Yang-Feldman Formalism on Noncommutative Minkowski Space

Dissertation
zur Erlangung des Doktorgrades
des Department Physik
der Universität Hamburg

vorgelegt von
Claus Döscher
aus Bremerhaven

Hamburg
2006

Gutachter der Dissertation:	Prof. Dr. K. Fredenhagen Prof. Dr. G. Mack
Gutachter der Disputation:	Prof. Dr. K. Fredenhagen Prof. Dr. J. Louis
Datum der Disputation:	15. Dezember 2006
Vorsitzender des Prüfungsausschusses:	Dr. K. Petermann
Vorsitzender des Promotionsausschusses:	Prof. Dr. G. Huber
Leiter des Department Physik:	Prof. Dr. R. Klanner
Dekan der Fakultät MIN:	Prof. Dr. A. Frühwald

Zusammenfassung:

Wir betrachten Quantenfeldtheorie auf nichtkommutativer Raumzeit. Dazu wählen wir einen Ansatz, welcher explizit dem nichtkommutativen Minkowskiraum zugeordnet ist, nämlich den Yang-Feldman Formalismus. Hier besteht der Ansatz darin, versuchen die Feldgleichung der Quantenfelderzu lösen. In diesem Zusammenhang betrachten wir zuerst eine Wechselwirkung in Form eines zusätzlichen Masse-Terms. Dies benutzen wir, um die Frage des Infrarot-Cutoffs und des adiabatischen Limes zu erörtern. Es werden Klassen von Abschneidefunktionen gefunden, welche den erwarteten Limes liefert. Des weiteren betrachten wir verschiedene wechselwirkende Modelle, das ϕ^3 Modell in vier und sechs Dimensionen, das ϕ^4 Modell und das Wess-Zumino Modell. Zu diesen berechnen wir Dispersionsrelationen und sehen, dass es extreme Unterschiede in den Größenordnungen im Vergleich von logarithmisch und quadratisch divergenten Modellen gibt. Integrale, welche durch Twist-Faktoren endlich gemacht werden, werden rigoros im Sinne der Theorie der oszillierenden Integrale berechnet.

Abstract:

We examine quantum field theory on noncommutative spacetime. For this we choose an approach which lives explicitly on the noncommutative Minkowski space, namely the Yang-Feldman formalism. Here the ansatz is to try to solve the field equation of the quantum fields. In this setting we first take a look at an additional mass term, and use this to discuss possible IR cutoffs. We find classes of IR cutoffs which indeed yield the expected limit. Furthermore, we look at interacting models, namely the ϕ^3 model in four and six dimensions, the ϕ^4 model and the Wess-Zumino model. For these we calculate dispersion relations. We see that there exist huge differences in the orders of magnitude between logarithmically and quadratically divergent models. Integrals which are made finite by twisting factors are calculated rigorously in the sense of the theory of oscillatory integrals.

Contents

Chapter 1

Introduction

Quantum field theory as it is given in standard textbooks today is modeled on Minkowski space $\mathbb{M}$ or on a curved spacetime which locally looks like $\mathbb{M}$. There is no interplay between the particles described by the quantum fields and the underlying classical spacetime. This is unsatisfactory, since one knows from general relativity that the metric, which describes the geometry of the spacetime, depends on the distribution of matter and on the other hand the motion of the matter depends on the metric. This cannot be described by the standard approaches to quantum field theory. One expects that the picture of the classical spacetime as a differential manifold with classical metric breaks down at very small length scales. This was already mentioned in [37] in 1934. By the incorporation of gravity one expects that distances below the order of magnitude of the Planck length,

$$\lambda_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \cdot 10^{-35} m,$$

become meaningless. A hope is that the divergences in quantum field theory, which make renormalization necessary and come from high momenta, i.e., small distances, disappear in the yet unknown new concept of spacetime.

A situation where the usual picture of spacetime breaks down is the measuring of the coordinates of an event to the order of magnitude of the Planck length. This was investigated in [13] semiclassically and it was shown that such an extreme precision measurement causes a gravitational collapse. This leads to the derivation of uncertainty relations for the position coordinates q^μ. These can be written as

$$\begin{aligned} \Delta q^0 \left(\Delta q^1 + \Delta q^2 + \Delta q^3\right) &\geq \lambda_P^2, \\ \Delta q^1 \cdot \Delta q^2 + \Delta q^2 \cdot \Delta q^3 + \Delta q^3 \cdot \Delta q^1 &\geq \lambda_P^2. \end{aligned}$$

These uncertainty relations can be realized, if one regards the coordinates as elements of an observable algebra, in which the different components of the position vector do not commute any longer:

$$[q^\mu, q^\nu] = i\lambda_P^2 Q^{\mu\nu}.$$

Here, $Q^{\mu\nu}$ is an element of the observable algebra unequal to zero. It is chosen in [13] to be a central element and to fulfill

$$\begin{aligned} Q^{\mu\nu}Q_{\mu\nu} &= 0, \\ \left(\frac{1}{8}Q^{\mu\nu}Q^{\rho\tau}\epsilon_{\mu\nu\rho\tau}\right)^2 &= \lambda_P^8 \mathbb{1}. \end{aligned}$$

This setting will be called noncommutative Minkowski space and is explained in more detail in section 2.1. Closely related to this is to choose $Q^{\mu\nu}$ to be a constant matrix, most often denoted by $\sigma^{\mu\nu}$. In this thesis we will work mostly in one of these settings and try to formulate interacting quantum field theory on it. The noncommutative Minkowski space should not be seen as the final concept for spacetime, but rather as an intermediate step towards it. There is still no direct interplay between the fields and the spacetime on which they live. The only remainder from gravity is the appearing of the gravitational constant G in the noncommutativity scale λ_P. The hope is that understanding the noncommutative Minkowski space and the formulation of quantum fields on it, helps to find a truly fundamental concept for spacetime. Sometimes we replace λ_P by λ_{nc} if we want to emphasize that we also consider different length scales associated to the underlying noncommutativity and are not restricted to the Planck length.

Noncommutative spacetime also arises in a certain limit of string theory with a constant background B-field [36, 39]. This setting can be described by a constant $\sigma^{\mu\nu}$ which maps a vector in time direction to zero, i.e., time and space still commute. But this is not compatible with the uncertainty relations mentioned above. Furthermore, Lorentz invariance is explicitly broken. We will not consider this setting here.

Free quantum fields can be defined in a straightforward way on noncommutative spacetime, as shown in section 2.2. But there are several different approaches to interacting quantum fields. While they are equivalent on commutative spacetime, they cease to be, if time does not commute with space any more. Section 2.3 gives an overview of the different approaches. It is not unclear which one is the most advantageous to choose since each has some weakness. Especially, there seems to be no connection between quantum field theory on noncommutative Minkowski and noncommutative Euclidean space.

Therefore, we choose a setting which works explicitly on the Minkowski version, namely the Yang-Feldman formalism. This is the most promising approach from our point of view. A phenomenon in noncommutative Euclidean spacetime is the mixing of UV and IR divergences, which is examined in section 2.4. Yet it is not clear, how this shows up on the noncommutative Minkowski space.

In Chapter 3 we introduce the Yang-Feldman formalism and look at possibilities to introduce an IR cutoff. This cutoff is necessary in order to keep us from manipulating expressions without a well-defined sense. We consider a mass term as interaction. This can be seen as a kind of toy model, since we already have an expectation of what the result should be. First, we do this for commutative spacetime and then have a closer look at the new situation on the noncommutative one. In Chapter 4 we look at interactions, namely the ϕ^3 model both in four and six dimensions, the ϕ^4 and the Wess-Zumino model. Some of these models are logarithmically divergent and the others quadratically. We look at their dispersion relations and see that the orders of magnitude of the modifications are rather moderate for the logarithmically divergent models but for the quadratically divergent ones considerably numbers of magnitude higher. In fact, for logarithmically divergent models the distortion of the group velocity is of the order of percentages. If one assumes that the Higgs model sees a noncommutative structure of spacetime and belongs to this class of divergence (possibly a supersymmetric extension of the model) this might be detectable in forthcoming colliders. The last Chapter 5 brings a conclusion and an outlook.

In the literature calculations for quantum field theory on noncommutative spacetime are often presented very vague and without well-defined objects. Here, we try to treat everything as rigorous as possible. For example integrals which are made finite by twisting factors are calculated using the theory of oscillatory integrals. To our knowledge, this has not been done before. The concept of oscillatory integrals is presented in Appendix B. In the whole setting we keep λ_{nc} finite and do not treat the fields as a formal power series in λ_{nc}.

Chapter 2

Quantum field theory on noncommutative spacetime

2.1 Noncommutative Minkowski space

In this section we present mainly the setting of noncommutative Minkowski space from [13]. However, the presentation given here is slightly simplified. For this section we set $\lambda_P = \lambda_{\text{nc}} = 1$.

$$\begin{aligned} [q^\mu, q^\nu] &=: iQ^{\mu\nu}, \\ [q^\mu, Q^{\nu\rho}] &= 0. \end{aligned}$$

Furthermore, we require $Q^{\mu\nu}$ to fulfill

$$\begin{aligned} Q^{\mu\nu}Q_{\mu\nu} &= 0, \\ \left(\frac{1}{8}Q^{\mu\nu}Q^{\rho\tau}\epsilon_{\mu\nu\rho\tau}\right)^2 &= \lambda_P^8 \mathbb{1}. \end{aligned}$$

These relations together with the commutation relations imply the uncertainty relations

$$\begin{aligned} \Delta q^0 \left(\Delta q^1 + \Delta q^2 + \Delta q^3\right) &\geq \lambda_P^2, \\ \Delta q^1 \cdot \Delta q^2 + \Delta q^2 \cdot \Delta q^3 + \Delta q^3 \cdot \Delta q^1 &\geq \lambda_P^2, \end{aligned} \tag{2.1}$$

but not vice versa. Since the uncertainty relation cannot be fulfilled by bounded operators, we will look at the Weyl realisations, i.e., instead of the q^μ we look at $e^{ik_\mu q^\mu}$. For these, the commutation relation becomes

$$e^{ik_\mu q^\mu} e^{il_\nu q^\nu} = e^{-\frac{i}{2}k_\mu Q^{\mu\nu} l_\nu} e^{i(k_\mu + l_\mu)q^\mu}. \tag{2.2}$$

The joint spectrum of the $Q^{\mu\nu}$ is a subset of

$$\Sigma := \left\{ \sigma \in \mathcal{T}^2_0(\mathbb{M}) \middle| \sigma^{\mu\nu}\sigma_{\mu\nu} = 0, \left(\frac{1}{8}\sigma^{\mu\nu}\sigma^{\rho\tau}\epsilon_{\mu\nu\rho\tau}\right)^2 = 1 \right\}.$$

Σ is a noncompact manifold and invariant under Lorentz transformations:

$$\sigma \in \Sigma, \Lambda \in L \Rightarrow \Lambda\sigma\Lambda^T \in \Sigma,$$

where L is the set of all Lorentz transformations and

$$(\Lambda\sigma\Lambda^T)^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau \sigma^{\rho\tau}.$$

Furthermore,

$$\forall \sigma, \sigma' \in \Sigma \quad \exists \Lambda \in L \text{ with } \sigma' = \Lambda\sigma\Lambda^T.$$

For $\sigma \in \Sigma$ we will define the (Euclidean) norm by

$$||\sigma||^2 := \frac{1}{2}\sum_{\mu<\nu} \sigma^{\mu\nu 2}.$$

This has the property that $||\sigma|| \geq 1 \; \forall \sigma \in \Sigma$. We define

$$\Sigma^{(1)} := \left\{ \sigma \in \Sigma \middle| ||\sigma|| = 1 \right\}.$$

$\Sigma^{(1)}$ is compact and invariant under rotations.

One possibility is to choose $Q^{\mu\nu}$ as the unit operator times a constant element $\sigma^{\mu\nu} \in \Sigma$. The corresponding C^*-algebra with the Weyl realizations will be denoted by $\mathcal{E}_\sigma$. The representations of $\mathcal{E}_\sigma$ are, by von Neumann uniqueness, all equivalent to the algebra of compact operators $\mathcal{K}$ on the Hilbert space $L^2(\mathbb{R}^2)$. In fact, if we choose the standard matrix

$$\sigma_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

we can identify $\mathcal{E}_{\sigma_0}$ with the setting of quantum mechanics on the two dimensional plane: $q^0 = X^1, q^1 = P_1, q^2 = X^2, q^3 = P_2$, where X^i and P_i are the usual position and momentum operators. σ_0 is an element of $\Sigma^{(1)}$.

A dense set of elements of $\mathcal{E}_\sigma$ is the set of symbols

$$f(q) := \frac{1}{(2\pi)^2}\int \mathrm{d}^4k \hat{f}(k) e^{-ik_\mu q^\mu},$$

where $\hat{f}$ is the Fourier transform of f and should be in $L^1(\mathbb{M})$. Most times we will choose $f \in \mathcal{S}(\mathbb{M})$. The above defines the Weyl correspondence: $\mathcal{W}$ is a map from $\mathcal{S}(\mathbb{M})$ to $\mathcal{E}_\sigma$ with $\mathcal{W}(f) = f(q)$. The product of two symbols is

$$f(q)g(q) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k_1 \mathrm{d}^4k_2 \hat{f}(k_1)\hat{g}(k_2) e^{-\frac{i}{2}k_1\sigma k_2} e^{-i(k_1+k_2)\cdot q} = (f \star_\sigma g)(q). \tag{2.3}$$

This means that the product on $\mathcal{E}_\sigma$ can be pulled down to the $\star_\sigma$ product[1] on $\mathcal{S}(\mathbb{M})$. In terms of the Weyl correspondence this relation is $\mathcal{W}(f)\mathcal{W}(g) = \mathcal{W}(f \star_\sigma g)$. The factor $e^{-\frac{i}{2}k_1\sigma k_2}$ is called twisting factor.

The product of (2.3) is often compared with the formal Moyal star product

$$\left(f \star_\sigma^M g\right)(x) := e^{\frac{i}{2}\partial_{y_1\mu} Q^{\mu\nu} \partial_{y_2\nu}} f(y_1)g(y_2)\Big|_{x=y_1=y_2}.$$

To ensure that $f \star_\sigma g = f \star_\sigma^M g$ one has to assume that f and g are analytic. But for analytic functions there exists no well-defined concept of locality.[2] So, one should treat the statement with care, that $f \star_\sigma^M g$ and therefore also $f \star_\sigma g$ are local products. However, if one takes $f \star^M_{\lambda_{\mathrm{nc}}^2\sigma} g$ and $f \star_{\lambda_{\mathrm{nc}}^2\sigma} g$ to be formal power series in λ_{nc} they are indeed equal.

The group of transformations acts on $\mathcal{E}_\sigma$ by $\tau_a(f(q)) = f(q - a\mathbb{1})$ for $a \in \mathbb{M}$. However, Lorentz symmetry is explicitly broken in this setting, since for $\Lambda \in L$ the operators $q'^\mu = \Lambda^\mu{}_\nu q^\nu$ fulfill

$$[q'^\mu, q'^\nu] = i(\Lambda\sigma\Lambda^T)^{\mu\nu}.$$

So, there exist distinct Lorentz frames, in which the noncommutativity matrix, say, equals σ_0.

To get rid of this explicit breaking of Lorentz invariance we look at the algebra of continuous functions from Σ to $L^1(\mathbb{R}^4)$ vanishing at infinity, endowed with the product

$$(F \times^\star G)(\sigma, k) = \int \mathrm{d}^4 l F(\sigma, k-l) G(\sigma, l) e^{-\frac{i}{2}k\sigma l},$$

norm $||F|| = \sup_{\sigma\in\Sigma} \int \mathrm{d}k |F(\sigma,k)|$ and the involution $F^*(\sigma, k) = \overline{F(\sigma, -k)}$. The C^*-closure of this algebra will be denoted by $\mathcal{E}$. It is isomorphic to $\mathcal{C}_0(\Sigma, \mathcal{K})$. If we denote the elements of the algebra $\mathcal{E}_\sigma$ by q^μ_σ we can see $q^\mu \in \mathcal{E}$ as a direct integral over the q^μ_σ. Furthermore, there exists the algebra of bounded continuous functions from Σ to $L^1(\mathbb{R}^4)$. The completion of this

[1] This is a nonlocal product. Hence, one can see this setting as an example of a nonlocal field theory.

[2] An analytic function is fixed globally by its derivatives at a single point.

algebra will be denoted $\tilde{\mathcal{E}}$ and can be viewed as a subset of the multiplier algebra $M(\mathcal{E})$ of $\mathcal{E}$. The algebra of bounded functions on Σ can be associated with the centre $\mathcal{Z}$ of $M(\mathcal{E})$.

The Weyl correspondence generalizes to continuous functions from Σ to $\mathcal{S}(\mathbb{M})$ (vanishing at infinity or bounded) by $\mathcal{W}(f)\mathcal{W}(g) = \mathcal{W}(f \star g)$, where the star product is pointwise in Σ:

$$\widehat{f \star g}(\sigma, k) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4 l \hat{f}(\sigma, k-l)\hat{g}(\sigma, l) e^{-\frac{i}{2}k\sigma l}.$$

Here, all Fourier transforms are at fixed σ. The symbol $f(\sigma, q) = \mathcal{W}(f)(\sigma)$ can be regarded as an element of $\mathcal{E}_\sigma$:

$$f(\sigma, q) = \frac{1}{(2\pi)^2} \int \mathrm{d}k \hat{f}(\sigma, k) e^{-ikq_\sigma}, \tag{2.4}$$

where q_σ denotes the element realized in $\mathcal{E}_\sigma$. If f does not depend on $\sigma \in \Sigma$, we will just write $f(q)$.

The Poincaré group acts on symbols in the following way:

$$\tau_{\Lambda,a}(f)(\sigma, q) = \det(\Lambda) f(\Lambda^{-1}\sigma\Lambda^{-1T}, \mathcal{U}_{\Lambda,a} q \mathcal{U}_{\Lambda,a}^{-1}), \tag{2.5}$$

where $\mathcal{U}_{\Lambda,a} q_\sigma \mathcal{U}_{\Lambda,a}^{-1} = q_{\Lambda^{-1}\sigma\Lambda^{-1T}} - \Lambda^{-1}a\mathbb{1}$. Loosely saying, one has to transform both q^μ and $Q^{\mu\nu}$. This induces an automorphism on $\mathcal{E}$. No distinct frame exists any more.[3]

Derivatives are defined as infinitesimal generators of translations:

$$\partial_{q^\mu} f(\sigma, q) = \partial_{a^\mu} f(\sigma, q + a\mathbb{1})\Big|_{a=0}. \tag{2.6}$$

We find a $\mathcal{Z}$-valued trace on $\mathcal{E}$, denoted by $\int \mathrm{d}^4 q$:

$$\int \mathrm{d}^4 q f(\sigma, q) = (2\pi)^2 \hat{f}(\sigma, 0).$$

This trace is cyclic, and on the product of two symbols it fulfills

$$\int \mathrm{d}^4 q f(\sigma, q) g(\sigma, q) = \int \mathrm{d}^4 x f(\sigma, x) g(\sigma, x).$$

[3] There exists the concept of twisted Poincaré symmetry [9, 42]. The commutation relations of the generators of the Poincaré algebra are deformed in order to make the relation $[q^\mu, q^\nu] = i\sigma^{\mu\nu}$ with fixed $\sigma^{\mu\nu}$ invariant under Poincaré transformations. However, this is essentially equivalent to leaving the Poincaré algebra untouched but transform $\sigma^{\mu\nu}$ like in (2.5) [20].

Formally, $\int \mathrm{d}^4 q e^{ikq} = (2\pi)^4 \delta(k)$. Using the Weyl correspondence, one can define the pointwise product for symbols $f(q)$ and $g(q)$,

$$f(q) \cdot_{\mathrm{pw}} g(q) = \mathcal{W}\left(\mathcal{W}^{-1}(f(q)) \cdot \mathcal{W}^{-1}(g(q))\right) = (f \cdot g)(q). \tag{2.7}$$

However, this product is rather artificial and lies somewhat outside the algebra structure of $\mathcal{E}$. Of course, the concepts of derivatives, the trace and the pointwise product also exist on $\mathcal{E}_\sigma$. In this case the trace has values in $\mathbb{C}$ instead of $\mathcal{Z}$.

States on these noncommutative spaces describe localizations.[4] To every state on $\mathcal{E}$ there exists a measure μ on Σ and a measurable function $\sigma \to \omega_\sigma$ on Σ, where each ω_σ is a state $\mathcal{E}_\sigma$, such that

$$\omega(f(Q, q)) = \int_\Sigma \mathrm{d}\mu(\sigma) \omega_\sigma(f(\sigma, q_\sigma)). \tag{2.8}$$

The uncertainty relations (2.1) are fulfilled if we set

$$\Delta q^\mu = \Delta_\omega q^\mu := \sqrt{\omega(q^{\mu 2}) - \omega(q^\mu)^2}.$$

A measure of localization of a state will be the quantity $\sum_\mu \left(\Delta_\omega q^\mu\right)^2$. However, this measure of localization is not Lorentz invariant. It was shown in [13] that it has the property

$$\sum_{\mu=0}^{3} \left(\Delta q^\mu\right)^2 \geq \sqrt{2} \int_\Sigma \mathrm{d}\mu(\sigma)(1 + ||\sigma||)$$

using the notation of (2.8). The quantity on the right-hand side has its minimum if the support of the measure $\mu(\sigma)$ is contained in the set $\Sigma^{(1)}$. For $\sigma \in \Sigma^{(1)}$ and $x \in \mathbb{M}$ we can find a unique optimal localized state $\omega_\sigma(x)$ on $\mathcal{E}_\sigma$ around x, i.e.,

$$\omega_\sigma(x)(q_\sigma^\mu) = x^\mu \text{ and } \sum_{\mu=0}^{3} \left(\Delta_{\omega_\sigma(x)} q_\sigma^\mu\right)^2 = 2. \tag{2.9}$$

Using the above mentioned identification with quantum mechanics on a plane these states can be identified with the ground state of the harmonic oscillator, shifted by a vector x in phase space. With these we can easily build optimal localized states on $\mathcal{E}$ around x. We just have to choose a measure μ with its support on $\Sigma^{(1)}$ and take the optimal localized states in $\mathcal{E}_\sigma$ around x.

[4] This is in accordance with the concept of noncommutative geometry from [12].

Note that there does not exist a Lorentz invariant optimal localized state, but rotational invariant ones.

In the commutative limit $\mathcal{E}$ reduces to the commutative algebra of functions on $\Sigma \times \mathbb{M}$. So, we get an additional manifold Σ, which has not been observed in nature yet. It has to be eliminated somehow since at the end the expectation values from the theory have to be real numbers and not functions on Σ. It is still an open question how to handle this problem. A natural idea would be to take a Lorentz invariant state on $\mathcal{E}$ and average over Σ. But since Σ is noncompact one cannot find such a state. However, there exist rotational invariant ones and averaging over $\Sigma^{(1)}$ would be the most reasonable choice in this setting.

In the following each of the settings $\mathcal{E}_\sigma, \mathcal{E}, \tilde{\mathcal{E}}$ or $M(\mathcal{E})$ will be called noncommutative Minkowski space, abbreviated by $\mathbb{M}_{\text{nc}}$. However, most of the time we will work in $\mathcal{E}_\sigma$ (and choose $\sigma \in \Sigma^{(1)}$) or $\tilde{\mathcal{E}}$, i.e., look at symbols which have no additional dependence on $\sigma \in \Sigma$. The noncommutativity scale will be absorbed into σ, i.e., $\lambda_{\text{nc}}^{-2}\sigma \in \Sigma$ actually. The continuation of the whole setting to higher even dimensions, like six, is straightforward.

2.2 Quantum fields

Now we look at quantum fields on noncommutative spacetime. Let Φ be a Wightman field and $\mathcal{F}$ the algebra of polynomials of the field. (Here we only consider hermitian scalar fields. The generalization to other fields is straightforward.) One can easily write down

$$\Phi(q) := \frac{1}{(2\pi)^2} \int \mathrm{d}^4k \hat{\Phi}(k) \otimes e^{-ik_\mu q^\mu}. \tag{2.10}$$

This lives formally on $\mathcal{F} \otimes \mathbb{M}_{\text{nc}}$. If $\Phi = \Phi_{\text{Free}}$ is the free field, it fulfills the Klein-Gordon equation:

$$(\Box_q + m^2)\Phi_{\text{Free}}(q) = 0.$$

We want to give (2.10) a precise meaning. In [4] $\Phi(q)$ was taken to be a functional on the subset of ($\mathcal{Z}$-valued) states of $\mathbb{M}_{\text{nc}}$ which are in the domain of all polynomials in the q^μ's. The functional takes values in $\mathcal{F}$. Note that for elements ω of this subset of states the function $k \to \omega\left(e^{ikq}\right)$ is in $\mathcal{S}$ (a function from Σ to $\mathcal{S}$). So, the Fourier transform can be defined and

$$\Phi(q)(\omega) = \omega(\Phi(q)) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4k \hat{\Phi}(k)\omega(e^{-ik_\mu q^\mu}) = \int \mathrm{d}^4k \hat{\Phi}(k)\check{f}_\omega(k) = \Phi(f_\omega) \tag{2.11}$$

with $\check{f}_\omega(k) := \frac{1}{(2\pi)^2}\omega(e^{-ik_\mu q^\mu})$. The last term has to be understood in the sense of Wightman fields as operator-valued distributions on $\mathbb{M}$.

Another definition is to consider multiplying by a symbol and taking the trace:[5]

$$\begin{aligned}\int \mathrm{d}^4 q f(q)\Phi(q) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4 q \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 \hat{f}(k_1)\hat{\Phi}(k_2)\otimes\left(e^{-\frac{i}{2}k_1 Q k_2}e^{-i(k_1+k_2)\cdot q}\right)\\ &= \int \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 \hat{f}(k_1)\hat{\Phi}(k_2)\otimes\left(e^{-\frac{i}{2}k_1 Q k_2}\delta(k_1+k_2)\right)\\ &= \int \mathrm{d}^4 k \hat{f}(-k)\hat{\Phi}(k) = \Phi(f).\end{aligned} \tag{2.12}$$

The result is similar to (2.11). Since more functions comply with (2.12) we adopt this point of view, but it hardly makes a difference in calculations which one we take. The tensor sign $\otimes$ between the $\mathcal{F}$ and $\mathbb{M}_{\mathrm{nc}}$ part will be dropped from now on.

However, as on commutative spacetime, products of fields are not well-defined:

$$\int \mathrm{d}^4 q f(q)\Phi(q)\Phi(q) = \frac{1}{(2\pi)^2}\int \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 \hat{\Phi}(k_1)\hat{\Phi}(k_2)\check{f}(k_1+k_2),$$

but $\check{f}(k_1+k_2)$ is not an element of $\mathcal{S}(\mathbb{M}^2)$, since it does not decrease rapidly in the direction $k_1 = -k_2$. Thus, the above expression is ill-defined. If the fields are free fields, $\Phi_0 := \Phi_{\mathrm{Free}}$, this can be cured by taking the so-called normal-ordered or Wick product, denoted by $:\Phi_0^n:$. This only applies to the field part, i.e.,

$$:\Phi_0(q)\Phi_0(q) := \frac{1}{(2\pi)^4}\int \mathrm{d}^4 k_1 \mathrm{d}^4 k_2 :\hat{\Phi}_0(k_1)\hat{\Phi}_0(k_2): e^{-ik_1 q}e^{-ik_2 q}. \tag{2.13}$$

The Wick product can be seen as the restriction of the product $\Phi_0(x)\Phi_0(y) - \Delta_+(x-y)$ to the diagonal $x = y$. In a certain sense the subtracted Δ_+ is a local subtraction (on commutative spacetime).

However, for higher products of fields a concept of locality is introduced in [4]which is more adapted to noncommutative spacetime. Some terms, which are subtracted using the usual Wick products, become nonlocal and finite. Thus, they should not be subtracted any more. This leads to the notion of

[5]Note that the map of the Weyl correspondence $\mathcal{W}$ is not positive. So, if $f \in \mathcal{S}$ is a positive function, the map $g(q) \to \int \mathrm{d}^4 q f(q)g(q)$ is not necessarily positive, and hence does not define a state.

quasiplanar Wick products, denoted by $\vdots\, \Phi_0^n \,\vdots$. The first order, where this differs from the usual Wick product, is three:

$$\vdots\, \Phi_0(q)\Phi_0(q)\Phi_0(q) \,\vdots\; =: \Phi_0(q)\Phi_0(q)\Phi_0(q) : + \frac{1}{(2\pi)^2} \int \mathrm{d}k \Delta_+(-k\sigma)\hat{\Phi}_0(k) e^{-ikq}.$$

As mostly we encounter the product of only two free fields it does not matter which product we take. However, in 4.5 we have a look at the ϕ^4 model. Here we have a product of three free fields and use the quasiplanar Wick product.

The generalizations from commutative to noncommutative spacetime are not always unique. For example

$$\int \mathrm{d}^4 q f_1(q)\Phi(q) f_2(q)\Phi(q)\dots f_n(q)\Phi(q) \tag{2.14}$$

and

$$\int \mathrm{d}^4 q f_1(q) f_2(q)\dots f_n(q)\Phi^n(q)$$

could both be seen as generalizations of

$$\int \mathrm{d}^4 x f_1(x) f_2(x)\dots f_n(x)\Phi^n(x).$$

While the last two expressions are ill-defined, it was shown in [45] that (2.14) is indeed well-defined. Among others we use terms like this to form an IR cutoff on noncommutative spacetime in sections 3.2.2 and 4.3.

2.3 Approaches to interactions

There exists a zoo of different approaches to interacting quantum field theory on noncommutative spacetime. To make clear, where our approach fits in, we give an overview of what has been done in this field of research. However, we do not claim it to be complete.

A first classification of the different approaches is by the treatment of λ_{nc}:

- Fields are treated as formal power series in λ_{nc}.
- λ_{nc} is not infinitesimally small but finite.

We take the latter point of view since we do not see a possibility to build a more fundamental concept of spacetime at small scales in the first approach. (Remember that the noncommutative Minkowski space is regarded as an intermediate model only.) Another classification is by the metric and commutativity of the time component:

- **Euclidean metric**
- **Minkowski metric**
  - Time still commutes with space.
  - Time and space do not commute.

As was already mentioned, on noncommutative spacetime there seems to be no connection between results on Euclidean and on Minkowski space, since the Osterwalder Schrader theorem [32, 33] is not applicable. Already it seems to be impossible to define the Wick rotation. In the Minkowskian setting, the uncertainty relations (2.1) do not hold if time and space still commute. Hence, we choose noncommuting time. However, this makes approaches which are equivalent on $\mathbb{M}$ inequivalent on $\mathbb{M}_{\mathrm{nc}}$.

The most important approaches are:

Modified Feynman Rules: This is the most prominent approach. It was first formulated in [18]. The usual Feynman rules are modified by adding at each vertex the twisting factor

$$e^{-\frac{i}{2}\sum_{a<b} k_a \sigma k_b}.$$

Here k_a are the incoming momenta at that vertex, numerated in clockwise direction. The twisting factor is invariant under cyclic permutation of the momenta. This modification of the Feynman rules is inspired by changing the usual action, e.g., of the ϕ^4 model to

$$\begin{aligned} S(\phi) &= \int \mathrm{d}^4 q \frac{1}{2} \partial_\mu \phi(q) \partial^\mu \phi(q) - \frac{m^2}{2}\phi(q)^2 - \frac{\lambda}{4}\phi(q)^4 \\ &= \int \mathrm{d}^4 x \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi(x). \end{aligned}$$

It was shown in [19] that unitarity is violated in this approach if time does not commute in space.[6] The modified Feynman rules approach is used for both Euclidean and Minkowski metric. For the latter concrete calculations seem to be very complicated if time does not commute with space.

[6] However, in the shown derivation given in [19] the Minkowski metric is used. In order to calculate the contribution of a Feynman graph, the loop integral is Wick rotated to an imaginary time component. To keep the twisting factor from becoming exponentially increasing in the imaginary time direction and making the integral nonconvergent the authors analytically continue the σ^{0j} components, too. This is a very questionable step.

Hamiltonian approach: This approach was first considered in [13]. A concept of integral at fixed time t is used to define an Hamiltonian $H(t)$. This is entered into the Dyson series, and time ordering with respect to the variable t can be applied. However, in the graph expansion the propagators are not the usual Feynman propagators. It was shown in [5] that this approach fulfills unitarity also for noncommuting time. In [6, 3] a UV-finite theory was developed. [29] investigates dispersion relations for this approach. However, the interaction part of the Hamiltonian is treated in a very different way than the free part, which is kind of unsatisfactory, and already at tree level the fields do not fulfill the equation of motion, see [2]. Furthermore, problems of the asymptotic behaviour at $t \to \infty$ appear, similar to those in nonlocal theories [24].

Yang-Feldman formalism: The ansatz is to solve the field equation of the quantum fields on noncommutative spacetime. This approach was already used on $\mathbb{M}_{\mathrm{nc}}$ in [5, 2] and it was shown that no problem with unitarity appears. We will analyse this approach in detail in the following chapters.

Adding a Grosse Wulkenhaar term: It was shown in [22] that adding a harmonic potential to the free scalar field action renders the ϕ^4 model renormalizable. This approaches works explicitly in the Euclidean setting. It seems unlikely that this result can be transferred to the Minkowski case.

There exist further differences on how gauge theory can be implemented. It seems that only $U(N)$ gauge groups can be defined in nonexpanded approaches. These gauge theories have severe IR divergences. More gauge groups can be defined if the Seiberg-Witten map [39] is used. This acts on formal power series in λ_{nc} and maps commutative gauge fields to noncommutative ones. The map is however not unique. We do not consider gauge theories here. For supersymmetric models approaches exist in which the fermionic variables θ do not anticommute any more [30]. This will not be considered, either.

Often one meets very formal calculations und loosely defined objects in the literature. This is kind of typical for this field. However, we try to treat everything as rigorous as possible.

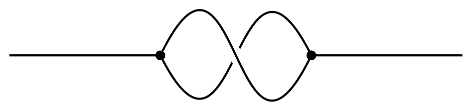

Figure 2.1: Nonplanar Feynman graph in the ϕ^3 model.

2.4 UV-IR mixing of divergences

We investigate the modified Feynman rules approach on Euclidean space. It was discovered that the contributions of some nonplanar graphs, which diverge on commutative space, become finite on noncommutative space if the incoming momentum is unequal to zero. An example for such a nonplanar graph in the ϕ^3 model is shown in figure 2.1. The internal loop integral renders finite due to the oscillating behaviour of the additional twisting factor. However, if the incoming momentum is zero, the internal loop integral would be UV divergent again. One could argue that an incoming momentum of zero is unphysical. But if the nonplanar graph is a subgraph of another one, like in figure 2.2, it is integrated over all incoming momenta of the internal nonplanar graph, even over zero. So, the integral over small momenta can give a UV divergence. This phenomenon is called UV-IR mixing of divergences. Such divergences can not be treated in the usual renormalization scheme. It was discovered in [31, 10, 11] that this is not a problem if the model is only logarithmically divergent. Here, we will show the reason at an example in the ϕ^3_4 model. This is compared with similar examples in the ϕ^3_6 and ϕ^4 model. Our calculation is different from the ones given in the above mentioned literature. The contributions of the nonplanar graphs will be calculated rigorously using the concept of oscillatory integrals, which is given in Appendix B. If the reader is not familiar with this concept it would be advisable to have a look at this appendix first. In particular, the generalized theorem of Fubini, theorem B.4.2, will play an important role.

We emphasize, that the following calculations are only valid in the Eu-

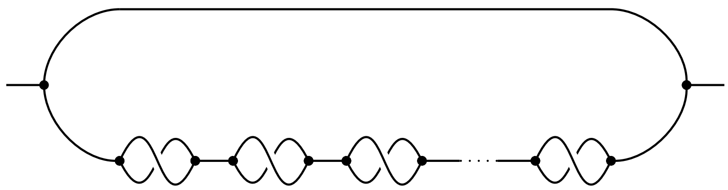

Figure 2.2: Feynman graph for the ϕ^3 model, possible candidate for showing UV-IR mixing.

clidean setting, since an analytic continuation would not be possible for the Minkowskian case due to the twisting factor. We will first have a look at the ϕ_4^3 model. For the nonplanar graph 2.1 the phases of the twisting factor of both vertices add and we get the (amputated) contribution:

$$F(k) = \int \mathrm{d}^4 l \frac{e^{ik\sigma l}}{\left((k-l)^2+m^2\right)\left(l^2+m^2\right)}. \tag{2.15}$$

This integral is not absolutely convergent but can be seen as an oscillatory integral.[7] It is easy to see, that indeed $a(k,l) = \frac{1}{((k-l)^2+m^2)(l^2+m^2)}$ is a symbol of order -4 and $\phi(k,l) = k\sigma l$ a phase function. So, the above defines a distribution in k. The singular support is contained in the set $\nabla_l \phi(k,l) = k\sigma = 0$. Since $(k\sigma)^2 = \lambda_{\mathrm{nc}}^4 k^2$ this is only the point $k=0$. We will calculate the above integral for $k \neq 0$. This calculation will be very detailed, since the usual techniques for transforming absolutely convergent integrals are a priori not applicable. We know that the integral is a $\mathcal{C}^\infty$-function outside 0 and we can use the result from section B.3 and see k as a fixed parameter.

We take a sequence of symbols $\{g_n\}$ like in proposition B.2.1 with g as in (B.6). So, the integral (2.15) is the limit of

$$\int \mathrm{d}^4 l \frac{e^{ik\sigma l}}{\left((k-l)^2+m^2\right)\left(l^2+m^2\right)} g_n(l).$$

This integral is now absolutely convergent and we can perform the usual transformations. We introduce Feynman parameters and write the integral

[7]It is not a simple Fourier transform since the variable k does not exclusively appear in the phase function but in the symbol as well.

as

$$\int_0^1 \mathrm{d}\alpha \int \mathrm{d}^4 l \frac{e^{ik\sigma l}}{\left((1-\alpha)\left((k-l)^2+m^2\right)+\alpha(l^2+m^2)\right)^2} g_n(l).$$

If we drop the g_n we can see the inner integral again as an oscillatory integral, now depending on an additional parameter α. We will introduce a different sequence of symbols, namely $g_n(l-(1-\alpha)k)$. It is easy to see that this has again the limit 1, as it only scales around a different point. (Remember that k and α can be seen as fixed for the inner integral.) Using this, we get for the oscillatory intergral (omitting the α integral for now):

$$\int \mathrm{d}^4 l \frac{e^{ik\sigma l}}{\left((1-\alpha)\left((k-l)^2+m^2\right)+\alpha(l^2+m^2)\right)^2} g_n(l-(1-\alpha)k).$$

Now, we make a variable transformation to $l' = l-(1-\alpha)k$ and get (dropping the prime again):

$$\int \mathrm{d}^4 l \frac{e^{ik\sigma l}}{\left(m^2+\alpha(1-\alpha)k^2+l^2\right)^2} g_n(l).$$

We define $b(k,\alpha) := m^2+\alpha(1-\alpha)k^2$. For $\alpha \in [0,1]$ we have $b(k,\alpha) \geq m^2$ and for bounded k it is bounded to above, too. We make a rotation in l such that the last component points in the direction of $k\sigma$. The rotational invariant g_n will be dropped. We use the coordinates $l = (\mathbf{l}, x)$ and get the oscillatory integral

$$\int \mathrm{d}^4 l \frac{e^{isx}}{\left(b+\mathbf{l}^2+x^2\right)^2}$$

with $s := |k\sigma| = \lambda_{\mathrm{nc}}^2 |k|$. As sx is a phase function in the x coordinate alone, we can use the generalized theorem of Fubini B.4.2 and perform the x integration first, which is an absolutely convergent integral. The result is

$$\frac{\pi}{2} \int \mathrm{d}^3 l e^{-s\sqrt{b+\mathbf{l}^2}} \left(\frac{s}{b+\mathbf{l}^2} + \frac{1}{(b+\mathbf{l}^2)^{3/2}} \right). \tag{2.16}$$

This integrand is now a Schwartz function in $\mathbf{l}$, as it should be according to the generalized theorem of Fubini. So, the integral is now absolutely convergent.

Now, we examine the behaviour of the integral at $|k| \to 0$. Since $b(k,\alpha)$ is bounded and greater or equal to m^2 we do not have to worry about this quantity. The integral over α, which is yet to be done, is over a compact set and does not change the divergent behaviour, either. Thus, we look at the above integral for small s. The first part of (2.16) stays finite, since after the angular integration we get (dropping prefactors from now on):

$$\int_0^\infty \mathrm{d}l e^{-s\sqrt{b+l^2}} \frac{sl^2}{b+l^2} = \int_0^\infty \mathrm{d}l e^{-\sqrt{bs^2+l^2}} \frac{l^2}{bs^2+l^2} \xrightarrow[s\to 0]{} \int_0^\infty \mathrm{d}l e^{-l} = 1.$$

After a similar variable transformation the second part gives

$$\int_0^\infty \mathrm{d}l e^{-\sqrt{bs^2+l^2}} \frac{l^2}{(bs^2+l^2)^{3/2}}.$$

For $s = 0$, this is now divergent at small l. Since the factor $e^{-\sqrt{bs^2+l^2}}$ is finite for small l and s, we can examine the divergent behaviour of the above integral by looking at

$$\int_0^1 \mathrm{d}l \frac{l^2}{(bs^2+l^2)^{3/2}} = -\frac{1}{\sqrt{1+bs^2}} + \mathrm{ArcCsch}(s\sqrt{b}).$$

The first part stays finite. $y = \mathrm{ArcCsch}(x)$ is the inverse function of $x = \frac{1}{\sinh(y)}$. The latter behaves for large y, hence small x, like $2e^{-y}$. Thus, $\mathrm{ArcCsch}(s\sqrt{b})$ behaves like $\log(2) - \frac{1}{2}\log(b) - \log(s)$. This shows that $F(k)$ behaves like $\log(|k|)$ for small k.

If the considered graph appears n times as a subgraph of another one, like in figure 2.2, we would have to calculate

$$\int \mathrm{d}^4 k F(k)^n \frac{1}{(k^2+m^2)^{n+1}((p-k)^2+m^2)},$$

where p is the outer momentum. This is finite for large k due to the $n+2$ propagators. The contribution of small k can be estimated by

$$C \cdot \int_0^K \mathrm{d}k k^3 \log(k)^n,$$

which is a finite integral. So, no IR divergence appears.

The situation is different for the ϕ_6^3 model, which is quadratically divergent. The calculation is the same until (2.16), except that the remaining integral is over five dimensions. The first part gives

$$\int_0^\infty \mathrm{d}l e^{-s\sqrt{b+l^2}} \frac{sl^4}{b+l^2} = \frac{1}{s^2} \int_0^\infty \mathrm{d}l e^{-\sqrt{bs^2+l^2}} \frac{l^4}{bs^2+l^2},$$

and the second

$$\int_0^\infty \mathrm{d}l e^{-\sqrt{b+l^2}} \frac{l^4}{(b+l^2)^{3/2}} = \frac{1}{s^2} \int_0^\infty \mathrm{d}l e^{-\sqrt{bs^2+l^2}} \frac{l^4}{(bs^2+l^2)^{3/2}}.$$

Both contributions diverge like $\frac{1}{s^2}$, since the remaining integrals are finite for $s = 0$. So, in six dimensions the contribution for small k of the graph in figure 2.2 behaves like

$$\int_0^K \mathrm{d}k k^3 \frac{1}{k^{2n}}.$$

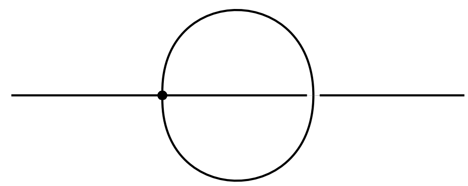

Figure 2.3: Nonplanar Feynman graph in the ϕ^4 model.

This is divergent for $n \geq 2$. Since the divergence is at small k, it is called an IR divergence. This is the so called UV-IR mixing.

Another model, where this can be seen, is the ϕ^4 model in four dimensions. Figure 2.3 shows a nonplanar tadpole graph, which is finite and gives the oscillatory integral[8]

$$\int \mathrm{d}^4 l \frac{1}{l^2 + m^2} e^{-ik\sigma l} = \int \mathrm{d}^4 l \frac{1}{x^2 + \mathbf{l}^2 + m^2} e^{-isx}.$$

The transformation to the second integral is similar as before. Again, we use the generalized Theorem of Fubini to perform the x integration first. We get

$$\begin{aligned}\pi \int \mathrm{d}^3 l \frac{1}{\mathbf{l}^2 + m^2} e^{-s\sqrt{\mathbf{l}^2+m^2}} &= 4\pi^2 \int_0^\infty \mathrm{d}l \frac{l^2}{\sqrt{l^2+m^2}} e^{-s\sqrt{l^2+m^2}} \\ &= 4\pi^2 \frac{1}{s^2} \int_0^\infty \mathrm{d}l \frac{l^2}{\sqrt{l^2+m^2 s^2}} e^{-\sqrt{l^2+m^2 s^2}}.\end{aligned}$$

So, for small s this behaves like $\frac{1}{s^2}$. If the graph appears as subgraphs in another one, like the one shown in figure 2.4, we can get an IR divergence. Thus, this quadratically divergent model shows UV-IR mixing, too.

Remark 2.4.1. Actually, $F(k)$ from (2.15) is a distribution in k. So, we have shown that its scaling degree at $k = 0$ is 0 for the ϕ_4^3 model and 2 for the ϕ_6^3 and the ϕ^4 model. We can use the concept of scaling degree, introduced by Steinmann [41], cf. Appendix A, to find a continuation of $F(k)^n$ to the

[8] This can also be seen as a simple Fourier transform.

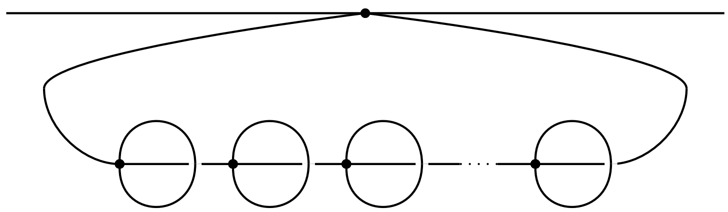

Figure 2.4: Feynman graph for the ϕ^4 model, showing UV-IR mixing.

origin. In the case of the ϕ_4^3 model there exists a unique continuation. For the other models the ambiguity can be described by $2n-4$ free parameters. As all powers of $F(k)$ appear in the Feynman graph calculus, this leads to an infinite number of free parameters in these quadratically divergent models.

Chapter 3

Yang-Feldman formalism

We introduce the Yang-Feldman formalism to solve perturbatively the field equation for a quantum field. The Yang-Feldman approach was mainly developed in [44, 27, 28]. Closely related to this approach are the retarded products. Steinmann [41] showed formally, i.e., without IR cutoff, how some divergences cancel and and the remaining can be put into free constants through continuation of distributions to the origin. The latter is equivalent to renormalization in the Feynman graph formalism. Due to the success of the Feynman graph formalism not much work was done in the Yang-Feldman formalism. Recent developments for retarded products can be found in [16]. The Yang-Feldman formalism was already used on the noncommutative Minkowski space in [5, 2, 4] and it was shown that no problems with unitarity appear. However, this approach is still underdeveloped both on commutative and noncommutative spacetime.

In section 3.1 we introduce the Yang-Feldman formalism for classical fields. We extend it to quantum fields in section 3.2. This makes an IR cutoff necessary. To take a closer look at this problem we investigate the two-point function for a mass term as interaction. We use this as a benchmark of the IR cutoff since we already have an anticipation of what the result should be, namely the same as for a free field of the shifted mass. A similar result is also missing for the commutative case, so we first have a look at this in section 3.2.1. Then we use this result to show how the correct limit is obtained on noncommutative spacetime in section 3.2.2.

We use a well known result of Epstein Glaser [17], which is for convenience stated in Appendix D. Epstein and Glaser used the theorem to calculate time ordered function. These are related to the n-point function on commutative spacetime. As time ordering is not well-defined on noncommutative spacetime, we have a look at the latter and in particular calculate the two-point function of the interacting field. The theorem from Epstein and Glaser lies restrictions on the support of the cutoff functions in momentum space. However, we will show that the class of cutoff functions can be extended. A large part of what we present in this Chapter has already been published in [14]. Some minor mistakes have been corrected.

3.1 Classical fields

The main approach in this formalism is to solve the field equation, which for a polynomial interaction $-\frac{\lambda}{a}\phi^a$ is

$$(\Box + m^2)\phi = -\lambda\phi^{a-1}. \tag{3.1}$$

For the time being, we assume that ϕ is some classical field on the commutative Minkowski space. We try to solve (3.1) perturbatively. The solution will be denoted by ϕ_{Int}. Let ϕ_{Free} be a field solving the free equation, i.e., $(\Box + m^2)\phi_{\text{Free}} = 0$. Now ϕ_{Int} is treated as a formal power series in the coupling λ:

$$\phi_{\text{Int}} = \sum_{n=0}^{\infty} \lambda^n \phi_n, \tag{3.2}$$

and for early times $t \to -\infty$ the interacting field should be approximated by the free field. Inserted into (3.1) and sorted by powers of λ we get

$$\begin{aligned} &(\Box + m^2)\phi_0 = 0, \\ &(\Box + m^2)\phi_n = -\sum_{k_1+\ldots+k_{a-1}=n-1} \phi_{k_1}\ldots\phi_{k_{a-1}} \qquad \text{for } n > 0. \end{aligned}$$

where a is the same as in (3.1). With the initial condition mentioned before, this is solved by

$$\begin{aligned} &\phi_0 = \phi_{\text{Free}}, \\ &\phi_n = -\Delta_R \times \sum_{k_1+\ldots+k_{a-1}=n-1} \phi_{k_1}\ldots\phi_{k_{a-1}} \qquad \text{for } n > 0. \end{aligned} \tag{3.3}$$

where $\times$ denotes the convolution.[1] The convolution with Δ_R is a priori only well-defined if the ϕ_k are elements of some test function space, e.g., $\mathcal{S}$.

We now want to generalize this formalism in two ways:

1. The fields should live on the noncommutative Minkowski space.

[1] If we had required the free and interacting fields to coincide at $t \to +\infty$ we would have had to take Δ_A instead of Δ_R.

2. They should be quantum fields.

The first point is straightforward. Using (2.6), the field equation becomes

$$(\Box_q + m^2)\phi(q) = (\Box_x + m^2)\phi(q+x)|_{x=0} = -\lambda\phi^{a-1}(q). \tag{3.4}$$

The boundary condition on ϕ_{Int} can be stated by requiring

$$\lim_{x^0\to-\infty} \omega_x(\phi_{\mathrm{Int}}(q) - \phi_{\mathrm{Free}}(q)) = 0,$$

where ω_x is the optimal-localized state given in (2.9). The solution is similar as before:

$$\begin{aligned}
\phi_0(q) &= \phi_{\mathrm{Free}}(q),\\
\phi_n(q) &= -\left(\Delta_R \times \sum_{k_1+\ldots+k_{a-1}=n-1} \phi_{k_1}\ldots\phi_{k_{a-1}}\right)(q) \qquad \text{for } n>0\\
&= -\int \mathrm{d}x\ \Delta_R(x) \sum_{k_1+\ldots+k_{a-1}=n-1} \phi_{k_1}(q-x)\ldots\phi_{k_{a-1}}(q-x).
\end{aligned} \tag{3.5}$$

A proof that this fulfills the boundary condition and that convolution with Δ_R is a well-defined process for symbols can be found in [45, 21].

The generalization to quantum fields gives rise to further problems, both in commutative and noncommutative spacetime. We will have a closer look at these in the following section.

3.2 Quantum fields, IR cutoff and adiabatic limit

We now want to generalize the solutions (3.3) or (3.5) to a quantum field $\Phi(x)$, i.e., an operator-valued distribution, or $\Phi(q)$, defined in the sense of (2.12). Now we face two problems:

a) The convolution of a distribution with the retarded propagator Δ_R is in general not well-defined.

b) The product of multiple Φ_{k_i} is a priori not well-defined, either.

The solution to the first problem will be to introduce a cutoff function. There are several possibilities to do this, and we will examine these in this section. The second one is more complicated and leads to the necessity of renormalization. We will have a closer look on these problems in Chapter 4, where several cases of interacting theories both on commutative and noncommutative spacetime are examined.

We will first have a look at problem a) on commutative spacetime. Let A be an operator-valued distribution. The convolution $B := \Delta_R \times A$ is not a well-defined distribution since we have for some test function f,

$$\int \mathrm{d}y f(y) \int \mathrm{d}x \Delta_R(x) A(y-x) = \int \mathrm{d}x \mathrm{d}y \Delta_R(x) A(y) f(x+y) = (\Delta_R \otimes A)(\tilde{f}) \tag{3.6}$$

with $\tilde{f}(x,y) = f(x+y)$. This function does not fall off fast in the direction $x = -y$, so it is not in $\mathcal{S}(\mathbb{M}^2)$. We will cure this now by introducing an additional cutoff function $g \in \mathcal{S}$ (this is called the infrared cutoff, or IR cutoff) and later let g approach 1 in some sense to be specified (this is called the adiabatic limit).

Looking at (3.6) there are two obvious ways to handle this: To multiply $f(x+y)$ by $g(x)$ or by $g(y)$, which is equivalent to multiplying Δ_R or A by g. A third possibility would be to take $g \in \mathcal{S}(\mathbb{M}^2)$ and $(\Delta_R \otimes A)(\tilde{f} \cdot g)$, but this is not considered here.

Multiplying by $g(y)$ can in our case be interpreted as a localization of the interaction:

$$(\Box + m^2)\phi(x) = -\lambda g(x) \phi^a(x).$$

There is no similar interpretation for taking $\Delta_R \cdot g$. Nevertheless, this cutoff was taken in [2] for fields on $\mathbb{M}_{\mathrm{nc}}$, as the multiplication of A by g is more complicated when A is not an operator-valued distribution on $\mathbb{M}$ but on $\mathbb{M}_{\mathrm{nc}}$.

The adiabatic limit will be taken in the following steps:

1. Introduce the cutoff g. The fields, which we get, will depend on the choice of g, e.g., using the second cutoff we would get

   $$\Phi_{n,g}(f) := -\int \mathrm{d}x \mathrm{d}y \Delta_R(x) \sum_{k_1+\ldots+k_{a-1}=n-1} \Phi_{k_1,g} \ldots \Phi_{k_{a-1},g}(y) f(x+y) g(y)$$

   with $\Phi_{0,g} = \Phi_0$.

2. Calculate the expectation values of $\Phi_{\mathrm{Int},g}$, which then also depend on g. Here we only have a look at the two-point function, i.e.,

   $$\langle \Phi_{\mathrm{Int},g}(f) \Phi_{\mathrm{Int},g}(h) \rangle .$$

3. The expectation values are a formal power series in the coupling constant. Insert a sequence of test functions with $g_a \to 1$ (which is equivalent to $\check{g}_a \to (2\pi)^2 \delta$) in an appropriate topology, and then calculate the limit of the expectation values in each order.

It turns out that it will be important to sum up all contributions to the expectation value of the same order before performing the adiabatic limit, because there will be no well-defined adiabatic limit for separate terms, as already seen by Epstein and Glaser [17].

The first two possibilities for an IR cutoff will be analysed and it turns out that only the second one gives a reasonable adiabatic limit. We test them by taking a mass term as interaction. The equation of motion for an additional mass term is

$$\left(\Box + m^2\right)\Phi = -\mu\Phi.$$

The advantage of taking a mass term is that first we do not face the problem of multiplying several distributions (problem b) on page 29). Second, we already have a precise expectation of what the outcome should be: The two-point function of the interacting field should be the same as the free field of mass square $m^2+\mu$, namely

$$\begin{aligned}\left\langle \Phi^{(m^2+\mu)}_{\mathrm{Free}}(f)\Phi^{(m^2+\mu)}_{\mathrm{Free}}(h)\right\rangle &= (2\pi)^2\int \mathrm{d}^4k\hat{\Delta}^{(m^2+\mu)}_{+}(k)\check{f}(k)\check{h}(-k)\\ =&2\pi\int \mathrm{d}^3k\frac{1}{2\omega^{(m^2+\mu)}_{\mathbf{k}}}\check{f}(\omega^{(m^2+\mu)}_{\mathbf{k}},\mathbf{k})\check{h}(-\omega^{(m^2+\mu)}_{\mathbf{k}},-\mathbf{k}).\end{aligned} \tag{3.7}$$

To contemplate Haag's theorem [23] is appropriate at this point. Haag's theorem says that representations of the CCR algebra for different masses are inequivalent. But since we are dealing only with expectation values and not with representations this theorem does not apply here.

Since we are working in perturbation theory, we have to treat everything as a formal power series in the coupling μ. Thus, we get at nth order for the right-hand side of (3.7):

$$\begin{aligned}\frac{2\pi}{n!}\int \mathrm{d}^3k\partial^n_\mu\Bigg(&\frac{1}{2\sqrt{m^2+\mu+\mathbf{k}^2}}\\ &\check{f}(\sqrt{m^2+\mu+\mathbf{k}^2},\mathbf{k})\check{h}(-\sqrt{m^2+\mu+\mathbf{k}^2},-\mathbf{k})\Bigg)\Bigg|_{\mu=0}.\end{aligned} \tag{3.8}$$

If we regard $\mu\to\Delta^{(m^2+\mu)}_{+}$ as a map into $\mathcal{S}'$, which is $\mathcal{C}^\infty$ around $\mu=0$, this equals

$$\frac{1}{n!}\int \mathrm{d}x\mathrm{d}yf(x)h(y)\partial^n_{m^2}\Delta^{(m^2)}_{+}(x-y) = \frac{(2\pi)^2}{n!}\int \mathrm{d}^4k\check{f}(k)\check{h}(-k)\partial^n_{m^2}\hat{\Delta}^{(m^2)}_{+}(k). \tag{3.9}$$

We calculate this to first order and with (3.8) we get :

$$\begin{aligned}
&(2\pi)^2 \int \mathrm{d}^4k \check{f}(k)\check{h}(-k)\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k) \\
&= 2\pi \int \mathrm{d}^3k \ \Big(-\frac{1}{4\omega_{\mathbf{k}}^3}\check{f}(\omega_{\mathbf{k}},\mathbf{k})\check{h}(-\omega_{\mathbf{k}},-\mathbf{k}) + \frac{1}{4\omega_{\mathbf{k}}^2}\partial_0\check{f}(\omega_{\mathbf{k}},\mathbf{k})\check{h}(-\omega_{\mathbf{k}},-\mathbf{k}) \\
&\qquad\qquad - \frac{1}{4\omega_{\mathbf{k}}^2}\check{f}(\omega_{\mathbf{k}},\mathbf{k})\partial_0\check{h}(-\omega_{\mathbf{k}},-\mathbf{k}) \Big). \quad (3.10)
\end{aligned}$$

Remark 3.2.1. For $s \in \mathbb{C}$ the Taylor series of $(1+x)^s$ converges absolutely around $x = 0$ with radius of convergence 1. Furthermore, if f and h have compact support, then $\check{f}$ and $\check{h}$ will be analytic. So, we see that

$$\Delta_+^{(m^2+\mu)} = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}\partial_{m^2}^n \Delta_+^{(m^2)}$$

does not only hold in the sense of power series, but also in the weak topology of $\mathcal{D}'$ for $|\mu| < m^2$.

We will now have a look at whether the two different possibilities of introducing a cutoff give the expected result on commutative spacetime first.

3.2.1 Adiabatic limit on commutative spacetime

We will first look at the cutoff introduced in [2] (there for fields on noncommutative spacetime). As the interacting and the free field coincide at zeroth order, we get the expected two-point function Δ_+ at this order. But already at first order this fails:

Proposition 3.2.2. *The cutoff defined by*

$$\Phi_{n,g}(x) = -\int \mathrm{d}^4z \Delta_R(z) g(z) \Phi_{n-1,g}(x-z)$$

does not give the correct adiabatic limit, i.e., (3.8) at first order on commutative spacetime.

Proof. At first order we have

$$\Phi_{1,g}(x) = -\int \mathrm{d}^4z \Delta_R(z) g(z) \Phi_0(x-z).$$

Thus, the two-point function at first order gives

$$
\begin{aligned}
&\langle \Phi_0(f)\Phi_{1,g}(h) + \Phi_{1,g}(f)\Phi_0(h)\rangle \\
&= - \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z f(x)h(y)\Delta_R(z)g(z)(\Delta_+(x-y+z) + \Delta_+(x-y-z)) \\
&= - \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\hat{\Delta}_+(k)\check{f}(k)\check{h}(-k)\left((\Delta_R \cdot g)^\vee(k) + (\Delta_R \cdot g)^\vee(-k)\right) \\
&= - \frac{1}{(2\pi)^5}\int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_\mathbf{k}}\check{f}(k_+)\check{h}(-k_+)C_g(k_+).
\end{aligned}
$$

Here $C_g(k) = \frac{1}{(2\pi)^2}((\check{\Delta}\times\check{g})(k)+(\check{\Delta}\times\check{g})(-k))$ is, as the sum of two terms which are convolutions of a distribution with a Schwartz function, a polynomially bounded $\mathcal{C}^\infty$-function, see [35]. Now we can deduce that, if the adiabatic limit of the above for any sequence $g_a \in \mathcal{S}$ is well-defined at all, it does not give $\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k)$: Choose $\check{f}$ / $\check{h}$ to vanish on the positive/negative mass-shell, but with derivatives in 0 direction unequal to zero on the shells. Then the above gives zero for all $g \in \mathcal{S}$, but $\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k)$ in general does not give zero for such an f and h, compare (3.10). $\square$

Now we look at the cutoff defined by

$$\Phi_{n,g}(x) = - \int \mathrm{d}^4z\Delta_R(z)g(x-z)\Phi_{n-1,g}(x-z).$$

This cutoff arises naturally if one changes the action to

$$\mathcal{S}[\Phi] = \int \mathrm{d}x \left(\partial_\nu\Phi(x)\partial^\nu\Phi(x) + \left(\frac{m^2}{2} + \frac{\mu}{2}g(x)\right)\Phi^2(x)\right).$$

Then the field equation becomes

$$\left(\Box + m^2\right)\Phi(x) = -\mu g(x)\Phi(x). \tag{3.11}$$

So with this cutoff the interaction can be localized to some bounded region in spacetime.

Again the zeroth order is trivial. We will do the first order calculation explicitly to show that it is important to first add all contributions of the same order before performing the adiabatic limit. At first order we get

$$
\begin{aligned}
&\langle \Phi_0(f)\Phi_{1,g}(h) + \Phi_{1,g}(f)\Phi_0(h)\rangle \\
=& -\int \mathrm{d}^4x_0\mathrm{d}^4x_1\mathrm{d}^4x_2\ f(x_0)h(x_2)g(x_1) \\
&\quad \cdot (\Delta_R(x_0-x_1)\Delta_+(x_1-x_2) + \Delta_+(x_0-x_1)\Delta_A(x_1-x_2)) \\
=& -(2\pi)^2\int \mathrm{d}^4k_0\mathrm{d}^4k_1\ \check{f}(k_0)\check{h}(-k_1)\check{g}(k_1-k_0) \\
&\quad \cdot \left(\hat{\Delta}_R(k_0)\hat{\Delta}_+(k_1) + \hat{\Delta}_+(k_0)\hat{\Delta}_A(k_1)\right).
\end{aligned}
\tag{3.12}
$$

To simplify this expression, we perform different transformations on each summand. We integrate out the zero component of the momentum appearing in $\hat{\Delta}_+$ using its δ-function, and for the remaining zero component we carry out a variable transformation to $x = \pm(k^0_{0/1} - \omega_{0/1})$ and get

$$
\begin{aligned}
\frac{1}{2\pi}\int &\frac{\mathrm{d}^3k_0}{2\omega_0}\frac{\mathrm{d}^3k_1}{2\omega_1}\mathrm{d}x\ \check{g}(\omega_1-\omega_0-x,\mathbf{k_1}-\mathbf{k_0}) \\
&\cdot\Bigg[\check{f}(\omega_0+x,\mathbf{k_0})\check{h}(-\omega_1,-\mathbf{k_1})\left(\frac{1}{x+i\epsilon}-\frac{1}{x+2\omega_0+i\epsilon}\right) \\
&\quad -\check{f}(\omega_0,\mathbf{k_0})\check{h}(-\omega_1+x,-\mathbf{k_1})\left(\frac{1}{x+i\epsilon}-\frac{1}{x-2\omega_1+i\epsilon}\right)\Bigg].
\end{aligned}
\tag{3.13}
$$

We assume that $\check{g}$ has only support in a closed subset of $R_1 = \{k \in \mathbb{M} | k^2 < (2m)^2\}$. Then the singularities $x = \mp 2\omega_{0/1}$ are not met, since the sum of two vectors on the positive mass shell has a square greater or equal to $(2m)^2$. Thus the second and fourth ϵ in the expression 3.13 can be dropped.

Now we make an expansion in x:

$$
\begin{aligned}
\check{f}(\omega_{\mathbf{k}}+x,\mathbf{k}) &= \check{f}(\omega_{\mathbf{k}},\mathbf{k}) + x\ \tilde{f}(x,\mathbf{k}), \\
\check{h}(-\omega_{\mathbf{k}}+x,-\mathbf{k}) &= \check{h}(-\omega_{\mathbf{k}},-\mathbf{k}) + x\ \tilde{h}(x,\mathbf{k}),
\end{aligned}
$$

where $\tilde{f}$ and $\tilde{h}$ are again functions in $\mathcal{S}$ satisfying $\tilde{f}(0,\mathbf{k}) = \partial_0\check{f}(\omega_{\mathbf{k}},\mathbf{k})$ and

$\tilde{h}(0,\mathbf{k}) = \partial_0 \check{h}(-\omega_{\mathbf{k}}, -\mathbf{k})$. With this (3.13) transforms to

$$
\begin{aligned}
\frac{1}{2\pi}\int \frac{\mathrm{d}^3k_0}{2\omega_0}\frac{\mathrm{d}^3k_1}{2\omega_1}\mathrm{d}x\ \check{g}(\omega_1-\omega_0-x,\mathbf{k_1}-\mathbf{k_0}) \\
\cdot\Big[-\check{f}(\omega_0+x,\mathbf{k_0})\check{h}(-\omega_1,-\mathbf{k_1})\frac{1}{2\omega_0+x} \\
-\check{f}(\omega_0,\mathbf{k_0})\check{h}(-\omega_1+x,-\mathbf{k_1})\frac{1}{2\omega_1-x} \\
+\check{f}(\omega_0,\mathbf{k_0})\check{h}(-\omega_1,-\mathbf{k_1})\left(\frac{1}{x+i\epsilon}-\frac{1}{x+i\epsilon}\right) \\
+\left(\tilde{f}(x,\mathbf{k}_0)\check{h}(-\omega_1,-\mathbf{k_1})-\check{f}(\omega_0,\mathbf{k_0})\tilde{h}(x,\mathbf{k}_1)\right)x\frac{1}{x+i\epsilon}\Big]
\end{aligned}
$$

The last but one term drops out. This cancellation only occurs because we have treated the sum of $\langle\Phi_0(f)\Phi_1(h)\rangle$ and $\langle\Phi_1(f)\Phi_0(h)\rangle$. The singularity of $\frac{1}{x+i\epsilon}$ in the last line is cancelled by the additional factor of x. Thus, with regard to the presupposed support of $\check{g}$, the remaining terms are smooth functions of $x, \mathbf{k}_0$ and $\mathbf{k}_1$. Then the adiabatic limit $\check{g}\to(2\pi)^2\delta$ can be carried out, e.g. in the topology of functions with compact support. The $\mathbf{k}_1$ and x integration then give $\mathbf{k}_1=\mathbf{k}_0$ and $x=0$. With the properties of $\tilde{f}$ and $\tilde{h}$ we then get exactly (3.10).

Remark 3.2.3. Here and in the following we replace distributions by a series of functions which approximate the distribution in the weak topology of $\mathcal{S}'$. An example would be $\hat{\Delta}_R(p)=\lim_{\epsilon\searrow 0}\frac{1}{(2\pi)^2}\frac{-1}{p^2-m^2+i\epsilon p_0}$. It is easy to see that this is actually independent of the choice of sequence. Indeed, we will sometimes change the sequence without further notice. For products of distributions in different variables we can take the product of the corresponding sequences, where the ϵ in each factor coincide.

Now we look at arbitrary orders. The rest of this subsection is structured in the following way

1. We calculate the two-point function of arbitrary order n.
2. We show that the theorem of Epstein and Glaser is applicable. With this theorem we know that an adiabatic limit is well-defined for a certain class ("class I")of sequences of functions and independent of the choice in class I. But we do not know, whether the result for the adiabatic limit is indeed (3.8).
3. Therefore, we calculate the result. For this we can take a special choice of sequence $\{g_a\}$ of class I. We split the two-point function into two parts $A(g_a)+B(g_a)$.

4. We show that $B(g_a) \to 0$ for $a \to \infty$.

5. We show that $A(g_a)$ yields (3.8) for $a \to \infty$. To do this we state a lemma, which is proved at the end of this subsection.

6. We enhance the class of functions giving the correct limit beyond class I given by the Epstein-Glaser theorem. The enhanced class will be called class II.

7. At the end we summarise the result of this section in a theorem.

To keep track of this schedule we indicate the different parts by boldface headings.

Part 1:

The field at mth order is

$$\Phi_{m,g}(y_0) = (-1)^m \int \prod_{i=1}^{m} \mathrm{d}y_i \Delta_R(y_0 - y_1) g(y_1) \dots \Delta_R(y_{m-1} - y_m) g(y_m) \Phi_0(y_m).$$

For the two-point function at nth order we get

$$\begin{aligned} &\sum_{m=0}^{n} \langle \Phi_{m,g}(f) \Phi_{n-m,g}(h) \rangle \qquad (3.14) \\ =&(-1)^n \int \prod_{i=0}^{n+1} \mathrm{d}^4 y_i \; f(y_0) h(y_{n+1}) \prod_{i=1}^{n} g(y_i) \sum_{m=0}^{n} \Delta_R(y_0 - y_1) \dots \Delta_R(y_{m-1} - y_m) \\ &\cdot \Delta_+(y_m - y_{m+1}) \Delta_A(y_{m+1} - y_{m+2}) \dots \Delta_A(y_n - y_{n+1}). \end{aligned}$$

We define

$$\begin{aligned} F_R(y_0, y_{n+1}; y_1, \dots, y_n) := (-1)^n \sum_{m=0}^{n} \Delta_R(y_0 - y_1) \dots \Delta_R(y_{m-1} - y_m) \\ \cdot \Delta_+(y_m - y_{m+1}) \Delta_A(y_{m+1} - y_{m+2}) \dots \Delta_A(y_n - y_{n+1}). \quad (3.15) \end{aligned}$$

So in momentum space the above is:

$$\int \prod_{i=1}^{n} \mathrm{d}^4 k_i \mathrm{d}p_1 \mathrm{d}p_2 \; \check{f}(p_1) \check{h}(p_2) \prod_{l=1}^{n} \check{g}(k_l) \hat{F}_R(p_1, p_2; k_1, \dots, k_n) \qquad (3.16)$$

with

$$\hat{F}_R(k_0, k_{n+1}; k_1, \dots, k_n) = \\ (-1)^n (2\pi)^2 \delta \left(\sum_{j=0}^{n+1} k_j \right) \sum_{m=0}^{n} \left[\hat{\Delta}_R(k_0) \dots \hat{\Delta}_R \left(\sum_{j=0}^{m-1} k_j \right) \right. \\ \left. \cdot \hat{\Delta}_+ \left(\sum_{j=0}^{m} k_j \right) \hat{\Delta}_A \left(\sum_{j=0}^{m+1} k_j \right) \dots \hat{\Delta}_A \left(\sum_{j=0}^{n} k_j \right) \right]. \quad (3.17)$$

Similar to F_R we define F_A just by exchanging Δ_R and Δ_A in (3.15).

Part 2:

Now we claim that these functions F_R and F_A fulfill the requirements of the Epstein-Glaser theorem, which is stated in Appendix D. To show that F_R has retarded support , we look at the mth summand. This is only unequal to zero if $y_j - y_{j+1} \in \bar{V}_+$ for $j = 0, \dots, m-1$ and $y_{j+1} - y_j \in \bar{V}_+$ for $j = m, \dots, n$. Since the sum of two elements in $\bar{V}_+$ is again in $\bar{V}_+$, we can add these terms and see that the support of each summand is in

$$\{(y_0, y_{n+1}; y_1, \dots, y_n) \in \mathbb{M}^{2+n} | \\ y_0 - y_j \in \bar{V}_+ \text{ for } j \leq m-1 \text{ and } y_{n+1} - y_j \in \bar{V}_+ \text{ for } j \geq m\}.$$

which is a subset of S_R (see (D.1)). So their sum, F_R, also has support in S_R. The proof for F_A works analogously.

We still have to show that $\hat{F}_R - \hat{F}_A$ vanishes on the set R_n (see (D.2)). To do this we replace each $\hat{\Delta}_{R/A}$ by $\hat{\Delta}_F - i\hat{\Delta}_{\mp}$ (compare (A.3) and (A.4)) in (3.17) and in the corresponding expression for $\hat{F}_A$ and then multiply out. $\hat{\Delta}_{\pm}$ have their support on the positive respectively negative mass shell, so $\hat{\Delta}_-(p)\hat{\Delta}_+(p+k) = 0$ if $k^2 < 4m^2$. Thus, after the substitutions all terms with a factor $\hat{\Delta}_-$ vanish on R_n as every summand has a factor of $\hat{\Delta}_+$. The remaining terms are all of the form (dropping the prefactors and the δ-function)

$$\hat{\Delta}_F(K_0) \dots \hat{\Delta}_F(K_{a-1}) \hat{\Delta}_+(K_a) \hat{\Delta}_{+/F}(K_{a+1}) \\ \dots \hat{\Delta}_{+/F}(K_{b-1}) \hat{\Delta}_+(K_b) \hat{\Delta}_F(K_{b+1}) \dots \hat{\Delta}_F(K_n)$$

with $K_j := \sum_{i=0}^{j} k_i$ and a and b the number of the first respectively last factor of $\hat{\Delta}_+$ in that term. In $\hat{F}_R - \hat{F}_A$ there are two terms of this form for given $a \leq b$. One is coming from the summand with $m = a$ in $\hat{F}_R$ the other one from the summand $m = b$ in $\hat{F}_A$. Their prefactors are equal, so these terms cancel in $\hat{F}_R - \hat{F}_A$. Therefore $\hat{F}_R - \hat{F}_A$ vanishes on R_n.

From the Epstein-Glaser theorem we can now deduce that the adiabatic limit exists if the sequence G_a has the correct properties. Here, G_a is of the form

$$G_a(k_1, \ldots, k_n) = g_a(k_1) \cdot \ldots \cdot g_a(k_n).$$

The condition that G_a tends to $(2\pi)^{2n}\delta^{(4n)}$ in the topology of $\mathcal{O}'_C(\mathbb{M}^n)$ is guaranteed if $\check{g}_a \to (2\pi)^2\delta^{(4)}$ in $\mathcal{O}'_C(\mathbb{M})$. To fulfill the condition supp $\check{G}_a \subset R_n$ we could demand that the support of $\check{g}_a$ lies in some convex subset of $\frac{1}{n}R_n$. An example for such a convex subset would be

$$V_n := \left\{ k \in \mathbb{M} \big| |k_0| < \frac{2m}{n} \right\}. \tag{3.18}$$

Furthermore, if these two conditions are fulfilled the limit is independent from the exact choice of sequence in this class. But it remains to be shown that indeed (3.8) is the adiabatic limit. So in the following calculation we assume that $\check{g}$ has the desired support, and later we will take a special choice of sequence g_a, namely one which scales with a.

Part 3:

Now, with a variable transformation and performing one integral to get rid of the δ-function, (3.16) becomes

$$\begin{aligned}(-1)^n(2\pi)^2 \int \prod_{i=0}^{n} \mathrm{d}^4k_i \; \check{f}(k_0)\check{h}(-k_n) \prod_{l=1}^{n} \check{g}(k_l - k_{l-1}) \\ \times \sum_{m=0}^{n} \hat{\Delta}_R(k_0)\ldots\hat{\Delta}_R(k_{m-1})\hat{\Delta}_+(k_m)\hat{\Delta}_A(k_{m+1})\ldots\hat{\Delta}_A(k_n).\end{aligned} \tag{3.19}$$

We use (A.2)

$$\begin{aligned}\hat{\Delta}_{R/A}(k_j) &= \frac{1}{(2\pi)^2}\frac{1}{2\omega_j}\left(\frac{1}{k_{0,j}+\omega_j \pm i\epsilon} - \frac{1}{k_{0,j}-\omega_j \pm i\epsilon}\right) \\ &= -\frac{1}{(2\pi)^2}\frac{1}{2\omega_j}\frac{1}{k_{0,j}-\omega_j \pm i\epsilon}\frac{2\omega_j}{k_{0,j}+\omega_j \pm i\epsilon}\end{aligned} \tag{3.20}$$

with $\omega_j = \omega_{\mathbf{k}_j}$. In each summand the momentum k_m is on the mass shell due to the $\hat{\Delta}_+(k_m)$. With the supposed support property of $\check{g}$ the integrand vanishes if some k_j lie on the negative mass shell. So we can actually drop the very last ϵ in (3.20). Define

$$T(k) := \frac{2\omega_{\mathbf{k}}}{k^0 + \omega_{\mathbf{k}}}.$$

Putting this together, (3.19) equals

$$\frac{1}{(2\pi)^{2n-1}} \int \prod_{i=0}^{n} \frac{\mathrm{d}^4 k_i}{2\omega_i} \check{f}(k_0)\check{h}(-k_n) \prod_{l=1}^{n} \check{g}(k_l - k_{l-1}) \sum_{m=0}^{n} \Bigg[\delta(k_{0,m} - \omega_m) \\ \cdot \prod_{j=0}^{m-1} \left(\frac{1}{k_{0,j} - \omega_j + i\epsilon} T(k_j) \right) \prod_{j=m+1}^{n} \left(\frac{1}{k_{0,j} - \omega_j - i\epsilon} T(k_j) \right) \Bigg]. \tag{3.21}$$

Now we make a variable transformation to $x_j = k_{0,j} - \omega_j$ and expand $\check{f}, \check{h}$ and T around $x_j = 0$ to nth order into a Taylor series, i.e.,

$$\check{f}(\omega_{\mathbf{k}} + x, \mathbf{k}) = \sum_{k=0}^{n} \frac{x^k}{k!} \partial_0^k \check{f}(\omega_{\mathbf{k}}, \mathbf{k}) + \frac{x^{n+1}}{(n+1)!} \tilde{f}(x, \mathbf{k}),$$

where $\tilde{f}$ is in $\mathcal{S}$, and similarly for $\check{h}(-k)$ and $T(k)$.

We insert these expansions into (3.21) and split the whole expression into a sum of $A(g) + B(g)$, where $A(g)$ contains all terms without $\tilde{f}$, $\tilde{h}$ or $\tilde{T}$, thus

$$A(g) = \frac{1}{(2\pi)^{2n-1}} \int \prod_{i=0}^{n} \frac{\mathrm{d}^3 k_i}{2\omega_i} \prod_{j=0}^{n} \mathrm{d}x_j \prod_{l=1}^{n} \check{g}(\omega_l - \omega_{l-1} + x_l - x_{l-1}, \mathbf{k}_l - \mathbf{k}_{l-1}) \\ \sum_{L_0=0}^{n} \frac{{x_0}^{L_0}}{L_0!} \partial_0^{L_0} \check{f}(k_{+,0}) \cdot \Bigg(\sum_{m=0}^{n} \delta(x_m) \prod_{j=0}^{m-1} \frac{1}{x_j + i\epsilon} \sum_{l_j=0}^{n} \frac{{x_j}^{l_j}}{l_j!} \partial_0^{l_j} T(k_{+,j}) \\ \prod_{j=m+1}^{n} \frac{1}{x_j - i\epsilon} \sum_{l_j=0}^{n} \frac{{x_j}^{l_j}}{l_j!} \partial_0^{l_j} T(k_{+,j}) \Bigg) \cdot \sum_{L_n=0}^{n} \frac{{x_n}^{L_n}}{L_n!} (-\partial_0)^{L_n} \check{h}(-k_{+,n}). \tag{3.22}$$

Part 4:
First we show that $B(g_a)$ vanishes for a special choice of sequence $g_a \to 1$. We choose a sequence which scales with a, i.e., $g_a(x) = g\left(\frac{x}{a}\right)$ for some $g \in \mathcal{S}$ with $g(0) = 1$.[2] Then $\check{g}_a(k) = a^4 \check{g}(ak)$. Furthermore we demand that the support of $\check{g}$ is a subset of $B_0\left(\frac{m}{n}\right) \subset V_n$ (this m being the mass). The terms in $B(g_a)$ all contain a factor

$$\delta(x_m) \prod_{l=1}^{n} \check{g}_a(\omega_l - \omega_{l-1} + x_l - x_{l-1}, \mathbf{k}_l - \mathbf{k}_{l-1}).$$

Thus, for the chosen sequence the integrand has support in $|\mathbf{k}_l - \mathbf{k}_{l-1}| < \frac{m}{n}\frac{1}{a}$ and, since $x_m = 0$ and $|\omega_l - \omega_{l-1}| \leq |\mathbf{k}_l - \mathbf{k}_{l-1}|$, also in $x_j < C\frac{1}{a}$ for some $C > 0$ and all j.

[2] It is easy to see that for such a sequence $\check{g}_a \to (2\pi)^2 \delta^{(4)}$ in $\mathcal{O}'_C(\mathbb{M})$

We consider the different terms in $B(g_a)$, coming from the expansion of $\check{f}, \check{h}$ and T and from splitting the sum over m. We integrate out x_m (thus, terms with factors x_m from the expansion already disappear), renumber the remaining x_j, and perform a transformation of variables to $\tilde{\mathbf{k}}_j = \mathbf{k}_j - \mathbf{k}_{j-1}$ for $j > 0$ ($\mathbf{k}_0$ remains). Then the terms have the form

$$\int \prod_{i=1}^{n} \mathrm{d}^3\tilde{k}_i \prod_{j=1}^{n} \mathrm{d}x_j \prod_{j=1}^{m} \frac{1}{x_j + i\epsilon} x^{b_j} \prod_{j=m+1}^{n} \frac{1}{x_j - i\epsilon} x^{b_j} U_a(x_1, \ldots, x_n, \tilde{\mathbf{k}}_1, \ldots, \tilde{\mathbf{k}}_n), \tag{3.23}$$

where U_a already includes the integration d^3k_0. This expression is always finite since $f \in \mathcal{S}$. Furthermore the following properties hold:

- At least one $b_j = n+1$, since every term in $B(g)$ contains $\tilde{f}$, $\tilde{h}$ or $\tilde{T}$.
- U_a is $\mathcal{C}^\infty$ and its support is contained in

$$\left\{(x_1, \ldots, x_n, \tilde{\mathbf{k}}_1, \ldots, \tilde{\mathbf{k}}_n) \middle| |x_j| < C/a, |\tilde{\mathbf{k}}_j| < C/a\right\}$$

  for some $C > 0$.
- The supremum of $|U_a|$ is bounded by $C_{0,0}a^{4n}$ for some $C_{0,0} > 0$. Every derivative with respect to some x_j gives an additional factor of a coming from the derivatives of $\check{g}_a$. Each multiplication with some x_j yields a factor $1/a$, since U_a has bounded support. In other words, for all n-multi-indices α, β there exists a constant $C_{\alpha,\beta} > 0$ with

$$\sup \left| x^\alpha D_x^\beta U_a \right| < C_{\alpha,\beta} a^{4n+|\beta|-|\alpha|}.$$

We have to prove that all terms vanish for $a \to \infty$. We show this only for the terms (3.23), where only one $b_j = n+1$ and the other b's equal 0. The remaining cases are similar. We use

$$\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x} \tag{3.24}$$

as distributions. Here $\mathcal{P}$ denotes the principal value of the fraction, i.e., the distribution

$$f \to \int \mathrm{d}x \mathcal{P}\frac{1}{x} f(x) = \frac{1}{2} \int \mathrm{d}x \frac{1}{x} \left(f(x) - f(-x)\right).$$

To make the calculation more comprehensible we demonstrate how we are dealing with the principal values in one dimension. Let $f(x) = 0$ for $x > X$. Then

$$\begin{aligned}\int \mathrm{d}x\mathcal{P}\frac{1}{x}f(x) \leq& \frac{1}{2}\left|\int_{-X}^{X}\mathrm{d}x\frac{1}{x}\left(f(x)-f(-x)\right)\right| &=& \frac{1}{2}\left|\int_{-X}^{X}\mathrm{d}x\frac{1}{x}\int_{-x}^{x}\mathrm{d}y\partial f(y)\right| \\ \leq& \frac{1}{2}\int_{-X}^{X}\mathrm{d}x\frac{1}{x}2x\sup_y|\partial f(y)| &=& 2X\sup_y|\partial f(y)|.\end{aligned}$$

Due to the $\check{g}_a$'s U_a will be of compact support. The terms we get using (3.24), omitting factors of $\pm i\pi$, are of the following form,[3]

$$\int\prod_{i=1}^{n}\mathrm{d}^3\tilde{k}_i\prod_{j=1}^{n}\mathrm{d}x_j\ x_1^n\prod_{j=2}^{r}\mathcal{P}\frac{1}{x_j}\prod_{j=r+1}^{n}\delta(x_j)U_a(x_1,\ldots,x_n,\tilde{\mathbf{k}}_1,\ldots,\tilde{\mathbf{k}}_n)$$

for some r after another relabelling of the x_j and of the arguments of U_a. Its absolut value can be estimated by

$$\begin{aligned}&\left|\int\prod_{i=1}^{n}\mathrm{d}^3\tilde{k}_i\prod_{j=1}^{r}\mathrm{d}x_jx_1^n\prod_{j=2}^{r}\mathcal{P}\frac{1}{x_j}U_a(x_1,\ldots,x_r,0,\ldots,0,\tilde{\mathbf{k}}_1,\ldots,\tilde{\mathbf{k}}_n)\right| \\ =&\Bigg|\left(\frac{1}{2^{r-1}}\prod_{j=2}^{r}\int_{-\frac{C}{a}}^{\frac{C}{a}}\mathrm{d}x_j\right)\prod_{i=1}^{n}\mathrm{d}^3\tilde{k}_i\mathrm{d}x_1\left(\prod_{j=2}^{r}\frac{1}{x_j}\int_{-x_j}^{x_j}\mathrm{d}y_j\right) \\ &\quad x_1^nD_x^\beta U_a(x_1,y_2\ldots,y_r,0,\ldots,0,\tilde{\mathbf{k}}_1,\ldots,\tilde{\mathbf{k}}_n)\Bigg| \\ \leq&\left(\frac{1}{2^{r-1}}\prod_{j=2}^{r}\int_{-\frac{C}{a}}^{\frac{C}{a}}\mathrm{d}x_j\right)\left(\frac{C}{a}\right)^{1+3n}2^{r-1}C_{\alpha,\beta}a^{4n-|\alpha|+|\beta|}\end{aligned}$$

where $\alpha = (n, 0, \ldots, 0)$ and $\beta = (0, 1, \ldots, 1, 0\ldots, 0)$ with $|\beta| = r - 1$. The last integrations introduce a factor $2\frac{C}{a}$ each. At the end we have

$$2^{r-1}C_{\alpha,\beta}\left(\frac{C}{a}\right)^{1+3n+r-1}a^{4n-n+r-1}\xrightarrow[a\to\infty]{}0.$$

This shows that the limit of $B(g_a)$ vanishes for this choice of a scaling sequence $\{g_a\}$.

Part 5:

It remains to treat $A(g)$, given by (3.22). We use

$$\partial_0^lT(k_+) = l!(-2\omega_{\mathbf{k}})^{-l}.$$

[3] Of course we have $\frac{x^{n+1}}{x\pm i\epsilon} = x^n$ as a distribution.

We get $x_m = 0$ from the δ-function, and for the remaining variables x_j we perform variable transformation depending on m to coordinates y_j with

$$y_1 = x_0 - x_1; \quad \dots \quad y_{m-1} = x_{m-2} - x_{m-1}; \quad y_m = x_{m-1};$$
$$y_{m+1} = -x_{m+1}; \quad y_{m+2} = x_{m+1} - x_{m+2}; \quad \dots \quad y_n = x_{n-1} - x_n.$$

Thus we have

$$x_j = \begin{cases} \sum_{t=j+1}^{m} y_t & \text{for } j < m, \\ -\sum_{t=m+1}^{j} y_t & \text{for } j > m, \end{cases}$$

and get

$$\begin{aligned} A(g) = {}& \frac{1}{(2\pi)^{2n-1}} \int \prod_{i=0}^{n} \frac{\mathrm{d}^3 k_i}{2\omega_i} \prod_{j=1}^{n} \mathrm{d}y_j \prod_{l=1}^{n} \check{g}(\omega_l - \omega_{l-1} - y_l, \mathbf{k}_l - \mathbf{k}_{l-1}) \\ & \cdot \sum_{m=0}^{n} \Bigg[(-1)^{n-m} \prod_{r=0}^{m-1} \frac{1}{\sum_{t=r+1}^{m} y_t + i\epsilon} \prod_{r=m+1}^{n} \frac{1}{\sum_{t=m+1}^{r} y_t + i\epsilon} \\ & \cdot \left(\sum_{L_0=0}^{n} \frac{\left(-\sum_{t=1}^{m} y_t\right)^{L_0}}{L_0!} (-\partial_0)^{L_0} \check{f}(k_{+,0}) \right) \left(\prod_{r=0}^{m-1} \sum_{l_r=0}^{n} \frac{\left(-\sum_{t=r+1}^{m} y_t\right)^{l_r}}{(2\omega_{r-1})^{l_r}} \right) \\ & \cdot \left(\prod_{r=m+1}^{n} \sum_{l_r=0}^{n} \frac{\left(\sum_{t=m+1}^{r} y_t\right)^{l_r}}{(2\omega_r)^{l_r}} \right) \left(\sum_{L_n=0}^{n} \frac{\left(\sum_{t=m+1}^{n} y_t\right)^{L_n}}{L_n!} \partial_0^{L_n} \check{h}(-k_{+,n}) \right) \Bigg]. \end{aligned} \tag{3.25}$$

To show that this gives the wanted result we need the following lemma:

Lemma 3.2.4. *For $n \in \mathbb{N}$ and $l_r \in \mathbb{N}_0, r = 0, \dots, n$,*

$$\sum_{m=0}^{n} (-1)^m \prod_{r=0}^{m-1} \frac{(-\sum_{t=r+1}^{m} y_t)^{l_r}}{\sum_{t=r+1}^{m} y_t + i\epsilon} \cdot \delta_0^{l_m} \cdot \prod_{r=m+1}^{n} \frac{(\sum_{t=m+1}^{r} y_t)^{l_r}}{\sum_{t=m+1}^{r} y_t + i\epsilon} = P_{l_0,\dots,l_n}(y_1, \dots, y_n), \tag{3.26}$$

where we have with $a := \sum_{t=0}^{n} l_t$:

I *: If $a < n$, then $P_{l_0,\dots,l_n} = 0$.*

II *: If $a = n$, then $P_{l_0,\dots,l_n} = 1$.*

III *: If $a > n$, then $P_{l_0,\dots,l_n}(y_1, \dots, y_n)$ is a homogeneous polynomial of degree $a - n$.*

We will give a proof of this lemma at the end of this section. In (3.25) we will pull all sums over $L_0, L_n, l_0, l_1, \dots, l_n$ to the front of the sum over m. Since there actually is no sum over l_m, we will have to introduce a factor $\delta_0^{l_m}$. Now we use Lemma 3.2.4 and get a polynomial $P_{L_0+l_0,l_1,\dots,l_{n-1},L_n+l_n}(y_1,\dots,y_n)$. What we arrive at is a $\mathcal{C}^\infty$-function of the variables $\mathbf{k}_i$ and y_j. We choose a sequence $\check{g}_a \to (2\pi)^2\delta$ with compact support. All integrations over y_j and $\mathbf{k}_i$ but one can be carried out. Because of the properties of P given in Lemma 3.2.4 only terms survive where

$$L_0 + \sum_{t=0}^{n} l_t + L_n = n.$$

Thus, we get as the limit of $A(g_a)$:

$$2\pi \sum_{L_0+\sum_{t=0}^n l_t+L_n=n} \int \mathrm{d}^3k\, \frac{(-1)^n}{(2\omega_{\mathbf{k}})^{2n+1-L_0-L_n}} \frac{1}{L_0!}\frac{1}{L_n!}(-\partial_0)^{L_0}\check{f}(k_+)\partial_0^{L_n}\check{h}(-k_+).$$

where the sum extends over all nonnegative integers L_i and l_j with given sum. The sums over the l_j's give a combinatorial factor and we can write

$$2\pi \sum_{a+b\le n} \int \mathrm{d}^3k\, \frac{(-1)^n}{(2\omega_{\mathbf{k}})^{2n+1-a-b}\cdot a!\cdot b!}\binom{2n-a-b}{n-a-b}(-\partial_0)^a\check{f}(k_+)\partial_0^b\check{h}(-k_+). \tag{3.27}$$

This quantity will be called $A_{\mathrm{adlim}}(n)$.

We show that $A_{\mathrm{adlim}}(n)$ equals (3.8) by induction.[4] The case $n=1$ is easily checked. Now we calculate $1/(n+1)\partial_{m^2}A_{\mathrm{adlim}}(n)$, which should yield $A_{\mathrm{adlim}}(n+1)$. We sort the terms of $1/(n+1)\partial_{m^2}A_{\mathrm{adlim}}(n)$ by $(-\partial_0)^c\check{f}\partial_0^d\check{h}$. There are three contributions, namely from $(a=c, b=d), (a=c-1, b=d)$ and $(a=c, b=d-1)$.[5] Their sum is

$$\begin{aligned}
2\pi \int \mathrm{d}^3k (-1)^n (-\partial_0)^c\check{f}(k_+)\partial_0^d\check{h}(-k_+)\frac{1}{n+1}& \\
\cdot\Bigg(\frac{1}{c!}\frac{1}{d!}\frac{2(-2n+1-c-d)}{(2\omega_{\mathbf{k}})^{2n+1-c-d+2}}\binom{2n-c-d}{n-c-d}& \\
-\frac{1}{(c-1)!}\frac{1}{d!}\frac{1}{(2\omega_{\mathbf{k}})^{2n+2-c-d}}\binom{2n+1-c-d}{n+1-c-d}\frac{1}{2\omega_{\mathbf{k}}}& \\
-\frac{1}{c!}\frac{1}{(d-1)!}\frac{1}{(2\omega_{\mathbf{k}})^{2n+2-c-d}}\binom{2n+1-c-d}{n+1-c-d}\frac{1}{2\omega_{\mathbf{k}}}\Bigg)&.
\end{aligned}$$

[4] The corresponding part in [14] contains some errors.

[5] Actually, this is only true if neither c nor d equals 0 or $n+1$. These cases can easily be checked separately.

Now we use

$$\binom{2n-c-d}{n-c-d}\frac{2(2n+1-c-d)}{n+1}+\binom{2n+1-c-d}{n+1-c-d}\frac{c+d}{n+1} \\ =\binom{2(n+1)-c-d}{n+1-c-d}$$

and get

$$2\pi\int \mathrm{d}^3k(-1)^{n+1}\frac{1}{n+1} \\ \cdot\frac{1}{c!}\frac{1}{d!}\frac{1}{(2\omega_{\mathbf{k}})^{2(n+1)+1-c-d}}\binom{2(n+1)-c-d}{n+1-c-d} \\ \cdot(-\partial_0)^c\check{f}(k_+)\partial_0^d\check{h}(-k_+).$$

The sum of these terms is equal to $A_{\mathrm{adlim}}(n+1)$, what had to be shown.
Part 6:
So far, we have shown that the correct adiabatic limit in nth order is obtained, if the sequence of functions $\{g_a\}$ fulfills the two properties

1. $\check{g}_a \xrightarrow[a\to\infty]{} (2\pi)^2\delta$ in $\mathcal{O}'_C(\mathbb{M})$,
2. $\mathrm{supp}\check{g}_a \subset V_n$.

Sequences with these two properties will be called sequences of class I,n. (The n will be omitted in most cases.) The second property is rather restrictive, as $\check{g}_a$ cannot be analytic, and therefore g_a cannot have compact support in position space.[6] Functions of compact support are needed if one wants to interpret (3.11) as the restriction of the interaction to a finite region.

We will now enlarge the class of sequences, which give the correct adiabatic limit. These will be called of class II,n. Suppose that a given sequence g_a of functions in $\mathcal{S}(\mathbb{M})$ can be decomposed into

$$g_a = g_a^0 + g_a^1, \tag{3.28}$$

where $\{g_a^0\}$ is of class I,n and $\{g_a^1\}$ has the property that

$$\prod_{t=1}^{r}\check{g}_a^1(k_t)\cdot\prod_{t=r+1}^{n}\check{g}_a^0(k_t)\xrightarrow[a\to\infty]{} 0 \text{ as a function in } \mathcal{S}(\mathbb{M}^n) \text{ for } 1\le r\le n. \tag{3.29}$$

[6]Instead of V_n we could have restricted the support of $\hat{g}_a$ to any other convex subset of $\frac{1}{n}R_n$, but the noncompactness of the support of g_a remains the same.

We observe that $\hat{F}_R(p_1, p_2; k_1, \ldots, k_n)$ is a distribution in $\mathcal{S}'(\mathbb{M}^{n+2})$. This will be needed in order to show that the correct adiabatic limit is obtained. We compute

$$\begin{aligned}
&\int \mathrm{d}^4p_1 \mathrm{d}^4p_2 \prod_{t=1}^{r} \mathrm{d}^4k_t \hat{F}_R(p_1, p_2, k_1, \ldots, k_n)\check{f}(p_1)\check{h}(p_2) \prod_{r=1}^{n} (\check{g}_a^0(k_r) + \check{g}_a^1(k_r)) \\
= &\int \mathrm{d}^4p_1 \mathrm{d}^4p_2 \prod_{t=1}^{r} \mathrm{d}^4k_t \hat{F}_R(p_1, p_2, k_1, \ldots, k_n)\check{f}(p_1)\check{h}(p_2) \prod_{r=1}^{n} \check{g}_a^0(k_r) \qquad (3.30)\\
&+ \int \mathrm{d}^4p_1 \mathrm{d}^4p_2 \prod_{t=1}^{r} \mathrm{d}^4k_t \hat{F}_R(p_1, p_2, k_1, \ldots, k_n)\check{f}(p_1)\check{h}(p_2)\Gamma_a(k_1, \ldots, k_n),
\end{aligned}$$

where Γ_a is a sum of $2^n - 1$ terms of products of $\check{g}_a^0$'s and $\check{g}_a^1$'s, where each has at least one factor of $\check{g}_a^1$. From (3.29) we deduce that Γ_a approaches 0 in $\mathcal{S}$, so the last line in (3.30) vanishes in the adiabatic limit and the remaining term gives the desired result as $\{g_a^0\}$ is of class I.

An example for a sequence of class II is easily constructed if we take an arbitrary function $g \in \mathcal{S}$ with $g(0) = 1$ and scale it, i. e. $g_a(x) := g(x/a)$. Then $\check{g}$ is normalized ($\int \mathrm{d}^4k\check{g}(k) = g(0) = 1$) and we have $\check{g}_a(k) = a^4\check{g}(ak)$. To prove that $\{g_a\}$ is of class II, we take a cutoff function $b \in \mathcal{S}$ with

$$b(k) = \begin{cases} 0 & \text{if } |k|^2 > 4m^2/n^2, \\ 1 & \text{if } |k|^2 < m^2/n^2, \end{cases}$$

and define

$$\begin{aligned}
\check{g}_a^0(k) &= b(k) \cdot \check{g}_a(k), \\
\check{g}_a^1(k) &= (1 - b(k)) \cdot \check{g}_a(k).
\end{aligned}$$

The sequence $\{g_a^0\}$ is clearly of class I. It remains to show that (3.29) holds. For this we first show that the growth of $\sup_k |k^\alpha D^\beta \check{g}_a^0(k)|$ is polynomially bounded in a (for all multi-indices α, β):

$$\begin{aligned}
\sup_k |k^\alpha D^\beta b(k) a^4 \check{g}(ak)| &\le a^4 |\beta|! \sum_{|\beta_{1,2}| \le |\beta|} \sup_k |k^\alpha D^{\beta_1} b(k)||D^{\beta_2}\check{g}(ak)| \\
&\le a^{4+|\beta|} |\beta|! \sum_{|\beta_{1,2}| \le |\beta|} \sup_k |k^\alpha D^{\beta_1} b(k)| \sup_{k'} |D^{\beta_2}\check{g}(k')|.
\end{aligned}$$

The summmands in the last line are finite since $b, \check{g} \in \mathcal{S}$. On the other hand

$\sup_k |k^\alpha D^\beta \check{g}_a^1(k)|$ falls off faster than any polynomial in a:

$$\begin{aligned}
&a^z \sup_k |k^\alpha D^\beta (1-b(k)) a^4 \check{g}(ak)| \\
\leq &a^{z+4}|\beta|! \sum_{|\beta_{1,2}|\leq|\beta|} \sup_{|k|\geq m/n} |D^{\beta_1}(1-b(k))||k^\alpha D^{\beta_2}\check{g}(ak)| \\
\leq &a^{z+4+|\beta|-|\alpha|}|\beta|! \sum_{|\beta_{1,2}|\leq|\beta|} \sup_{k'} |D^{\beta_1}(1-b(k'))| \sup_{|k|\geq m/n} |(ak)^\alpha (D^{\beta_2}\check{g})(ak)| \\
\leq &a^{z+4+|\beta|-|\alpha|}|\beta|! \left(\sum_{|\beta_1|\leq|\beta|} \sup_{k'} |D^{\beta_1} b(k')| + 1 \right) \cdot \sum_{|\beta_2|\leq|\beta|} \sup_{|k|\geq am/n} |s_{\alpha,\beta_2}(k)|,
\end{aligned}$$

where we have used that $1-b(k) = 0$ for $|k| < m/n$ and $s_{\alpha,\beta_2}(k) := k^\alpha D^{\beta_2}\check{g}(k)$ is a function in $\mathcal{S}$. As the supremum of this function is taken outside a ball of radius proportional to a, this falls off faster than $1/a^{z+4+|\beta|-|\alpha|}$ so the above approaches 0 with $a \to \infty$. To show that (3.29) is indeed fulfilled we have to establish that

$$\sup_{k_1,\ldots,k_n} |\prod_{t=1}^{r} k_t^{\alpha_t} D_t^{\beta_t} \check{g}_a^1(k_t) \cdot \prod_{t=r+1}^{n} k_t^{\alpha_t} D_t^{\beta_t} \check{g}_a^0(k_t)| \xrightarrow[a\to\infty]{} 0$$

for all multi-indices α_j, β_j. This is now obvious, since the growth of the last product is polynomially bounded and the first product falls off faster than any polynomial (and we have $r \geq 1$). So scaling functions in $\mathcal{S}$ yields indeed a sequence of class II, and among these there are also some with compact support in position space.

Part 7:

To put everything together we have shown in this section:

Theorem 3.2.5. *The IR cutoff*

$$\left(\Box + m^2\right) \Phi(x) = -\mu g(x)\Phi(x).$$

gives for the two-point function of order n,

$$\sum_{m=0}^{n} \langle \Phi_{m,g}(f)\Phi_{n-m,g}(h),$$

the correct adiabatic limit, i.e.,

$$\frac{(2\pi)^2}{n!} \int \mathrm{d}^4k \check{f}(k)\check{h}(-k)\partial_{m^2}^n \hat{\Delta}_+^{(m^2)}(k),$$

if the sequence of the cutoff functions $\{g_a\}$ is of class II,n.

It was important to consider the sum of all contributions to the two-point function of the same order when carrying out the adiabatic limit, as we already saw explicitly for the first order.

Now, we give the postponed proof for Lemma 3.2.4. It is the same proof already given in [14]. As we want to work by induction we slightly enhance the lemma to:

Lemma 3.2.6. *For $n \in \mathbb{N}$ and $l_r \in \mathbb{N}_0, r = 0, \ldots, n$*

$$\sum_{m=0}^{n}(-1)^m \prod_{r=0}^{m-1} \frac{(-\sum_{t=r+1}^{m} y_t)^{l_r}}{\sum_{t=r+1}^{m} y_t + i\epsilon} \cdot \delta_0^{l_m} \cdot \prod_{r=m+1}^{n} \frac{(\sum_{t=m+1}^{r} y_t)^{l_r}}{\sum_{t=m+1}^{r} y_t + i\epsilon} = P_{l_0,\ldots,l_n}(y_1, \ldots, y_n), \tag{3.31}$$

where we have with $a := \sum_{t=0}^{n} l_t$:

I *: If $a < n$, then $P_{l_0,\ldots,l_n} = 0$.*

II *: If $a = n$, then $P_{l_0,\ldots,l_n} = 1$.*

III *: If $a > n$, then $P_{l_0,\ldots,l_n}(y_1, \ldots, y_n)$ is a homogeneous polynomial of degree $a - n$ and further:*

> **IIIa** *: If $l_n = 0$, the term with highest power in y_n is $(-y_n)^{a-n}$.*
>
> **IIIb** *: If $n = 2$ and $l_0 = 0$, the term with highest power in y_1 is y_1^{a-n}.*

Proof. The case $n = 1$ is almost trivial, where we have of course $\frac{x_1}{x_1+i\epsilon} = 1$ as a distribution. For $n = 2$ the cases where (l_0, l_1, l_2) is equal to (a permutation of) $(1, 0, 0), (1, 1, 0), (\geq 2, \geq 1, 0)$ or $(\geq 1, \geq 1, \geq 1)$ are easily checked. For the case $(l_0, l_1, l_2) = (0, 0, 0)$ we compute

$$\begin{aligned}\frac{1}{y_1 + i\epsilon}\frac{1}{y_1 + y_2 + i\epsilon} - \frac{1}{y_2 + i\epsilon}\frac{1}{y_1 + i\epsilon} + \frac{1}{y_1 + y_2 + i\epsilon}\frac{1}{y_2 + i\epsilon}& \\ = \frac{1}{y_1 + y_2 + i\epsilon}\frac{y_1 + y_2}{y_1 y_2 + i(y_1 + y_2)\epsilon} - \frac{1}{y_1 y_2 + i(y_1 + y_2)\epsilon} &= 0.\end{aligned}$$

The remaining cases are permutations of $(b, 0, 0)$ with $b \geq 2$. We show it here for $l_1 = b$:

$$\begin{aligned}\frac{(y_1)^{b-1}}{y_1 + y_2 + i\epsilon} - \frac{(-y_2)^{b-1}}{y_1 + y_2 + i\epsilon} &= \frac{1}{y_1 + y_2 + i\epsilon}(y_1 + y_2) \sum_{k=0}^{b-2} y_1^k (-y_2)^{b-2-k} \\ &= \sum_{k=0}^{b-2} y_1^k (-y_2)^{b-2-k}.\end{aligned} \tag{3.32}$$

For the cases l_0 or $l_2 = b$ this can also be done and the parts IIIa and IIIb of the lemma are easily checked explicitly.

Now we want to work by induction. For this, we assume $n \geq 3$ and that the lemma has been proven for all lower orders. From the sum (3.31) we split off the terms with $m = n$,

$$(-1)^n \delta_0^{l_n} \prod_{r=1}^{n} \frac{(-\sum_{t=r+1}^{n} y_t)^{l_r}}{\sum_{t=r+1}^{n} y_t + i\epsilon} =: A,$$

and with $m = n-1$,

$$(-1)^{n-1} \delta_0^{l_{n-1}} \prod_{r=1}^{n-1} \frac{(-\sum_{t=r+1}^{n-1} y_t)^{l_r}}{\sum_{t=r+1}^{n-1} y_t + i\epsilon} \cdot \frac{y_n^{l_n}}{y_n + i\epsilon} =: B.$$

The remaining summands each have a factor

$$\begin{aligned} &\frac{(\sum_{t=m+1}^{n-1} y_t)^{l_{n-1}}}{\sum_{t=m+1}^{n-1} y_t + i\epsilon} \frac{(\sum_{t=m+1}^{n} y_t)^{l_n}}{\sum_{t=m+1}^{n} y_t + i\epsilon} \\ &= \delta_0^{l_{n-1}} \frac{1}{\sum_{t=m+1}^{n-1} y_t} \frac{y_n^{l_n}}{y_n + i\epsilon} - \delta_0^{l_n} \frac{1}{\sum_{t=m+1}^{n} y_t} \frac{(-y_n)^{l_{n-1}}}{y_n + i\epsilon} + P_{0,l_{n-1},l_n}(\sum_{t=m+1}^{n-1} y_t, y_n), \end{aligned} \tag{3.33}$$

where we used the induction hypothesis for $n = 2$. If we reinsert these terms into the remaining sum, we can split this into three parts, which we label according to the order in (3.33) by C, D and E. Now we can combine $A + D$ to

$$\begin{aligned} &- \delta_0^{l_n} \frac{(-y_n)^{l_{n-1}}}{y_n + i\epsilon} \sum_{m=0}^{n-1} (-1)^m \prod_{r=0}^{m-1} \frac{(-\sum_{t=r+1}^{m} y'_t)^{l'_r}}{\sum_{t=r+1}^{m} y'_t + i\epsilon} \cdot \delta_0^{l'_m} \cdot \prod_{r=m+1}^{n-1} \frac{(\sum_{t=m+1}^{r} y'_t)^{l'_r}}{\sum_{t=m+1}^{r} y'_t + i\epsilon} \\ &\qquad = -\delta_0^{l_n} \frac{(-y_n)^{l_{n-1}}}{y_n + i\epsilon} P_{l_0,\dots,l_{n-2},0}(y_1, \dots, y_{n-1} + y_n) \end{aligned} \tag{3.34}$$

with $l'_i = l_i$ and $y'_i = y_i$ for $i \leq n-2$ and $l'_{n-1} = 0$ and $x'_{n-1} = x_{n-1} + x_n$. The terms $B + C$ give

$$\begin{aligned} &\delta_0^{l_{n-1}} \frac{y_n^{l_n}}{y_n + i\epsilon} \sum_{m=0}^{n-1} (-1)^m \prod_{r=0}^{m-1} \frac{(-\sum_{t=r+1}^{m} y_t)^{l_r}}{\sum_{t=r+1}^{m} y_t + i\epsilon} \cdot \delta_0^{l_m} \cdot \prod_{r=m+1}^{n-1} \frac{(\sum_{t=m+1}^{r} y_t)^{l_r}}{\sum_{t=m+1}^{r} y_t + i\epsilon} \\ &\qquad = \delta_0^{l_{n-1}} \frac{y_n^{l_n}}{y_n + i\epsilon} P_{l_0,\dots,l_{n-2},0}(y_1, \dots, y_{n-1}). \end{aligned} \tag{3.35}$$

Now we have a closer look at

$$E=\sum_{m=0}^{n-2}(-1)^m\prod_{r=0}^{m-1}\frac{(-\sum_{t=r+1}^{m}y_t)^{l_r}}{\sum_{t=r+1}^{m}y_t+i\epsilon}\cdot\delta_0^{l_m}\cdot\prod_{r=m+1}^{n-2}\frac{(\sum_{t=m+1}^{r}y_t)^{l_r}}{\sum_{t=m+1}^{r}y_t+i\epsilon}\cdot P_{0,l_{n-1},l_n}(\sum_{t=m+1}^{n-1}y_t,y_n).$$

The last polynomial gives 0 if $l_{n-1}+l_n<2$. Otherwise, by IIIb, the term with highest power in $\sum_{t=m+1}^{n-1}y_t$ from P_{0,l_{n-1},l_n} is $(\sum_{t=m+1}^{n-1}y_t)^{l_{n-1}+l_n-2}$, and we can write it as

$$P_{0,l_{n-1},l_n}(\sum_{t=m+1}^{n-1}y_t,y_n)=\sum_{\alpha=0}^{l_{n-1}+l_n-2}(\sum_{t=m+1}^{n-2}y_t)^{l_{n-1}+l_n-2-\alpha}\tilde{P}_\alpha(y_{n-1},y_n),$$

where $\tilde{P}_\alpha(y_{n-1},y_n)$ is a homogeneous polynomial of degree α and $\tilde{P}_0=1$. If $l_n=0$, we can deduce from the explicit formula (3.32) that in each $\tilde{P}_\alpha(y_{n-1},y_n)$ we have a term $(-y_n)^\alpha$. Now in E we pull the sum over α to the front and for each summand use the induction hypothesis for $n-2$ to get

$$E=\sum_{\alpha=0}^{l_{n-1}+l_n-2}P_{l_0,\dots,l_{n-3},l_{n-2}+l_{n-1}+l_n-2-\alpha}(y_1,\dots,y_{n-2})\cdot\tilde{P}_\alpha(y_{n-1},y_n).\tag{3.36}$$

So E is a homogeneous polynomial of degree

$$\sum_{r=0}^{n}l_r-2-(n-2)-\alpha+\alpha=\sum_{r=0}^{n}l_r-n.$$

We have to check the following cases:

- $l_{n-1}=l_n=0$: $E=0$ and

$$A+D+B+C=\frac{1}{y_n+i\epsilon}\left[P_{l_0,\dots,l_{n-2},0}(y_1,\dots,y_{n-1})-P_{l_0,\dots,l_{n-2},0}(y_1,\dots,y_{n-1}+y_n)\right].$$

  These polynomials are of degree $\sum_{r=0}^{n}l_r-(n-1)$, if this is greater or equal to 0. If we expand the powers of $y_{n-1}+y_n$ of the second polynomial, we see that terms with no factor y_n vanish and from the remaining terms one factor is cancelled by the prefactor. So the remaining expression is of degree $\sum_{r=0}^{n}l_r-n$ and I to IIIa are easily checked.

- $l_{n-1} = 1, l_n = 0$: $E = B + C = 0$ and
$$A + D = \frac{y_n}{y_n + i\epsilon} P_{l_0,\dots,l_{n-2},0}(y_1, \dots, y_{n-1} + y_n).$$
This is of degree $\sum_{r=0}^n l_r - (n-1) = \sum_{r=0}^n l_r - n$. Again I to IIIa are easily checked.

- $l_{n-1} = 0, l_n = 1$: similar.

- $l_{n-1} \geq 2, l_n = 0$: $B+C = 0$. $A+D$ and E both vanish if $\sum_{r=0}^n l_r - n < 0$, so I is checked. Set $a' := \sum_{r=0}^{n-2} l_r$. To show II we assume $a' + l_{n-1} - n = 0$ from which $a' - n \leq -2$ follows. So from (3.34) we see that the polynomial in $A + D$ vanishes. In E only the term with $\alpha = 0$ gives a contribution, which is 1.
Now we want to show III and IIIa: We have $a' + l_{n-1} - n > 0$ and see that both $A + D$ and E are homogeneous polynomials of the right degree. We still have to show that not both are zero and they do not cancel each other. This is done by establishing IIIa. For that, we have to look at the cases:

    1. $a' - n < -2$: $A + D = 0$. The sum over α in E only goes to $\alpha = a' + l_{n-1} - n$ as for higher α the first polynomial in E vanishes. The term with highest degree in y_n comes from $\alpha = a' + l_{n-1} - n$ and is $(-y_n)^{a'+l_{n-1}-n}$.
    2. $a' - n = -2$: $A+D = 0$ and in E the term with highest $\alpha = l_{n-1} - 2$ gives just $(-y_n)^{a'+l_{n-1}-n}$. All other terms are of lower order in y_n.
    3. $a' - n > -2$: The highest degree of y_n in E is $l_{n-1} - 2 < a' + l_{n-1} - n$ whereas $A + D$ gives a term $(-y_n)^{l_{n-1}-1} \cdot (-y_n)^{a'-(n-1)} = (-y_n)^{a'+l_{n-1}-n}$.

- $l_{n-1} = 0, l_n \geq 2$: similar.

- $l_{n-1} \geq 1, l_n \geq 1$: $A + D = B + C = 0$, only E gives a contribution. I to III are again easily checked.

This completes the proof. □

3.2.2 Adiabatic limit on noncommutative spacetime

Now we look at the noncommutative case. Since the IR cutoff made by multiplying Δ_R by a function g already failed in the adiabatic limit on commutative spacetime, we do not consider this cutoff again.

The cutoff (3.11) can be viewed as coming from an interaction term

$$\mathcal{S}_{\text{Int}}(\Phi) = \frac{\mu}{2} \int \mathrm{d}x \left(g(x)\Phi^2(x) \right) , \tag{3.37}$$

If we take fields on $\mathbb{M}_{\text{nc}}$, there is no unique generalization. A straightforward way would be to simply take

$$\mathcal{S}_{\text{Int}}(\Phi) = \int \mathrm{d}q \; \mu 2 \operatorname{Tr} \left(g(q)\Phi(q)\Phi(q) \right) .$$

The limit $g_a(q) \to \mathbb{1}$ again corresponds to $\check{g}_a(k) \to (2\pi)^2\delta(k)$. Another possibility would be to take two $g(q)$ in the form of (2.14):

$$\mathcal{S}_{\text{Int}}(\Phi) = \int \mathrm{d}q \; \mu 2 \operatorname{Tr} \left(g(q)\Phi(q)g(q)\Phi(q) \right) .$$

This kind of cutoff is proposed in [45]. In the commutative limit this would correspond to taking g^2 in (3.37) instead of g. This would just be a relabelling and the adiabatic limit remains unchanged. These two possibilities take the cutoff as a multiplication of an element in $\mathbb{M}_{\text{nc}}$. A third one would be to take instead the pointwise product with a function $g \in \mathcal{S}$, compare (2.7).

First take the cutoffs by the algebra product. We look at both at once by considering a more generalized cutoff

$$\mathcal{S}_{\text{Int}}(\Phi) = \frac{\mu}{2} \operatorname{Tr} \left(g^{(1)}(q)\Phi(q)g^{(2)}(q)\Phi(q) \right) .$$

The $g^{(1)}$ and $g^{(2)}$ could be the same or one could even be $\mathbb{1}$ right from the beginning. Using the cyclicity of the trace we get the field equation

$$\left(\Box_q + m^2 \right) \Phi(q) = -\mu/2 \left(g^{(1)}(q)\Phi(q)g^{(2)}(q) + g^{(2)}(q)\Phi(q)g^{(1)}(q) \right) . \tag{3.38}$$

The solution at order m is

$$\begin{aligned} \Phi_{0,g}(q) =& \Phi_0(q) = \Phi_{\text{Free}}(q), \\ \Phi_{m,g}(q) =& -\frac{1}{2} \int \mathrm{d}^4 z \Delta_R(z) \Big(g^{(1)}(q-z)\Phi_{m-1,g}(q-z)g^{(2)}(q-z) \\ & + g^{(2)}(q-z)\Phi_{m-1,g}(q-z)g^{(1)}(q-z) \Big). \end{aligned}$$

Using the Weyl formula (2.2) this gives

$$\Phi_{m,g}(q) = \frac{(-1)^m}{(2\pi)^{2(m+1)}} \int \mathrm{d}^4p \prod_{j=1}^{m} (\mathrm{d}^4k_j \mathrm{d}^4l_j)\ \hat{\Phi}_0(p) e^{i(-p+\sum_{t=1}^{m}(k_t+l_t))q}$$
$$\cdot \prod_{r=1}^{m} \Bigg[\check{g}^{(1)}(k_r)\check{g}^{(2)}(l_r)\hat{\Delta}_R(p - \sum_{t=1}^{r}(k_t+l_t))$$
$$\cos\left(\frac{1}{2}\big\{(k_r - l_r)\sigma\big[-p + \sum_{t=1}^{r-1}(k_t+l_t)\big] + k_r\sigma l_r\big\}\right)\Bigg].$$

Thus, at nth order, the two-point function is

$$\begin{aligned}
&\sum_{m=0}^{n} \langle \Phi_{m,g}(f)\Phi_{n-m,g}(h)\rangle \qquad (3.39)\\
=&\frac{(-1)^n}{(2\pi)^{2(n-1)}} \int \prod_{j=1}^{n} (\mathrm{d}^4k_j \mathrm{d}^4l_j) \mathrm{d}^4p \mathrm{d}^4\tilde{p} \mathrm{d}^4p_1 \mathrm{d}^4p_2\\
&\check{f}(p_1)\check{h}(p_2) \prod_{j=1}^{n} \check{g}^{(1)}(k_j)\check{g}^{(2)}(l_j)\hat{\Delta}_+(p)\delta(p+\tilde{p})\\
&\cdot \sum_{m=0}^{n} \Bigg[\delta(p_1 - p + \sum_{t=1}^{m}(k_t+l_t))\ \delta(p_2 - \tilde{p} + \sum_{t=m+1}^{n}(k_t+l_t))\\
&\cdot \prod_{r=1}^{m} \hat{\Delta}_R(p - \sum_{t=r}^{m}(k_t+l_t))\\
&\quad \cdot \cos\left(\frac{1}{2}\big\{(k_r - l_r)\sigma\big[-p + \sum_{t=r+1}^{m}(k_t+l_t)\big] + k_r\sigma l_r\big\}\right)\\
&\cdot \prod_{r=m+1}^{n} \hat{\Delta}_R(\tilde{p} - \sum_{t=m+1}^{r}(k_t+l_t))\\
&\quad \cdot \cos\left(\frac{1}{2}\big\{(k_r - l_r)\sigma\big[-\tilde{p} + \sum_{t=m+1}^{r-1}(k_t+l_t)\big] + k_r\sigma l_r\big\}\right)\Bigg],
\end{aligned}$$

where we have relabelled in each summand the variables k_j and l_j from 1 to m:

$$k_1 \to k_m,\ k_2 \to k_{m-1}\ldots \qquad \text{and} \qquad l_1 \to l_m,\ l_2 \to l_{m-1}\ldots,$$

and use the higher indices for the parts coming from $\Phi_{\mathrm{Int}}(h)$. In order to extend the sum in the argument of the second cos to r, we have to subtract

$2k_r\sigma l_r$. As cos is even, these factors can now be pulled outside of the sum, and we obtain

$$\frac{1}{(2\pi)^{2n}}\int\prod_{j=1}^{n}(\mathrm{d}^4k_j\mathrm{d}^4l_j)\mathrm{d}^4p_1\mathrm{d}^4p_2\ \check{f}(p_1)\check{h}(p_2)\prod_{j=1}^{n}\check{g}^{(1)}(k_j)\check{g}^{(2)}(l_j)$$
$$\cdot\prod_{r=1}^{n}\cos\left(\frac{1}{2}\{(k_r-l_r)\sigma\big[p_1+\sum_{t=1}^{r}(k_t+l_t)\big]-k_r\sigma l_r\}\right)$$
$$\cdot\hat{F}_R(p_1,p_2;k_1+l_1,\dots,k_n+l_n),\quad(3.40)$$

where $\hat{F}_R$ is the same as in (3.17). The difference is that here the last n arguments are sums of variables k_j and l_j and it is multiplied by a product of cosine terms depending smoothly on the variables p_j, k_j and l_j. So, the last two lines are again a tempered distribution in the p's and infinitely differentiable in the k's and l's as long as each k_j+l_j lies inside V_n. In order to achieve this, we may for example require $\check{g}^1$ and $\check{g}^2$ to have support in a closed subset of V_{2n}. Obviously, the adiabatic limit exists and since the cosines give 1 there, it is the same as in the commutative case. The generalization to functions of class II (3.28) works similarly as before.

There is still the IR cutoff via the pointwise product (2.7) to be considered. For a mass term this would be

$$\Phi_{n,g}(q)=\frac{1}{(2\pi)^4}\int\mathrm{d}x\Delta_R(x)\int\mathrm{d}k_0\mathrm{d}k_1\check{g}(k_1)\hat{\Phi}_{n-1,g}(k_0+k_1)e^{-ik_0(q-x)}.\quad(3.41)$$

It is easy to see that all formulas are the same as in the commutative case, so in particular it gives the same adiabatic limit.

Thus, putting everything together, we have proven the following

Theorem 3.2.7. *The IR cutoff*

$$\left(\Box_q+m^2\right)\Phi(q)=-\mu/2\left(g^{(1)}(q)\Phi(q)g^{(2)}(q)+g^{(2)}(q)\Phi(q)g^{(1)}(q)\right),$$

gives for the two-point function of order n,

$$\sum_{m=0}^{n}\langle\Phi_{m,g}(f)\Phi_{n-m,g}(h),$$

the correct adiabatic limit, i.e.,

$$\frac{(2\pi)^2}{n!}\int\mathrm{d}^4k\check{f}(k)\check{h}(-k)\partial^n_{m^2}\hat{\Delta}_+^{(m^2)}(k),$$

if either

- *the sequences of the cutoff functions* $\{g_a^{(1)}\}$ *and* $\{g_a^{(2)}\}$ *are of class II,2n or*
- *the sequence of the cutoff functions* $\{g_a^{(1)}\}$ *is of class II,n and* $g^{(2)} = \mathbb{1}$.

The IR cutoff by taking the pointwise product (3.41) gives also the correct adiabatic limit if the sequence $\{g_a\}$ *is of class II,n.*

Nevertheless, taking the pointwise product seems a bit unnatural as it lies outside the algebra structure of $\mathbb{M}_{\mathrm{nc}}$. Since the IR cutoff can be seen as an intermediate technicality, this would not be too much of a drawback.

We will see in section (4.3) that in "truly interacting" models on noncommutative spacetime it is quite difficult to handle the adiabatic limit rigorously. The problem will be that no matter what kind of cutoff one takes it is not possible to pull out a common factor from the different contributions of the same order as in (3.40).

Chapter 4

Dispersion relations for interacting models

Now we will have a look at interacting quantum field theory. We want to calculate dispersion relations for interacting models on noncommutative spacetime. Dispersion relations are experimentally accessible. So, the comparison of theoretical predictions with experimental results can help to decide whether noncommutative spacetime is realized in the real world or not. In particular, we are going to compare the theoretical predictions of logarithmically and quadratically divergent models.

Section 4.1 gives the main definitions and concepts for calculating dispersion relations. If we regard interacting theories we have to consider a product of multiple Φ_k in the quantum version of (3.3). This brings problem b) on page 29 into play. To see more clearly, what additional features the noncommutativity of the spacetime brings in, we first have a look at the situation on commutative spacetime in section 4.2. We will see in section 4.3 that, despite of the successful treatment of a mass term in section 3.2.2, the adiabatic limit for truly interacting models on noncommutative spacetime is quite difficult to handle. In Section 4.4 we will have a look at the ϕ^3 model, both in four and six dimensions. The reason for looking at these different dimensions is that the ϕ_4^3 model is logarithmically divergent while ϕ_6^3 is quadratically divergent. Section 4.5 gives a short treatment of the ϕ^4 model, another quadratically divergent model. We will use quasiplanar Wick products to calculate the first order contribution to the dispersion relation. Finally we will look at the Wess-Zumino model in section 4.6, which is a supersymmetric model and logarithmically divergent. Parts of this chapter have already been published in [15].

4.1 Dispersion relations

We look at the two-point function of an interacting model

$$\langle \Phi_{\mathrm{Int}}(f)\Phi_{\mathrm{Int}}(h)\rangle.$$

Let T_a be the translation by the vector a:

$$T_a(f)(x) = f(x-a).$$

The two-point function should be translationally invariant in the adiabatic limit, i.e.,

$$\langle \Phi_{\mathrm{Int}}(T_a(f))\Phi_{\mathrm{Int}}(T_a(h))\rangle = \langle \Phi_{\mathrm{Int}}(f)\Phi_{\mathrm{Int}}(h)\rangle.$$

So, it can be written in the form

$$\langle \Phi_{\mathrm{Int}}(f)\Phi_{\mathrm{Int}}(h)\rangle = \int \mathrm{d}x\mathrm{d}y f(x)h(y)A(x-y)$$

with A a distribution. As we treat Φ_{Int} as a formal power series in the coupling constant λ, we have to do the same with A. In zeroth order we get $A_0 = \Delta_+$.

We are interested in the support of $\hat{A}$. From this support we can deduce how the p_0 component of the momentum depends on the spatial part $\mathbf{p}$. For example, the support of $\hat{\Delta}_+$ is the set $\{p \in \mathbb{M} | p_0 = \sqrt{m^2+\mathbf{p}^2}\}$. This dependence is called dispersion relation. Usually in interacting theories on commutative spacetime one expects it to be of the form

$$A(x) = \int_0^\infty \mathrm{d}\mu \rho(\mu)\Delta_+^{(\mu)}(x),$$

which is called the Källén-Lehmann spectral representation. $\Delta_+^{(\mu)}$ is the two-point function for mass $\sqrt{\mu}$. ρ is the spectral density and usually consists of a δ function at the physical mass m^2, which corresponds to one-particle states, some isolated parts in the vicinity of $(2m)^2$, corresponding to bound states, and a continuous part starting at $(2m)^2$, corresponding to multi-particle states.

It turns out that the two-point functions of models on noncommutative Minkowski space show a slightly different behaviour. The reason is that Lorentz symmetry is broken for a fixed noncommutativity matrix σ. We are mainly interested in the part corresponding to the one-particle states. Thus, we want to have a look at that part of the support of $\hat{A}$ which transforms to $\{p \in \mathbb{M} | p_0 = \sqrt{m^2+\mathbf{p}^2}\}$ for $\lambda \to 0$. In analogy to the free case, we expect the part of $\hat{A}(k)$ which corresponds to the one-particle spectrum in the free case to be of the form

$$\frac{1}{2\pi}\theta(k_0)F_Z(k,\lambda)\delta(F_M(k,\lambda)) \tag{4.1}$$

with the property that $F_M(k,0) = k^2 - m^2$ and $F_Z(k,0) = 1$. The support is of course the subset of $\mathbb{M}$ where $F_M(k,\lambda)$ vanishes.

As we are working in perturbation theory everything has to be treated as a formal power series in λ. We will only have a look at the first nonvanishing modification from the free case. Let this order be n:

$$\begin{aligned} F_M(k,\lambda) &= k^2 - m^2 - \lambda^n M(k) + O(\lambda^{n+1}), \\ F_Z(k,\lambda) &= 1 + \lambda^n Z(k) + O(\lambda^{n+1}). \end{aligned}$$

Thus, $M(k)$ is a mass and $Z(k)$ a field strength renormalization, both depending on the momentum. From now on all quantities will be regarded only up to order n in λ and the $O(\lambda^{n+1})$ will be dropped.

We are interested in the support of the two-point function of the interacting field. That is, we have to solve:

$$k_0^2 - \mathbf{k}^2 - m^2 - \lambda^n M(k_0, \mathbf{k})$$

Since we are working with formal power series, the equation has to be solved recursively by orders of λ. This gives[1]

$$k_0(\mathbf{k}) = \omega_{\mathbf{k}} + \lambda^n \frac{1}{2\omega_{\mathbf{k}}} M(\omega_{\mathbf{k}}, \mathbf{k}). \tag{4.2}$$

If we expand (4.1) around this solution as a power series in λ we get

$$\begin{aligned} &\frac{1}{2\pi}(1 + \lambda^n Z(k))\delta\left(k^2 - m^2 - \lambda^n M(k)\right) \\ =&\hat{\Delta}^{(m^2)}(k) + \lambda^n\left(Z(k_+)\hat{\Delta}^{(m^2)}(k) + M(k)\partial_{m^2}\hat{\Delta}^{(m^2)}(k)\right) \\ =&\hat{\Delta}^{(m^2)}(k) + \lambda^n\left((Z(k_+) - \partial_{m^2}M(k_+))\hat{\Delta}^{(m^2)}(k) + M(k_+)\partial_{m^2}\hat{\Delta}^{(m^2)}(k)\right). \end{aligned} \tag{4.3}$$

If only this expansion was known $Z(k_+)$ would not be uniquely determined since it could be absorbed into $\partial_0 M(k_+)$.

[1] The θ function in (4.1) cancels the negative solution for k_0. As usual $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

It will later turn out that the two-point function to the order of the first nonvanishing modification will be of the following form:

$$\begin{aligned}
&\hat{\Delta}^{(m^2)}(k) + \lambda^n \Sigma\left(k^2, (k\sigma)^2\right) \partial_{m^2} \hat{\Delta}^{(m^2)}(k) \\
&= \hat{\Delta}^{(m^2)}(k) + \lambda^n \Bigg(\Sigma\left(k_+^2, (k_+\sigma)^2\right) \partial_{m^2} \hat{\Delta}^{(m^2)}(k) \\
&\qquad\qquad - \partial_{m^2} \Sigma\left(m^2, (k_+\sigma)^2\right) \hat{\Delta}^{(m^2)}(k) \Bigg). \qquad (4.4)
\end{aligned}$$

If we assume that $M(k)$ and $Z(k)$ are of the form $M\left((k\sigma)^2\right)$ and $Z\left((k\sigma)^2\right)$, we can identify, by comparison with (4.3),

$$M(s) = -\Sigma(m^2, s) \qquad (4.5)$$

$$Z(s) = \partial^{(1,0)} \Sigma(m^2, s). \qquad (4.6)$$

In the commutative limit the assumed form of M and Z give a momentum independent mass and field strength renormalization. With this assumption Z is on the region under consideration, namely $s = (k_+\sigma)^2$, i.e., k on the mass shell, uniquely determined.

Remark 4.1.1. Formally, one can see the momentum dependent mass and field strength renormalization coming from the nonlocal terms

$$\lambda^n \frac{1}{(2\pi)^2} \int \mathrm{d}k M\left((k\sigma)^2\right) \hat{\Phi}(k) e^{-ikq}$$

and

$$\lambda^n \frac{1}{2(2\pi)^2} (\Box + m^2) \int \mathrm{d}k Z\left((k\sigma)^2\right) \hat{\Phi}(k) e^{-ikq}.$$

If we drop the initial interaction and only use these terms, the equation of motion becomes

$$\begin{aligned}
&(\Box + m^2) \left(\Phi(q) - \lambda^n \frac{1}{2(2\pi)^2} \int \mathrm{d}k Z\left((k\sigma)^2\right) \hat{\Phi}(k) e^{-ikq} \right) \\
&\qquad = -\lambda^n \frac{1}{(2\pi)^2} \int \mathrm{d}k M\left((k\sigma)^2\right) \hat{\Phi}(k) e^{-ikq}. \qquad (4.7)
\end{aligned}$$

The equation of motion (4.7) is solved to order n by

$$\begin{aligned}
\Phi_0(q) =& \Phi_{\text{Free}}(q), \\
\Phi_n(q) =& -\int \mathrm{d}x \Delta_R(x) \frac{1}{(2\pi)^2} \int \mathrm{d}k M\left((k\sigma)^2\right) \hat{\Phi}_0(k) e^{-ik(q-x)} \\
&+ \frac{1}{2(2\pi)^2} \int \mathrm{d}k Z\left((k\sigma)^2\right) \hat{\Phi}_0(k) e^{-ikq}.
\end{aligned}$$

The terms in between are all zero. So, the two-point function at order n gives

$$\begin{aligned}&\langle\Phi_0(f)\Phi_n(h)\rangle+\langle\Phi_n(f)\Phi_0(h)\rangle\\&\quad=-(2\pi)^2\int \mathrm{d}k\check{f}(k)\check{h}(-k)M\left((k\sigma)^2\right)\left[\hat{\Delta}_R(k)\hat{\Delta}_+(k)+\hat{\Delta}_+(k)\hat{\Delta}_A(k)\right]\\&\qquad\qquad+\int \mathrm{d}k\check{f}(k)\check{h}(-k)Z\left((k\sigma)^2\right)\hat{\Delta}_+(k).\end{aligned}\tag{4.8}$$

We saw in section 3.2.1 that (3.12) gives $(2\pi)^2\int \mathrm{d}k\check{f}(k)\check{h}(-k)\partial_{m^2}\Delta_+^{(m^2)}$ for $\check{g}\to(2\pi)^2\delta$. Since we are working without cutoff here, we set formally

$$\hat{\Delta}_R(k)\hat{\Delta}_+(k)+\hat{\Delta}_+(k)\hat{\Delta}_A(k)=-\frac{1}{(2\pi)^2}\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k).\tag{4.9}$$

Then (4.8) can be transformed to

$$\begin{aligned}&\int \mathrm{d}k\check{f}(k)\check{h}(-k)\left(M\left((k\sigma)^2\right)\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k)+Z\left((k\sigma)^2\right)\hat{\Delta}_+^{(m^2)}(k)\right)\\&=\int \mathrm{d}k\check{f}(k)\check{h}(-k)\Bigg[M\left((k_+\sigma)^2\right)\partial_{m^2}\hat{\Delta}_+^{(m^2)}(k)\\&\qquad\qquad+\left(Z\left((k_+\sigma)^2\right)-\partial_{m^2}M\left((k\sigma)^2\right)\right)\hat{\Delta}_+^{(m^2)}(k)\Bigg].\end{aligned}$$

This is exactly (4.3).

To calculate the dispersion relation we have to solve

$$0=k^2-m^2-\lambda^n M\left((k\sigma)^2\right),\tag{4.10}$$

and take that part which corresponds to $k_0>0$ in the free case. Note, that the solution of (4.10) is invariant under simultaneous Lorentz transformation of k and σ. Since we are working with formal power series, the equation has to be solved recursively by orders of λ.[2] This gives

$$k_0(\mathbf{k})=\omega_{\mathbf{k}}+\lambda^n\frac{1}{2\omega_{\mathbf{k}}}M\left((k_+\sigma)^2\right).\tag{4.11}$$

The group velocity is defined as the gradient with respect to $\mathbf{k}$ of the solution k_0. This is

$$\begin{aligned}\nabla k_0=&\frac{\mathbf{k}}{\omega_{\mathbf{k}}}+\lambda^n\left[\frac{\mathbf{k}}{2\omega_{\mathbf{k}}^3}M\left((k_+\sigma)^2\right)-\frac{1}{2\omega_{\mathbf{k}}}\left(\nabla(k_+\sigma)^2\right)M'\left((k_+\sigma)^2\right)\right]\\=&\frac{\mathbf{k}}{k_0}-\lambda^n\frac{1}{2k_0}\left(\nabla(k_+\sigma)^2\right)M'\left((k_+\sigma)^2\right).\end{aligned}$$

[2] If λ is not infinitesimal small and M large, (4.10) might not have a solution of the form $k_0(\mathbf{k})$. There might even be tachyonic solutions, i.e., with $k^2<0$.

In the last line we inserted the solution (4.11). From now on we take σ to be the standard noncommutativity matrix,

$$\sigma = \sigma_0 = \lambda_{\mathrm{nc}}^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and define for this $\mathbf{k}_\perp := (0, k_2, k_3)$ and $\mathbf{k}_{||} := (k_1, 0, 0)$. Then

$$(k\sigma)^2 = -\lambda_{\mathrm{nc}}^4 (k^2 + 2\mathbf{k}_\perp^2) \qquad \text{and} \qquad \nabla(k_+\sigma_0)^2 = -4\lambda_{\mathrm{nc}}^4 \mathbf{k}_\perp. \tag{4.12}$$

With (4.5) the group velocity is

$$\nabla k_0 = \frac{\mathbf{k}_{||}}{k_0} + \frac{\mathbf{k}_\perp}{k_0} \left[1 + 2\lambda_{\mathrm{nc}}^4 \lambda^n \partial^{(0,1)} \Sigma \left(m^2, (k_+\sigma)^2\right)\right]. \tag{4.13}$$

We define the distortion of the group velocity in perpendicular direction to be

$$\Delta v_\perp^{\mathrm{rel}} := 2\lambda_{\mathrm{nc}}^4 \lambda^n \partial^{(0,1)} \Sigma \left(m^2, (k_+\sigma)^2\right). \tag{4.14}$$

So,

$$\left|\Delta v_\perp^{\mathrm{rel}}\right| = \frac{|\nabla k_{0,\perp} - \mathbf{k}_\perp / k_0|}{|\mathbf{k}_\perp / k_0|}.$$

We will later calculate this quantity for different models on noncommutative spacetime. We want to emphasize that to calculate the dispersion relation (4.13) or the mass and field strength renormalization in first nontrivial order we only have to know $\Sigma\left(k^2, (k\sigma)^2\right)$ for k in the vicinity of the positive mass shell.

4.2 Interacting models on commutative spacetime

We want to use the Yang-Feldman formalism to calculate two-point functions of quantum fields of certain models. Before we will investigate quantum fields on noncommutative spacetime we first have a look at the situation on commutative spacetime. This gives the possibility to distinguish between features coming from the quantum structure of the fields and those coming from the noncommutativity of spacetime.

Mainly, we will calculate in this section the two point function of the ϕ_4^3 model. This is done in subsection 4.2.1. In subsection 4.2.2 we will have a short look at the ϕ^4 model. We will use the IR cutoff which already gave the correct adiabatic limit 3.2.1 and again make use of the Epstein-Glaser theorem.To apply this theorem, again we have to add all contributions of the same order before calculating the adiabatic limit. We will see that most divergences cancel. One remains, which can be handled by the continuation of a distribution to the origin using the concept of scaling degree of Steinmann [41]. This gives a free parameter. We will see that this corresponds to a mass renormalization.

4.2.1 ϕ^3 model on commutative spacetime

We investigate the ϕ^3_4 model. For the ϕ^3 interaction it is not possible to find a positive-definite energy. However, the perturbation series is well-defined. One can imagine that there exists an additional ϕ^4, which would make the energy indeed positive-definite, but is of sufficient higher order in the coupling λ, such that effects of this term are not visible at the calculated orders.

The field equation for the interacting field reads

$$(\Box + m^2)\Phi(x) = -\lambda\Phi^2(x).$$

The naive solution (3.3) has to be modified since it involves products of fields at the same point. Additionally, we have to introduce an IR cutoff. We choose that cutoff which was already successfully applied in section 3.2.1.

We will look at the solution for the interacting field up to second order. The first orders are

$$\begin{aligned}\Phi_0(x) =&\Phi_{\text{Free}}(x),\\ \Phi_1(x) =& -\int \mathrm{d}y \Delta_R(x-y) g(y) :\Phi_0^2(y): .\end{aligned}$$

Here we have taken the Wick-ordered product of two free fields at the same point. In the second order we get from (3.3):

$$\Phi_2(x) = -\int \mathrm{d}y \Delta_R(x-y) g(y) \left(\Phi_0(y)\Phi_1(y) + \Phi_1(y)\Phi_0(y)\right).$$

These fields can be represented by the graphs shown in figure 4.1. The two contributions to Φ_2 are represented by a sum of graphs. We will later see that, although the graphs represent the different summands in Φ_2 on their one, it will be more appropriate to always look at their sum. Taking the Wick ordered product in Φ_1 is graphically equivalent to subtract tadpoles right from the start, cf. figure 4.2. Graphs for the Yang-Feldman formalism are explained in more detail in Appendix C.

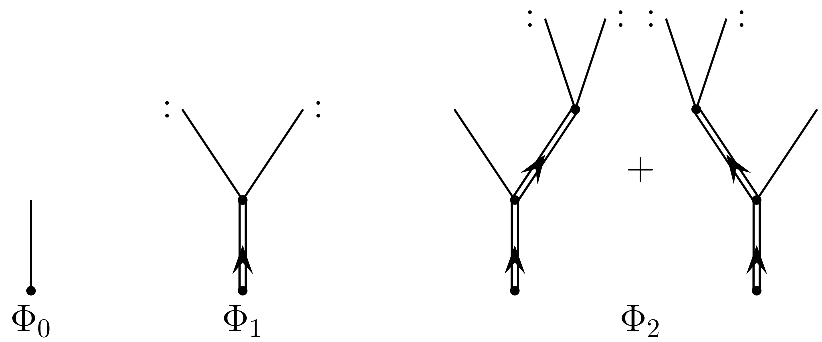

Figure 4.1: The first orders of the interacting field in the ϕ^3 model.

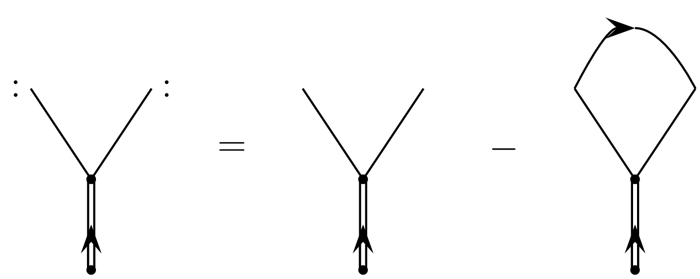

Figure 4.2: Subtraction of tadpoles.

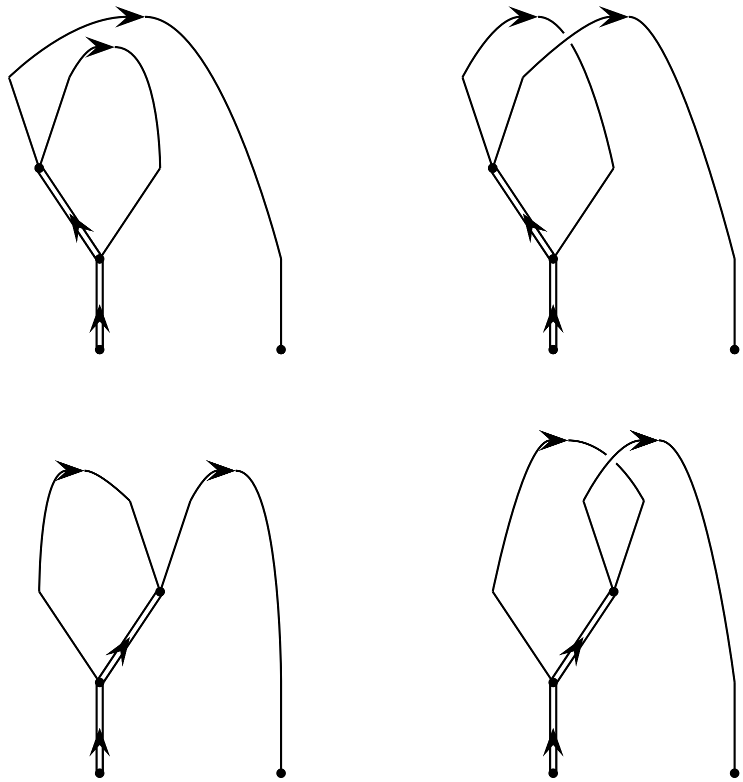

Figure 4.3: Second order contributions to two-point function of the ϕ^3 model, $\langle \Phi_2 \Phi_0 \rangle$.

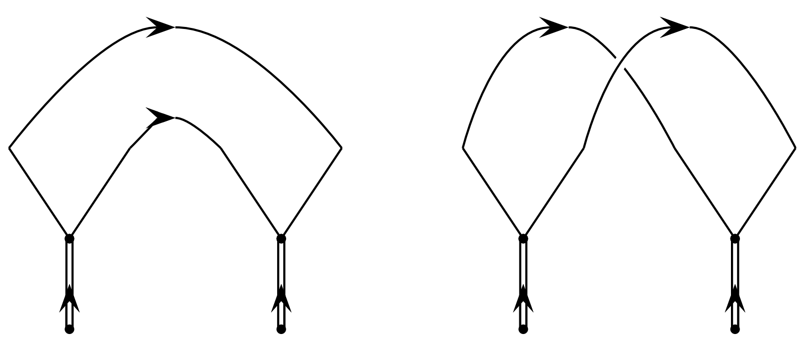

Figure 4.4: Second order contributions to two-point function of the ϕ^3 model, $\langle \Phi_1 \Phi_1 \rangle$.

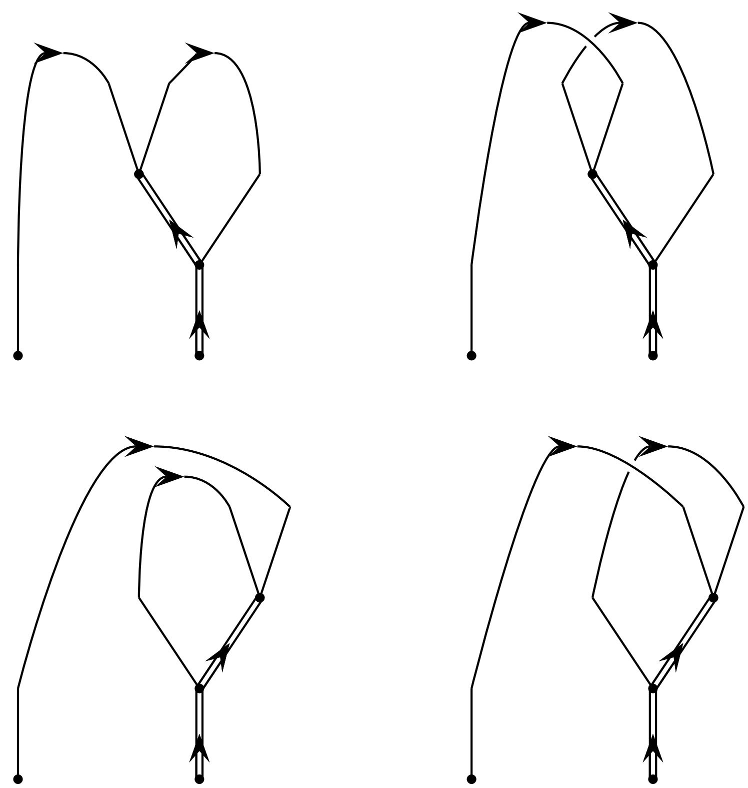

Figure 4.5: Second order contributions to two-point function of the ϕ^3 model, $\langle\Phi_0\Phi_2\rangle$.

We calculate the two-point function up to second order. The zeroth order ist that of the free case:

$$\langle \Phi_0(f)\Phi_0(h)\rangle = (2\pi)^2 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k).$$

The two-point function of first order, $\langle \Phi_1(f)\Phi_0(h) + \Phi_0(f)\Phi_1(h)\rangle$, vanishes since an odd number of fields is involved.

At second order we have

$$\langle \Phi_2(f)\Phi_0(h)\rangle + \langle \Phi_1(f)\Phi_1(h)\rangle + \langle \Phi_0(f)\Phi_2(h)\rangle. \tag{4.15}$$

The contribution from $\langle \Phi_2(f)\Phi_0(h)\rangle$, graphically represented by the sum of the graphs shown in figure 4.3, is

$$2\int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0)g(y_1)g(y_2)h(y_3)$$
$$\Delta_R(y_0-y_1)\Delta_R(y_1-y_2)\left[\Delta_+(y_1-y_2)+\Delta_-(y_1-y_2)\right]\Delta_+(y_2-y_3).$$

The $\langle \Phi_0(f)\Phi_2(h)\rangle$ contribution is

$$2\int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0)g(y_1)g(y_2)h(y_3)$$
$$\Delta_+(y_0-y_1)\left[\Delta_+(y_1-y_2)+\Delta_-(y_1-y_2)\right]\Delta_A(y_1-y_2)\Delta_A(y_2-y_3),$$

represented by the sum of graphs of figure 4.5, and the one from $\langle \Phi_0(f)\Phi_2(h)\rangle$ is

$$2\int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0)g(y_1)g(y_2)h(y_3)$$
$$\Delta_R(y_0-y_1)\Delta_+(y_1-y_2)\Delta_+(y_1-y_2)\Delta_A(y_2-y_3),$$

represented by the sum of graphs of figure 4.4. Here, the planar and non-planar graphs give the same contribution, which gives rise to the factors of 2 appearing in front of each integral. On noncommutative spacetime the contributions will be different.

Now, if we would treat the terms coming from the different graphs separately, we would, for example, in the lower left graph of figure 4.3 face the problem that

$$\Delta_R(x)\cdot\Delta_+(x) \qquad \text{with } x = y_1 - y_2$$

is a priori not well-defined. If we look at the set $\widetilde{\mathrm{WF}}(\Delta_R, \Delta_+)$ (compare Appendix A), we see that it contains elements of the form $(x, 0)$ if x lies on the forward lightcone. However, we can use[3]

$$\begin{aligned}\Delta_R(x)\left[\Delta_+(x) + \Delta_-(x)\right] &= i\theta(x_0)\left[\Delta_+(x) - \Delta_-(x)\right]\left[\Delta_+(x) + \Delta_-(x)\right] \\ &= i\theta(x_0)\left[\Delta_+^2(x) - \Delta_-^2(x)\right]. \end{aligned} \tag{4.16}$$

$\Delta_\pm^n$ are well-defined distributions for $n \in \mathbb{N}$. But for $n > 1$ there appears a problem due to the multiplication with $\theta(x_0)$ in $x = 0$, i.e., $(0,0) \in \widetilde{\mathrm{WF}}(\theta_0, \Delta_+)$. So, the problem of multiplication has been reduced to the origin, where the expression (4.16) is not well-defined. But we can use concept of scaling degreeto extend this distribution to the origin. Since $\mathrm{sd}(\Delta_\pm^2) = 4$ two such continuations differ by the multiple of a δ function. We will later show that this can be regarded as a mass renormalization.

Remark 4.2.1. We cannot use the concept of scaling degree at submanifolds [7] to uniquely extend $\Delta_R \cdot \Delta_+$ onto the forward lightcone. If x lies on the forward lightcone and v is a unit vector perpendicular to the lightcone at x pointing inwards, then $\Delta_R(x + \lambda v)$ is essentially constant for small $\lambda > 0$ and $\Delta_+(x + \lambda v)$ behaves essentially like $\frac{1}{\lambda}$. So $\Delta_R \cdot \Delta_+$ has the scaling degree 1 at the lightcone and since the codimension of the lightcone is 1 the continuation to x would be unique only up to a δ-term. Thus, the overall ambiguity corresponds to a function on the forward lightcone.

Let $T_R(x)$ be a continuation of $i\theta(x_0)\left[\Delta_+^2(x) - \Delta_-^2(x)\right]$ to the point $x = 0$ and define correspondingly $T_F(x) := T_R(x) - i\Delta_-^2(x)$ and $T_A(x) := T_F(x) - i\Delta_+^2(x)$. Then $T_A(x)$ is a continuation of $i\theta(-x_0)\left[\Delta_-^2(x) - \Delta_+^2(x)\right]$ and $T_A(x) = T_R(-x)$. Outside the origin $T_F(x) = \Delta_F^2(x)$. It is easy to see that all $T_{R/F/A}$ are Lorentz invariant and their Fourier transform $\hat{T}_{R/F/A}$ are $\mathcal{C}^\infty$ in the vicinity of the mass shell. Let

$$\mu := 2(2\pi)^2 \hat{T}_F(m, 0, 0, 0). \tag{4.17}$$

Since a different continuation differs by a δ-function in position space, it differs by a constant in momentum space. So, the parameter μ can be used to label the different continuations.

Now, we claim that the theorem of Epstein and Glaser (D.1.3) is applicable for the sum (4.15), which is

$$\int \prod_{j=0}^{3} \mathrm{d}y_j\, f(y_0) g(y_1) g(y_2) h(y_3) F_R(y_0, y_3; y_1, y_2)$$

[3] This was already discovered in [2].

with

$$\begin{aligned} F_R(y_0, y_3; y_1, y_2) = 2\Big[&\Delta_R(y_0 - y_1) T_R(y_1 - y_2)\Delta_+(y_2 - y_3) \\ &+ \Delta_R(y_0 - y_1)\Delta_+^2(y_1 - y_2)\Delta_A(y_2 - y_3) \\ &+ \Delta_+(y_0 - y_1) T_A(y_1 - y_2)\Delta_A(y_2 - y_3)\Big]. \end{aligned}$$

The distribution F_A will be defined by exchanging Δ_R with Δ_A and T_R with T_A respectively. Since T_R has retarded and T_A has advanced support the proof, that $F_{R/A}$ fulfills (D.1), is similar to the case for a mass term at second order, shown on page 37.

Fourier transformation gives

$$\begin{aligned} &\hat{F}_{R/A}(k_0, k_3; k_1, k_2) = \\ &\quad 8\pi^2 \delta\left(\sum_{j=0}^{3} k_j\right) \Big[\hat{\Delta}_{R/A}(k_0)\hat{T}_{R/A}(k_0 + k_1)\hat{\Delta}_+(k_0 + k_1 + k_2) \\ &\qquad +\hat{\Delta}_{R/A}(k_0)\widehat{\Delta_+^2}(k_0 + k_1)\hat{\Delta}_{A/R}(k_0 + k_1 + k_2) \\ &\qquad +\hat{\Delta}_+(k_0)\hat{T}_{A/R}(k_0 + k_1)\hat{\Delta}_{A/R}(k_0 + k_1 + k_2)\Big]. \end{aligned} \tag{4.18}$$

To show that $\hat{F}_R - \hat{F}_A$ vanishes on R_2, we replace $\hat{\Delta}_{R/A}$ by $\hat{\Delta}_F - i\hat{\Delta}_\mp$ and $\hat{T}_{R/A}$ by $\hat{T}_F - i\widehat{\Delta_\mp^2}$. The support of $\widehat{\Delta_+^2}$ lies above the positive $2m$-mass shell, the one of $\hat{\Delta}_-$ lies on the negative m-mass shell and the one of $\widehat{\Delta_-^2}$ is below the negative $2m$-mass shell. Each term containing one of these last two distributions can be dropped as all terms also contain $\hat{\Delta}_+$ or $\widehat{\Delta_+^2}$, so the multiplication with this gives zero on R_2, because a vector k from on or above the positive m-mass shell to somewhere on or below the negative m-mass shell has $k^2 \geq 4m^2$. Although the common support of $\hat{\Delta}_+$ and $\widehat{\Delta_+^2}$ is empty, their regions of support can be linked by an arbitraryíly small vector if one goes to high momenta, see figure 4.6. So, the terms with $\hat{\Delta}_+$ and $\widehat{\Delta_+^2}$ cannot be dropped a priori. We calculate $\hat{F}_R - \hat{F}_A$ on R_2 with the simplifications

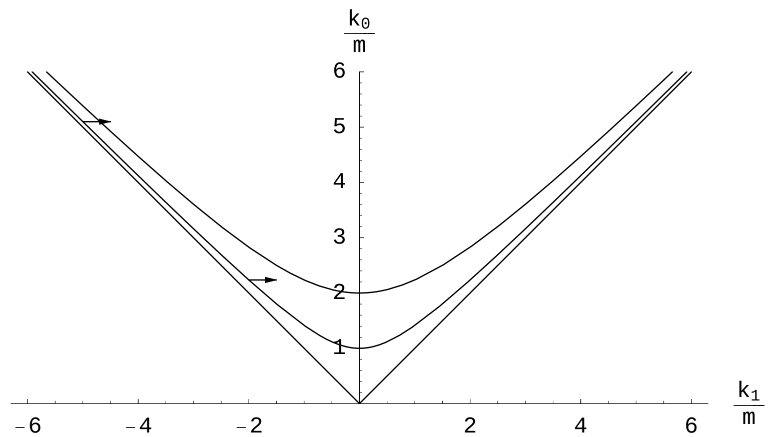

Figure 4.6: Small spacelike vector, connecting the support of $\hat{\Delta}_+$ with the support of $\widehat{\Delta_+^2}$ at high momenta.

mentioned before and get

$$
\begin{aligned}
&\hat{F}_R(k_0,k_3;k_1,k_2) - \hat{F}_A(k_0,k_3;k_1,k_2) = 8\pi^2\delta\left(K_3\right)\\
&\cdot\Big[\begin{array}{cccc}
 & \hat{\Delta}_F(K_0) & \hat{T}_F(K_1) & \hat{\Delta}_+(K_2) \\
+ & \hat{\Delta}_F(K_0) & \widehat{\Delta_+^2}(K_1) & \left(\hat{\Delta}_F - i\hat{\Delta}_+\right)(K_2) \\
+ & \hat{\Delta}_+(K_0) & \left(\hat{T}_F - i\widehat{\Delta_+^2}\right)(K_1) & \left(\hat{\Delta}_F - i\hat{\Delta}_+\right)(K_2) \\
- & \left(\hat{\Delta}_F - i\hat{\Delta}_+\right)(K_0) & \left(\hat{T}_F - i\widehat{\Delta_+^2}\right)(K_1) & \hat{\Delta}_+(K_2) \\
- & \left(\hat{\Delta}_F - i\hat{\Delta}_+\right)(K_0) & \widehat{\Delta_+^2}(K_1) & \hat{\Delta}_F(K_2) \\
- & \hat{\Delta}_+(K_0) & \hat{T}_F(K_1) & \hat{\Delta}_F(K_2)
\end{array} \Big]
\end{aligned}
$$

with $K_j = \sum_{i=0}^{j} k_i$. This gives 0, since, after multiplying out, each term has exactly one counter term with opposite sign. Hence all preliminaries for the Epstein-Glaser theorem are fulfilled. So, the adiabatic limit exists and is independent of the choice of sequence $\{g_a\}$. As for the mass term this can be of class II,2.

Note that it was important to consider all terms of order $n = 2$, including $\langle\Phi_1(f)\Phi_1(h)\rangle$. We will later ignore its contribution to the dispersion relations since, due to the $\widehat{\Delta_+^2}$ factor, the support of its adiabatic limit lies in momentum space above the $2m$-mass shell. But here we needed it, to be able to use the theorem of Epstein and Glaser and to show that the adiabatic limit exists and that it is independent of the sequence $\{g_a\}$.

To have a closer look on how the resulting two-point function of the adiabatic limit looks in the vicinity of the mass shell,[4] we choose a function $\check{f}$ which has support, say, in the set

$$\left\{ k \in \mathbb{M} \middle| \left(\frac{m}{2}\right)^2 < k^2 < \left(\frac{3m}{2}\right)^2 \text{ and } k_0 > 0 \right\}. \tag{4.19}$$

To be able to simplify the expressions in $\hat{F}_R(k_0, k_3; k_1, k_2)$ we further assume that $\check{f}$ has support in a compact subset of (4.19) and the sequence of functions $\{g_a\}$ has the property

$$\text{supp}\check{g}_a \subset B_{\epsilon/a}(0)$$

for some fixed $\epsilon > 0$ and $\check{g}_a \to (2\pi)^4 \delta^{(4)}$ in $\mathcal{O}'_C(\mathbb{M})$. Then, for a large enough (depending on the compact support of $\check{f}$), the vectors k_0, $k_0 + k_1$ and $k_0 + k_1 + k_2$ can neither reach the $2m$-mass shell nor the region above, if k_0 lies in the compact support of $\check{f}$ and k_1 and k_2 in the one of $\check{g}_a$. Thus, all expressions containing some factor $\widehat{\Delta^2_+}$ in $\hat{F}_R(k_0, k_3; k_1, k_2)$ can be dropped like the ones containing $\hat{\Delta}_-$ or $\widehat{\Delta^2_-}$. Note that, first, the adiabatic limit is independent of the special choice of $\{g_a\}$ in a certain class and, second, every $\check{f} \in \mathcal{S}$ with support (4.19) can be approximated by functions, which have as support a compact subset of (4.19) in the topology of $\mathcal{S}$. Thus, the Epstein-Glaser theorem assures that no information is lost by making the additional assumptions concerning the support of $\check{f}$ and $\check{g}_a$.

With these assumptions we get for the two-point function at second order

$$\begin{aligned} 8\pi^2 \int \prod_{j=0}^{3} \mathrm{d}^4 k_j \check{f}(k_0) \check{g}(k_1) \check{g}(k_2) \check{h}(k_3) \delta\left(\sum_{j=0}^{3} k_j\right) \\ \cdot \Big[\hat{\Delta}_R(k_0) \hat{T}_F(k_0 + k_1) \hat{\Delta}_+(k_0 + k_1 + k_2) \\ + \hat{\Delta}_+(k_0) \hat{T}_F(k_0 + k_1) \hat{\Delta}_A(k_0 + k_1 + k_2) \Big]. \end{aligned} \tag{4.20}$$

Now, after integrating out the δ-function and performing a variable transformation, this is

$$\begin{aligned} 8\pi^2 \int \prod_{j=0}^{2} \mathrm{d}^4 k_j \check{f}(k_0) \check{g}(k_1 - k_0) \check{g}(k_2 - k_1) \check{h}(-k_2) \\ \cdot \Big[\hat{\Delta}_R(k_0) \hat{T}_F(k_1) \hat{\Delta}_+(k_2) + \hat{\Delta}_+(k_0) \hat{T}_F(k_1) \hat{\Delta}_A(k_2) \Big]. \end{aligned}$$

[4] In section 4.1 we showed that to calculate dispersion relations it is only important to know the two-point function in the vicinity of the mass shell.

We insert the expressions for $\hat{\Delta}_{R/A/+}$ from Appendix A and make another transformation of variables (different for each summand) to get

$$\frac{1}{\pi}\int \frac{\mathrm{d}^3k_0}{2\omega_0}\mathrm{d}^4k_1\frac{\mathrm{d}^3k_2}{2\omega_2}\mathrm{d}x\check{g}(k_{0,1}-x-\omega_0,\mathbf{k}_1-\mathbf{k}_0)\check{g}(\omega_2-k_{0,1},\mathbf{k}_2-\mathbf{k}_1)\check{h}(k_3)$$
$$\cdot\Big[\check{f}(\omega_0+x,\mathbf{k}_0)\hat{T}_F(k_1)\check{h}(-k_{+,2})\left(\frac{1}{x+i\epsilon}-\frac{1}{x+2\omega_0+i\epsilon}\right)$$
$$-\check{f}(k_{+,0})\hat{T}_F(k_{0,1}+x,\mathbf{k}_1)\check{h}(-\omega_2+x,-\mathbf{k}_2)\left(\frac{1}{x+i\epsilon}-\frac{1}{x-2\omega_2+i\epsilon}\right)\Big].$$

Similar to the calculation following (3.13), we expand $\check{f}(\omega_0+x,\mathbf{k}_0)$, $\check{h}(-\omega_2+x,-\mathbf{k}_2)$ and $\hat{T}_F(k_{0,1}+x,\mathbf{k}_1)$ around $x=0$ ($\hat{T}_F$ is $\mathcal{C}^\infty$ around the mass shell) and get, after passing to the adiabatic limit,

$$2(2\pi)^3\int \mathrm{d}^3k\ \Bigg(\hat{T}_F(k_+)\Big[-\frac{1}{4\omega_{\mathbf{k}}^3}\check{f}(k_+)\check{h}(-k_+)+\frac{1}{4\omega_{\mathbf{k}}^2}\partial_0\check{f}(k_+)\check{h}(-k_+)$$
$$-\frac{1}{4\omega_{\mathbf{k}}^2}\check{f}(k_+)\partial_0\check{h}(-k_+)\Big]-\frac{1}{4\omega_{\mathbf{k}}^2}\partial_0\hat{T}_F(k_+)\check{f}(k_+)\check{h}(-k_+)\Bigg).$$

T_F is Lorentz invariant and $\hat{T}_F(k_+)=\frac{\mu}{2(2\pi)^2}$, cf. (4.17). If we compare this with (3.10) and (4.3), we see that the first terms gives a (constant) mass renormalization. Its value is

$$\lambda^2M=\lambda^2\mu$$

and depends on the choice of continuation we made during the calculation. The last term gives a (constant) field strength renormalization. It is

$$\lambda^2Z=\lambda^2\frac{(2\pi)^2}{\omega_{\mathbf{k}}}\partial_0\hat{T}_F(k_+)=\lambda^2\frac{2\sqrt{3}\pi-9}{72\pi^2m^2} \tag{4.21}$$

and independent of the chosen continuation. The method of continuation of distributions in position space corresponds to the introduction of counter terms in momentum space in the standard Feynman graph formalism.

4.2.2 ϕ^4 model on commutative spacetime

Here, we take a quick look at the ϕ^4 model. The calculation will be similar to the one of the last subsection. There, a cancellation took place such that multiplications like Δ_R with Δ_+, which are ill defined on the forward light-cone, dropped out. The main purpose of this subsection is to show, that a similar cancelation takes place for the ϕ^4 model.

The field equation is

$$(\Box + m^2)\Phi(x) = -\lambda\Phi^3(x),$$

and the interacting field is to first orders:

$$\begin{aligned}
\Phi_0(x) =& \Phi_{\text{Free}}(x), \\
\Phi_1(x) =& -\int \mathrm{d}y \Delta_R(y) g(x-y) : \Phi_0^3(x-y) :, \\
\Phi_2(x) =& -\int \mathrm{d}y \Delta_R(x-y) g(y) \\
& \big(: \Phi_0^2(y) : \Phi_1(y) + : \Phi_0(y)\Phi_1(y)\Phi_0(y) : + \Phi_1(y) : \Phi_0^2(y) : \big),
\end{aligned}$$

where the term $: \Phi_0(y)\Phi_1(y)\Phi_0(y) :$ is the continuation of $\Phi_0(x)\Phi_1(y)\Phi_0(z) - \Delta(x-z)\Phi_1(y)$ to the diagonal $x = y = z$. So, again all tadpoles are subtracted.

The two-point function at zero order is trivial. At first order it vanishes due to the Wick product in Φ_1. The contributions to the second order are

$$\begin{aligned}
\langle \Phi_2(f)\Phi_0(h) \rangle = 6 \int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0) g(y_1) g(y_2) h(y_3) \Delta_R(y_0 - y_1) \Delta_R(y_1 - y_2) \\
\left[\Delta_+^2(y_1 - y_2) + \Delta_+(y_1 - y_2)\Delta_-(y_1 - y_2) + \Delta_-^2(y_1 - y_2) \right] \Delta_+(y_2 - y_3),
\end{aligned}$$

$$\begin{aligned}
\langle \Phi_1(f)\Phi_1(h) \rangle = 6 \int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0) g(y_1) g(y_2) h(y_3) \\
\Delta_R(y_0 - y_1) \Delta_+^3(y_1 - y_2) \Delta_A(y_2 - y_3),
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi_0(f)\Phi_2(h) \rangle = 6 \int \prod_{j=0}^{3} \mathrm{d}y_j f(y_0) g(y_1) g(y_2) h(y_3) \Delta_+(y_0 - y_1) \\
\cdot \left[\Delta_+^2(y_1 - y_2) + \Delta_+(y_1 - y_2)\Delta_-(y_1 - y_2) + \Delta_-^2(y_1 - y_2) \right] \\
\cdot \Delta_A(y_1 - y_2) \Delta_A(y_2 - y_3).
\end{aligned}$$

Figure 4.7 shows graphs for some planar contributions.

Again $\Delta_R(x)\Delta_+^2(x)$ would be ill defined on the lightcone. However, as before, a cancellation takes place. For the $\langle \Phi_2(f)\Phi_0(h) \rangle$ terms it is

$$\begin{aligned}
\Delta_R(x) & \left[\Delta_+^2(x) + \Delta_+(x)\Delta_-(x) + \Delta_-^2(x) \right] \\
&= i\theta(x_0) \left[\Delta_+(x) - \Delta_-(x) \right] \left[\Delta_+^2(x) + \Delta_+(x)\Delta_-(x) + \Delta_-^2(x) \right] \\
&= i\theta(x_0) \left[\Delta_+^3(x) - \Delta_-^3(x) \right].
\end{aligned}$$

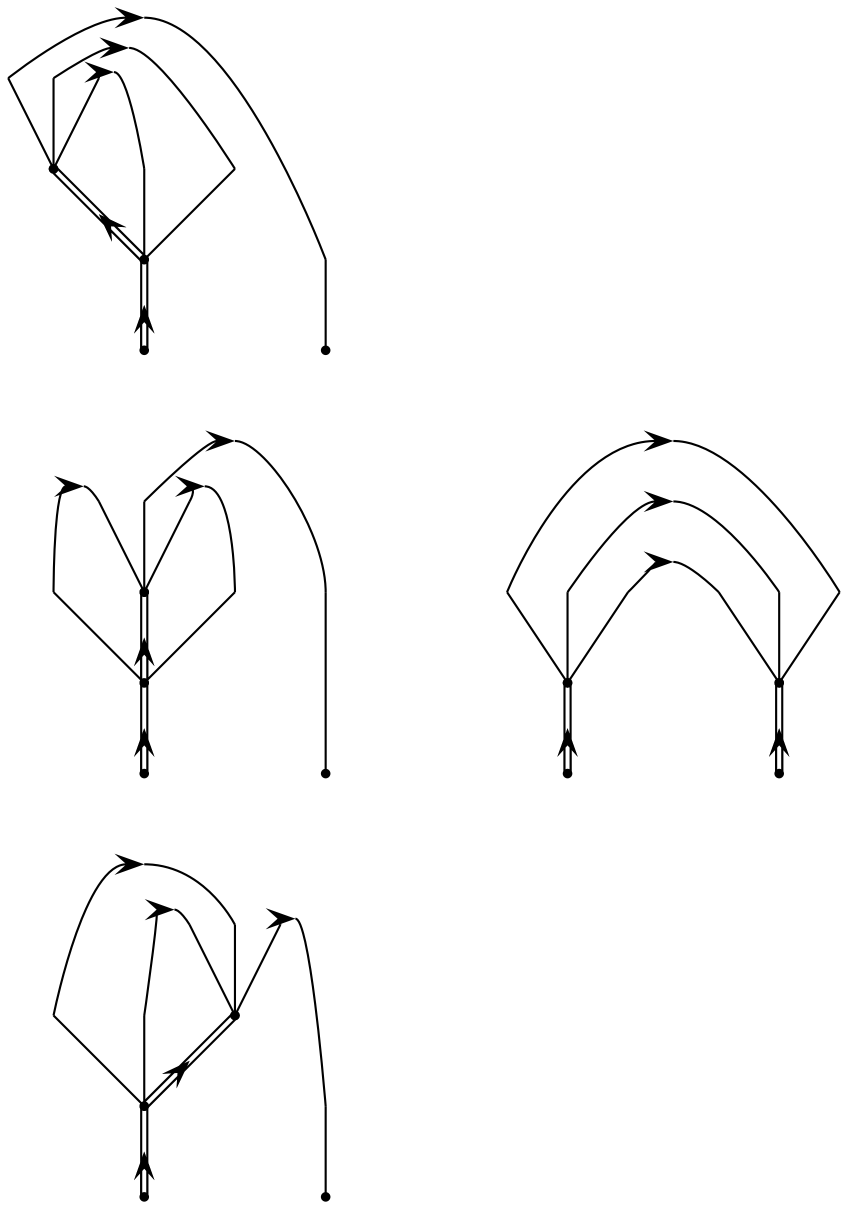

Figure 4.7: Second order contributions to the ϕ^4 two-point function, planar graphs to $\langle\Phi_2\Phi_0\rangle$ and $\langle\Phi_1\Phi_1\rangle$.

The product $\theta(x_0)\Delta_+^3(x)$ is ill defined only at the origin. As the scaling degree of Δ_+^3 is 6 we get three arbitrary constants. One can be dropped, using a symmetry condition. Now, we can choose a continuation of $i\theta(x_0)\left[\Delta_+^3(x) - \Delta_-^3(x)\right]$ and show correspondingly to case of the ϕ^3 model that the preliminaries for the Epstein-Glaser theorem are fulfilled.

4.3 Remarks on the adiabatic limit for interacting models on $\mathbb{M}_{\mathbf{nc}}$

In Chapter 3 we succeeded in finding suitable IR cutoffs for quantum field theory on $\mathbb{M}_{\mathrm{nc}}$ for an additional mass term. Unfortunately, the situation for truly interacting models is much more complicated and we have not been able to find a suitable cutoff yet. The problem is, that there appear additional twisting factors (even for planar graphs, see below), which depend on the momenta in the cutoff functions and are different for each graph. However, we have seen at several steps (adiabatic limit, cancellation of $\Delta_R \cdot \Delta_+$ divergences on the lightcone) that only the sum of all graphs of the same order shows good behaviour. But the different graphs have different twisting factors, and these cannot be pulled out as a common factor like in the case of a mass term in section 3.2.2. So, the cancelation of divergences might not take place in the adiabatic limit.[5] The adiabatic limit will probably very much depend on the special choice of sequence $g_a \to (2\pi)^2\delta$. At least we have not been able to find a suitable large class of sequences which give the same limit. But the reason might not be, that such a class does not exist, but only that the calculations are too complicated to find it.

We give examples of possible IR cutoffs in the ϕ^3 model on $\mathbb{M}_{\mathrm{nc}}$ and give their contribution for vertices, cf. Appendix C. The momenta of the vertex are labelled as in figure 4.8:

- For the cutoff by
$$g_1(q)\Phi_a(q)g_2(q)\Phi_b(q)g_3(q), \tag{4.22}$$
cf. (2.14), each vertex gives
$$(2\pi)^{-4}\int \mathrm{d}p_1\mathrm{d}p_2\check{g}_1(p_1)\check{g}_2(p_2)\check{g}_3(k_1+k_2-p_1-p_2-k_0)$$
$$e^{-\frac{i}{2}Q(-p_1,k_1,-p_2,k_2,k_1+k_2-k_0-p_1-p_2)},$$
with $Q(k_1,k_2,\dots,k_n) := \sum_{i<j} k_i\sigma k_j$. One or two of the three g_j's could be equal to $(2\pi)^2\delta$. These cutoffs can be combined to give, for example,

[5] Derivatives of the twisting factors in the momenta of the cutoff functions spoil this.

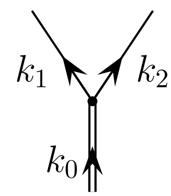

Figure 4.8: Typical vertex in the ϕ^3 model.

– $\frac{1}{3}\left(g(q)\Phi_a(q)\Phi_b(q) + \Phi_a(q)g(q)\Phi_b(q) + \Phi_a(q)\Phi_b(q)g(q)\right)$ or
– $g(q)\Phi_a(q)g(q)\Phi_b(q)g(q)$.

- Multiplying $\Phi_a(q)\Phi_b(q)$ by g, using the pointwise product (2.7) gives

$$\check{g}(k_1 + k_2 - k_0)e^{-\frac{i}{2}k_1\sigma k_2}$$

  at each vertex.

For these (and many others that we tried) the twisting factors of different graphs cannot be pulled outside the sum of all graphs of the same order. It is still an open problem, how to handle this problem. Note that, with the cutoff (4.22), the contribution from each graph is well-defined on its own, but this does not have to be the case if the adiabatic limit is regarded.

Therefore, we omit the cutoff in the next sections and perform formal calculations. That is, we set from the beginning $\check{g} = (2\pi)^2\delta$. In this case the twisting factors simplify, and the overall twisting factors can for each graph be calculated by looking at the crossing of contractions. Each crossing of lines of the kind shown in figure 4.9 gives the twisting factor

$$e^{ik_0\sigma k_1}.$$

This can be derived similarly as in the case for Feynman graphs, shown in [31]. So, without IR cutoff, no planar graphs have a twisting factor.

4.4 ϕ^3 model on $\mathbb{M}_{\mathbf{nc}}$

We want to look again at the ϕ^3 model, now on noncommutative spacetime, both in four and six dimensions. The reason to investigate these different dimensions is that, in four dimensions the ϕ^3 model is logarithmically divergent

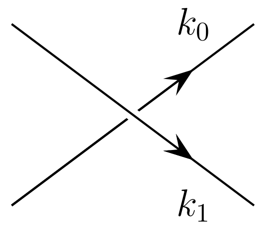

Figure 4.9: Crossing of lines.

but in six it is quadratically divergent. We want to compare their dispersion relations on noncommutative spacetime.

Since it turned out, that the IR cutoffs were quite difficult to handle rigorously (cf. section 4.3), we will make the calculation to some extent formal, i.e., without an IR cutoff. However, the factor coming from the nonplanar graphs will be calculated rigorously, using the technique of oscillatory integrals.

We want to calculate dispersion relations for the model in four dimensions first. The equation of motion for the ϕ^3 model is

$$\left(\Box_q + m^2\right)\Phi(q) = -\lambda\Phi(q)^2.$$

This gives for the interacting field

$$\begin{aligned}
\Phi_0(q) =& \Phi_{\mathrm{Free}}(q),\\
\Phi_1(q) =& -\int \mathrm{d}x\Delta_R(x) :\Phi_0^2(q-x):,\\
\Phi_2(q) =& -\int \mathrm{d}x\Delta_R(x)\left(\Phi_0(q-x)\Phi_1(q-x)+\Phi_1(q-x)\Phi_0(q-x)\right).
\end{aligned}$$

The Wick ordering, used in $\Phi_1(q)$, acts only on the field part, see (2.13). Thus, we get

$$\begin{aligned}
\Phi_2(q) = \frac{1}{(2\pi)^2}\int & \mathrm{d}k_0\mathrm{d}k_1\mathrm{d}k_2 e^{-ik_0q}\hat{\Delta}_R(k_0)\hat{\Delta}_R(k_0+k_1)\\
&\Big(\hat{\Phi}(-k_1):\hat{\Phi}(k_0+k_1+k_2)\hat{\Phi}(-k_2): e^{\frac{i}{2}k_1\sigma k_0}\\
&\quad + :\hat{\Phi}(k_0+k_1+k_2)\hat{\Phi}(-k_2):\hat{\Phi}(-k_1)e^{\frac{i}{2}k_0\sigma k_1}\Big)e^{\frac{i}{2}(k_0+k_1)\sigma k_2}.
\end{aligned}$$

The two-point function at first order vanishes again. At the second order we get three terms

$$\langle\Phi_2(f)\Phi_0(h)\rangle + \langle\Phi_1(f)\Phi_1(h)\rangle + \langle\Phi_0(f)\Phi_2(h)\rangle.$$

The term in the middle gives the contribution

$$(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_R(k)\hat{\Delta}_A(k) \int \mathrm{d}l \hat{\Delta}_+(k-l)\hat{\Delta}_+(l)\left(1+e^{-ik\sigma l}\right). \quad (4.23)$$

These correspond to the graphs shown in figure 4.4. Due to the factor $\hat{\Delta}_+(k-l)\hat{\Delta}_+(l)$ they are only unequal to zero for l and $k-l$ on the positive m-mass shell, hence k has to be on or above the $2m$-mass shell. This is also the reason why this contribution is well-defined, since the singularities of $\hat{\Delta}_{R/A}(k)$ are not met. Hence, the term (4.23) is interpreted as coming from the two-particle spectrum. It does not contribute to the dispersion relations that we want to calculate and will therefore not be discussed further.

The sum of the first and third term gives

$$\begin{aligned}
&(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k) \\
&\quad \Bigg(\hat{\Delta}_R(k)\Big[\hat{\Delta}_+ \times \hat{\Delta}_R(k) + \hat{\Delta}_R \times \hat{\Delta}_-(k) + \hat{\Delta}_+ \star_{2\sigma} \hat{\Delta}_R(k) + \hat{\Delta}_R \star_{2\sigma} \hat{\Delta}_-(k)\Big] \\
&+ \hat{\Delta}_A(k)\Big[\hat{\Delta}_- \times \hat{\Delta}_A(k) + \hat{\Delta}_A \times \hat{\Delta}_+(k) + \hat{\Delta}_- \star_{2\sigma} \hat{\Delta}_A(k) + \hat{\Delta}_A \star_{2\sigma} \hat{\Delta}_+(k)\Big]\Bigg).
\end{aligned}$$

The terms containing the convolution stem from the planar graphs, the ones containing the twisting from the nonplanar graphs. The planar graphs give up to a factor of $\frac{1}{2}$ the same as the second order contributions of the model in commutative spacetime, see (4.20), where the cutoff functions $\check{g}(k)$ have to be replaced by $(2\pi)^2\delta(k)$. Though in particular, the planar graphs have to be renormalized, i.e., from the continuation of a distribution to the origin a free mass renormalization, $\frac{1}{2}\lambda^2\mu$, enters.

The nonplanar contributions can be transformed to

$$(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\left(\hat{\Delta}_R(k)\hat{\Delta}_+(k)S_1(k) + \hat{\Delta}_+(k)\hat{\Delta}_A(k)S_2(k)\right), \quad (4.24)$$

where

$$\begin{aligned}
S_1(k) &= \int \mathrm{d}l \hat{\Delta}_+(l)\left(\hat{\Delta}_R(k-l) + \hat{\Delta}_R(k+l)\right)e^{ik\sigma l}, \\
S_2(k) &= \int \mathrm{d}l \hat{\Delta}_+(l)\left(\hat{\Delta}_A(k-l) + \hat{\Delta}_A(k+l)\right)e^{-ik\sigma l}.
\end{aligned} \quad (4.25)$$

We want to calculate these integrals and show that $S_1(k) = S_2(k)$ in the vicinity of the positive m-mass shell using the theory of oscillatory integrals.

So, we set $\Omega := \{k \in \mathbb{M} | k_0 > 0 \text{ and } \frac{m}{2} < \sqrt{k^2} < \frac{3m}{2}\}$. Due to the $\hat{\Delta}_+(l)$ the l integral will only be over the positive mass shell. There, $(k+l)^2 > m^2$ and $(k-l)^2 < m^2$. So the singularities of $\hat{\Delta}_R$ or $\hat{\Delta}_A$ are not met in the above integrals and we can savely set

$$\hat{\Delta}_{R/A}(p) = -\frac{1}{(2\pi)^2}\frac{1}{p^2 - m^2}.$$

We calculate

$$S_1(k) = -\frac{1}{(2\pi)^3}\int \mathrm{d}^3 l \frac{1}{2\omega_l}\left(\frac{1}{(k-l_+)^2 - m^2} + \frac{1}{(k+l_+)^2 - m^2}\right) e^{ik\sigma l_+}, \tag{4.26}$$

using the theory of oscillatory integrals given in appendix B. With the notion given there, we have $t = 3$,

$$\phi(k, \mathbf{l}) = k_\mu \sigma^{\mu\nu}(|\mathbf{l}|, \mathbf{l})_\nu \tag{4.27}$$

and

$$a(k,l) = -\frac{1}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{l}}}\left(\frac{1}{(k-l_+)^2 - m^2} + \frac{1}{(k+l_+)^2 - m^2}\right) e^{i(k\sigma)^0(\sqrt{\mathbf{l}^2+m^2} - |\mathbf{l}|)}.$$

a is an asymptotic symbol[6] on $\Omega \times \mathbb{R}^3$ of order -3. With this $S_1(k) = T_\phi(a)(k)$.

From Theorem B.1.4 we can see that $T_\phi(a)(k)$ is a $\mathcal{C}^\infty$-function of k on Ω since $\nabla_l \phi(k,l)$ is only zero for $k_\mu \sigma^{\mu\nu}$ lightlike and this is not possible on Ω. So, we can assume k to be fixed and consider ϕ as a phase function on $\{k\} \times \mathbb{R}^3$ and a as a symbol on $\{k\} \times \mathbb{R}^3$ and use (B.7).

For $k \in \Omega$ let Λ_k be a boost which takes the vector k to $\Lambda_k k = (\sqrt{k^2}, \mathbf{0})$. Let $g \in \mathcal{D}(\mathbb{R})$ have the property

$$g(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

and define

$$G_{k,n}(\mathbf{l}) := g\left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2}\right),$$

where $\overrightarrow{\Lambda}_k$ is only the vector part of the transformation, i.e., a 3×4 matrix and the square is the Euclidean square of a 3-vector. $G_{k,n}$ is a $\mathcal{C}^\infty$-function of $\mathbf{l}$ and for given k, n it has compact support in $\mathbf{l}$ and is in $\mathrm{Sym}(\{k\}, 3, 0)$ for all n.

[6]It is only asymptotic, since $|\mathbf{l}|$ is not differentiable at $\mathbf{l} = 0$, and one has to use $\sqrt{\mathbf{l}^2 + m^2} - |\mathbf{l}| \leq C(1 + |\mathbf{l}|)^{-1}$, cf. [35].

Lemma 4.4.1. *$G_{k,n} \to 1$ in $Sym(\{k\}, 3, 1)$ for $n \to \infty$.*

Proof. We have to show that for all multi-indices β

$$\sup_{\mathbf{l}} (1 + |\mathbf{l}|)^{|\beta|-1} \left| D_{\mathbf{l}}^{\beta} \left(g \left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2} \right) - 1 \right) \right| \xrightarrow[n\to\infty]{} 0. \tag{4.28}$$

It is easy to see that one can find positive constants d^β such that $\forall \beta$

$$\|D_{\mathbf{l}}^{\beta} l_+\|_{\text{Euclid}} \leq d^{\beta} (1 + |\mathbf{l}|)^{1-|\beta|}.$$

With these one can construct positive constants C_k^β, such that

$$\left| D_{\mathbf{l}}^{\beta} (\overrightarrow{\Lambda}_k l_+)^2 \right| \leq C_k^{\beta} (1 + |\mathbf{l}|)^{2-|\beta|}. \tag{4.29}$$

First, we show (4.28) for $|\beta| = 0$: $\left| g \left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2} \right) - 1 \right|$ is only unequal to zero if $(\frac{\overrightarrow{\Lambda}_k l_+}{n})^2 \geq 1$. We then get, with (4.29),

$$1 + |\mathbf{l}| \geq n \frac{1}{\sqrt{C_k^0}},$$

and with this

$$\sup_{\mathbf{l}} (1 + |\mathbf{l}|)^{-1} \left| g \left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2} \right) - 1 \right| \leq \sup_x |g(x) - 1| \sqrt{C_k^0} \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Now let β be unequal to zero: With (4.29) one can easily see that

$$\left| D_{\mathbf{l}}^{\beta} g \left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2} \right) \right| \leq \sum_{\gamma=1}^{|\beta|} \left| (\partial^{\gamma} g) \left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2} \right) \right| \frac{1}{n^{2\gamma}} \tilde{C}_{k,\beta}^{\gamma} (1 + |\mathbf{l}|)^{2\gamma - |\beta|},$$

where $\tilde{C}_{k,\beta}^{\gamma}$ are again positive constants. For each $\gamma \geq 1$ the function $\partial^{\gamma} g(x)$ is only unequal to 0 if $|x| < 2$. Now, we need the following estimate, which is also not hard to prove,

$$(\overrightarrow{\Lambda}_k l_+)^2 \geq a_k \cdot (1 + |\mathbf{l}|)^2 - b_k,$$

where a_k and b_k are again positive constants. So, if the argument of $\partial^{\gamma} g$, namely $\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2}$, has to be smaller than 2 we conclude

$$\frac{1 + |\mathbf{l}|}{n} \leq \sqrt{\frac{2 + \frac{b_k}{n^2}}{a_k}}.$$

Now, we can deduce

$$\begin{aligned}
&\sup_{\mathbf{l}}(1+|\mathbf{l}|)^{|\beta|-1}\left|D^\beta_{\mathbf{l}}\left(g\left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2}\right)-1\right)\right| \\
&\leq \sup_{\mathbf{l}}\sum_{\gamma=1}^{|\beta|}\left|\partial^\gamma g\left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2}\right)\right|\tilde{C}^\gamma_{k,\beta}\frac{(1+|\mathbf{l}|)^{2\gamma-1}}{n^{2\gamma}} \\
&\leq \sum_{\gamma=1}^{|\beta|}\sup_x|\partial^\gamma g(x)|\tilde{C}^\gamma_{k,\beta}\left(\frac{2+\frac{b_k}{n^2}}{a_k}\right)^{\gamma-\frac{1}{2}}\frac{1}{n}\xrightarrow[n\to\infty]{}0.
\end{aligned}$$

This completes the proof. □

With the above result, we conclude that $G_{k,n}(\mathbf{l})\cdot a(k,\mathbf{l})$ has compact support in $\mathbf{l}$ for the fixed k and approaches a in the topology of $\mathrm{Sym}(\{k\},3,-2)$. Now we want to calculate the integral (4.26). With the result from lemma 4.4.1 we see that it is the $n\to\infty$ limit of

$$-\frac{1}{(2\pi)^3}\int \mathrm{d}^3l\frac{1}{2\omega_{\mathbf{l}}}g\left(\frac{(\overrightarrow{\Lambda}_k l_+)^2}{n^2}\right)\left(\frac{1}{(k-l_+)^2-m^2}+\frac{1}{(k+l_+)^2-m^2}\right)e^{ik\sigma l_+}. \tag{4.30}$$

This integral is absolutely convergent, so the usual techniques for manipulating integrals are available. We perform a (k-dependent) nonlinear transformation on $\mathbf{l}$: $\mathbf{l}'=\overrightarrow{\Lambda}_k l_+$. The integration measure does not change. This transformation is chosen, such that $l_+=\Lambda_k^{-1}l'_+$. The prime will be dropped again and we get:

$$-\frac{1}{(2\pi)^3}\int \mathrm{d}^3l\frac{1}{2\omega_{\mathbf{l}}}g\left(\frac{\mathbf{l}^2}{n^2}\right)\left(\frac{1}{(k-\Lambda_k^{-1}l_+)^2-m^2}+\frac{1}{(k+\Lambda_k^{-1}l_+)^2-m^2}\right)\cdot e^{ik\sigma\Lambda_k^{-1}l_+}.$$

Now we calculate

$$(k\pm\Lambda_k^{-1}l_+)^2=(\Lambda_k^{-1}((\sqrt{k^2},\mathbf{0})\pm l_+))^2=k^2+m^2\pm 2\omega_{\mathbf{l}}\sqrt{k^2}.$$

The two fractions can be put together to give $\frac{2}{k^2-4\omega_{\mathbf{l}}^2}$. Define $\sigma':=\Lambda_k^{-1^T}\sigma\Lambda_k^{-1}$. σ' is again antisymmetric, so $(\sqrt{k^2},\mathbf{0})_\mu\sigma'^{\mu\nu}$ has vanishing time component. Let $\overrightarrow{(\sqrt{k^2},\mathbf{0})\sigma'}$ be its spatial part. Its length is $\sqrt{-((\sqrt{k^2},\mathbf{0})\sigma')^2}=\sqrt{-(k\sigma)^2}$. The expression in the exponent now becomes

$$k\sigma\Lambda_k^{-1}l_+=(\sqrt{k^2},\mathbf{0})\sigma' l_+=-\overrightarrow{(\sqrt{k^2},\mathbf{0})\sigma'}\cdot\mathbf{l}.$$

We use spherical coordinates for $\mathbf{l}$ where the z-axis is along $\overrightarrow{(\sqrt{k^2}, \mathbf{0})\sigma'}$. Then the exponent equals $\sqrt{-(k\sigma)^2}l\cos(\theta)$, and after performing the integration over ϕ and θ we get

$$\begin{aligned}
&-(2\pi)^{-2}\int_0^\infty \mathrm{d}l \int_0^\pi \mathrm{d}\theta\; g\left(\frac{l^2}{n^2}\right)\frac{l^2\sin(\theta)}{\omega_l(k^2-4\omega_l^2)}e^{l\sqrt{-(k\sigma)^2}\cos(\theta)}\\
=&-2(2\pi)^{-2}\int_0^\infty \mathrm{d}l\; g\left(\frac{l^2}{n^2}\right)\frac{l^2}{\omega_l(k^2-4\omega_l^2)}\frac{\sin(l\sqrt{-(k\sigma)^2})}{l\sqrt{-(k\sigma)^2}}.
\end{aligned}$$

For $n\to\infty$ this gives the value of $T_\phi(a)(k) = S_1(k)$, which is the absolutely convergent integral

$$S(k) := -2(2\pi)^{-2}\int_0^\infty \mathrm{d}l\; \frac{l}{\omega_l(k^2-4\omega_l^2)}\frac{\sin(l\sqrt{-(k\sigma)^2})}{\sqrt{-(k\sigma)^2}}. \tag{4.31}$$

It is straightforward to see that $S_2(k)$ gives the same result, since the only difference for $k\in\Omega$ is to replace $k\sigma$ by $-k\sigma$.

So, we have $S_1(k) = S_2(k) = S(k)$ for $k\in\Omega$. Then, we get for (4.24)

$$(2\pi)^4\int \mathrm{d}k\check{f}(k)\check{h}(-k)\left(\hat{\Delta}_R(k)\hat{\Delta}_+(k)+\hat{\Delta}_+(k)\hat{\Delta}_A(k)\right)S(k).$$

Now we can use equation (4.9) and compare with (4.4) to get

$$\Sigma\left(k^2,(k\sigma)^2\right) = S(k).$$

Actually, this is only the part coming from the nonplanar graphs. The contribution of the planar graphs is up to a factor of $\frac{1}{2}$ the same as in section 4.2.1 and gives momentum independent renormalizations.

We get, after a variable transformation, the following results:

$$\begin{aligned}
\lambda^2 M\left((k\sigma)^2\right) &= -\lambda^2\frac{2}{(2\pi)^2}\int_0^\infty \mathrm{d}l\frac{l}{\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)}\sin(l),\\
\lambda^2 Z\left((k\sigma)^2\right) &= \lambda^2\frac{2}{(2\pi)^2}\int_0^\infty \mathrm{d}l\frac{(k\sigma)^2 l}{\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)^2}\sin(l),\\
\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2) &= \lambda_{\mathrm{nc}}^4\lambda^2\frac{4}{(2\pi)^2}\int_0^\infty \mathrm{d}l\Bigg(\frac{3m^2 l}{\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)^2}\\
&\qquad -\frac{m^2 l}{2(-(k\sigma)^2m^2+l^2)^{\frac{3}{2}}(4l^2-3(k\sigma)^2m^2)^2}\Bigg)\sin(l).
\end{aligned}$$

We want to calculate these depending on the perpendicular momentum $k_\perp$, using (4.12) with $k^2 = m^2$. We use the parameters $\lambda_{\mathrm{nc}} = \lambda_P = 1$ (i.e. we use $c = \hbar = G = 1$ and use the Planck length for the scale of noncommutativity), $\sigma = \sigma_0$, $m = 10^{-17}$ and $\lambda = m$. The coupling λ is of mass dimension 1 in four spacetime dimensions. The orders of magnitude of the last two parameters are chosen, such that the identification of the scalar ϕ with the Higgs field is possible, see [34]. A mass of $m = 10^{-17}$ corresponds to approximately $m = 122\ GeV$ in common units. Actually, the value for the coupling would be $\lambda = 0.72 \cdot m$ for this mass (using the standard Higgs model and experimental results like the mass of the W-Boson). As we are only interested in the orders of magnitude we can savely set $\lambda = m$.

We use Mathematica to calculate the remaining absolutly convergent integrals numerically. $M\left((k\sigma)^2\right)$ is shown in figure 4.10. $Z\left((k\sigma)^2\right)$ is constant in the plotted region within machine precision. It gives

$$\lambda^2 Z\left((k\sigma)^2\right) \approx 1.32477 \cdot 10^{-3}.$$

The same contribution stems from the planar graphs and together they have the same value as the field strength renormalization in the commutative case (4.21). The distortion of the group velocity is shown in figure 4.11. All quantities have the behaviour, that their absolute values are largest for $k_\perp = 0$ and they tend to zero for $k_\perp \to \infty$. We see that the distortion of the group velocity is of the order of magnitude of percentages for small perpendicular momenta. This might be detectable if the Higgs boson is discovered in the next generation of colliders (LHC or ILC). The relative mass renormalization is almost -1 at $k_\perp = 0$. If we use the correct value of $\lambda = 0.72 \cdot m$ this would correspond to $\sqrt{m^2 + \lambda^2 M(k_\perp = 0)} \approx 85\ GeV$. This is not compatible with the experiment. However, we still have a mass renormalization from the planar part. With this, the mass at $k_\perp = 0$ can be set back to $122\ GeV$, but at higher perpendicular momenta we would have an increasing of the mass by almost a factor of $\sqrt{2}$.

We emphasize again that, in order to calculate the dispersion relation at the one-loop level, it is sufficient to know

$$S_{1/2}(k) = \int \mathrm{d}^4 l\ \hat{\Delta}_+(l) e^{\pm ik\sigma l} \left(\hat{\Delta}_{R/A}(k-l) + \hat{\Delta}_{R/A}(k+l)\right), \tag{4.32}$$

for k in the vicinity of the mass shell. However, when we want to calculate higher orders, the nonplanar fish-graphs shown in figures 4.3 and 4.5, which gave the contributions (4.24), may appear as subgraphs and have to be integrated over arbitrary k. Thus, there appears the problem that $\hat{\Delta}_{R/A}(k \pm l_+)$ can become singular. The singularities of $\hat{\Delta}_{R/A}$ lie on the m-mass shell. So:

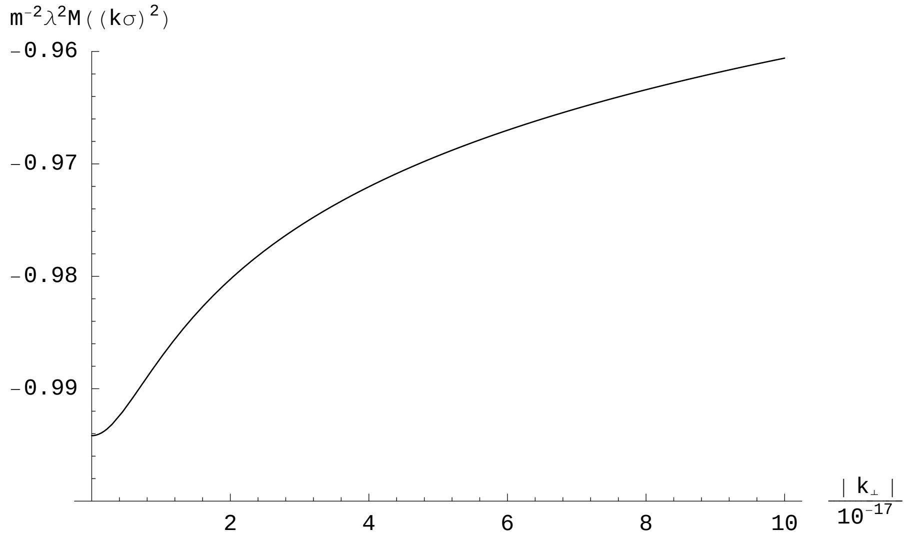

Figure 4.10: The relative mass correction $m^{-2}\lambda^2 M((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^3 model in four dimensions.

- The situation is smooth for $0 < k^2 < (2m)^2$ since neither $k+l_+$ nor $k-l_+$ can meet the mass shell.
- For $k^2 \geq (2m)^2$ and $k_0 > 0$ the vector $k-l_+$ can lie on the mass shell, for $k_0 < 0$ the other one.
- For k spacelike, both $k+l_+$ and $k-l_+$ can meet the mass shell. Second, one expects singularities for $k\sigma$ lightlike, since there $\nabla_l \phi(k,l) = 0$, cf. (4.27) and (B.3). For k this is the tilted lightcone around $k_\perp$. Thus, these overlap with the singularities from $\hat{\Delta}_{R/A}$.

Thus, it is yet unclear, how to handle the integrals in $S_{1/2}(k)$ outside the set $0 < k^2 < (2m)^2$, as the preliminaries for oscillatory integrals are not fulfilled. $\hat{\Delta}_{R/A}(k \pm l_+)$ cannot be treated as an asymptotic symbol, since the set where it may become singular is not compact in k. Thus, one has to extend the theory of oscillatory integrals to handle graphs of higher orders. Some ideas can be found at the end of Appendix B.

The calculation above was for four dimensions. The calculation in six dimensions is quite similar. We get a different prefactor and the l integration is two dimensions higher. So, instead of (4.26) we have

$$S_{1/2}(k) = -\frac{1}{(2\pi)^5} \int \mathrm{d}^5 l \frac{1}{2\omega_l} \left(\frac{1}{(k-l_+)^2 - m^2} + \frac{1}{(k+l_+)^2 - m^2} \right) e^{\pm ik\sigma l_+}.$$

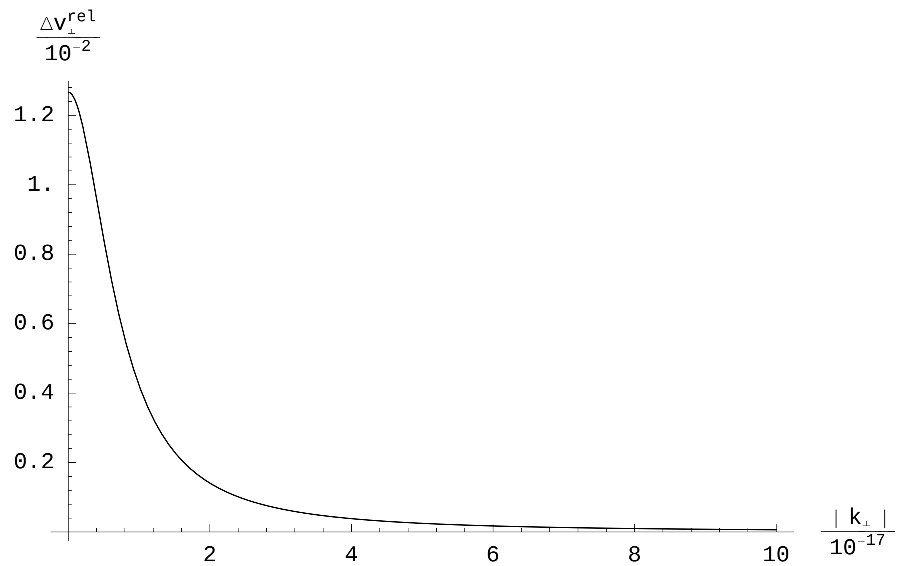

Figure 4.11: The distortion of the group velocity in perpendicular direction $\Delta v_\perp^{\text{rel}}((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^3 model in four dimensions.

The technique to calculate this oscillatory integral is analogously to the one before: Multiply the integrand by $g\left((\frac{\overrightarrow{\Lambda}_k l_+}{n})^2\right)$ (where Λ_k is correspondingly defined for six dimensions), make a variable transformation in l and use spherical coordinates in five dimensions, to get

$$
\begin{aligned}
&-\frac{1}{2(2\pi)^3}\int_0^\infty \mathrm{d}l \int_0^\pi \mathrm{d}\theta_3 \; g\left(\frac{l^2}{n^2}\right)\frac{l^4\sin^3(\theta_3)}{\omega_l(k^2-4\omega_l^2)}e^{l\sqrt{-(k\sigma)^2}\cos(\theta_3)}\\
=&\frac{-2}{(2\pi)^3}\int_0^\infty \mathrm{d}l \; g\left(\frac{l^2}{n^2}\right)\frac{l^4}{\omega_l(k^2-4\omega_l^2)}\left[\frac{\sin(l\sqrt{-(k\sigma)^2})}{\left(l\sqrt{-(k\sigma)^2}\right)^3}-\frac{\cos(l\sqrt{-(k\sigma)^2})}{\left(l\sqrt{-(k\sigma)^2}\right)^2}\right].
\end{aligned}
$$

The factor of $\sin^3(\theta_3)$ stems from the spherical volume element in five dimension. Thus, with $n\to\infty$, we get in six dimensions the improper Riemann integral

$$
\begin{aligned}
S(k) := \frac{-2}{(2\pi)^3}\int_0^\infty \mathrm{d}l \; &\frac{l^2}{-(k\sigma)^2\omega_l(k^2-4\omega_l^2)}\\
&\left[\frac{\sin(l\sqrt{-(k\sigma)^2})}{l\sqrt{-(k\sigma)^2}}-\cos(l\sqrt{-(k\sigma)^2})\right].
\end{aligned}
$$

This gives

$$\lambda^2 M\left((k\sigma)^2\right) = -\lambda^2 \frac{2}{(2\pi)^3}\int_0^\infty \mathrm{d}l \frac{l}{-(k\sigma)^2\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)} \cdot \left(\frac{\sin(l)}{l}-\cos(l)\right),$$

$$\lambda^2 Z\left((k\sigma)^2\right) = \lambda^2 \frac{2}{(2\pi)^3}\int_0^\infty \mathrm{d}l \frac{l}{-\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)^2} \cdot \left(\frac{\sin(l)}{l}-\cos(l)\right),$$

$$\begin{aligned}\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2) = \frac{4\lambda_{\mathrm{nc}}^4\lambda^2}{(2\pi)^3}\int_0^\infty \mathrm{d}l \Bigg(&\frac{3m^2 l}{-(k\sigma)^2\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)^2} \\ &- \frac{m^2 l}{-2(k\sigma)^2(-(k\sigma)^2m^2+l^2)^{\frac{3}{2}}(4l^2-3(k\sigma)^2m^2)^2} \\ &+ \frac{l}{(k\sigma)^4\sqrt{-(k\sigma)^2m^2+l^2}(4l^2-3(k\sigma)^2m^2)^2}\Bigg) \\ &\cdot \left(\frac{\sin(l)}{l}-\cos(l)\right).\end{aligned}$$

Again we plot $M\left((k\sigma)^2\right)$ and $Z\left((k\sigma)^2\right)$ and $\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2)$ depending on the perpendicular momentum $k_\perp$. We use the same parameters as in four dimensions except that we choose $\lambda = 1$ instead of $\lambda = m$, since the coupling has no mass dimension in six dimensions. $M\left((k\sigma)^2\right)$ is shown in figure 4.12, $Z\left((k\sigma)^2\right)$ in figure 4.12 and $\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2)$ in figure 4.14. The main difference compared to four dimensions is, that the order of magnitude is totally different. Instead of percentages $\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2)$ is of the order of 10^{65} and similarly for $M\left((k\sigma)^2\right)$. The reason for these differences in the order of magnitude stems to a great part from a factor $(k\sigma)^2$ in the denominator and the change in λ. The field strength renormalization is comparatively small because its integral is absolutely convergent and the factor $\frac{\sin(l)}{l} - \cos(l)$ is almost zero where the fraction in front under the integral sign is unequal to zero and vice versa.

However, the mass correction can again be brought to zero using the mass renormalization from the planar graph.[7] But it is questionable if the calculated result is reasonable since the use of perturbation methods seems not to be justifiable if the result differs that much from the unperturbed setting. (If we look as the mass renormalization it has to differ somewhere by

[7] In six dimensions there appears also a field strength renormalization in the ϕ^3 model.

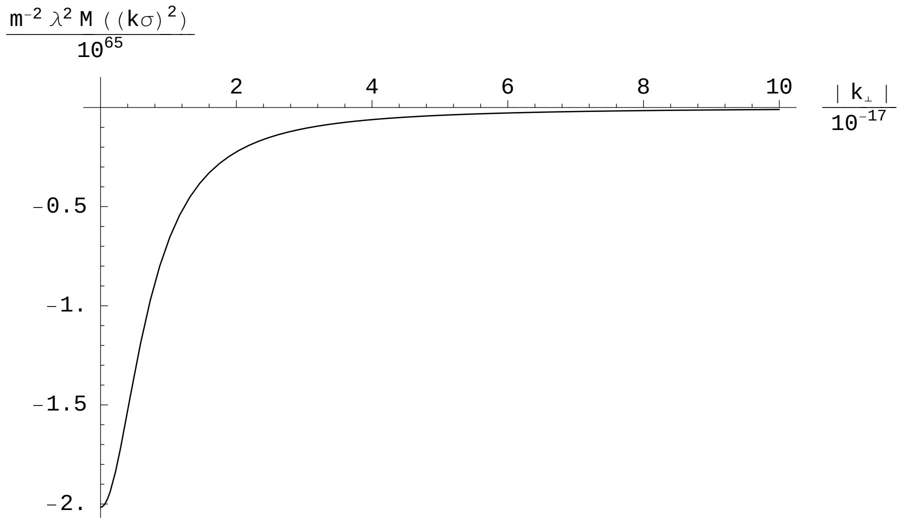

Figure 4.12: The relative mass correction $m^{-2}\lambda^2 M((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^3 model in six dimensions.

a factor of $10^{(65)}$ from the free case, either at small or at large $k_\perp$, no matter how the planar mass renormalization is chosen.)

4.5 ϕ^4 model on $\mathbb{M}_{\mathbf{nc}}$ using quasiplanar Wick products

Now, we investigate the ϕ^4 model on $\mathbb{M}_{\mathrm{nc}}$. Here, we use quasiplanar Wick products as defined in [4]. Quasiplanar Wick products are defined for free quantum fields on noncommutative spacetime. They are similar to the well-known Wick products for commutative spacetime. A product of multiple fields is defined in the limit of coinciding points by subtracting contractions. The subtracted contractions have to be local in a certain sense. The noncommutativity of spacetime leads to a different concept of locality. Thereby, some contractions, which are subtracted in the commutative spacetime, become nonlocal and remain finite in the limit of coinciding points, and are therefore not subtracted.

The field equation is

$$\left(\Box_q + m^2\right)\Phi(q) = -\lambda\Phi(q)^3.$$

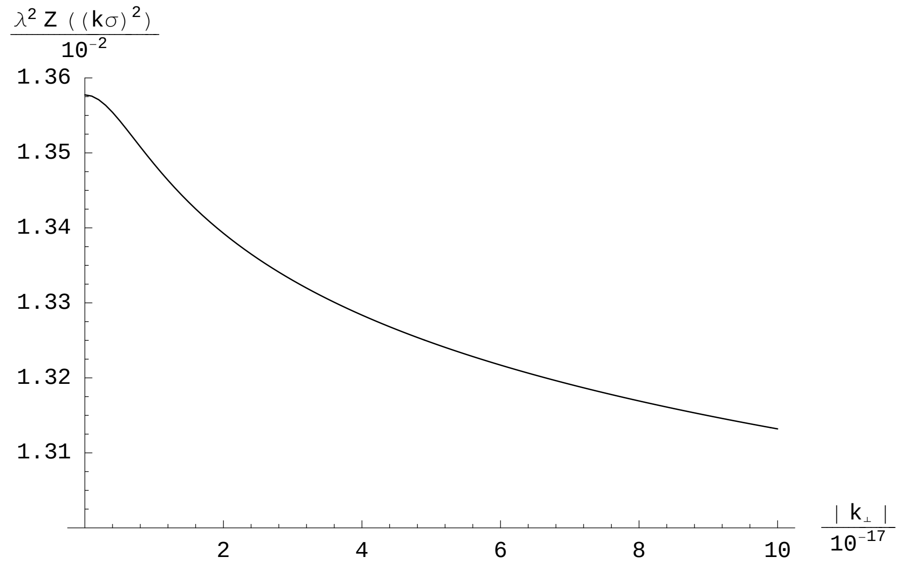

Figure 4.13: The field strength renormalization $\lambda^2 Z((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^3 model in six dimensions.

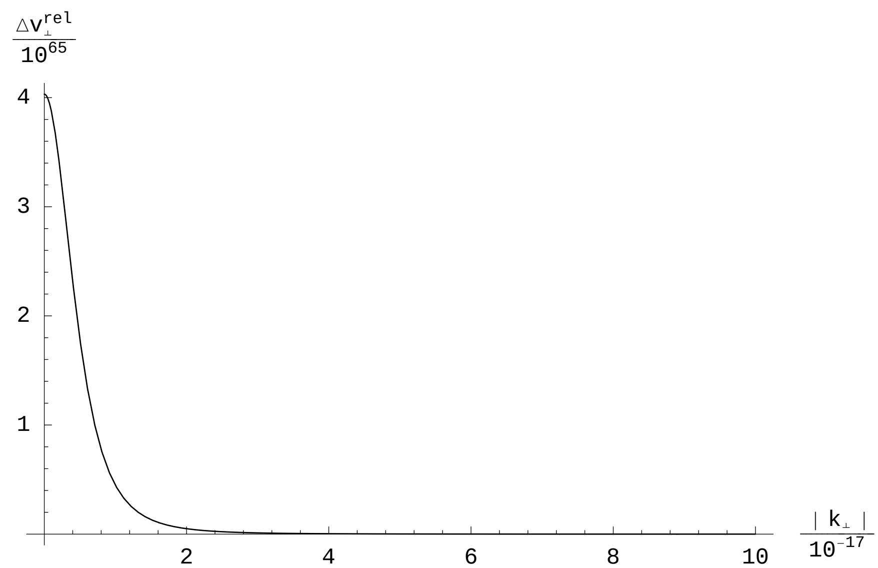

Figure 4.14: The distortion of the group velocity in perpendicular direction $\Delta v_\perp^{\text{rel}}((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^3 model in six dimensions.

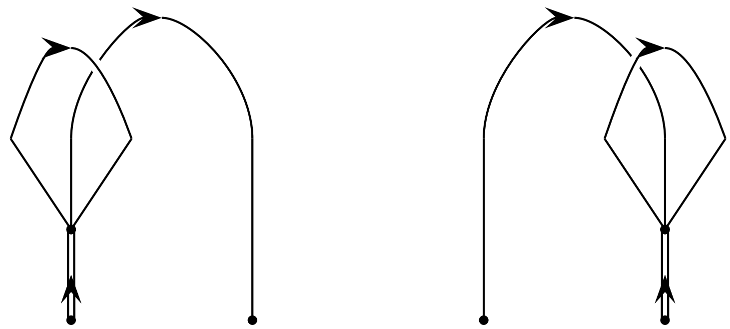

Figure 4.15: First order nonplanar contributions to the two-point function of the ϕ^4 model, $\langle \Phi_1 \Phi_0 \rangle$ and $\langle \Phi_0 \Phi_1 \rangle$.

This gives for the interacting field to first orders

$$\begin{aligned}
\Phi_0(q) =& \Phi_{\mathrm{Free}}(q), \\
\Phi_1(q) =& -\int \mathrm{d}x \Delta_R(x) \vdots\, \Phi_0^3(q-x) \,\vdots \\
=& -\Bigg(\int \mathrm{d}x \Delta_R(x) : \Phi_0^3(q-x) : \\
&+ \frac{1}{(2\pi)^2} \int \mathrm{d}x \Delta_R(x) \int \mathrm{d}k \Delta_+(-k\sigma) \hat{\Phi}_0(k) e^{-ik(q-x)} \Bigg). \qquad (4.33)
\end{aligned}$$

In this case, the first order contribution does not vanish anymore. The part (4.33) of $\Phi_1(q)$ gives

$$\begin{aligned}
&\langle \Phi_1(f) \Phi_0(h) \rangle + \langle \Phi_0(f) \Phi_1(h) \rangle \\
&\quad = -(2\pi)^2 \int \mathrm{d}k \check{f}(k) \check{h}(-k) \Delta_+(k\sigma) \left(\hat{\Delta}_R(k) \hat{\Delta}_+(k) + \hat{\Delta}_+(k) \hat{\Delta}_A(k) \right).
\end{aligned}$$

Here, we have used that, due to the $\hat{\Delta}_+(k)$, the whole integral is only evaluated for k on the mass shell. Then $k\sigma$ is spacelike and thus $\Delta_+(-k\sigma) = \Delta_+(k\sigma)$. We use equation (4.9) to transform the above to

$$\int \mathrm{d}k \check{f}(k) \check{h}(-k) \Delta_+(k\sigma) \partial_{m^2} \hat{\Delta}_+^{(m^2)}(k).$$

If we compare this with (4.4), we get in this case

$$\Sigma(k^2, (k\sigma)^2) = -\frac{1}{(2\pi)^2} \Delta_+(k\sigma).$$

It is well known[8] that for spacelike argument x

$$\Delta_+(x) = \frac{1}{(2\pi)^2}\frac{m}{\sqrt{-x^2}}\mathrm{BesselK}_1\left(m\sqrt{-x^2}\right).$$

$\mathrm{BesselK}_n$ is the modified Bessel function of the second kind to order n. Thus, we get

$$\begin{aligned}\lambda M\left((k\sigma)^2\right) &= \lambda\frac{1}{(2\pi)^4}\frac{m}{\sqrt{-(k\sigma)^2}}\mathrm{BesselK}_1\left(m\sqrt{-(k\sigma)^2}\right),\\ \Delta v_\perp^{\mathrm{rel}}((k\sigma)^2) &= -2\lambda_{\mathrm{nc}}^4\lambda\frac{1}{(2\pi)^4}\left(\frac{m}{2(-(k\sigma)^2)^{\frac{3}{2}}}\mathrm{BesselK}_1\left(m\sqrt{-(k\sigma)^2}\right)\right.\\ &\quad+\frac{m^2}{4\sqrt{-(k\sigma)^2}}\left[\mathrm{BesselK}_0\left(m\sqrt{-(k\sigma)^2}\right)\right.\\ &\quad\left.\left.+\mathrm{BesselK}_2\left(m\sqrt{-(k\sigma)^2}\right)\right]\right).\end{aligned}$$

Since in this case Σ is independent from its first argument, $Z\left((k\sigma)^2\right)$ is zero.

We calculate these for a set of parameters similar to the case in ϕ^3 theory, i.e. $\lambda_{\mathrm{nc}} = \lambda_P = 1$, $\sigma = \sigma_0$, $m = 10^{-17}$ and $\lambda = 1$. $M\left((k\sigma)^2\right)$ is shown in figure 4.16 and The distortion of the group velocity is shown in figure 4.17. Comparing with the ϕ_6^3 model, we see that the order of magnitude for the calculated quantities are equal. The sign is different. This latter is connected to the fact that here, the calculated quantities are of first order in λ and not in second order.

The setting used here differs slightly from the one given in [4, 2]. The reason is, that we have treated all quantities as a formal power series in the coupling λ. This was not done rigorously in the before mentioned publication. However, at the end we insert a finite λ. So higher orders can cancel each other and a finetuning process might still be possible. But as we mentioned before, the use of perturbation methods are questionable if the first order corrections are of this order of magnitude.

4.6 Wess-Zumino model

Now, we have a look at a supersymmetric model, namely the Wess-Zumino model, which is one of the simplest of this kind. Supersymmetric models have a better behaviour with respect to divergences because some divergent

[8] This could easily be calculated using the framework of oscillatory integrals, too.

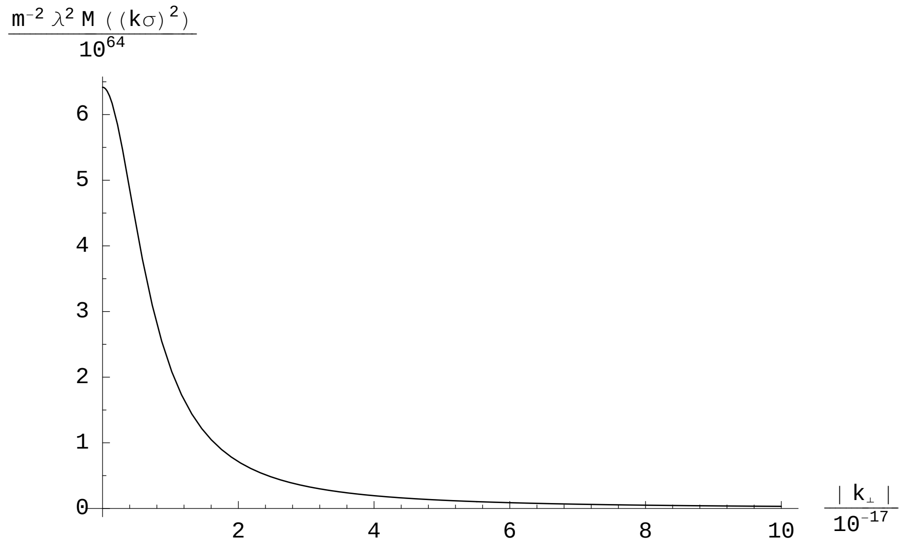

Figure 4.16: The relative mass correction $m^{-2}M((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^4 model using quasiplanar Wick products.

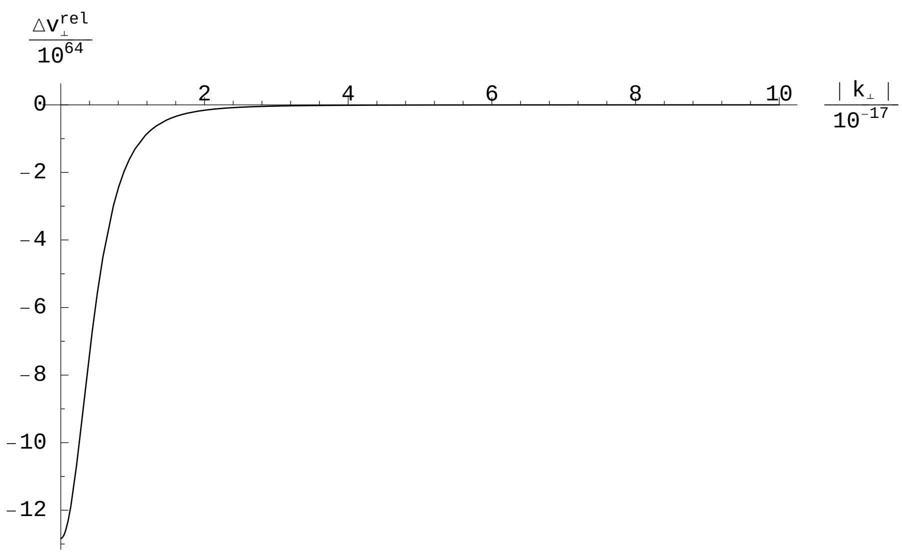

Figure 4.17: The distortion of the group velocity in perpendicular direction $\Delta v_\perp^{\mathrm{rel}}((k\sigma)^2)$ as a function of the perpendicular momentum $k_\perp$ for the ϕ^4 model using quasiplanar Wick products.

graphs, coming from different particle interactions, cancel each other. Supersymmetry takes care that the parameters have the right values, e.g., that fermions and bosons have the same mass parameter and that the coupling constants of the different interactions have a particular dependence in order to make the cancellation possible.[9] The calculation shown here is very close to the one given by Zahn in [15].

When working with supersymmetric models notation is a rather involved task. We use the conventions of [43] except for the metric. We keep our metric of signature $(+,-,-,-)$. Further changes have to be made to keep the zero component of the generator of translations, P_0, positive in representations of the supersymmetry algebra. We have to multiply σ^0 by -1.[10] The same happens to γ^0 and γ^5. The usual (anti-)commutation relations of these quantities do not change, except where the metric appears. In particular, we have $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

The chiral superfield is[11]

$$\Phi(q) = \phi(q) + \sqrt{2}\theta^\alpha \chi_\alpha(q) + \theta^2 F(q) \\ + i\theta^\alpha \sigma^\mu_{\alpha\dot\beta} \bar\theta^{\dot\beta} \partial_\mu \phi(q) - \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \chi^\alpha(q) \sigma^\mu_{\alpha\dot\beta} \bar\theta^{\dot\beta} - \frac{1}{4}\theta^2\bar\theta^2 \Box\phi(q),$$

where θ is a complex, χ a Weyl spinor and F an auxiliary field.

When dealing with complex quantum fields on noncommutative spacetime we have to mind that different orders of terms can give different twisting factors. E.g., the term $\phi^*\phi\phi^*\phi$ has only planar contractions, whereas $\phi^*\phi^*\phi\phi$ has also a nonplanar one.[12] To keep track of the orders in the field equations we derive them from a Lagrangian. We use the formalism introduced in [45] for classical fields on noncommutative spacetime.

[9] This follows from the nonrenormalization theorem, see for example [8].
[10] This follows from the fact that $\mathrm{Tr}(\sigma_0 \cdot P_0) = \mathrm{Tr}\{Q, \bar Q\}$ is a positive operator. With the conventions and signature of [43] only P^0 would be positive.
[11] The anticommutation relations among the fermionic variables θ and $\bar\theta$ are unchanged. Approaches, where even these are deformed, can be found, e.g., in [30].
[12] This was already noted in [1].

The action is

$$\begin{aligned}\int \mathrm{d}q^4 L(q) &= \int \mathrm{d}q^4 \left(\bar{\Phi}\Phi\Big|_{\theta^2\bar{\theta}^2} - \left[\frac{1}{2}m\Phi\Phi + \frac{1}{3}\lambda\Phi\Phi\Phi + \text{h.c.}\right]\Big|_{\theta^2} \right) \\ &= \int \mathrm{d}q^4 \Bigg(i\partial_\mu\bar{\chi}\bar{\sigma}^\mu\chi - \phi^*\Box\phi + F^*F \\ &\quad - \left[\left(m\left(\phi F - \frac{1}{2}\chi\chi\right) + \lambda\left(\phi\phi F - \chi\chi\phi\right)\right) + \text{h.c.}\right] \\ &\quad + \text{ total derivatives} \Bigg).\end{aligned}$$

As we have seen in section 2.1 $\int \mathrm{d}q^4$ is cyclic. So it does not matter whether we would have taken $\Phi\bar{\Phi}$ in the Lagrangian instead. The equations of motion are derived via the variation principle:

$$\text{variation } \delta\phi^* : \quad \Box\phi + mF^* + \lambda(\phi^*F^* + F^*\phi^*) - \lambda\bar{\chi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = 0 \tag{4.34}$$

$$\text{variation } \delta\bar{\chi}_{\dot{\alpha}} : \quad i\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\chi_\beta - m\bar{\chi}^{\dot{\alpha}} - \lambda(\phi^*\bar{\chi}^{\dot{\alpha}} + \bar{\chi}^{\dot{\alpha}}\phi^*) = 0 \tag{4.35}$$

$$\text{variation } \delta F^* : \quad F - m\phi^* - \lambda\phi^*\phi^* = 0. \tag{4.36}$$

To simplify the forthcoming calculation, we introduce Majorana spinors :

$$\psi := \begin{pmatrix}\chi_\alpha \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}, \quad \bar{\psi} := \psi^\dagger\gamma^0 = (\chi^\alpha, \bar{\chi}_{\dot{\alpha}}),$$

and the projectors

$$P_+ := \begin{pmatrix}\mathbb{1}_2 & 0 \\ 0 & 0\end{pmatrix} \text{ and } P_- := \begin{pmatrix}0 & 0 \\ 0 & \mathbb{1}_2\end{pmatrix}.$$

Thus, we have

$$P_\pm = \frac{1}{2}\left(\mathbb{1}_4 \mp i\gamma^5\right), \qquad \chi^\alpha\chi_\alpha = \bar{\psi}P_+\psi \quad \text{and} \quad \bar{\chi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}P_-\psi.$$

With this notation, (4.35) and its hermitian conjugate become

$$\begin{aligned}(i\partial\!\!\!/ - m\mathbb{1}_4)\psi &= \lambda\left(P_+(\psi\phi + \phi\psi) + P_-(\psi\phi^* + \phi^*\psi)\right), \\ \partial_\mu\bar{\psi}(-i\gamma^\mu - m\mathbb{1}_4) &= \lambda\left((\bar{\psi}\phi + \phi\bar{\psi})P_+ + (\bar{\psi}\phi^* + \phi^*\bar{\psi})P_-\right),\end{aligned} \tag{4.37}$$

where we have introduced the notation $\partial\!\!\!/ := \partial_\mu\gamma^\mu$ (and likewise for momenta k_μ). The contraction of the free fermion field is

$$\langle\hat{\bar{\psi}}_a(k_1)\hat{\psi}_b(k_2)\rangle = (2\pi)^2(-k\!\!\!/_1 + m\mathbb{1}_4)_{ba}\hat{\Delta}_+(k_1)\delta(k_1 + k_2).$$

Now, we insert the equation of motion of the auxiliary field F, (4.36), into (4.34) and get

$$(\Box + m^2)\phi = \lambda\bar{\psi}P_-\psi - m\lambda(\phi\phi + \phi^*\phi + \phi\phi^*) - \lambda^2(\phi^*\phi\phi + \phi\phi\phi^*). \quad (4.38)$$

Note that the last part is of order λ^2 and therefore does not appear in the calculation of ϕ_1.

So far, we have dealt with classical (bosonic or fermionic) fields on non-commutative spacetime. Now we replace them by quantum fields (here denoted by the same symbols). We have a closer look at the scalar field ϕ and want to calculate the two-point function $\langle\phi^*(f)\phi(h)\rangle$. From (4.38) we get

$$\hat{\phi}_1(k_1) = \hat{\Delta}_R(k_1)\int \mathrm{d}k_2 e^{-\frac{i}{2}k_2\sigma k_1}\Big[\hat{\bar{\psi}}_0(k_2)P_-\hat{\psi}_0(k_1-k_2)$$
$$- m\Big(\hat{\phi}_0(k_2)\hat{\phi}_0(k_1-k_2) + \hat{\phi}_0^*(k_2)\hat{\phi}_0(k_1-k_2) + \hat{\phi}_0(k_2)\hat{\phi}_0^*(k_1-k_2)\Big)\Big]. \quad (4.39)$$

As we are now dealing with quantum fields, we have to ask whether the products of fields on the right hand side of (4.38) are well-defined as they arise from multiplying distributions at the same point like, e.g., $\phi_0^*(q)\phi_0(q)$. We do not normal order these. Although the summands alone have divergences, their sum is well-defined. In fact, the vacuum expectation value of ϕ_1 vanishes:

$$\langle\phi_1(f)\rangle = \int \mathrm{d}k_1\mathrm{d}k_2\check{f}(k_1)\hat{\Delta}_R(k_1)\Big[(2\pi)^2\mathrm{Tr}\left(P_-(-\not{k}_2 + m\mathbb{1})\right)\hat{\Delta}_+(k_2)\delta(k_1)$$
$$- 2m(2\pi)^2\hat{\Delta}_+(k_2)\delta(k_1)\Big] = 0,$$

since $\mathrm{Tr}\left(P_-(-\not{k}_2 + m\mathbb{1})\right) = 2m$. This calculation seems rather formal but can be made rigorous, and the expression (4.39) equals the one where the right-hand side is normal ordered. In other words, the additional term needed for normal ordering the fermion fields cancels the ones for normal ordering the scalar fields. This is one example for how divergences cancel each other in supersymmetric field models.

As it is easy to see, the two-point function of the scalar field at first order vanishes:

$$\langle\phi_0^*(f)\phi_1(h)\rangle = \langle\phi_1^*(f)\phi_0(h)\rangle = 0.$$

Thus, we have to look at second order. ϕ_2 can be divided into three parts:

$$\phi_2 = \Delta_R \times \Big[\bar{\psi}_0P_-\psi_1 + \bar{\psi}_1P_-\psi_0 \qquad \text{(Yukawa part)}$$
$$- m\left(\phi_0\phi_1 + \phi_1\phi_0 + \phi_0^*\phi_1 + \phi_1^*\phi_0 + \phi_0\phi_1^* + \phi_1\phi_0^*\right) \qquad (\phi^3 \text{ part})$$
$$- \left(\phi_0^*\phi_0\phi_0 + \phi_0\phi_0\phi_0^*\right)\Big]. \qquad (\phi^4 \text{ part})$$

Note that the ϕ^4 part comes directly from the λ^2 part of (4.38). We do not normal order this part since the quadratical divergence coming from it cancels (partly) divergences coming from other parts as we will see later.

First, we calculate the fermion fields at first order. These appear in the Yukawa part. For this we define the Green's functions $S_R(x) := (-i\partial\!\!\!/ - m\mathbb{1})\Delta_R(x)$ and $\bar{S}_R(x) := (i\partial\!\!\!/ - m\mathbb{1})\Delta_R(x)$. These fulfill

$$(i\partial\!\!\!/_x - m\mathbb{1}_4)S_R(x-y) = (-i\partial\!\!\!/_x - m\mathbb{1}_4)\bar{S}_R(x-y)\gamma = \delta^{(4)}(x-y)\mathbb{1}_4,$$
$$\hat{S}_R(k) = (-k\!\!\!/ - m\mathbb{1})\hat{\Delta}_R(k) \quad \text{and} \quad \hat{\bar{S}}_R(k) = (k\!\!\!/ - m\mathbb{1})\hat{\Delta}_R(k).$$

The fermion field (4.37) gives at first order

$$\begin{aligned}\hat{\psi}_1(k_1) = \hat{S}_R(k_1) \int \mathrm{d}k_2 e^{-\frac{i}{2}k_2\sigma k_1} \\ \Big[P_+ \Big(\hat{\psi}_0(k_2)\hat{\phi}_0(k_1-k_2) + \hat{\phi}_0(k_2)\hat{\psi}_0(k_1-k_2)\Big) \\ + P_- \Big(\hat{\psi}_0(k_2)\hat{\phi}_0^*(k_1-k_2) + \hat{\phi}_0^*(k_2)\hat{\psi}_0(k_1-k_2)\Big)\Big]\end{aligned}$$

and

$$\begin{aligned}\hat{\bar{\psi}}_1(k_1) = \int \mathrm{d}k_2 e^{-\frac{i}{2}k_2\sigma k_1} \\ \Big[\Big(\hat{\bar{\psi}}_0(k_2)\hat{\phi}_0(k_1-k_2) + \hat{\phi}_0(k_2)\hat{\bar{\psi}}_0(k_1-k_2)\Big) P_+ \\ + \Big(\hat{\bar{\psi}}_0(k_2)\hat{\phi}_0^*(k_1-k_2) + \hat{\phi}_0^*(k_2)\hat{\bar{\psi}}_0(k_1-k_2)\Big) P_-\Big]\hat{\bar{S}}_R(k_1).\end{aligned}$$

There is no need to normal order the products of fields on the right-hand sides, since only products of commuting fields appear.[13]

We are only interested in the modification of the dispersion relations. So, we do not look at $\langle\phi_1^*(f)\phi_1(h)\rangle$ since all terms coming from these contain Δ_+^2 or $\Delta_+ \star_{2\sigma} \Delta_+$ and thus vanish in the vicinity of the m-mass shell.

It remains to examine the sum $\langle\phi_0^*(f)\phi_2(h)\rangle + \langle\phi_2^*(f)\phi_0(h)\rangle$, from which

[13] The products of different fields like the terms appearing in the expressions for $\hat{\psi}_1$ or $\hat{\bar{\psi}}_1$ do not make problems (at first order), since the field algebra parts of the factors live on a tensor product of Fock spaces. Thus, $\psi_0(x)\phi_0(x)$ is rather $\psi_0(x)\otimes\mathbb{1}\cdot\mathbb{1}\otimes\phi_0(x) = \psi_0(x)\otimes\phi_0(x)$ and this is a well-defined operator-valued distribution on $\mathcal{S}$. The situation for $\psi_0(q)\phi_0(q)$ is corresponding.

we will calculate the first summand now. The Yukawa part of ϕ_2 gives

$$
\begin{aligned}
&(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k)\hat{\Delta}_R(-k) \\
&\quad \int \mathrm{d}l \left(1+e^{-ik\sigma l}\right) \Big[\hat{\Delta}_+(l)\hat{\Delta}_R(-k-l)\mathrm{Tr}\left(P_-(\not{k}+\not{l}-m\mathbb{1})P_+(-\not{l}+m\mathbb{1})\right) \\
&\qquad\qquad + \hat{\Delta}_+(l)\hat{\Delta}_R(l-k)\mathrm{Tr}\left(P_+(\not{l}-\not{k}-m\mathbb{1})P_-(-\not{l}+m\mathbb{1})\right)\Big].
\end{aligned}
$$

We use $P_+P_- = P_-P_+ = 0$, so the terms which have factors of m drop out. After a short calculation we see that $\mathrm{Tr}\left(P_-\gamma^\mu P_+\gamma^\nu\right) = 2\eta^{\mu\nu}$. The remaining terms transform to

$$
\begin{aligned}
&-2(2\pi)^4 \int \mathrm{d}k\mathrm{d}l \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k)\hat{\Delta}_A(k)\hat{\Delta}_+(l) \\
&\qquad \Big[\hat{\Delta}_A(k+l)(l+k)\cdot l + \hat{\Delta}_A(k-l)(l-k)\cdot l\Big]\left(1+e^{-ik\sigma l}\right). \qquad (4.40)
\end{aligned}
$$

The contribution coming from the ϕ^3 part,

$$
\begin{aligned}
&3m^2(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k)\hat{\Delta}_A(k) \\
&\qquad \cdot \int \mathrm{d}l \hat{\Delta}_+(l)\left(\hat{\Delta}_A(k-l)+\hat{\Delta}_A(k+l)\right)\left(1+e^{-ik\sigma l}\right), \qquad (4.41)
\end{aligned}
$$

can be visualized by the graphs 4.5 given for the ϕ^3 calculation in section 4.4. The contribution we get here, has an additional factor of m^2 due to the coupling and a factor of 3 as each summand of the ϕ^3 part gives the same contribution. Remember that the fields in ϕ_1 can be seen as being normal ordered, so we get no tadpoles here.

Now, we take a look at the contribution coming from the ϕ^4 terms. The calculation is quite similar to the one given in section 4.5. The result is

$$
-(2\pi)^2 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k)\hat{\Delta}_A(k) \int \mathrm{d}l \hat{\Delta}_+(l) 2\left(1+e^{-ik\sigma l}\right). \qquad (4.42)
$$

The term without the twisting factor is quadratically divergent. In section 4.5 it was cancelled by using quasiplanar Wick ordering. Here it is cancelled by divergences appearing in other contributions, as we will see now.

The sum of (4.41), (4.40) and (4.42) gives

$$
\begin{aligned}
&(2\pi)^4 \int \mathrm{d}k \check{f}(k)\check{h}(-k)\hat{\Delta}_+(k)\hat{\Delta}_A(k) \int \mathrm{d}l \left(1+e^{-ik\sigma l}\right) \\
&\cdot \hat{\Delta}_+(l)\Big[\hat{\Delta}_A(k+l)(3m^2-2(l+k)\cdot l) + \hat{\Delta}_A(k-l)(3m^2-2(l-k)\cdot l) - 2(2\pi)^2\Big].
\end{aligned}
$$

The second line can be transformed, using $\hat{\Delta}_+(l)(m^2-l^2)=0$ and $-(2\pi)^2 = \hat{\Delta}_A(k\pm l)\left((k\pm l)^2-m^2\right)$, to

$$\hat{\Delta}_+(l)\Big[\hat{\Delta}_A(k+l)+\hat{\Delta}_A(k-l)\Big]\left(k^2+m^2\right).$$

The calculation of $\langle \phi_2^*(f)\phi_0(h)\rangle$ works quite similar and altogether we have

$$\begin{aligned}&\langle \phi_0^*(f)\phi_2(h)\rangle + \langle \phi_2^*(f)\phi_0(h)\rangle = \\ &(2\pi)^4\int \mathrm{d}k \check{f}(k)\check{h}(-k)\left(k^2+m^2\right)\Big(\hat{\Delta}_R(k)\hat{\Delta}_+(k)S_1(k)+\hat{\Delta}_+(k)\hat{\Delta}_A(k)S_2(k)\Big),\end{aligned}$$

with

$$\begin{aligned}S_1(k) &= \int \mathrm{d}l\hat{\Delta}_+(l)\Big(\hat{\Delta}_R(k-l)+\hat{\Delta}_R(k+l)\Big)\left(1+e^{ik\sigma l}\right),\\ S_2(k) &= \int \mathrm{d}l\hat{\Delta}_+(l)\Big(\hat{\Delta}_A(k-l)+\hat{\Delta}_A(k+l)\Big)\left(1+e^{-ik\sigma l}\right).\end{aligned}$$

Comparing with (4.24) we see that we almost get the same as in the case for the ϕ^3_4 model except for an additional factor of (k^2+m^2). (Here we also consider the planar contribution.) In the vicinity of the mass shell the contribution from the nonplanar graphs is

$$\Sigma_{\mathrm{WZ,np}}\left(k^2,(k\sigma)^2\right) = (k^2+m^2)\Sigma_{\phi^3,\mathrm{np}}\left(k^2,(k\sigma)^2\right),$$

where $\Sigma_{\phi^3,\mathrm{np}}$ is the result from section 4.4 and equals (4.31). Thus, from the nonplanar contributions we get

$$\begin{aligned}M((k\sigma)^2) &= -2m^2\Sigma_{\phi^3,\mathrm{np}}\left(m^2,(k\sigma)^2\right),\\ Z((k\sigma)^2) &= 2m^2\partial^{(1,0)}\Sigma_{\phi^3,\mathrm{np}}\left(m^2,(k\sigma)^2\right)+\Sigma_{\phi^3,\mathrm{np}}\left(m^2,(k\sigma)^2\right),\\ \Delta v_\perp^{\mathrm{rel}}((k\sigma)^2) &= 2\lambda_{\mathrm{nc}}^4\lambda^2m^2\partial^{(0,1)}\Sigma_{\phi^3,\mathrm{np}}\left(m^2,(k\sigma)^2\right).\end{aligned}$$

As before, we calculate these for the parameters $\lambda_{\mathrm{nc}}=\lambda_P=1$, $\sigma=\sigma_0$, $m=10^{-17}$ and $\lambda=1$. Apart from a factor of 2 the values for $\Delta v_\perp^{\mathrm{rel}}$ and M are the same as in the ϕ^3_4 model.[14] The field strength renormalization Z is the sum of the quantity of the ϕ^3_4 model and twice the the negative of its mass renormalization. Since the absolute value of the former is much smaller than the latter in the plotted region, Z looks almost like -2 times the plot shown in figure 4.10.

[14]Note that in the ϕ^3_4 model the coupling has a mass dimension and we chose $\lambda=m$. In the Wess-Zumino model the coupling has no mass dimension but a factor of m^2 enters through the prefactors of the calculated quantities.

Remark 4.6.1. In standard literature about supersymmetry, e.g., [40], one often finds the statement, that the Wess-Zumino model has only a field strength renormalization. Note that this is not a contradiction to our result since a different definition is used. With the definition invoked by [40] the field strength renormalization of the scalar field would be of the form

$$(1 + Z')\Box\phi + m^2\phi,$$

while our definition corresponds to

$$(1 + Z)\left(\Box\phi + (m^2 + M)\phi\right).$$

These are connected by setting $Z = Z'$ and $M = -Z'm^2/(1 + Z')$.

The distortion of the group velocity of this logarithmically divergent model is again quite moderate. If one identifies the field ϕ with the Higgs boson, the distortion might be detectable in future colliders, if the Higgs will be detected at all.

Chapter 5

Summary and outlook

We have seen that the distortion of the group velocity $\Delta v_\perp^{\text{rel}}$ is of drastically different order of magnitude in logarithmically divergent models (ϕ_4^3 and Wess-Zumino model) compared to quadratically divergent models (ϕ_6^3 and ϕ^4 model). It is of the order of percentages in the first kind and of order of 10^{65} for the latter kind. With these huge values for the quadratically divergent models, perturbation theory in λ might be inappropriate for investigating these except for very tiny coupling λ. The order of magnitude of the relative mass renormalization $M(k\sigma^2)/m^2$ is of order 1 in logarithmically divergent models. The mass renormalization can be used to fix the mass at vanishing perpendicular momentum $k_\perp$, but for higher values of $k_\perp$ the mass changes, and this change should be detectable. If we consider the noncommutative Minkowski space to correspond to $\mathcal{E}$, we still have to integrate over different $\sigma \in \Sigma$. Thus, there will be no distinct direction for $k_\perp$, but the mass will still depend on the momentum.

It is quite remarkable that the difference of orders of magnitude between models of different divergence class is so large while for models of the same class it is of order 1. There is no clear connection, for example, between the ϕ^3 model in six and the ϕ^4 model in four dimensions despite their quadratical divergence. It is worth while to further investigation, if there is some deeper reason behind this or if it is just by accident.

So far, no distortion of the group velocity has been detected on particles. Of course, the result depends on the concrete choice of the constants λ, m and λ_{nc}. We have chosen these to be compatible with the interpretation of the field with the Higgs boson. Although, the investigated models are not a possibility for the Higgs model, one could assume that the dispersion relation of the latter might also fall into this classification. If the Higgs is described by a logarithmically divergent (maybe supersymmetric) model, the distortion of the dispersion relation is quite mild but might be detectable in the LHC or

ILC (if the Higgs is detected at all).

That effects from noncommutativity are not observed on known particles might have its reason in that not all particles see this noncommutativity of spacetime. Remember, that this concept of spacetime was only thought to be an intermediate step. The uncertainty relations were derived in [13] by taking gravity into account in a detection process. So, it is not too naive to assume that the Higgs, which generates masses of other particles, couples differently to gravity and might see a noncommutative structure while other particles do not.

But there are still a lot of open conceptual problems to solve. An IR cut-off with a well-defined adiabatic limit for a reasonable class of sequences for interacting models on $\mathbb{M}_{\mathrm{nc}}$ would be desirable. Furthermore, if we do not see the coupling λ as infinitesimally small, contributions from different orders might cancel each other by a finetuning process. So, higher order contributions should be calculated. For this the concept of oscillatory integrals has to be extended. It would be interesting to see if some kind of UV-IR mixing appears also on $\mathbb{M}_{\mathrm{nc}}$ and whether it is harmless on logarithmically divergent models, too. Also, the treatment of massless fields or local gauge transformations are still problematic, see for example [46]. Results for noncommutative electrodynamics would be interesting since electrodynamics is experimentally tested to very high precision. The distortion of dispersion relations gives rise to further conceptional problems. Since the asymptotic behaviour is not that of a free field the LSZ formalism is not applicable. So it is not clear how to define the S-Matrix for interacting fields.

All in all there are a lot of open problems. If the colliders show the prescribed dispersion relations for the Higgs they are surely worth investigating. Another possibility would be to look at a change of the concept of the noncommutative spacetime. E.g., the commutator $Q^{\mu\nu}$ might not be a central element and be involved in the interaction. But these changes would make rigorous calculations probably very difficult. The best thing would be not to blindly test certain assumptions, but derive the setting from deeper concepts. However, this might be even harder.

Acknowledgments:

I am not a man of big words, but I would like to thank

- Prof. Dr. Klaus Fredenhagen for the competent supervision,
- Prof. Dr. Gerhard Mack and Prof. Dr. Jan Louis for taking the job of the co-referees,
- the Algebraic Quantum Field Theory group for the fun at work,
- in particular Jochen Zahn for fruitful collaboration,
- Dr. Martin Porrmann and Jochen Zahn for proofreading the manuscript and
- the Graduiertenkolleg "Zukünftige Entwicklungen in der Teilchenphysik" for financial support.

Hossa, and blue skies.

Appendix A

Conventions and useful formulae

In four-dimensional Minkowski space $\mathbb{M}$ we use the metric

$$\eta^{\mu\nu} = \mathrm{diag}(+, -, -, -)^{\mu\nu}$$

and indices running from 0 to 3 and analogously for higher dimensions. Mostly, we use upper indices for vectors in position space, like x^μ, and lower indices for vectors in momentum space, like p_μ. Tensors of type (r, s) on Minkowski space will be denoted by $\mathcal{T}^r_s(\mathbb{M})$. For $z \in \mathbb{C}$, $\bar{z}$ will denote the complex conjugate.

$\mathcal{S}$ denotes the Schwartz space, $\mathcal{D}$ is the space of smooth functions with compact support and $\mathcal{O}'_C$ are the distributions of rapid decrease. Dual spaces are generally denoted by a prime, e.g., $\mathcal{S}'$. The Fourier transform and its inverse in d dimensions are defined by

$$\begin{aligned} \hat{f}(k) &:= \frac{1}{(2\pi)^{\frac{d}{2}}} \int \mathrm{d}^d x f(x) e^{ikx}, \\ \check{f}(k) &:= \hat{f}(-k). \end{aligned}$$

For $f \in \mathcal{S}$ and $T \in \mathcal{S}'$ we have

$$\begin{aligned} \check{\hat{f}} &= \hat{\check{f}} = f, \\ \hat{T}(f) &:= T(\check{f}), \\ T(f) &= \hat{T}(\check{f}) = \check{T}(\hat{f}). \end{aligned}$$

The convolution for $f, g \in \mathcal{S}$ is

$$f \times g(x) := \int \mathrm{d}^d y f(x - y) g(y).$$

Hence,

$$\begin{aligned}
\widehat{f \cdot g}(k) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \hat{f} \times \hat{g}(k), \\
\widehat{f \times g}(k) &= \sqrt{2\pi}^{d} \hat{f}(k) \cdot \hat{g}(k).
\end{aligned}$$

Free field of mass m:

$$\begin{aligned}
\Phi(x) &:= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{\mathrm{d}^3 p}{2\omega_{\mathbf{p}}} \left(a(\mathbf{p}) e^{-ip_+ x} + a^\dagger(\mathbf{p}) e^{ip_+ x} \right), \\
\text{with} \quad \omega_{\mathbf{p}} &:= \sqrt{\mathbf{p}^2 + m^2}, \\
p_+ &:= (\omega_{\mathbf{p}}, \mathbf{p}).
\end{aligned}$$

The vacuum expectation value of $\Phi(f)\Phi(h)$ is denoted by $\langle \Phi(f)\Phi(h) \rangle$.
Delta-functions:

$$\begin{aligned}
\check{\delta}^{(d)} &= \frac{1}{(2\pi)^{\frac{d}{2}}} \mathbb{1}, \\
\Delta_+(x-y) &:= \langle \Phi(x)\Phi(y) \rangle, \qquad \text{(A.1)} \\
\hat{\Delta}_+(p) &= \frac{1}{(2\pi)^{\frac{d}{2}-1}} \theta(p_0)\delta(p^2 - m^2) = \frac{1}{(2\pi)^{\frac{d}{2}-1}} \frac{\delta(p_0 - \omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}}, \\
\Delta_-(x) &:= \Delta_+(-x) = \bar{\Delta}_+(x), \\
\Delta_R(x) &:= i\theta(x_0)(\Delta_+(x) - \Delta_-(x)), \\
\hat{\Delta}_R(p) &= \lim_{\epsilon \searrow 0} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{-1}{p^2 - m^2 + i\epsilon p_0}, \\
&= \lim_{\epsilon \searrow 0} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{2\omega_{\mathbf{p}}} \left(\frac{1}{p_0 + \omega_{\mathbf{p}} + i\epsilon} - \frac{1}{p_0 - \omega_{\mathbf{p}} + i\epsilon} \right). \qquad \text{(A.2)}
\end{aligned}$$

The definition of Δ_R is chosen such that

$$(\Box_x + m^2)\Delta_R(x-y) = \delta^{(d)}(x-y).$$

$$\begin{aligned}
\Delta_A(x) &:= \Delta_R(-x), \\
\Delta_F(x) &:= i\left(\theta(x_0)\Delta_+(x) + \theta(-x_0)\Delta_-(x)\right), \\
\Delta_R(x) &= \Delta_F(x) - i\Delta_-(x), \qquad \text{(A.3)} \\
\Delta_A(x) &= \Delta_F(x) - i\Delta_+(x). \qquad \text{(A.4)}
\end{aligned}$$

$B_x(r)$ denotes the open ball of radius r around x. $V_\pm$ is the full forward/backward lightcone. I_+^s is the set of nonnegative multi-indices of order length s.

On $\mathbb{M}_{\mathrm{nc}}$ we have the commutation relation:

$$[q^\mu, q^\nu] = iQ^{\mu\nu}.$$

In Weyl form this is

$$e^{ik_\mu q^\mu} e^{il_\nu q^\nu} = e^{-\frac{i}{2}k_\mu Q^{\mu\nu} l_\nu} e^{i(k_\mu + l_\mu)q^\mu}. \qquad \text{(copy of 2.2)}$$

Furthermore,

$$\int \mathrm{d}^4 q e^{ikq} = (2\pi)^4 \delta(k).$$

The starproduct or twisted convolution for $f, g \in \mathcal{S}(\mathbb{M}^d), \sigma \in \mathcal{T}_0^2(\mathbb{M}^d)$ is defined by its Fourier transform in the following way:

$$\widehat{f \star_\sigma g}(k) := \frac{1}{(2\pi)^2} \int \mathrm{d}l \hat{f}(k-l)\hat{g}(l) e^{-\frac{i}{2}k_\mu \sigma^{\mu\nu} l_\nu}.$$

The index σ at the star will mostly be omitted.

The wave front set of a distribution $T \in \mathcal{S}'(\mathbb{M})$ will be denoted $\mathrm{WF}(T)$. It is a subset of $\mathbb{M} \times (\mathbb{M}' \setminus \{0\})$. Define $\mathrm{WF}_x(T) := (\{x\} \times \mathbb{M}') \cap \mathrm{WF}(T)$. The set $\mathrm{WF}_x(T)$ can be seen as the set of singular directions of the distribution T at x. It is the set of all directions in which the Fourier transform of gT may not fall off faster than any polynomial for some function g with compact support and $g(x) \neq 0$. For an exact definition of the wave front set see, e.g., [35, 26]. The following properties hold:

$$\begin{aligned} \forall \lambda > 0 : \quad \lambda \cdot \mathrm{WF}_x(T) &= \mathrm{WF}_x(T), \\ \mathrm{WF}(T+S) &\subset \mathrm{WF}(T) \cup \mathrm{WF}(S). \end{aligned}$$

If $\mathrm{WF}_x(T) = \emptyset$ then T is $\mathcal{C}^\infty$ around x. If $\widetilde{\mathrm{WF}}(T,S) := \bigcup_{x \in \mathbb{M}} (\mathrm{WF}_x(T) + \mathrm{WF}_x(S))$ does not contain an element of the form $(x, 0)$, then the product $T \cdot S$ is a well-defined distribution and $\mathrm{WF}(T \cdot S) \subset \widetilde{\mathrm{WF}}(T,S)$.

$$\begin{aligned} \mathrm{WF}(\Delta_+) &= \{(x^\mu, k_\nu) | x \neq 0, x^\mu x_\mu = 0 \text{ and } \exists \lambda > 0 : \ k^\mu = -\lambda x^\mu\} \\ &\quad \cup \{(0, k_\nu) | k_\nu k^\nu = 0 \text{ and } k_0 < 0\}, \\ \mathrm{WF}(\Delta_+^2) &= \{(x^\mu, k_\nu) | x \neq 0, x^\mu x_\mu = 0 \text{ and } \exists \lambda > 0 : \ k^\mu = -\lambda x^\mu\} \\ &\quad \cup \{(0, k_\nu) | k_\nu k^\nu \geq 0 \text{ and } k_0 < 0\}, \\ \mathrm{WF}(\Delta_-) &= -\mathrm{WF}(\Delta_+), \\ \mathrm{WF}(\theta_0) &= \left\{(x^\mu, k_\nu) | x^0 = 0, k_0 \neq 0 \text{ and } \mathbf{k} = 0\right\}, \\ \mathrm{WF}(\Delta_R) &= \left\{(x^\mu, k_\nu) | x^0 > 0, x^\mu x_\mu = 0 \text{ and } \exists \lambda \neq 0 : \ k^\mu = -\lambda x^\mu\right\} \\ &\quad \cup \{(0, k_\nu) | k_\nu k^\nu \geq 0 \text{ and } k_0 \neq 0\}, \end{aligned}$$

where $\theta_0(x) = \theta(x_0)$.

For a distribution $T \in \mathcal{S}'(\mathbb{R}^d \setminus \{0\})$ or $T \in \mathcal{S}'(\mathbb{R}^d)$ the scaling degree $\mathrm{sd}(T)$ is defined as

$$\mathrm{sd}(T) := \inf\left\{\delta \in \mathbb{R} \middle| \lambda^\delta T(\lambda x) \xrightarrow[\lambda \searrow 0]{} 0\right\}.$$

Then we can deduce:

- If $\mathrm{sd}(T) < d$, then there exists a unique extension $\tilde{T} \in \mathcal{S}'(\mathbb{R}^d)$ of T such that $\mathrm{sd}(\tilde{T}) = \mathrm{sd}(T)$ and $\tilde{T} = T$ outside the origin.
- If $d \leq \mathrm{sd}(T) < \infty$, then there exist extensions $\tilde{T} \in \mathcal{S}'(\mathbb{R}^d)$ of T, such that $\mathrm{sd}(\tilde{T}) = \mathrm{sd}(T)$ and $\tilde{T} = T$ outside the origin. For two such extensions $\tilde{T}, \tilde{T}'$, there exists a polynomial P of degree $\mathrm{sd}(T) - d$ or smaller, such that $\tilde{T} - \tilde{T}' = P(\partial)\delta$.

For $\Delta_\pm$ in d dimensions we get $\mathrm{sd}(\Delta_\pm^n) = n(d-2)$ for $n \in \mathbb{N}$ More on the concept of scaling degree can be found in [41, 7].

Appendix B

Oscillatory integrals

When calculating nonplanar graphs in quantum field theory on noncommutative spacetime, we encounter many integrals which are not absolutely convergent[1] but made finite by an oscillating factor. An example for such an integral in one dimension is

$$\int_{\mathbb{R}} \mathrm{d}k \frac{1}{\sqrt{1+k^2}} \exp ik = \lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b \mathrm{d}k \frac{1}{\sqrt{1+k^2}} \exp ik,$$

where both limits exist and the result is independent of their order. However, this notion of an improper Riemann integral makes a priori only sense in one dimension. If one looks at higher dimensions a more sophisticated mathematical framework is needed, which is the theory of oscillatory integrals. The main definitions and results of this concept are given in this appendix.

We deal with not absolutely convergent integrals, so the usual rules of manipulating integrals are a priori not applicable. An example of a function not absolutely integrable for which the theorem of Fubini fails, is the following:

$$f(x,y) = \begin{cases} \frac{1}{4^n} & \text{if } \exists n \in \mathbb{N} \text{ with } 2^n - 1 < x < 2^{n+1} - 1 \\ & \text{and } 2^{n-1} - 1 < y < 2^n - 1, \\ -\frac{1}{2\cdot 4^n} & \text{if } \exists n \in \mathbb{N} \text{ with } 2^n - 1 < x < 2^{n+1} - 1 \\ & \text{and } 2^{n+1} - 1 < y < 2^{n+2} - 1, \end{cases}$$

which is kind of oscillating and decreasing at infinity. As easy to see, we

[1] We use the following terminology: An integral $\int \mathrm{d}x f(x)$ is said to be absolutely convergent if $\int \mathrm{d}x |f(x)| < \infty$ in the sense of Lebesgue integrals. In this case f is called absolutely integrable or measurable.

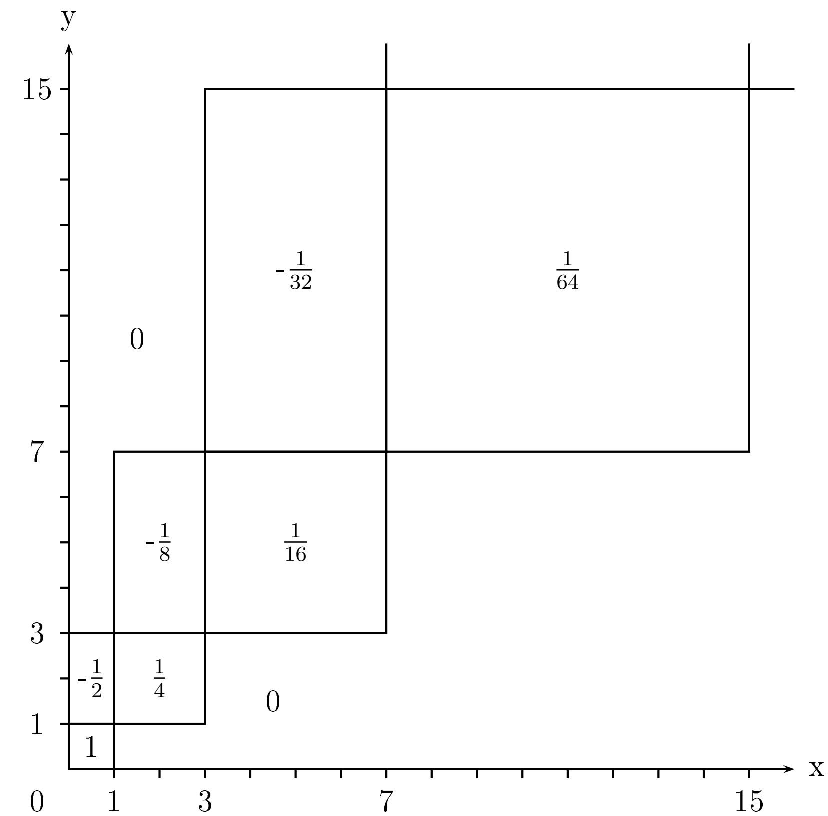

Figure B.1: Function f

have

$$a(x) := \int \mathrm{d}y f(x,y) = 0,$$

$$b(y) := \int \mathrm{d}x f(x,y) = \begin{cases} 1 & \text{if } 0 < y < 1, \\ 0 & \text{else.} \end{cases}$$

So,

$$\int \mathrm{d}x \; a(x) = 0 \neq 1 = \int \mathrm{d}y \; b(y).$$

This is similar to the case of an alternating series like $\sum_n (-1)^n \frac{1}{n}$. One can get an arbitrary value if one sums the terms in a different order. One can expect that accordingly in more than one dimensions one can find for a not integrable function, which is kind of oscillating, for each number a variable transformation, with which the integral of this function gives this number.[2]

[2]In one dimension the situation is different because there is essentially one way to go

However, oscillatory integrals, as defined below, have a well-defined sense. We give some important properties of these integrals which are useful to actually calculate one. Furthermore, a version of Fubini's theorem holds, shown in section B.4. The connection to the improper Riemann integral is given in proposition B.5.1.

B.1 Basic definitions and results

The theory of oscillatory integrals was to a great part developed by Hörmander [25]. However, to keep things from becoming unnecessary complicated, we use the theory as given in [35].

Let Ω be an open set in $\mathbb{R}^s$.

Definition B.1.1. A phase function on $\Omega \times \mathbb{R}^t$ is a continuous function $\phi : \Omega \times \mathbb{R}^t \to \mathbb{R}$ with

1. $\forall \lambda \geq 0, (k,l) \in \Omega \times \mathbb{R}^t$: $\phi(k, \lambda l) = \lambda \phi(k,l)$,
2. ϕ is $\mathcal{C}^\infty$ on $\Omega \times (\mathbb{R}^t \backslash \{0\})$,
3. $(\nabla_k \phi, \nabla_l \phi) \neq (0,0)$ on $\Omega \times (\mathbb{R}^t \backslash \{0\})$.

An example of a phase function is $k^\mu l_\mu$, which is used in Fourier transformation.

Definition B.1.2. A $\mathcal{C}^\infty$-function $a : \Omega \times \mathbb{R}^t \to \mathbb{C}$ is called symbol of order $r \in \mathbb{R}$ on $\Omega \times \mathbb{R}^t$ if $\forall K \subset \Omega$ compact and $\forall \alpha \in I_+^s, \beta \in I_+^t$ the seminorms

$$\|a\|_{K,\alpha,\beta} = \sup_{k \in K, l \in \mathbb{R}^t} (1 + |l|)^{|\beta| - r} |D_k^\alpha D_l^\beta a(k,l)| \tag{B.1}$$

are finite. The set of all such symbols equipped with the topology given by the seminorms will be denoted by $\mathrm{Sym}(\Omega, t, r)$.

A function $a : \Omega \times \mathbb{R}^t \to \mathbb{C}$ is called asymptotic symbol, if it can be written as $a = a_1 + a_2$ with $a_1 \in \mathrm{Sym}(\Omega, t, r)$ and a_2 having compact support in l and the map $k \to a_2(k, \cdot)$ is $\mathcal{C}^\infty$ as a map from Ω to $L^\infty(\mathbb{R}^t)$.

Loosely saying, derivatives in l have to lower the asymptotic polynomial behaviour of a and derivatives in k must not increase it. Hörmander [25] gave a generalized notion of symbols of type ρ, δ with $1 \geq \rho > 0$ and $1 > \delta \geq 0$ and, compared to (B.1), the exponent of $1 + |l|$ is $\rho|\beta| - \delta|\alpha| - r$. So, the derivatives in k are allowed to increase the order of the asymptotic polynomial behaviour. We do not need this generalized form.

---

to infinity. That is why the concept of improper Riemann integrals exists.

Remark B.1.3. If $r < r'$ then $\mathrm{Sym}(\Omega, t, r) \subset \mathrm{Sym}(\Omega, t, r')$ and the $\mathcal{C}^\infty$-functions of compact support are dense in $\mathrm{Sym}(\Omega, t, r)$ in the topology of $\mathrm{Sym}(\Omega, t, r')$. For $a_1 \in \mathrm{Sym}(\Omega, t, r_1)$ and $a_2 \in \mathrm{Sym}(\Omega, t, r_2)$ the product $a_1 \cdot a_2$ is in $\mathrm{Sym}(\Omega, t, r_1 + r_2)$ and accordingly for asymptotic symbols. $D_k^\alpha D_l^\beta a_1$ is in $\mathrm{Sym}(\Omega, t, r_1 - |\beta|)$.

Now we want to give a natural extension to expressions like

$$\int \mathrm{d}l \; a(k,l) e^{i\phi(k,l)}, \tag{B.2}$$

if the integral is not absolutely convergent:

Theorem B.1.4. *Let ϕ be a phase function. We can associate with ϕ a linear map from the asymptotic symbols to $\mathcal{D}'(\Omega)$, denoted by $T_\phi(a)$, which is uniquely determined by:*

1. *If a has compact support in l then $T_\phi(a)(k) = \int \mathrm{d}l \; a(k,l)e^{i\phi(k,l)}$ and is a $\mathcal{C}^\infty$-function of k.*
2. *The restriction of T_ϕ to $Sym(\Omega, t, r)$ is a continuous function from $Sym(\Omega, t, r)$ to $\mathcal{D}'(\Omega)$.*

Furthermore, the wave front set $WF(T_\phi(a))$ is contained in

$$\left\{(k, \nabla_k \phi(k,l)) | (k,l) \in \Omega \times \mathbb{R}^t \backslash \{0\} \text{ with } \nabla_l \phi(k,l) = 0\right\}. \tag{B.3}$$

We use the notion of $\int \mathrm{d}l \; a(k,l)e^{i\phi(k,l)}$ for the distribution $T_\phi(a)(k)$ even if the integral is not absolutely convergent. The case $s = 0$, where $\Omega = \mathbb{R}^0$ equals a single point, is allowed. In this case the functions only depend on l and $T_\phi(a) \in \mathbb{C}$.

Remark B.1.5. It is easy to see that the notion of asymptotic symbols can be generalized further. The function a could be split into more parts: $a = a_1 + a_2 + a_3 + \ldots$. For the additional terms, $k \to a_i(k, \cdot)$ should again be a $\mathcal{C}^\infty$-map into some "integrable space" having compact support in l. An example for such an "integrable space" would be $L^\infty(\mathbb{R}^t)$, which was already used for the original definition of asymptotic symbols in definition B.1.2, or the elements of $\mathcal{E}'(\mathbb{R}^t)$ which are $\mathcal{C}^\infty$ around $l = 0$.[3] The important point is that the integrals $\int \mathrm{d}k f(k) a_i(k,l) e^{i\phi(k,l)}$ should each be well-defined for $f \in \mathcal{D}(\Omega)$, one in the sense of oscillatory integrals, and their sum independent of the splitting. So one could even allow for some $k \to a_i(k, \cdot)$ to be distributions instead of $\mathcal{C}^\infty$-maps. This could, of course, increase the wave front set beyond (B.3).

[3] As the phase function does not have to be continuous in $l = 0$, $a_i(k.\cdot)$ should, e.g., not contain derivatives of δ-functions at that point.

Actually, in the following we are only going to treat symbols instead of asymptotic symbols. The extension to asymptotic symbols will be obvious.

B.2 Calculating oscillatory integrals

To actually calculate an oscillatory integral for a given phase function ϕ and some symbol a of order r, according to theorem B.1.4, a possibility is to find a sequence of symbols a_n with compact support in l which have as their limit a in the topology of symbols r' with $r' \geq r$.[4] Here, the following proposition is useful:

Proposition B.2.1. *Let g be a function in $\mathcal{S}(\mathbb{R}^t)$ with $g(0) = 1$. Then $\forall \epsilon > 0$ the sequence $g_n(l) := g(l/n)$ has the limit 1 for $n \to \infty$ in the topology of $Sym(\Omega, t, \epsilon)$.*

Proof. Let $0 < \epsilon < 1$. The cases with $\epsilon \geq 1$ follow easily. We have to prove that

$$\sup_l (1 + |l|)^{|\beta| - \epsilon} \left| D_l^\beta \left[g\left(\frac{l}{n}\right) - 1 \right] \right| \xrightarrow[n \to \infty]{} 0. \tag{B.4}$$

For $|\beta| = 0$ we write $g(l) = 1 + l\tilde{g}(l)$ with $\tilde{g} \in \mathcal{S}$. We make a variable transformation to $l' = \frac{l}{n}$ and use that (B.4) is smaller or equal to the sum of the suprema over the sets $|l'| > 1$ and $|l'| \leq 1$. The first gives

$$\sup_{|l'|>1} \frac{1}{n^\epsilon} \left(\frac{1}{n} + |l'| \right)^{-\epsilon} |l' \tilde{g}(l')| \leq \frac{1}{n^\epsilon} 2^{-\epsilon} \sup_{l'} |l' \tilde{g}(l')| \xrightarrow[n \to \infty]{} 0.$$

The supremum in the last expression is finite since $\tilde{g} \in \mathcal{S}$. The other term gives

$$\sup_{|l'|\leq 1} \frac{1}{n^\epsilon} \left(\frac{1}{n} + |l'| \right)^{-\epsilon} |l' \tilde{g}(l')|$$
$$\leq \frac{1}{n^\epsilon} \left(\sup_{l'} |\tilde{g}(l')| \right) \left(\sup_{|l'|\leq 1} |l'| \left(\frac{1}{n} + |l'| \right)^{-\epsilon} \right) \xrightarrow[n \to \infty]{} 0.$$

Here, the latter supremum in the last expression is smaller than 1 for all n since $l(\frac{1}{n} + l)^{-\epsilon}$ has its maximum on $[0, 1]$ at the point $l = 1$.

[4] Of course, the value of the oscillatory integral does not depend on whether we see it as a symbol of order r or r'. The only difference is, that such a sequence a_n might not exist in the topology of symbols of order r, compare with remark B.1.3.

Now we assume $|\beta| \geq 1$. Then $|\beta| - \epsilon > 0$. Furthermore

$$D_l^\beta g\left(\frac{l}{n}\right) = \frac{1}{n^{|\beta|}}\left(D^\beta g\right)\left(\frac{l}{n}\right)$$

and $D^\beta g \in \mathcal{S}$. The strategy will be similar to the above: Transform to l' and split the supremum. The first term gives

$$\begin{aligned}
&\sup_{|l'|>1} \frac{1}{n^{|\beta|}}(1+|l'|n)^{|\beta|-\epsilon}\left|\left(D^\beta g\right)(l')\right| \leq \sup_{|l'|>1} \frac{1}{n^\epsilon}\left(\frac{1}{n}+|l'|\right)^{|\beta|-\epsilon}\left|\left(D^\beta g\right)(l')\right| \\
\leq& 2^{|\beta|}\frac{1}{n^\epsilon}\sup_{|l'|>1}|l'|^{|\beta|}\left|\left(D^\beta g\right)(l')\right| \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

For the other one we have

$$\begin{aligned}
&\sup_{|l'|\leq 1} \frac{1}{n^{|\beta|}}(1+|l'|n)^{|\beta|-\epsilon}\left|\left(D^\beta g\right)(l')\right| \\
\leq& \frac{1}{n^\epsilon}\sup_{|l'|\leq 1}\left(\frac{1}{n}+|l'|\right)^{|\beta|-\epsilon}\sup_{|l'|}\left|\left(D^\beta g\right)(l')\right| \\
\leq& \frac{1}{n^\epsilon}2^{|\beta|-\epsilon}\sup_{|l'|}\left|\left(D^\beta g\right)(l')\right| \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

□

Remark B.2.2. With such a function g one can easily see that $g_n \cdot a$ has the limit a in the topology $\mathrm{Sym}(\Omega, t, r+\epsilon)$. So, we have for $f \in \mathcal{D}(\Omega)$

$$T_\phi(a)(f) = \lim_{n\to\infty}\int \mathrm{d}k\mathrm{d}l\ f(k)g\left(\frac{l}{n}\right)a(k,l)e^{i\phi(k,l)}.$$

From the proof of proposition B.2.1 we see, that $D^\beta g_n \to 0$ in the topology of $\mathrm{Sym}(\Omega, t, \epsilon)$. Hence,

$$\int \mathrm{d}l\ D^\beta g_n(l)a(k,l)e^{i\phi(k,l)} \xrightarrow[n\to\infty]{} 0 \tag{B.5}$$

as a distribution. Most of the time we will take g to be a function in $\mathcal{C}_0^\infty(\mathbb{R}^t)$ with

$$g(l) = \begin{cases} 1 & \text{if } |l| < 1 \\ 0 & \text{if } |l| > 2. \end{cases} \tag{B.6}$$

The restriction on the sequence g_n to be scaling can be loosened, but it is important, loosely saying, that the functions fall off more slowly when the region, where they almost equal 1, increases. An example of a sequence, where the derivatives do not fall off more slowly would be

$$\gamma_n(l) := \begin{cases} 1 & \text{if } |l| < n-1, \\ g(|l|+1-n) & \text{if } |l| \geq n+1, \end{cases}$$

with g as in (B.6) (with $t=1$). The sequence γ_n does not approach 1 in the topology of some $\mathrm{Sym}(\Omega, t, r)$.

A different method, to calculate the integrals, than multiplying with scaling functions stems from the following lemma and can be seen as the generalization of integration by parts:

Lemma B.2.3. *For every phase function ϕ on $\Omega \times \mathbb{R}^t$ there exist $A_\mu \in$ Sym$(\Omega, t, 0)$ and $B_\nu, C \in$ Sym$(\Omega, t, -1)$ such that*

$$Ve^{i\phi} = e^{i\phi} \text{ with } V = A_\mu \partial_l^\mu + B_\nu \partial_k^\nu + C.$$

Proof. See [35]. □

With this differential operator V, one can calculate

$$\begin{aligned} T_\phi(a)(f) &= \lim_{n\to\infty} \int \mathrm{d}k\mathrm{d}l\ f(k) g\left(\frac{l}{n}\right) a(k,l) V e^{i\phi(k,l)} \\ &= \lim_{n\to\infty} \int \mathrm{d}k\mathrm{d}l\ \left(V^{\mathrm{T}} f(k) g\left(\frac{l}{n}\right) a(k,l)\right) e^{i\phi(k,l)} \\ &= \lim_{n\to\infty} \int \mathrm{d}k\mathrm{d}l\ \left(V^{\mathrm{T}} f(k) a(k,l)\right) g\left(\frac{l}{n}\right) e^{i\phi(k,l)} \\ &= \int \mathrm{d}k\mathrm{d}l\ \left(V^{\mathrm{T}^{\lceil r+t+1 \rceil}} f(k) a(k,l)\right) e^{i\phi(k,l)}, \end{aligned}$$

where g is like in (B.6). The last integral is absolutely convergent, so the g was dropped. V^{T} denotes the transposed differential operator, i.e.,

$$V^{\mathrm{T}} = -A_\mu \partial_l^\mu - B_\nu \partial_k^\nu + C - (\partial_l^\mu A_\mu) - (\partial_k^\nu B_\nu)\,.$$

The passage from V to V^{T} in the first step was possible since the integrand has compact support and so the boundary terms vanish. In the second step we used (B.5) so the terms with derivatives of g vanish.

B.3 Further results

If the phase function ϕ and the symbol a are continuous functions of an additional parameter z with values in $\mathcal{C}^\infty(\Omega \times (\mathbb{R}^t \setminus \{0\}))$ and $\mathrm{Sym}(\Omega, t, r)$, then $T_\phi(a)(f)$ will also depend continuously on z. Thus, we can pass to limits under the integral sign. In particular, we can differentiate with respect to z under the integral sign, if this is possible for ϕ and a.

From the restriction on the wave front set of $T_\phi(a)$ given in Theorem B.1.4 we see that $T_\phi(a)(k)$ is a $\mathcal{C}^\infty$-function of k in the set

$$\Omega(\mathcal{C}^\infty) := \left\{k | k \in \Omega, \forall l \in \mathbb{R}^t \backslash \{0\} : \ \nabla_l \phi(k,l) \neq 0\right\}.$$

For given $k \in \Omega(\mathcal{C}^\infty)$ the function $\phi(k,\cdot)$ is a phase function on $\mathbb{R}^t$ and $a(k,\cdot)$ is a symbol of the same order r on $\mathbb{R}^t$. (We have $s = 0$ here.) So we can regard k as an additional parameter. The integral over l for this k is defined and we have:

$$T_\phi(a)(k) = T_{\phi(k,\cdot)}(a(k,\cdot)) = \int \mathrm{d}l \ a(k,l) e^{i\phi(k,l)}. \tag{B.7}$$

Furthermore, differentiation with respect to k can be performed under the integral sign:

$$\partial_k^\nu T_\phi(a)(k) = \int \mathrm{d}l \ \ \left(ia(k,l)\partial_k^\nu \phi(k,l) + \partial_k^\nu a(k,l)\right) e^{i\phi(k,l)}.$$

Now $a(k,l)\partial_k^\nu \phi(k,l)$ is an asymptotic symbol of order $r+1$ and the above is again defined as an oscillatory integral. Proofs can be found in [26] or [25].

B.4 Theorem of Fubini for oscillatory integrals

As we have seen above, in not absolutely convergent integrals the order of integration can in general not be interchanged. That this is nevertheless possible for oscillatory integrals show the following two generalizations of the theorem of Fubini.

Theorem B.4.1 (Theorem of Fubini for the "distributional variable" k)**.** *If Ω is of the form $\Omega = \Omega_1 \times \Omega_2$ and the phase function ϕ has the property $(\nabla_{k_1}\phi, \nabla_l \phi) \neq (0,0)$ we can perform the k_2-integration at the end:*

$$\int \mathrm{d}k\mathrm{d}l \ f(k) a(k,l) e^{i\phi(k,l)} = \int \mathrm{d}k_2 \left(\int \mathrm{d}k_1 \mathrm{d}l \ f(k_1,k_2) a(k_1,k_2,l) e^{i\phi(k_1,k_2,l)} \right),$$

where $f \in \mathcal{D}(\Omega)$ and on the left hand side the oscillatory integral is defined with a symbol and phase function depending on k_2 as an additional parameter.

Proof. See [25], (1.2.4). □

A new result, to our knowledge, is the following theorem of Fubini where we split the variable $l \in \mathbb{R}^t$ into two components:

Theorem B.4.2 (Theorem of Fubini for the "oscillatory variable" l)**.** *Let $l = (u, v) \in \mathbb{R}^{t_1} \times \mathbb{R}^{t_2}$, $t := t_1 + t_2$. and the phase function have the property that $\phi(k, u, v) = \phi_1(k, u) + \phi_2(v)$, where ϕ_1 is a phase function in the variables k and u. (ϕ_2 does not have to be a phase function and could even be zero.) Then, for $a \in Sym(\Omega, t, r)$ and $f \in \mathcal{D}(\Omega)$, the function*

$$H(v) := \int \mathrm{d}k\mathrm{d}u \; f(k)a(k, u, v)e^{i\phi_1(k,u)}$$

is in $\mathcal{S}(\mathbb{R}^{t_2})$ and furthermore

$$\int \mathrm{d}v H(v)e^{i\phi_2(v)} = \int \mathrm{d}k\mathrm{d}l \; f(k)a(k, l)e^{i\phi(k,l)}. \tag{B.8}$$

Proof. First, we show that $H \in \mathcal{S}(\mathbb{R}^{t_2})$, what is equivalent to

$$\sup_v |v^\alpha D_v^\beta H(v)| < \infty \tag{B.9}$$

for all multi-indices α, β. We have

$$v^\alpha D_v^\beta H(v) = \int \mathrm{d}k\mathrm{d}u \; f(k)v^\alpha D_v^\beta a(k, u, v)e^{i\phi_1(k,v)},$$

using the fact that the differentiation can be performed under the integral sign (see section B.3). In the following considerations it is important that we are dealing with symbols on $\Omega \times \mathbb{R}^t$. This means that derivatives with respect to u reduce the asymptotic polynomial behaviour for large v and vice versa. We note that:

- v^α is in $\mathrm{Sym}(\Omega, t, |\alpha|)$.
- If $a \in \mathrm{Sym}(\Omega, t, r)$ then $D_v^\beta a \in \mathrm{Sym}(\Omega, t, r - |\beta|)$.

So $v^\alpha D_v^\beta a$ is a symbol of order $r + |\alpha| - |\beta|$ and (B.9) is proved if we can show that

$$\sup_v \left| \int \mathrm{d}k\mathrm{d}u \; f(k)a(k, u, v)e^{i\phi_1(k,v)} \right| < \infty$$

for an arbitrary symbol a. We use that ϕ_1 is also a phase function on $\Omega \times \mathbb{R}^t$, so according to Lemma B.2.3 there exist symbols A^1_μ, A^2_ρ of order 0 and B_ν, C of order -1, all on $\Omega \times \mathbb{R}^t$, with

$$Ve^{i\phi_1(k,u)} = \left(A^1_\mu \partial^\mu_u + A^2_\rho \partial^\rho_v + B_\nu \partial^\nu_k + C\right) e^{i\phi_1(k,u)} = e^{i\phi_1(k,u)}.$$

As ϕ_1 does not depend on v the symbols A^2_ρ can be set to zero. With this we have

$$\begin{aligned}
&\sup_v \left| \int \mathrm{d}k\mathrm{d}u \; f(k)a(k,u,v)e^{i\phi_1(k,u)} \right| \\
=&\sup_v \lim_{n\to\infty} \left| \int \mathrm{d}k\mathrm{d}u \; f(k)a(k,u,v)g\left(\frac{u}{n}\right) V e^{i\phi_1(k,u)} \right| \\
=&\sup_v \lim_{n\to\infty} \left| \int \mathrm{d}k\mathrm{d}u \; \left(V^{\mathrm{T}} f(k)a(k,u,v)\right) g\left(\frac{u}{n}\right) e^{i\phi_1(k,u)} \right| \\
=&\sup_v \left| \int \mathrm{d}k\mathrm{d}u \; \left({V^{\mathrm{T}}}^{\lceil r+t+1\rceil} f(k)a(k,u,v)\right) e^{i\phi_1(k,u)} \right|.
\end{aligned}$$

From the second to third line it was important that A^2_ρ vanishes since we are not integrating over v. The last integral is absolutly integrable in k, u and v and the integrand is a continuous function of these variables, so the supremum of the integral over k and v has to be finite. Thus, $H \in \mathcal{S}(\mathbb{R}^{t_2})$ has been shown now.

To prove B.8 we use the same V as before and make use of Fubini's theorem for absolutely integrable functions:

$$\begin{aligned}
\int \mathrm{d}v H(v)e^{i\phi_2(v)} &= \int \mathrm{d}v \left(\int \mathrm{d}k\mathrm{d}u \; \left({V^{\mathrm{T}}}^{\lceil r+t+1\rceil} f(k)a(k,u,v)\right) e^{i\phi_1(k,u)} \right) e^{i\phi_2(v)} \\
&= \int \mathrm{d}k\mathrm{d}u\mathrm{d}v \; \left({V^{\mathrm{T}}}^{\lceil r+t+1\rceil} f(k)a(k,u,v)\right) e^{i\phi_1(k,u)}e^{i\phi_2(v)} \\
&= \int \mathrm{d}k\mathrm{d}l \; f(k)a(k,l)e^{i\phi(k,l)}.
\end{aligned}$$

□

B.5 Connection to other definitions

For one dimension there already exists a description on how to calculate expressions like (B.2), namely the improper Riemann integral. For this, a has to be decreasing, i.e., of order smaller than 0.[5] Oscillatory integrals are well-defined even if the value of the symbols a increases with l. The following proposition states that, if the improper Riemann is applicable too, the result equals the oscillatory integral. As we could not find a similar statement in [25] or [35], we give a proof.

[5]Functions which behave like $\frac{1}{\log l}$ for large $|l|$, which are of order 0, are also allowed. The following proposition can easily be generalized to this case.

Proposition B.5.1. *Let g be as in (B.6), $a \in Sym(\Omega, 1, r)$ with $r < 0$ and $\phi(k,l) = \phi(k) \cdot l$ a phase function*[6] *with $\phi(k) \neq 0$. Then the oscillatory integral equals the improper Riemann integral:*

$$\lim_{n\to\infty} \int_0^\infty \mathrm{d}l \; g\left(\frac{l}{n}\right) a(k,l) e^{i\phi(k)l} = \lim_{n\to\infty} \int_0^n \mathrm{d}l \; a(k,l) e^{i\phi(k)l}.$$

Proof. This is clear if $r < -1$ since then both integrals are absolutely convergent. In the other case the integral on the left-hand side equals

$$\int_0^n \mathrm{d}l a(k,l) e^{i\phi(k)l} + \int_n^{2n} \mathrm{d}l g\left(\frac{l}{n}\right) a(k,l) e^{i\phi(k)l}.$$

The first term already has the correct limit, so the proposition is proved if the other one approaches 0. To see this we make a variable transformation and integrate by parts:

$$\begin{aligned}
&\int_n^{2n} \mathrm{d}l g\left(\frac{l}{n}\right) a(k,l) e^{i\phi(k)l} \\
=&n \int_1^2 \mathrm{d}l g(l) a(k,nl) e^{i\phi(k)nl} \\
=&-\frac{i}{\phi(k)} \left[g(l) a(k,nl) e^{i\phi(k)nl}\right]_{l=1}^2 \\
&+\frac{i}{\phi(k)} \int_1^2 \mathrm{d}l \left(\partial g(l) a(k,nl) + g(l) n (\partial_l a)(k,nl)\right) e^{i\phi(k)nl}.
\end{aligned}$$

The boundary terms vanish in the limit $n \to \infty$ since a is a symbol of order $r < 0$ and hence decreasing at infinity. For the remaining integrals we have

$$\left|\int_1^2 \mathrm{d}l \partial g(l) a(k,nl) e^{i\phi(k)nl}\right| \leq \sup_l |\partial g(l)| \sup_{l\in[n,2n]} |a(k,l)| \xrightarrow[n\to 0]{} 0$$

and

$$\begin{aligned}
&\left|\int_1^2 \mathrm{d}l g(l) n (\partial_l a)(k,nl) e^{i\phi(k)nl}\right| \\
\leq &\int_n^\infty \mathrm{d}l \left| g\left(\frac{l}{n}\right) (\partial_l a)(k,l) e^{i\phi(k)l}\right| \\
\leq &\int_n^\infty \mathrm{d}l \sup_{l'} |g(l')| d_k (1+l)^{r-1} \xrightarrow[n\to 0]{} 0,
\end{aligned}$$

where the last integral is absolutely convergent. $\square$

[6] The most general form of a phase function in one dimension is $\theta(-l)\phi_+(k) \cdot l + \theta(l)\phi_-(k) \cdot l$. This case can easily be derived from the one given here.

To show how the last theorem and the theorem of Fubini can be used to calculate oscillatory integrals numerically, we give an example for $s = 0, t_1 = t_2 = 1$:

$$\int \mathrm{d}u\mathrm{d}v \; \frac{v^{17}u}{1+u^2+v^2}e^{iu} = \int \mathrm{d}v \left(\int \mathrm{d}u \; \frac{v^{17}u}{1+u^2+v^2}e^{iu} \right)$$
$$= i\pi \int \mathrm{d}v \; v^{17}e^{-\sqrt{1+v^2}}.$$

The absolute value of the first integrand increases in v-direction but the last integral is absolutely convergent and can be treated by the usual numerical methods.

Another prescription of how to interpret expressions like (B.2) is to multiply it with a function $f(k)$ and then perform first the k-integration and then the l-integration. This works, for example, for $\int \mathrm{d}l \; le^{ikl}$, since the remaining function of l is absolutely convergent. But this does not have to be the case, a and ϕ could for example not dependent on k at all. If it is absolutely convergent, then the result gives, of course, the same as the calculation with oscillatory integrals:

Proposition B.5.2. *Let a be a symbol, ϕ a phase function and $f \in \mathcal{D}(\Omega)$ with*

$$F(l) := \int \mathrm{d}k f(k) a(k,l) e^{i\phi(k,l)}$$

absolutely integrable. Then

$$\int \mathrm{d}l \; F(l) = T_\phi(a)(f).$$

Proof.

$$T_\phi(a)(f) = \lim_{n\to\infty} \int \mathrm{d}k\mathrm{d}l \; f(k) g\left(\frac{l}{n}\right) a(k,l) e^{i\phi(k,l)}$$
$$= \lim_{n\to\infty} \int \mathrm{d}l \; g\left(\frac{l}{n}\right) F(l) = \int \mathrm{d}l \; F(l).$$

□

If the phase function is of the kind $k^\mu l_\mu$ then the oscillatory integral is related to the Fourier transform. If a does not depend on k it can be seen as an element of $\mathcal{S}'$. Then the oscillatory integral gives exactly the Fourier transform of $a \in \mathcal{S}'$.

As we see in section 4.4 an extension of the theory of oscillatory integrals to the case where $a(k,l)$ is needed, if we want to calculate higher order diagrams in the Yang-Feldman formalism on $\mathbb{M}_{\mathrm{nc}}$. There are two natural approaches for such an extension:

1. The distributions a could be approximated by a sequence of symbols $(a_n)_{n\in\mathbb{N}}$, such that for each a_n the oscillatory integral is well-defined. The oscillatory integral for a can then be achieved from the limit $n \to \infty$ after integrating, if this is well-defined and to a large extend independent of the choice of the sequence.

2. One could regard the relation

$$\int \mathrm{d}k\mathrm{d}l f(k)a(k,l)e^{i\phi(k,l)} = \lim_{n\to\infty} \int \mathrm{d}k\mathrm{d}l f(k)g_n(l)a(k,l)e^{i\phi(k,l)} \quad \text{(B.10)}$$

   for a sequence g_n of symbols with compact support and approaching 1, as a definition. The right-hand side of (B.10), with finite n, is even defined for a a distribution. If the limit exists and is independent of the choice of the sequence g_n out of some large class of sequences, this would be a reasonable extension, too.

Appendix C

Graphs for Yang-Feldman formalism

Here, we show how to calculate the contributions to n-point functions with graphs like those shown in figures 4.3, 4.4 or 4.5.

The graphs for the Yang-Feldman formalism we present here are similar to those used in [2]. We analyse the r-point functions of the ϕ^a model. The field equation is

$$(\Box + m^2)\Phi = -\lambda\Phi^{a-1}.$$

The naive solution is given in (3.3). The fields are built up recursively as in figure 4.1.

The rules to calculate the contributions of graphs to the r point functions of the ϕ^a model in momentum space are as follows:

1. Draw r tree graphs of retarded propagators. The directions are upwards. At each vertex there should be at most $a-1$ branching outs consisting of other retarded propagators. (The empty tree is allowed.)

2. Add leaves such that each vertex has exactly $a-1$ branching outs. (The empty tree has one leaf.)

3. Connect each leaf by another one. The lines ("contractions") are directed from left to right. (Maybe there are additional rules, e.g., no tadpoles are allowed if Wick ordering is involved.) If this is not possible (e.g. the number of leaves could be odd) the contribution of this graph is zero.

4. Otherwise calculate the contribution of this graph to the r-point function in the following way:

(a) Numerate each retarded propagator and contraction by a different number. Each gets a momentum k flowing in the direction of the line.

(b) For the retarded propagator with number j write $\hat{\Delta}_R(k_j)$, for a contraction $(2\pi)^2\hat{\Delta}_+(k_j)$.

(c) For each root l with outgoing momentum k_{j_l} write $\check{f}_l(k_{j_l})$. (Incoming momenta are counted as the negative is outgoing.)

(d) The contribution coming from each vertex depends on the actual cutoff. If it is a formal calculation, i.e., without cutoff, each vertex with outgoing momenta $\{k_{j_l}\}$ gives

$$(2\pi)^{-2(a-3)}\delta(\sum_l k_{j_l}).$$

If the cutoff is defined by multiplying the field monomial with a cutoff function g like in section 4.2 it is

$$(2\pi)^{-2(a-2)}\check{g}(\sum_l k_{j_l}).$$

On noncommutative spacetime further twisting factors might arise. Some examples are given in section 4.3.

(e) Integrate over all momenta. The order of this contribution is the number of vertices.

A graphical example is given in figure C.1. Things can become more complicated, if fields of higher spins or multiple interactions are involved, see, e.g., the Wess-Zumino model in section 4.6. The topology of the graphs are similar to the graphs from Feynman rules. Note, that in the Yang-Feldman formalism each graph might not give a well-defined contribution on its own. It is necessary to sum over all graphs of the same order.

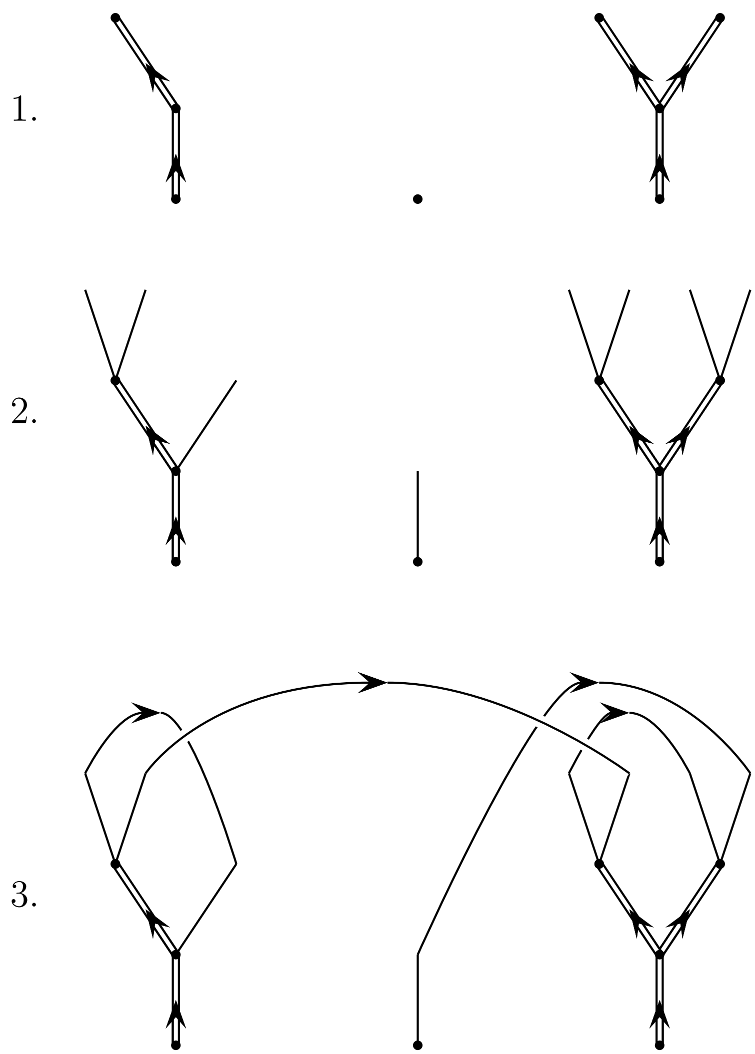

Figure C.1: Example of building up a contribution to the three-point function in the ϕ^3 model.

Appendix D

Theorem of Epstein and Glaser

Epstein and Glaser examine in [17] vacuum expectation values (l-point functions) of time-ordered products defined by retarded or advanced solutions. These, denoted by $F_{R/A}(p,q), p \in \mathbb{R}^{4l}; q \in \mathbb{R}^{4n}$, shall fulfill

$$\mathrm{supp} F_{R/A} \subset \left\{ (x;y) \in \mathbb{M}^{l+n} | \forall i \leq n \exists j \leq l \text{ with } x_j - y_i \in \bar{V}_\pm \right\} =: S_{R/A}. \tag{D.1}$$

$\bar{V}_\pm$ denotes the full closed forward/backward light cone. Furthermore, their Fourier transforms $\hat{F}_{R/A}(p;k)$ should be equal on the set

$$R_n := \left\{ k \in \mathbb{M}^n \middle| \left(\sum_{i \in I} k_i \right)^2 < 4m^2 \ \forall I \subset \{1, \ldots, n\} \right\} \tag{D.2}$$

for some $m \in \mathbb{R}$.

Then the following theorem holds:

Theorem D.1.3. *If a pair of tempered distributions $F_{R/A} \in \mathcal{S}'(\mathbb{M}^{l+n})$ has the support (D.1) and their Fourier transforms coincide for $k \in R_n$, then their Fourier transforms are tempered distributions in p and infinitely differentiable in k for all $k \in R_n$.*

Hence, we can choose an arbitrary sequence of test functions $\check{G}_a \in \mathcal{S}(\mathbb{M}^n)$ which have support in a closed subset of R_n and converge to $(2\pi)^{4n}\delta^{(4n)}$ in the topology of $\mathcal{O}'_C(\mathbb{M}^n)$[1] and the adiabatic limit,

$$\lim_{a \to \infty} \int \mathrm{d}k \mathrm{d}p \check{f}(p) \check{G}_a(k) \hat{F}_R(p,k),$$

[1] $\mathcal{O}'_C$ are the distributions of rapid decrease, see [38] for a rigorous definition. These act on smooth functions which are polynomially bounded. $\hat{F}_{R/A}$ is on R_n smooth and also polynomially bounded since it is a tempered distribution.

exists for all $f \in \mathcal{S}(\mathbb{M}^l)$ and is independent of the choice of the sequence $\check{G}_a$. As the Fourier transforms of F_R and F_A coincide in R_n, the adiabatic limit for such a G_a is the same for both.

Bibliography

[1] I. Y. Aref'eva, D. M. Belov and A. S. Koshelev, *A note on UV/IR for noncommutative complex scalar field* (2000) [arXiv:hep-th/0001215].

[2] D. Bahns, *Perturbative Methods on the Noncommutative Minkowski Space*, Dissertation, DESY-THESIS-2004-004 (2004).

[3] D. Bahns *Ultraviolet finiteness of the averaged Hamiltonian on the noncommutative Minkowski space* (2004) [arXiv:hep-th/0405224].

[4] D. Bahns, S. Doplicher and K. Fredenhagen, *Field theory on noncommutative spacetimes: Quasiplanar Wick products*, Phys. Rev. D **71**, 025022 (2005) [arXiv:hep-th/0408204].

[5] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, *On the unitarity problem in space/time noncommutative theories*, Phys. Lett. B **533**, 178 (2002) [arXiv:hep-th/0201222].

[6] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, *Ultraviolet finite quantum field theory on quantum spacetime*, Commun. Math. Phys. **237**, 221 (2003) [arXiv:hep-th/0301100].

[7] R. Brunetti and K. Fredenhagen, *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds*, Commun. Math. Phys. **208**, 623 (2000) [arXiv:math-ph/9903028].

[8] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, Institute of Physics Publishing, Bristol (1995).

[9] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*, Phys. Lett. B **604**, 98 (2004) [arXiv:hep-th/0408069].

[10] I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I. Scalars*, JHEP **0005**, 037 (2000) [arXiv:hep-th/9911098].

[11] I. Chepelev and R. Roiban, *Convergence theorem for noncommutative Feynman graphs and renormalization*, JHEP **0103**, 001 (2001) [arXiv:hep-th/0008090].

[12] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego (1994).

[13] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Commun. Math. Phys. **172**, 187-220 (1995) [arXiv:hep-th/0303037].

[14] C. Döscher and J. Zahn, *Infrared cutoffs and the adiabatic limit in noncommutative spacetime*, Phys. Rev. D **73**, 045024 (2006) [arXiv:hep-th/0512028].

[15] C. Döscher and J. Zahn, *Dispersion relations in the noncommutative ϕ^3 and Wess-Zumino model in the Yang-Feldman formalism* (2006) [arXiv:hep-th/0605062].

[16] M. Dütsch and K. Fredenhagen, *Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity*, Rev. Math. Phys. **16**, 1291 (2004) [arXiv:hep-th/0403213].

[17] H. Epstein and V. Glaser, *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré A **19**, 211 (1973).

[18] T. Filk, *Divergencies in a field theory on quantum space*, Phys. Lett. B **376**, 53 (1996).

[19] J. Gomis and T. Mehen, *Space-Time Noncommutatice Field Theories And Unitarity*, Nucl. Phys. B **591**, 265 (2000) [arXiv:hep-th/0005129].

[20] J. M. Gracia-Bondia, *Hidden symmetry and Hopf algebra* (2006) [arXiv:hep-th/0606107].

[21] J. M. Gracia-Bondia and J. C. Varilly, *Algebras of distributions suitable for phase space quantum mechanics. 1*, J. Math. Phys. **29**, 869 (1988).

[22] H. Grosse and R. Wulkenhaar, *Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base*, Commun. Math. Phys. **256**, 305 (2005) [arXiv:hep-th/0401128].

[23] R. Haag, *On quantum field theories*, Dan. Mat. Fys. Medd. **29**, No. 12 (1955).

[24] C. Hayashi, *Hamiltonian Formalism in Non-Local Field Theories*, Prog. Theor. Phys. **10**, 533 (1953).

[25] L. Hörmander, *Fourier Integral Operators. I*, Acta. Math. **127**, 79 (1971).

[26] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin (1990).

[27] G. Källén, *Mass- and charge-renormalizations in quantum electrodynamics without use of the interaction representation*, Ark. Fys. **2**, 187 (1950).

[28] G. Källén, *Formal integration of the equations of quantum theory in the Heisenberg representation*, Ark. Fys. **2**, 371 (1951).

[29] M. Kossow, *Infrarotverhalten einer ultraviolett-endlichen Quantenfeldtheorie auf nichtkommutativer Raumzeit*, Diplomarbeit, Universität Hamburg (2004).

[30] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, Class. Quant. Grav. **20**, 2905 (2003) [arXiv:hep-th/0104190].

[31] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002**, 020 (2000) [arXiv:hep-th/9912072].

[32] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions, I*, Commun. Math. Phys. **31**, 83 (1973).

[33] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions, II*, Commun. Math. Phys. **41**, 281 (1975).

[34] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books, Reading Massachusetts (1995).

[35] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York (1975).

[36] V. Schomerus, *D-branes and deformation quantization*, JHEP **9906**, 030 (1999) [arXiv:hep-th/9903205].

[37] E. Schrödinger, *Über die Unanwendbarkeit der Geometrie im Kleinen*, Naturwiss. **22**, 518 (1934).

[38] L. Schwartz, *Théorie des distributions, tome II*, Hermann, Paris (1957).

[39] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[40] M. F. Sohnius, *Introducing Supersymmetry*, Phys. Rep. **128**, 39 (1985).

[41] O. Steinmann, *Perturbation Expansions in Axiomatic Field Theory*, Springer, Berlin (1971).

[42] J. Wess, *Deformed coordinate spaces: Derivatives*, "Vrnjacka Banja 2003, Mathematical, theoretical and phenomenological challenges beyond the standard model" 122 (2003) [arXiv:hep-th/0408080].

[43] J. Wess and J. Bagger,*Supersymmetry and Subergravity*, Princeton University Press, Princeton (1992).

[44] C. N. Yang and D. Feldman, *The S Matrix In The Heisenberg Representation*, Phys. Rev. **79**, 972 (1950).

[45] J. Zahn, *Wirkungs- und Lokalitätsprinzip für nichtkommutative skalare Feldtheorien*, Diplomarbeit, DESY-THESIS-2003-041 (2003).

[46] J. Zahn, *Dispersion relations in quantum electrodynamics on the noncommutative Minkowski space*.PhD Thesis (2006).