials are a measure of the torsion and therefore of how much the manifold deviates from special holonomy.

At the level of the effective action, both fluxes and torsion can be considered independently. Only the conditions for a supersymmetric vacuum tie them together [33]. But fluxes and torsion are analogous in other senses, which explains the name of 'geometric flux' given to torsion. The compactification of Type IIA on a Calabi-Yau threefold has long been conjectured to be nonperturbatively dual to Heterotic on K3 $\times T^2$ [34]. It has also been conjectured that this duality can be extended to the case where fluxes are turned on along K3 $\times T^2$ on Heterotic's side. The dual is identified as Type IIA compactified on a manifold with SU(3)-structure (see [35] and references therein). So flux and torsion can be related by duality.

The effect on the low-dimensional theory of turning on fluxes and/or torsion is generically the gauging of isometries of the manifold spanned by the scalars of the ungauged theory [36]. Local supersymmetry dictates that such gauging be accompanied by the generation of a potential for the corresponding scalar fields [37]. As already mentioned, it is in this way that flux compactifications circumvent the moduli problem [38]. The flux parameters show up in the effective action as charges and masses for the scalars fields and in the structure constants of the gauge algebra.

1.2 Outline of the thesis

In this thesis, the compactification of heterotic low-energy supergravity on backgrounds with reduced structure group is studied. In particular, backgrounds with SU(2) and SU(3) structure are considered. In order to set the stage, the compactification of heterotic supergravity on K3 $\times T^2$ is reproduced in Chapter 2. Some background material is given, like heterotic supergravity in Section 2.1 and a description of the manifolds K3 and T^2 together with their moduli spaces in Section 2.2. The philosophy behind dimensional reduction and the derivation of the effective action after compactification on K3 $\times T^2$ is discussed in Section 2.3. As already mentioned, this is an $\mathscr{N}=2$ supergravity in four dimensions coupled to a number of vector- and hypermultiplets.

Chapter 3 is devoted to the compactification of heterotic supergravity on SU(2)-structure backgrounds. A characterization of manifolds with SU(2) structure in six dimensions is given in Section 3.1. The moduli space of these structures is discussed in Section 3.1.1 and an ansatz is constructed in Section 3.1.2 by expanding the exterior differential of the forms characterizing the SU(2) structure in terms of the forms themselves and imposing some consistency conditions [39, 40]. Two complementary cases are distinguished. The first can be realized by considering a K3 fibration over a torus base. The derivation of the corresponding effective action is performed in Section 3.2. The second case is more complicated in that some twisting is performed on the torus part. It was chosen to term this case 'K3 fibration over a twisted torus', although a twisted two-torus does not exist as a global manifold. Actually, this case can be made sense of as a Scherk-Schwarz type reduction on a $K3 \times S^1$ fibration over a circle and is discussed in Section 3.3. The effective action is obtained and again the spectrum and the moduli space is as in Chapter 2. In the end, it is argued that the effective theory for the general case is just a sum of the results for the two complementary cases discussed. The computation of the Killing prepotentials and the gauge algebra of the effective theory is performed in Section 3.4, where the consistency of the obtained effective Lagrangian with the general structure of $\mathscr{N} = 2$ supergravity is also checked. Ref. [41] is the result of this effort.

Chapter 4 deals with the compactification of heterotic supergravity on SU(3)-structure backgrounds. The bosonic part of this analysis has been already performed in the literature and the result of the reduction is an effective gauged $\mathscr{N} = 1$ supergravity [11, 42–44]. Here a different approach is followed, focusing on the fermionic terms of the action. The analysis completes the one already presented in [45]. Six-dimensional manifolds with SU(3) structure are discussed in Section 4.1. The results for the reduction of the bosonic sector are briefly recalled in Section 4.2.1, while the fermionic spectrum and the reduction of the kinetic terms for the fermions are discussed in Section 4.2.2. The computation of the gravitino mass term and the F-terms is performed in Section 4.2.3 and the D-term is computed in Section 4.2.4. Finally, the supersymmetry variations of fermionic fields and the conditions for a supersymmetric vacuum are analyzed in Section 4.3. These results have appeared in [46].

Some useful material is provided in several appendices. In Appendix A, the structure of $\mathscr{N} = 1$ and $\mathscr{N} = 2$ supergravity in four dimensions is presented. Appendix B provides the derivation of the line element in the space of four-dimensional metrics related to the SU(2) structure in terms of the variations of moduli fields. Finally, the concept of almost product structure is discussed in Appendix C.

1.3 Brief comment on notation

Indices $M, N, \ldots$ label ten-dimensional coordinates x^M and indices $\mu, \nu, \ldots$ denote four-dimensional spacetime coordinates x^μ. The internal six-dimensional coordinates are denoted by y^a and are eventually split as two coordinates z^i and four coordinates y^m.

The rank of a form is sometimes shown as a subindex, e.g. A_p denotes a p-form. When p takes a specific value, it is written in italics as in A_{1} or B_{2}. This is in order not to confuse, for example, the one-form $A_{\mathit{1}} = A_M \mathrm{d}x^M$ with its component A_1, i.e. A_M for $M = 1$.

The following shorthands are used throughout the thesis to denote integration of a scalar function f over an n-dimensional manifold $\mathcal{M}$, compact or not,

$$\int_n f(x) \equiv \int_{\mathcal{M}} f(x) = \int_{\mathcal{M}} f(x) \mathrm{vol}_n = \int \mathrm{d}^n x \sqrt{|g_n|} f(x) \ . \tag{1.1}$$

Here, $\mathrm{vol}_n = \sqrt{|g_n|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the volume form on $\mathcal{M}$ and g_n is the determinant of the metric. This makes implicit the invariant measure of integration and allows to write the volume of a compact $\mathcal{M}$ simply as $\int_{\mathcal{M}} 1$.

In order to avoid confusion, sometimes indices are replaced by the quantity they label. For example, the Killing vectors k_I^p in the covariant derivative $D_\mu v^p = \partial_\mu v^p + k_I^p \mathcal{A}_\mu^I$, where the index p refer to the vector multiplet scalars v^p and the index I counts the vectors $\mathcal{A}_\mu^I$, are written as $k_{\mathcal{A}^I}^{v^p}$.

Chapter 2

Heterotic compactification on $\mathrm{K3} \times T^2$

ΑΕΙ Ο ΘΕΟΣ ΓΕΩΜΕΤΡΕΙ

In this Chapter, the dimensional reduction of heterotic supergravity on the product manifold $\mathrm{K3} \times T^2$ is discussed in some detail based on [15, 47]. Heterotic supergravity, or in other words the low-energy limit of the heterotic string, is introduced in Section 2.1, while the relevant properties of the product manifold $\mathrm{K3} \times T^2$ are discussed in Section 2.2. The compactification procedure is briefly described in Section 2.3, and the four-dimensional spectrum and effective action are computed. Although this is known material, it sets the stage for the developments of Chapter 3.

2.1 Heterotic supergravity

Heterotic ten-dimensional supergravity describes the dynamics of the massless degrees of freedom of the heterotic string [4]. These massless modes are organized in multiplets of $\mathcal{N} = 1$ local supersymmetry in ten dimensions. Concretely, there is the gravitational multiplet and 496 vector multiplets transforming in the adjoint representation of either the gauge group $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32). The case $\mathrm{E}_8 \times \mathrm{E}_8$ will be assumed for concreteness in what follows.

The bosonic fields in the gravitational multiplet are the ten-dimensional metric g_{MN}, the Neveu-Schwarz two-form $B_2 = \frac{1}{2} B_{MN}\, \mathrm{d}x^M \wedge \mathrm{d}x^N$ and the dilaton Φ. Additionally, there are Yang-Mills fields $A^{\boldsymbol{a}}_M$ sitting in vector multiplets, where the index $\boldsymbol{a}$ labels the adjoint representation of $\mathrm{E}_8 \times \mathrm{E}_8$. As for the fermions, there is a left-handed gravitino ψ_M and a right-handed dilatino λ sitting in the gravitational multiplet, and there are also left-handed gauginos $\chi^{\boldsymbol{a}}$, the fermionic superpartners of the gauge vectors. The matrix-valued one-form $A^{\boldsymbol{a}}_M t_{\boldsymbol{a}} \mathrm{d}x^M$ can be denoted by A_1, where the matrices $t_{\boldsymbol{a}}$ are the generators of the algebra of $\mathrm{E}_8 \times \mathrm{E}_8$ in the adjoint representation. Analogously, it can be written $\chi = \chi^{\boldsymbol{a}} t_{\boldsymbol{a}}$. The trace on this adjoint is represented by Tr.

The action $\mathcal{S}_{\rm het}$ governing the dynamics of these fields can be split in three parts as [48]

$$\mathcal{S}_{\rm het} = \mathcal{S}_{\rm b} + \mathcal{S}_{\rm f} + \mathcal{S}_{\rm int} \ , \tag{2.1}$$

where $\mathcal{S}_{\rm b}$ involves only bosonic fields, $\mathcal{S}_{\rm f}$ represents the kinetic terms for the fermions and $\mathcal{S}_{\rm int}$ contains all the interaction terms. The first part is given by the expression

$$\mathcal{S}_{\rm b} = \tfrac{1}{2} \int_{10} {\rm e}^{-\Phi} \big(\mathcal{R}_{10} + \partial_M \Phi \partial^M \Phi - \tfrac{1}{4} {\rm Tr}\, F_{MN} F^{MN} - \tfrac{1}{12} H_{MNP} H^{MNP} \big) \ . \tag{2.2}$$

The first term in this action contains the Ricci scalar $\mathcal{R}_{10}$ for the ten-dimensional metric g_{MN} and the prefactor ${\rm e}^{-\Phi}$ makes clear that the action is written in string frame. The two-form $F_2 = \frac{1}{2} F_{MN}\, {\rm d}x^M \wedge {\rm d}x^N$ is the field strength for the gauge potential A_1 and is defined as

$$F_2 = {\rm d}A_1 + [A_1, A_1] \ . \tag{2.3}$$

There is also the three-form $H_3 = \frac{1}{3!} H_{MNP}\, {\rm d}x^M \wedge {\rm d}x^N \wedge {\rm d}x^P$, denoting the field strength of the NS two-form B_2. It is defined according to

$$H_3 = {\rm d}B_2 + \omega_{\rm YM} - \omega_{\rm L} \ , \tag{2.4}$$

where $\omega_{\rm YM}$ and $\omega_{\rm L}$ are Chern-Simons three-forms related to the Yang-Mills potential A_1 and the Lorentzian spin connection ω_1, respectively, and are given by the expressions

$$\begin{aligned} \omega_{\rm YM} &= {\rm Tr}\,(A_1 \wedge F_2 - \tfrac{1}{3} A_1 \wedge A_1 \wedge A_1) \ , \\ \omega_{\rm L} &= {\rm tr}\,(\omega_1 \wedge R_2 - \tfrac{1}{3} \omega_1 \wedge \omega_1 \wedge \omega_1) \ . \end{aligned} \tag{2.5}$$

In the last expression, $R_2 = {\rm d}\omega_1 + [\omega_1, \omega_1]$ denotes the curvature two-form or field strength of the spin connection. The inclusion of these forms in the definition of H_3 is dictated by the necessity of cancellation of gravitational and mixed anomalies [3].

On the other hand, the kinetic terms of the fermionic degrees of freedom have the form

$$\mathcal{S}_{\rm f} = - \int_{10} {\rm e}^{-\Phi} \big(\bar{\psi}_M \Gamma^{MNP} D_N \psi_P + \bar{\lambda} \Gamma^M D_M \lambda + {\rm Tr}\, \bar{\chi} \Gamma^M D_M \chi \big) \ . \tag{2.6}$$

In this expression, the derivatives $D_M = \partial_M + \cdots$ include terms that depend on the bosonic fields. The matrices Γ^M satisfy the Clifford algebra $\{\Gamma_M, \Gamma_N\} = 2 g_{MN} \mathbb{1}$ in ten dimensions and $\Gamma^{MN\cdots}$ denote antisymmetrized products thereof.

Finally, all the interaction terms are collected in the action

$$\begin{aligned} \mathcal{S}_{\rm int} = - \int_{10} {\rm e}^{-\Phi} \Big[& \tfrac{1}{\sqrt{2}} \partial_N \Phi (\bar{\psi}_M \Gamma^N \Gamma^M \lambda) - {\rm Tr}\, (F_{MN} \bar{\chi}) \Gamma^Q \Gamma^{MN} (\psi_Q + \tfrac{\sqrt{2}}{12} \Gamma_Q \lambda) \\ & + \tfrac{1}{12} H_{MNP} \big(\bar{\psi}_Q \Gamma^{QMNPR} \psi_R + 6 \bar{\psi}^M \Gamma^N \psi^P \\ & \qquad - \sqrt{2} \bar{\psi}_Q \Gamma^{MNP} \Gamma^Q \lambda + {\rm Tr}\, \bar{\chi} \Gamma^{MNP} \chi \big) + \cdots \Big] \ . \end{aligned} \tag{2.7}$$

where the dots stand for terms quartic in fermionic fields that will not be needed here.

The action $\mathcal{S}_{\text{het}}$ is locally supersymmetric. In other words, it is invariant with respect to local supersymmetry variations of the fields involved. The supersymmetry variation for a generic field Ψ is parametrized by an infinitesimal left-handed spinor ε and takes the form [1]

$$\delta_\varepsilon \Psi = [\bar{\varepsilon} Q, \Psi] \ , \tag{2.8}$$

where the supercharges Q form a **16** or Majorana-Weyl spinor representation of the Lorentz group and satisfy the anticommutation relations $\{Q_\alpha, \bar{Q}_\beta\} = 2P_M(\Gamma^M)_{\alpha\beta}$, with P_M the ten-dimensional momentum operator. The variations of the fermions ψ_μ, λ and χ are given by [48]

$$\begin{aligned} \delta_\varepsilon \psi_M &= D_M \varepsilon + \tfrac{1}{96} H_{NPQ} ({\Gamma_M}^{NPQ} - 9\delta_M^N \Gamma^{PQ}) \varepsilon \ , \\ \delta_\varepsilon \lambda &= \tfrac{\sqrt{2}}{48} H_{MNP} \Gamma^{MNP} \varepsilon \ , \\ \delta_\varepsilon \chi &= -\tfrac{1}{4} F_{MN} \Gamma^{MN} \varepsilon \ , \end{aligned} \tag{2.9}$$

up to terms involving fermionic fields.

2.2 The product manifold K3 $\times T^2$

In this Section, the product manifold K3 $\times T^2$ is briefly discussed. Clearly, this amounts to considering each one of the factors K3 and T^2 separately. Emphasis is made only on those properties that will be of later use. For a more detailed study of K3 see Ref. [13].

A two-dimensional torus is topologically a product of two circles, $T^2 \simeq S^1 \times S^1$. It can be parametrized by introducing two real coordinates z^i, $i = 1, 2$ together with the identifications $z^i \sim z^i + 1$. In consequence, there are two one-cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ in T^2 defined as the homology classes of the sets with constant z^2 and z^1, respectively. The pair of closed one-forms $v^i = \mathrm{d}z^i$ are dual to the cycles $\mathcal{C}_i$ and satisfy

$$\int_{\mathcal{C}_i} v^j = \delta^{ij} \ , \qquad \int_{T^2} v^i \wedge v^j = \epsilon^{ij} \ , \tag{2.10}$$

where $\epsilon^{ij} = -\epsilon^{ji}$ with $\epsilon^{12} = 1$. These forms generate the first integral cohomology of the torus, i.e. the lattice $H^1(T^2, \mathbb{Z})$. The torus is a flat manifold and has trivial holonomy, the latter meaning that any geometrical object parallelly transported along any closed path comes back to itself.

On the other hand, K3 is a four-dimensional compact Kähler manifold. This implies that K3 is complex, meaning that complex coordinates ζ^α with $\alpha = 1, 2$ can be defined on every patch in such a way that the transition functions for every pair of intersecting patches are holomorphic. Kählerity means that if $g_{\alpha\bar{\beta}}$ is a Hermitian metric on K3 then the associated Kähler form $\mathrm{i} g_{\alpha\bar{\beta}}\, \mathrm{d}\zeta^\alpha \wedge \mathrm{d}\bar{\zeta}^\beta$ is closed. As for every complex manifold, the forms on K3 can be classified by the complex type. The Hodge numbers $h^{p,q}$ counting the harmonic (p,q)-forms can be arranged in a so-called Hodge diamond. The latter in the case of K3 looks as follows,

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \; . \tag{2.11}$$

Since $h^{1,0} = h^{0,1} = 0$, there are no global one-forms on K3. This can in fact be considered one of K3's defining properties. It also follows from this diamond that the second cohomology of K3 is generated by the cohomology classes of $h^{2,0} + h^{1,1} + h^{0,2} = 22$ harmonic two-forms ω^A.*

But more than being Kähler, K3 is actually a hyperkähler manifold. This means that a triplet of complex structures $(I^x)_m{}^n$ with $x = 1, 2, 3$ and $m, n = 1, \ldots, 4$ can be defined satisfying

$$I^x I^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} I^z \; . \tag{2.12}$$

In fact, it can be easily checked that any linear combination

$$I = aI^1 + bI^2 + cI^3 \; , \qquad a^2 + b^2 + c^2 = 1 \; , \tag{2.13}$$

squares to $-\mathbb{1}$, providing K3 with a whole sphere of complex structures. By lowering the upper index on the complex structures I^x using the metric g_{mn} on K3, a triplet of self-dual two-forms

$$J^x_{mn} = (I^x)_m{}^p g_{pn} \tag{2.14}$$

is obtained. Since by definition the complex structures are integrable, it follows that the two-forms J^x are closed. The self-duality condition $J^x = *J^x$ then implies that they are also co-closed, i.e. $\mathrm{d}{*J^x} = 0$, and therefore harmonic. Additionally, using Eq. (2.12) the following relations can be derived,

$$J^x \wedge J^y = 2\delta^{xy} \mathrm{vol}_4 \; , \tag{2.15}$$

where vol_4 is the volume form of K3.

It can also be shown that with respect to any of the complex structures in Eq. (2.13), say for definiteness $I = I^3$, the corresponding two-form J^3 is of type $(1,1)$ and is in fact the Kähler form

$$J^3 = \mathrm{i} g_{\alpha\bar{\beta}} \, \mathrm{d}\zeta^\alpha \wedge \mathrm{d}\bar{\zeta}^{\bar{\beta}} \; . \tag{2.16}$$

*In contrast to this mathematically precise statement, it will be common practice in the rest of the thesis to speak of forms in the cohomology when in fact either cohomology classes or representative forms thereof is meant. This should be a harmless abuse of language.

The remaining combinations $J^1 \pm \mathrm{i}J^2$ are of type $(2,0)$ and $(0,2)$ with respect to I^3, respectively. The three two-forms J^x are the only linear combinations of the 22 harmonic forms ω^A that are self-dual. The remaining nineteen combinations are anti-self-dual and of type $(1,1)$ with respect to any of the complex structures (2.13).

K3 is the only nontrivial Calabi-Yau twofold,† since it is the only four-dimensional manifold with SU(2) holonomy. For a generic four-dimensional manifold, the spin group is the product $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ and the spinor representation is $\mathbf{4} = \mathbf{2} \otimes \mathbf{2}'$. If the holonomy group is SU(2), one of the factors $\mathbf{2}$ or $\mathbf{2}'$ must be broken according to $\mathbf{2} \to \mathbf{1} \oplus \mathbf{1}$, giving rise to two singlets of the holonomy group. These singlets are actually a spinor η and its conjugate η^{c}, since conjugation respects the spinor's chirality in four Euclidean dimensions. Being a singlet of the holonomy group, the spinor η must be covariantly constant with respect to the Levi-Civita connection. Adding the two flat directions of the torus factor, the existence of two linearly independent and covariantly constant spinors on K3 $\times T^2$ follows.

2.2.1 Geometric moduli of K3 $\times T^2$

The metric of K3 $\times T^2$ has a block-diagonal structure

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{K3}} + \mathrm{d}s^2_{T^2} = g_{mn}(y)\,\mathrm{d}y^m \mathrm{d}y^n + g_{ij}\,v^i v^j \ , \tag{2.17}$$

where y^m are four real coordinates on K3 and $v^i = \mathrm{d}z^i$ are the torus one-forms introduced before. The space of all possible K3 $\times T^2$ metrics splits as well,

$$\mathscr{M}^{\mathrm{geom}}_{\mathrm{K3}\times T^2} = \mathscr{M}^{\mathrm{geom}}_{\mathrm{K3}} \times \mathscr{M}^{\mathrm{geom}}_{T^2} \ , \tag{2.18}$$

with $\mathscr{M}^{\mathrm{geom}}_{\mathrm{K3}}$ being the space of all $g_{mn}(y)$ or K3 metrics and $\mathscr{M}^{\mathrm{geom}}_{T^2}$ being the space of possible metrics g_{ij} for the torus factor. These spaces are constructed in the following.

The metric g_{ij} contains three moduli, as follows from the number of independent components g_{11}, g_{12} and g_{22}. In order to find the space $\mathscr{M}^{\mathrm{geom}}_{T^2}$ that they describe, it is useful to define the 'zweibeins' or one-forms $\tilde{v}^i = A^i{}_j v^j$ such that

$$\mathrm{d}s^2_{T^2} = \tilde{v}^i \tilde{v}^i = (A^{\mathrm{T}}A)_{ij} v^i v^j = g_{ij} v^i v^j \ . \tag{2.19}$$

It is clear from this expression that both A and $A' = \mathcal{O}A$ with $\mathcal{O} \in \mathrm{O}(2)$ correspond to the same metric g_{ij}, since

$$A'^{\mathrm{T}} A' = A^{\mathrm{T}} \mathcal{O}^{\mathrm{T}} \mathcal{O} A = A^{\mathrm{T}} A \ . \tag{2.20}$$

This eliminates one 'unphysical' degree of freedom from the four components of the matrix A, leaving the right number of moduli. The moduli space $\mathscr{M}^{\mathrm{geom}}_{T^2}$ of possible T^2 metrics is therefore given by

$$\mathscr{M}^{\mathrm{geom}}_{T^2} = \frac{\mathrm{GL}(2)}{\mathrm{O}(2)} = \mathbb{R}^+ \times \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \simeq \mathbb{R}^+ \times \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \ . \tag{2.21}$$

† 'Twofold' refers to the number of complex dimensions.

The factor $\mathbb{R}^+$ in this expression describes the overall volume.

Nevertheless, there are still some discrete identifications that must be imposed on this space. It turns out that equivalent metrics are obtained for both A and $A' = A\mathcal{S}$ with $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, since in the second case the torus can be reparametrized as $z^i \to \mathcal{S}^i{}_j z^j$. These reparametrizations respect the identifications $z^i \sim z^i + 1$, and the redefined one-forms $v^i \to \mathcal{S}^i{}_j v^j$ span the same two-dimensional lattice $H^1(T^2,\mathbb{Z})$. Although it will not be made explicit in expression (2.21), it will be assumed that this $\mathrm{SL}(2,\mathbb{Z})$ subgroup has been modded out.

For the Ricci-flat metric $g_{mn}(y)$ of the K3 factor, the story is more complicated. In fact, such metrics are generally unknown, although their existence is guaranteed by Yau's theorem. Fortunately, to know the actual form of the metric is not necessary to perform the Kaluza-Klein reduction on this manifold. As for all Calabi-Yau compactifications, it is actually enough to find the geometric moduli space $\mathscr{M}^{\mathrm{geom}}_{\mathrm{K3}}$, together with an expression for the metric or line element on it in terms of the moduli. In what follows, the moduli space is derived, while the second task is deferred to Appendix B where actually a more general case is considered.

A K3 metric is determined by a choice of hyperkähler structure I^x up to an overall rescaling. Due to Eq. (2.14), this is equivalent to a choice of three self-dual forms J^x on $H^2(\mathrm{K3})$ satisfying (2.15). That this is indeed all one needs for determining the metric can be intuitively understood, since fixing the dual forms determines the Hodge star operator $*$ and therefore the metric g_{mn} up to normalization. Rescalings of the metric are then controlled by the normalization of the forms J^x, as is clear from Eq. (2.14).

All this can be made explicit as follows. Every harmonic two-form φ on K3 can be expanded as $\varphi = \varphi_A \omega^A$ with some constants φ_A. A scalar product on $H^2(\mathrm{K3})$ can then be introduced according to

$$(\varphi, \chi) = \int_{\mathrm{K3}} \varphi \wedge \chi = \varphi_A \eta^{AB} \chi_B \ , \qquad \forall \varphi, \chi \in H^2(\mathrm{K3}) \ , \tag{2.22}$$

where the intersection matrix η^{AB} for the two-forms ω^A is defined as

$$\eta^{AB} = \int_{\mathrm{K3}} \omega^A \wedge \omega^B \ . \tag{2.23}$$

It can be easily checked that $(\varphi, *\varphi) > 0$ for every form $\varphi \neq 0$ on a Riemannian manifold.‡ A self-dual form φ has thus $(\varphi,\varphi) = (\varphi, *\varphi) > 0$, while an anti-self-dual one has $(\varphi,\varphi) = -(\varphi, *\varphi) < 0$. Since there are three self-dual two-forms J^x and nineteen anti-self-dual ones on K3, it is concluded that the metric η^{AB} has signature $(3,19)$ and therefore $H^2(\mathrm{K3}) \simeq \mathbb{R}^{3,19}$. Integrating Eq. (2.15) over K3 one obtains

$$(J^x, J^y) = 2\delta^{xy} \mathrm{e}^{-\rho} > 0 \ , \tag{2.24}$$

‡Here, Riemannian is used in contrast to pseudo-Riemannian.

where $\mathrm{e}^{-\rho}$ denotes the volume of K3. From this equation it is already clear that the moduli space of four-dimensional hyperkähler structures is given by the choices of positive three-dimensional hyperplanes in $H^2(\mathrm{K3}) \simeq \mathbb{R}^{3,19}$, and that the complete metric moduli space $\mathscr{M}_{\mathrm{K3}}^{\mathrm{geom}}$ is obtained if additionally an $\mathbb{R}^+$ factor corresponding to the overall volume is included.

But some way to parametrize this space is needed. Since any closed and self-dual form is harmonic, the triplet of two-forms J^x can be expanded in the ω^A basis as

$$J^x = \mathrm{e}^{-\frac{1}{2}\rho} \xi_A^x \omega^A \tag{2.25}$$

for some real parameters ξ_A^x. From this expansion and Eqs. (2.22) and (2.24), it follows that the possible values of ξ_A^x are constrained to satisfy[§]

$$\eta^{AB} \xi_A^x \xi_B^y = 2\delta^{xy} \ . \tag{2.26}$$

The parameters ξ_A^x can be seen as three vectors labeled by the index x living in a 22-dimensional linear space isomorphic to $\mathbb{R}^{3,19}$ with metric η^{AB}. Condition (2.26) then simply states that these vectors are orthogonal to each other and of norm $\sqrt{2}$ each. They therefore define a positive three-dimensional hyperplane

$$\mathcal{H}^3 = \mathrm{span}(\xi^1, \xi^2, \xi^3) \subset \mathbb{R}^{3,19} \ . \tag{2.27}$$

Equation (2.26) provides six constraints on the 66 parameters ξ_A^x. Additionally, it is seen from the expansion (2.25) that an orthogonal transformation

$$\xi_A^x \to \mathcal{R}^x{}_y \xi_A^y \ , \qquad \mathcal{R} \in \mathrm{SO}(3) \tag{2.28}$$

leaving the hyperplane (2.27) invariant merely rotates the forms J^x among themselves. This means that values of the parameters ξ_A^x related by (2.28) lead essentially to the same hyperkähler structure and thus to the same K3 metric. The SO(3) transformation $\mathcal{R}$ in (2.28) leaves Eq. (2.26) invariant, and since it is parametrized by, say, the three Euler angles it removes three 'unphysical' degrees of freedom from the parameters ξ_A^x. It is concluded that the number of independent moduli comprised in ξ_A^x is precisely $66 - 6 - 3 = 57$. If the volume modulus ρ is added, the well-known number of 58 metric moduli for K3 is obtained.

It has been seen that the 'physical' values of ξ_A^x parametrize the space of hyperplanes $\mathcal{H}^3$ according to (2.27). This space is obtained by taking the group of isometries of η^{AB}, or in other words the group of transformations $\mathrm{SO}(3,19)$ leaving invariant Eq. (2.26), and dividing it by both the SO(3) in Eq. (2.28) acting within $\mathcal{H}^3$ and an SO(19) acting on the orthogonal hyperplane $\mathcal{H}_\perp^{19}$ defined by

$$\mathbb{R}^{3,19} = \mathcal{H}^3 \oplus \mathcal{H}_\perp^{19} \ . \tag{2.29}$$

[§]It was precisely in order to make this normalization condition and therefore the parameters ξ_A^x independent of ρ that a factor $\mathrm{e}^{-\frac{1}{2}\rho}$ was included in the expansion (2.25).

Additionally, an $\mathbb{R}^+$ factor corresponding to the volume modulus ρ must be included. The moduli space of K3 metrics is therefore given by

$$\mathscr{M}_{\rm K3}^{\rm geom} = \mathbb{R}^+ \times \frac{\rm SO(3,19)}{\rm SO(3) \times SO(19)} \ . \tag{2.30}$$

But once again, this is still not the whole story, as there are some discrete identifications that must be imposed on this space. Although it does not affect the counting of the number of moduli, there is a discrete $\mathrm{O}(3,19,\mathbb{Z})$ freedom in the definition of the harmonic basis. The reason for this is that the forms

$$\omega'^A = \mathcal{Z}^A{}_B \omega^B \ , \qquad \mathcal{Z} \in \mathrm{O}(3,19,\mathbb{Z}) \tag{2.31}$$

constitute an equivalent basis of harmonic forms for $H^2(\mathrm{K3},\mathbb{Z})$. In other words, two such sets of forms related by an $\mathrm{O}(3,19,\mathbb{Z})$ matrix $\mathcal{Z}$ define the same lattice $H^2(\mathrm{K3},\mathbb{Z})$. In view of Eq. (2.25), this freedom translates into the discrete equivalences

$$\xi^x_A \sim \xi^x_B \mathcal{Z}^B{}_A \ , \qquad \mathcal{Z} \in \mathrm{O}(3,19,\mathbb{Z}) \tag{2.32}$$

for the moduli ξ^x_A that must be modded out from the expression (2.30). It is precisely this equivalence relation that will allow to consider nontrivial fibrations of K3 in Chapter 3. Although the moduli space $\mathscr{M}_{\rm K3}^{\rm geom}$ will be written as in Eq. (2.30), the identifications (2.32) should be implicitly understood.

Another possibility is to parametrize the metrics g_{mn} directly in terms of the action of the Hodge star operator on two-forms. The Hodge dual of a harmonic form is harmonic as well, and therefore $*\omega^A$ can be expressed as linear combinations of the ω^B themselves. Let us introduce some numbers $M^A{}_B$ as the matrix elements of the Hodge star operator on the basis ω^A, namely

$$*\omega^A = M^A{}_B \omega^B \ . \tag{2.33}$$

As already explained, the matrix $M^A{}_B$ together with the volume modulus ρ must completely determine the metric g_{mn}. Since $**\omega^A = \omega^A$, it is seen that

$$M^A{}_C M^C{}_B = \delta^A_B \ , \tag{2.34}$$

and therefore the eigenvalues of $M^A{}_B$ can only be ± 1. The $+1$ (-1) eigenvalues correspond to the (anti-)self-dual linear combinations of the ω^A. By raising the lower index on $M^A{}_B$ with η^{AB}, one obtains the symmetric and positive-definite matrix¶

$$M^{AB} \equiv M^B{}_C \eta^{AC} = M^B{}_C \int_{\rm K3} \omega^A \wedge \omega^C = \int_{\rm K3} \omega^A \wedge *\omega^B \ . \tag{2.35}$$

¶Actually, it will prove useful to generalize this practice and use η^{AB} to raise and lower capital Latin indices.

Since $M^A{}_B$ contains the same information as the parameters ξ^x_A, an expression linking them can be written. Recalling the self-duality condition $*J^x = J^x$ and considering the action of the Hodge star operator on Eq. (2.25), it can be written,

$$\begin{aligned} \xi^x_A *\omega^A &= \xi^x_A \omega^A \\ \xi^x_A M^A{}_B \omega^B &= \xi^x_A \omega^A \\ (\xi^x_A M^A{}_B - \xi^x_B)\omega^B &= 0 \ . \end{aligned} \tag{2.36}$$

The forms ω^A are linearly independent and thus $\xi^x_A M^A{}_B = \xi^x_B$. This means that the matrix $M^A{}_B$ acts as the identity on the hyperplane $\mathcal{H}^3$ defined in Eq. (2.27). This result simply mirrors the fact that the forms J^x span the (+1)-eigenspace of the Hodge star operator acting on two-forms, as already explained. Clearly, $M^A{}_B$ must act as minus the identity on the orthogonal subspace $\mathcal{H}^{19}_\perp$, since the latter corresponds to the 19 anti-self-dual forms. From Eq. (2.26) it follows that the projection operator onto $\mathcal{H}^3$ is given by

$$P^A{}_B = \tfrac{1}{2}\eta^{AC}\xi^x_C\xi^x_B \ . \tag{2.37}$$

A linear operator $M^A{}_B$ that acts as the identity on $\mathcal{H}^3$ and as minus the identity on the orthogonal subspace $\mathcal{H}^{19}_\perp$ must necessarily be given by[‖]

$$\begin{aligned} M^A{}_B &= (+1)P^A{}_B + (-1)(\delta^A_B - P^A{}_B) \\ &= -\delta^A_B + \eta^{AC}\xi^x_C\xi^x_B \end{aligned} \tag{2.38}$$

if M^{AB} is to be symmetric. Notice that this expression is indeed invariant under the orthogonal transformations (2.28), as it should be.

In conclusion, the geometric moduli space of the product manifold K3 × T^2 is given by the product of the spaces (2.30) and (2.21) corresponding respectively to the factors K3 and T^2. In other words,

$$\mathscr{M}^{\text{geom}}_{\text{K3}\times T^2} = \mathbb{R}^+ \times \frac{\text{SO}(3,19)}{\text{SO}(3)\times\text{SO}(19)} \times \mathbb{R}^+ \times \frac{\text{SU}(1,1)}{\text{SU}(1)} \ . \tag{2.39}$$

Also, the factor $\mathscr{M}^{\text{geom}}_{\text{K3}}$ can be parametrized by the volume modulus ρ and either the parameters ξ^x_A satisfying (2.26) or the matrix $M^A{}_B$ defined by Eq. (2.33). The relation between these two sets of parameters is given by Eq. (2.38).

2.3 Compactification on K3 × T^2

The starting point of the compactification program is to assume that the ten-dimensional spacetime is actually a product[**]

$$\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times \mathcal{Y} \tag{2.40}$$

[‖] In this equation, as well as in other similar expressions, summation over the index x is understood.

[**] The possibility of a warped geometry is therefore left out.

of an extended four-dimensional spacetime and an 'internal' six-dimensional compact manifold $\mathcal{Y}$. Let us denote by x^μ the coordinates on Minkowskian spacetime $\mathcal{M}_{1,3}$ and by y^a the six real coordinates on $\mathcal{Y}$. A generic field $\Phi = \Phi(x,y)$ in ten dimensions depends on all ten coordinates. Suppose it satisfies an equation

$$\Delta_{10}\Phi(x,y) = 0 \ , \tag{2.41}$$

with a D'Alembertian-like differential operator Δ_{10}. Due to the product structure (2.40), this operator must split as $\Delta_{10} = \Delta_x + \Delta_y$, where each term acts on the displayed coordinates. Since the manifold $\mathcal{Y}$ is compact, the spectrum of the Laplacian-like operator Δ_y will be discrete and positive definite. A set of functions $f_n(y)$ on $\mathcal{Y}$ thus exists satisfying

$$\Delta_y f_n(y) = m_n^2 f_n(y) \ , \quad n = 0,1,2,\dots \ . \tag{2.42}$$

Since this is a complete set, $\Phi(x,y)$ can be expanded according to

$$\Phi(x,y) = \sum_{n=1}^{\infty} \phi_n(x) f_n(y) \ , \tag{2.43}$$

with coefficients ϕ_n that are functions on $\mathcal{M}_{1,3}$. Inserting this expansion in Eq. (2.41) and using Eq. (2.42), one obtains

$$(\Delta_x + m_n^2)\phi_n(x) = 0 \ , \quad n = 0,1,2,\dots \ . \tag{2.44}$$

This means that each component field $\phi_n(x)$ in four dimensions has a mass m_n. It is in this way that the geometry of the internal manifold shows up as physical parameters in the four-dimensional world.

If, for example, Δ_y is the Laplacian, $f_n(y)$ are harmonic functions on $\mathcal{Y}$. In this case, all eigenvalues m_n are of the order of $\mathcal{V}_{\mathcal{Y}}^{-1/6}$ or bigger, except for the zero-mode or constant function $f_{\rm o} = 1$ which has $m_{\rm o} = 0$. This illustrates how by making the volume $\mathcal{V}_{\mathcal{Y}}$ of the internal space small enough one can make all but the fields $\phi_{\rm o}$ corresponding to the zero-modes very heavy and therefore negligible in four dimensions. So, if one is interested in energies much smaller than $\mathcal{V}_{\mathcal{Y}}^{-1/6}$, only the light modes $\phi_{\rm o}(x)$ need to be kept in the expansion (2.43). This is the Kaluza-Klein program.

Let us denote by $\mathscr{L}_{10}[\Phi(x,y)]$ the Lagrangian in ten dimensions, which is a local functional of all ten-dimensional fields $\Phi(x,y)$. An effective four-dimensional theory is obtained by making the substitution

$$\Phi(x,y) \simeq \phi_{\rm o}(x) f_{\rm o}(y) \ , \tag{2.45}$$

or in other words if one truncates the expansion in Eq.(2.43) by disregarding the heavy modes and then integrates over the six-dimensional internal space $\mathcal{Y}$. The following effective Lagrangian in four-dimensions is obtained,

$$\mathscr{L}_4^{\rm eff}[\phi_{\rm o}(x)] = \int_{\mathcal{Y}} \mathscr{L}_{10}[\phi_{\rm o}(x) f_{\rm o}(y)] \ , \tag{2.46}$$

where g_6 is the determinant of the metric g_{ab} in $\mathcal{Y}$.

The metric $g_{ab}(y)$ is in fact part of the ten-dimensional graviton field. One should therefore include variations $\delta g_{ab}(x, y)$ among the fields $\Phi(x, y)$ as well. This means that the six-dimensional metric contributes with four-dimensional fields $\phi_{\rm o}$ given by the metric moduli parametrizing the class of manifolds $\mathcal{Y}$ under consideration.

If the ten-dimensional theory is supersymmetric, the question arises whether or not part of this supersymmetry is preserved by the compactification. This is the case if a spinor in ten dimensions have an expansion as in Eq. (2.45), giving thus rise to spinors in the four-dimensional theory. In other words, one needs spinor 'zero modes' $\eta_{\rm o}$ in the internal manifold $\mathcal{Y}$ so that, for example, each infinitesimal supersymmetry parameter ε in ten dimensions can be written as

$$\varepsilon = \varepsilon_{\rm o} \otimes \eta_{\rm o} + \bar{\varepsilon}_{\rm o} \otimes \bar{\eta}_{\rm o} \ , \tag{2.47}$$

leading to supersymmetry parameters $\varepsilon_{\rm o}$ in four dimensions. If one starts with a theory that has $\mathscr{N} = 1$ in ten dimensions, one should obtain an effective theory with $\mathscr{N} = \#\eta_{\rm o}$, i.e. as many copies of the $\mathscr{N} = 1$ supersymmetry algebra in four dimensions as there are internal spinors $\eta_{\rm o}$.

2.3.1 Effective theory

Now this program can be applied to the heterotic string (or rather to heterotic supergravity as described in Section 2.1) for the case where K3 × T^2 is chosen as the internal manifold $\mathcal{Y}$ in the product (2.40). Since there are two linearly independent global spinors in K3 × T^2, as explained in Section 2.2, this background is expected to preserve 8 supercharges. This means that as a result of the compactification, a low-energy effective $\mathscr{N} = 2$ supergravity in four dimensions must be obtained [14].

In this and in the next Chapter, the analysis is restricted to the bosonic sector of heterotic supergravity. This is the sector involving the metric g_{MN}, the NS two-form B_2 and the Yang-Mills one-form A_1. A compactification ansatz for these fields is written by expanding in terms of the harmonic one- and two-forms of K3 × T^2. As reviewed in Section 2.2, these are the pair of one-forms $v^i = \mathrm{d}z^i$ on the torus factor and the 22 harmonic two-forms ω^A of K3.

But before actually doing that, let us pause to comment on a consistency condition that all compactifications of the heterotic string must satisfy. From the definition of the three-form H_3 given in Eq. (2.4), the following Bianchi identity must be satisfied [49],

$$\mathrm{d}H_3 = \mathrm{tr}\,(R_2 \wedge R_2) - \mathrm{Tr}\,(F_2 \wedge F_2) \ . \tag{2.48}$$

Integrating this expression over the internal manifold K3 × T^2, recalling that the torus is flat and assuming that no Yang-Mills field is turned on on the torus factor, it is obtained

$$\int_{\mathrm{K3}} \mathrm{tr}\,(R_2 \wedge R_2) - \mathrm{Tr}\,(F_2 \wedge F_2) = 24 - \int_{\mathrm{K3}} \mathrm{Tr}\,(F_2 \wedge F_2) = \int_{\mathrm{K3}} \mathrm{d}H_3 = 0\ . \qquad (2.49)$$

This means that a gauge bundle $\mathcal{G} \subset \mathrm{E}_8 \times \mathrm{E}_8$ on K3 must be 'switched on' with instanton number canceling the curvature contribution $\int_{\mathrm{K3}} \mathrm{tr}\,(R_2 \wedge R_2) = 24$. The instantons have the effect of breaking the gauge symmetry $\mathrm{E}_8 \times \mathrm{E}_8$ down to a non-Abelian subgroup G. As already mentioned, the compactification on an SU(2)-holonomy manifold leads to an effective theory with $\mathscr{N} = 2$ local supersymmetry in four dimensions. The vector fields descending from the one-form A_1 reside in vector multiplets that also contain scalars. At a generic point of the moduli space of these scalars, the non-Abelian gauge symmetry G is further broken down to its maximal Abelian subgroup. In other words, the non-Abelian gauge symmetry G is spontaneously broken by nonzero vacuum expectation values of the scalar superpartners of the gauge vectors. The details of this breaking are model-dependent. Here it will simply be assumed that the consistency condition (2.49) is satisfied and that the gauge symmetry $\mathrm{E}_8 \times \mathrm{E}_8$ has been broken down to an Abelian subgroup $\mathrm{U}(1)^{n_\mathrm{g}}$ [15]. This means that only the Coulomb branch of the theory is analyzed. The number $n_\mathrm{g} = \dim G$ can be as high as the rank of $\mathrm{E}_8 \times \mathrm{E}_8$, namely $n_\mathrm{g} = 16$, and as low as zero. The Yang-Mills field A_1 therefore descends to n_g Abelian vectors or one-forms $A_1^\mathrm{a} = A_M^\mathrm{a}\mathrm{d}x^M$, with $\mathrm{a} = 1, \ldots, n_\mathrm{g}$.

Compactification ansatz and four-dimensional spectrum

The six internal coordinates y^a in $\mathcal{Y}$ split into two coordinates z^i for the torus and four coordinates y^m for K3. For the ten-dimensional metric, the following ansatz can be written,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{ij}\big(\mathrm{d}z^i + V_\mu^i\mathrm{d}x^\mu\big)\big(\mathrm{d}z^j + V_\nu^j\mathrm{d}x^\nu\big) + g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n\ , \qquad (2.50)$$

where a dependence of all metric components on x^μ is implicit. Notice that for convenience it has been chosen to order these coordinates as

$$y^a = (z^i, y^m)\ , \qquad i = 1, 2\ , \quad m = 1, \ldots, 4\ , \qquad (2.51)$$

corresponding to $T^2 \times \mathrm{K3}$ rather than $\mathrm{K3} \times T^2$. The ten-dimensional metric g_{MN} has thus the block form

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{ij}V_\mu^iV_\nu^j & V_\mu^i g_{ij} & 0 \\ g_{ij}V_\nu^j & g_{ij} & 0 \\ 0 & 0 & g_{mn}(y) \end{pmatrix}\ . \qquad (2.52)$$

The so-called Kaluza-Klein vectors V_μ^i must be included to account for the possibility of spacetime-dependent isometries

$$z^i \to z^i + \alpha^i(x) \qquad (2.53)$$

of the torus factor. The latter imply that $\mathrm{d}z^i \to \mathrm{d}z^i + (\partial_\mu \alpha^i)\mathrm{d}x^\mu$, which can be compensated in Eq. (2.50) by a transformation

$$V^i_\mu \to V^i_\mu - \partial_\mu \alpha^i \ . \tag{2.54}$$

From this, it should be no surprise that the vectors V^i_μ appear as gauge fields in the effective theory. No such vector V^m_μ is introduced for the K3 factor because there are no one-cycles on K3, as follows from the Hodge diamond (2.11). Finally, the dilaton $\Phi(x)$ is a function of the spacetime coordinates x^μ only, since a harmonic scalar on the internal manifold is just the constant function.

On the other hand, an ansatz for the NS two-form B_2 and the Yang-Mills one-forms A^a_1 in terms of the harmonic one- and two-forms available in K3 × T^2 can be written as

$$\begin{aligned} B_2 &= \tfrac{1}{2} B_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + B_{i\mu} \mathcal{E}^i \wedge \mathrm{d}x^\mu + \tfrac{1}{2} B_{ij} \mathcal{E}^i \wedge \mathcal{E}^j + b_A \omega^A \ , \\ A^a_1 &= A^a_\mu \mathrm{d}x^\mu + A^a_i \mathcal{E}^i \ , \end{aligned} \tag{2.55}$$

where the one-forms

$$\mathcal{E}^i = \mathrm{d}z^i + V^i_\mu \mathrm{d}x^\mu \tag{2.56}$$

have been introduced. The latter are invariant under the transformations (2.53) and (2.54). The expansion in terms of the forms $\mathcal{E}^i$ is also convenient because in the basis $(\mathrm{d}x^\mu, \mathcal{E}^i, \mathrm{d}y^m)$ the metric (2.50) is block-diagonal.

Taking a look at Eqs. (2.50) and (2.55), the spectrum of the effective four-dimensional theory can be already stated. In terms of their four-dimensional spin, these fields are the metric $g_{\mu\nu}$, a two-form $B_{\mu\nu}$, $4 + n_\mathrm{g}$ vectors V^i_μ, $B_{i\mu}$ and A^a_μ, one scalar arising from the dilaton Φ, $2(2 + n_\mathrm{g})$ scalars from $g_{ij} + B_{ij}$ and A^a_i, 58 scalars parametrizing the K3 metric g_{mn}, and finally twenty-two scalars b_A. Moreover, the two-form $B_{\mu\nu}$ in four-dimensions can be dualized to a scalar a, the axion. These fields organize in multiplets of $\mathscr{N} = 2$ supersymmetry in four dimensions as follows: the gravitational multiplet consisting of the metric and the graviphoton, $n_\mathrm{v} = 3 + n_\mathrm{g}$ vector multiplets containing each one a vector and a complex scalar, and finally $n_\mathrm{h} = 20$ hypermultiplets with four real scalars each (see the diagram in Figure 2.1).

The next step is to compute the effective action. This is done by substitution of the ansatz (2.55) for B_2 and A^a_1 and the metric of Eq. (2.50) into the bosonic action $\mathcal{S}_\mathrm{b}$ given in Eq. (2.2). The first two terms in this action involve the Ricci scalar and the dilaton. It will prove useful to compute them in a general fashion, as the final formula can be applied later to more general cases. Then one can just specialize to the present situation. Let us do that in the following.

Reduction of Ricci scalar and dilaton kinetic term

Consider the Lagrangian in D dimensions

$$\mathscr{L}_D = \tfrac{1}{2} \mathrm{e}^{-\Phi} \big(\mathcal{R}_D + \partial_M \Phi \partial^M \Phi \big) \ , \tag{2.57}$$

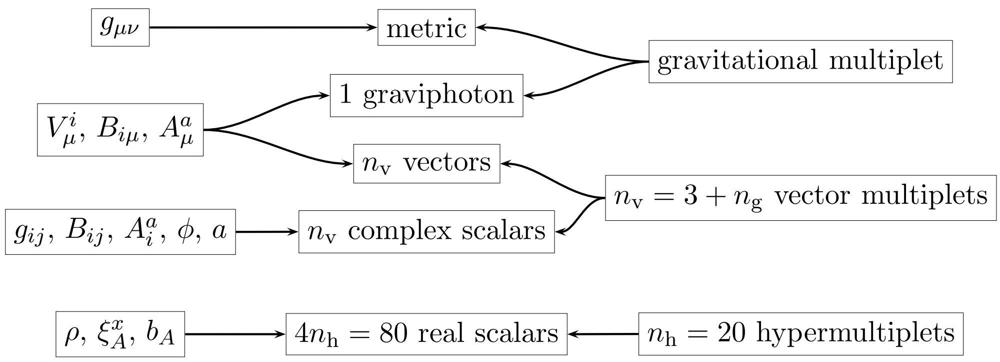

Figure 2.1: Four-dimensional fields and $\mathscr{N} = 2$ mutiplets.

where $\mathcal{R}_D$ is the Ricci scalar constructed out of a D-dimensional metric g_{MN} and Φ is a dilaton-like field. Although in principle one has in mind a space with signature $(1, D-1)$, the latter plays no role in the derivation and the final formula is valid for metrics of arbitrary signature. Let us split the set of D coordinates x^M into two subsets

$$x^M = (x^\mu, y^a) \ , \quad \mu = 0, \ldots, d-1 \ , \quad a = 1, \ldots, D-d \ . \tag{2.58}$$

In a compactification context, the coordinates x^μ correspond to the non-compact d-dimensional spacetime, while y^a are coordinates in the compact internal manifold. Nevertheless, we stress again that this is irrelevant for the present computation. The metric g_{MN} can be written in total generality in the following form

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{ab} V^a_\mu V^b_\nu & V^a_\mu g_{ab} \\ g_{ab} V^b_\nu & g_{ab} \end{pmatrix} \ , \tag{2.59}$$

or equivalently

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{ab} (\mathrm{d}y^a + V^a_\mu \mathrm{d}x^\mu)(\mathrm{d}y^b + V^b_\nu \mathrm{d}x^\nu) \ . \tag{2.60}$$

This is entirely general as long as $g_{\mu\nu}$, g_{ab} and V^a_μ depend on all coordinates x^μ and y^a. Let us nevertheless focus on cases where $g_{ab} = g_{ab}(x, y)$ and $V^a_\mu = V^a_\mu(x, y)$ might in principle depend on all D coordinates but the block $g_{\mu\nu} = g_{\mu\nu}(x)$ and the dilaton $\Phi = \Phi(x)$ depend only on the coordinates x^μ. The Ricci scalar for the metric g_{MN} given in Eq. (2.59) with the coordinate dependences just discussed can be computed and the result is [50, 51]

$$\begin{aligned} \mathcal{R}_D = \mathcal{R}_d - \tfrac{1}{4} g_{ab} V^a_{\mu\nu} V^{b,\mu\nu} + \tfrac{1}{4} \tilde{\mathcal{D}}_\mu g_{ab} \tilde{\mathcal{D}}^\mu g^{ab} \\ - \nabla^\mu (g^{ab} \tilde{\mathcal{D}}_\mu g_{ab}) - \tfrac{1}{4} (g^{ab} \tilde{\mathcal{D}}_\mu g_{ab})(g^{cd} \tilde{\mathcal{D}}^\mu g_{cd}) + \mathcal{R}_{D-d} \ . \end{aligned} \tag{2.61}$$

In this expression, $\mathcal{R}_d$ and $\mathcal{R}_{D-d}$ are the Ricci scalars corresponding to the metrics $g_{\mu\nu}$ and g_{ab}, respectively. The following definitions were also introduced,

$$\begin{aligned} V^a_{\mu\nu} &= \mathcal{D}_\mu V^a_\nu - \mathcal{D}_\nu V^a_\mu \ , \\ \mathcal{D}_\mu &= \partial_\mu - V^a_\mu \partial_a \ , \\ \nabla^\mu &= \mathcal{D}^\mu - g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} \ , \\ \tilde{\mathcal{D}}_\mu g_{ab} &= \mathcal{D}_\mu g_{ab} - g_{ac}\partial_b V^c_\mu - g_{bc}\partial_a V^c_\mu \ , \end{aligned} \tag{2.62}$$

where $\Gamma^\mu_{\rho\sigma}$ are the Christoffel symbols for the metric $g_{\mu\nu}$. Notice that $\nabla^\mu A_\mu$ for an arbitrary vector A_μ is just the divergence $A^\mu{}_{;\mu} = (\partial^\mu - g^{\rho\sigma}\Gamma^\mu_{\rho\sigma})A_\mu$ together with an additional term $-V^a_\mu \partial_a A_\mu$. It is useful to keep this in mind because the term containing ∇^μ must be integrated by parts, and the fact that it involves a covariant derivative makes the step easier.

The Lagrangian for the theory in d dimensions is obtained by integrating the Lagrangian $\mathscr{L}_D$ over the $(D-d)$-dimensional 'internal' space. The dimensionally-reduced Lagrangian corresponding to Eq. (2.57) is therefore

$$\mathscr{L}_d = \int_{D-d} \mathscr{L}_D = \tfrac{1}{2}\int_{D-d} \mathrm{e}^{-\Phi}\big(\mathcal{R}_D + \partial_M\Phi\partial^M\Phi\big) \ . \tag{2.63}$$

Now one just needs to substitute the expression (2.61) for the Ricci scalar. The only term that needs to be worked out is the first term in the second line of (2.61), since as already advanced an integration by parts of this term must be performed. This term and the one right next to it combine with the kinetic term for the dilaton in Eq. (2.57) to produce a single kinetic term for a redefined dilaton. The resulting Lagrangian is

$$\mathscr{L}_d = \tfrac{1}{2}\int_{D-d} \mathrm{e}^{-\Phi}\Big(\mathcal{R}_d + \mathcal{D}_\mu\phi\mathcal{D}^\mu\phi - \tfrac{1}{4}g_{ab}V^a_{\mu\nu}V^{b,\mu\nu} + \tfrac{1}{4}\tilde{\mathcal{D}}_\mu g_{ab}\tilde{\mathcal{D}}^\mu g^{ab}\Big) - \mathscr{V}_{D-d} \ , \tag{2.64}$$

where ϕ is a shifted dilaton

$$\phi = \Phi - \tfrac{1}{2}\ln g_{D-d} \tag{2.65}$$

and the 'potential' $\mathscr{V}_{D-d}$ arises from the curvature of the 'internal' space as

$$\mathscr{V}_{D-d} = -\tfrac{1}{2}\int_{D-d} \mathrm{e}^{-\Phi}\mathcal{R}_{D-d} \ . \tag{2.66}$$

If x^μ are spacetime coordinates then this is indeed a potential, because it does not contain derivatives with respect to x^μ.

Now the formula (2.64) can be used to perform the dimensional reduction of the first two terms in the action $\mathcal{S}_\mathrm{b}$ in Eq. (2.2) by setting $D = 10$ and $d = 4$. The general metric in Eq. (2.59) has in this case the particular form (2.52). In specializing to this case, some simplifications are therefore expected, since g_{ij} and V^i_μ are functions only of x^μ and $V^m_\mu = 0$. In other words, one must set in Eq. (2.64)

$$g_{ab}(x,y) = \begin{pmatrix} g_{ij}(x) & 0 \\ 0 & g_{mn}(x,y) \end{pmatrix} , \qquad V^a_\mu(x,y) = \begin{cases} V^i_\mu(x) \ , & \text{for } a = i \\ 0 \ , & \text{for } a = m \end{cases} . \tag{2.67}$$

It follows that $V^a_\mu \partial_a = V^i_\mu \partial_i + V^m_\mu \partial_m = V^i_\mu \partial_i$ vanishes, because nothing depends on the torus coordinates z^i. As seen from the definitions (2.62), in this case the derivatives $\mathcal{D}_\mu$ and $\tilde{\mathcal{D}}_\mu$ can be substituted by ordinary spacetime derivatives ∂_μ. Moreover, the potential $\mathcal{V}_6$ in (2.64) vanishes, because the torus is flat and the K3 metric $g_{mn}(y)$ is Ricci-flat. The result for the effective Lagrangian arising from the reduction of the Ricci scalar and dilaton kinetic term is therefore

$$\mathscr{L}_{4,g+\Phi} = \tfrac{1}{2} \int_{\mathrm{K3}\times T^2} \mathrm{e}^{-\Phi} \big(\mathcal{R}_4 + \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{4} g_{ij} V^i_{\mu\nu} V^{j,\mu\nu} + \tfrac{1}{4} \partial_\mu g_{ij} \partial^\mu g^{ij} + \tfrac{1}{4} \partial_\mu g_{mn} \partial^\mu g^{mn} \big) \ , \tag{2.68}$$

where $V^i_{\mu\nu} = \partial_\mu V^i_\nu - \partial_\nu V^i_\mu$ is the field strength of the vectors V^i_μ and the shifted dilaton is defined as

$$\phi = \Phi - \tfrac{1}{2} \ln g_6 = \Phi - \tfrac{1}{2} \ln g_2 - \tfrac{1}{2} \ln g_4 \ , \tag{2.69}$$

with $g_2 = \det g_{ij}$ and $g_4 = \det g_{mn}$.

In performing the integration over K3 × T^2, attention must be paid only to which coordinates each field depends on. Nothing depends on the torus coordinates and only the metric g_{mn} depends on the K3 coordinates y^m. The determinant g_4 is y-dependent, so it follows from Eq. (2.69) that also ϕ depends on y^m. Since in the Lagrangian (2.68) so far only the derivatives $\partial_\mu \phi$ appear, the shifted dilaton can be redefined as

$$\phi = \Phi - \tfrac{1}{2} \ln g_2 + \rho \ , \tag{2.70}$$

where $\mathrm{e}^{-\rho} = \int_{\mathrm{K3}} 1$ is the volume of K3. In other words, Eq. (2.69) has been truncated, leaving only the zero mode which is the constant function on K3. Now the integrals in Eq. (2.68) can be (almost) completely performed to obtain

$$\mathscr{L}_{4,g+\Phi} = \tfrac{1}{2} \mathrm{e}^{-\phi} \Big(\mathcal{R}_4 + \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{4} g_{ij} V^i_{\mu\nu} V^{j,\mu\nu} + \tfrac{1}{4} \partial_\mu g_{ij} \partial^\mu g^{ij} + \tfrac{1}{4} \mathrm{e}^{\rho} \int_{\mathrm{K3}} \partial_\mu g_{mn} \partial^\mu g^{mn} \Big) \ . \tag{2.71}$$

A formula for computing the last term in this Lagrangian in terms of the geometric moduli of K3 is derived in Appendix B. It can be used either Eq. (B.19) in terms of the volume modulus ρ and the parameters ξ^x_A or equivalently Eq. (B.21) if rather working with ρ and the matrix $M^A{}_B$ is preferred. Taking the second alternative and substituting the variations $\delta\rho$ and $\delta M^A{}_B$ in Eq. (B.21) by the spacetime derivatives $\partial_\mu \phi$ and $\partial_\mu M^A{}_B$, respectively, the result for the contribution to the effective Lagrangian is

$$\begin{aligned} \mathscr{L}_{4,g+\Phi} = \tfrac{1}{2} \mathrm{e}^{-\phi} \big(\mathcal{R}_4 + \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{4} g_{ij} V^i_{\mu\nu} V^{j,\mu\nu} + \tfrac{1}{4} \partial_\mu g_{ij} \partial^\mu g^{ij} \\ - \tfrac{1}{4} \partial_\mu \rho \partial^\mu \rho + \tfrac{1}{8} \partial_\mu M^A{}_B \partial^\mu M^B{}_A \big) \ . \end{aligned} \tag{2.72}$$

Reduction of Yang-Mills and NS two-form kinetic terms

The computation of the remaining terms in $\mathcal{S}_\mathrm{b}$ involving the Yang-Mills field and the NS two-form is a bit messy but otherwise straightforward. The first step is to compute the field strengths of B_2 and A_1^a. In doing this, $\mathrm{d}\mathcal{E}^i$ is needed, and taking the differential of Eq. (2.56) one obtains

$$\mathrm{d}\mathcal{E}^i = \partial_\mu V_\nu^i \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \tfrac{1}{2} V_{\mu\nu}^i \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{2.73}$$

Applying the d-operator to both expressions in (2.55) yields

$$\begin{aligned} \mathrm{d}B_2 = {} & \tfrac{1}{2}(\partial_\mu B_{\nu\rho} + B_{i\mu} V_{\nu\rho}^i)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\ & - \tfrac{1}{2}(B_{i\mu\nu} + B_{ij} V_{\mu\nu}^j)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathcal{E}^i \\ & + \tfrac{1}{2}\partial_\mu B_{ij}\,\mathrm{d}x^\mu \wedge \mathcal{E}^i \wedge \mathcal{E}^j + \partial_\mu b_A \,\mathrm{d}x^\mu \wedge \omega^A \ , \\ F_2^\mathrm{a} = \mathrm{d}A_1^\mathrm{a} = {} & \tfrac{1}{2}(F_{\mu\nu}^\mathrm{a} + A_i^\mathrm{a} V_{\mu\nu}^i)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \partial_\mu A_i^\mathrm{a}\,\mathrm{d}x^\mu \wedge \mathcal{E}^i \ , \end{aligned} \tag{2.74}$$

where $B_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu}$ and $F_{\mu\nu}^\mathrm{a} = \partial_\mu A_\nu^\mathrm{a} - \partial_\nu A_\mu^\mathrm{a}$ are the field strengths for the vectors $B_{i\mu}$ and A_μ^a, respectively. These expressions can now be used to compute H_3 as defined in Eq. (2.4). The restriction to terms with at most two-derivatives in the effective Lagrangian makes it enough to consider the Yang-Mills Chern-Simons form ω_YM only. Moreover, since the gauge fields are now Abelian, the cubic term in the expression for ω_YM in (2.5) vanishes. Therefore, it can be written

$$\begin{aligned} H_3 = {} & \mathrm{d}B_2 - \tfrac{1}{2} A_1^\mathrm{a} \wedge F_2^\mathrm{a} \\ = {} & \tfrac{1}{2}(\partial_\mu B_{\nu\rho} + B_{i\mu} V_{\nu\rho}^i - \tfrac{1}{2} A_\mu^\mathrm{a} F_{\nu\rho}^\mathrm{a} - \tfrac{1}{2} A_\mu^\mathrm{a} A_i^\mathrm{a} V_{\nu\rho}^i)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\ & - \tfrac{1}{2}(B_{i\mu\nu} + B_{ij} V_{\mu\nu}^j + \tfrac{1}{2} A_i^\mathrm{a} F_{\mu\nu}^\mathrm{a} + \tfrac{1}{2} A_i^\mathrm{a} A_j^\mathrm{a} V_{\mu\nu}^j + A_\mu^\mathrm{a} \partial_\nu A_i^\mathrm{a})\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathcal{E}^i \\ & + \tfrac{1}{2}(\partial_\mu B_{ij} + A_i^\mathrm{a} \partial_\mu A_j^\mathrm{a})\,\mathrm{d}x^\mu \wedge \mathcal{E}^i \wedge \mathcal{E}^j + (\partial_\mu b_A)\,\mathrm{d}x^\mu \wedge \omega^A \ . \end{aligned} \tag{2.75}$$

The following field redefinitions need also be performed,

$$B_{i\mu} \to B_{i\mu} + \tfrac{1}{2} A_i^\mathrm{a} A_\mu^\mathrm{a} \ , \qquad B_{\mu\nu} \to B_{\mu\nu} - \tfrac{1}{2}(B_{i\mu} V_\nu^i - B_{i\nu} V_\mu^i) \ , \tag{2.76}$$

since these are the fields with the correct gauge transformation properties. Making these substitutions in Eq. (2.75), the following expression is obtained

$$\begin{aligned} H_3 = {} & \mathrm{d}B_2 - \tfrac{1}{2} A_1^\mathrm{a} \wedge \mathrm{d}A_1^\mathrm{a} \\ = {} & \tfrac{1}{2}(\partial_\mu B_{\nu\rho} - \tfrac{1}{2} B_{i\mu} V_{\nu\rho}^i - \tfrac{1}{2} V_\mu^i B_{i\nu\rho} - \tfrac{1}{2} A_\mu^\mathrm{a} F_{\nu\rho}^\mathrm{a})\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\ & - \tfrac{1}{2}(B_{i\mu\nu} + A_i^\mathrm{a} F_{\mu\nu}^\mathrm{a} + C_{ij} V_{\mu\nu}^j)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathcal{E}^i \\ & + \tfrac{1}{2}(\partial_\mu B_{ij} + A_i^\mathrm{a} \partial_\mu A_j^\mathrm{a})\,\mathrm{d}x^\mu \wedge \mathcal{E}^i \wedge \mathcal{E}^j + (\partial_\mu b_A)\,\mathrm{d}x^\mu \wedge \omega^A \ , \end{aligned} \tag{2.77}$$

where $C_{ij} = B_{ij} + \tfrac{1}{2} A_i^\mathrm{a} A_j^\mathrm{a}$ has been defined.

Finally, the expressions for $F_2^{\rm a}$ and H_3 in Eqs. (2.74) and (2.77) can be substituted in the last two terms of the action $\mathcal{S}_{\rm b}$. The integration over K3 × T^2 is again almost trivial, since none of the fields depends on the internal coordinates. Only the forms ω^A depend on the K3 coordinates, but it is known how to integrate them. The result is the following effective Lagrangian

$$\begin{aligned}\mathscr{L}_{4,A+B} = -\tfrac{1}{2}\mathrm{e}^{-\phi}\Big[&\tfrac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} + \tfrac{1}{4}\big(F^{\rm a}_{\mu\nu} + A^{\rm a}_i V^i_{\mu\nu}\big)\big(F^{{\rm a},\mu\nu} + A^{\rm a}_j V^{j,\mu\nu}\big)\\ &+ \tfrac{1}{4}g^{ij}\big(B_{i\mu\nu} + A^{\rm a}_i F^{\rm a}_{\mu\nu} + C_{ik}V^k_{\mu\nu}\big)\big(B_j{}^{\mu\nu} + A^b_j F^{b,\mu\nu} + C_{jl}V^{l,\mu\nu}\big)\\ &+ \tfrac{1}{4}g^{ik}g^{jl}\big(\partial_\mu B_{ij} + A^{\rm a}_{[i}\partial_\mu A^{\rm a}_{j]}\big)\big(\partial^\mu B_{kl} + A^b_{[k}\partial^\mu A^b_{l]}\big)\\ &+ \tfrac{1}{2}g^{ij}\partial_\mu A^{\rm a}_i\partial^\mu A^{\rm a}_j + \tfrac{1}{2}\mathrm{e}^{\rho}M^{AB}\partial_\mu b_A\partial^\mu b_B\Big] \ ,\end{aligned} \tag{2.78}$$

where the three-form $\mathcal{H}_3 = \frac{1}{3!}\mathcal{H}_{\mu\nu\rho}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho$ in four dimensions is given by

$$\mathcal{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \tfrac{1}{2}B_{i\mu}V^i_{\nu\rho} - \tfrac{1}{2}V^i_\mu B_{i\nu\rho} - \tfrac{1}{2}A^{\rm a}_\mu F^{\rm a}_{\nu\rho} + \text{cyclic permutations} \tag{2.79}$$

and M^{AB} is defined in Eq. (2.35).

Effective action

The total effective Lagrangian $\mathscr{L}_4$ is the sum of the contributions $\mathscr{L}_{4,g+\Phi}$ and $\mathscr{L}_{4,A+B}$ given in Eqs. (2.72) and (2.78). But the resulting expression can be written in a compact form after some definitions are introduced. All $n_{\rm v}+1 = 4+n_{\rm g}$ four-dimensional vectors can be collectively denoted as

$$\mathcal{A}^I_\mu = (V^i_\mu, B_{i\mu}, A^{\rm a}_\mu) \ , \qquad I = 0,\ldots,n_{\rm v} \ , \tag{2.80}$$

and the corresponding field strengths are $\mathcal{F}^I_{\mu\nu} = \partial_\mu\mathcal{A}^I_\nu - \partial_\nu\mathcal{A}^I_\mu$, or compactly $\mathcal{F}^I_2 = \mathrm{d}\mathcal{A}^I_1$. From the $2(2+n_{\rm g})$ scalars g_{ij}, B_{12} and $A^{\rm a}_i$, an SO$(2,n_{\rm v}-1)$ matrix can be defined as

$$M^{IJ} = \begin{pmatrix} g^{ij} & -g^{ik}C_{kj} & -g^{ij}A^b_j \\ -C_{ki}g^{kj} & g_{ij} + A^{\rm a}_i A^{\rm a}_j + g^{kl}C_{ki}C_{lj} & A^b_i + C_{ki}g^{kj}A^b_j \\ -A^{\rm a}_i g^{ij} & A^{\rm a}_j + A^{\rm a}_i g^{ik}C_{kj} & \delta^{ab} + A^{\rm a}_i g^{ij}A^b_j \end{pmatrix} \ , \tag{2.81}$$

since it can be checked that $MLM = L$ is satisfied, with L being the SO$(2,n_{\rm v}-1)$ invariant metric

$$L_{IJ} = \begin{pmatrix} 0 & \delta^j_i & 0 \\ \delta^i_j & 0 & 0 \\ 0 & 0 & \delta^{ab} \end{pmatrix} = L^{IJ} \ . \tag{2.82}$$

The matrix $M_{IJ} = (LML)_{IJ}$ is therefore the inverse of M^{IJ}. Analogously, a matrix $\mathcal{M}^{PQ}$ can be defined from the 58 geometric moduli of K3 encoded in ρ and $M^A{}_B$ and the 22 fields

b_A as

$$\mathcal{M}^{PQ} = \begin{pmatrix} \mathrm{e}^{\rho} & \frac{1}{2}\mathrm{e}^{\rho}b^2 & -\mathrm{e}^{\rho}b^B \\ \frac{1}{2}\mathrm{e}^{\rho}b^2 & \mathrm{e}^{-\rho} + b_A M^{AB} b_B + \frac{1}{4}\mathrm{e}^{\rho}b^4 & -b_A M^{AB} - \frac{1}{2}\mathrm{e}^{\rho}b^2 b^B \\ -\mathrm{e}^{\rho}b^A & -M^{AB}b_B - \frac{1}{2}\mathrm{e}^{\rho}b^2 b^A & M^{AB} + \mathrm{e}^{\rho}b^A b^B \end{pmatrix} , \tag{2.83}$$

where $b^2 = \eta^{AB} b_A b_B$. If one index on $\mathcal{M}^{PQ}$ is lowered by means of the SO$(4,20)$ invariant metric

$$\mathcal{L}_{PQ} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \eta_{AB} \end{pmatrix} , \tag{2.84}$$

the resulting matrix $\mathcal{M}^P{}_Q$ satisfies $\mathcal{M}^{\mathrm{T}}\mathcal{L}\mathcal{M} = \mathcal{L}$ and is therefore an element of SO$(4,20)$. It is not difficult to check that in terms of the matrices M^{IJ} and $\mathcal{M}^{PQ}$ and the fields $\mathcal{A}^I_\mu$, the total effective Lagrangian $\mathscr{L}_4$ takes the compact form

$$\begin{aligned} \mathscr{L}_4 &= \mathscr{L}_{4,g+\Phi} + \mathscr{L}_{4,A+B} \\ &= \tfrac{1}{2}\mathrm{e}^{-\phi}\big(\mathcal{R}_4 + \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \tfrac{1}{4}M_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} \\ &\qquad + \tfrac{1}{8}\partial_\mu M^I{}_J\partial^\mu M^J{}_I + \tfrac{1}{8}\partial_\mu\mathcal{M}^P{}_Q\partial^\mu\mathcal{M}^Q{}_P\big) , \end{aligned} \tag{2.85}$$

where now the four-dimensional three-form $\mathcal{H}_3$ in Eq. (2.79) can be written as

$$\mathcal{H}_3 = \mathrm{d}B_2 - \tfrac{1}{2}L_{IJ}\mathcal{A}^I_1 \wedge \mathcal{F}^J_2 . \tag{2.86}$$

At this point, a Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^{\phi}g_{\mu\nu}$ of the four-dimensional metric must be performed to obtain a Lagrangian with a canonical Einstein-Hilbert term,

$$\begin{aligned} \mathscr{L}_4 = \tfrac{1}{2}\mathcal{R}_4 - \tfrac{1}{4}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{24}\mathrm{e}^{-2\phi}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \tfrac{1}{8}\mathrm{e}^{-\phi}M_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} \\ + \tfrac{1}{16}\partial_\mu M^I{}_J\partial^\mu M^J{}_I + \tfrac{1}{16}\partial_\mu\mathcal{M}^P{}_Q\partial^\mu\mathcal{M}^Q{}_P . \end{aligned} \tag{2.87}$$

Furthermore, the two-form $B_{\mu\nu}$ can be dualized to a scalar a. To see how this works, isolate the term in the effective Lagrangian (2.85) involving the three-form $\mathcal{H}_3$, that is

$$\mathscr{L}_{4,\mathcal{H}} = -\tfrac{1}{4}\mathrm{e}^{-2\phi}\mathcal{H}_3 \wedge *\mathcal{H}_3 . \tag{2.88}$$

Taking the exterior derivative of (2.86), it is seen that the three-form must satisfy $\mathrm{d}\mathcal{H}_3 + \frac{1}{2}L_{IJ}\mathcal{F}^I_2 \wedge \mathcal{F}^J_2 = 0$. To enforce this constraint, a Lagrange multiplier a is introduced and the Lagrangian $\mathscr{L}_{4,\mathcal{H}}$ is consequently modified according to

$$\mathscr{L}'_{4,\mathcal{H}} = -\tfrac{1}{4}\mathrm{e}^{-2\phi}\mathcal{H}_3 \wedge *\mathcal{H}_3 + \tfrac{1}{2}a\big(\mathrm{d}\mathcal{H}_3 + \tfrac{1}{2}L_{IJ}\mathcal{F}^I_2 \wedge \mathcal{F}^J_2\big) . \tag{2.89}$$

The second term can be integrated by parts and the equation of motion for $\mathcal{H}_3$ that follows is

$$\frac{\partial\mathscr{L}'_{4,\mathcal{H}}}{\partial\mathcal{H}_3} = \tfrac{1}{2}(\mathrm{e}^{-2\phi}*\mathcal{H}_3 - \mathrm{d}a) = 0 , \tag{2.90}$$

with solution $\mathcal{H}_3 = \mathrm{e}^{2\phi} * \mathrm{d}a$. Substituting this back into $\mathcal{L}'_{4,\mathcal{H}}$, it is obtained

$$\mathscr{L}_{4,a} = -\tfrac{1}{4}\mathrm{e}^{2\phi}\mathrm{d}a \wedge *\mathrm{d}a + \tfrac{1}{4} a L_{IJ} \mathcal{F}^I_2 \wedge \mathcal{F}^J_2 \ . \tag{2.91}$$

After dualizing the two-form $B_{\mu\nu}$ by means of the replacement $\mathscr{L}_{4,\mathcal{H}} \to \mathscr{L}_{4,a}$, the Lagrangian (2.87) can be written as

$$\begin{aligned}\mathscr{L}_4 = \tfrac{1}{2}\mathcal{R}_4 - \tfrac{1}{8}\mathrm{e}^{-\phi} M_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{16} a L_{IJ}\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\ + \frac{\partial_\mu s \partial^\mu \bar{s}}{(s-\bar{s})^2} + \tfrac{1}{16}\partial_\mu M^I{}_J \partial^\mu M^J{}_I + \tfrac{1}{16}\partial_\mu \mathcal{M}^P{}_Q \partial^\mu \mathcal{M}^Q{}_P \ ,\end{aligned} \tag{2.92}$$

where a complex heterotic dilaton or axion-dilaton s has been defined as

$$s = \tfrac{1}{2}a - \tfrac{\mathrm{i}}{2}\mathrm{e}^{-\phi} \ . \tag{2.93}$$

It is useful to express the $2(2+n_\mathrm{g})$ scalars g_{ij}, B_{12} and A^a_i parametrizing the matrix M_{IJ} in terms of $n_\mathrm{v} - 1 = 2 + n_\mathrm{g}$ complex fields u, t and n^a as follows,

$$\begin{aligned} g_{ij} &= \tfrac{1}{2}\Big[(t-\bar{t}) - \frac{(n^a-\bar{n}^a)(n^a-\bar{n}^a)}{u-\bar{u}}\Big]\frac{1}{u-\bar{u}}\begin{pmatrix} 2u\bar{u} & u+\bar{u} \\ u+\bar{u} & 2\end{pmatrix} \ , \\ B_{12} &= -\tfrac{1}{2}\Big[(t+\bar{t}) - \frac{(n^a+\bar{n}^a)(n^a-\bar{n}^a)}{u-\bar{u}}\Big] \ , \\ A^a_1 &= \sqrt{2}\,\frac{\bar{u}n^a - u\bar{n}^a}{u-\bar{u}} \ , \qquad A^a_2 = \sqrt{2}\,\frac{n^a-\bar{n}^a}{u-\bar{u}} \ . \end{aligned} \tag{2.94}$$

The complex axion-dilaton s can be appended to this set to obtain the n_v complex fields

$$v^p = (s, u, t, n^a) \ , \qquad p = 1, \ldots, n_\mathrm{v} \ . \tag{2.95}$$

The Lagrangian in Eq. (2.92) can now be cast into a final form in terms of the complex scalars v^p and the 80 real scalars q^u parametrizing the matrix $\mathcal{M}^P{}_Q$. The expression is

$$\begin{aligned}\mathscr{L}_4 = \tfrac{1}{2}\mathcal{R}_4 + \tfrac{1}{4} I_{IJ}(v)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{8} R_{IJ}(v)\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\ - G_{p\bar{q}}(v)\partial_\mu v^p \partial^\mu \bar{v}^{\bar{q}} - h_{uv}(q)\partial_\mu q^u \partial^\mu q^v \ .\end{aligned} \tag{2.96}$$

The metric $G_{p\bar{q}}(v)$ in the kinetic term for the complex scalars v^p is Kähler, since it can be shown that $G_{p\bar{q}} = \partial_p \partial_{\bar{q}} K$ for a Kähler potential

$$K = -\ln \mathrm{i}(s-\bar{s}) - \tfrac{1}{4}\ln[(u-\bar{u})(t-\bar{t}) - (n^a-\bar{n}^a)^2] \ . \tag{2.97}$$

The gauge kinetic functions are given by

$$I_{IJ}(v) = \frac{s-\bar{s}}{2\mathrm{i}} M_{IJ} \ , \qquad R_{IJ}(v) = \frac{s+\bar{s}}{2} L_{IJ} \ . \tag{2.98}$$

The effective Lagrangian in Eq. (2.96) has indeed the form (A.1) for the bosonic sector of four-dimensional $\mathscr{N} = 2$ minimal supergravity coupled to $n_\mathrm{v} = 3 + n_\mathrm{g}$ Abelian vector multiplets and $n_{\rm h} = 20$ hypermultiplets in the ungauged case. One of the vectors in the set of $n_{\rm v} + 1 = 4 + n_{\rm g}$ vectors $\mathcal{A}^I_\mu$ is the graviphoton. The latter forms, together with the metric $g_{\mu\nu}$, the bosonic content of the gravitational multiplet (see Figure 2.1). The remaining $n_{\rm v}$ vectors are paired with the $n_{\rm v}$ complex scalars v^p to form the corresponding number of vector multiplets. The complex scalars v^p span the moduli space of the vector multiplet sector

$$\mathscr{M}_{\rm v} = \frac{\rm SU(1,1)}{\rm U(1)} \times \frac{{\rm SO}(2, n_{\rm v} - 1)}{{\rm SO}(2) \times {\rm SO}(n_{\rm v} - 1)} \ , \tag{2.99}$$

where the first factor is spanned by the heterotic dilaton s and the second factor contains the geometric moduli space $\mathscr{M}^{\rm geom}_{T^2}$. As explained in Appendix A, this is a special Kähler manifold, and holomorphic projective coordinates X^I can be introduced as

$$X^0 = \tfrac{1}{2} \ , \quad X^1 = -\tfrac{1}{2} u \ , \quad X^2 = -\tfrac{1}{2} s \ , \quad X^3 = \tfrac{1}{2} t \ , \quad X^a = \tfrac{1}{\sqrt{2}} n^a \ . \tag{2.100}$$

It is not difficult to check that the Kähler potential in Eq. (2.97) can be written as

$$K = - \ln \left({\rm i} \bar{X}^I \partial_I \mathscr{F} - {\rm i} X^I \partial_{\bar{I}} \bar{\mathscr{F}} \right) \tag{2.101}$$

for a prepotential

$$\mathscr{F}(X) = \frac{X^2 (X^1 X^3 + \frac{1}{2} X^a X^a)}{X^0} = \tfrac{1}{4} s (ut - n^a n^a) \ . \tag{2.102}$$

Finally, there are 80 scalars q^u sitting in $n_{\rm h} = 20$ hypermultiplets. They span the quaternionic manifold

$$\mathscr{M}_{\rm h} = \frac{\rm SO(4,20)}{\rm SO(4) \times SO(20)} \supset \mathscr{M}^{\rm geom}_{\rm K3} \ . \tag{2.103}$$

Although they have been ignored here, it should nevertheless be mentioned that there are also the scalars parametrizing the gauge bundle $\mathcal{G}$ needed to satisfy the consistency condition (2.49). They sit in additional hypermultiplets, therefore enlarging the moduli space $\mathscr{M}_{\rm h}$ in Eq. (2.103).

Chapter 3

Heterotic on SU(2)-structure backgrounds

ΣΥΝ ΑΘΗΝΑ ΚΑΙ ΧΕΙΡΑ ΚΙΝΕΙ

This Chapter is devoted to the compactification of heterotic supergravity on SU(2)-structure backgrounds. It will be seen that six-dimensional manifolds with SU(2) structure are generalizations of $\mathrm{K3} \times T^2$, and their properties are analyzed in Section 3.1. The reason for focusing on this class of manifolds is that they also lead to effective $\mathscr{N} = 2$ locally supersymmetric theories in four dimensions. The geometric moduli spaces and an ansatz for these backgrounds are discussed in Sections 3.1.1 and 3.1.2, respectively. Two cases are distinguished. The first one can be realized by considering fibrations of K3 over T^2 and is discussed in Section 3.2. The second one has been termed 'K3 fibrations over a twisted torus' and is analyzed in Section 3.3. The effective action is obtained in both cases and turns out to be a gauged version of the supergravity obtained for compactifications on $\mathrm{K3} \times T^2$. The gauge algebra and the prepotentials for the general case are computed in Section 3.4, where the consistency of the results with the general action of $\mathscr{N} = 2$ supergravity is also verified.

3.1 Manifolds with SU(2) structure

As already explained in Section 2.3, obtaining a supersymmetric effective theory after compactification demands the existence of global nowhere-vanishing spinors on the internal manifold. If the heterotic string, having $\mathscr{N} = 1$ supersymmetry in ten dimensions or 16 supercharges, is compactified on a six-dimensional background that possesses N internal spinors, the reduction procedure is expected to preserve $4N$ supercharges. This leads to a theory with $\mathscr{N} = N$ supersymmetry in four dimensions. Therefore, two internal spinors are needed in order to obtain an $\mathscr{N} = 2$ supersymmetric theory.

The existence of such spinors restricts the class of possible manifolds. It implies concretely a reduction of the structure group. Consider the bundle of all orthonormal frames in a six-dimensional manifold $\mathcal{Y}$. For a generic manifold, the transition functions of this bundle are the orthogonal transformations SO(6), since the transition functions must in any case preserve the orthonormality of the frames. The fibers are isomorphic to the group SO(6) acting on them and in consequence this is a principal bundle. For a generic manifold, the structure group is therefore SO(6). However, it can happen that a subbundle can be defined such that the transition functions take values on a subgroup $G \subset \mathrm{SO}(6)$. If this is the case, it is said that the structure group has been reduced to G, or in other words that the manifold $\mathcal{Y}$ belongs to the class of G-structure manifolds [22].

The spinor representation of SO(6) is a **4**, i.e. the fundamental representation of the spin group Spin(4) $\simeq$ SU(4). The global nowhere-vanishing spinors must appear as singlets in the decomposition of SU(4) in representations of the reduced structure group G. If one is interested in an effective theory that has $\mathcal{N} = 2$ supersymmetry, two of these singlets are needed, and the right decomposition is

$$\mathbf{4} \to \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \ . \tag{3.1}$$

In other words, $\mathcal{Y}$ must be an SU(2)-structure manifold. What the decomposition (3.1) therefore tells us is that *a six-dimensional manifold $\mathcal{Y}$ has structure group* SU(2) *if it possesses a pair of globally defined and nowhere-vanishing* SO(6) *spinors η_i that are linearly independent everywhere on $\mathcal{Y}$.*

If additionally the spinors η_i happen to be covariantly constant with respect to the Levi-Civita connection, any spinor parallelly transported around a closed path must come back to itself up to at most a transformation in the 'unbroken' SU(2). In this case $\mathcal{Y}$ has SU(2) holonomy and must be the product manifold $\mathrm{K3} \times T^2$. Manifolds with SU(2) structure are thus generalizations of $\mathrm{K3} \times T^2$ for which the two spinors η_i are *not required* to satisfy parallel transport with respect to the Levi-Civita connection but maybe only with respect to a different, torsionful connection. This explains the name of torsional geometries given to these backgrounds.

It will be assumed in the following that the two spinors are normalized as $\eta_i^\dagger \eta_j = \delta_{ij}$. Using the spinors η_i and the SO(6) Clifford algebra γ_a with $a = 1, \ldots, 6$, one can construct a triplet of self-dual two-forms J^x and a complex one-form $v^1 + \mathrm{i} v^2$ on $\mathcal{Y}$ as follows,

$$\begin{gathered} J^1_{ab} + \mathrm{i} J^2_{ab} = \mathrm{i} \eta_2^\dagger \gamma_{ab} \eta_1 \ , \qquad J^3_{ab} = -\tfrac{\mathrm{i}}{2} (\eta_1^\dagger \gamma_{ab} \eta_1 + \eta_2^\dagger \gamma_{ab} \eta_2) \ , \\ v^1_a + \mathrm{i} v^2_a = \eta_2^{\mathrm{c}\dagger} \gamma_a \eta_1 \ , \end{gathered} \tag{3.2}$$

where γ_{ab} denotes the antisymmetrized product of two γ-matrices. Under an SU(2) transformation that rotates the pair of spinors η_i into each other, the two-forms J^x transform as a vector of the corresponding SO(3) while the v^i remain invariant.

The two-forms J^x and the one-forms v^i characterize completely the SU(2) structure, since they contain all the information on the global spinors [33]. As can be verified from Eq. (3.2), these forms are closed if and only if the spinors η_i are covariantly constant with respect to the Levi-Civita connection. For a generic SU(2)-structure manifold, the departure from SU(2) holonomy is therefore measured by the failure of $\mathrm{d}J^x$ and $\mathrm{d}v^i$ to vanish. In case the forms J^x and v^i are closed, the manifold $\mathcal{Y}$ is $\mathrm{K3} \times T^2$. Then the two-forms J^x related to the hyperkähler structure on the K3 factor together with the torus one-forms $v^i = \mathrm{d}z^i$ as discussed in Section 2.2 are recovered.

Making use of Fierz identities and Eq. (3.2), the following relations among the components of the two- and one-forms can be derived,

$$g^{ab} v^i_a v^j_b = \delta^{ij} \ , \qquad g^{ab} v^i_a J^x_{bc} = 0 \ , \qquad J^x \wedge J^y = 2\delta^{xy} \iota_{v^1} \iota_{v^2} \mathrm{vol}_6 \ . \tag{3.3}$$

Here, vol_6 is the volume form on $\mathcal{Y}$, g^{ab} is the inverse metric and ι_{v^i} represents the interior product with respect to the vectors $v^{ia} = g^{ab} v^i_b$.*

Although a generic SU(2)-structure manifold $\mathcal{Y}$ cannot be written as a product manifold, the existence of the one-forms v^i does allow to introduce an almost product structure, i.e. a globally defined tensor $\mathcal{P}^a{}_b$ satisfying $\mathcal{P}^a{}_c \mathcal{P}^c{}_b = \delta^a_b$. This is achieved by setting

$$\mathcal{P}^a{}_b = 2 v^{ia} v^i_b - \delta^a_b \ , \tag{3.4}$$

as can be easily checked making use of the first relation in (3.3). This tensor can naturally be viewed as an endomorphism $\mathcal{P} : T\mathcal{Y} \to T\mathcal{Y}$ of the tangent bundle of $\mathcal{Y}$, and as discussed in Appendix C it actually splits the tangent space over every point of $\mathcal{Y}$ into a direct sum of a two- and a four-dimensional subspaces. This can be seen as follows. From the definition (3.4) and the normalization condition in Eq. (3.3), it can be verified that $\mathcal{P}(v^i) = v^i$. It is also not difficult to check that $\mathcal{P}(w) = -w$ for every w orthogonal to both vectors v^i. The subspace formed by all vectors w is clearly four-dimensional. The two-dimensional subspace is thus spanned by the two vectors v^i. This can be made explicit by writing Eq. (3.4) in the form $\mathcal{P} = \mathcal{P}_2 - \mathcal{P}_4$, where

$$(\mathcal{P}_2)^a{}_b = v^{ia} v^i_b \ , \qquad (\mathcal{P}_4)^a{}_b = \delta^a_b - v^{ia} v^i_b \ . \tag{3.5}$$

It is clear that $\mathcal{P}_2$ projects on the two-dimensional subspace generated by the vectors v^i and $\mathcal{P}_4 = \mathbb{1} - \mathcal{P}_2$ projects on the orthogonal subspace.

If the almost product structure in Eq. (3.4) is integrable, every neighborhood of $\mathcal{Y}$ can be written as $\mathcal{U}_2 \times \mathcal{U}_4$ such that $\mathcal{P}^a{}_b$ acts as the identity on the tangent space to $\mathcal{U}_2$ and as minus the identity on the tangent space to $\mathcal{U}_4$. In other words, such that

$$\mathcal{P}_2(T\mathcal{U}) = T\mathcal{U}_2 \ , \qquad \mathcal{P}_4(T\mathcal{U}) = T\mathcal{U}_4 \ . \tag{3.6}$$

This means that 'separating coordinates' can be introduced on every neighborhood of $\mathcal{Y}$ and the metric can be given the block-diagonal structure

$$\mathrm{d}s^2 = g_{mn}(y,z)\, \mathrm{d}y^m \mathrm{d}y^n + g_{ij}(y,z)\, \mathrm{d}z^i \mathrm{d}z^j \ , \tag{3.7}$$

*In the following, v^i will denote both the one-forms $v^i = v^i_a \mathrm{d}y^a$ and the corresponding dual vectors $v^i = v^{ia}\partial_a$.

where z^i and y^m are local coordinates respectively on $\mathcal{U}_2$ and $\mathcal{U}_4$ and the following is satisfied,

$$\mathcal{P}(\partial_i) = \partial_i \ , \qquad \mathcal{P}(\partial_m) = -\partial_m \ . \tag{3.8}$$

For such a metric, $\mathcal{P}_{ab}$ defined as $g_{ac}\mathcal{P}^c{}_b$ is symmetric and $\mathcal{P}^a{}_b$ is called 'metric-compatible'. The set of neighborhoods $\mathcal{U}_2$ and $\mathcal{U}_4$ represent foliations of the manifold $\mathcal{Y}$, and it can happen that the maximal leaves obtained by patching together the neighborhoods $\mathcal{U}_2$ and $\mathcal{U}_4$ constitute embedded submanifolds $\mathcal{Y}_2$ and $\mathcal{Y}_4$ of $\mathcal{Y}$, respectively [52]. In the following, it will simply be assumed that the almost product structure (3.4) is integrable. In this case, it is customary to call it local product structure.†

Since the vectors v^i span the (+1)-eigenspace of $\mathcal{P}^a{}_b$, they can be written in 'separating coordinates' as $v^i = v^{ij}\partial_j$. It follows from the block-diagonal structure (3.7) that the one-forms are given by $v^i = v^i{}_k \mathrm{d}z^k$ with $v^i{}_k = v^{ij}g_{kj}$. The second condition in (3.3) now implies that the two-forms J^x have legs only along $\mathcal{U}_4$, or in other words that

$$J^x = \tfrac{1}{2} J^x_{mn}(y,z)\, \mathrm{d}y^m \wedge \mathrm{d}y^n \ , \tag{3.9}$$

though as explicitly shown the components J^x_{mn} may still depend on both set of coordinates y^m and z^i. It can be checked that the last condition in (3.3) becomes

$$J^x \wedge J^y = 2\delta^{xy}\mathrm{vol}_4 \ , \tag{3.10}$$

with vol_4 being the volume form on $\mathcal{U}_4$. Raising an index on the two-forms J^x with the metric, one obtains a triplet of almost complex structures I^x satisfying

$$I^x I^y = -\delta^{xy}\mathbb{1} + \epsilon^{xyz} I^z \ . \tag{3.11}$$

Since the spinors need not be covariantly constant, these almost complex structures are in general not integrable and thus they do not form a hyperkähler structure on $\mathcal{Y}$ as they do on $\mathrm{K3} \times T^2$. Nevertheless, it turns out that they locally define a hyperkähler structure on $\mathcal{U}_4$ as they do on K3.

3.1.1 Geometric moduli space of SU(2)-structures

The space of possible geometrical deformations of manifolds with SU(2) structure has been discussed thoroughly in Ref. [53]. Here the results are summarized. The recipe is to project out all doublets of the SU(2) structure group. The quantities surviving the projection should lead to light modes in four dimensions. This yields in particular the right number of light gravitini in four dimensions. That the doublets should be projected out was more or less evident already from the decomposition (3.1).

† More about this issue is discussed in Appendix C. Integrability of the almost product structure for the concrete examples of SU(2)-structure manifolds that will be considered in this thesis is trivially guaranteed by the very definition of such examples as fibrations.

The local product structure $\mathcal{P}$ given by Eq. (3.4) is rigid. This means that no geometrical deformation changing this structure is allowed and therefore $\mathcal{P}$ contributes no light degrees of freedom to the moduli space. The reason for this rigidity can be seen as follows. From the definition (3.4) it is apparent that the local product structure is completely determined by a two-form $v^1 \wedge v^2$ decomposable as the product of two one-forms. The nontrivial deformations of such a two-form must have one leg in $\mathcal{U}_2$ and the other leg along $\mathcal{U}_4$, otherwise it defines the same splitting $\mathcal{U}_2 \times \mathcal{U}_4$ and therefore the same local product structure. But as shown in Ref. [53], the one-forms in $T^*\mathcal{U}_4$ are doublets of the SU(2) structure group, while $T^*\mathcal{U}_2$ contains only singlets. It follows that the deformations of the local product structure are doublets of SU(2) and therefore must be projected out, leaving us with a rigid $\mathcal{P}$.

Only deformations of both the two- and the four-dimensional component of $\mathcal{Y}$ separately are thus possible, and the total space of geometrical deformations of SU(2)-structures has a product form

$$\mathscr{M}^{\text{geom}}_{\text{SU}(2)} = \mathscr{M}^{\text{geom}}_2 \times \mathscr{M}^{\text{geom}}_4 \ , \tag{3.12}$$

where $\mathscr{M}^{\text{geom}}_2$ and $\mathscr{M}^{\text{geom}}_4$ are the spaces of allowed geometrical deformations of $\mathcal{U}_2$ and $\mathcal{U}_4$, respectively. The deformations of $\mathcal{U}_2$ are given by redefinitions of the one-forms $v^i \to \tilde{v}^i = A^i{}_j v^i$, where A is an arbitrary 2×2 real matrix, i.e. an element of

$$\text{GL}(2) = \mathbb{R} \times \text{SL}(2) \simeq \mathbb{R} \times \text{SU}(1,1) \ . \tag{3.13}$$

Looking at the definition of the v^i in Eq. (3.2), it is seen that such a redefinition of the one-forms corresponds to a redefinition of the Clifford algebra and thus to a change of the metric $g_{ij} \to \tilde{g}_{ij}$. This is even more explicit if the first relation in (3.3) is used to write

$$\tilde{g}^{kl} \tilde{v}^i_k \tilde{v}^j_l = g^{kl} v^i_k v^j_l = \delta^{ij} \ . \tag{3.14}$$

From this, it follows that if A is an $\text{SO}(2) \simeq \text{U}(1)$ matrix then $\tilde{g}_{ij} = g_{ij}$ and this subgroup therefore needs to be modded out. Moreover, both A and $-A$ define the same metric as well, and only those redefinitions with, say, $\det A > 0$ must be considered. This has the effect of modifying the factor $\mathbb{R}$ in Eq. (3.13) to include only the positive reals $\mathbb{R}^+$. The space of geometric deformations of $\mathcal{U}_2$ is in consequence

$$\mathscr{M}^{\text{geom}}_2 = \mathbb{R}^+ \times \frac{\text{SU}(1,1)}{\text{U}(1)} \ . \tag{3.15}$$

On the other hand, the analysis for the four-dimensional component goes in similar lines to the one already discussed for K3 in Section 2.2.1 and it can be summarized as follows. Recall that $\mathcal{U}_4$ is characterized by a triplet of self-dual two-forms J^x satisfying Eq. (3.10). Concentrate first on the space of two-forms $\Lambda^2_p \mathcal{U}_4$ over a point p in $\mathcal{Y}$. This space is six-dimensional, since it is given by all 4×4 antisymmetric matrices. A scalar product (φ, χ) can be introduced in this space according to

$$\varphi \wedge \chi = (\varphi, \chi) \text{vol}_4 \ , \qquad \forall \varphi, \chi \in \Lambda^2_p \mathcal{U}_4 \ , \tag{3.16}$$

and it has signature $(3,3)$, since $(\varphi, *\varphi) \geq 0, \forall \varphi \in \Lambda^2_p \mathcal{U}_4$ and there must be three self-dual and three anti-self-dual forms over a point. Using this scalar product, the condition (3.10) translates into

$$(J^x, J^y) = 2\delta^{xy} \; . \tag{3.17}$$

The forms J^x therefore define a positive three-dimensional hyperplane in $\Lambda^2_p \mathcal{U}_4$. From all the endomorphisms of $\Lambda^2_p \mathcal{U}_4$, the ones preserving the orthonormalization condition (3.17) form an $\mathrm{SO}(3,3)$ subgroup. But in obtaining the nontrivial deformations of the forms J^x it is also needed to mod out both the $\mathrm{SO}(3)$ subgroup acting on the hyperplane orthogonal to the forms J^x and the $\mathrm{SO}(3)$ subgroup that merely rotates the forms J^x into themselves. Considering also the single parameter entering the choice of volume form vol_4, it is concluded that the possible choices of self-dual two-forms over a point p of $\mathcal{Y}$ parametrize the space

$$\mathscr{M}^{\mathrm{geom}}_{4,p} = \mathbb{R}^+ \times \frac{\mathrm{SO}(3,3)}{\mathrm{SO}(3) \times \mathrm{SO}(3)} \simeq \frac{\mathrm{GL}^+(4)}{\mathrm{SO}(4)} \; . \tag{3.18}$$

This space indeed captures all degrees of freedom for the 4×4 symmetric and positive-definite matrix $g_{mn}(p)$.

Now this result needs to be extended to the whole manifold $\mathcal{Y}$. At first sight it would seem that since the space $\Lambda^2 \mathcal{U}_4 = \cup_{p\in\mathcal{Y}} \Lambda^a_p \mathcal{U}_4$ of two-forms on $\mathcal{Y}$ is infinite-dimensional there will be an infinite number of moduli. Remember though that a Kaluza-Klein reduction is to be performed on these backgrounds, and therefore only light modes need to be kept. This space can therefore be truncated to a finite-dimensional subset $\Lambda^2_{\mathrm{finite}} \mathcal{U}_4$. There must be three self-dual forms on this space that are singlets of $\mathrm{SU}(2)$. These are the forms J^x defined in Eq. (3.2). Any other self-dual two-form that is a singlet of $\mathrm{SU}(2)$ must be expressible in terms of these. The coefficients of this expansion may depend on the point of the manifold, but the truncation precisely means that only zero-modes or constant coefficients are possible. Thus there are only three self-dual forms in $\Lambda^2_{\mathrm{finite}} \mathcal{U}_4$. However, in principle nothing constrains the number $n-3$ of anti-self-dual forms, so that the dimension of $\Lambda^2_{\mathrm{finite}} \mathcal{U}_4$ is n. Performing a similar analysis as the one leading to Eq. (3.18), it is concluded that the space of deformations of the component $\mathcal{U}_4$ is

$$\mathscr{M}^{\mathrm{geom}}_{4} = \mathbb{R}^+ \times \frac{\mathrm{SO}(3,n-3)}{\mathrm{SO}(3) \times \mathrm{SO}(n-3)} \; . \tag{3.19}$$

Parametrizing this space is completely analogous to the parametrization discussed in Section 2.2.1 for K3. Denote by ω^A the n two-forms spanning $\Lambda^2_{\mathrm{finite}} \mathcal{U}_4$. An intersection matrix η^{AB} can be defined for these forms by writing

$$\omega^A \wedge \omega^B = \eta^{AB} \mathrm{e}^{\rho} \mathrm{vol}_4 \; , \tag{3.20}$$

since the only possible four-form surviving the projection must be proportional to the volume form on $\mathcal{U}_4$. For convenience, a volume modulus ρ is introduced. Due to the same argument as before, the matrix η^{AB} defined in this way has signature $(3, n-3)$. The three self-dual two-forms can be expanded as

$$J^x = \mathrm{e}^{-\frac{1}{2}\rho}\xi^x_A\omega^A \tag{3.21}$$

by introducing $3n$ parameters ξ^x_A. Using this expansion in Eq. (3.10), a constraint on the possible values of ξ^x_A is obtained in the form

$$\eta^{AB}\xi^x_A\xi^y_B = 2\delta^{xy} \; . \tag{3.22}$$

The three orthogonal and normalized vectors ξ^x_A therefore span a three-dimensional positive hyperplane

$$\mathcal{H}^3 = \mathrm{span}(\xi^1, \xi^2, \xi^3) \subset \Lambda^2_{\mathrm{finite}}\mathcal{U}_4 \simeq \mathbb{R}^{3,n-3} \; . \tag{3.23}$$

The set of these hyperplanes is precisely the second factor of the moduli space (3.19). An orthogonal transformation rotating the three vectors ξ^x_A among themselves is clearly a redundancy that must be modded out. The number of physical degrees of freedom in ξ^x_A is in consequence $3n - 6 - 3 = 3(n-3)$.

Since the space $\Lambda^2_{\mathrm{finite}}\mathcal{U}_4$ is preserved by the Hodge star operator, the following expansion can be written,

$$*\omega^A = M^A{}_B\omega^B \; . \tag{3.24}$$

Again, $M^A{}_B$ has eigenvalues $+1$ (-1) corresponding to the (anti-)self-dual linear combinations of the forms ω^A. The $+1$-eigenspace is spanned by the forms J^x and thus corresponds to the hyperplane $\mathcal{H}^3$. The orthogonal hyperplane $\mathcal{H}^{n-3}_\perp$ is then the -1-eigenspace. Due to Eq. (3.22), a projector on $\mathcal{H}^3$ can be constructed as $P^A{}_B = \frac{1}{2}\xi^{xA}\xi^x_B$ with $\xi^{xA} = \eta^{AB}\xi^x_B$. Since

$$\omega^A \wedge *\omega^B = M^B{}_C\eta^{AC} = M^{AB}\mathrm{e}^\rho \mathrm{vol}_4 \tag{3.25}$$

must be symmetric, this is enough to fix the form of the matrix $M^A{}_B$ as

$$\begin{aligned} M^A{}_B &= (+1)P^A{}_B + (-1)(\delta^A_B - P^A{}_B) \\ &= -\delta^A_B + \eta^{AC}\xi^x_C\xi^x_B \; . \end{aligned} \tag{3.26}$$

3.1.2 Ansatz for the SU(2)-structure backgrounds

It has been seen that in an SU(2)-structure manifold $\mathcal{Y}$ there is a pair of one-forms v^i, together with some number n of two-forms. The latter have been denoted by ω^A, in analogy to the harmonic forms of K3. Also, every neighborhood of $\mathcal{Y}$ can be written as a product $\mathcal{U}_2 \times \mathcal{U}_4$ of two- and four-dimensional components in such a way that v^i is in $T^*\mathcal{U}_2$ while ω^A belongs to $\Lambda^2\mathcal{U}_4$. In other words, there are local coordinates (z^i, y^m) such that

$$v^i = v^i{}_j \mathrm{d}z^j \; , \tag{3.27a}$$

$$\omega^A = \tfrac{1}{2}\omega^A_{mn}\mathrm{d}y^m \wedge \mathrm{d}y^n \; . \tag{3.27b}$$

However, up to this point this says nothing about the dependence of the components ${v^i}_j$ and ω^A_{mn} of these forms on the coordinates z^i and y^m.

In contrast to the K3 $\times T^2$ case, for a generic SU(2)-structure manifold $\mathcal{Y}$ the forms v^i and ω^A need not be closed. Nevertheless, their exterior derivatives must be expressible in terms of all possible exterior products of these forms with themselves. The most general closure algebra that can be written in this way is thus

$$\mathrm{d}v^i = \theta^i v^1 \wedge v^2 \ , \tag{3.28a}$$

$$\mathrm{d}\omega^A = T^A_{iB} v^i \wedge \omega^B \ , \tag{3.28b}$$

for some constant coefficients θ^i and T^A_{iB}. In principle, one might think of adding a term $\theta^i_A \omega^A$ to the r.h.s. of Eq. (3.28a), but a simple inspection of (3.27) rules out such a term, since it is impossible that $\mathrm{d}v^i$ has a part in $\Lambda^2\mathcal{U}_4$ if v^i is in $T^*\mathcal{U}_2$. Therefore $\theta^i_A = 0$.

Eq. (3.28a) implies that the components ${v^i}_j$ can only depend on the coordinates z^i. If this were not the case, $\mathrm{d}v^i$ would contain a term in $T^*\mathcal{U}_2 \wedge T^*\mathcal{U}_4$ and as it has been seen there is no such possibility in the r.h.s. of Eq. (3.28a). In fact, the reasoning goes really the other way round: it is precisely because the forms in $T^*\mathcal{U}_2 \wedge T^*\mathcal{U}_4$ (among others) must be projected out that $\mathrm{d}v^i$ and also $\mathrm{d}\omega^A$ must be expressible only in terms of (products of) v^i and ω^A.

On the other hand, the components ω^A_{mn} might very well depend on both sets of coordinates y^m and z^i. But considering the splitting

$$\mathrm{d} = \mathrm{d}_2 + \mathrm{d}_4 = \mathrm{d}z^i \wedge \partial_i + \mathrm{d}y^m \wedge \partial_m \tag{3.29}$$

of the exterior differential and Eqs. (3.27b) and (3.28b), it is not difficult to check that $\mathrm{d}_4\omega^A = 0$. In other words, the forms ω^A are actually closed on each slice $\mathcal{U}_4$ with constant z^i. This is also true for the forms J^x, since they are just the self-dual combinations of the forms ω^A. This means that there is locally a hyperkähler structure on $\mathcal{U}_4$. Now if all the leaves $\mathcal{U}_4$ combine to form a maximal leaf $\mathcal{Y}_4$, the latter must be a K3 and the number of two-forms is restricted to $n = 22$. This is the reason why the manifolds $\mathcal{Y}$ that will be considered here are constructed as K3 fibrations over some two-dimensional space $\mathcal{Y}_2$.

The possible values of θ^i and T^A_{iB} in the closure algebra of Eq. (3.28) are restricted by the nilpotency of exterior differentiation and by Stokes' theorem. Taking the exterior differential of Eq. (3.28a) leads to an identity, but the same operation on (3.28b) yields

$$\theta^i T^A_{iB} = \epsilon^{jk} T^A_{jC} T^C_{kB} \ . \tag{3.30}$$

Considering T^A_{iB} as a pair of matrices $T_i = (T^A_{iB})$, the last condition can be rewritten compactly as the commutation relation

$$[T_1, T_2] = \theta^i T_i \ . \tag{3.31}$$

On the other hand, Stokes' theorem implies that $\int_{\mathcal{Y}} \mathrm{d}(v^i \wedge \omega^A \wedge \omega^B) = 0$, which after making use of (3.28) imposes the additional constraint

$$\epsilon^{ij}(T^A_{jC}\eta^{CB} + T^B_{jC}\eta^{CA}) = \eta^{AB}\theta^i \ . \tag{3.32}$$

The intersection matrix η^{AB} is defined in Eq. (3.20), and as already mentioned it has signature $(3, n-3)$. Eq (3.32) can be written in matrix form as[‡]

$$T_i\eta + \eta T_i^{\mathrm{T}} = -\epsilon_{ij}\theta^j\eta \ , \tag{3.33}$$

and this can be conveniently rewritten as

$$(T_i + \tfrac{1}{2}\epsilon_{ij}\theta^j \mathbb{1})\eta + \eta(T_i + \tfrac{1}{2}\epsilon_{ij}\theta^j\mathbb{1})^{\mathrm{T}} = 0 \ . \tag{3.34}$$

This equation implies that the metric η^{AB} is invariant under transformations generated by the matrices $\Theta_i = T_i + \frac{1}{2}\epsilon_{ij}\theta^j\mathbb{1}$, and the latter must therefore be in the algebra of $\mathrm{SO}(3, n-3)$. Another way to see this is by taking the trace of Eq. (3.34) and concluding that Θ^i is traceless. In conclusion, the matrices T_i parametrizing how much $\mathrm{d}\omega^A$ deviates from zero are constrained to have the form

$$T_i = -\tfrac{1}{2}\epsilon_{ij}\theta^j\mathbb{1} + \Theta_i \qquad \text{with} \qquad \mathrm{tr}\,\Theta^i = 0 \ . \tag{3.35}$$

It is easy to check that the matrices Θ_i satisfy the same commutation relation (3.31) as the matrices T_i do, namely

$$[\Theta_1, \Theta_2] = \theta^i\Theta_i \ . \tag{3.36}$$

If the expression (3.35) for T_i is substituted back into (3.28), it is obtained

$$\mathrm{d}v^i = \theta^i v^1 \wedge v^2 \ , \tag{3.37a}$$

$$\mathrm{d}\omega^A = \tfrac{1}{2}\theta^i\epsilon_{ij}v^j \wedge \omega^A + \Theta^A_{iB}v^i \wedge \omega^B \ . \tag{3.37b}$$

In the following, two possible cases are studied separately. The first case corresponds to $\theta^i = 0$ but nonzero Θ_i. The one-forms v^i are therefore closed. It is shown in Section 3.2 that this case can be realized by considering manifolds $\mathcal{Y}$ constructed as nontrivial fibrations of K3 over a two-torus. The twisting of this fibered space is controlled by the parameters Θ^i. If they go to zero, the fibration becomes trivial and the product manifold $\mathrm{K3}\times T^2$ is recovered.

On the other hand, there is the case complementary to the first one. It sets $\Theta_i = 0$ but allows for a nonvanishing θ^i. The latter implies that the one-forms v^i are not closed and the local structure of this two-dimensional space corresponds to a twisted two-torus. As it turns out though, a twisted torus does not exist as a compact manifold, and this is the reason for the quotation marks in the label 'K3 fibration over a twisted torus' given here to this case. Nevertheless, sense can be made of the compactification of heterotic supergravity on this background in a Scherk-Schwarz fashion. The discussion is deferred to Section 3.3.

It should come as no surprise that the effective theory obtained for the general case with both θ^i and Θ_i nonvanishing is nothing but a 'sum' of the results for the two cases above.

[‡]Hopefully, the matrix $\eta = (\eta^{AB})$ will not be confused with the spinors η_i. The latter do not show up in the rest of the Chapter.

3.2 Compactification on K3 fibration over a torus

Roughly speaking, a K3 fibration over a torus is constructed by giving a dependence to the K3 metric g_{mn} on the torus coordinates z^i. As discussed in Section 2.2.1, the K3 metric is not known explicitly, but it is intimately connected to the harmonic two-forms ω^A generating the second integral cohomology $H^2(\mathrm{K3}, \mathbb{Z})$. The same effect is therefore obtained if these forms are made z-dependent. Let us set

$$\omega^A(z) = \gamma^A{}_B(z)\omega^B \ , \tag{3.38}$$

where $\omega^A = \omega^A(0)$ is a fixed choice for the harmonic forms on K3 and $\gamma = (\gamma^A{}_B)$ is a z-dependent matrix defined as

$$\gamma(z) = \exp(z^i \Theta_i) \ . \tag{3.39}$$

In going once around each of the cycles $\mathcal{C}_i$ of the torus by making $z^i \to z^i + 1$, the basis of harmonic forms $\omega^A(z)$ on the K3 fibers picks up the corresponding monodromies

$$\gamma_i = \exp \Theta_i \ . \tag{3.40}$$

If sense is to be made of the fibration, it must be required that the basis $\gamma^A_{iB}\omega^B$ be equivalent to the basis ω^A. For these two bases to define the same lattice $H^2(\mathrm{K3}, \mathbb{Z})$, they must be related as in Eq. (2.31), or in other words γ_i must be in $\mathrm{SO}(3, 19, \mathbb{Z})$. In particular, this implies that Θ_i must be in the algebra of $\mathrm{SO}(3, 19)$. It also follows that the matrix $\gamma(z)$ in Eq. (3.39) is in $\mathrm{SO}(3, 19)$ and therefore leaves the matrix η^{AB} invariant,

$$\gamma^A{}_C(z)\eta^{CD}\gamma^B{}_D(z) = \eta^{AB} \ . \tag{3.41}$$

Moreover, the monodromies γ_i must commute with each other, and as a consequence the same must be true for the generators Θ_i. Condition (3.36) with $\theta^i = 0$ is therefore fulfilled. This construction is schematically shown in Figure 3.1.

The forms $\omega^A(z)$ are certainly closed on every K3 fiber, that is $\mathrm{d}_4\omega^A = 0$. But due to the z-dependence, they fail to be closed in the whole fibered space. Taking the derivative of Eq. (3.38), it is obtained

$$\mathrm{d}\omega^A(z) = \mathrm{d}_2\omega^A(z) = \Theta^A_{iB}\mathrm{d}z^i \wedge \omega^B(z) \ . \tag{3.42}$$

Since the one-forms $v^i = \mathrm{d}z^i$ on the torus are closed, the closure algebra (3.37) with $\theta^i = 0$ is indeed satisfied. Setting $\Theta_i = 0$ eliminates the z-dependence and trivializes the fibration. The parameters Θ_i therefore measure how much the manifold $\mathcal{Y}$ deviates from $\mathrm{K3} \times T^2$.

The three self-dual two-forms J^x also depend on the torus coordinates, as can be seen from the expansion (3.21),

$$J^x(z) = \mathrm{e}^{-\frac{1}{2}\rho}\xi^x_A\omega^A(z) \ . \tag{3.43}$$

This expansion makes manifest another way of seeing this fibration. In one 'frame', the basis forms $\omega^A(z)$ vary with z^i while the moduli ρ and ξ^x_A are constant. But there is also

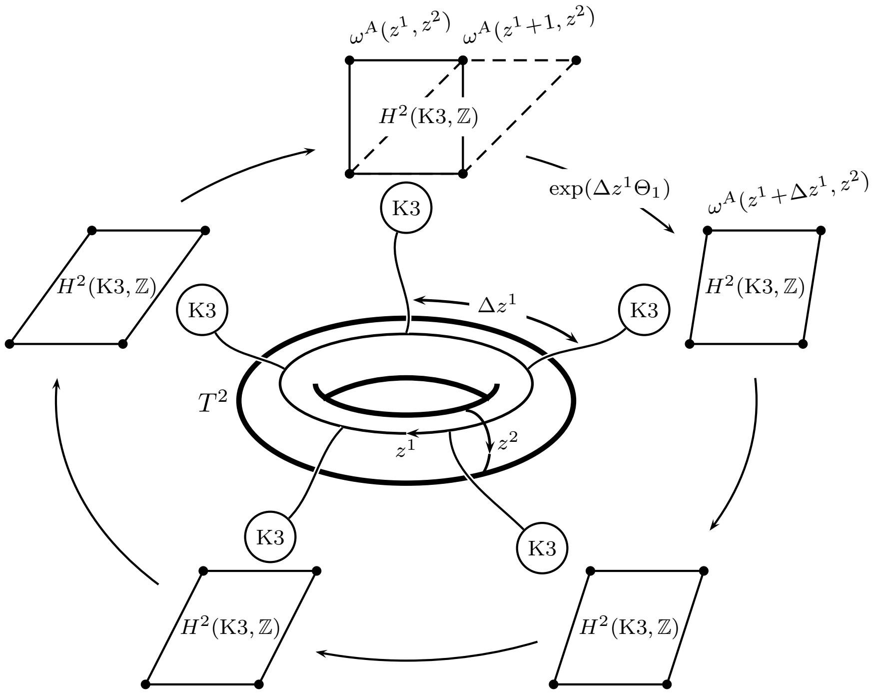

Figure 3.1: Structure of a K3 fibration over a torus.

the possibility of leaving the forms ω^A constant and transferring the z-dependence to the moduli $\rho(z)$ and $\xi^x_A(z)$ in such a way that one has the same forms $J^x(z)$. These are but two equivalent ways or 'frames' to express the same thing.

In the following, it will be useful to consider the second point of view. As already expressed, this means setting

$$J^x(z) = \mathrm{e}^{-\frac{1}{2}\rho(z)}\xi^x_A(z)\omega^A \ . \tag{3.44}$$

Comparing with Eq. (3.43) and using (3.38) it must be set

$$\mathrm{e}^{-\frac{1}{2}\rho(z)}\xi^x_A(z) = \mathrm{e}^{-\frac{1}{2}\rho}\xi^x_B\gamma^B{}_A(z) \ . \tag{3.45}$$

Deciding what z-dependence corresponds to $\rho(z)$ and what to $\xi^x_A(z)$ is based on the requirement that $\xi^x_A(z)$ must satisfy the orthonormalization condition

$$\eta^{AB}\xi^x_A(z)\xi^y_B(z) = 2\delta^{xy} \ . \tag{3.46}$$

Since $\gamma \in \mathrm{SO}(3,19,\mathbb{Z}) \subset \mathrm{SO}(3,19)$ and these transformations respect the scalar product η^{AB}, the right answer in this case is

$$\rho(z) = \rho \ , \qquad \xi^x_A(z) = \gamma^B{}_A(z)\xi^x_B \ . \tag{3.47}$$

In particular, the volume of the K3 fiber is not affected.

3.2.1 Effective theory

Now the dimensional reduction of heterotic supergravity on a K3 fibration over a torus characterized by the matrices Θ_i can be performed. The procedure is quite analogous to the one in Section 2.3 for K3$\times T^2$. In particular, the ansatz for the ten-dimensional bosonic fields g_{MN}, B_2 and A_1^a has the same form as in Eqs. (2.52) and (2.55). The metric has the block-diagonal form

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{ij}V_\mu^i V_\nu^j & V_\mu^i g_{ij} & 0 \\ g_{ij}V_\nu^j & g_{ij} & 0 \\ 0 & 0 & g_{mn}(y,z) \end{pmatrix} , \tag{3.48}$$

where the difference shows up exclusively in the z-dependence of the K3 metric. For the NS two-form and gauge potential one has

$$\begin{aligned} B_2 &= \tfrac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + B_{i\mu}\mathcal{E}^i \wedge \mathrm{d}x^\mu + \tfrac{1}{2}B_{ij}\mathcal{E}^i \wedge \mathcal{E}^j + b_A(z)\omega^A , \\ A_1^a &= A_\mu^a \mathrm{d}x^\mu + A_i^a \mathcal{E}^i , \end{aligned} \tag{3.49}$$

with $\mathcal{E}_i = \mathrm{d}z^i + V_\mu^i \mathrm{d}x^\mu$ as before. The difference in this ansatz is solely in the z-dependence given to the b_A moduli,

$$b_A(z) = \gamma^B{}_A(z) b_B . \tag{3.50}$$

From these expressions, it is clear that the spectrum of this theory coincides with that of K3$\times T^2$ compactification. Moreover, since the only changes involve scalars that in K3$\times T^2$ compactification sit in hypermultiplets, namely the scalars ξ_A^x (or equivalently $M^A{}_B$) and b_A, the vector-multiplet sector should not be affected.

The first step is to use the metric (3.48) and compute the Ricci scalar. This can be done by applying the general formula (2.64) from Chapter 2 to the metric given in Eq. (3.48). Considering that only g_{mn} depends on the internal coordinates, the first two terms in the ten-dimensional action $\mathcal{S}_\mathrm{b}$ of Eq. (2.2) lead to the Lagrangian

$$\begin{aligned} \mathscr{L}_{4,g+\Phi} = \tfrac{1}{2}\mathrm{e}^{-\phi}\big(&\mathcal{R}_4 + \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}g_{ij}V_{\mu\nu}^i V^{j,\mu\nu} + \tfrac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij} \\ &+ \tfrac{1}{4}\mathcal{V}_\mathcal{Y}^{-1}\int_\mathcal{Y} \mathcal{D}_\mu g_{mn}\mathcal{D}^\mu g^{mn}\big) - \mathscr{V}_6 , \end{aligned} \tag{3.51}$$

where

$$\mathcal{V}_\mathcal{Y} = \int_\mathcal{Y} 1 = \int_{T^2}\int_{\mathrm{K3}} 1 = \int_{T^2} \mathrm{e}^{-\rho} = \mathrm{e}^{-\rho}\int_{T^2} 1 \tag{3.52}$$

is the volume of the internal manifold $\mathcal{Y}$ and the derivative is defined as $\mathcal{D}_\mu = \partial_\mu - V_\mu^i\partial_i$. Furthermore, $\mathscr{V}_6$ is related to the curvature of the fibration, and according to Eq. (2.66) is given by

$$\mathscr{V}_6 = -\tfrac{1}{2}\mathrm{e}^{-\phi}\mathcal{V}_\mathcal{Y}^{-1}\int_\mathcal{Y} \mathcal{R}_6 , \tag{3.53}$$

with $\mathcal{R}_6$ being the Ricci scalar for the internal metric

$$g_{ab} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{mn}(y,z) \end{pmatrix} . \tag{3.54}$$

Notice that this six-dimensional metric is again of the form (2.59) for $D = 6$ and $d = 2$ if one translates

$$\begin{aligned} &x^M \to y^a , && x^\mu \to z^i , && y^a \to y^m , \\ &g_{MN} \to g_{ab} , && g_{\mu\nu} \to g_{ij} , && g_{ab} \to g_{mn} , \end{aligned} \qquad V_\mu^a \to V_i^m = 0 , \tag{3.55}$$

or in other words if one considers a total six-dimensional space with external 'spacetime' coordinates z^i and internal four-dimensional manifold parametrized by y^m. Recall that both the base torus and each K3 fiber has vanishing Ricci scalar. Equation (2.61) can thus be used to compute

$$\int_{\mathcal{Y}} \mathcal{R}_6 = \tfrac{1}{4} \int_{\mathcal{Y}} g^{ij} \partial_i g_{mn} \partial_j g^{mn} . \tag{3.56}$$

The difference in the Lagrangian $\mathscr{L}_{4,g+\Phi}$ with respect to the one obtained in the compactification on K3 $\times\, T^2$ is only in the second line of Eq. (3.51). Those two terms involve only the moduli comprised in the K3 metric. As already explained, those scalars sit in hypermultiplets of the effective theory. The contribution to the four-dimensional effective Lagrangian $\mathscr{L}_{\mathrm{h}}$ involving scalars in hypermultiplets and arising from $\mathscr{L}_{4,g+\Phi}$ in Eq. (3.51) is therefore given by

$$\mathscr{L}_{\mathrm{h},g} = \tfrac{1}{8} \mathrm{e}^{-\phi} \mathcal{V}_{\mathcal{Y}}^{-1} \int_{\mathcal{Y}} \big(\mathcal{D}_\mu g_{mn} \mathcal{D}^\mu g^{mn} + g^{ij} \partial_i g_{mn} \partial_j g^{mn} \big) . \tag{3.57}$$

The first term in this expression is a kinetic term with a modified spacetime derivative $\mathcal{D}_\mu = \partial_\mu - V_\mu^i \partial_i$. This already makes clear how the z-dependence introduced via the fibration indeed induces a gauging, turning ordinary spacetime derivatives into covariant ones. On the other hand, the second term in $\mathscr{L}_{\mathrm{h},g}$ gives rise to a potential, since it involves no spacetime derivative.

Equation (3.57) can be easily expressed in terms of the modulus ρ and either ξ_A^x or $M^A{}_B$. This can be done by using the line element in the space of K3 metrics computed in Appendix B and given in Eqs. (B.19) and (B.21). One just needs to substitute, for example, $\delta\rho$ by $\mathcal{D}_\mu \rho(z)$ and $\partial_i \rho(z)$, etc. But before doing that, let us compute the following derivatives for $\xi_A^x(z)$ as defined in Eq. (3.47),

$$\begin{aligned} \partial_i \xi_A^x(z) &= \partial_i \gamma^B{}_A(z) \xi_B^x = \gamma^B{}_A(z) \Theta_{iB}^C \xi_C^x , \\ \mathcal{D}_\mu \xi_A^x(z) &= \gamma^B{}_A(z) \partial_\mu \xi_B^x - V_\mu^i \partial_i \gamma^B{}_A(z) \xi_B^x = \gamma^B{}_A(z) \big(\partial_\mu \xi_B^x - V_\mu^i \Theta_{iB}^C \xi_C^x \big) . \end{aligned} \tag{3.58}$$

It turns out that all the z-dependence drops out because all factors of $\gamma(z)$ cancel each other. The reason for this is Eq. (3.41). The integral on the torus is in consequence trivial.

Additionally, a Weyl rescaling of the spacetime metric $g_{\mu\nu} \to \mathrm{e}^{\phi} g_{\mu\nu}$ must be performed. The final result is

$$\begin{aligned}\mathscr{L}_{\mathrm{h},g} &= -\tfrac{1}{8}\partial_\mu\rho\partial^\mu\rho + \tfrac{1}{4}(\eta^{AB} - \tfrac{1}{2}\xi^{yA}\xi^{yB})D_\mu\xi^x_A D^\mu\xi^x_B - \mathscr{V}_{\mathrm{h},g} \\ &= -\tfrac{1}{8}\partial_\mu\rho\partial^\mu\rho + \tfrac{1}{16}D_\mu M^A{}_B D^\mu M^B{}_A - \mathscr{V}_{\mathrm{h},g} \ ,\end{aligned} \tag{3.59}$$

where the covariant derivatives are given by

$$\begin{aligned}D_\mu\xi^x_A &= \partial_\mu\xi^x_A - V^i_\mu\Theta^B_{iA}\xi^x_B \ , \\ D_\mu M^A{}_B &= \partial_\mu M^A{}_B - V^i_\mu(M^A{}_C\Theta^C_{iB} - \Theta^A_{iC}M^C{}_B) \ ,\end{aligned} \tag{3.60}$$

and the potential is

$$\begin{aligned}\mathscr{V}_{\mathrm{h},g} &= \tfrac{1}{8}\mathrm{e}^{\phi}g^{ij}\left(\xi^x_A\Theta^A_{iB}\xi^{yB}\xi^x_C\Theta^C_{jD}\xi^{yD} + 2\xi^x_A\Theta^A_{iB}\Theta^B_{jC}\xi^{xC}\right) \\ &= -\tfrac{1}{16}\mathrm{e}^{\phi}g^{ij}(M^A{}_C\Theta^C_{iB} - \Theta^A_{iC}M^C{}_B)(M^B{}_D\Theta^D_{jA} - \Theta^B_{jD}M^D{}_A) \ .\end{aligned} \tag{3.61}$$

Now the terms in the ten-dimensional action involving the NS two-form B_2 and the one-form A^a_1 need to be worked out. As already noted, the only difference in the ansatz for these fields with respect to the $\mathrm{K3}\times T^2$ case is in the term $b_A(z)\omega^A$ in the expansion of B_2. This term contributes only to the hypermultiplet sector of the effective theory. If it is substituted into the kinetic term for the NS two-form B_2 in the action $\mathcal{S}_\mathrm{b}$, a contribution to $\mathscr{L}_\mathrm{h}$ is generated in the form

$$\mathscr{L}_{\mathrm{h},b} = -\tfrac{1}{4}\mathrm{e}^{-\phi}\mathcal{V}_{\mathcal{Y}}^{-1}\int_{T^2}\left(\mathcal{D}_\mu b_A(z)\mathcal{D}^\mu b_B(z) + g^{ij}\partial_i b_A(z)\partial_j b_B(z)\right)\int_{\mathrm{K3}}\omega^A\wedge *\omega^B \ . \tag{3.62}$$

The derivatives of $b_A(z)$ as defined in Eq. (3.50) can be computed and substituted here. Once again, the factors of $\gamma(z)$ cancel and all the z-dependence drops out. The integral over the torus is trivial. Recalling that by definition M^{AB} is given by Eq. (2.35) and performing the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^{\phi}g_{\mu\nu}$, it is finally obtained

$$\mathscr{L}_{\mathrm{h},b} = -\tfrac{1}{4}\mathrm{e}^{\rho}M^{AB}D_\mu b_A D^\mu b_B - \mathscr{V}_{\mathrm{h},b} \ , \tag{3.63}$$

where the covariant derivative takes the form

$$D_\mu b_A = \partial_\mu b_A - V^i_\mu\Theta^B_{iA}b_B \tag{3.64}$$

and the potential is

$$\mathscr{V}_{\mathrm{h},b} = \tfrac{1}{4}\mathrm{e}^{\phi}g^{ij}M^{AB}\Theta^C_{iA}\Theta^D_{jB}b_C b_D \ . \tag{3.65}$$

At the end of the computation, the total effective Lagrangian is

$$\begin{aligned}\mathscr{L}_4 = \tfrac{1}{2}\mathcal{R}_4 + \tfrac{1}{4}I_{IJ}(v)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{8}R_{IJ}(v)\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\ - G_{p\bar{q}}(v)\partial_\mu v^p\partial^\mu\bar{v}^{\bar{q}} - h_{uv}(q)D_\mu q^u D^\mu q^v - \mathscr{V}_\mathrm{h}(q) \ ,\end{aligned} \tag{3.66}$$

where the last two terms correspond precisely to the Lagrangian $\mathscr{L}_\mathrm{h} = \mathscr{L}_{\mathrm{h},g} + \mathscr{L}_{\mathrm{h},b}$ for the hypermultiplet sector and are given by the sum of Eqs. (3.57) and (3.63). As already mentioned, the vector multiplet sector is exactly as in the $\mathrm{K3} \times T^2$ compactification. Also the metric $h_{uv}(q)$ of the σ-model for hypermultiplet scalars q^u is the same. The moduli spaces $\mathscr{M}_\mathrm{v}$ and $\mathscr{M}_\mathrm{h}$ are therefore as in Eqs. (2.99) and (2.103). The difference is that now some scalars q^u in hypermultiplets q^u are charged with respect to the Abelian Kaluza-Klein vectors V^i_μ. The covariant derivatives of hypermultiplet scalars are generically of the form given in Eqs. (3.60) and (3.64), that is

$$D_\mu q^u = \partial_\mu q^u + k^{q^u}_{\mathcal{A}^I} \mathcal{A}^I_\mu = \partial_\mu q^u + k^{q^u}_{V^i} V^i_\mu \ . \tag{3.67}$$

In our case, the Killing vectors have the expressions

$$k^{\rho}_{V^i} = 0 \ , \qquad k^{\xi^x_A}_{V^i} = -\Theta^B_{iA} \xi^x_B \ , \qquad k^{b_A}_{V^i} = -\Theta^B_{iA} b_B \ . \tag{3.68}$$

It is seen that the torsion parameters Θ_i are indeed the charge matrices. A potential $\mathscr{V}_\mathrm{h}(q)$ is also generated and it is given by the sum of the contributions in Eqs. (3.61) and (3.65),

$$\begin{aligned} \mathscr{V}_\mathrm{h} = & -\tfrac{1}{16} \mathrm{e}^\phi g^{ij} \big[(M^A{}_C \Theta^C_{iB} - \Theta^A_{iC} M^C{}_B)(M^B{}_D \Theta^D_{jA} - \Theta^B_{jD} M^D{}_A) - 4 M^{AB} \Theta^C_{iA} \Theta^D_{jB} b_C b_D \big] \\ = & \ \tfrac{1}{8} \mathrm{e}^\phi g^{ij} \big(\xi^x_A \Theta^A_{iB} \xi^{yB} \xi^x_C \Theta^C_{jD} \xi^{yD} + 2 \xi^x_A \Theta^A_{iB} \Theta^B_{jC} \xi^{xC} \\ & \qquad + 2 b_A \Theta^A_{iB} \Theta^B_{jC} b^C + 2 b_A \Theta^A_{iB} \xi^{yB} b_C \Theta^C_{jD} \xi^{yD} \big) \ . \end{aligned} \tag{3.69}$$

Here the matrices Θ_i appear as masses for the moduli fields. That the Lagrangian in Eq. (3.66) is indeed consistent with the general form given in Eq. (A.1) for gauged $\mathscr{N} = 2$ supergravity will be verified in Section 3.4.

If a matrix $\mathcal{M}^P{}_Q$ is introduced as defined in Eq. (2.83), the last two terms in the Lagrangian (3.66) can be written as

$$\begin{aligned} \mathscr{L}_\mathrm{h} &= -\tfrac{1}{16} \mathrm{tr}\big(D_\mu \mathcal{M} D^\mu \mathcal{M} \big) - \mathscr{V}_\mathrm{h} \ , \\ \mathscr{V}_\mathrm{h} &= -\tfrac{1}{16} \mathrm{e}^\phi g^{ij} \mathrm{tr}\big([\mathcal{M}, \mathcal{T}_i][\mathcal{M}, \mathcal{T}_j] \big) \ , \end{aligned} \tag{3.70}$$

where the covariant derivative is

$$D_\mu \mathcal{M} = \partial_\mu \mathcal{M} - V^i_\mu [\mathcal{M}, \mathcal{T}_i] \tag{3.71}$$

and the matrices $\mathcal{T}_i = (\mathcal{T}^P_{iQ})$ are given by

$$\mathcal{T}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Theta_i \end{pmatrix} \ . \tag{3.72}$$

These matrices are in the algebra of $\mathrm{SO}(4,20)$, which is the isometry group of the moduli space $\mathscr{M}_\mathrm{h}$ in Eq. (2.103). Moreover, $\exp \mathcal{T}_i$ is an element of the U-duality group $\mathrm{SO}(4,20,\mathbb{Z})$ of the full heterotic string theory compactified on K3, since $\exp \Theta_i$ is in $\mathrm{SO}(3,19,\mathbb{Z})$.

3.3 Compactification on 'K3 fibration over twisted torus'

Now let us consider a nonzero θ^i. For simplicity, it can be set $\Theta_i = 0$ and leave the discussion of the general case to Section 3.3.2. This is a complementary situation to the one analyzed in Section 3.2, and it will be seen that the general case is just a 'sum' of the results for the two cases. The closure relations (3.37) for $\Theta_i = 0$ and a generic θ^i take the form

$$\mathrm{d}v^i = \theta^i v^1 \wedge v^2 \;, \tag{3.73a}$$

$$\mathrm{d}\omega^A = \tfrac{1}{2}\theta^i \epsilon_{ij} v^j \wedge \omega^A \;. \tag{3.73b}$$

Intuitively, the first of these equations says already that the parameters θ^i introduce some torsion in the torus base. The equation is indeed the extrapolation to two dimensions of the algebra $\mathrm{d}v^i = \theta^i_{jk} v^j \wedge v^k$ satisfied by the m one-forms v^i, $i = 1, \ldots, m$ defining a twisted m-torus. The latter is constructed as a two-torus successively fibered over circles in much the same way a K3 fibration over a torus was considered in Section 3.2. A two-torus does not really exist though, which explains the quotation marks in the title. Nevertheless, sense can be made of this case as will be explained below.

Eq. (3.73a) can be solved locally in the following way. First notice that

$$\mathrm{d}(\epsilon_{ij}\theta^i v^j) = \epsilon_{ij}\theta^i \mathrm{d}v^j = \epsilon_{ij}\theta^i\theta^j v^1 \wedge v^2 = 0 \;. \tag{3.74}$$

A coordinate z^1 can therefore be introduced such that

$$\epsilon_{ij}\theta^i v^j = -\vartheta \mathrm{d}z^1 \;, \tag{3.75}$$

where $\vartheta = (\delta_{ij}\theta^i\theta^j)^{1/2}$ was defined and a minus sign was included for convenience. It is also not difficult to see that

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{-\vartheta z^1}\delta_{ij}\theta^i v^j) &= \mathrm{e}^{-\vartheta z^1}\delta_{ij}\theta^i \mathrm{d}v^j - \mathrm{e}^{-\vartheta z^1}\delta_{ij}\theta^i \vartheta \mathrm{d}z^1 \wedge v^j \\ &= \vartheta^2 \mathrm{e}^{-\vartheta z^1} v^1 \wedge v^2 + \mathrm{e}^{-\vartheta z^1}\delta_{ij}\theta^i \epsilon_{kl}\theta^k v^l \wedge v^j = 0 \;. \end{aligned} \tag{3.76}$$

A second coordinate can thus be introduced as

$$\mathrm{e}^{-\vartheta z^1}\delta_{ij}\theta^i v^j = \vartheta \mathrm{d}z^2 \;. \tag{3.77}$$

Now Eqs. (3.75) and (3.77) can be inverted to obtain

$$\vartheta v^i = \epsilon_{ij}\theta^j \mathrm{d}z^1 + \theta^i \mathrm{e}^{\vartheta z^1}\mathrm{d}z^2 \;, \tag{3.78}$$

or more explicitly

$$\begin{aligned} \vartheta v^1 &= \theta^2 \mathrm{d}z^1 + \theta^1 \mathrm{e}^{\vartheta z^1}\mathrm{d}z^2 \;, \\ \vartheta v^2 &= -\theta^1 \mathrm{d}z^1 + \theta^2 \mathrm{e}^{\vartheta z^1}\mathrm{d}z^2 \;. \end{aligned} \tag{3.79}$$

From the two parameters θ^1 and θ^2 actually only one has physical significance, since a new set of one-forms $\tilde{v}^i = A^i{}_j v^j$ can always be defined with A an O(2) matrix. The wedge product $\tilde{v}^1 \wedge \tilde{v}^2 = v^1 \wedge v^2$ is invariant and therefore

$$\mathrm{d}\tilde{v}^i = \tilde{\theta}^i \tilde{v}^1 \wedge \tilde{v}^2 \tag{3.80}$$

with redefined parameters $\tilde{\theta}^i = A^i{}_j \theta^j$. By such a rotation, one can always set one of the two components of θ^i to zero, with the nonzero one being positive. Let us therefore set $\theta^1 = 0$ and $\theta^2 = \vartheta$ on Eq. (3.79). The result is

$$v^1 = \mathrm{d}z^1 \ , \qquad v^2 = \mathrm{e}^{\vartheta z^1} \mathrm{d}z^2 \ , \tag{3.81}$$

or in terms of the components $v^i = v^i{}_j \mathrm{d}z^j$,

$$v^i{}_j(z^1) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\vartheta z^1} \end{pmatrix} \ . \tag{3.82}$$

Furthermore, Eq. (3.73b) reads

$$\mathrm{d}\omega^A = -\tfrac{1}{2}\vartheta \mathrm{d}z^1 \wedge \omega^A \ . \tag{3.83}$$

Eqs. (3.81) and (3.83) can be satisfied if one considers the product space $\mathrm{K3} \times S^1_2$, where the circle S^1_2 is parametrized by the coordinate z^2, and fibers it over another circle S^1_1 parametrized by the coordinate z^1. According to the second equality in (3.81), the length L_2 of the S^1_2 factor in the fiber varies as

$$L_2 \sim \int_{S^1_2} v^2 = \mathrm{e}^{\vartheta z^1} \ . \tag{3.84}$$

On the other hand, Eq. (3.83) can be satisfied if one sets

$$\omega^A(z^1) = \mathrm{e}^{-\frac{1}{2}\vartheta z^1} \omega^A \ . \tag{3.85}$$

This is just a rescaling of the forms ω^A. Since the wedge product of two such forms is proportional to the volume form on K3, the K3 volume $\mathcal{V}_{\mathrm{K3}} = \mathrm{e}^{-\rho}$ must vary on the base circle according to

$$\mathcal{V}_{\mathrm{K3}}(z^1) = \mathrm{e}^{-\vartheta z^1} \mathcal{V}_{\mathrm{K3}} \ , \tag{3.86}$$

or equivalently

$$\rho(z^1) = \rho + \vartheta z^1 \ . \tag{3.87}$$

Notice that the total volume of the fiber, $\mathcal{V}_{\mathrm{K3}} L_2$, is constant.

Another way to arrive at Eq. (3.87) is to consider the expansion of $J^x(z^1)$ in the two frames,

$$J^x(z^1) = \mathrm{e}^{-\frac{1}{2}\rho} \xi^x_A \omega^A(z^1) = \mathrm{e}^{-\frac{1}{2}\rho - \frac{1}{2}\vartheta z^1} \xi^x_A \omega^A = \mathrm{e}^{-\frac{1}{2}\rho(z^1)} \xi^x_A(z^1) \omega^A \ . \tag{3.88}$$

Since the z^1-dependence in Eq. (3.85) is juts a rescaling, it must go all to the modulus ρ. This is a consequence of the moduli ξ^x_A having fixed norm. In conclusion,

$$\rho(z^1) = \rho + \vartheta z^1 \ , \qquad \xi^x_A(z^1) = \xi^x_A \ . \tag{3.89}$$

Since the dependence of the volumes of each factor on the fiber is monotonic, there is no way the fibers can be patched after going once around the base circle. Therefore, this fibration does not exist as a compact manifold. Nevertheless, one can still make sense of this background. As it turns out, one can exploit the fact that heterotic supergravity compactified on K3 $\times S^1$ has a global symmetry that serves to make the necessary identifications. This is just the Scherk-Schwarz program.

3.3.1 Effective theory

One can now proceed to the computation of the effective action for this background. A metric ansatz can be written in analogy to Eq. (2.50), only this time the forms $\mathrm{d}z^i$ on the torus are replaced by the twisted forms v^i satisfying Eq. (3.73a),

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{ij}(v^i + V^i_\mu\mathrm{d}x^\mu)(v^j + V^j_\nu\mathrm{d}x^\nu) + g_{mn}(y,z^1)\mathrm{d}y^m\mathrm{d}y^n \\ &= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \tilde{g}_{ij}(\mathrm{d}z^i + \tilde{V}^i_\mu\mathrm{d}x^\mu)(\mathrm{d}z^j + \tilde{V}^j_\nu\mathrm{d}x^\nu) + g_{mn}(y,z^1)\mathrm{d}y^m\mathrm{d}y^n \ . \end{aligned} \tag{3.90}$$

In this expression, g_{ij} is a z-independent metric for the twisted torus, and by substituting $v^i = v^i{}_j\mathrm{d}z^j$ with $v^i{}_j$ defined in Eq. (3.82) the following z-dependent quantities were defined,

$$\begin{aligned} \tilde{g}_{ij}(z^1) &= g_{kl}v^k{}_i v^l{}_j = \begin{pmatrix} g_{11} & \mathrm{e}^{\vartheta z^1}g_{12} \\ \mathrm{e}^{\vartheta z^1}g_{12} & \mathrm{e}^{2\vartheta z^1}g_{22} \end{pmatrix} \ , \\ \tilde{V}^i_\mu(z^1) &= (v^{-1})^i{}_j V^j_\mu = \begin{pmatrix} V^1_\mu \\ \mathrm{e}^{-\vartheta z^1}V^2_\mu \end{pmatrix} \ . \end{aligned} \tag{3.91}$$

Here the matrix $v^{-1} = \mathrm{diag}(1, \mathrm{e}^{-\vartheta z^1})$ is the inverse of Eq. (3.82). The metric for the basis $(\mathrm{d}x^\mu, \mathrm{d}z^i, \mathrm{d}y^m)$ can therefore be written in the block form

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + \tilde{g}_{ij}\tilde{V}^i_\mu\tilde{V}^j_\nu & \tilde{V}^i_\mu(z^1)\tilde{g}_{ij}(z^1) & 0 \\ \tilde{g}_{ij}(z^1)\tilde{V}^j_\nu(z^1) & \tilde{g}_{ij}(z^1) & 0 \\ 0 & 0 & g_{mn}(y,z^1) \end{pmatrix} \ . \tag{3.92}$$

Since the metric of the two-dimensional factor is affected, some differences are expected to appear in the vector multiplet sector with respect to the K3 $\times T^2$ compactification.

Once again, the first two-terms involving the Ricci scalar and the dilaton in the action $\mathcal{S}_\mathrm{b}$ in Eq. (2.2) can be computed by making use of formula (2.64) with the metric g_{MN}

given in Eq. (3.92). The result is

$$\mathscr{L}_{4,g+\Phi} = \tfrac{1}{2}\mathcal{V}_{\mathcal{Y}}^{-1}\int_{\mathcal{Y}} \mathrm{e}^{-\phi}\big(\mathcal{R}_4 + \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}\tilde{g}_{ij}\tilde{V}^i_{\mu\nu}\tilde{V}^{j,\mu\nu} + \tfrac{1}{4}\tilde{\mathcal{D}}_\mu\tilde{g}_{ij}\tilde{\mathcal{D}}^\mu\tilde{g}^{ij} \\ + \tfrac{1}{4}\mathcal{D}_\mu g_{mn}\mathcal{D}^\mu g^{mn}\big) - \mathscr{V}_6 \ , \tag{3.93}$$

where the four-dimensional dilaton was defined as $\phi = \Phi - \frac{1}{2}\ln\mathcal{V}_{\mathcal{Y}}$ and $\mathcal{V}_{\mathcal{Y}}$ is the volume of $\mathcal{Y}$. The derivative $\mathcal{D}_\mu$ is given by

$$\mathcal{D}_\mu = \partial_\mu - \tilde{V}^i_\mu\partial_i = \partial_\mu - \tilde{V}^1_\mu\partial_{z^1} = \partial_\mu - V^1_\mu\partial_{z^1} \ , \tag{3.94}$$

since nothing depends on z^2. The remaining quantities in the Lagrangian of Eq. (3.93) are computed according to the definitions (2.62). The field strengths $\tilde{V}^i_{\mu\nu} = \mathcal{D}_\mu\tilde{V}^i_\nu - \mathcal{D}_\nu\tilde{V}^i_\mu$ for the vectors $\tilde{V}^i_\mu$ turn out to be

$$\begin{aligned} \tilde{V}^1_{\mu\nu} &= V^1_{\mu\nu} = \partial_\mu V^1_\nu - \partial_\nu V^1_\mu \ , \\ \tilde{V}^2_{\mu\nu} &= \mathrm{e}^{\vartheta z^1}V^2_{\mu\nu} = \mathrm{e}^{\vartheta z^1}\big(\partial_\mu V^2_\nu - \partial_\nu V^2_\mu + \vartheta V^1_\mu V^2_\nu - \vartheta V^1_\nu V^2_\mu\big) \ . \end{aligned} \tag{3.95}$$

This means that the vector V^1_μ is neutral, while the vector V^2_μ has charge ϑ with respect to the U(1) gauge field V^1_μ. On the other hand, the derivative $\tilde{\mathcal{D}}_\mu\tilde{g}_{ij}$ can also be computed from the expression $\tilde{\mathcal{D}}_\mu\tilde{g}_{ij} = \mathcal{D}_\mu\tilde{g}_{ij} - \tilde{g}_{ik}\partial_j\tilde{V}^k_\mu - \tilde{g}_{kj}\partial_i\tilde{V}^k_\mu$, and the result is

$$\begin{aligned} \tilde{\mathcal{D}}\tilde{g}_{11} &= D_\mu g_{11} = \partial_\mu g_{11} + 2\vartheta V^2_\mu g_{12} \ , \\ \tilde{\mathcal{D}}\tilde{g}_{12} &= \mathrm{e}^{\vartheta z^1}D_\mu g_{12} = \mathrm{e}^{\vartheta z^1}\big(\partial_\mu g_{12} - \vartheta V^1_\mu g_{12} + \vartheta V^2_\mu g_{22}\big) \ , \\ \tilde{\mathcal{D}}\tilde{g}_{22} &= \mathrm{e}^{2\vartheta z^1}D_\mu g_{22} = \mathrm{e}^{2\vartheta z^1}\big(\partial_\mu g_{22} - 2\vartheta V^1_\mu g_{22}\big) \ . \end{aligned} \tag{3.96}$$

There is also the potential $\mathscr{V}_6$, that according to Eq. (2.66) is related to the curvature of $\mathcal{Y}$ as

$$\mathscr{V}_6 = -\tfrac{1}{2}\mathrm{e}^{-\phi}\mathcal{V}_{\mathcal{Y}}^{-1}\int_{\mathcal{Y}}\mathcal{R}_6 \ , \tag{3.97}$$

where $\mathcal{R}_6$ is the Ricci scalar of the metric

$$g_{ab} = \begin{pmatrix} g_{11} & \mathrm{e}^{\vartheta z^1}g_{12} & 0 \\ \mathrm{e}^{\vartheta z^1}g_{12} & \mathrm{e}^{2\vartheta z^1}g_{22} & 0 \\ 0 & 0 & g_{mn}(y,z^1) \end{pmatrix} . \tag{3.98}$$

Once again, this Ricci scalar can be computed by applying the formula (2.61) with $D = 6$ and $d = 1$ if one translates

$$\begin{aligned} & x^M \to y^a \ , && \\ & x^\mu \to z^1 \ , && g_{ab} \to \begin{pmatrix} \mathrm{e}^{2\vartheta z^1}g_{22} & 0 \\ 0 & g_{mn}(y,z^1) \end{pmatrix} \ , \\ & y^a \to (z^2, y^m) \ , && \\ & g_{MN} \to g_{ab} \ , && V^a_\mu \to \begin{pmatrix} V^2_1 \\ V^m_1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\vartheta z^1}\dfrac{g_{12}}{g_{22}} \\ 0 \end{pmatrix} \ . \\ & g_{\mu\nu} \to g_{11} \ , && \end{aligned} \tag{3.99}$$

The other necessary ingredient is that the dependence of the K3 fiber metric g_{mn} on the coordinate z^1 is through an overall factor of $\exp(-\frac{1}{2}\vartheta z^1)$. This means that $\partial_{z^1} g_{mn} = -\frac{1}{2}\vartheta g_{mn}$. Putting all this together, it is found

$$\mathcal{R}_6 = -\tfrac{5}{4}\vartheta^2 g^{11} \ , \tag{3.100}$$

which upon substitution into Eq. (3.97) gives

$$\mathscr{V}_6 = \tfrac{5}{8}\mathrm{e}^{-\phi}\vartheta^2 g^{11} \ . \tag{3.101}$$

Now this potential and the derivatives (3.95) and (3.96) can be substituted in the Lagrangian of Eq. (3.93). The term involving the metric g_{mn} of the K3 fiber can be computed by using the line element Eq. (B.21). The variation of the parameters ξ^x_A or equivalently $M^A{}_B$ must be substituted by the spacetime derivatives $\partial_\mu \xi^x_A$ or $\partial_\mu M^A{}_B$. In the case of the modulus ρ, its variation $\delta\rho$ must be substituted by the derivative

$$\mathcal{D}_\mu \rho(z^1) = \partial_\mu \rho(z^1) - V^1_\mu \partial_{z^1}\rho(z^1) = \partial_\mu\rho - \vartheta V^1_\mu = D_\mu \rho \ , \tag{3.102}$$

where Eq. (3.87) was used. All the z^1-dependence in the integrand of Eq. (3.93) cancels and the integral over the internal manifold $\mathcal{Y}$ is trivial. The final result for the Lagrangian is

$$\begin{aligned}\mathscr{L}_{4,g+\Phi} = \tfrac{1}{2}\mathrm{e}^{-\phi}\big(\mathcal{R}_4 + \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}g_{ij}V^i_{\mu\nu}V^{j,\mu\nu} + \tfrac{1}{4}D_\mu g_{ij}D^\mu g^{ij} - \tfrac{5}{4}\vartheta^2 g^{11} \\ - \tfrac{1}{4}D_\mu\rho D^\mu\rho + \tfrac{1}{8}\partial_\mu M^A{}_B \partial^\mu M^B{}_A\big) \ .\end{aligned} \tag{3.103}$$

The next step is to compute the terms in the ten-dimensional action $\mathcal{S}_\mathrm{b}$ that depend on the NS two-form B_2 and the gauge one-forms A^a_1. An ansatz for these fields can be written in analogy to the K3 $\times\, T^2$ case as

$$\begin{aligned}B_2 &= \tfrac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu + B_{i\mu}\mathcal{E}^i\wedge\mathrm{d}x^\mu + \tfrac{1}{2}B_{ij}\mathcal{E}^i\wedge\mathcal{E}^j + b_A(z^1)\omega^A \ , \\ A^a_1 &= A^a_\mu \mathrm{d}x^\mu + A^a_i\mathcal{E}^i \ ,\end{aligned} \tag{3.104}$$

where now the one-forms are defined as

$$\mathcal{E}^i = v^i + V^i_\mu \mathrm{d}x^\mu \tag{3.105}$$

and the $b_A(z^1)$ moduli have a z^1-dependence given by

$$b_A(z^1) = \mathrm{e}^{-\frac{1}{2}\vartheta z^1} b_A \ . \tag{3.106}$$

The latter is a consequence of transferring the z^1-dependence (3.85) from the forms ω^A to the b_A fields as in the former Section. The main difference in computing the field strengths comes from the derivative of the forms $\mathcal{E}_i$. Taking the exterior differential of Eq. (3.105) and recalling the expressions (3.81), it is obtained

$$\begin{aligned}\mathrm{d}\mathcal{E}^1 &= \tfrac{1}{2}V^1_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \ , \\ \mathrm{d}\mathcal{E}^2 &= \tfrac{1}{2}V^2_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu - \vartheta\epsilon_{ij}V^i_\mu\mathrm{d}x^\mu\wedge\mathcal{E}^j + \vartheta\mathcal{E}^1\wedge\mathcal{E}^2 \ ,\end{aligned} \tag{3.107}$$

where the field strengths $V^i_{\mu\nu}$ were defined in Eq. (3.95). These expressions can now be used in the computation of the exterior differentials of the ansatz (3.104). Performing here also the field redefinitions (2.76), the result for $F^{\rm a}_{2} = {\rm d}A^{\rm a}_{1}$ and $H_{3} = {\rm d}B_{2} - \frac{1}{2}A^{\rm a}_{1} \wedge F^{\rm a}_{2}$ is

$$\begin{aligned} F^{\rm a}_{2} &= \tfrac{1}{2}(F^{\rm a}_{\mu\nu} + A^{\rm a}_i V^i_{\mu\nu})\,{\rm d}x^\mu \wedge {\rm d}x^\nu + D_\mu A^{\rm a}_i\,{\rm d}x^\mu \wedge \mathcal{E}^i + \vartheta A^{\rm a}_2\,\mathcal{E}^1 \wedge \mathcal{E}^2 \ , \\ H_{3} &= \tfrac{1}{2}(\partial_\mu B_{\nu\rho} - \tfrac{1}{2}B_{i\mu}V^i_{\nu\rho} - \tfrac{1}{2}V^i_\mu B_{i\nu\rho} - \tfrac{1}{2}A^{\rm a}_\mu F^{\rm a}_{\nu\rho} + 2\vartheta B_{2\mu}V^1_\nu V^2_\rho)\,{\rm d}x^\mu \wedge {\rm d}x^\nu \wedge {\rm d}x^\rho \\ &\quad - \tfrac{1}{2}(B_{i\mu\nu} + A^{\rm a}_i F^{\rm a}_{\mu\nu} + C_{ij}V^j_{\mu\nu})\,{\rm d}x^\mu \wedge {\rm d}x^\nu \wedge \mathcal{E}^i + \tfrac{1}{2}(D_\mu B_{ij} + A^{\rm a}_i D_\mu A^{\rm a}_j)\,{\rm d}x^\mu \wedge \mathcal{E}^i \wedge \mathcal{E}^j \\ &\quad + \mathcal{D}_\mu b_A(z^1)\,{\rm d}x^\mu \wedge \omega^A + \partial_{z^1} b_A(z^1)\,\mathcal{E}^1 \wedge \omega^A \ . \end{aligned} \tag{3.108}$$

The field strengths and the covariant derivatives in these expressions are given by

$$\begin{aligned} F^{\rm a}_{\mu\nu} &= \partial_\mu A^{\rm a}_\nu - \partial_\nu A^{\rm a}_\mu \ , \\ B_{i\mu\nu} &= \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} + \vartheta\epsilon_{ij}V^j_\mu B_{2\nu} - \vartheta\epsilon_{ij}V^j_\nu B_{2\mu} \ , \\ D_\mu A^{\rm a}_i &= \partial_\mu A^{\rm a}_i + \vartheta\epsilon_{ij}A^{\rm a}_2 V^j_\mu \ , \\ D_\mu B_{12} &= \partial_\mu B_{12} - \vartheta B_{2\mu} - \vartheta V^1_\mu B_{12} \ , \\ \mathcal{D}_\mu b_A(z^1) &= (\partial_\mu - V^1_\mu \partial_{z^1}) b_A(z^1) = {\rm e}^{-\frac{1}{2}\vartheta z^1}(\partial_\mu b_A + \tfrac{1}{2}\vartheta V^1_\mu b_A) = {\rm e}^{-\frac{1}{2}\vartheta z^1} D_\mu b_A \ . \end{aligned} \tag{3.109}$$

The field strengths $F^{\rm a}_{2}$ and H_{3} in Eq. (3.108) can be substituted in the last two terms of the action $\mathcal{S}_{\rm b}$ to obtain the following contribution to the effective Lagrangian,

$$\begin{aligned} \mathscr{L}_{4,A+B} = -\tfrac{1}{2}{\rm e}^{-\phi}\Big[&\tfrac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} + \tfrac{1}{4}\big(F^{\rm a}_{\mu\nu} + A^{\rm a}_i V^i_{\mu\nu}\big)\big(F^{{\rm a},\mu\nu} + A^{\rm a}_j V^{j,\mu\nu}\big) \\ &+ \tfrac{1}{4}g^{ij}\big(B_{i\mu\nu} + A^{\rm a}_i F^{\rm a}_{\mu\nu} + C_{ik}V^k_{\mu\nu}\big)\big(B_j{}^{\mu\nu} + A^b_j F^{b,\mu\nu} + C_{jl}V^{l,\mu\nu}\big) \\ &+ \tfrac{1}{4}g^{ik}g^{jl}\big(D_\mu B_{ij} + A^{\rm a}_{[i}D_\mu A^{\rm a}_{j]}\big)\big(D^\mu B_{kl} + A^b_{[k}D^\mu A^b_{l]}\big) + \tfrac{1}{2}g^{ij}D_\mu A^{\rm a}_i D^\mu A^{\rm a}_j \\ &+ \tfrac{1}{2}{\rm e}^\rho M^{AB}D_\mu b_A D^\mu b_B + \tfrac{1}{2}\vartheta^2 g_2^{-1}A^{\rm a}_2 A^{\rm a}_2 + \tfrac{1}{8}\vartheta^2 g^{11}{\rm e}^\rho M^{AB}b_A b_B\Big] \ , \end{aligned} \tag{3.110}$$

where this time the three-form is given by

$$\mathcal{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \tfrac{1}{2}B_{i\mu}V^i_{\nu\rho} - \tfrac{1}{2}V^i_\mu B_{i\nu\rho} - \tfrac{1}{2}A^{\rm a}_\mu F^{\rm a}_{\nu\rho} + 2\vartheta B_{2\mu}V^1_\nu V^2_\rho + \text{cyclic perm.} \ . \tag{3.111}$$

The total effective action $\mathscr{L}_4$ is obtained by adding the two contributions $\mathscr{L}_{4,g+\Phi}$ and $\mathscr{L}_{4,A+B}$. The result can again be written compactly in terms of appropriately defined quantities. Let us denote all vectors by $\mathcal{A}^I_\mu$ as in Eq. (2.80). The corresponding field strengths are

$$\mathcal{F}^I_{\mu\nu} = (V^i_{\mu\nu}, B_{i\mu\nu}, F^{\rm a}_{\mu\nu}) \ , \qquad I = 0,\dots,n_{\rm v} \ , \tag{3.112}$$

with $V^i_{\mu\nu}$, $B_{i\mu\nu}$ and $F^{\rm a}_{\mu\nu}$ defined in Eqs. (3.95) and (3.109) but reproduced here for clarity,

$$\begin{aligned}
V^1_{\mu\nu} &= \partial_\mu V^1_\nu - \partial_\nu V^1_\mu \ , \\
V^2_{\mu\nu} &= \partial_\mu V^2_\nu - \partial_\nu V^2_\mu + \vartheta V^1_\mu V^2_\nu - \vartheta V^1_\nu V^2_\mu \ , \\
B_{1\mu\nu} &= \partial_\mu B_{1\nu} - \partial_\nu B_{1\mu} + \vartheta V^2_\mu B_{2\nu} - \vartheta V^2_\nu B_{2\mu} \ , \\
B_{2\mu\nu} &= \partial_\mu B_{2\nu} - \partial_\nu B_{2\mu} - \vartheta V^1_\mu B_{2\nu} + \vartheta V^1_\nu B_{2\mu} \ , \\
F^{\rm a}_{\mu\nu} &= \partial_\mu A^{\rm a}_\nu - \partial_\nu A^{\rm a}_\mu \ .
\end{aligned} \tag{3.113}$$

A general expression for these field strengths is $\mathcal{F}^I = \mathrm{d}\mathcal{A}^I + f^I_{JK}\mathcal{A}^J \wedge \mathcal{A}^K$, where the constants f^I_{JK} are given by

$$f^1_{01} \equiv f_{V^1 V^2}^{V^2} = \vartheta \ , \qquad f^3_{03} \equiv f_{V^1 B_2}^{B_2} = -\vartheta \ , \qquad f^2_{13} \equiv f_{V^2 B_2}^{B_1} = \vartheta \ , \tag{3.114}$$

and the rest vanishing. These constants satisfy the Jacobi identity and are in fact the structure constants of the gauge algebra, as will be verified in Section 3.4. Also, the three-form $\mathcal{H}_3$ in Eq. (3.111) can be written as $\mathcal{H}_3 = \mathrm{d}B_2 - \frac{1}{2}\omega_{\rm CS}$, where the Chern-Simons three-form is

$$\omega_{\rm CS} = L_{IJ}\mathcal{A}^I \wedge \mathcal{F}^J - \tfrac{1}{3} f_{IJK}\mathcal{A}^I \wedge \mathcal{A}^J \wedge \mathcal{A}^K \ . \tag{3.115}$$

Here $f_{IJK} = L_{IL} f^L_{JK}$ is completely antisymmetric. In terms of these quantities, the total effective action takes the form

$$\begin{aligned}
\mathscr{L}_4 &= \mathscr{L}_{4,g+\Phi} + \mathscr{L}_{4,A+B} \\
&= \tfrac{1}{2}\mathrm{e}^{-\phi}\Big\{ \mathcal{R}_4 + \partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \tfrac{1}{4}M_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{8}D_\mu M^I{}_J D^\mu M^J{}_I \\
&\qquad - \tfrac{1}{4}D_\mu\rho D^\mu\rho + \tfrac{1}{8}\partial_\mu M^A{}_B \partial^\mu M^B{}_A - \tfrac{1}{2}M^{AB}D_\mu b_A D^\mu b_B \\
&\qquad\quad - \vartheta^2 g_2^{-1}\Big[\big(g_{22} + \tfrac{1}{2}A^{\rm a}_2 A^{\rm a}_2\big) + \tfrac{1}{4}g_{22}\big(1 + \tfrac{1}{2}\mathrm{e}^\rho M^{AB} b_A b_B\big)\Big]\Big\} \ ,
\end{aligned} \tag{3.116}$$

where the matrix M^{IJ} has been defined in Eq. (2.81). The last line in this expression is a potential for the scalar fields.

The next steps are to dualize the three-form $\mathcal{H}_3$ into the axion a and to perform the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^\phi g_{\mu\nu}$. The axion-dilaton s and the complex moduli v^p can again be defined according to Eqs. (2.93), (2.94) and (2.95). A matrix $\mathcal{M}^P{}_Q$ can also be introduced as in Eq. (2.83). Two equivalent final forms for the Lagrangian $\mathscr{L}_4$ can be written as

$$\begin{aligned}
\mathscr{L}_4 &= \tfrac{1}{2}\mathcal{R}_4 - \tfrac{1}{8}\mathrm{e}^{-\phi}M_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{16}aL_{IJ}\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\
&\qquad + \frac{\partial_\mu s\partial^\mu\bar{s}}{(s-\bar{s})^2} + \tfrac{1}{16}D_\mu M^I{}_J D^\mu M^J{}_I + \tfrac{1}{16}D_\mu\mathcal{M}^P{}_Q D^\mu\mathcal{M}^Q{}_P - \mathscr{V}(s,M,\mathcal{M}) \\
&= \tfrac{1}{2}\mathcal{R}_4 + \tfrac{1}{4}I_{IJ}(v)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{8}R_{IJ}(v)\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\
&\qquad - G_{p\bar{q}}(v)D_\mu v^p D^\mu\bar{v}^{\bar{q}} - h_{uv}(q)D_\mu q^u D^\mu q^v - \mathscr{V}(v,q) \ .
\end{aligned} \tag{3.117}$$

The differences in this final expression with respect to the effective Lagrangian (2.96) of the compactification on $\mathrm{K3}\times T^2$ are the covariant derivatives $D_\mu v^p$ and $D_\mu q^u$, the non-Abelian field strengths (3.113) and the potential $\mathscr{V}(v,q)$. In other words, for this SU(2)-structure background one also obtains a gauged version of the supergravity corresponding to $\mathrm{K3}\times T^2$. The covariant derivatives for the complex scalars v^p are

$$\begin{aligned} D_\mu s &= \partial_\mu s \ , \\ D_\mu u &= \partial_\mu u + \vartheta(V^1_\mu u + V^2_\mu) \ , \\ D_\mu t &= \partial_\mu t - \vartheta(V^1_\mu t - B_{2\mu}) \ , \\ D_\mu n^a &= \partial_\mu n^a \ . \end{aligned} \tag{3.118}$$

From these expressions, the following nonvanishing holomorphic Killing vectors $k^p_I \equiv k^{v^p}_{\mathcal{A}^I}$ as appearing in the general expression $D^p_\mu = \partial_\mu v^p - k^{v^p}_{\mathcal{A}^I}\mathcal{A}^I_\mu$ can be read off,

$$k^u_{V^1} = \vartheta u \ , \qquad k^u_{V^2} = \vartheta \ , \qquad k^t_{V^1} = -\vartheta t \ , \qquad k^t_{B_2} = \vartheta \ . \tag{3.119}$$

On the other hand, the covariant derivatives $D_\mu q^u = \partial_\mu - k^{q^u}_{\mathcal{A}^I}\mathcal{A}^I_\mu$ of scalars in hypermultiplets have the form

$$\begin{aligned} D_\mu \rho &= \partial_\mu \rho - \vartheta V^1_\mu \ , \\ D_\mu \xi^x_A &= \partial_\mu \xi^x_A \ , \\ D_\mu b_A &= \partial_\mu b_A + \tfrac{1}{2} V^1_\mu b_A \ , \end{aligned} \tag{3.120}$$

and the corresponding nonzero Killing vectors are

$$k^\rho_{V^1} = \vartheta \ , \qquad k^{b_A}_{V^1} = -\tfrac{1}{2}\vartheta b_A \ . \tag{3.121}$$

Equivalently, the covariant derivatives of the matrices $M^I{}_J$ and $\mathcal{M}^P{}_Q$ can be given instead. One can check that they conform to the expressions

$$\begin{aligned} D_\mu M^I{}_J &= \partial_\mu M^I{}_J - f^I_{KL}\mathcal{A}^K_\mu M^L{}_J + f^L_{KJ}\mathcal{A}^K_\mu M^I{}_L \ , \\ D_\mu \mathcal{M} &= \partial_\mu \mathcal{M} - V^1_\mu[\mathcal{M},\mathcal{T}] \ , \end{aligned} \tag{3.122}$$

where the matrix $\mathcal{T} = (\mathcal{T}^P{}_Q)$ is defined as

$$\mathcal{T} = \begin{pmatrix} -\vartheta & 0 & 0 \\ 0 & \vartheta & 0 \\ 0 & 0 & 0_{22} \end{pmatrix} \ . \tag{3.123}$$

Since this matrix satisfies $\mathcal{T}\mathcal{L} + \mathcal{L}\mathcal{T} = 0$, it is in the algebra of SO(4, 20).

Finally, the generated potential is given by the expressions

$$\begin{aligned} \mathscr{V} &= \tfrac{1}{2}\mathrm{e}^\phi \vartheta^2 g_2^{-1}\Big[\big(g_{22} + \tfrac{1}{2}A^{\mathrm{a}}_2 A^{\mathrm{a}}_2\big) + \tfrac{1}{4} g_{22}\big(1 + \tfrac{1}{2}\mathrm{e}^\rho M^{AB} b_A b_B\big)\Big] \\ &= \tfrac{1}{24}\mathrm{e}^\phi M^{IL}\big(M^{JM}M^{KN} - 3L^{JM}L^{KN}\big) f_{IJK} f_{LMN} - \tfrac{1}{16}\mathrm{e}^K \mathrm{tr}\left([\mathcal{M},\mathcal{T}]^2\right) \ . \end{aligned} \tag{3.124}$$

That this potential is indeed consistent with the gaugings characterized by the Killing vectors (3.119) and (3.121) will be established in Section 3.4.

3.3.2 General case

For the general case, both sets of parameters θ^i and Θ_i are present in the closure algebra (3.37). It is not difficult to guess that this case can be realized by considering a fibration over a base circle whose fiber is a five-dimensional space constructed as a K3 fibration over another circle. Let us make this more explicit in the following.

Consider first a K3 fibration over a circle S^1_2. As explained in Section 3.2, it can be done by giving a dependence to the harmonic two-forms of K3 as

$$\omega^A(z^2) = \gamma^A_{2B}(z^2)\omega^B \ , \tag{3.125}$$

for a z^2-dependent matrix $\gamma_2(z^2) = \exp(z^2\Theta_2)$. The monodromy after going once around the circle is given by the matrix $\gamma_2(1) = \exp\Theta_2$. If this matrix is in $\mathrm{SO}(3,19,\mathbb{Z})$, the two bases $\omega^A(1)$ and $\omega^A(0)$ are equivalent and lead to the same K3. It is through this identification that the fibration is realized.

As a second step, consider a further fibration of this five-dimensional manifold over a circle S^1_1. This can be done be giving the following z^1-dependences to the two-forms $\omega^A(z^2)$ and to the one-form $v^2 = \mathrm{d}z^2$ in the base circle of the first fibration,

$$\begin{aligned} v^2(z^1) &= \mathrm{e}^{\vartheta z^1}\mathrm{d}z^2 \ , \\ \omega^A(z^1,z^2) &= \mathrm{e}^{-\frac{1}{2}\vartheta z^1}\gamma^A_{1B}(z^1)\omega^B(z^2) = \mathrm{e}^{-\frac{1}{2}\vartheta z^1}\gamma^A_{1B}(z^1)\gamma^B_{2C}(z^2)\omega^C \ . \end{aligned} \tag{3.126}$$

Taking the exterior derivative of these expressions one obtains

$$\mathrm{d}v^2 = \vartheta \mathrm{d}z^1 \wedge v^2 \ , \tag{3.127a}$$

$$\mathrm{d}\omega^A = -\tfrac{1}{2}\vartheta\mathrm{d}z^1 \wedge \omega^A + \Theta^A_{1B}\mathrm{d}z^1 \wedge \omega^B + \tilde{\Theta}^A_{2B}(z^1)\mathrm{d}z^2 \wedge \omega^B \ , \tag{3.127b}$$

where the following matrix has been defined,

$$\tilde{\Theta}_2(z^1) = \gamma_1(z^1)\Theta_2\gamma_1(z^1)^{-1} \ . \tag{3.128}$$

Comparing the expression for $\mathrm{d}\omega^A$ with Eq. (3.73a), it should be clear that the third term in Eq. (3.127b) must be of the form $\Theta^A_{2B}v^2 \wedge \omega^B = \Theta^A_{2B}\mathrm{e}^{\vartheta z^1}\mathrm{d}z^2 \wedge \omega^B$. This is the case if and only if

$$\tilde{\Theta}_2(z^1) = \mathrm{e}^{\vartheta z^1}\Theta_2 \ . \tag{3.129}$$

Equating this and (3.128), taking the derivative with respect to z^1 and setting $z^1 = 0$ in the end, the following commutator follows,

$$[\Theta_1, \Theta_2] = \vartheta\Theta_2 \ . \tag{3.130}$$

The closure algebra (3.37) is indeed satisfied with matrices Θ_i in the algebra of $\mathrm{SO}(3,19)$ and satisfying the right commutation relation with parameter ϑ.

The effective four-dimensional action for this background can be derived with no extra effort. The vector multiplet is exactly as in the compactification corresponding to $\vartheta \neq 0$ but $\Theta_i = 0$ performed in Section 3.3.1. The hypermultiplet sector is also easy to analyze. The moduli of the K3 metric and the b_A fields must be given the following z-dependences,

$$\begin{aligned} \rho(z) &= \rho + \vartheta z^1 \ , \\ \xi^x_A(z) &= \gamma^B_{1A}(z^1)\gamma^C_{2B}(z^2)\xi^x_C \ , \\ b_A(z) &= \mathrm{e}^{-\frac{1}{2}\vartheta z^1}\gamma^B_{1A}(z^1)\gamma^C_{2B}(z^2) b_C \ . \end{aligned} \tag{3.131}$$

The covariant derivatives can be computed by acting with $\mathcal{D}_\mu = \partial_\mu - V^i_\mu \partial_i$ on these expressions, and the result is

$$\begin{aligned} D_\mu \rho &= \partial_\mu \rho + \vartheta V^1_\mu \ , \\ D_\mu \xi^x_A &= \partial_\mu \xi^x_A - V^i_\mu \Theta^B_{iA}\xi^x_B \ , \\ D_\mu b_A &= \partial_\mu b_A - \tfrac{1}{2} V^1_\mu b_A - V^i_\mu \Theta^B_{iA} b_B \ . \end{aligned} \tag{3.132}$$

The Killing vectors arising from these expressions are just the sum of the Killing vectors obtained in Section 3.2.1 for $\vartheta = 0$ and those of Section 3.3.1 for $\Theta_i = 0$.

The potential generated in the general case can also be stated easily, and it is

$$\mathscr{V} = \tfrac{1}{24}\mathrm{e}^{\phi} M^{IL}\big(M^{JM}M^{KN} - 3L^{JM}L^{KN}\big) f_{IJK} f_{LMN} - \tfrac{1}{16}\mathrm{e}^{\phi} g^{ij}\,\mathrm{tr}\left([\mathcal{M},\mathcal{T}_i][\mathcal{M},\mathcal{T}_j]\right) \ , \tag{3.133}$$

where the structure constants are given in Eq. (3.114) and the matrics $\mathcal{T}_i$ are defined as

$$\mathcal{T}_1 = \begin{pmatrix} -\vartheta & 0 & 0 \\ 0 & \vartheta & 0 \\ 0 & 0 & \Theta_1 \end{pmatrix} \ , \qquad \mathcal{T}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Theta_2 \end{pmatrix} \ . \tag{3.134}$$

3.4 Gauge algebra and Killing prepotentials

The Killing vectors in Eq. (3.119) can be used to determine the gauge algebra. If everything is correct, the latter must have structure constants f^I_{IJ} as given in Eq. (3.114). Let us denote the generators by $T_I = (X_i, Y^j, Z_{\mathrm{a}})$, so that the generic element in the algebra is $\mathcal{A}^\mu = \mathcal{A}^I_\mu T_I = V^i_\mu X_i + B_{j\mu} Y^j + A^{\mathrm{a}}_\mu Z_{\mathrm{a}}$. The generators T_I must therefore satisfy the commutation relations $[T_I, T_J] = f^K_{IJ} T_K$, or explicitly

$$[X_1, X_2] = -\vartheta X_2 \ , \qquad [X_1, Y^2] = \vartheta Y^2 \ , \qquad [X_2, Y^2] = -\vartheta Y^1 \ , \tag{3.135}$$

and the rest vanishing.

As discussed in Appendix A and shown in Eq. (A.4), the tangent vectors $T^{\mathrm{v}}_I = k^{v^p}_{\mathcal{A}^I}\partial_{v^p}$ and $T^{\mathrm{h}}_I = k^{q^u}_{\mathcal{A}^I}\partial_{q^u}$ must furnish two realizations of this algebra. They generate subgroups of the isometry groups of the special Kähler manifold $\mathscr{M}_{\text{v}}$ and the quaternionic manifold $\mathscr{M}_{\text{h}}$, respectively. Using the Killing vectors in Eq. (3.119) and omitting the superscript "v" one obtains the generators

$$X_1 = \vartheta u \frac{\partial}{\partial u} - \vartheta t \frac{\partial}{\partial t} \ , \qquad X_2 = \vartheta \frac{\partial}{\partial u} \ , \qquad Y^2 = \vartheta \frac{\partial}{\partial t} \ . \tag{3.136}$$

The commutators of these vectors can be computed easily and the result is

$$[X_1, X_2] = -\vartheta X_2 \ , \qquad [X_1, Y^2] = \vartheta Y^2 \ , \qquad [X_2, Y^2] = 0 \ . \tag{3.137}$$

This algebra is indeed obtained by setting the central charge Y^1 to zero in the algebra (3.135). The generator Y^1 is a central charge because it commutes with the rest of the generators.

On the other hand, taking the sum of the Killing vectors in Eqs. (3.68) and (3.121) one can construct another realization of this algebra. Omitting the superscript "h", one has the nonzero generators

$$\begin{aligned} X^1 &= \vartheta \frac{\partial}{\partial \rho} - \tfrac{1}{2} \vartheta b_A \frac{\partial}{\partial b_A} - \xi^x_A \Theta^A_{1B} \frac{\partial}{\partial \xi^x_B} - b_A \Theta^A_{1B} \frac{\partial}{\partial b_B} \ , \\ X^2 &= -\xi^x_A \Theta^A_{2B} \frac{\partial}{\partial \xi^x_B} - b_A \Theta^A_{2B} \frac{\partial}{\partial b_B} \ . \end{aligned} \tag{3.138}$$

The only nontrivial commutation relation in this case is $[X^1, X^2] = -\vartheta X^2$. In its derivation, it is crucial to have the commutator (3.130). This can be obtained from the algebra (3.135) by setting to zero the central charge Y^1 and the generator Y^2.

The Killing prepotential $\mathcal{P}_I \equiv \mathcal{P}_{\mathcal{A}^I}$ for the vector multiplet sector is a real quantity that satisfies Eq. (A.12), namely

$$k^{v^p}_{\mathcal{A}^I} = -\mathrm{i} G^{p\bar{q}} \frac{\partial \mathcal{P}_{\mathcal{A}^I}}{\partial \bar{v}^{\bar{q}}} \ . \tag{3.139}$$

This equation can be easily solved and the following expressions are found,

$$\begin{aligned} \mathcal{P}_0 \equiv \mathcal{P}_{V^1} &= -\vartheta \mathrm{e}^K (s - \bar{s})(\bar{u} t - u \bar{t}) \ , \\ \mathcal{P}_1 \equiv \mathcal{P}_{V^2} &= -\vartheta \mathrm{e}^K (s - \bar{s})(t - \bar{t}) \ , \\ \mathcal{P}_3 \equiv \mathcal{P}_{B_2} &= -\vartheta \mathrm{e}^K (s - \bar{s})(u - \bar{u}) \ . \end{aligned} \tag{3.140}$$

The computation of the triplet of Killing prepotentials $\mathcal{P}^x_I \equiv \mathcal{P}^x_{\mathcal{A}^I}$ corresponding to the quaternionic manifold $\mathscr{M}_{\text{h}}$ and satisfying Eq. (A.13) can also be performed. First of all, a 4×24 matrix Z can be introduced as

$$Z = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{\frac{1}{2}\rho} & -\mathrm{e}^{-\frac{1}{2}\rho} + \frac{1}{2}\mathrm{e}^{\frac{1}{2}\rho} b^2 & -\mathrm{e}^{\frac{1}{2}\rho} b^A \\ 0 & -\xi^x_A b^A & \xi^{xA} \end{pmatrix} \ . \tag{3.141}$$

This matrix satisfies $2(Z^{\mathrm{T}}Z)^{PQ} = \mathcal{M}^{PQ} + \mathcal{L}^{PQ}$, with $\mathcal{M}^{PQ}$ and $\mathcal{L}^{PQ}$ given in Eqs. (2.83) and (2.84), respectively. As a second step, a 4×4 matrix of one-forms

$$Z\mathcal{L}^{-1}\mathrm{d}Z^{\mathrm{T}} = \tfrac{1}{2}\begin{pmatrix} 0 & \mathrm{e}^{\frac{1}{2}\rho}\xi^y_A \mathrm{d}b^A \\ -\,\mathrm{e}^{\frac{1}{2}\rho}\xi^x_A \mathrm{d}b^A & \xi^x_A \mathrm{d}\xi^{yA} \end{pmatrix} , \tag{3.142}$$

is computed, from which in turn the SU(2) connection

$$\omega^x = -\tfrac{1}{2}\mathrm{tr}(Z\mathcal{L}^{-1}\mathrm{d}Z^{\mathrm{T}}\Sigma^x) \tag{3.143}$$

follows. The three 4×4 matrices Σ^x are the self-dual 't Hooft matrices defined in [37]. The curvature for this connection is found by using the expression

$$K^x = \mathrm{d}\omega^x + \tfrac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z \ . \tag{3.144}$$

Substituting ω^x and K^x together with the sum of the Killing vectors (3.68) and (3.121) in Eq. (A.13), the following solution for the prepotentials is found,

$$\mathcal{P}^x_{V^i} = \tfrac{1}{2}(-1)^{x+1}\big(\mathrm{e}^{\frac{\rho}{2}} b_A T^A_{iB}\xi^{xB} - \tfrac{1}{2}\epsilon^{xyz}\xi^y_A T^A_{iB}\xi^{zB}\big) \ , \tag{3.145}$$

where the matrices T_i are given in terms of the parameters ϑ and Θ_i in Eq. (3.35). This result can also be written as the integral expression

$$\mathcal{P}^x_{V^i} = \tfrac{1}{2}(-1)^x \epsilon_{ij}\mathrm{e}^{\rho}\Big(\int_{\mathcal{Y}} J^x \wedge \mathrm{d}B \wedge v^j - \tfrac{1}{2}\epsilon^{xyz}\int_{\mathcal{Y}} J^y \wedge \mathrm{d}J^z \wedge v^j\Big) \ . \tag{3.146}$$

Finally, it can be checked that these Killing prepotentials (3.145) actually satisfy

$$\mathcal{P}^x_{\mathcal{A}^I} = k^{q^u}_{\mathcal{A}^I}\omega^x_u \ . \tag{3.147}$$

That the effective Lagrangian in Eq. (2.96) is in agreement with the general form (A.1) of $\mathcal{N} = 2$ supergravity has already been established. It has been shown that the effective theories obtained from the compactification on the analyzed SU(2)-structure backgrounds and described by the Lagrangians (3.66) and (3.66) share the same spectrum and the same scalar manifolds $\mathscr{M}_{\mathrm{v}}$ and $\mathscr{M}_{\mathrm{h}}$ with the effective theory corresponding to $\mathrm{K3}\times T^2$. Therefore, the consistency of these Lagrangians with $\mathcal{N} = 2$ gauged supergravity is established if the potentials $\mathscr{V}$ and the gaugings characterized by the Killing vectors obtained are in agreement with Eq. (A.14), that is

$$\mathscr{V}_{\mathcal{N}=2} = \mathrm{e}^K \bar{X}^I X^J \big(G_{p\bar{q}} k^{v^p}_{\mathcal{A}^I} k^{\bar{v}^q}_{\mathcal{A}^J} + 4h_{uv}k^{q^u}_{\mathcal{A}^I}k^{q^v}_{\mathcal{A}^J}\big) + \big[\tfrac{1}{2}(I^{-1})^{IJ} + 4\mathrm{e}^K X^I \bar{X}^J\big]\mathcal{P}^x_{\mathcal{A}^I}\mathcal{P}^x_{\mathcal{A}^J} \ . \tag{3.148}$$

It follows from the Killing vectors (3.68) and (3.121) that scalars in hypermultiplets are charged with respect to the vectors V^i_μ exclusively. This is the reason why the only nonzero prepotentials for the quaternionic manifold are $\mathcal{P}^x_{V^i}$ or equivalently $\mathcal{P}^x_0$ and $\mathcal{P}^x_1$. The only possibly nonzero contribution from the second term in Eq. (3.148) is thus for $I, J = 0, 1$, corresponding to the two vectors V^i_μ. But it turns out that for $I, J = 0, 1$ the expression inside square brackets vanishes. The second term in Eq. (3.148) thus cancels.

The Kähler metric $G_{p\bar{q}}$ can be obtained from the Kähler potential K in Eq. (2.97) and the quaternionic metric h_{uv} can be read off from the Lagrangians (3.59) and (3.63). The expressions for h_{uv} are

$$h_{\rho\rho} = \tfrac{1}{8} \; , \qquad h_{\xi^x_A \xi^y_B} = -\tfrac{1}{4}(\eta^{AB} - \tfrac{1}{2}\xi^z_A \xi^z_B)\delta^{xy} \; , \qquad h_{b_A b_B} = \tfrac{1}{4}\mathrm{e}^{\rho} M_{AB} \; . \tag{3.149}$$

The remaining ingredients are the Killing prepotentials (3.68), (3.121) and (3.119), and the holomorphic projective coordinates X^I given in Eq. (2.100). Substituting all these quantities in Eq. (3.148), an expression is obtained that is in total agreement with the potential (3.133). This completes the verification that the bosonic part of the effective theories obtained from the reduction on the analyzed SU(2)-structure backgrounds are indeed consistent with the general form of $\mathcal{N} = 2$ gauged supergravity.

Chapter 4

Heterotic on SU(3)-structure backgrounds: fermionic approach

ΜΟΛΩΝ ΛΑΒΕ

In this Chapter, the heterotic compactification on manifolds with SU(3) structure is revisited. The reduction of the bosonic sector has been known for a while, but here a fermionic approach is taken. This means that the attention is on terms of the four-dimensional action involving fermionic fields. The couplings of the effective four-dimensional supergravity can be in fact more reliably computed in this way, because they enter linearly in the fermionic terms, while in the bosonic sector they enter quadratically. The developments closely follow Ref. [45], correcting the results presented there.

Six-dimensional SU(3)-structure manifolds are discussed in Section 4.1. The compactification of heterotic supergravity on these backgrounds is then analyzed in Section 4.2. First the results for the bosonic sector are recalled and then, in more detail, the fermionic spectrum and the reduction of the relevant fermionic terms in the ten-dimensional action are presented. The fermionic kinetic terms in the effective action are needed in order to find the appropriate normalization for the fermions. They also provide a check of the Kähler potential known from the reduction of the bosonic sector. The computation of the gravitino mass term and the F- and D-terms comes next, and from them the holomorphic superpotential W and its derivatives are obtained. Finally, the supersymmetry variations of the four-dimensional fermionic fields are worked out in Section 4.3, together with a discussion of the conditions for a supersymmetric vacuum.

4.1 Manifolds with SU(3) structure

Demanding that a lesser amount of supersymmetry be preserved by the compactification broadens the class of possible internal manifolds $\mathcal{Y}$. If instead of two, only one global nowhere-vanishing internal spinor is required to exist, the effective four-dimensional theory obtained by compactification of the heterotic string is expected to be $\mathcal{N} = 1$ supersymmetric. In this case, the spinor representation **4** of Spin(6) $\simeq$ SU(4) decomposes as

$$\mathbf{4} \to \mathbf{3} + \mathbf{1} \ . \tag{4.1}$$

The singlet in this decomposition is the global spinor. The structure group of the manifold is therefore reduced to SU(3). Manifolds with SU(3) structure can in consequence be defined by the existence of one global nowhere-vanishing spinor η. This spinor can be split into two Weyl spinors of opposite chiralities η_+ and η_-.

If the spinor η happens to be covariantly constant with respect to the Levi-Civita connection, the manifold $\mathcal{Y}$ must have SU(3) holonomy, since due to the decomposition (4.1) every spinor undergoing parallel transport along a closed path must come to itself up to at most an SU(3) transformation. In other words, $\mathcal{Y}$ is a Calabi-Yau threefold. Manifolds with SU(3) structure are therefore generalizations of Calabi-Yau manifolds in the same way that SU(2)-structure manifolds are generalizations of K3 $\times$ T^2. For the generic SU(3)-structure manifold, the global spinor is parallel with respect to a connection with nonvanishing torsion.

The compactification of the heterotic string on Calabi-Yau spaces has long been known. The effective theory in four dimensions is obtained by expansion of the ten dimensional fields in terms of the harmonic $(1,1)$- and $(1,2)$-forms of the Calabi-Yau. This leads to an $\mathcal{N} = 1$ supergravity coupled to vector and chiral multiplets. For a generic SU(3)-structure manifold, the light modes can be identified by expanding the fields in ten dimensions in terms of a finite set of forms. This set of forms can be constructed by projecting out all **3** and $\bar{\mathbf{3}}$ representations of the structure group SU(3) in analogy to the SU(2)-structure case discussed in Chapter 3 where the doublets were projected out.

The SU(3) structure can also be characterized by a number of forms. These forms are defined analogously to the SU(2)-structure case by using the Clifford algebra γ_a and constructing spinor bilinears. Concretely, a two-form J and a three-form Ω can be defined as follows,

$$J_{ab} = \mp \mathrm{i}\eta_\pm^\dagger \gamma_{ab}\eta_\pm \ , \qquad \Omega_{abc} = -\mathrm{i}\eta_-^\dagger \gamma_{abc}\eta_+ \ , \tag{4.2}$$

If the spinors are normalized as $\eta_\pm^\dagger \eta_\pm = 1$, one can make use of Fierz identities to obtain the following relations satisfied by these forms,

$$J \wedge J \wedge J = \tfrac{3}{4}\mathrm{i}\, \Omega \wedge \bar{\Omega} \ , \qquad J \wedge \Omega = 0 \ . \tag{4.3}$$

It can also be shown that by raising one index on the two-form J_{ab} by means of the metric g_{ab}, an almost complex structure $I_a{}^b$ can be defined satisfying $I_a{}^b I_b{}^c = -\delta_a^c$. With respect to this almost complex structure, the two-form J and the three-form Ω are respectively of type $(1,1)$ and $(3,0)$.

For a Calabi-Yau space, the forms J and Ω are the Kähler form and the holomorphic three-form defined as

$$J = \mathrm{i} g_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}} \ , \qquad \Omega = \sqrt{g_3} \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \ , \tag{4.4}$$

where z^α are three complex coordinates and $g_{\alpha\bar{\beta}}$ is a hermitian metric on the Calabi-Yau. These forms are certainly harmonic, but for a generic SU(3)-structure manifold $\mathcal{Y}$ the forms J and Ω need not be closed. In fact, from the definitions (4.2) one can show that these forms are closed if and only if the spinor η is covariantly constant with respect to the Levi-Civita connection, i.e. if $\mathcal{Y}$ is Calabi-Yau. The exterior differentials $\mathrm{d}J$ and $\mathrm{d}\Omega$ therefore represent how much the manifold $\mathcal{Y}$ deviates from the Calabi-Yau condition and are a measure of the torsion. It has been shown that five torsion classes $\mathcal{W}_1, \dots, \mathcal{W}_5$ can be introduced and that $\mathrm{d}J$ and $\mathrm{d}\Omega$ can be expanded as follows [22],

$$\begin{aligned} \mathrm{d}J &= -\tfrac{3}{2}\mathrm{Im}(\mathcal{W}_1 \bar{\Omega}) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \ , \\ \mathrm{d}\Omega &= \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega \ , \end{aligned} \tag{4.5}$$

with the constraints

$$J \wedge J \wedge \mathcal{W}_2 = \Omega \wedge \mathcal{W}_3 = 0 \ , \qquad J \wedge \mathcal{W}_3 = 0 \tag{4.6}$$

arising from the relations (4.3). As can be deduced from these expressions, $\mathcal{W}_1$ is a zero-form, $\mathcal{W}_4$ and $\mathcal{W}_5$ are one-forms, $\mathcal{W}_2$ is a two-form and $\mathcal{W}_3$ is a three-form. Each one of them can be characterized by its SU(3) transformation properties. For a Calabi-Yau, all of them vanish. The vanishing of only a subset define also special classes of manifolds inside the broader group of SU(3)-structure manifolds. For example, $\mathcal{Y}$ is a complex manifold if and only if $\mathcal{W}_1$ and $\mathcal{W}_2$ are zero. Only in this case is the almost complex structure ${I_a}^b$ integrable. Projecting out all triplets of the structure group SU(3) amounts to consider $\mathcal{W}_4 = \mathcal{W}_5 = 0$.

4.2 Compactification on SU(3)-structure backgrounds

As already explained, the dimensional reduction on an SU(3)-structure background can be performed by projecting out all triplets or representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ of the SU(3) structure group. In particular, no one-form in $\mathcal{Y}$ survives this projection, but there could be a number $h^{1,1}$ of $(1,1)$-forms and $h^{1,2}$ of $(1,2)$-forms with respect to the almost complex structure I. Let us denote the former by ω_i and the latter by ρ_m. There is also one $(3,0)$-form Ω and its complex conjugate, unique up to rescaling. The compactification ansatz for the ten-dimensional fields is constructed in terms of these forms. Note that the numbers $h^{1,1}$ and $h^{1,2}$ have been labeled in that way because for a Calabi-Yau these are the corresponding Hodge numbers, but it should be clear that they do not count the harmonic forms for a generic SU(3)-structure manifold $\mathcal{Y}$.

4.2.1 Bosonic sector

Let us first discuss the bosonic spectrum. Since this is fairly known material, the exposition should be brief. There being no one-forms on $\mathcal{Y}$ available for expansion, the dilaton Φ, the metric g_{MN} and the NS two-form field B_{MN} in ten dimensions contribute only with a metric $g_{\mu\nu}$, a two-form $B_{\mu\nu}$ and scalars Φ, g_{ab} and B_{ab} in four dimensions. More explicitly, the ten-dimensional two-form B_2 can be expanded as

$$B_2 = \tfrac{1}{2} B_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + b^i \omega_i \ , \tag{4.7}$$

with the result that the scalar part B_{ab} gives rise to $h^{1,1}$ moduli fields b^i. Moreover, the two-form $B_{\mu\nu}$ can be dualized to a scalar a, the axion, as already seen in Chapter 2.

The expansion of the internal part g_{ab} of the metric can be done by considering separately $g_{\alpha\bar\beta}$ and $g_{\alpha\beta}$, where the indices $\alpha, \bar\alpha$ refer to the $\mathbf{3}$ and the $\bar{\mathbf{3}}$ of SU(3) or to the almost complex structure I. In analogy to the Calabi-Yau case, the metric deformations $\delta g_{\alpha\bar\beta}$ and $\delta g_{\alpha\beta}$ are given by the expressions

$$\delta g_{\alpha\bar\beta} = -\mathrm{i}\, v^i(x) (\omega_i)_{\alpha\bar\beta} \ , \qquad \delta g_{\alpha\beta} \ = \frac{\mathrm{i}}{\|\Omega\|^2}\, \bar z^m(x) (\bar\rho_m)_{\alpha\bar\gamma\bar\delta}\, \Omega_\beta{}^{\bar\gamma\bar\delta} \ . \tag{4.8}$$

In these expansions, the three-form Ω is the $(3,0)$-form in $\mathcal{Y}$, and it differs from $\boldsymbol{\Omega}$ defined above in the normalization, $\Omega = \|\Omega\| \boldsymbol{\Omega}$ with $\|\Omega\|^2 = \frac{1}{3!}\Omega_{\alpha\beta\gamma}\bar\Omega^{\alpha\beta\gamma}$. The $h^{1,1}$ Kähler moduli v^i are real and the $h^{1,2}$ complex structure moduli z^m are complex. The real scalars v^i are complexified by introducing $t^i = b^i + iv^i$, with the fields b^i stemming from the expansion (4.7).

For the Yang-Mills one-forms $A_1^{\mathbf{a}}$ the story is more complicated. The details are model dependent, since one must specify a gauge bundle $\mathcal{G}$ to satisfy the consistency condition

$$\int_{\mathcal{Y}} \mathrm{tr} \Big[(R_2 \wedge R_2) - \mathrm{Tr}\,(F_2 \wedge F_2) \Big] = 0 \tag{4.9}$$

that arises from integrating Eq. (2.48) on the internal manifold $\mathcal{Y}$. For concreteness, one can analyze the standard embedding. This means that one identifies the spin connection of the manifold with an SU(3) subgroup of the gauge group. In this way, the consistency condition is satisfied. If one takes the case of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, this leads to the decomposition of one of the E_8 factors as

$$\mathrm{E}_8 \to \mathrm{SU}(3) \times \mathrm{E}_6 \ . \tag{4.10}$$

The **248** adjoint representation of E_8 decomposes as

$$\mathbf{248} \to (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \overline{\mathbf{27}}) \ . \tag{4.11}$$

The four-dimensional vectors $A_\mu^{\mathbf{a}}$ and scalars $A_a^{\mathbf{a}}$ descending from $A_M^{\mathbf{a}}$ give rise to the following fields. From the decomposition (4.11), it is clear that after projecting out all the triplets in $A^{\mathbf{a}}_{\mu}$ only the vectors A^{a}_{μ} in the adjoint of the unbroken gauge group E_6 survive. For the scalars $A^{\mathbf{a}}_{a}$, or equivalently $A^{\mathbf{a}}_{\alpha}$ and $A^{\mathbf{a}}_{\bar\alpha}$, one must consider the representations $\mathbf{3}\times\mathbf{248}$ and $\bar{\mathbf{3}}\times\mathbf{248}$, because of the SU(3) transformation properties of the spatial indices α and $\bar\alpha$. After projecting out the triplets, one obtains scalar fields $A_{\alpha\beta}$ and $A_{\alpha\bar\beta}$ in the $\overline{\mathbf{27}}$ and the $\mathbf{27}$ of the gauge group E_6, respectively. These scalars can now be decomposed in exactly the same fashion as the components of the internal metric, namely

$$A_{\alpha\bar\beta} = A^i(\omega_i)_{\alpha\bar\beta}\ , \qquad A_{\alpha\beta} = \frac{1}{\|\Omega\|^2}A^m(\bar\rho_m)_{\alpha\bar\gamma\bar\delta}\,\Omega_\beta{}^{\bar\gamma\bar\delta}\ . \tag{4.12}$$

The gauge transformation properties of A_μ, A^i and A^m will be kept implicit in most of the following, meaning that gauge indices will be suppressed.

All four-dimensional bosonic fields organize in multiplets of $\mathscr{N}=1$ supersymmetry. Explicitly, the metric $g_{\mu\nu}$ sits in the gravitational multiplet, while the vectors A_μ and the scalars sit in vector and chiral multiplets, respectively. The precise structure is given in Table 4.1, after the fermions have been analyzed.

The expansions for the ten-dimensional fields can be substituted in the bosonic action $\mathcal{S}_{\rm b}$ of Eq. (2.2). The resulting effective Lagrangian is well-known and has been derived for Calabi-Yau compactifications in [11, 42, 43] and for manifolds with SU(3) structure in [31, 32, 54–57]. Here only the results are summarized.

The effective four-dimensional action is a gauged $\mathscr{N}=1$ supergravity, and as explained in Appendix A it is characterized by the Kähler potential K, the holomorphic gauge kinetic function f and the holomorphic superpotential W. An important step in the derivation is the Weyl rescaling $g_{\mu\nu}\to e^{\phi}g_{\mu\nu}$, where $\phi=\Phi-\ln\mathcal{V}$ is the four-dimensional dilaton and $\mathcal{V}$ is the volume of the internal manifold $\mathcal{Y}$. An expression for $\mathcal{V}$ can be written as

$$\mathcal{V} = \tfrac{1}{6}\int_{\mathcal{Y}} J\wedge J\wedge J\ , \qquad J = v^i\omega_i\ , \tag{4.13}$$

since the two-form J has components $J_{\alpha\bar\beta}=\mathrm{i}g_{\alpha\bar\beta}$. The gauge kinetic function f depends only on the axion-dilaton s defined in Eq.(2.93) and has the simple expression

$$f(s) = \mathrm{i}s\ . \tag{4.14}$$

The metric on the field space of the complex scalars s, t^i and z^m in chiral multiplets is block-diagonal and the different pieces are

$$g_{s\bar s} = -\frac{1}{(s-\bar s)^2}\ , \qquad g_{ij} = \frac{1}{4\mathcal{V}}\int_{\mathcal{Y}}\omega_i\wedge *\omega_j\ , \qquad g_{m\bar n} = \frac{\int_{\mathcal{Y}}\rho_m\wedge\bar\rho_n}{\int_{\mathcal{Y}}\Omega\wedge\bar\Omega}\ . \tag{4.15}$$

Each block can be shown to be a Kähler metric on its own and the Kähler potential is therefore a sum of three contributions

$$K = K_s + K_J + K_\Omega\ , \tag{4.16}$$

where

$$\begin{aligned} e^{-K_s} &= i(s-\bar{s}) = e^{-\phi} , \\ e^{-K_J} &= \tfrac{1}{6}\int_{\mathcal{Y}} J\wedge J\wedge J = \mathcal{V} , \\ e^{-K_\Omega} &= i\int_{\mathcal{Y}} \Omega\wedge\bar{\Omega} . \end{aligned} \tag{4.17}$$

The manifold $\mathscr{M}_J$ spanned by the complexified Kähler moduli t^i and the manifold $\mathscr{M}_\Omega$ spanned by the complex structure moduli z^m are actually special Kähler. As explained in Appendix A, this means that each of the Kähler potentials K_J and K_Ω determining the respective metrics g_{ij} and $g_{m\bar{n}}$ takes a special form in terms of a holomorphic prepotential.

Let us define the quantities

$$\varkappa_i = \frac{1}{\mathcal{V}}\int_{\mathcal{Y}} \omega^i\wedge J\wedge J , \qquad \varkappa_{ij} = \frac{1}{\mathcal{V}}\int_{\mathcal{Y}} \omega_i\wedge\omega_j\wedge J . \tag{4.18}$$

Due to Eq. (4.13), it can be checked that $\varkappa_i v^i = \varkappa_{ij} v^i v^j = 6$. For the space $\mathscr{M}_J$ of Kähler deformations, one can derive

$$*\omega_i = -J\wedge\omega_i + \varkappa_i J\wedge J , \tag{4.19}$$

and substituting this into the metric g_{ij} in Eq. (4.15) one obtains

$$g_{ij} = -\tfrac{1}{4}\varkappa_{ij} + \tfrac{1}{16}\varkappa_i\varkappa_j . \tag{4.20}$$

On the other hand, the forms Ω and ρ_m associated to the space $\mathscr{M}_\Omega$ are related by

$$\frac{\partial\Omega}{\partial z^m} = -\frac{\partial K_\Omega}{\partial z^m}\Omega + \rho_m , \tag{4.21}$$

and their Hodge duals are given by

$$*\Omega = -i\Omega , \qquad *\rho_m = i\rho_m . \tag{4.22}$$

One can also derive the following expression from $\Omega = \|\Omega\|\boldsymbol{\Omega}$ and Eqs. (4.17) and (4.3),

$$\|\Omega\| = e^{-\frac{1}{2}K_\Omega + \frac{1}{2}K_J} . \tag{4.23}$$

Finally, the metric for the matter fields A^i and A^m is also block-diagonal, and the respective blocks are found to be

$$Z_{ij} = e^{\frac{1}{3}(K_\Omega - K_J)} g_{ij} = \|\Omega\|^{-\frac{2}{3}} g_{ij} , \qquad Z_{m\bar{n}} = e^{\frac{1}{3}(K_J - K_\Omega)} g_{m\bar{n}} = \|\Omega\|^{\frac{2}{3}} g_{m\bar{n}} , \tag{4.24}$$

after the rescalings

$$A^i \to \tfrac{1}{2}\|\Omega\|^{-\frac{1}{3}} A^i , \qquad A^m \to \tfrac{1}{2}\|\Omega\|^{\frac{1}{3}} A^m . \tag{4.25}$$

4.2.2 Fermionic spectrum and kinetic terms

A left-handed Majorana-Weyl spinor $\hat{\varepsilon}$ in ten dimensions can be decomposed thanks to the existence of the global spinor η as

$$\hat{\varepsilon} = \varepsilon \otimes \eta_- + \bar{\varepsilon} \otimes \eta_+ \ , \tag{4.26}$$

where ε is a Weyl spinor with positive chirality in four dimensions.* For a right-handed ten-dimensional spinor, one simply needs to switch η_- and η_+ in this expression.

In order to find the fermionic spectrum in four dimensions, the transformation properties of the ten-dimensional fermionic fields with respect to the SU(3) structure group are needed. Recall that the massless fermionic fields in ten-dimensions are a left-handed gravitino $\hat{\psi}_M$, a right-handed dilatino $\hat{\lambda}$ and left-handed gauginos $\hat{\chi}^{\mathbf{a}}$ in the adjoint of $\mathrm{E}_8 \times \mathrm{E}_8$. The components $\hat{\psi}_\mu$ of the gravitino $\hat{\psi}_M$ give rise to a four-dimensional spin-$\frac{3}{2}$ field and transform as $\mathbf{1} \oplus \mathbf{3}$ with respect to the SU(3) structure group. This makes manifest the claim made above that in obtaining an $\mathcal{N} = 1$ supersymmetric effective theory one must project out the triplets of SU(3), since only one light gravitino is required to survive. This singlet can be decomposed as

$$\hat{\psi}_\mu = \psi_\mu \otimes \eta_- + \bar{\psi}_\mu \otimes \eta_+ \ , \tag{4.27}$$

where ψ_μ is the four-dimensional gravitino.

The components $\hat{\psi}_\alpha$ and $\hat{\psi}_{\bar{\alpha}}$ have more complex transformation properties. After projecting out the triplets, representations $\mathbf{8} + \mathbf{1}$ and $\mathbf{6}$ remain. An ansatz for their decomposition is

$$\hat{\psi}_\alpha = \xi^i \otimes (\omega_i)_{\alpha\bar{\beta}} \gamma^{\bar{\beta}} \eta_+ + \frac{1}{\|\Omega\|^2} \bar{\zeta}^m \otimes (\bar{\rho}_m)_{\alpha\bar{\beta}\bar{\gamma}} \Omega_\delta{}^{\bar{\beta}\bar{\gamma}} \gamma^\delta \eta_- \ . \tag{4.28}$$

The right-handed dilatino transforms as $\mathbf{1} \oplus \mathbf{3}$, and the singlet can be expressed as

$$\hat{\lambda} = \lambda \otimes \eta_+ + \bar{\lambda} \otimes \eta_- \ . \tag{4.29}$$

Recalling the decomposition (4.11) of the adjoint representation of E_8, it can be seen that the gauginos $\hat{\chi}^{\mathbf{a}}$ contribute with a spinor χ in the adjoint of the unbroken gauge group $\mathrm{E}_6 \times \mathrm{E}_8$ arising from

$$\hat{\chi} = \chi \otimes \eta_- + \bar{\chi} \otimes \eta_+ \ . \tag{4.30}$$

Additionally, there are fields in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ of E_6 arising from

$$\hat{\chi}_\alpha = \chi^i \otimes (\omega_i)_{\alpha\bar{\beta}} \gamma^{\bar{\beta}} \eta_+ + \frac{1}{\|\Omega\|^2} \bar{\chi}^m \otimes (\bar{\rho}_m)_{\alpha\bar{\beta}\bar{\gamma}} \Omega_\delta{}^{\bar{\beta}\bar{\gamma}} \gamma^\delta \eta_- \ , \tag{4.31}$$

where α labels the $\mathbf{3}$ in (4.11).

All the fermionic fields in four dimensions organize in multiplets of $\mathcal{N} = 1$ together with the bosonic fields as shown in Table 4.1.

*In the rest of this Chapter, hats are placed over the ten-dimensional fermionic fields in order to distinguish them from the corresponding four-dimensional fields.

multiplet	multiplicity	bosons	fermions
gravitational	1	$g_{\mu\nu}$	ψ_μ
vector	$\dim(\mathrm{E}_6 \times \mathrm{E}_8)$	A_μ	χ
chiral	$h^{1,1}$	t^i	ξ^i
	$h^{1,2}$	z^m	ζ^m
	$27h^{1,1}$	A^i	χ^i
	$27h^{1,2}$	A^m	χ^m
	1	s	λ

Table 4.1: $\mathscr{N} = 1$ multiplets.

Let us now turn to the computation of the kinetic terms for the fermions. Due to supersymmetry, this really adds no new information, merely checking the consistency of the couplings obtained for the bosonic sector. Nevertheless, in order to compute the superpotential and D-terms via fermionic couplings it is mandatory to know the right normalization of the fermions, and this is dictated by the kinetic terms.

The kinetic terms are obtained from those terms in the action $\mathcal{S}_\mathrm{f}$ given in Eq. (2.6) with a spacetime derivative D_μ. The Clifford algebra in ten-dimensions $\{\Gamma_M, \Gamma_N\} = 2g_{MN}\mathbb{1}$ can be satisfied by decomposing the Γ-matrices as

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1} \ , \qquad \Gamma^\alpha = \gamma^5 \otimes \gamma^\alpha \ , \qquad \Gamma^{\bar\alpha} = \gamma^5 \otimes \gamma^{\bar\alpha} \ , \tag{4.32}$$

where the four-dimensional γ-matrices are conventionally taken as

$$\gamma^\mu = -\mathrm{i}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \ , \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ . \tag{4.33}$$

In these expressions, $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar\sigma^\mu = (\mathbb{1}, -\sigma^i)$, with the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \ , \qquad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{4.34}$$

On the other hand, the six dimensional γ-matrices γ^α and $\gamma^{\bar\alpha}$ satisfy $\{\gamma^\alpha, \gamma^{\bar\beta}\} = 2g^{\alpha\bar\beta}\mathbb{1}$ with other anticommutators vanishing. The spinor singlets of definite chirality $\eta_\pm$ are annihilated by the corresponding set of γ-matrices, namely

$$\gamma^\alpha \eta_+ = 0 \ , \qquad \gamma^{\bar\alpha}\eta_- = 0 \ . \tag{4.35}$$

As a consequence of this, terms of the form $\eta_+^\dagger \gamma^\alpha \cdots \gamma^{\bar\beta}\eta_+$ or $\eta_-^\dagger \gamma^{\bar\alpha} \cdots \gamma^\beta \eta_-$vanish unless they have an equal number of γ-matrices with holomorphic and antiholomorphic indices. In that case, they can be computed by repeated application of $\{\gamma^\alpha, \gamma^{\bar\beta}\} = 2g^{\alpha\bar\beta}\mathbb{1}$ like, for example,

$$\eta_+^\dagger \gamma^\gamma \gamma^{\bar\alpha\beta}\gamma^{\bar\delta}\eta_+ = 4g^{\gamma\bar\alpha}g^{\beta\bar\delta} - 2g^{\beta\bar\alpha}g^{\gamma\bar\delta} \ . \tag{4.36}$$

One can now insert the expansions (4.27)–(4.31) into the action $\mathcal{S}_\mathrm{f}$ and compute the kinetic terms in the effective Lagrangian for the fermions. The only remaining point that is still worth mentioning is that one encounters the integrals

$$\int_{\mathcal{Y}} (\omega_i)_{\alpha\bar{\beta}}(\omega_j)_{\gamma\bar{\delta}}\big(g^{\alpha\bar{\delta}}g^{\gamma\bar{\beta}} - 2g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\big) = 4\mathcal{V}\big(g_{ij} + \tfrac{1}{2}\varkappa_{ij}\big) \ ,$$
$$\frac{1}{\|\Omega\|^2}\int_6 (\rho_m)_{\bar{\alpha}\beta\gamma}(\bar{\rho}_n)_{\delta\bar{\epsilon}\bar{\zeta}}\bar{\Omega}_{\bar{\sigma}}{}^{\beta\gamma}\Omega_\tau{}^{\bar{\epsilon}\bar{\zeta}}\big(g^{\delta\bar{\alpha}}g^{\tau\bar{\sigma}} - 2g^{\delta\bar{\sigma}}g^{\tau\bar{\alpha}}\big) = -\frac{4i}{\|\Omega\|^2}\int_{\mathcal{Y}} \rho_m \wedge \bar{\rho}_n = -4\mathcal{V}g_{m\bar{n}} \ , \tag{4.37}$$

The resulting Lagrangian is

$$\begin{aligned}\mathscr{L}_\mathrm{kin} = \mathrm{e}^{-\phi}\big[&\epsilon^{\mu\rho\nu\lambda}\bar{\psi}_\mu\bar{\sigma}_\lambda D_\rho\psi_\nu + \mathrm{i}\varkappa_i\xi^i\sigma^{[\mu}\bar{\sigma}^{\nu]}D_\mu\psi_\nu + \mathrm{i}\varkappa_i\bar{\psi}_\mu\bar{\sigma}^{[\mu}\sigma^{\nu]}D_\nu\bar{\xi}^i \\ &+ 16\mathrm{i}\big(g_{ij} + \tfrac{1}{2}\varkappa_{ij}\big)\bar{\xi}^i\bar{\sigma}^\mu D_\mu\xi^j - 16\mathrm{i}g_{\bar{m}n}\bar{\zeta}^m\bar{\sigma}^\mu D_\mu\zeta^n - 2\mathrm{i}\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda \\ &- 2\mathrm{i}\bar{\chi}\bar{\sigma}^\mu D_\mu\chi - 16\mathrm{i}g_{ij}\bar{\chi}^i\bar{\sigma}^\mu D_\mu\chi^j - 16\mathrm{i}g_{\bar{m}n}\bar{\chi}^m\bar{\sigma}^\mu D_\mu\chi^n\big] \ .\end{aligned} \tag{4.38}$$

The next step is to perform the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^\phi g_{\mu\nu}$. Since the matrices σ^μ are defined with a vierbein, they must be rescaled accordingly. In addition, all fermionic fields must be rescaled. The appropriate expressions are [1]

$$\begin{aligned}&\sigma^\mu \to \mathrm{e}^{-\frac{\phi}{2}}\sigma^\mu \ , \quad \psi_\mu \to \mathrm{e}^{\frac{\phi}{4}}\psi_\mu \ , \quad \xi^i \to \mathrm{e}^{-\frac{\phi}{4}}\xi^i \ , \quad \zeta^m \to \mathrm{e}^{-\frac{\phi}{4}}\zeta^m \ , \\ &\lambda \to \mathrm{e}^{-\frac{\phi}{4}}\lambda \ , \quad \chi \to \mathrm{e}^{-\frac{\phi}{4}}\chi \ , \quad \chi^i \to \mathrm{e}^{-\frac{\phi}{4}}\chi^i \ , \quad \chi^m \to \mathrm{e}^{-\frac{\phi}{4}}\chi^m \ .\end{aligned} \tag{4.39}$$

But the kinetic terms in the Lagrangian (4.38) are not diagonal. They can be diagonalized by redefining the gravitino including a mixing with the fermions ξ^i as follows,

$$\psi_\mu \to \psi_\mu + \tfrac{1}{2}\sigma_\mu\varkappa_i\bar{\xi}^i \ . \tag{4.40}$$

Inserting (4.39) and (4.40) into (4.38) one arrives at

$$\begin{aligned}\mathscr{L}_\mathrm{kin} = 2\epsilon^{\mu\rho\nu\lambda}\bar{\psi}_\mu\bar{\sigma}_\lambda D_\rho\psi_\nu &+ \mathrm{i}\big(16g_{ij} - 3\varkappa_i\varkappa_j + 8\varkappa_{ij}\big)\bar{\xi}^i\bar{\sigma}^\mu D_\mu\xi^j - 16\mathrm{i}g_{\bar{m}n}\bar{\zeta}^m\bar{\sigma}^\mu D_\mu\zeta^n \\ &- 2\mathrm{i}\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda - 2\mathrm{i}\bar{\chi}\bar{\sigma}^\mu D_\mu\chi - 16\mathrm{i}g_{ij}\bar{\chi}^i\bar{\sigma}^\mu D_\mu\chi^j - 16\mathrm{i}g_{\bar{m}n}\bar{\chi}^m\bar{\sigma}^\mu D_\mu\chi^n \ .\end{aligned} \tag{4.41}$$

To bring the kinetic terms of all fermions to the standard form (A.20) dictated by $\mathscr{N} = 1$ supergravity, the fermionic fields need to be further rescaled as follows,

$$\begin{aligned}&\psi_\mu \to \tfrac{1}{\sqrt{2}}\psi_\mu \ , \quad \xi^i \to \tfrac{1}{4}(\xi^i - \tfrac{1}{12}v^i\varkappa_j\xi^j) \ , \quad \zeta^m \to \tfrac{1}{4}\zeta^m \ , \quad \lambda \to \tfrac{1}{\sqrt{2}}\mathrm{e}^\phi\lambda \ , \\ &\chi \to \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\phi}{2}}\chi \ , \quad \chi^i \to \tfrac{1}{4}\mathrm{e}^{\frac{1}{6}(K_\Omega - K_J)}\chi^i \ , \quad \chi^a \to \tfrac{1}{4}\mathrm{e}^{\frac{1}{6}(K_J - K_\Omega)}\chi^a \ .\end{aligned} \tag{4.42}$$

Notice that using the relation $\varkappa_i v^i = 6$, the combination $\varkappa_i\xi^i$ can be seen to transform in the much simpler way

$$\varkappa_i\xi^i \to \tfrac{1}{8}\varkappa_i\xi^i \ . \tag{4.43}$$

It is straightforward to check that after substitution of (4.42) into (4.41) it is obtained

$$\begin{aligned}\mathscr{L}_{\rm kin} = \epsilon^{\mu\rho\nu\lambda}\bar{\psi}_\mu\bar{\sigma}_\lambda D_\rho\psi_\nu - {\rm i}g_{i\bar{j}}\bar{\xi}^{\bar{i}}\bar{\sigma}^\mu D_\mu\xi^j - {\rm i}g_{\bar{a}b}\bar{\zeta}^a\bar{\sigma}^\mu D_\mu\zeta^b \\ - {\rm i}g_{s\bar{s}}\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda - {\rm i}({\rm Re}f)\bar{\chi}\sigma^\mu D_\mu\chi - {\rm i}Z_{ij}\bar{\chi}^i\bar{\sigma}^\mu D_\mu\chi^j - {\rm i}Z_{\bar{m}n}\bar{\chi}^m\bar{\sigma}^\mu D_\mu\chi^n\ ,\end{aligned} \tag{4.44}$$

in accord with the standard form of $\mathscr{N}=1$ supergravity.†

4.2.3 Gravitino mass term and F-terms

It has been seen that the kinetic terms in the four-dimensional effective theory can be cast into a form consistent with the Kähler potential (4.16) and the gauge kinetic function (4.14). There are two ways to compute the remaining couplings. One could reduce the bosonic ten-dimensional action (2.2) and derive the scalar potential $\mathcal{V}_{\mathscr{N}=1}$. Then, from the supergravity relation (A.18), one could infer the superpotential W and the D-terms. However, this procedure is problematic since W and its derivatives enter quadratically in $\mathcal{V}_{\mathscr{N}=1}$ and thus cannot be computed reliably within the approximation used. However, in the fermionic couplings of (A.21), both W and its derivatives appear linearly and therefore can be obtained more easily [57, 58]. Concretely, W can be computed from the gravitino mass term, while the derivatives of W can be computed from the couplings of the gravitino to the chiral fermions or F-terms.

The contributions to the gravitino mass term and to the F-terms arise from two different sources. On one hand, they come from the reduction of $S_{\rm f}$ in Eq. (2.6) when no spacetime derivative D_μ is present and the internal derivative D_a acts on the spinor η in the expansion of the fermions (4.27)–(4.31). In this case, the contribution will be proportional to certain torsion components of the SU(3)-structure manifold. The second possibility is that such terms arise from the reduction of $\mathcal{S}_{\rm int}$ given in Eq. (2.7) when a nonzero background value for H_3 is present. Let us start with this second case.

Contribution from H_3-flux

As already mentioned, in this case the contribution to the gravitino mass term arises from the first term in the second line of Eq. (2.7) when both gravitino factors have external

†Note that in the derivation of (4.44) all terms where spacetime derivatives act on bosonic terms have been ignored. They should combine into appropriate covariant derivatives as given in [1] which, however, was not explicitly checked.

spacetime indices. Inserting the decomposition (4.27) and using Eq. (4.2) one finds

$$\begin{aligned}\mathscr{L}_{m_{3/2},\text{flux}} &= -\tfrac{1}{24}\int_{\mathcal{Y}} \mathrm{e}^{-\Phi} H_{abc}\bar{\hat{\psi}}_\mu \Gamma^{\mu abc\nu}\hat{\psi}_\nu \\ &= \tfrac{1}{48}\bar{\psi}_\mu \bar{\sigma}^{[\mu}\sigma^{\nu]}\bar{\psi}_\nu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} H_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\eta_-^\dagger \gamma^{\bar{\alpha}\bar{\beta}\bar{\gamma}}\eta_+ + \text{h.c.} \\ &= -\tfrac{1}{8}\bar{\psi}_\mu \bar{\sigma}^{[\mu}\sigma^{\nu]}\bar{\psi}_\nu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} \Omega\wedge H_3 + \text{h.c.}\ ,\end{aligned} \tag{4.45}$$

where only the contribution to the gravitino mass is displayed. Performing the Weyl rescaling $g_{\mu\nu} \to \mathrm{e}^{\phi} g_{\mu\nu}$ and using Eqs. (4.39), (4.42) and (4.23), it is obtained

$$\mathscr{L}_{m_{3/2},\text{flux}} = -\tfrac{1}{4}\bar{\psi}_\mu \bar{\sigma}^{\mu\nu}\bar{\psi}_\nu \mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}} \Omega\wedge H_3 + \text{h.c.}\ , \tag{4.46}$$

where $\bar{\sigma}^{\mu\nu} = \frac{1}{4}\bar{\sigma}^{[\mu}\sigma^{\nu]}$. Comparison with (A.21) leads to the following contribution to the superpotential arising from the H_3-flux,

$$W_{\text{flux}} = \tfrac{1}{4}\int_{\mathcal{Y}} \Omega\wedge H_3\ , \tag{4.47}$$

a result computed previously in [59–61]. This derivation provides an independent check on the Kähler potential (4.16).

Let us now proceed to the computation of the derivatives of W, or in other words the F-terms. They arise from the same ten-dimensional term in $\mathcal{S}_{\text{int}}$ as before, only that this time one must choose one of the ten-dimensional gravitino factors to carry an internal index a. There is also an additional contribution coming from the insertion of the gravitino redefinition (4.40) in Eq. (4.46). Inserting the decompositions (4.27) and (4.28), one finds

$$\begin{aligned}\mathscr{L}'_{F\text{-term,flux}} =& -\tfrac{1}{96}\varkappa_i\xi^i\sigma^\mu\bar{\psi}_\mu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} H_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\eta_-^\dagger\gamma^{\bar{\alpha}\bar{\beta}\bar{\gamma}}\eta_+ \\ &+ \frac{1}{4\|\Omega\|}\zeta^m\sigma^\mu\bar{\psi}_\mu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} (\rho_m)_{\bar{\alpha}\beta\gamma} H_{\delta\bar{\epsilon}\bar{\zeta}}\bar{\Omega}^{\delta\beta\gamma}\Omega^{\bar{\alpha}\bar{\epsilon}\bar{\zeta}} + \text{h.c.}\ .\end{aligned} \tag{4.48}$$

In computing this expression, the following property derived in [62] was used

$$g^{\alpha\bar{\beta}}(\omega_i)_{\alpha\bar{\beta}} = \tfrac{\mathrm{i}}{2}\varkappa_i\ . \tag{4.49}$$

This contraction is therefore independent of the internal coordinates. Using Eqs. (4.2), (4.22), (4.23) and performing the rescalings one obtains

$$\mathscr{L}'_{F\text{-term,flux}} = -\tfrac{\mathrm{i}}{4\sqrt{2}}\mathrm{e}^{\frac{K}{2}}\Big(\xi^i\sigma^\mu\bar{\psi}_\mu\,\tfrac{\mathrm{i}}{4}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge H_3 + \zeta^m\sigma^\mu\bar{\psi}_\mu\int_{\mathcal{Y}}\rho_m\wedge H_3\Big) + \text{h.c.}\ . \tag{4.50}$$

It is straightforward to check from Eqs. (A.19), (4.17), (4.13) and (4.21) that in the absence of torsion (or in other words for $\mathrm{d}\omega_i = 0$) the Kähler derivatives of W_{flux} as obtained in Eq. (4.47) are given by

$$D_iW_{\text{flux}} = \tfrac{\mathrm{i}}{4}\varkappa_i W_{\text{flux}}\ ,\qquad D_mW_{\text{flux}} = \tfrac{1}{4}\int_{\mathcal{Y}}\rho_m\wedge H_3\ . \tag{4.51}$$

The F-terms in Eq. (4.50) can then be written as

$$\mathscr{L}'_{F\text{-term,flux}} = -\tfrac{\mathrm{i}}{\sqrt{2}} e^{\frac{K}{2}} \left(\xi^i \sigma^\mu \bar{\psi}_\mu D_i W_{\text{flux}} + \zeta^m \sigma^\mu \bar{\psi}_\mu D_m W_{\text{flux}}\right) + \text{h.c.} \,, \tag{4.52}$$

which is consistent with (A.21).

Finally, there is also a gravitino-dilatino coupling which is obtained from the appropriate term in Eq. (2.7),

$$\begin{aligned} \mathscr{L}''_{F\text{-term,flux}} &= \tfrac{\sqrt{2}}{24} \int_{\mathcal{Y}} \mathrm{e}^{-\Phi} H_{abc} \hat{\bar{\psi}}_\mu \Gamma^{abc} \Gamma^\mu \hat{\lambda} \\ &= \tfrac{1}{2\sqrt{2}} \lambda \sigma^\mu \bar{\psi}_\mu \mathrm{e}^{-\phi} \mathcal{V}^{-1} \int_{\mathcal{Y}} \Omega \wedge H_3 + \text{h.c.} \,, \end{aligned} \tag{4.53}$$

where the decompositions (4.30), (4.29) and (4.27) have been used. Following a similar procedure as for the other F-terms that include the rescalings, one obtains

$$\mathscr{L}''_{F\text{-term,flux}} = -\tfrac{\mathrm{i}}{\sqrt{2}} \lambda \sigma^\mu \bar{\psi}_\mu \mathrm{e}^{\frac{K}{2}} D_s W_{\text{flux}} \,, \tag{4.54}$$

where it was used

$$D_s W_{\text{flux}} = \mathrm{i}\mathrm{e}^{\phi} W_{\text{flux}} \,, \tag{4.55}$$

as can be derived from Eqs. (A.19), (4.14), (4.15) and (4.17). The total contribution to the F-terms arising from H_3-flux is the sum of Eqs. (4.52) and (4.54),

$$\begin{aligned} \mathscr{L}_{F\text{-term,flux}} &= \mathscr{L}'_{F\text{-term,flux}} + \mathscr{L}''_{F\text{-term,flux}} \\ &= -\tfrac{\mathrm{i}}{\sqrt{2}} e^{\frac{K}{2}} \left(\xi^i \sigma^\mu \bar{\psi}_\mu D_i W_{\text{flux}} + \zeta^m \sigma^\mu \bar{\psi}_\mu D_m W_{\text{flux}} + \lambda \sigma^\mu \bar{\psi}_\mu \mathrm{e}^{\frac{K}{2}} D_s W_{\text{flux}}\right) + \text{h.c.} \,. \end{aligned} \tag{4.56}$$

This result is again in agreement with supergravity if it is compared with (A.21).

Contribution from torsion

In addition to H_3-flux, also the torsion of the manifold $\mathcal{Y}$ gives a contribution to the superpotential W. As already explained, the intrinsic torsion is measured by the five torsion classes $\mathcal{W}_1, \ldots, \mathcal{W}_5$, or the exterior derivatives $\mathrm{d}J$ and $\mathrm{d}\Omega$. This contribution to the superpotential can be computed from Eq. (2.6) when an internal derivative D_a acts on the internal spinor η. These derivatives have been determined in Ref. [63]. One can expand $D_a \eta_\pm$ in the basis $(\eta_\pm, \gamma^a \eta_\mp)$ and define the tensors q_a, q'_a and q_{ab} via

$$D_a \eta_\pm = (q_a \pm \mathrm{i} q'_a) \eta_\pm \pm \mathrm{i} q_{ab} \gamma^b \eta_\mp \,. \tag{4.57}$$

All q's are real, with q_a and q'_a transforming in the $\mathbf{3} \oplus \bar{\mathbf{3}}$ of SU(3) and q_{ab} containing the representations $\mathbf{36} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{8} \oplus \mathbf{8}$. Going to holomorphic indices and using Eqs. (4.2) and (4.5), one can express q_{ab} via the torsion classes as follows [63],

$$\begin{aligned} q_{\alpha\beta} &= -\tfrac{\mathrm{i}}{16} (\mathcal{W}_3)_{\alpha\bar{\gamma}\bar{\delta}} \Omega_\beta{}^{\bar{\gamma}\bar{\delta}} - \tfrac{1}{4} \Omega_{\alpha\beta\gamma} \overline{\mathcal{W}}_4^\gamma \,, \\ q_{\alpha\bar{\beta}} &= \tfrac{1}{4} g_{\alpha\bar{\beta}} \overline{\mathcal{W}}_1 - \tfrac{\mathrm{i}}{4} (\overline{\mathcal{W}}_2)_{\alpha\bar{\beta}} \,. \end{aligned} \tag{4.58}$$

Equipped with (4.57) and (4.58), the contribution to W due to torsion can be computed. Let us start again with the contribution to the gravitino mass term. It arises from the first term in Eq. (2.6) with the derivative in an internal direction. Inserting the decomposition (4.27) in the kinetic term for the ten-dimensional gravitino one obtains

$$\begin{aligned}\mathscr{L}_{m_{3/2},\text{torsion}} &= -\int_{\mathcal{Y}} \mathrm{e}^{-\Phi}\hat{\bar{\psi}}_\mu \Gamma^{\mu a \nu} D_a \hat{\psi}_\nu \\ &= \tfrac{1}{2}\bar{\psi}_\mu \bar{\sigma}^{[\mu}\sigma^{\nu]}\bar{\psi}_\nu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} \eta_-^\dagger \gamma^{\bar{\alpha}} D_{\bar{\alpha}}\eta_+ + \text{h.c.}\ .\end{aligned} \tag{4.59}$$

Using Eqs. (4.57), (4.58) and (4.5) it is found

$$\int_{\mathcal{Y}} \eta_-^\dagger \gamma^{\bar{\alpha}} D_{\bar{\alpha}}\eta_+ = \tfrac{3\mathrm{i}}{2}\int_{\mathcal{Y}} \mathcal{W}_1 = \tfrac{\mathrm{i}}{4}\int_{\mathcal{Y}} \Omega \wedge \mathrm{d}J\ . \tag{4.60}$$

Performing the usual rescalings and using (4.23) yields

$$\mathscr{L}_{m_{3/2},\text{tor}} = -\bar{\psi}_\mu \bar{\sigma}^{\mu\nu}\bar{\psi}_\nu \mathrm{e}^{\frac{K}{2}} W_{\text{tor}} + \text{h.c.}\ , \tag{4.61}$$

with

$$W_{\text{tor}} = \tfrac{\mathrm{i}}{4}\int_{\mathcal{Y}} \Omega \wedge \mathrm{d}J\ . \tag{4.62}$$

Together with the contribution from H_3-flux computed in Eq. (4.47), this leads to the total superpotential

$$W = W_{\text{flux}} + W_{\text{tor}} = \tfrac{1}{4}\int_{\mathcal{Y}} \Omega \wedge (H_3 + \mathrm{i}\mathrm{d}J)\ . \tag{4.63}$$

As it was done for the H_3-flux contribution, let us now focus on the gravitino-fermion couplings in order to determine the torsion contribution to the F-terms. For the fermions in the chiral multiplets, this contribution arises from the kinetic term of the ten-dimensional gravitino in $\mathcal{S}_\mathrm{f}$ when one of the gravitino factors has an internal index, the other has an external one, and the derivative is internal. After insertion of (4.27) in the kinetic term for $\hat{\psi}_M$ one obtains

$$\begin{aligned}\mathscr{L}'_{F\text{-term,tor}} &= -\int_{\mathcal{Y}} \mathrm{e}^{-\Phi}\big[\hat{\bar{\psi}}_a \Gamma^{ab\nu} D_b \hat{\psi}_\nu + \hat{\bar{\psi}}_\mu \Gamma^{\mu ab} D_a \hat{\psi}_b\big] \\ &= 2\mathrm{i}\xi^i \sigma^\mu \bar{\psi}_\mu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} (\omega_i)_{\alpha\bar{\beta}}\eta_-^\dagger \gamma^{\bar{\beta}}\gamma^{\alpha\bar{\gamma}}\mathrm{i}q_{\bar{\gamma}\delta}\gamma^\delta \eta_- \\ &\qquad + \frac{2\mathrm{i}}{\|\Omega\|}\zeta^m \sigma^\mu \bar{\psi}_\mu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}} (\rho_m)_{\bar{\beta}\gamma\delta}\bar{\Omega}^{\gamma\delta}_{\bar{\epsilon}}\eta_-^\dagger \gamma^{\alpha\bar{\beta}}\gamma^{\bar{\epsilon}}\mathrm{i}q_{\alpha\zeta}\gamma^\zeta \eta_- + \text{h.c.}\ .\end{aligned} \tag{4.64}$$

The term containing $D_a\psi_b$ in the first line can be integrated by parts. The result is equal to the first term. This is convenient because only the action of the internal derivative D_a on ψ_μ needs to be considered. All one needs is therefore the derivative of the internal spinors

$D_a\eta_\pm$ and not more complex expressions involving the derivatives of the internal forms that would arise in computing $D_a\psi_b$. Substituting (4.58) in (4.64) yields

$$\begin{aligned}\mathscr{L}'_{F\text{-term,tor}} =& -\mathrm{i}\xi^i\sigma^\mu\bar{\psi}_\mu \mathrm{e}^{-\phi}\mathcal{V}^{-1}\Big(-5\mathrm{i}\int_{\mathcal{Y}}(\omega_i)_{\alpha\bar{\beta}}g^{\alpha\bar{\beta}}\mathcal{W}_1 + \int_{\mathcal{Y}}(\omega_i)_{\alpha\bar{\beta}}(\mathcal{W}_2)_{\delta\bar{\gamma}}g^{\alpha\bar{\gamma}}g^{\delta\bar{\beta}}\Big)\\ &-\frac{\mathrm{i}}{\|\Omega\|}\zeta^m\sigma^\mu\bar{\psi}_\mu\mathrm{i}\mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}}\rho_m\wedge\mathcal{W}_3 + \text{h.c.}\ .\end{aligned} \tag{4.65}$$

Using (4.49) it can be written

$$\begin{aligned}-\mathrm{i}\int_{\mathcal{Y}}(\omega_i)_{\alpha\bar{\beta}}g^{\alpha\bar{\beta}}\mathcal{W}_1 &= \tfrac{1}{2}\varkappa_i\int_{\mathcal{Y}}\mathcal{W}_1 = \tfrac{1}{12}\varkappa_i\int_{\mathcal{Y}}\mathcal{W}_1 J\wedge J\wedge J\\ &= \tfrac{1}{12}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge\mathrm{d}J\ .\end{aligned} \tag{4.66}$$

In the last step, $\mathcal{W}_1 J\wedge J\wedge J = \mathrm{d}\Omega\wedge J$ was used, which is a consequence of (4.5) together with $\mathcal{W}_2\wedge J\wedge J = 0$. Analogously, one finds

$$\begin{aligned}\int_{\mathcal{Y}}(\omega_i)_{\alpha\bar{\beta}}(\mathcal{W}_2)_{\delta\bar{\gamma}}g^{\alpha\bar{\gamma}}g^{\delta\bar{\beta}} &= \int_{\mathcal{Y}}\mathcal{W}_2\wedge J\wedge\omega_i\\ &= \int_{\mathcal{Y}}\mathrm{d}\Omega\wedge\omega_i - \int_{\mathcal{Y}}\mathcal{W}_1 J\wedge J\wedge\omega_i\\ &= \int_{\mathcal{Y}}\Omega\wedge\mathrm{d}\omega_i - \tfrac{1}{6}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge\mathrm{d}J\ .\end{aligned} \tag{4.67}$$

In going from the first to the second line, $\mathcal{W}_2\wedge J\wedge\omega_i = \mathrm{d}\Omega\wedge\omega_i - \mathcal{W}_1 J\wedge\omega_i + \cdots$ was used, which also follows from Eq. (4.5). In the last step, $\mathcal{W}_1 J\wedge J\wedge\omega_i$ was substituted by twice the expression (4.66). Finally, it can also be seen from (4.5) that

$$\int_{\mathcal{Y}}\rho_m\wedge\mathcal{W}_3 = \int_{\mathcal{Y}}\rho_m\wedge\mathrm{d}J. \tag{4.68}$$

Inserting (4.66), (4.67) and (4.68) into (4.65) and performing the usual rescalings one obtains

$$\begin{aligned}\mathscr{L}'_{F\text{-term,tor}} =& -\tfrac{\mathrm{i}}{4\sqrt{2}}\xi^i\sigma^\mu\bar{\psi}_\mu\mathrm{e}^{\frac{K}{2}}\Big(-\tfrac{\mathrm{i}}{24}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge\mathrm{i}\mathrm{d}J + \int_{\mathcal{Y}}\Omega\wedge\mathrm{d}\omega_i\Big)\\ &-\tfrac{\mathrm{i}}{4\sqrt{2}}\int_4\zeta^m\sigma^\mu\bar{\psi}_\mu\mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}}\rho_m\wedge\mathrm{i}\mathrm{d}J + \text{h.c.}\ .\end{aligned} \tag{4.69}$$

There is also a contribution $\mathscr{L}''_{F\text{-term,tor}}$ arising from the insertion of the shifted gravitino (4.40) into Eq. (4.59). Adding this contribution to Eq. (4.69) leads to

$$\begin{aligned}\mathscr{L}_{F\text{-term,tor}} =& -\tfrac{\mathrm{i}}{4\sqrt{2}}\xi^i\sigma^\mu\bar{\psi}_\mu\,\mathrm{e}^{\frac{K}{2}}\Big(\tfrac{\mathrm{i}}{3}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge\mathrm{i}\mathrm{d}J + \int_{\mathcal{Y}}\Omega\wedge\omega_i\Big)\\ &-\tfrac{\mathrm{i}}{4\sqrt{2}}\zeta^m\sigma^\mu\bar{\psi}_\mu\mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}}\rho_m\wedge\mathrm{i}\mathrm{d}J + \text{h.c.}\ .\end{aligned} \tag{4.70}$$

Combining this result with the H_3-flux contribution obtained in (4.50) and (4.54) yields

$$\begin{aligned}\mathscr{L}_{F\text{-term}} &= \mathscr{L}_{F\text{-term,flux}} + \mathscr{L}_{F\text{-term,tor}} \\ &= -\tfrac{\mathrm{i}}{4\sqrt{2}}\xi^i\sigma^\mu\bar\psi_\mu\,\mathrm{e}^{\frac{K}{2}}\Big[\tfrac{\mathrm{i}}{4}\varkappa_i\int_{\mathcal{Y}}\Omega\wedge\big(H_3+\tfrac{4}{3}\mathrm{i}\mathrm{d}J\big)+\int_{\mathcal{Y}}\Omega\wedge\mathrm{d}\omega_i\Big] \\ &\quad -\tfrac{\mathrm{i}}{4\sqrt{2}}\zeta^m\sigma^\mu\bar\psi_\mu\,\mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}}\rho_m\wedge(H_3+\mathrm{i}\mathrm{d}J) \\ &\quad -\tfrac{\mathrm{i}}{4\sqrt{2}}\lambda\sigma^\mu\bar\psi_\mu\,\mathrm{e}^{\frac{K}{2}}\,\mathrm{i}\mathrm{e}^{\phi}\int_{\mathcal{Y}}\Omega\wedge H_3+\text{h.c.}\ .\end{aligned} \tag{4.71}$$

However, this is not yet in the standard supergravity form, since the gravitino-dilatino coupling received no contribution from the torsion. This can be remedied by the following redefinitions

$$\xi^i\to\xi^i-\tfrac{1}{12}v^i\varkappa_j\xi^j+v^i\mathrm{e}^{\phi}\lambda\ ,\qquad \lambda\to-\tfrac{1}{2}\lambda+\tfrac{1}{8}\mathrm{e}^{-\phi}\varkappa_j\xi^j\ . \tag{4.72}$$

One can show that these transformations leave invariant the kinetic terms (4.44) and the total contribution $\mathscr{L}_{F\text{-term,flux}}$ to the F-terms from H_3-flux in Eq. (4.56). Inserting (4.72) into Eq.(4.71) one finally obtains

$$\mathscr{L}_{F\text{-term}} = -\tfrac{\mathrm{i}}{\sqrt{2}}\big(\xi^i\sigma^\mu\bar\psi_\mu\mathrm{e}^{\frac{K}{2}}D_iW+\zeta^m\sigma^\mu\bar\psi_\mu\mathrm{e}^{\frac{K}{2}}D_mW+\lambda\sigma^\mu\bar\psi_\mu\mathrm{e}^{\frac{K}{2}}D_sW\big)+\text{h.c.}\ , \tag{4.73}$$

where

$$\begin{aligned} D_iW &= \tfrac{\mathrm{i}}{4}\varkappa_iW+\tfrac{1}{4}\int_{\mathcal{Y}}\Omega\wedge\mathrm{d}\omega_i\ , \\ D_mW &= \tfrac{1}{4}\int_{\mathcal{Y}}\rho_m\wedge(H_3+\mathrm{i}\mathrm{d}J)\ , \\ D_sW &= \mathrm{i}\mathrm{e}^{\phi}W\ ,\end{aligned} \tag{4.74}$$

with W given in Eq. (4.63). This establishes the consistency with $\mathscr{N}=1$ supergravity.

4.2.4 D-terms

Finally, one can compute the D-terms in the effective action. As can be seen from Eq. (A.21), in the fermionic action they appear in the coupling of the gravitino to the gaugino. This contribution to the action comes from the reduction of the similar coupling between the ten-dimensional gravitino $\hat\psi_M$ and the ten-dimensional gaugino $\hat\chi$ in Eq. (2.7). Performing the reduction of the relevant term is straightforward and leads to

$$\begin{aligned}\mathscr{L}_{D\text{-term}} &= \tfrac{1}{2}\int_{\mathcal{Y}}\mathrm{e}^{-\Phi}\mathrm{Tr}\big(F_{ab}\hat{\bar\chi}\big)\Gamma^\mu\Gamma^{ab}\hat\psi_\mu \\ &= -\mathrm{i}\psi_\mu\sigma^\mu\bar\chi^{\mathrm{a}}\mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}}F^{\mathrm{a}}_{\alpha\bar\beta}g^{\alpha\bar\beta}+\text{h.c.} \\ &= \psi_\mu\sigma^\mu\bar\chi^{\mathrm{a}}\mathrm{e}^{-\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}}F^{\mathrm{a}}\wedge *J+\text{h.c.}\ .\end{aligned} \tag{4.75}$$

In the last step, the relation

$$\int_{\mathcal{Y}} F^{\mathrm{a}}_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}} = \tfrac{\mathrm{i}}{2} \int_{\mathcal{Y}} F^{\mathrm{a}} \wedge J \wedge J = \mathrm{i} \int_{\mathcal{Y}} F^{\mathrm{a}} \wedge *J \tag{4.76}$$

was used. After performing the usual rescalings one obtains

$$\mathscr{L}_{D\text{-term}} = \frac{1}{2} \psi_\mu \sigma^\mu \bar{\chi}^{\mathrm{a}} \mathcal{V}^{-1} \int_{\mathcal{Y}} F^{\mathrm{a}} \wedge *J \ . \tag{4.77}$$

Comparing with (A.21) and recalling from (4.14) that $\mathrm{Re} f = \mathrm{e}^{-\phi}$, it is concluded that the D-term is given by

$$\mathcal{D}^{\mathrm{a}} = -\mathrm{e}^{\phi} \mathcal{V}^{-1} \int_{\mathcal{Y}} F^{\mathrm{a}} \wedge *J \ . \tag{4.78}$$

4.3 Supersymmetry transformations

For completeness, one can additionally obtain the supersymmetry transformations of the fermions in the effective four-dimensional theory. This can be done by substitution of the reduction ansatz into the supersymmetry transformations of the ten-dimensional fields, Eq. (2.9). The form of these transformations for a generic $\mathcal{N} = 1$ theory in four dimensions is given in Eq. (A.22). From them it is seen that the gravitino supersymmetry transformation gives directly the superpotential W, in analogy to the gravitino mass term in the action. On the other hand, the transformations of the chiral fermions are proportional to the derivatives of the superpotential with respect to the corresponding scalar superpartners. In this they are similar to the F- and D-terms in the action.

Let us start with the gravitino. The supersymmetry transformation of the gravitino in ten dimensions is given by

$$\delta\hat{\psi}_M = D_M \hat{\epsilon} + \tfrac{1}{96} H_{NPQ} \big(\Gamma_M{}^{NPQ} - 9 \delta^N_M \Gamma^{PQ} \big) \hat{\epsilon} \ , \tag{4.79}$$

which implies

$$\delta\hat{\psi}_\mu = D_\mu \hat{\varepsilon} + \tfrac{1}{96} H_{abc} \Gamma_\mu \Gamma^{abc} \hat{\varepsilon} \ . \tag{4.80}$$

However, the correct four-dimensional gravitino is only obtained after the shift given in Eq. (4.40). The latter can be interpreted at the level of the ten-dimensional gravitino as [54]

$$\delta\hat{\psi}'_\mu \equiv \delta\hat{\psi}_\mu + \tfrac{1}{2} \Gamma_\mu \Gamma^a \delta\hat{\psi}_a \ . \tag{4.81}$$

Also from Eq. (4.79) one has

$$\begin{aligned} \Gamma^a \hat{\psi}_a &= \Gamma^a D_a \hat{\varepsilon} + \tfrac{1}{96} H_{bcd} (\Gamma^a \Gamma_a{}^{bcd} - 9 \Gamma^b \Gamma^{cd}) \hat{\varepsilon} \\ &= \Gamma^a D_a \hat{\varepsilon} - \tfrac{1}{16} H_{abc} \Gamma^{abc} \hat{\varepsilon} \ , \end{aligned} \tag{4.82}$$

where $\Gamma^a\Gamma_a{}^{bcd} = 3\Gamma^{bcd}$ was used. After substitution of (4.80) and (4.82) into Eq. (4.81) it is obtained

$$\delta\hat{\psi}'_\mu = D_\mu\hat{\varepsilon} + \tfrac{1}{2}\Gamma_\mu\Gamma^a D_a\hat{\varepsilon} - \tfrac{1}{48}H_{abc}\Gamma_\mu\Gamma^{abc}\hat{\varepsilon} \ . \tag{4.83}$$

Inserting (4.27) and acting with the projector $\mathcal{V}^{-1}\int_{\mathcal{Y}} \mathbf{1}\otimes\eta_-^\dagger$ yields (omitting the prime)

$$\begin{aligned}\delta\psi_\mu =& D_\mu\varepsilon + \tfrac{\mathrm{i}}{2}\sigma_\mu\bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}\eta_-^\dagger\gamma^{\bar{\alpha}}D_{\bar{\alpha}}\eta_+ - \tfrac{\mathrm{i}}{48}\sigma_\mu\bar{\varepsilon}\mathcal{V}^{-1}\int_6 H_{abc}\eta_-^\dagger\gamma^{abc}\eta_+ \\ =& D_\mu\varepsilon - \tfrac{\mathrm{i}}{8}\sigma_\mu\bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}\Omega\wedge(H_3 + \mathrm{i}\mathrm{d}J) \ ,\end{aligned}$$

where in the second step Eq. (4.60) was used. After performing the rescalings (4.39) and (4.42) one indeed obtains

$$\delta\psi_\mu = D_\mu\varepsilon + \tfrac{\mathrm{i}}{2}\sigma_\mu\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}W \ , \tag{4.84}$$

with the superpotential W given by the expression (4.63).

Let us now turn to the supersymmetry transformations of the chiral fermions ξ^i. In order to do so, it is useful to compute

$$(\gamma^5\otimes\eta_-\eta_-^\dagger)\,\Gamma^a\delta\hat{\psi}_a \ . \tag{4.85}$$

Inserting (4.27) and using (4.82) it follows that

$$\begin{aligned}\delta\bar{\xi}^{\bar{i}}\otimes(\omega_i)_{\alpha\bar{\beta}}\,\eta_-\eta_-^\dagger\gamma^{\bar{\beta}}\gamma^\alpha\eta_- &= \bar{\varepsilon}\otimes\eta_-\eta_-^\dagger\gamma^{\bar{\alpha}}D_{\bar{\alpha}}\eta_+ + \tfrac{1}{16}\bar{\varepsilon}\otimes H_{abc}\eta_-\eta_-^\dagger\gamma^{abc}\eta_+ \ , \\ 2\delta\bar{\xi}^{\bar{i}}\otimes(\omega_i)_{\alpha\bar{\beta}}g^{\alpha\bar{\beta}}\eta_- &= \tfrac{3\mathrm{i}}{2}\bar{\varepsilon}\otimes\mathcal{W}_1\eta_- + \tfrac{1}{16}\bar{\varepsilon}\otimes\mathrm{i}H_{abc}\Omega^{abc}\eta_- \ .\end{aligned} \tag{4.86}$$

Using the same projector as above one gets

$$\varkappa_i\delta\bar{\xi}^{\bar{i}} = \tfrac{\mathrm{i}}{4}\bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}\Omega\wedge(\tfrac{3}{2}H_3 + \mathrm{i}\mathrm{d}J) \ , \tag{4.87}$$

which after the rescalings of the fields reads

$$\varkappa_i\delta\bar{\xi}^{\bar{i}} = \tfrac{\mathrm{i}}{\sqrt{2}}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}}\Omega\wedge(3H_3 + 2\mathrm{i}\mathrm{d}J) \ . \tag{4.88}$$

This is not yet in the desired form dictated by Eq. (A.22), since the mixing (4.72) with the dilatino has not been taken into account yet.

For the ten-dimensional dilatino, the supersymmetry transformation is given by

$$\delta\hat{\lambda} = \tfrac{\sqrt{2}}{48}H_{MNP}\Gamma^{MNP}\hat{\varepsilon} \ , \tag{4.89}$$

which after applying the decompositions (4.30) and (4.29) and the projection leads to

$$\delta\bar{\lambda} = -\tfrac{\mathrm{i}\sqrt{2}}{8}\bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}\Omega\wedge H_3 \ . \tag{4.90}$$

After applying the rescalings this reads

$$\delta\bar{\lambda} = -\tfrac{\mathrm{i}\sqrt{2}}{8}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}\mathrm{e}^{-\phi}\int_{\mathcal{Y}}\Omega\wedge H_3 \ . \tag{4.91}$$

As in the computation of the F-terms, there is no torsion contribution in this transformation. But the mixing with ξ^i must still be performed. Going to new field variables as dictated by the field redefinition (4.72) one obtains

$$\begin{aligned} \varkappa_i\delta\bar{\xi}^i =& \tfrac{\mathrm{i}}{\sqrt{2}}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}\int_{\mathcal{Y}}\Omega\wedge(3H_3+\mathrm{i}\mathrm{d}J) \ , \\ \delta\bar{\lambda} =& -\tfrac{\mathrm{i}\sqrt{2}}{8}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}\mathrm{e}^{-\phi}\int_{\mathcal{Y}}\Omega\wedge(H_3+\mathrm{i}\mathrm{d}J) \ . \end{aligned} \tag{4.92}$$

From (4.74) one can write

$$\begin{aligned} g^{ij}\varkappa_i D_j W =& 3\mathrm{i}\int_{\mathcal{Y}}\Omega\wedge(H_3+\mathrm{i}\mathrm{d}J)+2\int_{\mathcal{Y}}\Omega\wedge\mathrm{d}J \\ =& \mathrm{i}\int_{\mathcal{Y}}\Omega\wedge(3H_3+\mathrm{i}\mathrm{d}J) \ , \end{aligned} \tag{4.93}$$

and also

$$g^{s\bar{s}}D_sW = \mathrm{i}\mathrm{e}^{-\phi}W \ . \tag{4.94}$$

The following supersymmetry transformations are thus obtained,

$$\begin{aligned} \delta\bar{\xi}^i =& \tfrac{1}{\sqrt{2}}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}g^{ij}D_jW \ , \\ \delta\bar{\lambda} =& \tfrac{1}{\sqrt{2}}\bar{\varepsilon}\mathrm{e}^{\frac{K}{2}}g^{s\bar{s}}D_sW \ , \end{aligned} \tag{4.95}$$

in agreement with Eq. (A.22).

For the supersymmetry transformations of the ζ^m one evaluates

$$\mathcal{V}^{-1}\int_{\mathcal{Y}}(\gamma^5\otimes\eta_-\eta_-^\dagger)\,(\rho_m)_{\bar{\alpha}\gamma\delta}\bar{\Omega}^{\beta\gamma\delta}\Gamma^{\bar{\alpha}}\delta\hat{\psi}_\beta \ . \tag{4.96}$$

Using the decomposition (4.28) and Eqs. (4.15) and (4.23), this expression can be written as

$$\begin{aligned} \delta\bar{\zeta}^m\mathcal{V}^{-1}\frac{1}{\|\Omega\|^2}\int_{\mathcal{Y}}(\rho_n)_{\bar{\alpha}\gamma\delta}\bar{\Omega}^{\beta\gamma\delta}(\bar{\rho}_m)_{\beta\bar{\varepsilon}\bar{\zeta}}\Omega_\lambda{}^{\bar{\varepsilon}\bar{\zeta}}\eta_-^\dagger\gamma^{\bar{\alpha}}\gamma^\lambda\eta_- =& 8\mathrm{i}\delta\bar{\zeta}^m\frac{\int_{\mathcal{Y}}\rho_n\wedge\bar{\rho}_m}{\int_{\mathcal{Y}}\Omega\wedge\bar{\Omega}} \\ =& 8g_{n\bar{m}}\delta\bar{\zeta}^m \ . \end{aligned} \tag{4.97}$$

On the other hand, using Eq. (4.79) in (4.96) leads to

$$\begin{aligned} \bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}(\rho_n)_{\bar{\alpha}\gamma\delta}\bar{\Omega}^{\beta\gamma\delta}\Big[\eta_-^\dagger\gamma^{\bar{\alpha}}D_\beta\eta_+ + \tfrac{1}{96}(H_{abc}\eta_-^\dagger\gamma^{\bar{\alpha}}\gamma_\beta{}^{abc}\eta_+ - 9H_{\beta bc}\eta_-^\dagger\gamma^{\bar{\alpha}}\gamma^{bc}\eta_+)\Big] \\ = \tfrac{1}{2}\bar{\varepsilon}\mathcal{V}^{-1}\int_{\mathcal{Y}}\rho_n\wedge(H_3+\mathrm{i}\mathrm{d}J) \ , \end{aligned} \tag{4.98}$$

where use was made of Eqs. (4.58), (4.5) and

$$H_{abc}\eta_-^\dagger\gamma^{\bar\alpha}\gamma_\beta{}^{abc}\eta_+ = -6\mathrm{i}H^{\bar\alpha\varepsilon\zeta}\Omega_{\beta\varepsilon\zeta}\ , \qquad H_{\beta ab}\eta_-^\dagger\gamma^{\bar\alpha}\gamma^{ab}\eta_+ = \mathrm{i}H_{\beta\bar\varepsilon\bar\zeta}\Omega^{\bar\alpha\bar\varepsilon\bar\zeta}\ . \tag{4.99}$$

Equating (4.97) and Eq. (4.98) yields

$$\delta\bar\zeta^m = \tfrac{1}{16}\bar\varepsilon\mathcal{V}^{-1}\|\Omega\|^{-1}g^{\bar m n}\int_{\mathcal{Y}}\rho_n\wedge(H_3+\mathrm{i}\mathrm{d}J)\ , \tag{4.100}$$

which after the Weyl rescaling can be written as

$$\bar\delta\zeta^m = \tfrac{1}{\sqrt2}\bar\varepsilon\mathrm{e}^{\frac{K}{2}}g^{\bar m n}D_nW\ , \tag{4.101}$$

with W given once again by (4.63).

Finally, the transformation of the gauginos can be computed. The ten-dimensional variation is

$$\delta\hat\chi^{\mathbf{a}} = -\tfrac14 F^{\mathbf{a}}_{MN}\Gamma^{MN}\hat\varepsilon\ . \tag{4.102}$$

Inserting the decomposition of the ten-dimensional gaugino given in Eq. (4.30) leads to

$$\delta\chi^{\mathrm{a}} = F^{\mathrm{a}}_{\mu\nu}\sigma^{\mu\nu}\varepsilon + \varepsilon\mathcal{V}^{-1}\int_{\mathcal{Y}}F^{\mathrm{a}}_{\alpha\bar\beta}g^{\alpha\bar\beta}\ . \tag{4.103}$$

Substituting (4.76) and performing the rescalings it is obtained

$$\delta\chi^{\mathrm{a}} = F^{\mathrm{a}}_{\mu\nu}\sigma^{\mu\nu}\varepsilon + \mathrm{i}\varepsilon\mathrm{e}^{\phi}\mathcal{V}^{-1}\int_{\mathcal{Y}}F^{\mathrm{a}}\wedge *J\ . \tag{4.104}$$

Comparing with Eq. (4.78), the agreement with the supergravity expression (A.22) is established.

Supersymmetry conditions for the vacuum

With the supersymmetry transformations for the fermions at hand, the conditions which lead to a supersymmetric background in a flux compactification can be discussed. In the case of the heterotic string, Strominger has shown that for a supersymmetric vacuum the background must allow for a non-vanishing torsion [20]. Moreover, the internal manifold has to be complex and the fundamental two-form J, the Yang-Mills field strength F_2^{a} and the three-form flux H_3 have to satisfy the following conditions‡

$$J^{\alpha\bar\beta}F^{\mathrm{a}}_{\alpha\bar\beta} = 0\ , \qquad H_3 = \mathrm{i}(\partial-\bar\partial)J\ . \tag{4.105}$$

Strominger's analysis was made on backgrounds which allow for a warp factor Δ. Demanding the vanishing of the gravitino supersymmetry transformation, he shows that Δ is equal

‡In [20], the condition for the H_3 flux includes a factor of $\frac12$. This is because his normalization for H_3 is half the one used here.

to the dilaton. Since here no warping is considered, the assumption of a constant dilaton is consistent with Strominger's result in the limit of no warp factor.

On a supersymmetric vacuum, the supersymmetry transformations of the fermionic fields vanish, and in particular this must be true for the chiral fermions,

$$\delta_\varepsilon \xi^i = \delta_\varepsilon \zeta^m = \delta_\varepsilon \lambda = \delta_\varepsilon \chi^a = 0 \ . \tag{4.106}$$

Take for example the transformation of the gaugino in Eq. (4.104). Setting it to zero and considering Eq. (4.76) lead to the vanishing of the contraction $F^a_{\alpha\bar{\beta}} J^{\alpha\bar{\beta}}$. Hence Strominger's condition on the Yang-Mills field strength is obtained.

The vanishing of the supersymmetry transformation of the dilatino and the ξ^i chiral fermions as given in Eq. (4.95) requires $D_s W = D_i W = 0$. Considering the expressions for these derivatives given in Eq. (4.74) these conditions are indeed equivalent to

$$\begin{aligned} &\int_{\mathcal{Y}} \Omega \wedge (H_3 + \mathrm{i d}J) = 0 \ , \\ &\int_{\mathcal{Y}} \Omega \wedge \mathrm{d}\omega_i = \int_{\mathcal{Y}} \mathcal{W}_1 J^2 \wedge \omega_i + \int_{\mathcal{Y}} \mathcal{W}_2 \wedge J \wedge \omega_i = 0 \ . \end{aligned} \tag{4.107}$$

From the second expression it follows that on a supersymmetric background the torsion classes $\mathcal{W}_1$ and $\mathcal{W}_2$ vanish. As already mentioned when the torsion classes were introduced, this is equivalent to having a complex manifold $\mathcal{Y}$.

On the other hand, the first condition in Eq. (4.107) says that $H_3 + \mathrm{id}J$ on a supersymmetric background can only be a sum of $(3,0)$, $(2,1)$ and $(1,2)$ pieces. The $(3,0)+(0,3)$ part of $\mathrm{d}J$ is proportional to $\mathcal{W}_1$, as can be checked from Eq. (4.5), and therefore vanishes. This result together with reality of H_3 requires that the combination $H_3 + \mathrm{id}J$ must be actually of type $(2,1)+(1,2)$.

Now set to zero the transformation of the ζ^a chiral fermions in Eq. (4.101). In view of Eq. (4.74) it means that

$$\int_{\mathcal{Y}} \rho_m \wedge (H_3 + \mathrm{id}J) = 0 \ . \tag{4.108}$$

This in turn implies the vanishing of the $(1,2)$ part of $H_3 + \mathrm{id}J$. Since the two-form J is a $(1,1)$-form and the NS-flux is $H_3 = H_{(2,1)} + H_{(1,2)}$, one can write the last result for a complex manifold as $H_{(1,2)} = -\mathrm{i}\bar{\partial} J$. Considering also the conjugate, this leads to

$$H_3 = \mathrm{i}(\partial - \bar{\partial})J \tag{4.109}$$

on a supersymmetric vacuum. The condition on the three-form flux and with it all the supersymmetry conditions for the heterotic string obtained by Strominger in [20] are verified.

Chapter 5

Conclusions

ΟΠΕΡ ΕΔΕΙ ΔΕΙΞΑΙ

In this thesis, the low-energy four-dimensional theories arising from the compactification of the heterotic string on some classes of reduced structure backgrounds have been obtained. In particular, the bosonic terms of heterotic supergravity have been dimensionally reduced *a là* Kaluza-Klein assuming that the internal manifold has SU(2) structure group. Manifolds with SU(2) structure in six dimensions are characterized by the existence of two global nowhere-vanishing spinors that are covariantly constant with respect to a connection with torsion. If the torsion vanishes, the spinors are constant with respect to the Levi-Civita connection and the manifold has SU(2) holonomy. Such manifolds are therefore generalizations of $\mathrm{K3} \times T^2$. The existence of the two spinors guarantees that the dimensional reduction preserves part of the supersymmetry in ten dimensions. Concretely, effective actions with $\mathcal{N} = 2$ local supersymmetry are obtained.

The SU(2) structure can be characterized equivalently by a pair of real one-forms v^i and a triplet of self-dual two-forms J^x. If and only if the torsion vanishes, these forms are closed, corresponding to the harmonic one-forms of the torus and the hyperkähler structure on K3. In other words, $\mathrm{d}v^i$ and $\mathrm{d}J^x$ are a measure of the torsion or how much the manifold deviates from $\mathrm{K3} \times T^2$. The one-forms v^i also allow to define an almost product structure, smoothly splitting the tangent space of the manifold over each point into a two-dimensional and a four-dimensional space. For a generic SU(2)-structure manifold, the expansion is done in terms of a finite set of forms corresponding to light modes. This set is obtained by projecting out all doublets of the structure group SU(2). As a result, one is left with the pair of one-forms v^i and a set of two-forms ω^A, three linear combinations of which are the self-dual J^x and the rest are anti-self-dual. The almost product structure is rigid, and the only allowed deformations correspond to the local two-dimensional and four-dimensional subspaces. If the four-dimensional local subspaces extend to form embedded four-manifolds, the latter must be copies of K3 and the forms ω^A must reduce to the harmonic two-forms on each K3 slice.

An ansatz for the SU(2)-structure backgrounds can be written by expanding $\mathrm{d}v^i$ and $\mathrm{d}\omega^A$ in terms of all possible exterior products of v^i and ω^A. Nilpotency of exterior differentiation and Stokes' theorem impose constraints on the possible values of the parameters in this expansion characterizing the torsion. Two complementary cases can be distinguished. The first can be realized by considering a K3 fibration over a torus. The second introduces some torsion in the torus part as well and can be realized as $\mathrm{K3} \times S^1$ fibered over a circle. The latter construction is ill-defined as six-manifold, but one can make sense of this fibration if one exploits the global symmetry of heterotic supergravity compactified on K3. The lifting to the full string theory is not clear, but the obtained low-energy supergravity is consistent.

In both cases, the low-energy supergravity has the same field content and scalar manifolds as the theory arising from the compactification on $\mathrm{K3} \times T^2$. The difference is that some isometries of the scalar manifolds are gauged. For the case of the K3 fibration over a torus, only isometries of the quaternionic manifold spanned by scalars in hypermultiplets are gauged. In the second case, the gauging affects also isometries of the special Kähler manifold spanned by scalars in vector multiplets. A potential is generated in each case for the corresponding scalars. As usual, the torsion parameters appear as charges and masses in the effective action. The general case is just a sum of the results for the two complementary cases. The gauge algebra and all Killing prepotentials have been determined and the conformity of the obtained actions to the general form of $\mathcal{N} = 2$ gauged supergravity has been established.

Additionally, the reduction of fermionic terms in the ten-dimensional heterotic action on SU(3)-structure manifolds has been revisited. SU(3)-structure manifolds in six dimensions are characterized by the existence of one global nowhere-vanishing spinor and generalize Calabi-Yau threefolds. The low-energy effective theory is an $\mathcal{N} = 1$ gauged supergravity. The relevant couplings, namely the Kähler potential, the gauge kinetic function and the superpotential, were obtained by computing the kinetic terms for the fermions, the gravitino mass term and the F- and D-terms. The results have been further checked by computing the supersymmetry transformations of the fermions.

Appendix A

$\mathcal{N}=2$ and $\mathcal{N}=1$ supergravity theories in four dimensions

In this Appendix, the general structure of theories with $\mathcal{N}=2$ and $\mathcal{N}=1$ local supersymmetry in four dimensions is recalled. This is done in order to facilitate the verification that the results in the main text indeed have these structures.

$\mathcal{N}=2$ SUGRA

A theory with $\mathcal{N}=2$ local supersymmetry describes the dynamics of a gravitational multiplet coupled to some numbers $n_{\rm v}$ and $n_{\rm h}$ of vector- and hypermultiplets, respectively. In the following, only the bosonic sector is considered. The gravitational multiplet consists of the metric $g_{\mu\nu}$ and a graviphoton $\mathcal{A}^0_\mu$. Each vector multiplet contains a vector $\mathcal{A}^p_\mu$ and a complex scalar v^p, with $p=1,\ldots,n_{\rm v}$. Finally, each hypermultiplet contains four real scalars, summing up to $4n_{\rm h}$ scalar fields q^u. All vectors can be labeled collectively as $\mathcal{A}^I_\mu$ with $I=0,1,\ldots,n_{\rm v}$.

The most general bosonic Lagrangian describing the dynamics of these fields can be written as [37]

$$\begin{aligned}\mathscr{L}_{\rm b} = \tfrac{1}{2}\mathcal{R}_4 + \tfrac{1}{4}I_{IJ}(v)\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J,\mu\nu} + \tfrac{1}{8}R_{IJ}(v)\epsilon^{\mu\nu\rho\lambda}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\lambda} \\ - G_{p\bar{q}}D_\mu v^p D^\mu \bar{v}^q - h_{uv}D_\mu q^u D^\mu q^v - \mathscr{V}_{\mathcal{N}=2}\ .\end{aligned} \tag{A.1}$$

In this expression, $\mathcal{R}_4$ is the Ricci scalar and the two-forms $\mathcal{F}^I_{\mu\nu}$ are the field strengths for the vector $\mathcal{A}^I_\mu$. Generically, these correspond to a non-Abelian gauge algebra with structure constants f^I_{JK} and one has

$$\mathcal{F}^I_{\mu\nu} = \partial_\mu\mathcal{A}^I_\nu - \partial_\nu\mathcal{A}^I_\mu + f^I_{JK}\mathcal{A}^J_\mu\mathcal{A}^K_\nu\ . \tag{A.2}$$

The generators of the gauge algebra T_I satisfy $[T_I,T_J]=f^K_{IJ}T_K$. The covariant derivatives for the scalars have the form

$$D_\mu v^p = \partial_\mu + k^p_I\mathcal{A}^I_\mu\ , \qquad D_\mu q^u = \partial_\mu + k^u_I\mathcal{A}^I_\mu\ , \tag{A.3}$$

where the Killing vectors k_I^p and k_I^u must lead to the following representations of the gauge algebra

$$T_I^{\mathrm{v}} = k_I^p \frac{\partial}{\partial v^p} \ , \qquad T_I^{\mathrm{h}} = k_I^u \frac{\partial}{\partial q^u} \ . \tag{A.4}$$

Local $\mathscr{N}=2$ supersymmetry imposes constraints on the scalar metrics $G_{p\bar{q}}$ and h_{uv}. In particular, the space $\mathscr{M}_{\mathrm{v}}$ spanned by the complex scalars v^p is a special Kähler manifold. This means that the metric $G_{p\bar{q}}$ on this space can be written as

$$G_{p\bar{q}} = \frac{\partial^2 K}{\partial v^p \partial \bar{v}^q} \tag{A.5}$$

for a real Kähler potential $K(v,\bar{v})$. Moreover, this Kähler potential can be written in terms of a holomorphic prepotential $\mathscr{F}(X)$ as

$$K = -\ln\left(\mathrm{i}\bar{X}^I \mathscr{F}_I - \mathrm{i}X^I \bar{\mathscr{F}}_I\right) \ , \tag{A.6}$$

where the $X^I(v)$ are $n_{\mathrm{v}}+1$ holomorphic functions of the complex scalars v^p and $\mathscr{F}_I = \partial_I \mathscr{F}$ is the derivative of $\mathscr{F}(X)$ with respect to X^I. The prepotential $\mathscr{F}(X)$ is a homogeneous function of degree two. It can happen that the quantities $\mathscr{F}_I$ in Eq. (A.6) are not the derivative of a prepotential $\mathscr{F}(X)$. But by a symplectic rotation of the vector $(X^I, \mathscr{F}_I)$ that leaves invariant the Kähler potential (A.6) one can go to a new basis $(X'^I, \mathscr{F}'_I)$ where a prepotential $\mathscr{F}'(X')$ does exist such that $\mathscr{F}'_I = \partial_I \mathscr{F}'$.

The gauge kinetic couplings $I_{IJ}(v)$ and $R_{IJ}(v)$ can also be expressed in terms of the function $\mathscr{F}(X)$ and its derivatives. It turns out that

$$I_{IJ} = \mathrm{Im}\mathcal{N}_{IJ} \ , \qquad R_{IJ} = \mathrm{Re}\mathcal{N}_{IJ} \ , \tag{A.7}$$

where the matrix $\mathcal{N}_{IJ}$ is given by

$$\mathcal{N}_{IJ} = \bar{\mathscr{F}}_{IJ} + 2\mathrm{i}\frac{\mathrm{Im}\mathscr{F}_{IK}\mathrm{Im}\mathscr{F}_{JL}X^K X^L}{\mathrm{Im}\mathscr{F}_{KL}X^K X^L} \tag{A.8}$$

and the quantities $\mathscr{F}_{IJ} = \partial_I \partial_J \mathscr{F}$ are the second derivatives of the prepotential.

On the other hand, the scalars in hypermultiplets q^u span a quaternionic manifold $\mathscr{M}_{\mathrm{h}}$. This implies the existence of three almost complex structures $(\mathcal{I}^x)_u{}^v$, $x=1,2,3$, satisfying the quaternionic algebra

$$\mathcal{I}^x \mathcal{I}^y = -\delta^{xy}\mathbb{1} + \mathrm{i}\epsilon^{xyz}\mathcal{I}^z \ . \tag{A.9}$$

The metric h_{uv} can then be used to lower one index on these structures and obtain a triplet of two-forms

$$K^x_{uv} = (\mathcal{I}^x)_u{}^w h_{wv} \ . \tag{A.10}$$

The holonomy group of a quaternionic manifold is $\mathrm{Sp}(2)\times\mathrm{Sp}(n_{\mathrm{h}})$, and K^x_{uv} can be identified as the field strength of the $\mathrm{Sp}(2)\simeq\mathrm{SU}(2)$ connection ω^x_u. This means that

$$K^x = \mathrm{d}\omega^x + \tfrac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z \ . \tag{A.11}$$

The transformations $\delta v^p = \Lambda_I k_I^p$ and $\delta q^u = \Lambda_I k_I^u$ must be isometries of the respective manifolds $\mathscr{M}_{\rm v}$ and $\mathscr{M}_{\rm h}$, therefore the name of Killing vectors. The Killing equations can be solved in terms of four Killing prepotentials, $\mathcal{P}_I$ and $\mathcal{P}_I^x$. For the special Kähler manifold $\mathscr{M}_{\rm v}$, the holomorphic Killing vectors k_I^p must satisfy

$$k_I^p = {\rm i} G^{p\bar{q}} \frac{\partial \mathcal{P}_I}{\partial \bar{v}^q} \ . \tag{A.12}$$

for a real function $\mathcal{P}_I(v, \bar{v})$. On the other hand, the Killing vectors k_I^u for the quaternionic manifold $\mathscr{M}_{\rm h}$ must conform to

$$k_I^u K_{uv}^x = -\frac{\partial \mathcal{P}_I^x}{\partial q^v} - \epsilon^{xyz} \omega_v^y \mathcal{P}_I^z \ . \tag{A.13}$$

Finally, the potential $\mathscr{V}_{\mathscr{N}=2}$ is constrained to have the following form in terms of the Killing vectors and prepotentials,

$$\mathscr{V}_{\mathscr{N}=2} = {\rm e}^K \bar{X}^I X^J \big(G_{p\bar{q}} k_I^p \bar{k}_J^q + 4 h_{uv} k_I^u k_J^v \big) + \big[\tfrac{1}{2} (I^{-1})^{IJ} + 4 {\rm e}^K X^I \bar{X}^J \big] \mathcal{P}_I^x \mathcal{P}_J^x \ . \tag{A.14}$$

The ungauged theory is obtained by setting f_{JK}^I, k_I^p and k_I^u to zero. In this case the vectors $\mathcal{A}^I$ are Abelian, all the scalars are neutral and the potential $\mathscr{V}_{\mathscr{N}=2}$ vanishes.

$\mathscr{N} = 1$ SUGRA

An $\mathscr{N} = 1$ supergravity in four dimensions describes the dynamics of a gravitational multiplet coupled to some number of vector and chiral multiplets. The gravitational multiplet is constituted by the metric $g_{\mu\nu}$ and a spin-$\frac{3}{2}$ field or gravitino ψ_μ. The latter is a left-handed Weyl spinor. Let us denote the components of the vector multiplets by $(A_\mu^{\rm a}, \chi^{\rm a})$, with vectors $A_\mu^{\rm a}$ and gauginos $\chi^{\rm a}$. The components of the chiral multiplets can be collectively denoted by (Φ^I, Ξ^I) with Φ^I being complex scalars and Ξ^I being the corresponding spin-$\frac{1}{2}$ superpartners.

The Lagrangian for such a theory can be decomposed as follows [1]

$$\mathscr{L} = \mathscr{L}_{\rm b} + \mathscr{L}_{\rm f} + \mathscr{L}_{\rm int} + \cdots \ , \tag{A.15}$$

where terms which are irrelevant for our analysis are being neglected. The piece $\mathscr{L}_{\rm b}$ includes only bosonic fields and is given by

$$\mathscr{L}_{\rm b} = \tfrac{1}{2} \mathcal{R}_4 - \tfrac{1}{4} ({\rm Re} f) F_{\mu\nu}^{\rm a} F^{{\rm a}\mu\nu} - \tfrac{1}{8} ({\rm Im} f) \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu}^{\rm a} F_{\rho\lambda}^{\rm a} - g_{I\bar{J}} \partial_\mu \Phi^I \partial^\mu \bar{\Phi}^{\bar{J}} - \mathscr{V}_{\mathscr{N}=1} \ , \tag{A.16}$$

where $\mathcal{R}_4$ is the Ricci scalar, $f = f(\Phi)$ is the holomorphic gauge kinetic function, $F_{\mu\nu}$ is the field strength for the vectors A_μ, and $g_{I\bar{J}}$ is the Kähler metric

$$g_{I\bar{J}}(\Phi, \bar{\Phi}) = \frac{\partial}{\partial \Phi^I} \frac{\partial}{\partial \bar{\Phi}^{\bar{J}}} K(\Phi, \bar{\Phi}) \tag{A.17}$$

with the Kähler potential $K(\Phi,\bar\Phi)$. The scalar potential $\mathscr{V}_{\mathscr{N}=1}(\Phi,\bar\Phi)$ is given as a function of the superpotential $W=W(\Phi)$,

$$\mathscr{V}_{\mathscr{N}=1}(\Phi,\bar\Phi)=\mathrm{e}^K(D_IWg^{I\bar J}\overline{D}_{\bar J}\overline{W}-3|W|^2)+\tfrac{1}{2}(\mathrm{Re}f)^{-1}\mathrm{Tr}\mathcal{D}^{\mathrm{a}}\mathcal{D}^{\mathrm{a}}\ , \tag{A.18}$$

where

$$D_IW=\frac{\partial W}{\partial\Phi^I}+\frac{\partial K}{\partial\Phi^I}W \tag{A.19}$$

is the Kähler derivative of the superpotential and $\mathcal{D}$ is the D-term. The second term in (A.15) comprises the kinetic terms for the fermions,

$$\mathscr{L}_{\mathrm{f}}=\epsilon^{\mu\nu\rho\lambda}\bar\psi_\mu\bar\sigma_\nu D_\rho\psi_\lambda-\mathrm{i}g_{\bar IJ}\bar\Xi^{\bar I}\bar\sigma^\mu D_\mu\Xi^J-\mathrm{i}(\mathrm{Re}f)\bar\chi^{\mathrm{a}}\bar\sigma^\mu D_\mu\chi^{\mathrm{a}}. \tag{A.20}$$

Finally, one also needs the gravitino mass term, the gravitino-fermion couplings and the Yukawa couplings. They are given by

$$\begin{aligned}\mathscr{L}_{\mathrm{int}}=-\bar\psi_\mu\bar\sigma^{\mu\nu}\psi_\nu\,\mathrm{e}^{\frac{K}{2}}W-\tfrac{\mathrm{i}}{\sqrt2}\Xi^I\sigma^\mu\bar\psi_\mu\mathrm{e}^{\frac{K}{2}}D_IW&\\-\tfrac12(\mathrm{Re}f)\mathcal{D}^{\mathrm{a}}\psi_\mu\sigma^\mu\bar\chi^{\mathrm{a}}-\tfrac12\Xi^I\Xi^JD_ID_JW+\mathrm{h.c.}\ ,&\end{aligned} \tag{A.21}$$

where $\bar\sigma^{\mu\nu}=\frac14\bar\sigma^{[\mu}\sigma^{\nu]}$. The first two terms in this expression are precisely the gravitino mass term $\mathscr{L}_{m_{3/2}}$ and the F-terms $\mathscr{L}_{F\text{-term}}$, respectively. The third one gives the D-term $\mathscr{L}_{D\text{-term}}$.

The supersymmetry transformations of the gravitino and the fermions in the chiral multiplets, excluding terms depending on the fermionic fields in the r.h.s., are given by

$$\begin{aligned}\delta\psi_\mu&=D_\mu\epsilon+\tfrac{\mathrm{i}}{2}\sigma_\mu\bar\epsilon\mathrm{e}^{\frac{K}{2}}W\ ,\\ \delta\bar\Xi^{\bar I}&=\tfrac{1}{\sqrt2}\bar\epsilon\mathrm{e}^{\frac{K}{2}}g^{\bar IJ}D_JW\ ,\\ \delta\chi^{\mathrm{a}}&=F^{\mathrm{a}}_{\mu\nu}\sigma^{\mu\nu}\epsilon-\mathrm{i}\epsilon\mathcal{D}^{\mathrm{a}}\ .\end{aligned} \tag{A.22}$$

Appendix B

Line element in the space of $\mathcal{Y}_4$ metrics

In this appendix, a derivation of an expression for the line element

$$\delta s^2 = \tfrac{1}{4}\int_{\mathcal{Y}_4} \delta g_{mn}\delta g^{mn} = -\tfrac{1}{4}\int_{\mathcal{Y}_4} g^{mn}g^{pq}\delta g_{mp}\delta g_{nq} \tag{B.1}$$

in the space of metrics g_{mn} of a four-dimensional $\mathcal{Y}_4$ in terms of the variations of moduli fields is given. These moduli are ρ and ξ^x_A as defined by

$$J^x = \mathrm{e}^{-\frac{1}{2}\rho}\xi^x_A\omega^A \ . \tag{B.2}$$

Equivalently, one can replace ξ^x_A by the matrix $M^A{}_B$ defined by $*_4\omega^A = M^A{}_B\omega^B$ and satisfying

$$M^{AB} = M^A{}_C\eta^{BC} = \int_{\mathcal{Y}_4} \omega^A \wedge *_4\omega^B = -\delta^A_B + \eta^{AC}\xi^x_C\xi^x_B \ . \tag{B.3}$$

Here ω^A are a set of two-forms on $\mathcal{Y}_4$ and J^x are the triplet of self-dual two-forms associated with the triplet of almost complex structures I^x. Recall that the latter satisfy

$$I^xI^y = -\delta^{xy}\mathbb{1} + \epsilon^{xyz}I^z \ , \tag{B.4}$$

which due to the relation $J^x_{mn} = (I^x)_m{}^p g_{pn}$ implies

$$J^x \wedge J^y = 2\delta^{xy}\mathrm{vol}_4 \ , \tag{B.5}$$

with vol_4 the volume form in $\mathcal{Y}_4$.

Although in the main text the final formula is applied to the case where $\mathcal{Y}_4$ is in fact a K3, the following derivation holds more generally. In particular, $\mathcal{Y}_4$ can be just a local embedding of a four-dimensional neighborhood $\mathcal{U}_4$ in a six-dimensional $\mathcal{Y}_6$, with the unique assumption that there is a hyperkähler structure (B.4) or equivalently (B.5) on $\mathcal{U}_4$ satisfying $\mathrm{d}_4J^x = 0$, where d_4 is the exterior derivative restricted to $\mathcal{U}_4$.

Due to Eq. (B.5) we can set

$$\eta^{AB}\xi^x_A\xi^y_B = 2\delta^{xy} \ , \tag{B.6}$$

with η^{AB} being the intersection matrix of the ω^A. If we have a global $\mathcal{Y}_4$ then

$$\eta^{AB} = \int_{\mathcal{Y}_4} \omega^A \wedge \omega^B \tag{B.7}$$

and $e^{-\rho}$ is the volume of $\mathcal{Y}_4$.

In the following, it will prove convenient to work in matrix notation and set

$$J^x = (J^x)_{mn} \ , \qquad I^x = (I^x)_m{}^n \ , \qquad \text{and} \qquad g = g_{mn} \ . \tag{B.8}$$

One can therefore write, for example, $I^x g = J^x$. Acting on the left of this equality with I^x and using Eq. (B.4) yields $g = -I^1 J^1 = -I^2 J^2 = -I^3 J^3$. It follows that the variation δg is given by

$$\delta g = -I^1 \delta J^1 - \delta I^1 J^1 = -I^2 \delta J^2 - \delta I^2 J^2 = -I^3 \delta J^3 - \delta I^3 J^3 \ . \tag{B.9}$$

The variation of Eq. (B.4) yields

$$\delta I^3 = \delta I^1 I^2 + I^1 \delta I^2 \ , \tag{B.10}$$

and similar expressions with the indices cyclically permuted. From this expression, and making repeated use of (B.4), it follows that

$$\begin{aligned} \delta I^3 J^3 &= I^1 (\delta I^1 J^1 - \delta I^2 J^2) g^{-1} I^1 \\ &= (\delta J^1 + I^3 \delta J^2) g^{-1} J^1 \ . \end{aligned} \tag{B.11}$$

In the last step, the second equality in (B.9) was used. Substituting Eq. (B.11) into the last equality of (B.9) one obtains

$$\delta g = -I^3 \delta J^3 - (\delta J^1 + I^3 \delta J^2) g^{-1} J^1 \ . \tag{B.12}$$

This expresses δg in terms of δJ^x. A similar expression can be given with I^x and J^x cyclically permuted. The physical variations of the J^x are all independent, with the exception of the volume modulus $\delta\rho$ that rescales all them at the same time

$$\delta J^x = -\tfrac{1}{2} J^x \delta\rho \ . \tag{B.13}$$

Using the cyclic symmetry of (B.12) one has, for example $\delta J^1 g^{-1} J^1 = I^1 \delta J^1$. Inserted back into (B.12) yields

$$\delta g = -I^1 \delta J^1 - I^2 \delta J^2 - I^3 \delta J^3 = -I^x \delta J^x \tag{B.14}$$

for all physical variations other than (B.13). Restoring the indices this result reads

$$\delta g_{mn} = -(I^x)_m{}^p (\delta J^x)_{pn} \ . \tag{B.15}$$

Eq. (B.14) can now be applied in the computation of δs^2 in Eq. (B.1),

$$\begin{aligned}\delta s^2 &= -\tfrac{1}{4}\int_{\mathcal{Y}_4} g^{mn}g^{pq}\delta g_{mp}\delta g_{nq} = -\tfrac{1}{4}\int_{\mathcal{Y}_4}\mathrm{tr}(g^{-1}\delta g g^{-1}\delta g)\\ &= -\tfrac{1}{4}\int_{\mathcal{Y}_4}\mathrm{tr}(g^{-1}I^x\delta J^x g^{-1}I^y\delta J^y) = -\tfrac{1}{4}\int_{\mathcal{Y}_4}\mathrm{tr}(g^{-1}\delta J^x g^{-1}\delta J^x) \\ &= -\tfrac{1}{2}\int_{\mathcal{Y}_4}\delta J^x\wedge *_4\delta J^x\ .\end{aligned} \tag{B.16}$$

In deriving the last equalities, use was again made of $I^x\delta J^y g^{-1} = \delta J^y g^{-1} I^x$.

The next step is to express the (independent) physical variations δJ^x in terms of variations of the moduli $\delta\xi^x_A$. In particular, one needs to take into account the fact that variations which simply rotate the J^x into themselves do not take us to a different point of the moduli space. For such 'unphysical' variations, one must certainly have $\delta g_{mn} = 0$. It is therefore required that the 'physical' variations $\delta_{\mathrm{phys}}\xi^x_A$ be orthogonal to the ξ^x_A. In other words, they have to satisfy

$$\eta^{AB}\xi^x_A\delta_{\mathrm{phys}}\xi^y_B = 0\ . \tag{B.17}$$

Notice that such variations automatically respect the constraint (B.6) and thus Eq. (B.17) represents the only nine constraints that must be imposed on the variations of the $3n$ parameters ξ^x_A. This leaves $3(n-3)$ degrees of freedom. If $\mathcal{Y}_4$ is indeed K3 then the number of forms is $n = 22$ and there are 57 physical degrees of freedom in ξ^x_A.

The operator which projects onto the subspace orthogonal to the ξ^x_A is given by $P_A{}^B = \delta^A_B - \frac{1}{2}\xi^y_A\xi^{yB}$ and one has $\delta_{\mathrm{phys}}\xi^x_A = P_A{}^B\delta\xi^x_B$. Thus the physically inequivalent variations of J^x (apart from the variation of the volume) can be written as

$$\delta J^x = \mathrm{e}^{-\frac{\rho}{2}}\delta_{\mathrm{phys}}\xi^x_B\omega^B = \mathrm{e}^{-\frac{\rho}{2}}(P_B{}^A\delta\xi^x_A)\omega^B = \mathrm{e}^{-\frac{\rho}{2}}(\delta^A_B - \tfrac{1}{2}\xi^{yA}\xi^y_B)\delta\xi^x_A\omega^B\ , \tag{B.18}$$

with $\xi^{xA} = \eta^{AB}\xi^x_B$ and $\delta\xi^x_A$ unrestricted since the unphysical part is being projected out.

Now Eq. (B.18) can be substituted into the last line in (B.16) and use can be made of Eq. (B.3). If the contribution (B.13) due to a volume rescaling is also added, one obtains

$$\delta s^2 = -\tfrac{1}{4}\mathrm{e}^{-\rho}(\delta\rho)^2 + \tfrac{1}{2}\mathrm{e}^{-\rho}(\eta^{AB} - \tfrac{1}{2}\xi^{yA}\xi^{yB})\delta\xi^x_A\delta\xi^x_B\ . \tag{B.19}$$

Making use of Eq. (B.3), the last result can be written in terms of $\delta M^A{}_B$. One can take the variation of Eq. (B.3), but recall that one is interested in physical variations of the parameters ξ^x_A and therefore $\delta M^A{}_B = \delta_{\mathrm{phys}}\xi^{xA}\xi^x_B + \xi^{xA}\delta_{\mathrm{phys}}\xi^x_B$. Recalling Eq. (B.17), it follows that

$$\begin{aligned}\delta M^A{}_B\delta M^B{}_A &= 4\delta_{\mathrm{phys}}\xi^{xA}\delta_{\mathrm{phys}}\xi^x_A\\ &= 4(\eta^{AB} - \tfrac{1}{2}\xi^{yA}\xi^{yB})\delta\xi^x_A\delta\xi^x_B\ .\end{aligned} \tag{B.20}$$

Comparing this with Eq. (B.19), it is finally concluded that

$$\delta s^2 = -\tfrac{1}{4}\mathrm{e}^{-\rho}(\delta\rho)^2 + \tfrac{1}{8}\mathrm{e}^{-\rho}\delta M^A{}_B\delta M^B{}_A\ . \tag{B.21}$$

Appendix C

Almost product structures

Let M be a manifold of dimension $n = p + q$. An almost product structure is a globally-defined tensor $P^a{}_b$, $a, b = 1, \ldots, n$ satisfying $P^a{}_b P^b{}_c = \delta^a_c$. This tensor allows to split the tangent space T_xM to each point x of the manifold as $T_xM = V_x \oplus W_x$, where V_x and W_x are p- and q-dimensional subspaces, respectively. The subspaces V_x and W_x vary smoothly with x, and actually define p- and q-dimensional distributions*

$$V = \cup_{x\in M} V_x \ , \qquad W = \cup_{x\in M} W_x \tag{C.1}$$

on M. This means that on each patch $U \subset M$ one can define p vector fields v_i, $i = 1, \ldots, p$ and q vector fields w_m, $m = 1, \ldots, q$ in such a way that V is generated by the v_i and W is generated by the w_m. The fact that P, and therefore the distributions V and W, are defined globally means that the vectors v_i and w_m defined in a patch U and the vectors $\tilde{v}_i$ and $\tilde{w}_m$ defined in a patch $\tilde{U}$ are related in the intersection $U \cap \tilde{U}$ by

$$\tilde{v}_i = A_i{}^j v_j \ , \qquad \tilde{w}_m = B_m{}^n w_n \ . \tag{C.2}$$

For generic coordinates x^a, $a = 1, \ldots, n$ on M, the tangent space is generated by the set of vectors $\partial_a \equiv \partial/\partial x^a$. In going from a patch U with coordinates x^a to a patch $\tilde{U}$ with coordinates $\tilde{x}^a$, this basis transforms according to

$$\tilde{\partial}_a = \mathcal{A}_a{}^b \partial_b \ , \qquad \text{with} \qquad \mathcal{A}_a{}^b = \frac{\partial x^b}{\partial \tilde{x}^a} \ . \tag{C.3}$$

If instead of the coordinate basis ∂_a one takes the basis $\{v_i, w_m\}$, the transition matrix is

$$\mathcal{A}_a{}^b = \begin{pmatrix} A_i{}^j & 0 \\ 0 & B_m{}^n \end{pmatrix} \ , \tag{C.4}$$

as seen from Eq. (C.2).

*A p-dimensional distribution is just a subbundle of the tangent bundle [52]. It assigns to each point of the manifold a p-dimensional subspace of the tangent space over that point. This is done smoothly over the manifold. In the intersection of two patches the p-dimensional fibers must of course coincide.

In the same way as with the tangent space, the cotangent space T_x^*M can be split over each point as $V_x^* \oplus W_x^*$, and one can define the dual forms $v^i \in V^*$ and $w^m \in W^*$ satisfying

$$v^i(v_j) = \delta^i_j \ , \quad w^m(w_n) = \delta^m_n \quad \text{and} \quad v^i(w_m) = w^m(v_i) = 0 \ . \tag{C.5}$$

The tensor $P_a{}^b$ is called 'metric-compatible' if $P_{ab} = P_a{}^c g_{cb}$ is symmetric. This means that the metric (understood as a symmetric element of $T^*M \otimes T^*M$) must have the block-diagonal form

$$\mathrm{d}s^2 = g_{ij} v^i v^j + g_{mn} w^m w^n \ . \tag{C.6}$$

Of course, this does not mean that g_{ab} as defined by $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$ in a coordinate basis is block-diagonal.

The almost product structure P is integrable if on every neighborhood or patch U one can find coordinates $x^a = \{y^i, z^m\}$ such that one can choose the vector fields v_i and w_m generating the distributions V and W as

$$v_i = \frac{\partial}{\partial y_i} \ , \qquad w_m = \frac{\partial}{\partial z^m} \ . \tag{C.7}$$

This means of course that the transition functions $\partial x^b / \partial \tilde{x}^a$ have indeed the form (C.4), with $A_i{}^j$ depending only on y^i and $B_m{}^n$ depending only on z^m. It is equivalent to integrability of the system of n partial differential equations

$$\begin{aligned} \frac{\partial f}{\partial y^i} &= v_i(f) \ , \\ \frac{\partial f}{\partial z^m} &= w_m(f) \ , \end{aligned} \tag{C.8}$$

with $f = f(y,z)$. If the almost product structure is integrable then one has a block-diagonal metric in a coordinate basis, that is

$$\mathrm{d}s^2 = g_{ij}(y,z)\mathrm{d}y^i\mathrm{d}y^j + g_{mn}(y,z)\mathrm{d}z^m\mathrm{d}z^n \ , \tag{C.9}$$

where, as explicitly shown, the blocks depend generically on all the coordinates.

Lets assume that one has an almost product structure on M that is integrable. If (and only if) one can define a projection $\pi : M \to N$ for some manifold N such that $\pi_*(V_x) = T_{\pi(x)}N$ for all $x \in M$ then M is a fibered space with base N. If (and only if) there is additionally another projection $\pi' : M \to N'$ such that $\pi'_*(W_x) = T_{\pi'(x)}N'$ for all $x \in M$ then M is topologically the product manifold $M = N \times N'$. Fibered spaces are therefore examples of manifolds where one has an integrable almost product structure but one does not have a global product structure.

Bibliography

[1] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton, USA: Univ. Pr., 1992.

[2] B. S. DeWitt and G. Esposito, "An introduction to quantum gravity," *Int. J. Geom. Meth. Mod. Phys.* **5** (2008) 101–156, `arXiv:0711.2445 [hep-th]`.

[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Vol. 1 & 2.* Cambridge, Uk: Univ. Pr., 1987.

[4] J. Polchinski, *String theory.* Cambridge, Uk: Univ. Pr., 1998.

[5] K. Becker, M. Becker, and J. H. Schwarz, *String theory and M-theory: A modern introduction.* Cambridge, UK: Cambridge Univ. Pr., 2007.

[6] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, `arXiv:hep-th/9503124`.

[7] T. Kaluza, "On the Problem of Unity in Physics," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1921** (1921) 966–972.

[8] O. Klein, "Quantum theory and five-dimensional theory of relativity," *Z. Phys.* **37** (1926) 895–906.

[9] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[10] B. R. Greene, "String theory on Calabi-Yau manifolds," `arXiv:hep-th/9702155`.

[11] P. Candelas and X. de la Ossa, "Moduli Space of Calabi-Yau Manifolds," *Nucl. Phys.* **B355** (1991) 455–481.

[12] E. Witten, "Dimensional Reduction of Superstring Models," *Phys. Lett.* **B155** (1985) 151.

[13] P. S. Aspinwall, "K3 surfaces and string duality," `arXiv:hep-th/9611137`.

[14] J. Louis and K. Foerger, "Holomorphic couplings in string theory," *Nucl. Phys. Proc. Suppl.* **55B** (1997) 33–64, `arXiv:hep-th/9611184`.

[15] D. Lust, "Dual string vacua with N = 2 supersymmetry in four dimensions," *Nucl. Phys. Proc. Suppl.* **62** (1998) 375–403.

[16] M. Grana, "Flux compactifications in string theory: A comprehensive review," *Phys. Rept.* **423** (2006) 91–158, `arXiv:hep-th/0509003`.

[17] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," *Phys. Rept.* **445** (2007) 1–193, `arXiv:hep-th/0610327`.

[18] F. Denef, M. R. Douglas, and S. Kachru, "Physics of string flux compactifications," *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 119–144, `arXiv:hep-th/0701050`.

[19] B. Wecht, "Lectures on Nongeometric Flux Compactifications," *Class. Quant. Grav.* **24** (2007) S773–S794, `arXiv:0708.3984 [hep-th]`.

[20] A. Strominger, "Superstrings with Torsion," *Nucl. Phys.* **B274** (1986) 253.

[21] M. Rocek, "Modified Calabi-Yau manifolds with torsion,". Submitted to Proc. of Mirror Symmetry Workshop, MSRI, Berkeley, CA, May 1991.

[22] S. Chiossi and S. Salamon, "The intrinsic torsion of SU(3) and G_2 structures," `arXiv:math/0202282`.

[23] N. Hitchin, "Generalized Calabi-Yau manifolds," *Quart. J. Math. Oxford Ser.* **54** (2003) 281–308, `arXiv:math/0209099`.

[24] G. Lopes Cardoso *et al.*, "Non-Kaehler string backgrounds and their five torsion classes," *Nucl. Phys.* **B652** (2003) 5–34, `arXiv:hep-th/0211118`.

[25] K. Becker, M. Becker, K. Dasgupta, and P. S. Green, "Compactifications of heterotic theory on non-Kaehler complex manifolds. I," *JHEP* **04** (2003) 007, `arXiv:hep-th/0301161`.

[26] K. Becker, M. Becker, P. S. Green, K. Dasgupta, and E. Sharpe, "Compactifications of heterotic strings on non-Kaehler complex manifolds. II," *Nucl. Phys.* **B678** (2004) 19–100, `arXiv:hep-th/0310058`.

[27] M. Becker, L.-S. Tseng, and S.-T. Yau, "Moduli space of torsional manifolds," *Nucl. Phys.* **B786** (2007) 119–134, `arXiv:hep-th/0612290`.

[28] M. Becker, L.-S. Tseng, and S.-T. Yau, "Heterotic Kahler/non-Kahler Transitions," `arXiv:0706.4290 [hep-th]`.

[29] M. Becker, L.-S. Tseng, and S.-T. Yau, "New Heterotic Non-Kahler Geometries," `arXiv:0807.0827 [hep-th]`.

[30] K. Becker and S. Sethi, "Torsional Heterotic Geometries," *Nucl. Phys.* **B820** (2009) 1–31, `arXiv:0903.3769 [hep-th]`.

[31] S. Gurrieri, A. Lukas, and A. Micu, "Heterotic String Compactifications on Half-flat Manifolds II," *JHEP* **12** (2007) 081, `arXiv:0709.1932 [hep-th]`.

[32] S. Gurrieri, J. Louis, A. Micu, and D. Waldram, "Mirror symmetry in generalized Calabi-Yau compactifications," *Nucl. Phys.* **B654** (2003) 61–113, `arXiv:hep-th/0211102`.

[33] J. P. Gauntlett, D. Martelli, and D. Waldram, "Superstrings with intrinsic torsion," *Phys. Rev.* **D69** (2004) 086002, `arXiv:hep-th/0302158`.

[34] S. Ferrara, J. A. Harvey, A. Strominger, and C. Vafa, "Second quantized mirror symmetry," *Phys. Lett.* **B361** (1995) 59–65, `arXiv:hep-th/9505162`.

[35] J. Louis and A. Micu, "Heterotic-type IIA duality with fluxes," *JHEP* **03** (2007) 026, `arXiv:hep-th/0608171`.

[36] H. Samtleben, "Lectures on Gauged Supergravity and Flux Compactifications," *Class. Quant. Grav.* **25** (2008) 214002, `arXiv:0808.4076 [hep-th]`.

[37] L. Andrianopoli *et al.*, "N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," *J. Geom. Phys.* **23** (1997) 111–189, `arXiv:hep-th/9605032`.

[38] B. de Carlos, S. Gurrieri, A. Lukas, and A. Micu, "Moduli stabilisation in heterotic string compactifications," *JHEP* **03** (2006) 005, `arXiv:hep-th/0507173`.

[39] B. Spanjaard, "Compactifications of IIA supergravity on SU(2)-structure manifolds,". DESY-THESIS-2008-016.

[40] R. A. Reid-Edwards and B. Spanjaard, "N=4 Gauged Supergravity from Duality-Twist Compactifications of String Theory," *JHEP* **12** (2008) 052, `arXiv:0810.4699 [hep-th]`.

[41] J. Louis, D. Martinez-Pedrera, and A. Micu, "Heterotic compactifications on SU(2)-structure backgrounds," *JHEP* **09** (2009) 012, `arXiv:0907.3799 [hep-th]`.

[42] A. Strominger, "Yukawa Couplings in Superstring Compactification," *Phys. Rev. Lett.* **55** (1985) 2547.

[43] L. J. Dixon, V. Kaplunovsky, and J. Louis, "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String," *Nucl. Phys.* **B329** (1990) 27–82.

[44] G. Lopes Cardoso, G. Curio, G. Dall'Agata, and D. Lust, "BPS action and superpotential for heterotic string compactifications with fluxes," *JHEP* **10** (2003) 004, `arXiv:hep-th/0306088`.

[45] I. Benmachiche, "Heterotic and type II orientifold compactifications on SU(3) structure manifolds,". DESY-THESIS-2006-016.

[46] I. Benmachiche, J. Louis, and D. Martinez-Pedrera, "The effective action of the heterotic string compactified on manifolds with SU(3) structure," *Class. Quant. Grav.* **25** (2008) 135006, `arXiv:0802.0410 [hep-th]`.

[47] J. Louis and A. Micu, "Heterotic string theory with background fluxes," *Nucl. Phys.* **B626** (2002) 26–52, `arXiv:hep-th/0110187`.

[48] E. A. Bergshoeff and M. de Roo, "The Quartic Effective Action of the Heterotic String and Supersymmetry," *Nucl. Phys.* **B328** (1989) 439–468.

[49] L. J. Romans and N. P. Warner, "Some Supersymmetric Counterparts of the Lorentz Chern-Simons Term," *Nucl. Phys.* **B273** (1986) 320.

[50] J. Maharana and J. H. Schwarz, "Noncompact symmetries in string theory," *Nucl. Phys.* **B390** (1993) 3–32, `arXiv:hep-th/9207016`.

[51] N. Kaloper and R. C. Myers, "The O(dd) story of massive supergravity," *JHEP* **05** (1999) 010, `arXiv:hep-th/9901045`.

[52] T. Frankel, "The geometry of physics: An introduction,". Cambridge, UK: Univ. Pr. (1997) 654 p.

[53] H. Triendl and J. Louis, "Type II compactifications on manifolds with SU(2) x SU(2) structure," `arXiv:0904.2993 [hep-th]`.

[54] M. Grana, J. Louis, and D. Waldram, "Hitchin functionals in N = 2 supergravity," *JHEP* **01** (2006) 008, `arXiv:hep-th/0505264`.

[55] M. Grana, J. Louis, and D. Waldram, "SU(3) x SU(3) compactification and mirror duals of magnetic fluxes," *JHEP* **04** (2007) 101, `arXiv:hep-th/0612237`.

[56] A. Micu, "Heterotic compactifications and nearly-Kaehler manifolds," *Phys. Rev.* **D70** (2004) 126002, `arXiv:hep-th/0409008`.

[57] S. Gurrieri, A. Lukas, and A. Micu, "Heterotic on half-flat," *Phys. Rev.* **D70** (2004) 126009, `arXiv:hep-th/0408121`.

[58] H. Jockers and J. Louis, "D-terms and F-terms from D7-brane fluxes," *Nucl. Phys.* **B718** (2005) 203–246, `arXiv:hep-th/0502059`.

[59] K. Behrndt and S. Gukov, "Domain walls and superpotentials from M theory on Calabi- Yau three-folds," *Nucl. Phys.* **B580** (2000) 225–242, arXiv:hep-th/0001082.

[60] M. Becker and D. Constantin, "A note on flux induced superpotentials in string theory," *JHEP* **08** (2003) 015, arXiv:hep-th/0210131.

[61] G. Lopes Cardoso, G. Curio, G. Dall'Agata, and D. Lust, "BPS action and superpotential for heterotic string compactifications with fluxes," *JHEP* **10** (2003) 004, arXiv:hep-th/0306088.

[62] A.-K. Kashani-Poor and R. Minasian, "Towards reduction of type II theories on SU(3) structure manifolds," *JHEP* **03** (2007) 109, arXiv:hep-th/0611106.

[63] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," *JHEP* **08** (2004) 046, arXiv:hep-th/0406137.