Gauge covariant multiscale analysis of complex systems

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Abstract

A fundamental physical theory is characterized by a minimum of a priori structure. Observable physical phenomena at low energy scales can be explained as a result of complexity reduction [1]. Thus the building of effective theories is of fundamental importance. Renormalization group transformations and multigrid methods as a sort of it aim at constructing effective theories. These should reproduce the physics at a larger scale providing for locality and gauge covariance. In this thesis block spins (variables of the effective theory) are constructed with respect to locality requirements. They are used to solve discretized elliptical differential equations by means of multigrid methods. A generalized formalism is presented in form of a twogrid. It does not assume a Hilbert space a priori. The gauge covariance of the resulting block spin is discussed. The locality requirement of an effective theory may also lead to nonunitary parallel transporters. Higgs fields may be interpreted as a part of such a nonunitary parallel transporter in a fifth dimension. They may arise as a result of renormalization group transformations. The renormalization group flow of the Higgs potential is computed in a one-dimensional example. (This dimension will later be interpreted as extra dimension.) Starting with a pure gauge theory at the UV-fixpoint the flow moves towards an IR-fixpoint in the asymptotic behaviour. This is computed exactly. Spontaneous symmetry breaking can lead to a splitting of the masses of the fermions of different flavor. The possibilities of symmetry breaking are discussed. Based on joint work with G. Mack and T. Prüstel a gauge theoretic model in five dimensions is introduced, which contains the standard model as effective theory. It explains flavor mass splitting as result of a Higgs mechanism and provides the possibility to compute the Cabibbo-Kobayashi-Maskawa matrix in principle.

Zusammenfassung

Eine fundamentale physikalische Theorie zeichnet sich durch ein Minimum an a-priori-Struktur aus. Der Beobachtung zugängliche physikalische Phänomene auf niedrigeren Energieskalen können als Resultat einer Komplexitätsreduktion erklärt werden [1]. Somit ist die Bildung effektiver Theorien von zentraler Bedeutung. Renormierungsgruppentransformationen und Mehrgitterverfahren, als eine Form dieser, zielen auf die Bildung von effektiven Theorien. Diese sollten die Physik auf einer gröberen Skala wiedergeben und dabei Lokalität und Eichkovarianz gewährleisten. In dieser Arbeit werden unter Berücksichtigung von Lokalitätsforderungen Blockspins (Variablen der effektiven Theorie) konstruiert. Sie finden ihre Anwendung in der Lösung diskretisierter elliptischer Differentialgleichungen, die mit Hilfe von Mehrgitterverfahren gelöst werden können. Ein allgemeiner Formalismus, der nicht von einer a-priori-Hilbertraumstruktur ausgeht, wird hier anhand eines Zweigitters vorgestellt. Die Eichkovarianz des resultierenden Blockspins wird diskutiert. Die Forderung nach Lokalität einer effektiven Theorie kann aber auch zu nichtunitären Paralleltransportern führen. Higgsfelder können als Teil eines solchen nichtunitären Paralleltransporters in eine fünfte Dimension interpretiert werden und durch Renormierungsgruppentransformationen entstehen. In einem eindimensionalen Beispiel wird der Renormierungsgruppenfluss des Higgspotentials exakt berechnet. (Diese Dimension soll später als Extradimension interpretiert werden.) Beginnend mit einer reinen Eichtheorie am UV-Fixpunkt nähert sich der Fluss im asymptotischen Verhalten dem IR-Fixpunkt, der berechnet wird. Spontane Symmetriebrechung kann zur Massenaufspaltung von Fermionen unterschiedlichen Flavors führen. Die Möglichkeiten der Symmetriebrechung werden hier diskutiert. Basierend auf einer gemeinsamen Arbeit mit G. Mack und T. Prüstel wird ein eichtheoretisches Modell in fünf Dimensionen vorgestellt, dessen effektive Theorie das Standardmodell liefert. Es erklärt die Flavor-Massenaufspaltung als Folge eines Higgsmechanismus und bietet die Möglichkeit, die Cabibbo-Kobayashi-Maskawa-Matrix im Prinzip zu berechnen.

Contents

Chapter 1

Introduction

It is generally agreed that a physical theory is the more fundamental the less a priori structure is assumed. Gauge theories are of great importance because of their poor a priori structure. Starting with a theory that is assumed to be fundamental one wants to know the features of its effective theory at observable energy scales. Following this, Adler for example obtained the Einstein action as effective phenomena. This leads to the assumption that in spite of complexity reduction that is obtained by building effective theories emergent phenomena may appear that do not have to be assumed a priori. The Higgs mechanism is an example.

Multiscale analysis aims at constructing effective theories. Given a local theory which describes physics at some possibly arbitrarily small scale, one wishes to construct an effective theory with fewer degrees of freedom that describes physics at a coarser scale. This means that it reproduces the results of the original theory for all measurements whose resolution is not higher than indicated by the coarse scale. The procedure may be repeated. Wilson's real space renormalization group is a form of multiscale analysis [50].

The central issue in this approach is locality. The effective theory has to be local. Otherwise it is unmanageable and useless. The first and essential step in the definition of an effective theory is the specification of its variables. These are functions of the variables of the original theory. For historical reasons they are called block spins [27]. They are obtained as image of a so-called restriction operator. The block spin must be chosen in such a way that the resulting effective theory is local. This is an art. No general recipe is known.

Gauge theories live on vector bundles whose fibres are vector spaces $\mathbb{V}_z$ attached to sites of space or space-time, continuous or discrete. We will concentrate on the discrete case, including unstructured grids. There is no natural a priori identification of different fibres. Instead, there are parallel transporters $U(\mathcal{C}) : \mathbb{V}_z \to \mathbb{V}_w$ along paths $\mathcal{C}$ from z to w. In conventional gauge field theories they are unitary maps when fibres are equipped with a suitable scalar product.

It is a central hypothesis of this thesis that the requirement of locality of the effective theory may require the introduction of nonunitary parallel transporters as block spins at some scale when one starts with an ordinary gauge theory in five or more dimensions, possibly with matter fields but without Higgs fields.

A general theory of gauge field theories with nonunitary parallel transporters was recently presented in the Ph.D. thesis of T. Prüstel [41]. The nonunitary parallel transporters contain, besides a unitary factor, another factor which may be interpreted as a Higgs field.

Before turning to the renormalization group flow for gauge field theories, the solution of inhomogeneous linear equations

$$Lu = f \tag{1.1}$$

on vector bundles on a grid is considered in chapter 2. Solving such equations with local relaxation methods one faces the problem of critical slowing down. Basically the convergence decreases with the number of iteration steps. This motivates the introduction of multigrid methods: An error that cannot be reduced any more can be opened up for further iterations on a coarser lattice. This is to be repeated in a multigrid algorithm. Elliptic inhomogeneous linear equations are of great importance in engineering. Accordingly, there is a continued effort to enlarge the domain of applicability of the well developed existing theory. There are difficulties to solve linear equations in physics as well. The Euklidean Dirac equation in an external gauge field is an example, which is important in lattice gauge theory.
Besides this, much can be learned in this laboratory about the proper choice of a restriction operator $\mathcal{C}$ and the preservation of locality properties in the effective theory. The restriction operator maps the fields of the original lattice to the effective scale. The effective theory will involve an effective problem operator L^1 on a coarse grid. There is an important issue which has never been systematically studied in gauge field theories of elementary particles, in spite of its importance. By contrast, it is a central theme in the multiscale approach to linear equations. This is the truncation of the effective action or problem operator $\hat{L}^1$ which is defined by L once the restriction operator is chosen. It is impractical because it has exponential tails and thus needs to be truncated. One seeks exact solutions of eq.(1.1). Therefore the error made by the approximation of $\hat{L}^1$ by L^1 has to be corrected. This is done with the help of so-called Schwarz iterations. A measure of quality of the truncation is introduced and shown to govern the convergence of the Schwarz iteration.

Even the standard model exhibits features that are not completely understood. The explanation of the hierarchy of the quark masses or the computation of the Cabibbo-Kobayashi-Maskawa-matrix belongs here. The wish to describe such features as emergent phenomena motivates the study of renormalization group flows. In chapter 3, the study of renormalization group transformations of gauge theories with nonunitary parallel transporters is begun with the study of a one-dimensional model. One imagines that only the extra dimension of a five-dimensional theory is retained. The locality requirement becomes trivial, but the renormalization group flow of the Higgs potential can be studied in detail. A qualitative explanation is added showing how locality properties can become restrictive when the extra dimensions are added.

If the nonunitary parallel transporters (PT) are invertible maps between fibres, PT along closed loops and their inverses generate a noncompact holonomy group H which contains the compact gauge group G. Only G is a local symmetry. It is an interesting question as to what patterns of spontaneous symmetry breaking can arise from Higgs fields made out of gauge fields in extra directions. This will be discussed in chapter 4 as result of a collaboration with M. de Riese and G. Mack. The general answer is not known, but for the Higgs field studied here, results are obtained. First, H determines according to which representation of G the Higgs field transforms. It is in general not the adjoint representation, unless H can be interpreted as a complex group. Further, the possible unbroken symmetry subgroups are maximal compact subgroups of parabolic subgroups of H. Group theorists know how to classify them [22]. With the help of existing software written by M. de Riese [43] lists of examples are compiled.

Models with extra dimensions became an interesting topic in contemporary physics [3]. Of course, on effective scales they should come by the standard model's properties. Chapter 5 presents a model constructed in collaboration with G. Mack and T. Prüstel [31], which explains how Higgs fields of the type studied here can generate flavor mass splitting by spontaneous symmetry breaking of G. It lives on a five-dimensional space-time with a finite extent d in the fifth direction. Left- and right-handed fermions are attached to opposite boundaries L and R. The Higgs potential is supposed to be determined by the result of a renormalization group flow which stops when the scale equals d. The Cabibbo-Kobayashi-Maskawa matrix C is originally a dynamical field which fluctuates around a constant value determined by the ground state of the full theory. It is not fixed by the Higgs mechanism and would remain undetermined if it were not for explicit symmetry breaking on the boundaries. The model predicts axigluons.

Chapter 6 sums up the results. The appendix shows technical details that help to understand some issues outlined in this work.

Chapter 2

Solving linear problems for disordered systems

There are different methods to solve discretized elliptic linear differential equations [46]. Most of them are not applicable to disordered systems, which are not approximately translation invariant. We introduce a scheme that resists and could be applied to a large range of problems. It is based on a multigrid procedure that is demonstrated by the twogrid. The most important condiment is the construction of a local block spin that also gauge covariant.

2.1 Statement of the problem

2.1.1 Multigrid procedures to solve linear equations

We want to solve discretized differential equations of the form

$$Lu = f. \tag{2.1}$$

We concentrate at first on elliptic problems with a positive operator L. Some such problems can be solved by Algebraic Multigrid: a black box problem solver. But its applicability is so far mostly restricted to scalar equations with special properties. We are seeking much more general procedures. Usual local iteration methods are applied to improve an approximate solution u_{approx} of equation (2.1) step by step. The solution of such problems using local relaxation methods shows the feature of critical slowing down. This means basically that the modes associated with small eigenvalues of the operator L do not converge well. The multigrid method deals with this. An error that cannot be reduced any more on some scale can become again a candidate for local relaxation methods on a coarser scale. Concerning the Laplacian for example, the smooth modes, those to small eigenvalues, converge badly an their smoothness can be decreased relatively to a lattice by enlarging the lattice spacing. Thus one goes to larger scales to solve an effective version of eq.(2.1) and interpolates the result back to the fine scale. One has to find good restriction operators $\mathcal{C}$ to convert (2.1) into a coarse grid equation and good interpolators to go back to the original scale. We will discuss such choices for very general problems.

The most important requirement on $\mathcal{C}$ is that nailing down $\mathcal{C}u = 0$ will eliminate critical slowing down. Ellipticity will not be required to start with, but spectral properties of local approximations L^N to L turn out to be important to make the construction of $\mathcal{C}$ practical.

2.1.2 Locality and coarse graining

The elimination of degrees of freedom is the center point of building an effective theory. For linear problems this complexity reduction can be obtained by retaining weighted sums of fine grid variables as variables of the effective theory on the coarse grid. They are called block spin, see eq.(2.3). In general the block spin construction is not unique and may lead to nonlocality of the effective theory. The building of an effective theory is connected with the question of separating high and low frequencies in such a way that locality and the physics at the coarser scale is preserved. In this chapter we want to discuss how the demand of locality can be satisfied when we go to an effective theory. The main task is to find an appropriate choice of a block spin that represents the fields at the larger scale.

A chosen block spin leads to a decomposition into high (u^h) and low (u^l) frequency parts of a field u.

$$u = u^l + u^h. \tag{2.2}$$

The low frequency part is involved in the effective theory. The exact solution to an equation $Lu = f$ on a fine lattice is

$$u = Gf$$

with $G = L^{-1}$. $G(x, y)$ is the Green's function. The high frequency solution u^h defines a new Green's function, what we will call $\Gamma(x, y)$. Its decay length has not to be bigger than one lattice spacing of the coarse grid. If this is, so we can write

$$u^l = \mathcal{J}U$$

where $\mathcal{J}$ is an interpolator and U an element of the vector space on the coarse grid. We consider subsystems of the original lattice when we go to coarser scales. This will be explained in section 2.3. Equation (2.1) on the coarse grid reads

$$L^1 U = F,$$

where the effective operator L^1 should have good locality properties. The decomposition (2.2) is linear. This linear problem is equivalent to a free field theory without translation invariance. In gauge theory we will not have such a linear decomposition.

Let us have a look at the Green's function

$$G(x, y) = \frac{1}{Z} \int Du\ u(x)u(y)e^{-\frac{1}{2}\langle u, Lu\rangle},$$

which involves the Gaussian measure $\frac{1}{Z}Due^{-\frac{1}{2}\langle u,Lu\rangle}$ with covariance L^{-1}. The Green's function can be identified with a correlation function in statistical systems. In cases of interest it decays slowly. The field on the coarser scale is obtained by linear restriction, applying a so-called restriction operator $\mathcal{C}$

$$U = \mathcal{C}u, \ U(x) = \sum_{z\in x} \mathcal{C}_{xz}u(z). \tag{2.3}$$

Here the sites z are elements of a grid Λ^0 and x label those of coarse grid $\Lambda^1 \subset \Lambda^0$. The Green's function of the high frequencies is then given by

$$\Gamma(x,y) = \frac{1}{Z}\int Du\ u(x)u(y)e^{-\frac{1}{2}\langle u,Lu\rangle}\delta(\mathcal{C}u),$$

keeping the block spin fixed. A proper choice of a restriction operator leads to the fact that the correlation length of Γ is short. Thus locality properties are improved by keeping the block spin fixed. Γ will appear later as Green's function of the high frequency problem in the compatible relaxation scheme.

A key issue will be how to construct a restriction operator $\mathcal{C}$ that assures good locality properties. In lattice gauge theory one is interested in the construction of gauge covariant block spins. This question will be examined in section 2.5.

2.2 Technical preliminaries

This section introduces the mathematical preliminaries to solve discretized linear equations. We introduce the notion of a linear system and the equation defined on it. Possible choices of a scalar product on the space of solutions and properties of the operators are discussed. Preconditioners are introduced. They convert the problem into a Hilbert space problem. We will focus on its desirable properties.

2.2.1 The problem

We seek solutions $u \in \mathcal{V}$ of well posed linear equations

$$Lu = f \tag{2.4}$$

where $f \in \mathcal{V}'$. The vector space $\mathcal{V}'$ is the dual of a finite dimensional real or complex vector space $\mathcal{V}$, and L is an invertible linear operator $L : \mathcal{V} \mapsto \mathcal{V}'$. Elements of the dual $\mathcal{V}'$ are linear maps $f : v \mapsto \langle f, v\rangle$ from $\mathcal{V}$ to $\mathbb{R}$ or $\mathbb{C}$. Eq.(2.4) is therefore equivalent to

$$a(u,v) = \langle f,v\rangle \ \text{ for all } v \in \mathcal{V}, \tag{2.5}$$

where a is the nondegenerate bilinear form on $\mathcal{V}$ defined by

$$a(u,v) = \langle Lu, v\rangle.$$

In the literature, eq.(2.5) is called the weak form of eq.(2.4). We are interested in spaces $\mathcal{V}$ which contain real or complex functions or vector valued generalizations thereof, as follows.

There is a finite but large set of nodes $z, w, ... \in \Lambda^0$, and to every node z there is a finite dimensional vector space $\mathbb{V}_z$ with dual $\mathbb{V}'_z$. In addition there is a finite set of directed edges $\langle zw \rangle$, and whenever $\langle zw \rangle$ is an edge or $z = w$, a linear map

$$L_{zw} : \mathbb{V}_w \mapsto \mathbb{V}'_z$$

is defined. For convenience, we set $L_{zw} = 0$ otherwise. $\mathcal{V}$ consists of functions

$$u : z \mapsto u(z) \in \mathbb{V}_z .$$

Accordingly, $\mathcal{V}'$ consists of functions $f : z \mapsto f(z) \in \mathbb{V}'_z$, and

$$\langle f, u \rangle = \sum_z \langle f(z), u(z) \rangle,$$
$$(Lu)(z) = \sum_w L_{zw} u(w).$$

Often, nodes $z, w... \in \Lambda^0$ are sites of structured or unstructured grids. But our interest includes cases which are very different from discretized differential equations. To give an example of an interesting problem, the nodes may be the atoms of a protein whose folding process is to be studied.

By abuse of terminology, we will always speak of a grid, and we include the edges as part of the specification of the grid. A grid together with a specification of vector spaces $\mathbb{V}_z$ and linear maps L_{zw} will be called a LINEAR SYSTEM Λ^0 for short. Concerning the vector spaces there are several cases

i) If all $\mathbb{V}_z \equiv \mathbb{V}$ are the same, f and u are vector valued functions. In the special cases $\mathbb{V} = \mathbb{R}$ or $\mathbb{C}$, they are ordinary real or complex functions.

ii) If all $\mathbb{V}_z$ are isomorphic (i.e. have the same dimension) but cannot be identified in a natural way, then f, u are sections of vector bundles. This situation is encountered in lattice gauge theories of elementary particle physics, and for some other disordered systems.

iii) Different $\mathbb{V}_z$ may have different dimensions. f, u are sections of fibre bundles with fibres $\mathbb{V}'_z$ and $\mathbb{V}_z$. When we start from a problem of type i) or ii), multigrid methods may lead to problems of type iii) on coarser grids, because different numbers of coarse grid variables may be needed at different sites of the coarse grid. We wish to admit this possibility because there may be problems without translation invariance which are "difficult in the vicinity of some hot spots".

We will start with the case of positive L. We discuss more general situations later on.

We wish to construct a fast converging iterative procedure by exclusive reference to the structure furnished by the equation (2.4). A scalar product which turns $\mathcal{V}$ into Hilbert space is not part of the given structure, nor is the identification of different fibres $\mathbb{V}_z$. On the contrary, it is the core of the problem how to convert the linear equation (2.4) into a Hilbert space problem with a small condition number of the problem operator.

The condition number k of a positive operator in a Hilbert space is the ratio of the largest and smallest eigenvalue. More generally it is the ratio of largest and smallest singular value.

2.2.2 Hilbert spaces

Before proceeding, let us recall some results from linear algebra concerning dual spaces, and fix notation.

$\mathcal{V}'$ consists of linear maps $f : v \mapsto \langle f, v\rangle \in \mathbb{R}$ or $\mathbb{C}$ and

$$\langle \lambda_1 f_1 + \lambda_2 f_2, u\rangle = \bar{\lambda}_1\langle f_1, u\rangle + \bar{\lambda}_2\langle f_2, u\rangle$$

where $\bar{\lambda}$ is the complex conjugate of the number λ; $\bar{\lambda} = \lambda$ for $\lambda \in \mathbb{R}$.

By definition an operator $L : \mathcal{V} \mapsto \mathcal{V}'$ is positive semidefinite if $\langle Lu, u\rangle \geq 0$ for all u, and positive if $\langle Lu, u\rangle > 0$ for all $u \neq 0$. We write

$$A \leq B$$

if $B - A$ is positive semidefinite. The adjoint of a map $A : \mathcal{V} \mapsto \mathcal{W}$ of vector spaces is a map $A^* : \mathcal{W}' \mapsto \mathcal{V}'$, defined by $\langle A^* f, u\rangle = \langle f, Au\rangle$ for $f \in \mathcal{W}', u \in \mathcal{V}$. A positive operator $L : \mathcal{V} \mapsto \mathcal{V}'$ is automatically self-adjoint.

A positive operator $D : \mathcal{V} \mapsto \mathcal{V}'$ equips $\mathcal{V}$ with a scalar product $\langle\cdot,\cdot\rangle_D$,

$$\langle u, v\rangle_D = \langle Du, v\rangle$$

thereby converting it into a Hilbert space. In this notation, the above bilinear form $a(u, v) = \langle u, v\rangle_L$.

A Hilbert space is identified in a natural way with its dual. The Hilbert space adjoint A^+ of an operator in a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_D$ is again an operator in the same Hilbert space with the defining property $\langle A^+ u, v\rangle_D = \langle u, Av\rangle_D$. Equivalently

$$A^+ = D^{-1} A^* D \ .$$

A Hilbert space operator A is positive if $\langle Au, u\rangle_D > 0$ for $u \neq 0$. A positive A is automatically self-adjoint, $A = A^+$. We take it for granted that a Hilbert space problem $Au = f$ can be solved by fast converging iteration if the condition number is not too large.

2.2.3 Preconditioners

Here we introduce preconditioners. They are applied in multigrid analysis to lessen the condition number and thus lead to better convergence. The search for a good twogrid or multigrid procedure can be interpreted as search for good preconditioners.

The standard approaches to solving the linear equation (2.4) consist of two steps:

1.) Convert the problem into a Hilbert space problem with the help of a preconditioner S.

2.) Solve the Hilbert space problem with the help of Krylov subspace methods (sketched below), such as conjugate gradient.

A preconditioner S is an invertible linear map

$$S : \mathcal{V}' \mapsto \mathcal{V}.$$

It is required to be positive if L is positive. In this case $SL : \mathcal{V} \mapsto \mathcal{V}$ is a positive self-adjoint operator in the Hilbert space $\mathcal{V}$ with scalar product $a = \langle \cdot, \cdot \rangle_L$, i.e. $\langle SLu, u \rangle_L > 0$ if $u \neq 0$. Often S is not given explicitly, but must be computed as an inverse of an operator $B = S^{-1} : \mathcal{V} \mapsto \mathcal{V}'$.

Iteration methods

Iteration methods are techniques for improving the accuracy of an approximate solution. Let u_{approx} be an approximate solution of the equation $Lu = f$. We write then $Lu_{\text{approx}} = f$. The error is denoted by $e \in \mathcal{V}$, $e = u_{\text{approx}} - u$, and obeys and thus

$$Le = f - Lu_{\text{approx}} = r, r \in \mathcal{V}'.$$

Now assume we have a linear algorithm for the approximate solution of $Le = r$. We can find an approximation to the error

$$e \approx Sr.$$

Thus the new, corrected approximate solution is given by

$$u_{\text{new}} \approx Sr + u.$$

The operator S is referred to as the preconditioner. The convergence rate depends only on the condition number $k(SL) = \frac{\lambda_{\max}(SL)}{\lambda_{\min}(SL)}$.

One can introduce methods for accelerating the convergence of the basic iteration. If u^n denotes the approximate solution after n iterations of the solution process, one can write the iteration as

$$u^{n+1} = u^n + \tau S(f - Lu^n)$$

with a so-called damping factor τ. It is an extra scalar parameter, which may be chosen to improve convergence. This scheme is often called Richardson's method. Its convergence can be accelerated again by taking linear combinations of the obtained iterates. This is done by Krylov subspace methods.

i) Krylov methods consists of first generating a Krylov subspace, which is given by

$$K_n(r, L, S) = \text{span}\{Sr, (SL)Sr, (SL)^2Sr, ...(SL)^{n-1}Sr\}.$$

ii) An approximate solution to the linear system is then obtained by minimizing the error in various norms, over all potential solutions in $K_n(r, L, S)$.

A special Krylov subspace method is the conjugate gradient method. It is generally only suitable for symmetric positive definite matrices and strongly recommended for such problems. The conjugate gradient uses a three-term recurrence relation. This means that only a small number of vectors have to be kept during the iteration. [45]

A good preconditioner should have the following properties:

i) The computation of $Sf \ (f \in \mathcal{V}')$ is manageable.

ii) The condition number k of SL is not too large. Loosely speaking this means that S is an approximate inverse of L.

Given a good preconditioner S, the solution u of the linear equation (2.4) can be determined by fast converging iterative solution of the equation

$$SLu = Sf.$$

Remark: If L is a positive operator, Conjugate Gradient is an appropriate Krylov subspace method for solving the Hilbert space problem $SLu = Sf$. Its rate of convergence is determined by the condition number k of SL.

We will seek to construct good preconditioners by Schwarz methods.
The strategy which we pursue is to start with a bad preconditioner D^{-1} and to cure it by adding coarse grid corrections. An operator $D : \mathcal{V} \mapsto \mathcal{V}'$ will be called *diagonal* if $D_{zw} = 0$ for $z \neq w$. Diagonal preconditioners will be our simplest and most important starting point. An obvious candidate is

$$D_{zz} = L_{zz} \ .$$

This is positive if L is positive.

High and low frequency problem

The essence of a multigrid procedure can be seen by considering a twogrid. One obtains a multigrid method from a twogrid method by using a twogrid once more to solve the low frequency problem, and so on and so on. We will concentrate on twogrid problems.

The basic idea of the twogrid cure is as follows. Suppose that for $u \in \mathcal{V}$

$$\lambda_{min}\langle Du, u\rangle \ \leq \ \langle Lu, u\rangle \ \leq \ \lambda_{max}\langle Du, u\rangle,$$

so that the condition number of the operator $\mathbb{L} = D^{-1}L$ is $\lambda_{max}(\mathbb{L})/\lambda_{min}(\mathbb{L})$. In the most transparent variant (compatible relaxation), one introduces an "energy cutoff" ϵ and one splits $\mathcal{V}$ into a sum

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1,$$

which is approximately orthogonal with respect to the scalar product $a(\cdot,\cdot) = \langle\ ,\ \rangle_L$ and such that:

1. $\epsilon\langle Du, u\rangle \ \le\ \langle Lu, u\rangle \ \le\ \lambda_{max}\langle Du, u\rangle$ for $u \in \mathcal{V}_0$. The determination of $u \in \mathcal{V}_0$ is the high frequency problem.

2. $\mathcal{V}_1$ consists of interpolates of elements $U \in \mathcal{V}^1$ of sections on a coarse grid, and finding the solution $u \in \mathcal{V}_1$ of the equation $\langle Lu, v\rangle = \langle f, v\rangle$ for all $v \in \mathcal{V}_1$ is equivalent to finding the solution $U \in \mathcal{V}^1$ of a coarse grid equation

$$L^1 U = F. \tag{2.6}$$

3. There is a positive and easily inverted coarse grid version $D^1 : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}$ of D such that

$$\lambda_{min}\langle D^1 U, U\rangle \ \le\ \langle L^1 U, U\rangle \ \le\ C\epsilon\langle D^1 U, U\rangle \text{ for } U \in \mathcal{V}^1$$

   with $C \approx 1$.

Then the problem has been approximately factored in such a way that the condition number is factored, $\lambda_{max}/\lambda_{min} = (\lambda_{max}/\epsilon)(\epsilon/\lambda_{min})$. There is a high frequency problem with condition number λ_{max}/ϵ on the Hilbert space $\mathcal{V}_0$ with scalar product provided by D, and a low frequency problem with condition number $C\epsilon/\lambda_{min}$ on the Hilbert space $\mathcal{V}^1$ with scalar product provided by D^1.

If $\mathcal{V}_0$ and $\mathcal{V}_1$ are only approximately orthogonal, one needs a further iteration, to be constructed by invoking a Schwarz method, which calls for approximate solutions of high and low frequency problems in alternation.

The coarse grid equation (2.6) is a linear equation on the coarse grid. It is also called the effective problem, and L^1 is called the effective problem operator. One may say that the problem has been reduced to an effective problem on the coarse grid when the high frequency problem is mastered. It is possible to admit energy cutoffs $\epsilon(x)$, which depend on the block x. Convergence results depend on a lower bound on them.

2.3 Twogrid

Now we are ready to perform the step towards an effective problem. We introduce a block lattice Λ^1 by composition of sites on the fine grid to a so-called block. Types of blocks are classified according to their overlap. The construction of an operator L with generalized Neumann boundary conditions on block boundaries is introduced. It is our crucial tool to construct a restriction operator. Then we recall the construction of projectors that are used for Schwarz methods.

Given a Linear System Λ^0 we construct a coarse grid with nodes $x, y, ...$ edges $\langle xy\rangle$, and vector spaces $\mathbb{V}_x$. Linear maps $L^1_{xy} : \mathbb{V}_y \mapsto \mathbb{V}'_x$ will also be constructed. L^1 will be invertible, so that we obtain another Linear System Λ^1. We call this the block lattice. There is an associated vector space $\mathcal{V}^1$ of sections

$$U : x \mapsto U(x) \in \mathbb{V}_x$$

and its dual space $\mathcal{V}^{1\prime}$, which consists of sections $F : x \mapsto F(x) \in \mathbb{V}'_x$. Given $F \in \mathcal{V}^{1\prime}$, the solution of the coarse grid problem is $U \in \mathcal{V}$ such that $L^1 U = F$. The nodes x are themselves grids, called blocks to begin with. They are obtained from the original grid by domain decomposition. They consist of some subsets $z \in x$ of the nodes of the original grid together with all the edges between distinct nodes $z, w \in x$.

Given x, the boundary ∂x of x consists of all edges $\langle zw\rangle$ whose target z is in x, and whose source w is not. $z \in x$ is called a *boundary point* if it is target of an edge in the boundary of x.

In Algebraic Multigrid (AMG) [46], the construction of a suitable decomposition is a major part of the analysis. Here we imagine at first that some decomposition has been chosen. Then we will investigate convergence properties, and from this criteria for good choices of decompositions will emerge. Examples suggest much less sensitivity to the choice of a decomposition than for AMG.

There are two different kinds of domain decomposition.

1. Non-overlapping blocks. Different blocks share neither nodes nor edges.

2. Different blocks overlap in boundary points and possible edges between them.

Somewhat paradoxically, case 1 leads to generalizations of what is known as Schwarz methods with overlap, while case 2 leads to sub-structuring methods, also known as Schwarz methods without overlap. What is known in the literature [49] as balanced Neumann Neumann, outlined in appendix B. It illustrates possibility 2.

The construction of the vector spaces $\mathbb{V}_x$ associated with coarse grid points x is more subtle.

First, the blocks x must be converted into Linear Systems by specification of a collection $\{L^{N,x}\}$ of operators $\mathcal{V} \mapsto \mathcal{V}'$, such that $L^{N,x}_{zw} = 0$, unless $z = w \in x$ or $\langle zw\rangle$ is an edge in x. They are to be constructed from L.

In the case of blocks which do not overlap, it is required that

$$L^N = \sum_x L^{N,x} \leq L \ , \tag{2.7}$$

i.e. $L - L^N$ is a positive semidefinite operator. Note that in this case, edges in boundaries of blocks do not belong to any block. We say that L^N does not couple different blocks.

In the case of blocks, which overlap in boundaries, it is required that

$$\sum_x L^{N,x} = L \ . \tag{2.8}$$

Collections of operators $L^{N,x}$, which fulfill these conditions, will be called *problem operators with generalized Neumann boundary conditions.* The same name is given to L^N in case of non-overlapping blocks.

Let us motivate the name. For structured grids, there is often a unique choice of L^N which respects the symmetry of the grid. In particular, in the classical case of the standard discretization $-\Delta$ of the negative Laplacian, $-\Delta^N$ is defined by the requirement that $-\Delta^N u = -\Delta u$ for all u such that $u(z) = u(w)$ for all edges $\langle zw \rangle$ in boundaries of (non-overlapping) blocks x. It obeys

$$0 \leq -\Delta^N \leq -\Delta, \tag{2.9}$$

and it has a zero mode for every block.

For the general class of problems, which we consider here, there is no natural definition of a problem operator with Neumann boundary conditions. It may be impossible to find operators which obey the analog of bounds (2.9). The square L of the Dirac operator for an electron in an external magnetic field can serve as an example to see the difficulty. To the best of the author's knowledge Neumann methods for such a general class of problem operators L have never been considered before. We seek operators L^N with the aforementioned properties. In general, the choice of the $L^{N,x}$ is neither unique, nor is $L^{N,x}$ positive semidefinite. It will be a main task to deal with generalized eigenmodes of $L^{N,x}$ with eigenvalues below some small positive ϵ which will be called the *energy cutoff*. The construction of $L^{N,x}$ will be the topic of section 2.3.4. To define generalized eigenmodes, one needs in general a preconditioner D^{-1} which does not couple blocks. Any positive diagonal D satisfies this condition. Problem operators with Dirichlet boundary conditions can also be used as preconditioners. In special cases (positive semidefinite $L^{N,x}$ and ϵ small enough), the only relevant generalized eigenmodes are zero modes, and they are actually independent of the choice of D.

Given $L^{N,x}$ and D, the vector spaces $\mathbb{V}_x$ will be constructed as detailed in section 2.3.5, together with restriction maps

$$\mathcal{C}_x : \mathcal{V} \mapsto \mathbb{V}_x$$

such that $\mathcal{C}_x u$ depends only on $u(z)$ for $z \in x$. Using this, a restriction operator

$$\begin{aligned} \mathcal{C} : \mathcal{V} &\mapsto \mathcal{V}^1 \\ \mathcal{C}u(x) &= \mathcal{C}_x u \end{aligned}$$

is defined. It maps sections on the original grid to sections on the coarse grid.

Given the restriction operator $\mathcal{C}$, one needs to select an interpolation operator

$$\mathcal{J} : \mathcal{V}^1 \mapsto \mathcal{V}$$

such that the operator $\mathcal{C}\mathcal{J} : \mathcal{V}^1 \mapsto \mathcal{V}^1$ is easy to invert, and the support of functions in $\mathcal{J}\mathbb{V}_x$ extends to a reasonably small neighborhood of the subdomain x. We discuss in section 2.4.1 how to choose $\mathcal{J}$.

Finally, a decomposition of $\mathcal{V}$ is chosen in the form

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_0 + \mathcal{V}_1 \\ \mathcal{V}_1 &= \mathcal{J}\mathcal{V}^1, \end{aligned}$$

together with scalar products $\tilde{a}_0$ and $\tilde{a}_1$ on $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively. These scalar products serve as substitutes for the scalar product $a(u, v) = \langle Lu, v\rangle$ in the weak form (2.5) of the original equation.

Given $\tilde{a}_0$ and $\tilde{a}_1$, this defines operators L^1, L_0 such that

$$\begin{aligned} \tilde{a}_1(\mathcal{J}U, \mathcal{J}V) &= \langle L^1 U, V\rangle, \quad L^1 : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}, \\ \tilde{a}_0(u, v) &= \langle L_0 u, v\rangle, \quad L_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0'. \end{aligned}$$

In addition, preconditioners

$$\begin{aligned} S^1 :& \ \mathcal{V}^{1\prime} \mapsto \mathcal{V}^1, \\ S_0 :& \ \mathcal{V}_0' \mapsto \mathcal{V}_0. \end{aligned}$$

are specified. The preconditioner S^1 serves to solve the coarse grid (or low frequency) problem, which is to compute $(L^1)^{-1}F$ for $F \in \mathcal{V}^{1\prime}$, and S_0 serves to solve the so-called high frequency problem which is to compute $L_0^{-1}f$, for $f \in \mathcal{V}'$. $L_0^{-1}f$ is defined for $f \in \mathcal{V}'$ because $\mathcal{V}_0'$ is a factor space of $\mathcal{V}'$, i.e. consists of equivalence classes $[f]$ of elements f of $\mathcal{V}'$. $L_0^{-1}f = L_0^{-1}[f]$ depends only on the equivalence class of f.

The additive or multiplicative Schwarz method will be used in a standard way to reduce the original problem to the solution of the high frequency and low frequency problems. It involves operators $T_i : \mathcal{V} \mapsto \mathcal{V}_i$ such that for all $v \in \mathcal{V}_i$

$$\tilde{a}_i(T_i u, v) = \langle Lu, v\rangle.$$

Explicitly,

$$\begin{aligned} T_1 &= \mathcal{J}(L^1)^{-1}\mathcal{J}^* L, \\ T_0 &= L_0^{-1} L. \end{aligned} \tag{2.10}$$

To verify the first formula, write $T_1 u = \mathcal{J}U$ with $U = (L^1)^{-1}\mathcal{J}^* Lu$ and compute $\tilde{a}_1(T_1 u, \mathcal{J}V) = \langle L^1 U, V\rangle = \langle \mathcal{J}^* Lu, V\rangle = \langle Lu, \mathcal{J}V\rangle$. **q.e.d.**

We recall some standard results. Given the split

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1$$

and scalar products $\tilde{a}_0$ on $\mathcal{V}_0$ and $\tilde{a}_1$ on $\mathcal{V}_1$, operators $T_i : \mathcal{V} \mapsto \mathcal{V}_i$ are defined for $i = 0, 1$ by

$$\tilde{a}_i(T_i u, v) = a(u, v) \text{ for all } v \in \mathcal{V}_i$$

2.3.1 Additive Schwarz method

One introduces

$$T_{as} = T_0 + T_1 : \mathcal{V} \mapsto \mathcal{V}$$

This is well defined since $\mathcal{V}_i \subset \mathcal{V}$. The corresponding preconditioner $S_{as} : \mathcal{V}' \mapsto \mathcal{V}$ is defined by $T_{as} = S_{as}L$. If the scalar product $\tilde{a}_1$ comes from a positive operator $D_1 : \mathcal{V}_1 \mapsto \mathcal{V}_1'$ and $\tilde{a}_0$ comes from a positive operator $D_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0'$ then $T_0 = D_0^{-1}L$ and $T_1 = D_1^{-1}L$. Therefore,

$$S_{as} = D_0^{-1} + D_1^{-1}.$$

Now remember that S_{as} is a good preconditioner if $T = SL$ has reasonably small condition number as an operator in the Hilbert space $\mathcal{V}$ with scalar product $a = \langle \cdot, \cdot \rangle_L$. The standard convergence theory [49] gives bounds on the largest and smallest eigenvalue. In the case of the decomposition in only two subspaces the result simplifies as follows

Theorem 1 *Let there exist*
(i) a constant C_0 such that for all $u \in \mathcal{V}$ there exists a decomposition $u = u_0 + u_1$, $u_i \in \mathcal{V}_i$, such that

$$\tilde{a}_0(u_0, u_0) + \tilde{a}_1(u_1, u_1) \leq C_0^2 a(u, u) \ ;$$

(ii) a constant ω such that for $i = 0, 1$

$$a(u, u) \leq \omega \tilde{a}_i(u, u) \textit{ for all } u \in \textit{Range } T_i.$$

Then the minimal and maximal eigenvalues of T obey

$$C_0^{-2} \leq \lambda_{min} \leq \lambda_{max} \leq 2\omega.$$

2.3.2 Multiplicative Schwarz method

In this method, the same operators T_0 and T_1 are used as above, and one sets

$$T_{ms} = I - (I - T_1)(I - T_0) = T_0 + T_1 - T_1 T_0 \ .$$

Therefore, if the scalar products come from positive operators D_0, D_1 as above, then $T_{ms} = S_{ms}L$ with

$$S_{ms} = D_0^{-1} + D_1^{-1} - D_0^{-1} L D_1^{-1}$$

The standard convergence result is a variant of a result of Bramble Pasciak, Wang and Xu [7], as follows [49]. It is only useful if $0 < \omega < 2$. The inequality $\omega < 2$ can always be achieved by rescaling the scalar products $\tilde{a}_i$. The result is stated in terms of the norm $|| \cdot ||_a$ in the Hilbert space $\mathcal{V}$ with scalar product a

Theorem 2 *Under the same conditions as in Theorem 1,*

$$||I - T_{ms}||_a^2 \leq 1 - \frac{2-\hat{\omega}}{1+2\hat{\omega}^2 C_0^2}$$

where $\hat{\omega} = max(1, \omega)$.

The standard procedure is to use this for simple iterations. Starting with an initial guess u^0, the n-th iterate u^n $(n = 0, 1, ...)$ gets updated to

$$u^n \mapsto u^{n+1} = (I - T_{ms})u^n.$$

But one can also use Krylov subspace methods. There exists also a symmetric version (which is suitable for use with Conjugate gradient)

$$T_{sms} = I - (I - T_{ms}^+)(I - T_{ms}) = T_{ms} + T_{ms}^+ - T_{ms}^+ T_{ms}\ .$$

2.3.3 Twogrid schemes

We will now present a list of twogrid schemes. Given the restriction operator $\mathcal{C} : \mathcal{V} \to \mathcal{V}^1$, the space $\mathcal{V}^1$ is fixed. Different schemes differ in the choice of $\mathcal{V}_0$, the scalar products $\tilde{a}_0$ and $\tilde{a}_1$, and the preconditioners. The interpolation operator $\mathcal{J}$ enters only into the Schwarz method, and determines its convergence.

Non-overlapping blocks

1. *Compatible relaxation.*

   The name compatible relaxation refers to the use of updating sweeps on the original grid which leave $\mathcal{C}u$ unchanged.

$$\begin{aligned} \mathcal{V}_0 &= \{u \in \mathcal{V}, \mathcal{C}u = 0\}, \\ \tilde{a}_0(u,v) &= \langle Lu, v\rangle,\ u,v \in \mathcal{V}_0 \\ &= \langle [L + \kappa \mathcal{C}^* D^1 \mathcal{C}]u, v\rangle, \\ \langle S_0^{-1}u, v\rangle &= \langle Du, v\rangle.\ u,v \in \mathcal{V}_0 \end{aligned} \tag{2.11}$$

   Eq.(2.11) holds for any κ and for any D^1. Concerning the coarse grid problem, there are several variants.

   i) Choose

$$\tilde{a}_1(\mathcal{J}U, \mathcal{J}V) = \langle L^1 U, V\rangle,\ L^1 = \mathcal{J}^* L \mathcal{J}$$

   and a preconditioner S^1. There are two choices for S^1. Either $S^1 = (D^1)^{-1}$ with

$$(D^1)^{-1} = \mathcal{C} D^{-1} \mathcal{C}^* \tag{2.12}$$

   or one solves the coarse grid problem by setting up another twogrid procedure, and so on if desired. This leads to a **multigrid procedure**.

ii)

$$\tilde{a}_1(\mathcal{J}U, \mathcal{J}V) = \langle D^1 U, V\rangle$$

with D^1 from eq.(2.12). This amounts to making very simple coarse grid corrections. No preconditioner S^1 is needed in this case since D^1 is easy to invert. Very simple coarse grid corrections are frequently used in domain decomposition. They are not in a multigrid spirit.

2. *Softened compatible relaxation.*

   Choose $\kappa > \epsilon$ and set

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{V}\ , \\ \tilde{a}_0(u,v) &= \langle [L + \kappa \mathcal{C}^* D^{1\prime} \mathcal{C}]u, v\rangle,\ u, v \in \mathcal{V}_0, \\ S_0 &= (D + \kappa \mathcal{C}^* D^{1\prime} \mathcal{C})^{-1} \end{aligned} \tag{2.13}$$

   with $D^{1\prime} : \mathcal{V}^1 \to \mathcal{V}^1$ constructed together with the restriction operator $\mathcal{C}$ in section 2.3.5. The coarse grid is treated the same way as for compatible relaxation.

3. *Block Jacobi.*

   $\mathcal{V}_0 = \{u \in \mathcal{V}, \mathcal{C}u = 0\}$ as in the compatible relaxation scheme but

$$\begin{aligned} \tilde{a}_1(\mathcal{J}U, \mathcal{J}V) &= \epsilon \langle D^1 U, V\rangle, \\ \tilde{a}_0(u,v) &= \lambda_{max}\langle Du, v\rangle, \\ S_0 &= D^{-1}(D - \mathcal{C}^* D^1 \mathcal{C}) D^{-1} \end{aligned}$$

   with D^1 from eq.(2.12). S_0 is the exact inverse of the operator $L_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0'$ associated with a_0, i.e. it is a perfect preconditioner. $L^1 = D^1$ can also be inverted exactly.

4. *Orthogonal spaces.*

   Let $\mathcal{V}_1^\perp$ be the orthogonal complement of $\mathcal{V}_1 = \mathcal{J}\mathcal{V}^1$ with respect to $a(\cdot,\cdot)$, and let P_1 be the orthogonal projector on $\mathcal{V}_1$. Set

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{V}_1^\perp = (I - P_1)\mathcal{V}, \\ \tilde{a}_0(u,v) &= \langle Lu, v\rangle, \\ \tilde{a}_1(\mathcal{J}U, \mathcal{J}V) &= \langle L^1 U, V\rangle \end{aligned}$$

   with $L^1 = \mathcal{J}^* L \mathcal{J}$. In such a scheme, the Schwarz iteration converges in one step, but every iteration of the high frequency problem involves an application of P_1 which requires the solution of a coarse grid problem, because

$$P_1 = \mathcal{J}(L^1)^{-1}\mathcal{J}^* L.$$

   P_1 equals the operator T_1 in equation (2.10). Often this will be too expensive unless the high frequency iteration converges fast. For this purpose one needs a good preconditioner S_0 for the high frequency problem. Suitable choices would be

$$S_0 = (D + \kappa \mathcal{C}^* D^1 \mathcal{C})^{-1} \text{ or}$$
$$S_0 = (L^N + \kappa \mathcal{C}^* D^1 \mathcal{C})^{-1}$$

with an operator $L^N \leq L$ that does not couple between blocks.
For instance, one may use the "historical" **additive Schwarz method with overlap**. One selects neighborhoods $\mathcal{N}(x)$ of the blocks x in the original grid. One sweeps through the blocks x in some order. For each x, one updates $u(z)$ for $z \in \mathcal{N}(x)$ in such a way that the cost functional $\frac{1}{2}\langle Lu, u\rangle - \langle f, u\rangle$ is minimized, subject to the constraint that $u(z)$ for $z \notin \mathcal{N}(x)$ remain unchanged. This amounts to solving inhomogeneous Dirichlet problems on $\mathcal{N}(x)$, cp. [45].

The coarse grid problem could be solved in the same ways as discussed above. In particular one may use a multigrid procedure.

Alternatively one could use $\tilde{a}^1(U, V) = \langle D^1 U, V\rangle$. Again, this is not in a multigrid spirit.

Our main interest will be in the first two methods.

Blocks overlapping in boundaries

5. *Orthogonal spaces and Neumann Neumann methods.*

   Let $\mathcal{V}_1^\perp$ be the orthogonal complement of $\mathcal{V}_1$ with respect to $a(\cdot,\cdot)$, and let P_1 be the orthogonal projector on $\mathcal{V}_1$, as above.

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{V}_1^\perp = (I - P_1)\mathcal{V}, \\ \tilde{a}_0(u, v) &= \langle Lu, v\rangle. \end{aligned}$$

   The high frequency problem is solved by an additive Schwarz method, cp. section 2.3.1. This method reduces to the balanced Neumann Neumann method in case $L^{N,x}$ are positive semidefinite, and the energy cutoff ϵ is chosen so small that only the zero modes are relevant for defining the coarse grid problem.

   A possible preconditioner is also given by

$$S_0 = L^N + \kappa \mathcal{C}^* D^1 \mathcal{C}.$$

In the sequel of this chapter we concentrate on non-overlapping blocks.

2.3.4 Generalized Neumann boundary conditions

We would like to explain the construction of the operator with generalized Neumann boundary conditions. This preserves a generalization of the equality $-\triangle^N \leq -\triangle$ that is valid for the Laplacian. This mechanism is also applicable to "tight" Dirichlet boundary conditions, as outlined in appendix A. In subsection 2.3.6 a suitable construction of the Helmholtz operator with Neumann boundary conditions is proposed. It fulfills the requirement $L^N \leq L$.

Assume that $\mathcal{V}$ is made into a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_D$. Then $\mathbb{L} = D^{-1}L$ is selfadjoint in this Hilbert space if $L = L^*$, and $\mathbb{L}^+ = D^{-1}L^*$. We define positive semidefinite operators

$$^wP_z = D_{zz}(\mathbb{L}^+_{wz}\mathbb{L}_{wz})^{\frac{1}{2}} : \mathbb{V}_z \mapsto \mathbb{V}'_z.$$

Let X be the union of all blocks x and ∂X consists of all edges between different blocks. ∂x consists of edges with target in block x and source not in x. Then we define

$$L^N_{zw} = \begin{cases} 0 & \text{for } \langle zw\rangle \in \partial X \\ L_{zw} & \text{for } z \neq w \text{ and } \langle zw\rangle \notin \partial X \\ L_{zz} - \sum_{z':\langle z',z\rangle\in\partial X} {}^{z'}P_z & \text{for } z = w. \end{cases}$$

where $\langle zw\rangle$ stands for the edge between z and w. This operator fulfills the requirement for blocks that do not overlap in boundaries

$$L^N = \sum L^{N,x} \leq L,$$

if L is positive definite with respect to the scalar product. This is shown in detail in appendix A.

2.3.5 The restriction operator $\mathcal{C}$

Given a Linear System, a domain decomposition into blocks x, a definition of the problem operator with Neumann boundary conditions, a preconditioner D which does not couple blocks, and an "energy" cutoff ϵ. We may consider generalized eigenmodes $\phi^\alpha_x \in \mathcal{V}$ of $L^{N,x}$, defined by

$$\begin{aligned} D^{-1}L^{N,x}\phi^\alpha_x(z) &= E^\alpha_x\phi^\alpha_x(z) \text{ for } z \in x, \\ \phi^\alpha_x(z) &= 0 \text{ for } z \notin x. \end{aligned}$$

We may imagine that the modes are ortho-normalized,

$$\langle\phi^\alpha_x, \phi^\beta_y\rangle_D = \delta^{\alpha\beta}\delta_{xy}. \tag{2.14}$$

Given x, let there be $n_x \geq 0$ such eigenmodes with eigenvalues $E^\alpha_x < \epsilon$, and let them be labeled by $\alpha = 1, ..., n_x$. Then $\mathbb{V}_x$ shall consist of column vectors

$$\mathbb{V}_x \ni c_x = (c^1_x, ...c^{n_x}_x)^T$$

made of real or complex numbers. Let $\phi^x_\alpha \in \mathcal{V}'$ be the dual basis, viz. $(\phi^x_\alpha, \phi^\beta_y) = \delta_\alpha^{\ \beta}\delta_x^{\ y}$. For $u \in \mathcal{V}$ we define

$$c_x = \mathcal{C}_x u \text{ by } c^\alpha_x = \langle\phi^x_\alpha, u\rangle\ ,\ \alpha = 1, ...n_x. \tag{2.15}$$

$\mathcal{C}$ maps the functions $\phi^\alpha_x \in \mathcal{V}, \alpha = 1, ...n_\alpha$ bijectively into a basis of $\mathbb{V}_x$. Therefore we may define a scalar product $\langle\ ,\ \rangle_{D^{1\prime}}$ on $\mathcal{V}^1 = \{\mathbb{V}_x\}$ by

$$\langle\mathcal{C}\phi^\alpha_x, \mathcal{C}\phi^\beta_x\rangle_{D^{1\prime}} = \langle\phi^\alpha_x, \phi^\beta_x\rangle_D \tag{2.16}$$

for $\alpha, \beta = 1, ...n_x$. This defines an operator

$$D^{1\prime} : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}.$$

Let $E_x : \mathbb{V}_x \mapsto \mathbb{V}_x$ be the matrix made of the eigenvalues,

$$(E_x c_x)^\alpha = E_x^\alpha c_x^\alpha.$$

and let $E : \mathcal{V}^1 \mapsto \mathcal{V}^1$ be the operator defined by

$$(EU)(x) = E_x U(x) \ .$$

Then

$$\mathcal{C}_x D^{-1} L^{N,x} = E_x \mathcal{C}_x$$

and in case of non-overlapping blocks,

$$\mathcal{C} D^{-1} L^N = E\mathcal{C}.$$

2.3.6 Examples

To illustrate the spirit of the construction of the restriction operator, we give an informal discussion of some examples.

The anisotropic Laplacian

Consider a two-dimensional quadratic grid. Let ∇ be the finite difference derivative ($\nabla_\mu u(z) = u(z + \hat{\mu}) - u(z)$ for lattice spacing 1 and $\mu = 1, 2$) and let

$$L = -\sum_{\alpha=1}^{2} a_{\alpha\beta}(z) \nabla_\alpha \nabla_\beta,$$

be the anisotropic Laplacian, where $a(z)$ is a positive 2×2-matrix which depends smoothly on z, and which may have a large ratio $\frac{\lambda_2}{\lambda_1}$ of eigenvalues for some or all z.
To bring out the ideas as clearly as possible, suppose that we have inside of some cube of side length s around $\hat{x}$

$$L = -\lambda_1 \nabla_1^2 - \lambda_2 \nabla_2^2$$

with large ratio $\frac{\lambda_2}{\lambda_1}$. The operator L^N with traditional Neumann boundary conditions on the boundary of the cube is well defined. For suitably positioned fairly large blocks, the eigenmodes are $\cos(k_1 z_1 + k_2 z_2)$ with $k_i \approx \frac{n_i \pi}{s}$ with non-negative integer n_i and the eigenvalues are $\lambda_1 k_1^2 + \lambda_2 k_2^2$. Thus if we choose $\epsilon = \lambda_2 (\frac{\pi}{s})^2$, then the eigenmodes with eigenvalues $< \epsilon$ are

$$\cos k_1 z_1, \quad k_1 = \frac{n_1 \pi}{s} \text{ with } n_1 = 0, ... \left[\sqrt{\frac{\lambda_2}{\lambda_1}}\right], \tag{2.17}$$

where $[n]$ is the largest integer $\leq n$. Thus, the number of modes increases proportional $\sqrt{\frac{\lambda_2}{\lambda_1}}$. From these modes, the restriction operator $\mathcal{C}$ is to be built according to section 2.3.5.

Let us try to explain why this will work by comparing with another approach which is based on the adjustment of the shape of the blocks. If it were not for the discreteness of the grid, we could rescale z_1 to remove the anisotropy.

$$z_1 \to z_1' = \sqrt{\frac{\lambda_2}{\lambda_1}} z_1, \quad z_2 \to z_2' = z_2$$

where $\nabla = \sqrt{\frac{\lambda_2}{\lambda_1}} \nabla'$ and

$$L = -\lambda_2 (\nabla_1'^2 + \nabla_2^2).$$

We can treat this problem locally by choosing as a block a square of the extent s in z_1' and in z_2', and an energy cutoff $\epsilon = \lambda(\frac{\pi}{s})^2$ so that the only eigenmode $< \epsilon$ of L^N is the constant mode. The new square is the image of a rectangle in z_1, z_2 of extent s in z^2 and $\sqrt{\frac{\lambda_1}{\lambda_2}} s$ in z_1. There are $\sqrt{\frac{\lambda_2}{\lambda_1}}$ of such rectangles inside of the original square in z_1, z_2. The constant modes in these rectangle multiplied with arbitrary factors combine to modes in the original rectangle, which are constant in z_2 and piecewise constant in z_1. Suitable such modes can be regarded as piecewise constant approximations to our modes (2.17). Thus the two ways of treating the problem locally are quite similar.

If it were not for the z-dependence of $a(z)$ the whole problem would reduce to solving the Laplace equation and could be solved be the second approach. The block spins would be block averages of u. It is well known that compatible relaxation with such a choice of a restriction operator works well.

The advantage of our approach is that it is a general purpose procedure for a wide range of problems. One does not need to inspect the problem first in order to find a suitable choice of ingredients in the twogrid procedure. Besides, in the present example, the z-dependence of a would create practical problems when one wants to implement the second procedure.

The Helmholtz operator

When one considers solutions of a wave equation with periodic time dependence $e^{-i\omega t}$ one is lead to the Helmholtz equation. Its inhomogeneous version reads

$$(\triangle + \omega^2)u = f$$

where $\triangle$ is the lattice Laplacian. Note that the original problem is a hyperbolic problem, and that the Helmholtz operator is neither positive nor elliptic because the equation $-k^2 + \omega^2 = 0$ has nontrivial solutions k.

To get a well defined problem, one needs to specify boundary conditions such as the

Sommerfeld radiation condition, which involves the normal derivation of the solution at the boundary (ideally at ∞),

$$(\triangle_n + i\omega)u = o\big(r^{-\frac{d-1}{2}}\big)$$

in d dimensions. The Helmholtz equation has been studied in the literature [9]. We recall some heuristics of which we expect that it can be made precise applying our procedure using the Helmholtz operator with suitable generalized Neumann boundary conditions. Details have not been worked out yet. The key feature of the problem is that there are very many low lying modes, but not so many that the multigrid procedure would not lead to a consecutive thinning of degrees of freedom at successive scales. In our approach, the dimension of the fibres $\mathbb{V}_x$ will increase with the scale, but nevertheless the total number of degrees of freedom is expected to decrease.

The low lying eigenmodes of the Helmholtz operator are plane waves e^{ikx} with eigenvalues

$$\lambda_k = \omega^2 - k^2$$

of small modulus $|\lambda_k| < \epsilon$. Because $\omega^2 - k^2 \approx 2\omega(\omega - |k|)$ this is equivalent to

$$|k| = \omega - \frac{\epsilon}{2\omega} \quad \ldots \quad \omega + \frac{\epsilon}{2\omega}.$$

Their number is approximately proportional to the integral $\Omega_d \int_{\omega-\frac{\epsilon}{2\omega}}^{\omega+\frac{\epsilon}{2\omega}} drr^{-1} \approx \Omega_d d\epsilon\omega^{d-2}$, where Ω_d is the volume of the unit sphere S^{d-1}.

The problem will be to find a similar estimate for the number of low lying modes of the Helmholtz operator with generalized Neumann boundary conditions on block boundaries.

A proposal for the squared operator with generalized Neumann boundary conditions is as follows. Consider

$$L_H = (\triangle + \omega)^2 u = f.$$

Expanding the square this reads

$$(\triangle^2 + 2\triangle\omega^2 + \omega^4)u = f.$$

An adequate Operator with Neumann boundary conditions is given by

$$L_H^N = (\triangle^2)^N + \omega^4 + 2\omega^2\triangle^D$$

where ^{D} stands for Dirichlet boundary conditions, as $\triangle^2$ is positive and $\triangle$ negative definite. It fulfills $L_H^N \leq L_H$ due to $-\triangle^N \leq -\triangle \leq -\triangle^D$.

2.4 Interpolation operators

We wish to construct an interpolation operator $\mathcal{J}$ that can be used in practical applications. We construct it as approximation to another ideal interpolator that is canonically determined by the restriction operator $\mathcal{C}$ and that is well known in the literature. It was introduced in the framework of quantum field theory in the framework of a rigorous renormalization group approach [14]. Our interpolation operator is defined to be a local approximation to the ideal interpolation operator which has exponential tails. We will explain how the practical interpolation operator is determined. A measure of its quality will be introduced. Given the problem operator, the interpolation operator $\mathcal{J}$ will determine the effective problem operator L^1 on the coarse grid. Again L^1 is an approximate to an ideal coarse grid operator $\hat{L}^1$.

2.4.1 Ideal interpolation operator

We turn to the consideration of the interpolation operator $\mathcal{J} : \mathcal{V}^1 \mapsto \mathcal{V}$ from the coarse grid to the fine grid. Let

$$\mathcal{V}_{\mathcal{C}} = \{u \in \mathcal{V} : \ \mathcal{C}u = 0\}.$$

This is the space $\mathcal{V}_0$ of the compatible relaxation scheme.

We will construct interpolation operators $\mathcal{J}$ as approximations to an *ideal interpolation operator* $\mathcal{A}$ which maps $\mathcal{V}^1$ to the orthogonal complement of $\mathcal{V}_{\mathcal{C}}$. If we could use the split $\mathcal{V} = \mathcal{V}_{\mathcal{C}} + \mathcal{A}\mathcal{V}^1$ then the multiplicative Schwarz iteration would converge in a single step.

This requirement does not fix $\mathcal{A}$ uniquely. If we substitute $\mathcal{A} \mapsto \mathcal{A}R$ where, R is any invertible map $\mathcal{V}^1 \mapsto \mathcal{V}^1$, then $\mathcal{A}\mathcal{V}^1$ stays unchanged. We will later use this freedom to advantage. For now we eliminate it by imposing the subsidiary condition

$$\mathcal{C}\mathcal{A} = I.$$

$\mathcal{C}\mathcal{A}$ maps $\mathcal{V}^1$ to $\mathcal{V}^1$.

We give a formula for $\mathcal{A}$. Define an *ideal coarse grid problem operator* $\hat{L}^1 : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}$ by

$$(\hat{L}^1)^{-1} = \mathcal{C}L^{-1}\mathcal{C}^* \ . \tag{2.18}$$

Proposition 1

$$\mathcal{A} = L^{-1}\mathcal{C}^*\hat{L}^1 \ .$$

maps $\mathcal{V}^1$ to the orthogonal complement of $\mathcal{V}_{\mathcal{C}}$ in the Hilbert space $\mathcal{V}$ with scalar product $a = \langle \cdot, \cdot \rangle_L$ and satisfies $\mathcal{C}\mathcal{A} = I$. Moreover, the ideal problem operator $\hat{L}^1$ can be recovered from it,

$$\hat{L}^1 = \mathcal{A}^* L \mathcal{A}.$$

Needless to say, these formulae do not give a practical way of computing $\mathcal{A}$ and $\hat{L}^1$. But we will see later that $\mathcal{A}U$ can be computed by fast converging compatible relaxation.

PROOF of Proposition 1. Let $v \in \mathcal{V}_{\mathcal{C}}$ so that $\mathcal{C}v = 0$. Then

$$\langle \mathcal{A}U, v\rangle_L = \langle L\mathcal{A}U, v\rangle = \langle \mathcal{C}^*\hat{L}^1 U, v\rangle = \langle \hat{L}^1 U, \mathcal{C}v\rangle = 0.$$

This proves the first part of the proposition. The assertion $\mathcal{C}\mathcal{A} = I$ is obvious from eq.(2.18). Finally we compute

$$\mathcal{A}^* L \mathcal{A} = \mathcal{A}^*\mathcal{C}^*\hat{L}^1 = \hat{L}^1 \ . \qquad \textbf{q.e.d.}$$

Let $\mathcal{A}(z,x) : V_x \mapsto V_z$ be the "integral kernel" of $\mathcal{A}$, viz.

$$(\mathcal{A}U)(z) = \sum_x \mathcal{A}(z,x)U(x) \ .$$

It can be proven under fairly general assumptions, without use of translation invariance, that $\mathcal{A}_{zx}$ decays exponentially with distance between z and x, in appropriate norms and with distance measured in path length on the coarse grid. The precise decay length may depend on ϵ. In the classical case $L = -\Delta$, $\epsilon > 0$ small, the operators $\mathcal{C}$ and $\mathcal{A}$ were introduced and studied 20 years ago by Kupiainen and Gawedzki [14] for use in a rigorous renormalization group approach for quantum field theory. In this classical case, the decay properties on hypercubic grids with hypercubes as blocks can be obtained by Fourier analysis. The decay length is one coarse grid lattice spacing. The results were later extended to gauge theories, where translation invariance cannot be used, by Balaban and collaborators [6]. The fast decay could also be deduced from the fast convergence of the compatible relaxation.

In spite of this, the exponential tails of $\mathcal{A}(z,x)$ preclude its practical use. Moreover, one knows from the study of the classical case $L = -\Delta$ that the decay properties of $\mathcal{A}$ can be much improved by making use of the freedom of replacing $\mathcal{A}$ by $\mathcal{A}R$. This will cure another problem at the same time. The strategy outlined in section 2.2.3 requires a good upper bound on $(D^1)^{-1}L^1$ to make the low frequency problem well conditioned. D^1 is the coarse grid version of the original "bad" preconditioner D, cp. eq.(2.12). There is no guarantee that $(D^1)^{-1}\hat{L}^1$ will satisfy such a bound.

With this motivation, we introduce a one-parameter family of ideal interpolation operators $\mathcal{A}_\kappa$ and, later on, of ideal coarse grid problem operators $\hat{L}^1_\kappa$. The parameter κ shall be in the range

$$\epsilon < \kappa \leq \infty,$$

where ϵ is the energy cutoff. Set

$$\begin{aligned} G_\kappa &= \mathcal{C}L^{-1}\mathcal{C}^* + \frac{1}{\kappa}(D^1)^{-1} \ , \\ \mathcal{A}_\kappa &= L^{-1}\mathcal{C}^* G_\kappa^{-1} \ . \end{aligned} \tag{2.19}$$

These formulae are useless for practical calculation, but there is a useful alternative form which can be used as equivalent definitions.

Proposition 2 *For $\kappa < \infty$ and D^1 as defined in eq.(2.12),*

$$\mathcal{A}_\kappa = \kappa(L+\kappa\mathcal{C}^* D^1 \mathcal{C})^{-1}\mathcal{C}^* D^1 \tag{2.20}$$

Moreover $\mathcal{A}_\kappa = \mathcal{A}R_\kappa$ with $R_\kappa = 1 - \frac{1}{\kappa}(D^1)^{-1}G_\kappa^{-1}$, and $\mathcal{A} = \lim_{\kappa\mapsto\infty}\mathcal{A}_\kappa$.

Proof of proposition 2: Starting from the definitions (2.19)f, we compute $(L+\kappa\mathcal{C}^* D^1\mathcal{C})\mathcal{A}_\kappa = \mathcal{C}^* G_\kappa^{-1} + \kappa\mathcal{C}^*(D^1)[G_\kappa - \frac{1}{\kappa}(D^1)^{-1}]G_\kappa^{-1} = \kappa\mathcal{C}^* D^1$. This proves the first formula. The limiting value of $\mathcal{A}_\kappa$ is immediate from the definitions. The assertion $\mathcal{A}_\kappa = \mathcal{A}R_\kappa = \mathcal{A}\big(1 - \frac{1}{\kappa}(D^1)^{-1}G_\kappa^{-1}$ is derived by inserting G_κ. It follows $\mathcal{A}_\kappa G_\kappa = \mathcal{A}G_\kappa - \frac{1}{\kappa}\mathcal{A}(D^1)^{-1} \Leftrightarrow \mathcal{A}(\hat{L}^1)^{-1} = L^{-1}\mathcal{C}$. **q.e.d.**

Corollary 1 (*High frequency propagators) Define the self-adjoint high frequency propagators Γ_κ and $\Gamma : \mathcal{V}' \mapsto \mathcal{V}$ by*

$$\begin{aligned}\Gamma_\kappa &= (L+\kappa\mathcal{C}^* D^1\mathcal{C})^{-1}\\ \Gamma &= \lim_{\kappa\mapsto\infty}\Gamma_\kappa.\end{aligned}$$

$0 < \kappa < \infty$. *Then*

$$\mathcal{A}_\kappa = \kappa\Gamma_\kappa\mathcal{C}^* D^1, \tag{2.21}$$

$$\Gamma\mathcal{C}^* = 0\ , \quad \mathcal{C}\Gamma = 0, \tag{2.22}$$

$$\Gamma_\kappa L = 1 - \mathcal{A}_\kappa\mathcal{C}\ , \tag{2.23}$$

so that $\langle u, \Gamma_k L v\rangle = \langle u, v\rangle$ for $u, v \in \mathcal{V}_\mathcal{C}$. In particular

$$\Gamma L v = v \text{ for all } v \in \mathcal{V}_\mathcal{C}.$$

The last assertion shows that Γ restricted to $\mathcal{V}_\mathcal{C}$ is the inverse of L_0 in the compatible relaxation scheme.

Proof of corollary 1 is immediate from the proposition. Multiply both sides of eq.(2.23) with $(L+\kappa\mathcal{C}^* D^1\mathcal{C})$; divide eq.(2.21) by κ and take the limit. Γ maps $\mathcal{V}_\mathcal{C}$ to $\mathcal{V}_\mathcal{C}$, because $\mathcal{C}\Gamma = 0$. **q.e.d.**

Lemma 1 *Define the family of ideal coarse grid problem operators by*

$$\hat{L}^1_\kappa = \mathcal{A}^*_\kappa L \mathcal{A}_\kappa.$$

so that $\hat{L}^1 = \lim_{\kappa\to\infty}\hat{L}^1_\kappa$ and $L^1_\kappa = R^*_\kappa \hat{L} R_\kappa$. It satisfies the upper bound

$$\hat{L}^1_\kappa \leq \kappa D^1.$$

Proof of lemma 1. Inserting the definitions one computes

$$\mathcal{A}^*_\kappa L \mathcal{A}_\kappa = G_\kappa^{-1} - \frac{1}{\kappa}G_\kappa^{-1}(D^1)^{-1}G_\kappa^{-1} \leq G_\kappa^{-1}.$$

It is immediate from its definition that $G_\kappa \geq \frac{1}{\kappa}(D^1)^{-1}$. This implies the stated bound. **q.e.d.**

For illustration we show in figure 2.1 the integral kernel $\mathcal{A}(z,x)$ of $\mathcal{A}_\kappa$ for various values of κ in the classical case $L = -\Delta$ on a quadratic lattice in two dimensions, with 8×8 squares as blocks. $\mathcal{C}u(x)$ is the average of $u(z)$ over the 8×8 square x in this example. A single block $x = 0$ is considered, and $\mathcal{A}_\kappa(z,0)$ is shown for z on a line which passes close to the center of x. One sees that the constraint $\mathcal{CA} = I$ for $\kappa \mapsto \infty$ forces oscillations into $\mathcal{A}$ which worsen the decay properties. The constraint requires that the block averages $\mathrm{av}_{z\in y}\mathcal{A}(z,x) = 1$ for $y = x$ and $= 0$ for all other blocks y. The falloff properties are improved if κ is above and close to ϵ, where ϵ is the first nonzero eigenvalue of $-\Delta^N$ in the example. [21, 17]

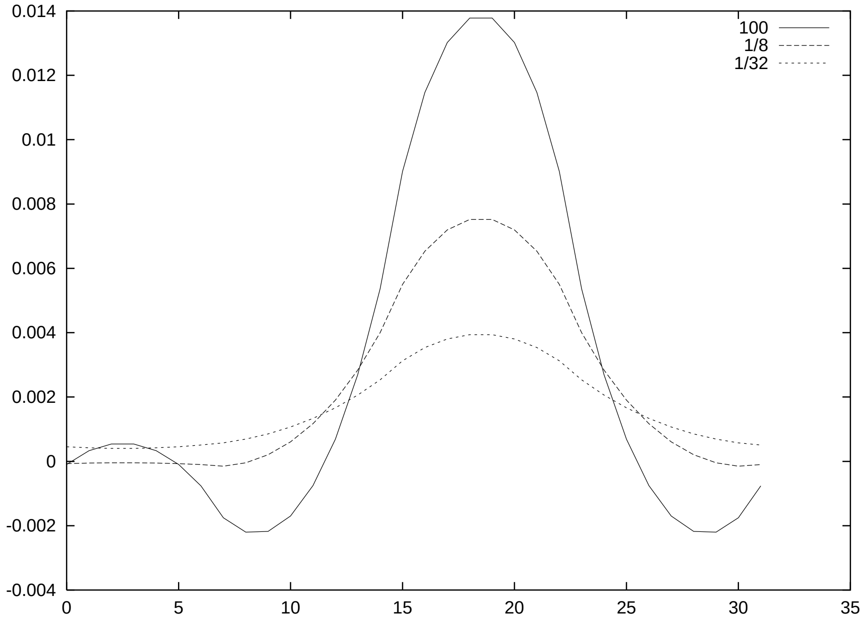


Figure 2.1: This figure shows cross sections through $\mathcal{A}$-kernels for 8×8-blocks with κ values of 100 (top), $\frac{1}{8}$ (center) and $\frac{1}{32}$ (bottom) in units $a = 1$. Note that the kernel oscillates for high κ values [18].

2.4.2 Determination of practical interpolators

Finally we are ready to consider practical approximations $\mathcal{J}$ to $\mathcal{A}_\kappa$, which differ from $\mathcal{A}_\kappa$ in their support properties. Let $\mathcal{J}_x : \mathbb{V}_x \mapsto \mathcal{V}$ be defined by $\mathcal{J}U(z) = \sum_x \mathcal{J}_x U(x)$.

We choose neighborhoods $\mathcal{N}(x)$ of blocks x in the original grid, and require that $\mathcal{J}_x$ vanishes outside $\mathcal{N}(x)$.

We distinguish two cases.

i) $\mathcal{N}(x)$ consists of entire blocks, viz. x and some neighbors.

ii) It consists smaller neighborhoods $\mathcal{N}(x)$.

In case i) we may continue to use formula (2.20) for the kernel $\mathcal{A}_\kappa(z,x)$ of $\mathcal{A}_\kappa$ when $z \in \mathcal{N}(x)$; for $z \notin \mathcal{N}(x)$ the kernel vanishes. More generally, let us consider the problem of finding a good interpolation operator $\mathcal{J}$ up to right multiplication $\mathcal{J} \mapsto \mathcal{J}R$ with a *diagonal* R. Multiplication with diagonal R does not change the locality properties of either $\mathcal{J}$ or $L^1 = \mathcal{J}^* L \mathcal{J}$; it takes $L^1 \mapsto R^* L^1 R$. We may use this freedom to prescribe the diagonal part of L^1, e.g.

$$L^1_{xx} = \epsilon D^1_{xx}.$$

We formulate the problem of finding an interpolation operator as an extremal problem. If we impose support properties as subsidiary conditions, practical approximations to the ideal interpolation operator will be obtained. The iterative solution of the extremal problem converges fast because it is equivalent to a compatible relaxation; moreover, it involves a local problem operator with Dirichlet boundary conditions.

Let for example be

$$S_0 = D^{-1}[D - \mathcal{C}^* D^1 \mathcal{C}]D^{-1} : \mathcal{V}' \mapsto \mathcal{V}_{\mathcal{C}},$$

with D^1 from eq.(2.12). This is suitable preconditioner, regarded as a map of $\mathcal{V}'$. It satisfies $\mathcal{C}S_0 = 0, S_0\mathcal{C}^* = 0$, and $\langle S_0 D v_0, v_0\rangle = \langle v_0, v_0\rangle$ if $\mathcal{C}v_0 = 0$. This is easily verified. An ideal interpolation kernel $\mathcal{J} = \mathcal{A}R$ has the defining property $L\mathcal{J} = \mathcal{C}^* L^1$ for some L^1. Therefore $S_0 L \mathcal{J} = 0$. Since $D^1 > 0$ this is true if and only if $\mathrm{tr}(D^1)^{-1}\mathcal{J}^* L S_0 L \mathcal{J} = 0$. We may therefore determine $\mathcal{J}$ as solution of the extremal problem

$$\mathrm{Tr}(D^1)^{-1}\mathcal{J}^* L S_0 L \mathcal{J} = \mathrm{Min}$$

subject to some subsidiary condition which excludes $\mathcal{J} = 0$. Here Tr is the trace in $\mathcal{V}^1$, i.e. $\mathrm{Tr}\Phi = \sum_x \mathrm{tr}\,\Phi_{xx}$, tr is the trace in $\mathbb{V}_x$. As a subsidiary condition we could take

$$\mathrm{tr}\left(\mathcal{C}D^{-1}L\mathcal{J}\right)_{xx} = 1.$$

Without support properties as further subsidiary conditions, the extremal problem has general solution $\mathcal{J} = L^{-1}\mathcal{C}^* D^1 \lambda$, with $\lambda : \mathcal{V}^1 \mapsto \mathcal{V}^1$ arbitrary, except that $\mathrm{tr}\,\lambda_{xx} = 1$. In particular, $\mathcal{J} = \mathcal{A}_\kappa R$ with a suitably normalized *diagonal* R furnishes a family of solutions parameterized by κ. Remembering that the support properties of $\mathcal{A}_\kappa$ depend in general on κ, we expect that the imposition of support properties as additional subsidiary condition will remove the arbitrariness, except for right multiplication with a suitable *diagonal* R. Given that $\mathcal{A}_\kappa$ has good enough support properties, the unique extremum is expected to be close to $\mathcal{A}_\kappa R$ for some $\kappa \approx \epsilon$, with a diagonal R. Let $L^{D,x}$ be the problem operator restricted to $\mathcal{N}(x)$ with homogeneous Dirichlet boundary conditions on the boundary of $\mathcal{N}(x)$. Hence, $L\mathcal{J}_x = L^{D,x}\mathcal{J}_x$ if $\mathcal{J}_x$ has the indicated support properties. Variation with Lagrange multipliers leads to $S_0 L^{D,x}\mathcal{J}_x = D^{-1}\mathcal{C}^*\gamma$ with some γ. But $\mathcal{C}S_0 = 0$ and $\mathcal{C}D^{-1}\mathcal{C}^* = (D^1)^{-1}$, therefore $\gamma = 0$. An equation $S_0\Phi = 0$ requires that $\Phi = \mathcal{C}^*\Psi$. Therefore we are lead to the equation

$$L^{D,x}\mathcal{J}_x = (\mathcal{C}^*\lambda)_x \tag{2.24}$$

for some $\lambda : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}$. Explicitly, $(\mathcal{C}^*\lambda)_x = \sum_y \mathcal{C}_y \lambda_{yx}$. Assuming that for all neighboring blocks y to x the support of $\mathcal{J}_x$ fills out only part of y, eq.(2.24) cannot be true unless $\lambda_{yx} \propto \delta_{yx}$. Therefore we may determine $\mathcal{J}_x$ up to multiplication with diagonal R from the right by solving the inhomogeneous equation

$$L^{D,x}\mathcal{J}_x = \left(\mathcal{C}^* D^1\right)_x$$

This implies that $(L^{D,x}\mathcal{J}_x)(z) = 0$ for $z \notin x$, i.e. $\mathcal{J}_x$ is a "harmonic" function on *supp* $\mathcal{J}_x \setminus x$.

Next we define a measure of the quality of an interpolation operator. As earlier, let $\mathcal{V}'_{\mathcal{C}}$ be the dual of the space $\mathcal{V}_{\mathcal{C}} = \{u \in \mathbb{C} : \mathcal{C}u = 0\}$. This equals the high frequency space $\mathcal{V}_0$ of the compatible relaxation scheme. Recall also that we have the freedom of substituting $\mathcal{J} \mapsto \mathcal{J}R$. This freedom may be eliminated by prescribing $\mathcal{C}\mathcal{J}$.

Lemma 2 *Suppose $\mathcal{C}\mathcal{J} = R$. Then*

$$\mathcal{J} = \mathcal{A}R + \Gamma h$$

with $h = L\mathcal{J}$ mod $\mathcal{C}^*\mathcal{V}^{1\prime} \ \in \mathcal{V}'_{\mathcal{C}}$. Here $\Gamma = L_0^{-1}$ is the high frequency propagator (2.22) in the compatible relaxation scheme.

Proposition 3 *Under the same condition, let $L^1 = \mathcal{J}^* L \mathcal{J}$. Then L^1 is related to the ideal coarse grid problem operator $\hat{L}^1$ by*

$$L^1 = R^* \hat{L}^1 R + h^* \Gamma h,$$

where $\hat{L}^1$ is the exact effective problem operator defined by $\mathcal{C}L^{-1}\mathcal{C}^* = (L^1)^{-1}$ In particular, $L^1 \geq R^* \hat{L}^1 R$.

Definition 1 (Quality of coarse grid problem operators)
Under the same conditions, L^1 is said to have quality at least δ, $0 < \delta \leq 1$ if

$$\frac{1}{2-\delta} R^* \hat{L}^1 R \leq L^1 \leq \frac{1}{\delta} R^* \hat{L}^1 R.$$

Definition 2 (Quality of interpolation operators)
Under the same conditions, the quality δ of the interpolation operator $\mathcal{J}$ is the largest number δ in the interval $0 < \delta \leq 1$ such that

$$h^* \Gamma h \leq (1-\delta) L^1.$$

It is easy to show that there is always such a $\delta > 0$.

Proposition 4 *If $\mathcal{J}$ has quality δ then $L^1 = \mathcal{J}^* L \mathcal{J}$ has quality at least δ.*

Proof of Lemma 2: Let $\mathcal{V}_0 = \mathcal{V}_\mathcal{C}$ and $L_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0'$ be as in the compatible relaxation scheme. We have $\mathcal{C}(\mathcal{J} - \mathcal{A}R)U = 0$ for any $U \in \mathcal{V}^1$. Therefore

$$u = (\mathcal{J} - \mathcal{A}R)U \in \mathcal{V}_0.$$

Since $L_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0'$ is invertible, $u = L_0^{-1}h$ with $h = L_0 u \in \mathcal{V}_0'$. This proves the first assertion. Since $\mathcal{V}_0$ is a subspace of $\mathcal{V}$, its dual $\mathcal{V}_0'$ is a factor space of $\mathcal{V}'$, and by definition of L_0 we have

$$L_0 u = Lu \bmod \mathcal{C}^*\mathcal{V}^{1\prime} = L\mathcal{J}U \bmod \mathcal{C}^*\mathcal{V}^{1\prime}$$

because $L\mathcal{A}RU = \mathcal{C}^*\hat{L}^1RU \in \mathcal{C}^*\mathcal{V}^{1\prime}$. **q.e.d.**

Proof of Proposition 3: We compute $L^1 = (R^*\mathcal{A}^* + h^*\Gamma)L(\mathcal{A}R + \Gamma h) = R^*\mathcal{A}^*L\mathcal{A}R + h^*\Gamma L\Gamma h$ since $L\mathcal{A} = \mathcal{C}^*\hat{L}^1$ and $\Gamma\mathcal{C}^* = 0$. It follows that $L^1 = R^*\hat{L}^1R + h^*\Gamma L_0\Gamma h = R^*\hat{L}^1R + h^*\Gamma h$. **q.e.d.**

Proof of Proposition 4: The lower bound on L^1 follows because $L^1 \geq R^*\hat{L}^1R$ by proposition 3. To verify the upper bound on L^1 we start from the assertion of proposition 3, $L^1 = R^*\hat{L}^1R + h^*\Gamma h$. By the definition of the quality of $\mathcal{J}$ it follows that $L^1 \leq R^*\hat{L}^1R + (1-\delta)L^1$, i.e. $\delta L^1 \leq R^*\hat{L}^1R$. **q.e.d.**

2.4.3 Compatible relaxation

The convergence of a twogrid scheme involves three issues.

i) Convergence of the iterative solution of the high frequency problem.

ii) Convergence of the iterative solution of the low frequency problem.

iii) Convergence of the Schwarz iteration which is needed when $\mathcal{V}_0$ and $\mathcal{V}_1$ are not exactly orthogonal.

The first issue involves the condition number of L_0. In the (softened) compatible relaxation scheme $\langle u, L_0 v\rangle = \langle u, L + \kappa\mathcal{C}^*D^{1\prime}\mathcal{C}v\rangle$ for $v \in \mathcal{V}_0 \subseteq \mathcal{V}$. The crucial issue is to ensure by construction of the restriction operator $\mathcal{C}$ that the lower bound $L \geq \lambda_{\min}D$ gets improved to $L_0 \geq \epsilon D$. This is ensured by proposition 5 below.

Proposition 5 *Let the restriction operator $\mathcal{C}$ be constructed as in section 2.3.5 with "energy cutoff" ϵ, and let $D^{1\prime} : \mathcal{V}^1 \mapsto \mathcal{V}^{1\prime}$ be the diagonal operator with defining property (2.16). Let $\kappa \geq \epsilon$. Then*

$$L + \kappa\mathcal{C}^*D^{1\prime}\mathcal{C} \geq \epsilon D$$

on $\mathcal{V}$.

Proposition 5 shows that the construction of the restriction operator $\mathcal{C}$ works. The constraint $\mathcal{C}u = 0$ nails down those modes that are slow to converge.

PROOF of proposition 5: Since $L \geq L^N$ it suffices to establish that

$$\langle u, [L^N + \kappa \mathcal{C}^* D^{1\prime} \mathcal{C}] u \rangle \geq \epsilon \langle u, Du \rangle .$$

Expand u in the complete set of eigenvectors ϕ_x^α of $D^{-1}L^N$ with eigenvalues E_x^α, viz. $u = \sum_{x,\alpha} u_\alpha^x \phi_x^\alpha$ with $u_\alpha^x \in \mathbb{C}$. Because L^N does not couple blocks

$$\sum_x \sum_{\alpha,\beta} \bar{u}_\beta^x u_\alpha^x \{ \langle \phi_x^\beta, L^N \phi_x^\alpha \rangle + \kappa \langle \mathcal{C} \phi_x^\beta, D^{1\prime} \mathcal{C} \phi_x^\alpha \rangle \}$$

$\mathcal{C}$ annihilates ϕ_x^α with $\alpha > n_x$, and for the remaining modes we may use eq.(2.16) to evaluate the second term. Thus,

$$= \sum_x \sum_{\alpha,\beta=1}^{n_x} \bar{u}_\beta^x u_\alpha^x \tilde{E}_x^\alpha \langle \phi_x^\beta, D \phi_x^\alpha \rangle$$

with $\tilde{E}_x^\alpha = E_x^\alpha + \kappa$ if $\alpha = 1, ...n_x$ and $\tilde{E}_x^\alpha = E_x^\alpha$ otherwise. The matrix element is proportional $\delta^{\alpha\beta}$ by eq.(2.14) and $\tilde{E}_x^\alpha \geq \epsilon$ by construction. Therefore the above expression is $\geq \epsilon \langle u, Du \rangle$. **q.e.d.**

The second issue involves upper and lower bounds on the effective problem operator L^1 on the coarse grid (compare the discussion in section 2.2.3 on how to construct a good preconditioner). In the coarse grid problem it is to be solved by simple iteration, the condition number of L^1 governs the convergence of the coarse grid iteration. If we want to solve it by a twogrid procedure (and so on and so on), the upper and lower bounds can serve as a substitute for the bounds $\lambda_{\min} D \leq L \leq \lambda_{\max} D$ from which we started. Proposition 6 and 7 below give lower and upper bounds on L^1.

Proposition 6 *Suppose $\lambda_{min} D \leq L$ and the interpolation operator obeys $\mathcal{C}\mathcal{J} = R$. Let D^1 as in eq.(2.12). Then*

$$L^1 \geq \lambda_{min} R^* D^1 R.$$

Proposition 7 *Suppose the interpolation operator is constructed as an approximation to $\mathcal{A}_\kappa$ in the sense that $R = \mathcal{C}\mathcal{J} = \mathcal{C}\mathcal{A}_\kappa$, $\kappa \geq \epsilon$ (ons thinks of $R = R_\kappa$ as $\mathcal{A}_\kappa = \mathcal{A} R_\kappa$), and the effective problem operator has quality $\delta > 0$. Then*

$$L^1 \leq \frac{\kappa}{\delta} D^1.$$

The second result suggests to choose $\kappa \approx \epsilon$. We were unable to prove a useful bound in case $\kappa = \infty$. This suggests that the freedom to improve locality properties by choice of R is essential.

PROOF of proposition 6: By proposition 3

$$L^1 = R^* \hat{L}^1 R + h^* \Gamma h \geq R^* \hat{L}^1 R$$

where $\hat{L}^1$ is the exact effective problem operator defined by $\mathcal{C}L^{-1}\mathcal{C}^* = (L^1)^{-1}$. Thus

$$(L^1)^{-1} \le R^{-1}(\hat{L}^1)^{-1}R^{*-1} = R^{-1}\mathcal{C}L^{-1}\mathcal{C}^*R^{*-1}$$

and for all $F \in \mathcal{V}^{1\prime}$

$$\langle F, (L^1)^{-1}F\rangle \le \langle R^{*-1}F, \mathcal{C}L^{-1}\mathcal{C}^*R^{*-1}F\rangle \le \lambda_{\min}^{-1}\langle R^{*-1}F, \mathcal{C}D^{-1}\mathcal{C}^*R^{*-1}F\rangle,$$

since $L^{-1} \le \lambda_{\min}^{-1}D^{-1}$ by hypothesis. By definition of D^1 of eq.(2.12) this equals

$$= \lambda_{\min}^{-1}\langle F, R^{-1}(D^1)^{-1}R^{-1}F\rangle.$$

This implies $L^1 \ge \lambda_{\min}R^*D^1R$. **q.e.d.**

PROOF of proposition 7: According to proposition 3

$$L^1 = R^*\hat{L}^1R + h^*\Gamma h$$

and by the definition 2 of the quality of the interpolation operator

$$h^*\Gamma h \le (1-\delta)L^1.$$

This combines to $L^1 \le \delta^{-1}R^*\hat{L}^1R$. By choice of R, $R^*\hat{L}^1R = \hat{L}^1_\kappa$. It follows now from lemma 1 that $L^1 \le \frac{\kappa}{\delta}D^1$. **q.e.d.**

Finally, we need to examine the convergence of the Schwarz iteration. For conceptual clarity we separated the issues, and the Schwarz iteration involves operators

$$T_0 = L_0^{-1}L, \qquad\qquad T_1 = \mathcal{J}(L^1)^{-1}\mathcal{J}^*L.$$

Because of the appearance of the inverse L_0^{-1} of L_0 and $(L^1)^{-1}$ of L^1, their application involves the exact solution of a high frequency problem and of a low frequency problem, respectively. Proposition 8 below asserts convergence properties of the symmetrical multiplicative Schwarz method with these operators. If the error propagation operator has norm $||E|| < 1$, then the error decreases by a factor $||E||$ in every step of the iteration.

In practice one uses approximate solutions of the high frequency and low frequency problem. The author proved convergence results for this case in [33] in somewhat less general setting.

Proposition 8 (Convergence rate of the symmetrical multiplicative Schwarz twogrid method)
Let $\mathcal{V} = \mathcal{V}_0 + \mathcal{J}\mathcal{V}^1$ with $\mathcal{V}_0 = \{u \in \mathcal{V} : \mathcal{C}v = 0\}$, $a_0(u,v) = \langle Lu, v\rangle$ and $a_1(\mathcal{J}U, \mathcal{J}V) = \langle L^1U, V\rangle$ for some L^1. Suppose that the coarse grid problem operator L^1 has quality at least δ, then the error propagation operator $E = 1 - T_{sms}$, $T_{sms} = (1-T_0)(1-T_1)(1-T_0)$ for the symmetrical multiplicative Schwarz twogrid method has norm

$$||E||_a \le (1-\delta)$$

with respect to the scalar product $a(\cdot,\cdot) = \langle\cdot,\cdot\rangle_L$.

Appeal to the standard convergence theory (of section 2.3.2) would give a poorer bound. Therefore, we use a more direct estimate for the proof.

Proof of proposition 8: We split an arbitrary $v \in \mathcal{V}$ as follows

$$v = v_0 + v_1 \ , \quad v_0 \in \mathcal{V}_0, v_1 \in \mathcal{A}\mathcal{V}^1$$

By construction of $\mathcal{A}$ this is an orthogonal split with respect to the scalar product a. Explicitly,

$$v_1 = \mathcal{A}\mathcal{C}v \ , \ v_0 = (1 - \mathcal{A}\mathcal{C})v.$$

Because the Schwarz construction is invariant under $\mathcal{J} \mapsto \mathcal{J}R$, we may assume without loss of generality that

$$\mathcal{C}\mathcal{J} = I. \tag{2.25}$$

We exhibit $(1 - T_0)$ and $(1 - T_1)$ as block matrices. $T_0 = \Gamma L$ where $\Gamma = L_0^{-1}$ in the compatible relaxation scheme. Thus,

$$\begin{aligned} T_0 v &= \Gamma L(1 - \mathcal{A}\mathcal{C})v + \Gamma L\mathcal{A}\mathcal{C}v \\ &= (1 - \mathcal{A}\mathcal{C})v + \Gamma\mathcal{C}^*\hat{L}^1\mathcal{C}v \\ &= v_0 \end{aligned}$$

because $\Gamma\mathcal{C}^* = 0$. Thus,

$$(1 - T_0)v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} .$$

Since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ we only need the low frequency component of $T_1 v_1$, $v_1 = \mathcal{A}V$ from the block matrix giving $(1 - T_1)v$. We have from the explicit formula (2.10) for T_1 in section 2.3

$$\begin{aligned} T_1\mathcal{A}V &= \mathcal{J}(L^1)^{-1}\mathcal{J}^*L\mathcal{A}V = \mathcal{J}(L^1)^{-1}\mathcal{J}^*\mathcal{C}^*\hat{L}^1V \\ &= \mathcal{A}(L^1)^{-1}\hat{L}^1V + \Gamma h(L^1)^{-1}\hat{L}^1V \\ &= \mathcal{A}(L^1)^{-1}\hat{L}^1V + \text{high frequency part.} \end{aligned}$$

By normalization convention (2.25), $V = \mathcal{C}v$ if $v = \mathcal{A}V$. Therefore

$$E = 1 - (1 - T_0)(1 - T_1)(1 - T_0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}[1 - (L^1)^{-1}\hat{L}^1]\mathcal{C} \end{pmatrix} .$$

Thus,

$$\begin{aligned} \langle v, Ev \rangle_L &= \langle Lv, Ev \rangle \\ &= \langle L\mathcal{A}V, \mathcal{A}[1 - (L^1)^{-1}\hat{L}^1]V \rangle = \langle \hat{L}^1V, [1 - (L^1)^{-1}\hat{L}^1]V \rangle \\ &= \langle [\hat{L}^1 - \hat{L}^1(L^1)^{-1}\hat{L}^1]V, V \rangle. \end{aligned}$$

On the other hand, $\langle v, v \rangle_L = \langle \hat{L}^1V, V \rangle + \langle v_0, v_0 \rangle \geq \langle \hat{L}^1V, V \rangle$.

From the assumption on the quality of L^1 it follows that $-(1-\delta)\hat{L}^1 \leq [\hat{L}^1 - \hat{L}^1(L^1)^{-1}\hat{L}^1] \leq (1-\delta)\hat{L}^1$. Thus $|\langle v, Ev\rangle_L| \leq (1-\delta)\langle v, v\rangle_L$. **q.e.d.**

It is interesting that only the coarse grid problem operator itself enters, not the quality of the interpolation. Thus if we had a way of finding a good approximation L^1 to the ideal coarse grid problem operator $\hat{L}^1$ other than with the help of a good interpolation operator, this would be good enough. Renormalization is a way to improve coarse grid problem operators.

This property of the symmetrical multiplicative Schwarz method comes from the fact that the last factor $(1-T_0)$ (post smoothing) which involves an exact solution of a high frequency problem corrects any imperfections of the interpolation.

Note also that the ideal coarse grid problem operator $\hat{L}^1$ has quality 1. If we could use it, the multiplicative Schwarz method would converge in one step; it is actually not essential to use the symmetrical method for this.

2.5 Comments on the gauge covariance of the procedure

One wants to obtain gauge covariant fields on the coarser grid. Here we recall some basics of the idea of gauge covariance. Parallel transporters are introduced.

We adhere to the philosophy that the essence of gauge theories is the validity of the Naheinformationsprinzip: There is no a priori possibility to compare vectors in different fibres $\mathbb{V}_z$ of bundles $\mathcal{V}$. This means that there is neither a natural identification of different fibres $\mathbb{V}_z \neq \mathbb{V}_w$ nor a preferred basis in the fibre. To make any kind of comparison, one needs a connection. It yields parallel transporters. They are linear maps

$$U(\mathcal{C}) : \mathbb{V}_z \mapsto \mathbb{V}_w$$

associated with paths $\mathcal{C}$ from z to w. Sometimes one introduces a basis $\{e_\alpha(z) \in \mathbb{V}_z; \alpha = 1, ...n_z = \dim \mathbb{V}_z\}$. This is called a moving frame. Sometimes moving frames within a class are singled out, for instance orthonormal basis when $\mathbb{V}_z$ is equipped with a scalar product.

Gauge transformations are transformations of moving frames (within the class)

$$e_\alpha(z) \mapsto e'_\alpha(z) = e_\beta(z) S^\beta{}_\alpha(z) \tag{2.26}$$

with an invertible matrix $S(z)$. For orthonormal bases $S(z)$ needs to be unitary. The matrices $S(z)$ form the gauge group. We admit situations where it depends on x.
Gauge covariance means independence of a choice of a moving frame (within the class). When everything is formulated without any reference to a moving frame, then gauge covariance in the above sense is manifest. Vector potentials A depend on a choice of a moving frame and they transform in a certain way under changes of moving frames, i.e.

gauge transformations. But connections can also be defined without any reference to a moving frame [29]. The parallel transporters $U(\mathcal{C})$ of eq.(2.26) make no reference to a moving frame.

Our procedure for solving linear equations is gauge covariant in the above sense. It uses neither an identification of fibres nor a preferred basis. When the twogrid procedure is repeatedly applied the procedure on all the coarser grids is manifestly gauge covariant in the same sense. With the exception of a subtle point in the construction of the restriction operator $\mathcal{C}$ which will be analyzed below we work without any moving frames.

This gives an additional flexibility compared to block spin procedures as have been used by Balaban et al. [6] in the rigorous approach to the construction of Euklidean gauge field theories. The dimension of the fibres $\mathbb{V}_x$ on coarse grids is dynamically determined in the present theory (given an energy cutoff) and is a priori arbitrary. It follows that the gauge groups are different on different scales.

Let us discuss the difference in more detail. Given a moving frame, parallel transporters $U(\mathcal{C})$ along paths $\mathcal{C}$ from z to w can be converted to parallel transport matrices $\sqcup(\mathcal{C})$ via

$$U(\mathcal{C})e_\alpha(z) = e_\beta(w) \sqcup^\beta{}_\alpha (\mathcal{C}).$$

They transform under gauge transformations according to

$$\sqcup(\mathcal{C}) \mapsto S(w) \sqcup (\mathcal{C}) S(z)^{-1}.$$

The parallel transport matrices determine the vector potentials, $\sqcup(\mathcal{C}) = 1 - A_\mu(z)\delta z$ for infinitesimal paths from z to $z + \delta z^\mu$. There are corresponding formulas on the lattice within the semi-commutative differential calculus and geometry of Dimakis and Müller-Hoissen [10]. In the traditional block spin approach, parallel transport matrices $\sqcup^1$ on the coarse grid are constructed from parallel transport matrices $\sqcup$ on the fine grid. To do so, sites x of the coarse grid are identified with sites $\hat{x}$ of the fine grid, e.g. the centers of hypercubes. The fibres $\mathbb{V}_x$ on the coarse grid have the same dimension as fibres $\mathbb{V}_z$ on the fine grid. One identifies the fibres $\mathbb{V}_x = \mathbb{V}_{\hat{x}}$. In this way, *a moving frame on the fine grid determines a moving frame on the coarse grid*, and gauge transformations on the fine grid induce gauge transformations on the coarse grid

$$\sqcup^1(\mathcal{C}^1) \mapsto S(\hat{y}) \sqcup^1 (\mathcal{C}^1) S(\hat{x})^{-1}$$

for coarse grid paths from x to y. For our procedure one would have to adapt the number of eigenmodes that are taken to the coarse grid, thus one would have to fix the energy cutoff.

In the present approach, the dimensions of the of the fibres $\mathbb{V}_z$ and $\mathbb{V}_x$ on the fine and coarse grids need not be the same, and it is not true in general that a moving frame on the fine grid determines a moving frame on the coarse grid. Moreover the dimension of $\mathbb{V}_x$ may depend on x. The remarks at the beginning of this subsection ought to convince the reader that there is a meaningful notion of gauge covariance nevertheless.

Parallel transporters in the linear problem

On discrete spaces, paths $\mathcal{C}$ are composed from links $\langle wz\rangle$ and parallel transporters along paths are composed from parallel transporters $U(\langle wz\rangle) = U_{wz}$ along links according to the rule $U(\mathcal{C}_2 \circ \mathcal{C}_1) = U(\mathcal{C}_2)U(\mathcal{C}_1)$. Given the problem operator $L : \mathcal{V} \to \mathcal{V}'$ of the linear problem and a diagonal operator $D : \mathcal{V} \to \mathcal{V}'$, parallel transporters can be identified as

$$U_{wz} = -D_{ww}^{-1} L_{wz} : \mathbb{V}_z \to \mathbb{V}_w . \tag{2.27}$$

The parallel transporters can be used to write down the random walk expansion of the solution u of the linear equation $Du = f$, viz. [35]

$$u(z) = \sum_w \sum_{\mathcal{C}:w\to z} U(\mathcal{C}) D_{ww}^{-1} f(w).$$

This is the iterative solution based on the (poor) preconditioner D^{-1}.

Covariance properties of the restriction operator

In the construction of the restriction operator, a basis in the fibre $\mathbb{V}_x$ on the coarse grid was used. This basis is in general not unique. Let us show that the maps

$$\hat{L}^1_{yx} : \mathbb{V}_x \mapsto \mathbb{V}_y$$

which define the ideal coarse grid problem operator $\hat{L}^1$ are unaffected by the arbitrariness.

The arbitrariness comes from the freedom in the choice of phases of the generalized eigenmodes ϕ_x^α, and from possibly degenerate eigenmodes. This leaves the arbitrariness

$$\phi_x^\alpha \mapsto \phi_x'^\alpha = \phi_x^\beta S_\beta{}^\alpha(x) \tag{2.28}$$

with a unitary matrix S with the property that $S_\beta{}^\alpha(x) = 0$ unless $E_x^\alpha = E_x^\beta$. This is of the form of gauge transformations of moving frames. It induces gauge transformations of the dual basis ϕ_α^x of the dual space $\mathbb{V}_x'$.

If we use a more elegant notation than in section 2.3.5 regarding the eigenmodes ϕ_x^α themselves as the basis vectors of $\mathbb{V}_x$, then the formula for the block spin reads

$$\mathcal{C}u(x) = \sum_{\alpha=1}^{n_x} \phi_x^\alpha \langle \phi_\alpha^x, u\rangle \in \mathbb{V}_x$$

and this is invariant under gauge transformations (2.28). Therefore $\mathcal{C}$ is a well defined operator mapping $\mathcal{V} \mapsto \mathcal{V}^1 = \{\mathbb{V}_x\}$. The construction of the ideal problem operator $\hat{L}^1$ involves only $\mathcal{C}$ and the problem operator L. Therefore $\hat{L}^1$ is well defined and invariant under gauge transformations (2.28).

An example with degenerate eigenmodes: The SU(2)-covariant Laplace equation

Consider a hypercubic lattice as fine grid, and suppose the fibres $\mathbb{V}_z$ of $\mathcal{V}$ are two-dimensional irreducible representation spaces of SU(2), equipped with the scalar product (i.e. defined by an operator D) which makes the representation unitary. Let unitary parallel transporters $U_{wz} : \mathbb{V}_z \mapsto \mathbb{V}_w$ be given for w nearest neighbor of z. Then the SU(2)-covariant Laplacian $\triangle$ is defined. Its brother $\triangle^{N,x}$ with Neumann boundary conditions acts on $\phi \in \mathcal{V}$ according to

$$(\triangle^{N,x}\phi)(z) = \sum_{w \in x w \text{ n.n. } z} (U_{zw}\phi(w) - \phi(z)) \text{ for } z \in x.$$

The sum goes over nearest neighbors w of z which are in the same block x.
It was first shown by Kalkreuter [24] that there is to any solution $\phi \in \mathcal{V}'$ of the eigenvalue equation

$$(-\triangle^{N,x}\phi)(z) = E_x\phi(z)$$

a second solution $\phi' = \bar{\phi}\epsilon$ with the same eigenvalue E_x. $\bar{\phi}$ is the complex conjugate of ϕ and ϵ is the antisymmetric tensor which equips $\mathbb{V}_z$ with an antisymmetric bilinear form. In the limit $U_{zw} \to \mathbb{I}$, $-\triangle$ becomes the ordinary Laplacian, and the two solutions with eigenvalue $E_x = 0$ (zero modes) are constants on the block x, and vanish outside.

More generally, if we retain only the lowest eigenvalue, the fibres $\mathbb{V}_x$ of $\mathcal{V}^1$ are two-dimensional and the traditional identification $\mathbb{V}_x = \mathbb{V}_{\hat{x}}$ could be made. This procedure was described in [26].

Following Kalkreuter the kernel of the restriction operator can be regarded as solution of the eigenvalue equation

$$(-\triangle^{N,x}\mathcal{C}^*)(z,x) = E_x\mathcal{C}(z,x)$$

with the subsidiary condition $\mathcal{C}^*(z,x) = 0$ if $z \notin x$. The identification leads to the covariance condition

$$\mathcal{C}(x,\hat{x}) = r(x)\mathbb{I}, \quad \text{with } r(x) > 0 \text{ real} \tag{2.29}$$

for gauge groups U(1) and SU(2) at the block center $\hat{x}$. Then (2.29) ensures the transformation property

$$\mathcal{C}(x,z) \mapsto g(\hat{x})\mathcal{C}(x,z)g(z)^{-1}$$

under gauge transformations.

2.6 Generalization to other operators

So far, we considered positive problem operators L. When one wishes to consider an inhomogeneous linear equation $Au = F$ with A that is not positive, possibly not even self-adjoint, one may form the positive operator $L = A^*A$ and consider the equation $Lu = A^*f$. It is generally agreed that it is not a good idea to proceed in this way, because the condition number gets squared, and becomes, therefore, much larger.

The most essential step in a multigrid procedure for the solution $u \in \mathcal{V}$ of a linear equation is the formulation of a high frequency problem on a subspace $\mathcal{V}_0$ of $\mathcal{V}$. It can be solved by fast converging iteration. This requires improving the lower bound on the problem operator $D^{-1}L$.
To appreciate this remember that the construction of a good interpolator of a solution U of a coarse grid equation (from a poor approximation like $\mathcal{C}^*U$) is a high frequency problem, and a good coarse grid problem operator can be computed from a good interpolation operator.

In this section we show how a suitable high frequency problem can be formulated whose problem operator does not involve a square of A. The restriction operator $\mathcal{C}$ of the squared problem will be used to define it. Krylov subspaces will be exhibited, which can be used to obtain an iterative solution. The main tool is the consideration of what we call Dirac operators. Their classical counterpart is the Euklidean version of Dirac's operator for an electron in an exterior electromagnetic field with vector potential a_μ. It is made from the Weyl operator $A = i\hbar \sum_{\mu=1}^{4} \sigma^\mu(\partial_\mu - iea_\mu)$, with Pauli matrices $\sigma^\mu(\mu = 1, 2, 3)$ and $\sigma^4 = i\mathbb{I}$ (2×2 unit matrix), according to formula (2.30) below.

2.6.1 The Dirac formalism

We introduce a formalism that allows to treat operators A that appear in the discretized differential equation, which need not be positive, self-adjoint or symmetric. The construction of subspaces of high frequencies involve a positive L.

Let A be an arbitrary operator

$$A : \mathcal{V} \mapsto \mathcal{V}'.$$

We assume again that a positive diagonal operator $D : \mathcal{V} \mapsto \mathcal{V}'$ has been selected, that defines a scalar product $\langle \cdot, \cdot \rangle_D$ on $\mathcal{V}$. It will be used to construct preconditioners in a similar way as before. Let $D^{-1}A := \mathbb{A} : \mathcal{V} \mapsto \mathcal{V}$. According to Dirac we introduce the self-adjoint operators

$$\not{A} := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad \not{D} := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} : \qquad \underbrace{\mathcal{V} \oplus \mathcal{V}}_{=: \mathcal{W}} \quad \mapsto \quad \underbrace{\mathcal{V}' \oplus \mathcal{V}'}_{=: \mathcal{W}'} \tag{2.30}$$

with $A^* : \mathcal{V} \mapsto \mathcal{V}'$ the adjoint of A. The scalar product can be written as follows

$$\langle \not{A}\,(u \oplus v), (u \oplus v) \rangle_D = \langle Av \oplus A^*u, (u \oplus v) \rangle_D = \langle Av, u \rangle_D + \langle u, Av \rangle_D.$$

with $u \oplus v \in \mathcal{V} \oplus \mathcal{V}$.

Eigenequations

The operator $L := A^* D^{-1} A$ is positive and self-adjoint since D is positive by hypothesis. It follows that $\mathbb{L} = D^{-1} L = D^{-1} A^* D^{-1} A$ is a positive operator on the Hilbert space.

Theorem 3 *To every eigenvalue $\lambda^2 > 0$ of the quadratic operator $\mathbb{L}$ there is a pair of eigenvalues $\pm\lambda$ of the Dirac eigenvalue problem $\not{A} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \not{D} \begin{pmatrix} u \\ v \end{pmatrix}$ with operator A.*

Proof: The generalized eigenequation of $\not{A}$

$$\not{D}^{-1} \not{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

can be written explicitly

$$\begin{aligned} Av &= \lambda Du \\ A^* u &= \lambda Dv. \end{aligned} \tag{2.31}$$

The first equation is solved by $u = \lambda^{-1} D^{-1} A v$. Inserting into (2.31) leads to

$$A^* D^{-1} A v = L v = \lambda^2 D v.$$

Thus (λ^2, v) is generalized eigenvalue and eigenvector of the squared operator L, more precisely eigenvalue to $\mathbb{L} = D^{-1} L$ and the eigenvector to $\not{D}^{-1} \not{A}$ is $\begin{pmatrix} \pm\lambda^{-1} D^{-1} v \\ v \end{pmatrix}$.
q.e.d.

It's easy to show, that

$$\not{A} \not{D} \not{A} = \begin{pmatrix} A D^{-1} A^* & 0 \\ 0 & L \end{pmatrix}.$$

Formulation of the high frequency problem

The operator A acts on $\mathcal{V}$ and so does L. We use the restriction operator $\mathcal{C}$ of the squared problem operator to define

$$\mathcal{V}_0 = \{ v \in \mathcal{V}, \mathcal{C} v = 0 \}.$$

The Dirac operator $\not{A}$ acts on $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}$ and we define its high frequency subspace

$$\mathcal{W}_0 = D^{-1} A \mathcal{V}_0 \oplus \mathcal{V}_0.$$

A high frequency Dirac operator $\not{A}_0$ is defined by the requirement that $\langle \xi, \not{A}_0 \eta \rangle = \langle \xi, \not{A} \eta \rangle$ for $\xi, \eta \in \mathcal{W}_0$. Because $\mathcal{W}_0$ is a subspace of $\mathcal{W}$, its dual space $\mathcal{W}_0'$ is a quotient space of $\mathcal{W}'$. The following lemma characterizes it. It uses the exact interpolation operator $\mathcal{A} : \mathcal{V}^1 \to \mathcal{V}$ of the squared problem.

Lemma 3 *The dual space* $\mathcal{W}_0'$ *of* $\mathcal{W}_0$ *consists of the equivalence class* $[f \oplus g], f, g \in \mathcal{V}'$, *where* $[f \oplus g] = 0$ *if* $g = A\mathcal{A}G, G \in \mathcal{V}^1$ *and* $f = \mathcal{C}^*F, F \in \mathcal{V}^{1\prime}$.

Proof: We must show that $\langle g, D^{-1}Av_0\rangle = 0$ and $\langle f, u_0\rangle = 0$ if $f = \mathcal{C}^*F$ and $g = A\mathcal{A}G$ and $u_0, v_0 \in \mathcal{V}_0$. We compute $\langle A\mathcal{A}G, D^{-1}Av_0\rangle = \langle L\mathcal{A}G, v_0\rangle = \langle \mathcal{C}^*\hat{L}^1G, v_0\rangle$ by proposition 1. This equals $\langle \hat{L}^1G, \mathcal{C}v_0\rangle = 0$. Similarly $\langle \mathcal{C}^*F, u_0\rangle = \langle F, \mathcal{C}u_0\rangle = 0$. **q.e.d.**

According to lemma 3, this reads

$$\begin{aligned} Av &= g \bmod A\mathcal{A} \\ A^*u &= f \bmod \mathcal{C}^*. \end{aligned}$$

Since $A^*D^{-1}A\mathcal{A} = L\mathcal{A} = \mathcal{C}^*\hat{L}^1$, the solution of the high frequency Dirac problem involves the solution of

$$\begin{aligned} A^*D^{-1}Av &= A^*D^{-1}g \bmod \mathcal{C}^* \\ \text{that is} \quad L_0 v &= A^*D^{-1}g \bmod \mathcal{C}^*. \end{aligned}$$

Thus by solving the high frequency problem for $\not{A}$ one obtains $v = \Gamma A^*D^{-1}g$, where Γ is the high frequency Green's function for the squared problem.

Having related the solution of the high frequency Dirac problem to the solution of the high frequency problem for the squared operator L, we can now use theorem 5 to convert the bound $L_0 \geq \epsilon D$ for the problem operator of the latter problem to a lower bound $\sqrt{\epsilon}$ on the modolus of the generalized eigenvalues of $\not{A}_0$. This shows that the choice of $\mathcal{W}_0$ achieves what it should.

Krylov subspaces of $\mathcal{W}_0$

If $v_1, v_2, v_3, ...$ is a sequence of elements $v_i \in \mathcal{V}_0$ then $\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} D^{-1}Av_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} D^{-1}Av_2 \\ v_3 \end{pmatrix}, ...$ is a sequence of elements of $\mathcal{W}_0$. One could choose

$$v_{n+1} = S_0 A v_n$$

where

$$S_0 = D^{-1}(D - \mathcal{C}^*D^1\mathcal{C})D^{-1}$$

and D^1 is the coarse grid version of D (for the squared problem) defined in eq.(2.12), viz. $(D^1)^{-1} = \mathcal{C}D^{-1}\mathcal{C}^*$. S_0 projects on $\mathcal{V}_0$, compare the operator S_0 for the block Jacobi method in section 2.3.

Concerning the low frequency problem one could introduce suitable restriction operators $\not{\mathcal{C}}$ and interpolation operators $\not{\mathcal{A}}$. Unfortunately this had not yet lead to any restrictions on the effectice operator A^1, that should be choosen with good locality properties.

Chapter 3

Renormalization group flow for gauge theories with nonunitary parallel transporters

After an introduction to holonomy groups the basic idea of an interpretation of Higgs fields as nonunitary parallel transporters is presented. In principle, they are restricted to an extra dimension. In a one-dimensional model that retains only the nonunitary extra direction we will explain how Higgs fields and thus a Higgs potential appears out of a usual gauge theory by performing renormalization group transformations. What we can learn from this model will be explained with respect to more space-time dimensions.

3.1 Holonomy groups and unitary gauge group

In standard differential geometry on connected differentiable manifolds $\mathcal{M}$ every connection defines a holonomy group [29]. One chooses a point $\hat{z}$ on $\mathcal{M}$ arbitrarily. The connection defines parallel transporters $U(\mathcal{C})$ along piecewise smooth curves $\mathcal{C}$ on $\mathcal{M}$. Let $-\mathcal{C}$ be the path backwards and $\mathcal{C}_2 \circ \mathcal{C}_1$ the path that is obtained by composition of path $\mathcal{C}_1$ with $\mathcal{C}_2$. Conventional parallel transporters obey

$$U(-\mathcal{C}) = U(\mathcal{C})^{-1}, \tag{3.1}$$

$$U(\mathcal{C}_2 \circ \mathcal{C}_1) = U(\mathcal{C}_2)U(\mathcal{C}_1). \tag{3.2}$$

The holonomy group H consists of parallel transporters $U(\mathcal{C})$ along closed loops $\mathcal{C}$ from $\hat{z}$ to $\hat{z}$. H is independent of $\hat{z}$ up to isomorphy. Because of eq.(3.2) the product of two elements of H is also an element of H. Thus H is a semigroup. We wish to discuss what happens if we do not demand eq.(3.1).
In discrete theories curves $\mathcal{C} = b_n \circ ... \circ b_1$ are composed from edges b_i, and we can give up $U(-b) = U(b)^{-1}$. Without further assumptions except (3.2) the parallel transporters $\mathcal{C}$ along closed paths from $\hat{z}$ to $\hat{z}$ make up a semigroup that does not need to be a group, and that can depend on $\hat{z}$. This leads to theories that are much more general than usual gauge theories. They may be relevant for a generalized theory of complex systems [35]. In the following chapters we focus attention on theories in which eq.(3.1) need not be hold,

but in which it is still true that parallel transporters $U(\mathcal{C})$ are invertible maps between fibres of a vector bundle. We will see that such theories are useful to explain features of elementary particle physics. They are natural generalizations of standard gauge theories which can incorporate Higgs fields with a geometrical interpretation.

Let U be a connection in a vector bundle, then $U(\mathcal{C}) : \mathbb{V}_{\hat{z}} \to \mathbb{V}_{\hat{z}}$ are maps from the fiber $V_{\hat{z}}$ onto itself, for closed paths $\mathcal{C}$. If we make the assumption that these $U(\mathcal{C})$ are invertible, then the holonomy group H is defined to be the group that is generated by $U(\mathcal{C})$ and $U(\mathcal{C})^{-1}$. The inverse $U(\mathcal{C})^{-1}$ does not need to be a parallel transporter itself. This holonomy group is again independent of the choice of $\hat{z}$ up to isomorphy. In the following, we denote a parallel transporter that does not need to fulfill eq.(3.1) by $\phi \in \mathrm{H}$, thus $\phi^{-1} \in \mathrm{H}$, but ϕ^{-1} does not need to be a parallel transporter.

In the theories which we will examine here all matter fields must form representation spaces of H, so they can be parallel transported. But only a compact subgroup G of H will be a group of local symmetries. We call this the unitary gauge group, or gauge group for short. We remark that parallel transporters which do not obey eq.(3.1) are also useful in general relativity. In this case the group G of local symmetries is noncompact and isomorphic to SL(2,ℂ), see [41].

3.2 Higgs fields as nonunitary parallel transporters

It has been proposed to consider lattice gauge theories with nonunitary parallel transporters ϕ in order to obtain a geometric interpretation of Higgs fields [38, 41]. The fields $\phi(z, z \pm \mu)$ are attached to links on a lattice. They may be subject to a polar decomposition into a selfadjoint factor and a unitary factor. The unitary factor is like a conventional lattice gauge field and the selfadjoint factor is interpreted as a Higgs field. In defect models of the Higgs one imagines that nonunitary parallel transporters occur across defects in space-time. In a five-dimensional theory they would be associated with parallel transporters in one direction, the fifth.

3.2.1 The idea of nonunitary extra dimensions

Many features of the standard model cannot be explained. Think of the fact that the mass of the top quark is more than three thousand times bigger than the mass of the up quark. Thus it is a dream to compute quark masses and the Cabibbo-Kobayashi-Maskawa matrix from a theory that retains the standard model as effective theory.

Based on the work of Prüstel and Mack [38] the idea to abandon the unitarity of parallel transporters across defects has already been examined in [2]. In lattice gauge theory this means for a parallel transporter $\phi(z, z + a_{0\mu})$, where $a_{0\mu}$ is the unit vector in space-time direction μ, that

$$\phi^*(z, z + a_{0\mu}) \neq \phi^{-1}(z, z + a_{0\mu}). \tag{3.3}$$

This proposal goes back to the ideas in differential geometry presented in section 3.1. Their principles occur also in the differential calculus of Dimakis and Müller-Hoissen [10], that are relevant for the mathematical description of general gauge theories. It leads to a geometric interpretation of Vierbein fields as parts of gauge fields. The Vierbein of general relativity then originates in the nonunitary parallel transport in the first four directions. The Higgs comes from the nonunitary parallel transport in a further direction [41]. This parallel transport in extra direction(s) can be interpreted as the one between right-and left-handed fermions.

A parallel transporter that fulfills eq.(3.3) must be an element of a group, including nonunitary matrices, that is the holonomy group H. And due to the fact that a parallel transporter along a path composed with the one in the reverse direction changes the field at the starting point, we have to consider holonomy groups that may be noncompact and larger than the gauge group that we denoted by G. The elements $\phi \in$ H transform under the unitary gauge group according to

$$\phi(z, z + a_{0\mu}) \mapsto u(z)\phi(z, z + a_{0\mu})u(z + a_{0\mu})^{-1} \tag{3.4}$$

with $u(z), u(z + a_{0\mu}) \in$ G. G is the local symmetry.

3.2.2 The bilayered membrane

Here we will sketch a possible model with Higgs field as nonunitary parallel transporter in a fifth direction, namely the bilayered membrane, which is illustrated in figure 3.1.

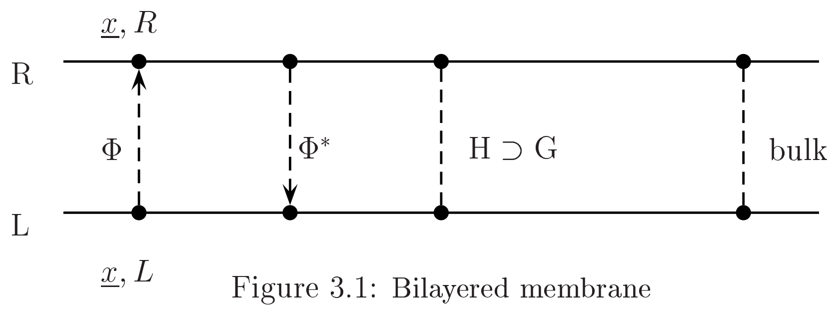


Figure 3.1: Bilayered membrane

The bilayered membrane illustrates the world consisting of two world sheets. Here left- and right-handed fermions live on boundaries on different sides of a domain wall. So they sit on different points $(\underline{x}, R)$ and $(\underline{x}, L)$ in a five-dimensional space-time. The Higgs fields are interpreted as nonunitary parallel transporters Φ along this fifth extra dimension, for example across a defect. Such a model has been examined in [2]. As in the standard model we have a two-dimensional representation space $\tilde{V}_L$ of the left-handed gauge group $\mathrm{G}_L = \mathrm{SU}(2) \times \mathrm{U}(1)$ and two one-dimensional representation spaces $\tilde{V}_R^D$ and $\tilde{V}_R^U$ of the right-handed gauge group U(1). Ignoring color the Higgs field is a mapping $\Phi : (\tilde{V}_R^D \oplus \tilde{V}_R^U) \times \mathbb{C}^3 \mapsto \tilde{V}_L \times \mathbb{C}^3$, where $\mathbb{C}^3$ is the flavor space.

We could admit as many fermions in the bulk as on the boundary. In section 3.4 we will explain how nonunitarity appears naturally after a renormalization group step, even if we start with unitary Higgs fields. The bilayered membrane with a distance between the world sheets equal to a lattice spacing is then interpreted as the endpoint of a renormalization group (RG-)flow. The renormalization group flow should then determine the Higgs potential V, starting from an ordinary gauge theory. The minima of the Higgs potential V then give the quark masses and allow us to calculate the Cabibbo-Kobayashi-Maskawa matrix in principle as it will be shown in section 5.3. This leads to the study of RG-flows in theories with nonunitary parallel transporters in an extra dimension.

The existence of a gauge invariant local Higgs potential is a special property of the theory. A model of the type of the bilayered membrane will be presented in the chapter 5. There we will see how an exponential mass hierarchy as we observe it in the standard model can be explained. This leads to the possibility to explain the big differences of the fermion masses.

3.2.3 The Higgs potential and its essential G-biinvariance property

The Higgs potential $V(\phi)$ is a real function of $\phi \in$ H with the property

$$V(\phi^*) = \overline{V(\phi)}.$$

The path integral [40] involves the Haar measure $d\phi$ on H. One integrates over the lattice gauge fields $\phi(z, z + a_{0\mu})$. The use of the Haar measure is a technical convenience. In case H = GL(n,$\mathbb{C}$), one could use an Euklidean measure instead. Noninvertible matrices will form a set of Euklidean measure zero, and the logarithm of the factor which distinguishes the Haar measure from the Euklidean measure could be incorporated into the Higgs potential. The two treatments are therefore equivalent.
We assume

$$V(\phi) \to \infty \quad \text{for} \quad \|\phi\| \to \infty \tag{3.5}$$

so that $\exp\big(-V(\phi)\big)$ is integrable and square integrable on the holonomy group H. In the action, terms $\sum V(\phi(z, z+a_{0\mu}))$ appear. They are attached to links between z and $z+a_{0\mu}$. Under G-gauge invariance $\phi(z, z+a_{0\mu}) \mapsto u(z)\phi(z, z+a_{0\mu})u(z+a_{0\mu})^{-1}$ for $u(z), u(z+a_{0\mu}) \in$ G. The action is required to be invariant under G-gauge transformations. Therefore the Higgs potential must obey

$$V(u_1\phi u_2) = V(\phi)$$

for all $u_1, u_2 \in$ G, $\phi \in$ H. This is called the G-*biinvariance property.* We now assume that we can parameterize an element $\phi \in$H as follows

$$\phi = \zeta u_1 d(\eta) u_2, u_1, u_2 \in \mathrm{G}, d(\eta) = \begin{pmatrix} e^{\eta_1} & & \\ & \ddots & \\ & & e^{\eta_r} \end{pmatrix}, \zeta \in \mathbb{C}. \tag{3.6}$$

We will show later on (section 4.1) by group theoretical means, that a parametrization of the form (3.6) is always possible when H is a semi simple group and G is its maximal compact subgroup. In this case $\zeta = 1$. It is also true when H and G are reductive, e.g. semi simple except for an abelian factor, and their semi simple factors are related in the aforementioned way. In certain cases, including those that are examined in this chapter, $d(\eta)$ diagonal makes up the complete noncompact Cartan subgroup A of the holonomy group H. It follows from this parameterization and the G-biinvariance property that

$$V(\zeta a) = V(\zeta d(\eta)) \tag{3.7}$$

for $\zeta \in \mathbb{C}$ and $a = u_1 d(\eta) u_2$.

We consider the special case G = SU(2) and H = GL(2,$\mathbb{C}$) so that $a \in$ SL(2,$\mathbb{C}$). The following lemma is valid.

Lemma 4 (Properties of SU(2)-biinvariant functions)
Suppose $\mathrm{SU}(2) \subset \mathrm{H} \subseteq \mathrm{GL}(2,\mathbb{C})$.
Let V be a function of $\phi \in H$ so that $\phi = \zeta a$ with $0 \neq \zeta \in \mathbb{C}$ and $a \in \mathrm{SL}(2,\mathbb{C})$, which is SU(2)*-biinvariant in the sense that $V(\phi) = V(u_1 \phi u_2)$ for all $u_1, u_2 \in \mathrm{SU}(2)$. Then*

$$V(\zeta a) = V(\zeta a^{-1}) = V(\zeta \epsilon a^T \epsilon^T).$$

ϵ is the antisymmetric tensor in two dimensions.

PROOF: Every element $a \in$ SL(2,$\mathbb{C}$) may be decomposed as $a = u_1 d(\eta) u_2$ with $u_1, u_2 \in$ SU(2), $\eta \in \mathbb{R}$, and $d(\eta) = \begin{pmatrix} e^{-\frac{\eta}{2}} & 0 \\ 0 & e^{\frac{\eta}{2}} \end{pmatrix}$.

There exists $w \in$ SU(2) such that $w d(\eta) w^{-1} = d(\eta)^{-1}$, viz. $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It follows that

$$\begin{aligned} V(\zeta a) &= V(\zeta d(\eta)) = V(\zeta w d(\eta) w^{-1}) = V(\zeta d(\eta)^{-1}) = V(\zeta u_2^{-1} d(\eta)^{-1} u_1^{-1}) \\ &= V(\zeta a^{-1}). \end{aligned}$$

The last equation follows from

$$\epsilon \phi^T \epsilon^{-1} = \phi^{-1}, \tag{3.8}$$

which is true for all $\phi \in$ SL(2,$\mathbb{C}$). We can use this for $\phi = a$, since $\det a = 1$. **q.e.d.**

This generalizes to other groups H. The essential biinvariance property leads to a very important simplification: The Higgs potential can be written as a function of η:

$$V(\phi) = V(\zeta u_1 d(\eta) u_2) = V(\zeta d(\eta)) = \mathcal{V}(\zeta, \eta). \tag{3.9}$$

For the groups H that are of interest in this chapter, to every permutation π of $\{1, ..., r\}$ there exists $w_\pi \in$ G such that

$$w_\pi d(\eta) w_\pi^* = d(\pi(\eta)) \tag{3.10}$$

with $\pi(\eta_1, ..., \eta_r) = (\eta_{\pi(1)}, ..., \eta_{\pi(r)})$. The elements (w_π) form the Weyl group W of the Lie algebra $\mathfrak{a}$ of the Cartan subgroup A of the holonomy group H. This Weyl group is a discrete group of symmetries of $\mathcal{V}$. The symmetry comes from the invariance under G-gauge transformations. Thus it follows from the biinvariance property that the Higgs potential has the discrete symmetry

$$\mathcal{V}(\zeta, \pi(\eta)) = \mathcal{V}(\zeta, \eta). \tag{3.11}$$

We will discuss later on, in section 4.2.1, how this property generalizes to other pairs (H,G).

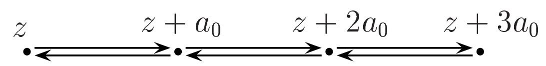
3.3 A soluble model in one dimension

We will now study the renormalization group flow of the action in a one-dimensional theory with fermions which transform according to the two-dimensional representation of SU(2) with one spinor degree of freedom. In one dimension the unitary factor in the polar decomposition of $\phi(x, \mu)$ can be gauged away, so that ϕ can be regarded as a Higgs field. The renormalization group flow shows interesting features, of which we expect that they generalize to higher dimensions.

Here we assume that only the nonunitary extra dimension is retained. Since we regard only space-like dimensions we do not have time reflections in our model. We will study the real space renormalization group (RG-)flow of the model.

It turns out that a well defined effective action is only obtained when the side length of the blocks is odd in units of the lattice spacing, e.g. $l = 3$ (otherwise the Boltzmann factor has no logarithm). The gauge group considered here is G = SU(2). We also consider the case with G = U(2). The holonomy group is chosen as a subgroup of GL(2,$\mathbb{C}$). We consider this group itself, $\mathbb{R}_+$ SL(2,$\mathbb{C}$), $\mathbb{R}_+$ SU(2) and SL(2,$\mathbb{C}$). Starting with the well known fermionic lattice action, we integrate out fermions and bosons. This leads to a generalized version of the lattice action on the block lattice. It is shown how to compute the renormalization group flow exactly. Further, we compute the asymptotic behavior of the Higgs potential.

In particular, we consider now a one-dimensional Euclidean lattice with fermions that is denoted by Λ^0 with lattice spacing a_0, $z \in a_0\mathbb{Z}$. We will later set $a_0 = 1$.



The Fermi fields $\psi(z)$ and $\bar{\psi}(z)$ are sitting on the sites. The Fermi fields considered here have one spin and two color components.

With every link $(z, z + a_0)$ and its adjoint $(z + a_0, z)$ between neighboring sites there is associated a possibly nonunitary lattice gauge field, or parallel transporter $\phi(z, z+a_0) \in$ H and its adjoint

$$\phi(z, z+a_0) = \phi(z+a_0, z)^*. \tag{3.12}$$

These elements $\phi \in \mathrm{H} \subset \mathrm{GL}(2,\mathbb{C})$ transform according to the fundamental representation of the gauge group $\mathrm{G} = \mathrm{SU}(2)$ and are integrated using the Haar measure on H.

$$\begin{aligned}\phi(z, z+a_0) &\mapsto u(z)\phi(z, z+a_0)u(z+a_0)^{-1},\\ \psi(z) &\mapsto u(z)\psi(z),\\ \bar{\psi}(z) &\mapsto \bar{\psi}(z)u(z)^{-1}.\end{aligned}$$

For simplicity, we assume $a_0 = 1$. We start with an initial action of the form

$$S^0(\bar{\psi}, \psi, \phi) = \sum_z S^0(z, z+1) = \sum_z \big(S_F^0(z, z+1) - V^0(\phi(z, z+1))\big)$$

with an initial Higgs potential $V^0(\phi)$, that is a real function of $\phi \in$ H and fulfills the biinvariance property

$$V(u_1 \phi u_2) = V(\phi) \quad \text{for} \quad u_1, u_2 \in \mathrm{SU}(2)$$

and the fermionic action

$$S_F^0(z, z+1) = \mathfrak{z}\bar{\psi}(z)\phi(z, z+1)\psi(z+1) + \bar{\mathfrak{z}}\bar{\psi}(z+1)\phi(z+1, z)\psi(z). \tag{3.13}$$

The constant factor $\mathfrak{z}$ is inserted for later convenience. The resulting Boltzmann factor is

$$B^0(\bar{\psi}, \psi, \phi) = e^{S^0} = \prod_z B_F^0(z, z+1)e^{-V^0(\phi(z,z+1))}.$$

Real space renormalization group (RG-)flow

After n RG-steps, the model lives on a chain $\Lambda^n \subset \Lambda^0$ of lattice spacing $a_n = l^n a_0$ with fields $\psi^n, \bar{\psi^n}, \phi^n(z, z+a_n)$.

The RG-steps act on the Fermi fields via decimation, so that $\psi^n(z) = \psi(z)$ for $z \in a_n\mathbb{Z}$. The other Grassmann variables on sites in between are integrated out. The effective actions will remain strictly local. There will be sums of contributions of the sites of the block lattices,

$$S^n(\psi^n, \bar{\psi^n}, \phi) = \sum_{z \in \Lambda^n} S^n(z, z+a_n).$$

There are two ways to calculate the effective action of a one-dimensional problem after n RG-steps. In each case we have to integrate out the fermionic and the bosonic degrees of freedom. In either case the final result is an effective Boltzmannian $B^n = e^{S^n}$, which is a product of factors

$$B^n(z, z+a_n) = e^{S^n(z,z+a_n)}$$

attached to links $(z, z+a_n)$ of the block lattice Λ^n.

The first possibility consists of studying the RG-flow by performing individual RG-steps in order to compute B^{n+1} from B^n. Subsequently, the resulting recursion relation will be solved. Examining the first step shows that the block lattice spacing must be chosen odd. We choose $a_1 = 3a_0$. It also shows that the effective actions will have a more general form than our initial action.
To perform the general step $n \to n+1$, the most general form of the local Boltzmannian $B^n(z, z+a_n)$ must be found. It will involve coefficients c_i^n which depend on the block spin ϕ^n. The result is given in eq.(3.22). The coefficients of the initial Boltzmannian are found in eq.(3.23).

The RG-step will be performed in two parts.

First, the decimation of the fermions will be performed. Since the Higgs potential V does not depend on the fermions the factor e^{-V} can be dropped for this purpose. Thus, one computes from B^n, without the factor e^{-V} a semi-effective Boltzmannian $\hat{B}^{n+1}$ which depends on the Fermi fields on Λ^{n+1}, but involves the Bose fields on Λ^n. It has the same general form, eq.(3.21), as B^{n+1} but with coefficients $\hat{c}_i^n$ which depend on ϕ^n. Second, the effective Boltzmannian B^{n+1} is computed from $\hat{B}^{n+1} = \hat{B}_F^{n+1} e^{-\sum V^n}$ by integrating ϕ^n subject to the constraint that the block spin ϕ^{n+1} takes a prescribed value. This problem is solved by noting that the integral is a convolution product on H. As a result (theorem 5), one obtains the coefficients c_i^{n+1}, which are functions on H as convolution products of factors c_j^n. The convolution product can be converted into an ordinary product by harmonic analysis on H (corollary 4). The resulting recursion relations for the coefficients can be solved (theorem 6). Exploiting again the structure of the coefficients as convolution products, the asymptotic behavior of the Higgs potential for large n can be determined by applying a group theoretical analogon of the central limit theorem. The effective action $S^n(z, z+a_n)$ is obtained from the Boltzmannian by taking the logarithm (see proposition 10).

The second possibility consists in first performing all the fermion integrations. The factor e^{-V} may again be dropped for this purpose. As a result (corollary 3) one obtains an iterated semi-effective Boltzmannian $\hat{\hat{B}}^n$, which depends on the Fermi fields on the block lattice Λ^n but on the Bose fields ϕ on the original lattice Λ^0. The combinatorics which describes how new terms in the Boltzmannian are obtained by fermion decimation is interesting. A graphical representation is presented. It is used to discuss how the results could be generalized to fermions in higher dimensional representations of H. From $\hat{\hat{B}}^n e^{-\sum V^0}$ one can obtain B^n by integrating the Bose fields ϕ, again subject to the constraint of prescribed block spin ϕ^n. One can show that the integral is again a convolution product, and the result is the same as before. We will not go into detail.

3.3.1 Integration of the fermions

The integration of the fermions itself leads to a so-called semi-effective Boltzmannian, or, after taking the logarithm, to a semi-effective action. The integration of the bosons follows as the next step. Here we perform the fermionic integration step by step. We see why an odd lattice spacing of the block lattice is necessary. Corollary 3 then gives the semi-effective Boltzmannian after n RG-steps.

Euklidean fermion fields ψ_α^σ at one site x are generators of a Grassmann algebra $\mathfrak{A}$ and thus satisfy the anticommutator relation

$$\{\psi_\alpha^\sigma, \psi_\beta^\tau\} = 0; \quad \alpha, \beta = 1, 2 : \sigma, \tau = 1, \ldots s. \tag{3.14}$$

The dimension s stands for the spinor degree of freedom and one associates to each lattice site x a complex vector space $\mathbb{V}_x$. With a local internal symmetry one gets $\mathbb{V}_x = (\mathbb{V}_x)_s \otimes (\mathbb{V}_x)_d$, where $(\mathbb{V}_x)_s$ is s-dimensional and $(\mathbb{V}_x)_d$ carries a d-dimensional representation of the gauge group G. Then $\mathfrak{A}$ is the Grassmann algebra over the vector space $\mathcal{V} = \bigoplus_x \mathbb{V}_x$. [12]
We limit our attention to fermions with one spinor degree of freedom $s = 1$ which transform according to the two-dimensional representation of the gauge group SU(2).

$$\psi(z) \mapsto u(z)\psi(z) \qquad \bar{\psi}(z) \mapsto \bar{\psi}(z)u(z)^{-1}, \quad u(z) \in \mathrm{SU}(2). \tag{3.15}$$

Integration of Grassmann variables at one site z with $\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi_\gamma\epsilon_{\gamma\delta}\psi_\delta$ gives according to the Berezin integration formula

$$\int d^2\psi\psi = 0; \int d^2\psi\frac{1}{2}(\psi^T\epsilon\psi) = \int d^2\psi\psi_\alpha\psi_\beta = \epsilon_{\alpha\beta}. \tag{3.16}$$

The need for odd lattice spacing in one dimension

We start with the initial fermionic action given in eq.(3.13), set for simplicity $\mathfrak{z} = 1$, $a_0 = 1$ and consider a chain of sites $z-1, z, z+1$.

$$\underset{\bullet}{z-1} \rightleftarrows \underset{\bullet}{z} \rightleftarrows \underset{\bullet}{z+1}$$

Here an initial Higgs potential is not involved.

Proposition 9 *The Boltzmann factor which is obtained by integrating out $\psi(z)$ and $\bar{\psi}$ at a single site z is of the form*

$$\begin{aligned}
&\hat{B}_F(z-1, z+1) \\
=&\Big[\frac{1}{2}\sum_{\pm}\big(\bar{\psi}(z\pm 1)\phi^1_-(z\pm 1, z\mp 1)\bar{\psi}(z\mp 1) + \bar{\psi}(z\pm 1)\phi^1_-(z\pm 1, z\pm 1)\bar{\psi}(z\pm 1)\big)\Big] \\
&\times\Big[\frac{1}{2}\sum_{\pm}\big(\psi(z\pm 1)\phi^1_+(z\pm 1, z\mp 1)\psi(z\mp 1) + \psi(z\pm 1)\phi^1_+(z\pm 1, z\pm 1)\psi(z\pm 1)\big)\Big],
\end{aligned} \tag{3.17}$$

with

$$\phi^1_-(x,y)_z = -\phi(x,z)\epsilon\phi^T(y,z) = \phi(x,z)\epsilon^T\phi^T(y,z) \quad and$$
$$\phi^1_+(x,y)_z = -\phi^T(z,x)\epsilon\phi(z,y) = \phi^T(z,x)\epsilon^T\phi(z,y) \tag{3.18}$$

This does not define an effective action due to the absence of 1 *in the Boltzmann factor, because the lowest order term is of fourth order in Fermi fields. There are also terms of higher orders in the Fermi fields, but the sum terminates because of the anti-commutativity (3.14).*

Proof:
The Boltzmann factor is a product of factors associated with links. The factors which contain $\psi(z)$ or $\bar\psi(z)$ are associated with links that contain z as initial or endpoint. Their product is

$$\begin{aligned} B_F(z-1,z+1) &= \exp\Big(\sum_\pm(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)+\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1))\Big)\\ &= \prod_\pm \exp\big(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1)\big), \end{aligned}$$

The exponential has to be evaluated to fourth order. The higher terms vanish because they contain more than two factors $\psi(z)$ or $\bar\psi(z)$ at the same site.

$$\begin{aligned} B_F(z-1,z+1) =&1+\sum_\pm(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)+\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1))\\ &+\frac{1}{2}\Big(\sum_\pm(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)+\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1))\Big)^2\\ &+\frac{1}{3!}\Big(\sum_\pm(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)+\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1))\Big)^3\\ &+\frac{1}{4!}\Big(\sum_\pm(\bar\psi(z\pm1)\phi(z\pm1,z)\psi(z)+\bar\psi(z)\phi(z,z\pm1)\psi(z\pm1))\Big)^4 \end{aligned}$$

Integration via eq.(3.16) leads to the result. **q.e.d.**

The Boltzmannian of proposition 9 does not contain terms of zeroeth order in the Fermi fields. Thus we are not able to take the logarithm to obtain an effective action by integrating out the fermions at all even sites. That means $l^1 = 2$ is forbidden.

Semi-effective Boltzmann factors

The following corollary gives the semi-effective Boltzmann factor for block length $l = 3$ after one RG-step. Eq.(3.25) gives the recursion relations for the semi-effective Boltzmannian. Here an initial Higgs potential V and thus the factor $e^{-\sum V}$ is also not involved. It will be involved when the integration of the bosonic degrees of freedom is performed.

Corollary 2 *Suppose we integrate out $\psi(z)$ and $\bar{\psi}(z)$ at a single site z, then we integrate out the fermions at the proximate site $z+1$. We start again with the initial action*

$$S_F^0(z,z+3) = \sum_{z'=z,z+1,z+2} \bar{\psi}(z')\phi(z',z'+1)\psi(z'+1) + \bar{\psi}(z'+1)\phi(z'+1,z')\psi(z').$$

The resulting semi-effective Boltzmann factor is

$$\begin{aligned}
&\hat{B}_F^1(z,z+3) = \\
&\quad \frac{1}{4} tr\big(\phi(z+2,z+1)\epsilon^T\phi^T(z+2,z+1)\epsilon^T\big) \times tr\big(\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z+2)\epsilon^T\big) \\
&\quad +\frac{1}{2} tr\big(\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z+2)\epsilon^T\big) \\
&\quad \times\bar{\psi}(z)\phi(z,z+1)\epsilon\phi^T(z+2,z+1)\epsilon^T\phi(z+2,z+3)\psi(z+3) \\
&\quad +\frac{1}{2} tr\big(\phi(z+2,z+1)\epsilon^T\phi^T(z+2,z+1)\epsilon^T\big) \\
&\quad \times\bar{\psi}(z+3)\phi(z+3,z+2)\epsilon\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad +\frac{1}{8} tr\big(\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z+2)\epsilon^T\big) \\
&\quad \times\bar{\psi}(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar{\psi}(z) \\
&\quad \times\psi(z+3)\phi^T(z+2,z+3)\epsilon^T\phi(z+2,z+3)\psi(z+3) \\
&\quad +\frac{1}{8} tr\big(\phi(z+2,z+1)\epsilon^T\phi^T(z+2,z+1)\epsilon^T\big) \\
&\quad \times\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad \times\bar{\psi}(z+3)\phi(z+3,z+2)\epsilon^T\phi^T(z+3,z+2)\bar{\psi}(z+3) \\
&\quad +\frac{1}{4}\bar{\psi}(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar{\psi}(z) \\
&\quad \times\psi(z+3)\phi^T(z+2,z+3)\epsilon^T\phi(z+2,z+3)\psi(z+3) \\
&\quad \times\bar{\psi}(z+3)\phi(z+3,z+2)\epsilon\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad +\frac{1}{4}\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad \times\bar{\psi}(z+3)\phi(z+3,z+2)\epsilon^T\phi^T(z+3,z+2)\bar{\psi}(z+3) \\
&\quad \times\bar{\psi}(z)\phi(z,z+1)\epsilon\phi^T(z+2,z+1)\epsilon^T\phi(z+2,z+3)\psi(z+3) \\
&\quad +\frac{1}{16}\bar{\psi}(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar{\psi}(z) \\
&\quad \times\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad \times\bar{\psi}(z+3)\phi(z+3,z+2)\epsilon^T\phi^T(z+3,z+2)\bar{\psi}(z+3) \\
&\quad \times\psi(z+3)\phi^T(z+2,z+3)\epsilon^T\phi(z+2,z+3)\psi(z+3) \\
&\quad + \quad \bar{\psi}(z+3)\phi(z+3,z+2)\epsilon\phi^T(z+1,z+2)\epsilon^T\phi(z+1,z)\psi(z) \\
&\quad \times\bar{\psi}(z)\phi(z,z+1)\epsilon\phi^T(z+2,z+1)\epsilon^T\phi(z+2,z+3)\psi(z+3).
\end{aligned}$$

These nine terms are obtained by evaluating the fermionic initial action and integrating out the Grassmann variables $\bar{\psi}_1(z+1), \bar{\psi}_2(z+1), \bar{\psi}_1(z+2), \bar{\psi}_2(z+2), \psi_1(z+1), \psi_2(z+$

1), $\psi_1(z+2), \psi_2(z+2)$. Obviously the block spin

$$\phi^1(z,z+3) = \phi(z,z+1)\epsilon\phi^T(z+2,z+1)\epsilon^T\phi(z+2,z+3)$$

is involved.

Corollary 3 *Assume we want to have the semi-effective Boltzmann factor after n renormalization group steps. This means that we integrate out all fermions at the sites between z and $z+3^n$. The recursion relation for the effective parallel transporters ϕ^{n-1} referring to the next finer lattice that is obtained after $n-1$ renormalization group steps is*

$$\begin{aligned}&\phi^n(z,z+3^n) = \\ &\quad \phi^{n-1}(z,z+3^{n-1})\epsilon(\phi^{n-1})^T(z+2\times 3^{n-1},z+3^{n-1})\epsilon^T\phi^{n-1}(z+2\times 3^{n-1},z+3\times 3^{n-1}),\end{aligned}$$

$$\begin{aligned}&\phi^n(z+3^n,z) = \\ &\quad \phi^{n-1}(z+\underbrace{3\times 3^{n-1}}_{=3^n},z+2\times 3^{n-1})\epsilon(\phi^{n-1})^T(z+3^{n-1},z+2\times 3^{n-1})\epsilon^T\phi^{n-1}(z+3^{n-1},z).\end{aligned}$$

Referring to this original lattice the iterated semi-effective Boltzmann factor $\hat{\hat{B}}_F(z,z+3^n)$ is

$$\begin{aligned}
\hat{\hat{B}}_F(z,z+3^n) = \quad & 2^{-(3^n-1)}\times\prod_{i=0}^{\frac{3^n-3}{2}}\Big(\mathrm{tr}\big(\phi(z+1+2i,z+2+2i)\epsilon^T\phi^T(z+1+2i,z+2+2i)\epsilon^T\big)\\
&\times\mathrm{tr}\big(\phi(z+2+2i,z+1+2i)\epsilon^T\phi^T(z+2+2i,z+1+2i)\epsilon^T\big)\Big)\\
+\quad & 2^{-\frac{3^n-1}{2}}\times\Big(\prod_{i=0}^{\frac{3^n-3}{2}}\mathrm{tr}\big(\phi(z+1+2i,z+2+2i)\epsilon^T\phi^T(z+1+2i,z+2+2i)\epsilon^T\big)\Big)\\
&\times\bar\psi(z)\phi^n(z,z+3^n)\psi(z+3^n)\\
+\quad & 2^{-\frac{3^n-1}{2}}\times\Big(\prod_{i=0}^{\frac{3^n-3}{2}}\mathrm{tr}\big(\phi(z+2+2i,z+1+2i)\epsilon^T\phi^T(z+2+2i,z+1+2i)\epsilon^T\big)\Big)\\
&\times\bar\psi(z+3^n)\phi^n(z+3^n,z)\psi(z)\\
+\quad & 2^{-3^n}\times\Big(\prod_{i=0}^{3^n-3}\mathrm{tr}\big(\phi(z+1+i,z+2+i)\epsilon^T\phi^T(z+1+i,z+2+i)\epsilon^T\big)\Big)\\
&\times\bar\psi(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar\psi(z)\\
&\times\psi(z+3^n)\phi^T(z+3^n-1,z+3^n)\epsilon^T\phi(z+3^n-1,z+3^n)\psi(z+3^n)\\
+\quad & 2^{-3^n}\times\Big(\prod_{i=0}^{3^n-3}\mathrm{tr}\big(\phi(z+2+i,z+1+i)\epsilon^T\phi^T(z+2+i,z+1+i)\epsilon^T\big)\Big)\\
&\times\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z)
\end{aligned}$$

$$
\begin{aligned}
&\quad \times\bar{\psi}(z+3^n)\phi(z+3^n,z+3^n-1)\epsilon^T\phi^T(z+3^n,z+3^n-1)\bar{\psi}(z+3^n)\\
+\ & 2^{-\frac{3^n+1}{2}}\times\Big(\prod_{i=0}^{\frac{3^n-5}{2}}\mathrm{tr}\big(\phi(z+2+2i,z+3+2i)\epsilon^T\phi^T(z+2+2i,z+3+2i)\epsilon^T\big)\Big)\\
&\quad \times\bar{\psi}(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar{\psi}(z)\times\bar{\psi}(z+3^n)\phi^n(z+3^n,z)\psi(z)\\
&\quad \times\psi(z+3^n)\phi^T(z+3^n-1,z+3^n)\epsilon^T\phi(z+3^n-1,z+3^n)\psi(z+3^n)\\
+\ & 2^{-\frac{3^n+1}{2}}\times\Big(\prod_{i=0}^{\frac{3^n-5}{2}}\mathrm{tr}\big(\phi(z+3+2i,z+2+2i)\epsilon^T\phi^T(z+3+2i,z+2+2i)\epsilon^T\big)\Big)\\
&\quad \times\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z)\times\bar{\psi}(z)\phi^n(z,z+3^n)\psi(z+3^n)\\
&\quad \times\bar{\psi}(z+3^n)\phi(z+3^n,z+3^n-1)\epsilon^T\phi^T(z+3^n,z+3^n-1)\bar{\psi}(z+3^n)\\
+\ & 2^{-(3^n+1)}\times\prod_{i=0}^{\frac{3^n-5}{2}}\Big(\mathrm{tr}\big(\phi(z+2+2i,z+3+2i)\epsilon^T\phi^T(z+2+2i,z+3+2i)\epsilon^T\big)\\
&\qquad \times\phi(z+3+i,z+2+i)\epsilon^T\phi^T(z+3+i,z+2+i)\epsilon^T\big)\Big)\\
&\quad \times\psi(z)\phi^T(z+1,z)\epsilon^T\phi(z+1,z)\psi(z)\times\bar{\psi}(z)\phi(z,z+1)\epsilon^T\phi^T(z,z+1)\bar{\psi}(z)\\
&\quad \times\psi(z+3^n)\phi^T(z+3^n-1,z+3^n)\epsilon^T\phi(z+3^n-1,z+3^n)\psi(z+3^n)\\
&\quad \times\bar{\psi}(z+3^n)\phi(z+3^n,z+3^n-1)\epsilon^T\phi^T(z+3^n,z+3^n-1)\bar{\psi}(z+3^n)\\
+\ & \bar{\psi}(z)\phi^n(z,z+3^n)\psi(z+3^n)\times\bar{\psi}(z+3^n)\phi^n(z+3^n,z)\psi(z). \qquad (3.19)
\end{aligned}
$$

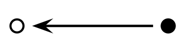
After n or $n+1$ RG-steps we still have a sum of nine terms in the semi-effective Boltzmannian, which is what we already had in the initial Boltzmannian after the expansion of the initial action. In our one-dimensional model with fermions of one spinor degree of freedom only these nine terms are reproduced after each RG-step. The most general semi-effective, or even effective Boltzmannian that is strictly local has the same form. It is introduced in eq.(3.22).

Observe the structure of the nine terms in eq.(3.19). Each is a product of factors which together contain all the Fermi fields and depend on ϕ on the fine lattice Λ^0 through block spins ϕ^n only, and gauge invariant factors which are functions of ϕ-fields on individual links on the fine lattice.

Notation of graphical representation of terms appearing

The graphical representation of terms appearing in the Boltzmann factor as well as in the resulting action is now introduced to simplify the following expressions.

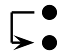
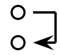
$\bar{\psi}(z)\phi^n(z,z+a_n)\psi(z+a_n) \qquad = L^n(z,z+a_n)$



$\bar{\psi}(z+a_n)\phi^n(z+a_n,z)\psi(z) \qquad = \bar{L}^n(z,z+a_n)$

$\frac{1}{4}\bar{\psi}\epsilon\bar{\psi}(z)\psi\epsilon\psi(z+a_n) \qquad = k^n(z,z+a_n)$

$$\tfrac{1}{4}\bar{\psi}\epsilon\bar{\psi}(z+a_n)\psi\epsilon\psi(z) \qquad = \bar{k}^n(z, z+a_n) \tag{3.20}$$

We write $\bar{\psi}\epsilon\bar{\psi}(z)$ for $\bar{\psi}(z)\epsilon\bar{\psi}(z)$ and $\psi\epsilon\psi(z)$ for $\psi(z)\epsilon\psi(z)$.

Lemma 5 *For all $\phi \in \mathrm{GL}(2,\mathbb{C})$ the following equation is valid:*

$$\epsilon\phi^T\epsilon^{-1} = [\det\phi]\phi^{-1}.$$

Theorem 4 *For any two-dimensional representation of the fermions with parallel transporters $\phi \in \mathrm{H} \supseteq \mathrm{GL}(2,\mathbb{C})$, the terms above fulfill the equations*

$$(L^n)^2 = -2[\det(\phi^n)]k^n \quad \textit{and} \quad (\bar{L}^n)^2 = -2[\det(\phi^{n*})]\bar{k}^n.$$

Proof: This follows from $\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi_\gamma\epsilon_{\gamma\delta}\psi_\delta$ and lemma 5. **q.e.d.**

A generalized version of the action and the Boltzmannian

It is clear from the results obtained so far that the RG-flow will produce Boltzmannians of a more general form than the Boltzmannian B^0 with which we started. The result (3.19) for the iterated semi-effective Boltzmannian suggests that nine terms will be produced by the fermion decimation. Indeed, the most general G-gauge invariant local effective Boltzmannian will be a sum of nine terms. Expressions with fewer terms will not be reproduced by the RG-flow, except if $\mathrm{H} \subseteq \mathrm{SL}(2,\mathbb{C})$. SL(2,$\mathbb{C}$) is a special case in the sense that for all $\phi \in \mathrm{SL}(2,\mathbb{C})$, $\det\phi = 1$. This simplifies the expression (3.19) for the iterated semi-effective Boltzmannian as one can see by inspection. The formula will involve the fermion bilinears L^n and quadrilinears k^n defined in (3.20). For technical reasons, we will also need the most general form of the semi-effective Boltzmannian $\hat{B}^n$. The most general strictly local semi-effective Boltzmannian, denoted by $\hat{B}$, then has the form

$$\begin{aligned}\hat{B}^n(z, z+a_n) =& \hat{c}_0^n + \hat{c}_1^n L^n + \hat{\bar{c}}_1^n \bar{L}^n + \hat{c}_2^n k^n + \hat{\bar{c}}_2^n \bar{k}^n \\ &+ \hat{c}_3^n L^n \bar{L}^n + \hat{c}_4^n k^n \bar{L}^n + \hat{\bar{c}}_4^n \bar{k}^n L^n + \hat{c}_5^n k^n \bar{k}^n.\end{aligned} \tag{3.21}$$

Strictly local means that we have only sums of contributions of this form from the lattice sites in the action or products of those in the semi-effective Boltzmannian. This semi-effective Boltzmannian $\hat{B}^{n+1}$ is obtained by starting with the effective Boltzmann B^n, effective after the n-th RG-step, and by integrating out only the fermionic degrees of freedom. The action is obtained by taking the logarithm and has the same form and is given in section 3.3.3. The coefficients c_i^n are still functions of the bosonic variables on the fine lattice. Except for the SL(2,$\mathbb{C}$)-case the Boltzmannian and the action involves four-, six- and eight-fermion interaction terms as k^n or $L^n\bar{L}^n$, $k^n\bar{L}^n$ and $k^n\bar{k}^n$.

The most general strictly local effective action and the effective Boltzmannian have the same basic form, but here we name the coefficients c_i.

$$\begin{aligned}B^n(z, z+a_n) =& c_0^n + c_1^n L^n + \bar{c}_1^n \bar{L}^n + c_2^n k^n + \bar{c}_2^n \bar{k}^n \\ &+ c_3^n L^n \bar{L}^n + c_4^n k^n \bar{L}^n + \bar{c}_4^n \bar{k}^n L^n + c_5^n k^n \bar{k}^n\end{aligned} \tag{3.22}$$

where $c_0^n = \exp\big(-V^n(\phi^n(z,z+a_n))\big)$ is, by definition, the exponentiated initial Higgs potential, if there is one. The coefficients c_0^n, c_3^n and c_5^n are selfadjoint.

Generally, the coefficients c_i^n are functions of $\phi^n(z,z+a_n)$

$$
\begin{aligned}
c_i^n &= c_i^n(z,z+a_n) = c_i^n[\phi^n(z,z+a_n)], \\
\bar{c}_i^n &= \bar{c}_i^n(z,z+a_n) = c_i^n[\phi^n(z,z+a_n)^*] = c_i^n[\phi^n(z+a_n,z)].
\end{aligned}
$$

They possess the reality property

$$
c_i^n(\phi^*) \;=\; \bar{c}_i^n(\phi).
$$

The RG-flow will determine the dependence of c_i^n on the number n of RG-steps. Again the action is obtained by taking the logarithm and has the same form. Starting with the initial fermionic action (3.13), which can now be written as

$$
\Rightarrow S_F^0(z,z+1) = \mathfrak{z}L + \bar{\mathfrak{z}}\bar{L},
$$

the initial Boltzmann factor is then

$$
\begin{aligned}
B_F^0(z,z+1) = & 1 + \mathfrak{z}L + \bar{\mathfrak{z}}\bar{L} - \mathfrak{z}^2[\det\phi]k - \bar{\mathfrak{z}}^2[\det\phi^*]\bar{k} \\
& + \mathfrak{z}\bar{\mathfrak{z}}L\bar{L} - \mathfrak{z}^2\bar{\mathfrak{z}}[\det\phi]k\bar{L} - \bar{\mathfrak{z}}^2\mathfrak{z}[\det\phi^*]\bar{k}L \\
& + \mathfrak{z}^2\bar{\mathfrak{z}}^2[\det\phi][\det\phi^*]k\bar{k}.
\end{aligned} \tag{3.23}
$$

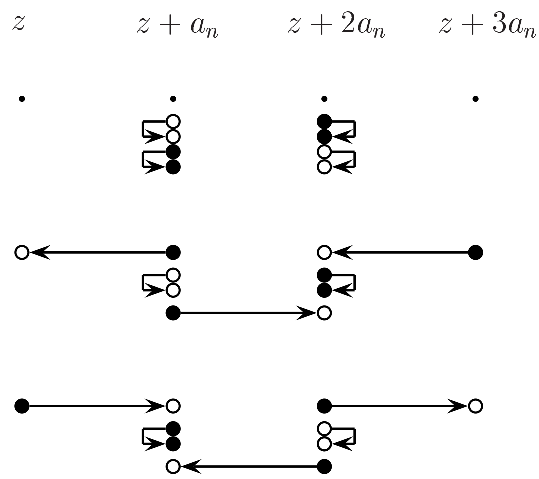
Performing the fermion integration graphically

For the following integration of the bosons and the setting of the recursion relations we now show how the different terms of the Boltzmannian match and form a term of the effective expression upon performing the fermion integration . The integration leads to the following relations, where $\int$ is the short notation for $\int d^2\psi(z+a_n)d^2\bar{\psi}(z+a_n)d^2\psi(z+2a_n)d^2\bar{\psi}(z+2a_n)$. That means we integrate out the four Grassmann variables at each site $z+a_n$ and $z+2a_n$.

$\int (k\bar{k})_2 \qquad\qquad = 1$

$\int L_1(\bar{L}k)_2 L_3 = L^1 \qquad = L^1$

$\int \bar{L}_1(L\bar{k})_2\bar{L}_3 \qquad = \bar{L}^1$



$$\int k_1 k_2 k_3 = k^1 \qquad = k^1$$

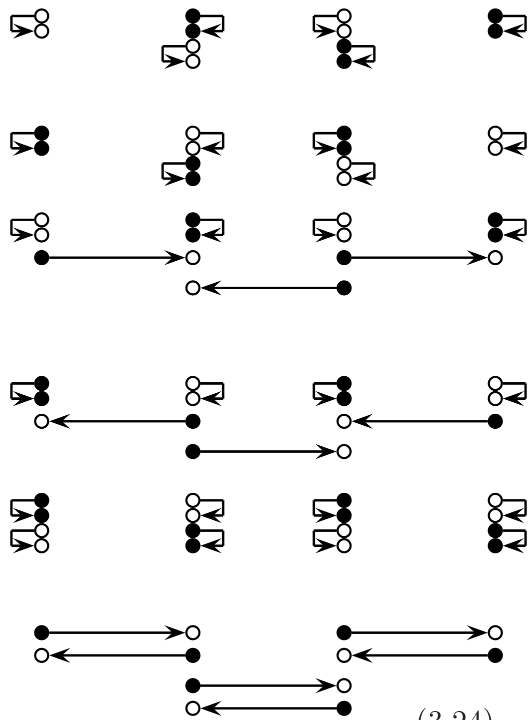
$$\int \bar{k}_1 \bar{k}_2 \bar{k}_3 \qquad = \bar{k}^1$$

$$\int (\bar{L}k)_1 L_2 (\bar{L}k)_3 \qquad = \bar{L}^1 k^1$$

$$\int (L\bar{k})_1 \bar{L}_2 (L\bar{k})_3 \qquad = L^1 \bar{k}^1$$

$$\int (k\bar{k})_1 (k\bar{k})_3 \qquad = k^1 \bar{k}^1$$

$$\int (L\bar{L})_1 (L\bar{L})_2 (L\bar{L})_3 \qquad = (L^1 \bar{L}^1)$$



(3.24)

with $(k)_i = k(z + ia_n - 1, z + ia_n)$ and $(L)_i = L(z + ia_n - 1, z + ia_n)$.

Recursion relations for the coefficients in the semi-effective Boltzmannian

Let us admit an effective Boltzmannian as in eq.(3.21) with coefficients c_i. Performing only the fermion integration leads to the following recursion relation for the semi-effective coefficients that are denoted by $\hat{c}_i$ after the $(n+1)$-st RG-step

$$\hat{c}_i^{n+1}(z, z+3a_n) = \sum_{jkl} d^i_{jkl} c^n_j(z, z+a_n) c^n_k(z+a_n, z+2a_n) c^n_l(z+2a_n, z+3a_n)$$

with one non vanishing term $d^i_{jkl} = 1$ in the sum $\sum_{jkl}$ for $i = 0, 1, \bar{1}, 2, \bar{2}, 3, 4, \bar{4}, 5$. Explicitly

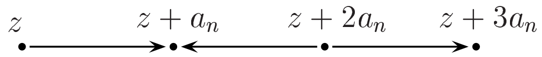
$$\begin{aligned}
\hat{c}_0^{n+1}(z, z+3a_n) &= c_0^n(z, z+a_n) c_5^n(z+a_n, z+2a_n) c_0^n(z+2a_n, z+3a_n) \\
\hat{c}_1^{n+1}(z, z+3a_n) &= c_1^n(z, z+a_n) c_4^n(z+a_n, z+2a_n) c_1^n(z+2a_n, z+3a_n) \\
\hat{\bar{c}}_1^{n+1}(z, z+3a_n) &= \bar{c}_1^n(z, z+a_n) \bar{c}_4^n(z+a_n, z+2a_n) \bar{c}_1^n(z+2a_n, z+3a_n) \\
\hat{c}_2^{n+1}(z, z+3a_n) &= c_2^n(z, z+a_n) c_2^n(z+a_n, z+2a_n) c_2^n(z+2a_n, z+3a_n) \\
\hat{\bar{c}}_2^{n+1}(z, z+3a_n) &= \bar{c}_2^n(z, z+a_n) \bar{c}_2^n(z+a_n, z+2a_n) \bar{c}_2^n(z+2a_n, z+3a_n) \\
\hat{c}_3^{n+1}(z, z+3a_n) &= c_3^n(z, z+a_n) c_3^n(z+a_n, z+2a_n) c_3^n(z+2a_n, z+3a_n) \\
\hat{c}_4^{n+1}(z, z+3a_n) &= c_4^n(z, z+a_n) c_1^n(z+a_n, z+2a_n) c_4^n(z+2a_n, z+3a_n) \\
\hat{\bar{c}}_4^{n+1}(z, z+3a_n) &= \bar{c}_4^n(z, z+a_n) \bar{c}_1^n(z+a_n, z+2a_n) \bar{c}_4^n(z+2a_n, z+3a_n) \\
\hat{c}_5^{n+1}(z, z+3a_n) &= c_5^n(z, z+a_n) c_0^n(z+a_n, z+2a_n) c_5^n(z+2a_n, z+3a_n).
\end{aligned} \tag{3.25}$$

This means that $d^0_{050} = d^1_{141} = d^{\bar{1}}_{\bar{1}\bar{4}\bar{1}} = d^2_{222} = d^{\bar{2}}_{\bar{2}\bar{2}\bar{2}} = d^3_{333} = d^4_{414} = d^{\bar{4}}_{\bar{4}\bar{1}\bar{4}} = d^5_{505} = 1$ and otherwise $d^i_{jkl} = 0$. This can be seen directly from the graphical representation.

The block spin

In the semi-effective Boltzmannian $\hat{B}^{n+1}$, the fermionic factors (but not the coefficients $\hat{c}_i^n$) depend on ϕ^n only through the following block spin

$$\phi^{n+1}(z, z+3a_n) = \quad \phi^n(z, z+a_n)\epsilon\phi^n(z+2a_n, z+a_n)^T\epsilon^T\phi^n(z+2a_n, z+3a_n). \qquad (3.26)$$



This block spin is gauge covariant in the sense described in section 2.5. And further it is gauge covariant in the sense that a gauge transformation on the finer lattice determines the transformation on the coarser lattice. It transforms according to

$$\phi^{n+1}(z, z+3a_n) \mapsto u(z)\phi^{n+1}(z, z+3a_n)u(z+3a_n)^{-1}$$

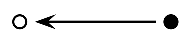
for $u(z), u(z+3a_n) \in \mathrm{G} = \mathrm{SU}(2)$.

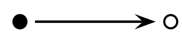
Further application of the graphical representation

The graphical integration can also be easily applied to fermions that transform according to higher dimensional representations of the gauge group. We focus now on three-dimensional representations. In this case we have to perform a fermionic integration over six, instead of four, Grassmann variables. Thus the evaluation of the Boltzmann factor stops after sixteen terms. We have in short notation

$$\begin{aligned} B^n(z, z+a_n) =& c_0^n + c_1^n L^n + \bar{c}_1^n \bar{L}^n + c_2^n L^n \bar{L}^n + c_3^n (L^n)^2 + \bar{c}_3^n (\bar{L}^n)^2 \\ &+ c_4^n (L^n)^2 \bar{L}^n + \bar{c}_4^n L^n (\bar{L}^n)^2 + c_5^n (L^n)^2 (\bar{L}^n)^2 + c_6^n k^n + \bar{c}_6^n \bar{k}^n \\ &+ c_7^n k^n \bar{L}^n + \bar{c}_7^n \bar{k}^n L^n + c_8^n k^n (\bar{L}^n)^2 + \bar{c}_8^n \bar{k}^n (L^n)^2 + c_9^n k^n \bar{k}^n \end{aligned} \qquad (3.27)$$

but with a new defined 6-fermion interaction term k^n.

$\bar{\psi}(z)\phi^n(z, z+a_n)\psi(z+a_n) \qquad = L^n(z, z+a_n)$ 

$\bar{\psi}(z+a_n)\phi^n(z+a_n, z)\psi(z) \qquad = \bar{L}^n(z, z+a_n)$ 

$\frac{1}{6^2}\epsilon_{\alpha\beta\gamma}\bar{\psi}_\alpha\bar{\psi}_\beta\bar{\psi}_\gamma(z)\epsilon_{\alpha\beta\gamma}\psi_\alpha\psi_\beta\psi_\gamma(z+a_n) \qquad = k^n(z, z+a_n)$ 

$\frac{1}{6^2}\epsilon_{\alpha\beta\gamma}\bar{\psi}_\alpha\bar{\psi}_\beta\bar{\psi}_\gamma(z+a_n)\epsilon_{\alpha\beta\gamma}\psi_\alpha\psi_\beta\psi_\gamma(z) \qquad = \bar{k}^n(z, z+a_n)$ 

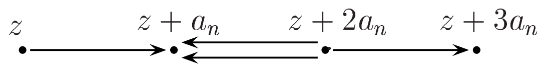
The integration involves the antisymmetric tensor of third order. We sum over the indices.

$$\int d^3\psi \frac{1}{6}\psi_\alpha\psi_\beta\psi_\gamma = \epsilon_{\alpha\beta\gamma},$$

with $\psi_\alpha\psi_\beta\psi_\gamma = \frac{1}{6}\epsilon_{\alpha\beta\gamma}(\epsilon_{lmn}\psi_l\psi_m\psi_n)$. It can be seen easily that $L^3 = -[\det\phi]k$. The block spin is

$$\begin{aligned}&\phi^1_{\alpha\beta}(z, z+3a_n) = \\ &\phi_{\alpha b}(z, z+a_n)\epsilon_{bdf}\epsilon_{cge}\phi_{cd}(z+2a_n, z+a_n)\phi_{ef}(z+2a_n, z+a_n)\phi_{g\beta}(z+2a_n, z+3a_n),\end{aligned} \tag{3.28}$$

that can be illustrated via



The block spin of eq.(3.28) can also be written as

$$\begin{aligned}\phi^1(z, z+3a_n) =& 2\phi(z, z+a_n)\phi(z+2a_n, z+a_n)\phi(z+2a_n, z+a_n)\phi(z+2a_n, z+3a_n)\\ &- 2[\mathrm{tr}\,\phi(z+2a_n, z+a_n)]\phi(z, z+a_n)\phi(z+2a_n, z+a_n)\phi(z+2a_n, z+3a_n)\\ &+ [\mathrm{tr}\,\phi(z+2a_n, z+a_n)]^2\phi(z, z+a_n)\phi(z+2a_n, z+3a_n)\\ &- [\mathrm{tr}\,\phi(z+2a_n, z+a_n)^2]\phi(z, z+a_n)\phi(z+2a_n, z+3a_n).\end{aligned}$$

Remark: If we would start with pure gauge theory the parallel transporters would be elements of the unitary group SO(3). Also the block spin ϕ^1 would be an element of SO(3), its middle factor is just an antisymmetric tensor product of two elements of SO(3).

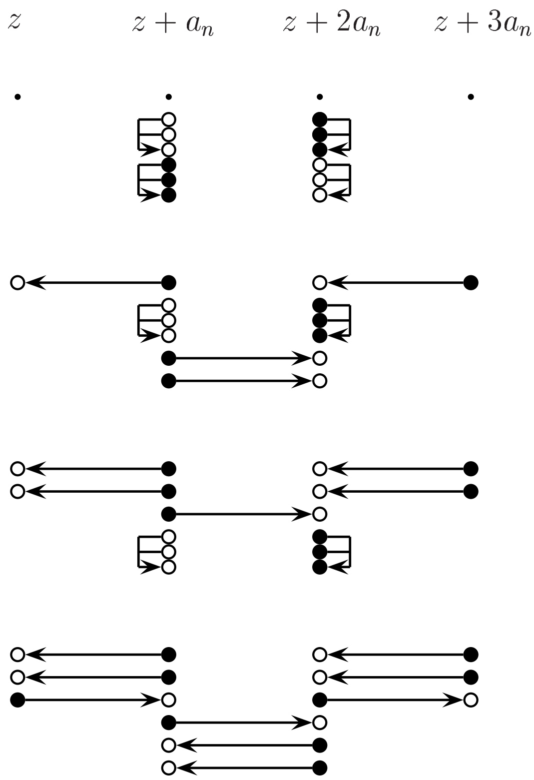
Nevertheless we admit nonunitary parallel transporters. We now apply the graphical representation to see the matching scheme of the terms that are appearing in the Boltzmannian (3.27). Here, the integral $\int$ stands for $\int d^3\psi(z+a_n)d^3\bar{\psi}(z+a_n)d^3\psi(z+2a_n)d^3\bar{\psi}(z+2a_n)$.

$\int(k\bar{k})_2 \qquad = 1$

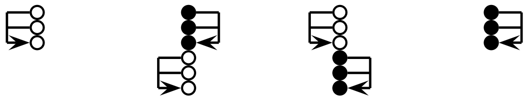
$\int L_1(\bar{L}^2k)_2L_3 \qquad = L^1$

$\int(L^2)_1(\bar{L}k)_2(L^2)_3 \qquad = (L^1)^2$

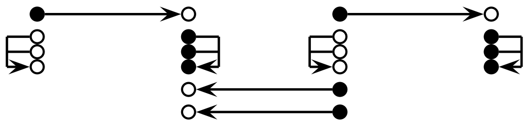
$\int(L^2\bar{L})_1(L\bar{L}^2)_2(L^2\bar{L})_3 \qquad = (L^1)^2\bar{L}^1$



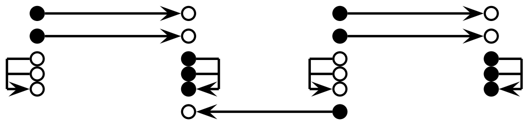
$\int k_1 k_2 k_3 \qquad = k^1$



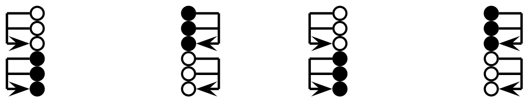
$\int (k\bar{L})_1 (L^2)_2 (k\bar{L})_3 \qquad = k^1 \bar{L}^1$



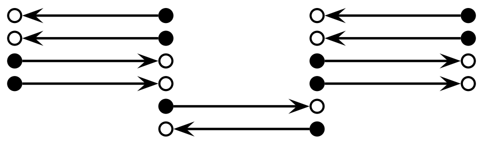
$\int (k\bar{L}^2)_1 (L)_2 (k\bar{L}^2)_3 \qquad = k^1 (\bar{L}^1)^2$



$\int (k\bar{k})_1 (k\bar{k})_3 \qquad = k^1 \bar{k}^1$



$\int (L^2\bar{L}^2)_1 (L\bar{L})_2 (L^2\bar{L}^2)_3 \qquad = (L^1)^2 (\bar{L}^1)^2$



We can read the recursion relations for the semi-effective Boltzmannian $B^{n+1}(z, z+3a_n)$ after the $(n+1)$ RG-step in dependence on the coefficients c_i^n of the effective or most general Boltzmannian after n RG-steps (3.27) out of the graphical illustration, e.g.

$$\hat{c}_0^{n+1}(z, z+3a_n) = c_0^n(z, z+a_n) c_9^n(z+a_n, z+2a_n) c_0^n(z+2a_n, z+3a_n).$$

3.3.2 Integration of the bosons

We now return to the theory with two components of fermions. The generalization to more dimensional representations is straightforward by applying the graphical representation as we have seen. In principle, we now integrate out the three lattice gauge fields $\phi_1 = \phi(z, z+a_n), \phi_2 = \phi(z+a_n, z+2a_n)$ and $\phi_3 = \phi(z+2a_n, z+3a_n)$ that were connected by the neighboring fermions that we already integrated out, holding the block spin fixed. What we obtain are convolution products. To convert these convolution products into ordinary products harmonic analysis is applied. Details of the harmonic analysis are presented in appendix C. The block spin ϕ^1 of the first block-lattice is denoted by Φ.

Integration of the Higgs field

We have to integrate out the parallel transporters $\phi_1 = \phi(z, z+1), \phi_2 = \phi(z+2, z+1)$ and $\phi_3 = \phi(z+2, z+3)$ while keeping the block spin fixed. The constraint is imposed with the help of the delta function in the expression below. The semi-effective Boltzmann factor $\hat{B}^1$ still depends on the Higgs fields ϕ on the links of the original lattice. Thus the expression

$$\begin{aligned} B^1 &= \int d\phi_1 d\phi_2 d\phi_3 \delta(\phi_1 \epsilon \phi_2^{*T} \epsilon^T \phi_3 \Phi^{-1}) \hat{B}^1 \\ &= \int d\phi_1 d\phi_2 d\phi_3 \delta(\phi_1 \epsilon \phi_2^{*T} \epsilon^T \phi_3 \Phi^{-1}) \hat{B}_1^0 \hat{B}_2^0 \hat{B}_3^0, \end{aligned}$$

with $\hat{B}_1^0 = \hat{B}(z, z+1)$, $\hat{B}_2^0 = \hat{B}(z+1, z+2)$ and $\hat{B}_3^0 = \hat{B}(z+2, z+3)$, has to be evaluated. The block spin $\phi^1(z, z+3)$ is denoted by Φ.

Remark: Without loss of generality, we assume that the original action is living on the lattice Λ^0. This can always be achieved by rescaling.

Remember the non-vanishing d^i_{jkl} with $\sum_{jkl} d^i_{jkl} = 1$ that brought us to the semi-effective coefficients

$$\hat{c}_i(z, z+3) = \sum_{jkl} d^i_{jkl} c_j(z, z+1) c_k(z+1, z+2) c_l(z+2, z+3) \tag{3.29}$$

as shown in section 3.3.1. Thus we have to compute coefficients of the form

$$\begin{aligned} c_i^1[\Phi] &= c_i^1(z, z+3) = \int d\phi_1 d\phi_2 d\phi_3 \hat{c}_i(z, z+3) \delta(\phi_1 \epsilon \phi_2^{*T} \epsilon^T \phi_3 \Phi^{-1}) \\ &= \sum_{jkl} \int d\phi_1 d\phi_2 d\phi_3 d^i_{j\bar{k}l} c_j(z, z+1) c_k(z+1, z+2) c_l(z+2, z+3) \delta(\phi_1 \epsilon \phi_2^{*T} \epsilon^T \phi_3 \Phi^{-1}) \\ &= \sum_{jkl} \int d\phi_1 d\phi_2 d\phi_3 d^i_{j\bar{k}l} c_i[\phi_1] c_j[\phi_2] c_k[\phi_3] \delta(\phi_1 \epsilon \phi_2^{*T} \epsilon^T \phi_3 \Phi^{-1}). \end{aligned}$$

The coefficients c_i^1 are SU(2)-biinvariant functions. An application of lemma 4 of section 3.2.3 is used to perform a change of variables: $\phi_2 \to \phi_2' = \epsilon \phi_2^{*T} \epsilon^T$. In this way, we derive

$$\begin{aligned} c_i^1[\Phi] &= \sum_{jkl} \int d\phi_1 d\phi_2' d\phi_3 d^i_{jkl} c_j[\phi_1] c_k[\phi_2'] c_l[\phi_3] \delta(\phi_1 \phi_2' \phi_3 \Phi^{-1}) \\ &= \sum_{jkl} d^i_{jkl} c_j * \bar{c}_k * c_l[\Phi] \\ &= \sum_{jkl} d^i_{jkl} c_j * c_{\bar{k}} * c_l[\Phi] \end{aligned}$$

with $d\phi_2' = d\phi_2$, $c_i[\phi_2'] = c_i[\epsilon(\phi_2^*)^T \epsilon^T] = c_i[\bar{\zeta}_2 \epsilon (a_2^*)^T \epsilon^T] = c_i[\bar{\zeta}_2 (a_2^*)^{-1}] = c_i[\phi_2^*] = \bar{c}_i[\phi_2]$. The explicit recursion relations that are derived from this are presented in theorem 5.

Theorem 5 (RG-flow for general H)
The coefficients $c_i^n[\Phi]$ in the effective Boltzmann factor obey the following recursion rela-

tions,

$$
\begin{array}{rcl}
c_0^{n+1}[\Phi] &=& c_0 * c_5 * c_0[\Phi] \ , \\
c_1^{n+1}[\Phi] &=& c_1^n * \bar{c}_4^n * c_1^n[\Phi] \ , \\
c_2^{n+1}[\Phi] &=& c_2^n * \bar{c}_2^n * c_2^n[\Phi] \ , \\
c_3^{n+1}[\Phi] &=& c_3^n * c_3^n * c_3^n[\Phi] \ , \\
c_4^{n+1}[\Phi] &=& c_4^n * \bar{c}_1^n * c_4^n[\Phi] \ , \\
c_5^{n+1}[\Phi] &=& c_5^n * c_0^n * c_5^n[\Phi] \ ,
\end{array}
$$

where $*$ *is the convolution product on the group* H*, and* $\Phi \in$ H*.* ($\bar{c}_i^n = c_i^n$ *for* $i = 0, 3, 5$).

All the coefficients are SU(2)*-biinvariant functions on* H*, viz.* $c_i[u_1 \Phi u_2] = c_i[\Phi]$ *for* $\Phi \in$ H*,* $u_1, u_2 \in$ SU(2).

The result of theorem 5 is an immediate consequence of lemma 9 in appendix C and of lemma 4.

We can apply harmonic analysis that is presented explicitly in appendix C to convert the convolution products into ordinary products. G-biinvariant functions, here the Higgs potential as well as the coefficients are SU(2)-biinvariant functions on the holonomy group H. They can be expanded in spherical functions P^{Ξ}. They are associated with unitary representations Ξ of the holonomy group H. Therefore, we have the following corollary of the recursion relations.

Corollary 4 *Let us denote by* Ξ *those representations which contribute to the harmonic expansion of square integrable functions on the group which are* G*-biinvariant. Write* $\bar{c}_i = c_{\bar{i}}$*. Then,*

$$
c_i^{n+1}[\Phi] = \int d\Xi \ \tilde{c}_i^{n+1}(\Xi) \ P^{\Xi}(\Phi) \ ,
$$

with Fourier coefficients

$$
\tilde{c}_i^{n+1}(\Xi) = \sum_{jkl} d^i{}_{jkl} \tilde{c}_j^n(\Xi) \tilde{c}_k^n(\Xi) \tilde{c}_l^n(\Xi) \ ,
$$

where

$$
\tilde{c}_i^n(\Xi) = \int_H d\Phi \ c_i^n[\Phi] P^{\Xi}(\Phi^{-1})
$$

The sum over $j, k, l = 0, 1, \bar{1}, 2, \bar{2}, 3, 4, \bar{4}, 5$ *has only one non-vanishing term, which is equal to* 1*. The non-vanishing coefficients are*

$$
d^0{}_{050}, \ d^1{}_{1\bar{4}1}, \ d^2{}_{2\bar{2}2}, \ d^3{}_{333}, \ d^4{}_{4\bar{1}4}, \ d^5{}_{505}; \quad d^{\bar{1}}{}_{\bar{1}4\bar{1}}, \ d^{\bar{2}}{}_{\bar{2}2\bar{2}}, \ d^{\bar{4}}{}_{\bar{4}1\bar{4}}.
$$

Finally we present the solutions of the recursion relations in terms of the initial values c_j^0, that appear in the initial Boltzmannian (3.23). We assumed the most generalized form of a strictly local Boltzmann factor as given in eq.(3.22) for $n = 0$.

Theorem 6 (Solution of the flow equations)
For temporary use we introduce the notation $(A)^{*N} = A * A * ... * A$ (N factors) for the N-fold convolution product. In this notation

$$\begin{aligned}
c_0^n[\Phi] &= c_0^0 * (c_5^0 * c_0^0)^{*N}[\Phi], \\
c_1^n[\Phi] &= c_1^0 * (\bar{c}_4^0 * c_1^0)^{*N}[\Phi] \ , \\
c_2^n[\Phi] &= c_2^0 * (\bar{c}_2^0 * c_2^0)^{*N}[\Phi] \ , \\
c_3^n[\Phi] &= c_3^0 * (c_3^0 * c_3^0)^{*N}[\Phi] \ , \\
c_4^n[\Phi] &= c_4^0 * (\bar{c}_1^0 * c_4^0)^{*N}[\Phi], \\
c_5^n[\Phi] &= c_5^0 * (c_0^0 * c_5^0)^{*N}[\Phi] \ ,
\end{aligned}$$

with $N = \frac{1}{2}(3^n - 1)$.

The Fourier coefficients $\tilde{c}_j^n(\Xi)$ obey the same relations with ordinary products in place of convolution products.

The proof is immediate from theorem 5.

3.3.3 Taking the logarithm

The most general form of an action between the lattice sites z and $z + a_n$ has in principle the same form as the Boltzmannian

$$S^n(z, z + a_n) = a_0^n + a_1^n L^n + \bar{a}_1^n \bar{L}^n + a_2^n k^n + \bar{a}_2^n \bar{k}^n + a_3^n L^n \bar{L}^n + a_4^n k^n \bar{L}^n + \bar{a}_4^n \bar{k}^n L^n + a_5^n k^n \bar{k}^n \tag{3.30}$$

again with coefficients that are functions of the form

$$a_i^n = a_i^n[\phi^n(z, z + a_n)] = a_i^n[\phi^n(z + a_n, z)^*].$$

In particular $a_0^n = -V^n\big(\phi^n(z, z + a_n)\big)$ is the negative effective Higgs potential. By straightforward computation (Taylor expansion) one deduces the following proposition.

Proposition 10 (Effective action from the effective Boltzmannian)
Retaining the assumption $L^2 = -2[\det \phi]k$, the coefficients a_i^n in the effective action (3.30) are obtained from the coefficients c_j^n in the effective Boltzmannian (3.22) as follows

$$\begin{aligned}
a_0^n =& \ln(c_0^n) \\
a_1^n =& b_1 \\
a_2^n =& b_2 + b_1^2[\det \phi] \\
a_3^n =& b_3 - b_1 \bar{b}_1 \\
a_4^n =& \bar{b}_4 - \bar{b}_1 b_2 + 2 b_1 b_3 [\det \phi] - 2 b_1^2 \bar{b}_1 [\det \phi] \\
& + \big(8 b_1 \bar{b}_1 b_3 - 2 b_3^2 - 6 b_1^2 \bar{b}_1^2\big)[\det \phi][\det \phi^*]
\end{aligned}$$

where $b_i = \frac{c_i^n}{c_0^n}$ and $\bar{a}_2^n$, $\bar{a}_4^n$ are obtained by complex conjugation.

.

3.3.4 An example: The case SL(2,ℂ)

We now specialize to Higgs fields $\phi \in$ SL(2,ℂ). We first give the initial values of the coefficients. They simplify in this case because $\det \phi = 1$. The recursions relations will also simplify.

Let us admit an initial fermionic action

$$\begin{aligned} S_F^0(z,z+1) &= \mathfrak{z}\bar\psi(z)\phi(z,z+1)\psi(z+1) + \bar{\mathfrak{z}}\bar\psi(z+1)\phi(z+1,z)\psi(z) \\ &= \mathfrak{z}L + \bar{\mathfrak{z}}\bar L. \end{aligned} \tag{3.31}$$

Expanding the Boltzmann factor gives for arbitrary holonomy groups the following initial coefficients c_i^0 in the representation of eq.(3.22)

$$c_i^0 = b_i^0 e^{-V^0} \text{ and} \tag{3.32}$$

$$\begin{aligned} b_0^0 &= 1 && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && 1 \\ b_1^0 &= \mathfrak{z} && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && \mathfrak{z} \\ b_2^0 &= -\mathfrak{z}^2 \det\phi && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && -\mathfrak{z}^2 \\ b_3^0 &= \mathfrak{z}\bar{\mathfrak{z}} && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && \mathfrak{z}\bar{\mathfrak{z}} \\ b_4^0 &= -\mathfrak{z}^2\bar{\mathfrak{z}} \det\phi && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && -\mathfrak{z}^2\bar{\mathfrak{z}} \\ b_5^0 &= \mathfrak{z}^2\bar{\mathfrak{z}}^2 \det\phi \det\phi^* && \overset{\mathrm{SL}(2,\mathbb{C})}{=} && \mathfrak{z}^2\bar{\mathfrak{z}}^2 \end{aligned} \tag{3.33}$$

with the abbreviations $\phi = \phi(z,z+1)$, $V^0 = V(\phi(z,z+1))$.

Proposition 11 *Consider the model with holonomy group $H =$ SL(2,ℂ). Starting from the initial action (3.31), the effective action after the first, and therefore after every RG-step, retains the form (3.31). In particular*

$$\begin{aligned} S^1(\psi^1,\bar\psi^1,\phi^1) &= \sum_{z\in\Lambda^1} S^1(z,z+3) \\ S^1(z,z+3) &= \mathfrak{z}^1\bar\psi^1(z)\phi^1(z,z+3)\psi^1(z+3) + \bar{\mathfrak{z}}^1\bar\psi^1(z+3)\phi^1(z+3,z)\psi^1(z) \\ &\quad -V^1(\phi^1(z,z+3)). \end{aligned}$$

The semi-effective Boltzmannian for $H =$ SL(2,ℂ) after one RG-step has the form

$$\begin{aligned} \hat B^1(z,z+3) &= (\mathfrak{z}\bar{\mathfrak{z}})^2 e^{S_F^1(z,z+3)} e^{-V(\phi(z,z+1))-V(\phi(z+1,z+2))-V(\phi(z+2,z+3))}, \\ S_F^1(z,z+3) &= \mathfrak{z}^1\bar\psi^1(z)\phi^1(z,z+3)\psi^1(z+3) + \bar{\mathfrak{z}}^1\bar\psi^1(z+3)\phi^1(z+3,z)\psi^1(z) \end{aligned}$$

with $\mathfrak{z}^1 = -(\mathfrak{z})^2\bar{\mathfrak{z}}^{-1}$. The effective Higgs potential V^1 is described in theorem 7 below. This is equal to coefficients $\hat c_i^1 = (\mathfrak{z}\bar{\mathfrak{z}})^2 b_i^1 e^{-3V^0}$, [1] *and b_i^1 as in eq.(3.33) after having replaced $\mathfrak{z}$ by $\mathfrak{z}^1$ ($\det\phi = 1$).*

[1] We admit $V^0(\phi(z,z+1)) = V^0(z+1,z+2)$. The bosonic integration has to follow, so that this leads to a three-fold convolution product.

PROOF: Because $\mathfrak{z}$ are constants and $\det\phi = 1$, the starting coefficients c_i^0, eq.(3.32), which determine the original Boltzmannian (3.34), become ϕ-independent except for overall factors $e^{-V^0(\phi(z,z+1))}$ common to all of them. They may be pulled out. It follows from eq.(3.29) that the coefficients $\hat{c}_i^1$ have the same form as the coefficients in the original Boltzmannian (3.34) except for the common overall-factor

$$(\mathfrak{z}\bar{\mathfrak{z}})^2 e^{-V^0(\phi(z,z+1))-V^0(\phi(z+1,z+2))-V^0(\phi(z+2,z+3))}$$

and replacement of $\mathfrak{z}$ by $\mathfrak{z}^1$ as defined in the proposition. The factor $(\mathfrak{z}\bar{\mathfrak{z}})^2$ remains in the Boltzmannian. Note that $(\mathfrak{z}^1\bar{\mathfrak{z}}^1)^2 = (\mathfrak{z}\bar{\mathfrak{z}})^2$.

Since the original Boltzmannian

$$\begin{aligned} B_F^0(z,z+1) &= 1+\mathfrak{z}L+\bar{\mathfrak{z}}\bar{L}-\mathfrak{z}^2k+\mathfrak{z}\bar{\mathfrak{z}}L\bar{L}-\bar{\mathfrak{z}}^2\bar{k} \\ &\quad -\mathfrak{z}^2\bar{\mathfrak{z}}k\bar{L}-\mathfrak{z}\bar{\mathfrak{z}}^2\bar{k}L+\mathfrak{z}^2\bar{\mathfrak{z}}^2k\bar{k}\ , \end{aligned} \tag{3.34}$$

with $L^2 = -2k$ comes from the action (3.31), the same will be true with appropriate adjustments for the semi-effective Boltzmannian, as stated in the proposition. **q.e.d.**

We now turn to the integration of the bosons. In particular, we want to describe the flow of the Higgs potential. Harmonic analysis taught us for SL(2,$\mathbb{C}$)

$$\begin{aligned} \tilde{c}_0^1 &= \int \sinh^2\eta d\eta\ \ c_0^1 \frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta} \\ &= \int \sinh^2\eta d\eta\ \ (\mathfrak{z}\bar{\mathfrak{z}})^2(e^{-V^0})^{*3}\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta} \end{aligned}$$

and thus we get

$$c_0^1 = \frac{1}{8\pi}\int_{-\infty}^{\infty} \tilde{c}_0^1\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta} \tag{3.35}$$

because of $c_0^1 = e^{-V^1(\phi^1)}$. This generalizes to n renormalization group steps as follows.

Theorem 7 (Flow of the Higgs potential for H = SL(2,$\mathbb{C}$))
Given the initial Higgs potential $V(\phi)$, define

$$b(\rho) = \int \sinh^2\eta d\eta\ \ e^{-V(d(\eta))}\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta}$$

for real ρ. Then after n RG-steps,

$$e^{-V^n(d(\eta))} = \frac{1}{8\pi}\int_{-\infty}^{\infty}\rho^2 d\rho\ \ [\mathfrak{z}\bar{\mathfrak{z}}]^{2n}\,[b(\rho)]^N\,\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta}$$

with $N = 3^n$.

The factor $[\mathfrak{z}\bar{\mathfrak{z}}]^{2n}$ shows the essential role of the fermions. Because of the recursion relation $\mathfrak{z}^1 = (\mathfrak{z})^2(\bar{\mathfrak{z}})^{-1}$ we have $(\mathfrak{z}^1)^2(\bar{\mathfrak{z}}^1)^2 = (\mathfrak{z})^2(\bar{\mathfrak{z}})^2$.

In this "trivial" SL(2,$\mathbb{C}$) model one can see that the integration of the "high frequency components" of the fermions does not affect the flow of the Higgs potential. Further, the integration of the Higgs field does not produce 4-fermion, 6-fermion or 8-fermion interactions. This fact can be understood because the fermionic variables $\in$ SL(2,$\mathbb{C}$) are transformed independently, so that the fermionic part of the action is invariant under SL(2,$\mathbb{C}$)-gauge transformations. On the other hand, the Higgs potential is not invariant under such gauge transformations.

3.3.5 Asymptotic behavior after many RG-steps

Theorem 6 gives the behavior of the coefficients of the effective Boltzmannian after n steps in the form of a convolution product of $2N+1$ factors $c_i^0[\Phi]$ with $N = 3^n$.

The central limit theorem gives the asymptotic behavior of such convolution products in the limit $N \to \infty$, provided the coefficients $c_j^0[\Phi]d\Phi$ are normalizable measures, or negatives of such. Assuming $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$, this is always the case for c_0^0 and for c_0^5, which enter the formula for the exponentiated Higgs potential $c_0^n = e^{-V^n}$. It is true for the other coefficients in all cases except for holonomy group GL(2,$\mathbb{C}$) and gauge group SU(2).

We need a central limit theorem for G-bicovariant functions on the noncompact groups H. The principle is derived as follows. Under appropriate conditions ("good functions f") the Fourier coefficient $\tilde{f}(\Xi)$ for normalized measures $fd\Phi$ has an isolated absolute maximum at a special representation Ξ_0. The prototype of such a result, which stands behind the central limit theorem on $\mathbb{R}$, is lemma 6 below. As a result, $\tilde{f}(\Xi)^N$, which is the Fourier coefficient of $f[\Phi]^{*N}$, is well approximated by a Gaussian, and therefore $f[\Phi]^{*N}$ can be accurately computed as inverse transform of a Gaussian. For the applicability of the central limit theorem the SU(2)-biinvariance is of great importance. Basically it leads to the simplification that one only needs the real component of the representation Ξ, that is ρ for SL(2,$\mathcal{C}$) and σ for $\mathbb{R}_+$ SU(2), as it is outlined in appendix C. In the following subsections we focus on the exponentiated Higgs potential.

Lemma 6 (Fourier Transform of normalized measures $\mathbb{R}$) *Let f be an integrable and square integrable positive definite function on the real line with $\int f(x)dx = 1$. Then the modulus of its Fourier transform $\tilde{f}(k) = \int e^{ikx} f(x)dx$ has a unique absolute maximum at $k = 0$ with $\tilde{f}(0) = 1$.*

PROOF: From the definition of the Fourier transform and from $\int f dx = 1$ it follows that

$$|\tilde{f}(k)|^2 = |\tilde{f}(0)|^2 - \int dx\ dy\ [1 - \cos k(x-y)] f(x) f(y).$$

The second term is negative semi-definite and for any $k \neq 0$ it can only vanish if f is either a δ-function or a series of δ-functions with supports whose distance is an integer multiple of $2\pi/k$. This is excluded by the assumption that f is square integrable. Therefore, the only absolute maximum is at $k = 0$. The assertion $\tilde{f}(0) = 1$ is a restatement of $\int f(x)dx = 1$. **q.e.d.**

The group $\mathbf{H} = \mathbf{SL(2,\mathbb{C})}$

We assume that $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$ and consider the SU(2)-biinvariant function

$$f(\Phi) = e^{-V(\Phi)} = f(d(\eta)),$$

which is positive definite. By our assumptions, the additive constant in V can be chosen so that $f(\Phi)d\Phi$ is a normalized measure on SL(2,$\mathbb{C}$).

The Fourier coefficient is

$$\begin{aligned}\tilde{f}(\rho) &= \int d\Phi f(\Phi) P^{\rho}(\Phi^{-1}) \\ &= \int \sinh^2 \eta d\eta \;\; f(d(\eta)) \frac{2\sin\frac{1}{2}\eta\rho}{\rho \sinh\eta}.\end{aligned}$$

It is real, obeys $\tilde{f}(\rho) = \tilde{f}(-\rho)$ and has an isolated absolute maximum 1 at $\rho = 0$. This follows from the fact that $\lim_{\rho\to 0} \frac{2\sin\frac{1}{2}\eta\rho}{\rho\sinh\eta} = 1$, while $|\frac{2\sin\frac{1}{2}\eta\rho}{\rho\eta}| \leq 1$ and $|\eta^{-1}\sinh\eta| > 1$ for $\eta \neq 0$.

If follows from the unique maximum property that

$$\tilde{f}(\rho) = e^{-\frac{b}{2}\rho^2} + ...$$

with

$$b = -\frac{\partial^2}{\partial\rho^2}\tilde{f}(\rho)\big|_{\rho=0} = -\int d\Phi f(\Phi) \frac{\partial^2}{\partial\rho^2} P^{\rho}(\eta)\big|_{\rho=0} \;. \tag{3.36}$$

Theorem 8 *The coefficient b in the power series expansion of $P^{\rho}(\eta)$ is*

$$b = \frac{1}{12}\int \sinh^2\eta d\eta \;\; f(d(\eta))\eta^2. \tag{3.37}$$

Proof: The power series expansion of $P^{\rho}(\eta)$ is

$$P^{\rho}(\eta) = \frac{2}{\rho\eta}(\frac{1}{2}\rho\eta - \frac{1}{3!}\frac{1}{8}\rho^3\eta^3 + ...) = 1 - \frac{1}{4!}\eta^2\rho^2 + ... \;\; .$$

With b defined as in eq.(3.36) we obtain the result. **q.e.d.**

Theorem 9 *With $B = Nb$, $N = 3n$,*

$$\tilde{f}(\rho)^N \sim e^{-\frac{B}{2}\rho^2} \quad \textit{and}$$

$$e^{-V^n(d(\eta))} = f^{*N}(\eta) \quad \sim \int \frac{\rho^2 d\rho}{16\pi} e^{-\frac{B}{2}\rho^2} \frac{2\sin\frac{1}{2}\eta\rho}{\rho\sinh\eta},$$

where $e^{-V^n} = (f)^{*N}$ is a n-fold convolution product, we get the following flow of the Higgs potential for large n

$$e^{-V^n(d(\eta))} \quad = \frac{1}{(2\pi B)^{\frac{3}{2}}} \frac{\eta}{4\sinh\eta} e^{-\frac{1}{8B}\eta^2}.$$

PROOF: The integral can be interpreted as a Fourier transformation. Using

$$g(\eta) = \int d\rho e^{-\frac{B}{2}\rho^2 + i\frac{\rho\eta}{2}} = \frac{1}{(2\pi B)^{\frac{1}{2}}} e^{-\frac{1}{8B}\eta^2}$$

one gets

$$\begin{aligned} f^{*N}(\eta) &= \big[2i\frac{2}{i}\big(\frac{\partial}{\partial\eta}g(\eta) - \frac{\partial}{\partial\eta}g(-\eta)\big)\big]\frac{1}{\sinh\eta}\frac{1}{16\pi} \\ &= \frac{1}{16\pi\sinh\eta}\big[-4\frac{2\eta}{8B}\frac{1}{(2\pi B)^{\frac{1}{2}}}e^{-\frac{1}{8B}\eta^2} - 4\frac{2\eta}{8B}\frac{1}{(2\pi B)^{\frac{1}{2}}}e^{-\frac{1}{8B}\eta^2}\big] \\ &= -\frac{1}{(2\pi B)^{\frac{3}{4}}}\frac{\eta}{4\sinh\eta}e^{-\frac{1}{8B}\eta^2}. \qquad \textbf{q.e.d.} \end{aligned}$$

The group $\mathbf{H} = \mathbb{R}_+\ \mathbf{SU(2)}$

We specify the representations which enter into the decomposition of SU(2)-biinvariant functions by $s = i\sigma$. SU(2)-biinvariant functions f of $ru \in \mathbb{R}_+$ SU(2) depend only on r, and the harmonic analysis amounts to a Mellin transformation (see appendix C.1) which in turn is related to a standard Fourier transformation

$$\begin{aligned} \tilde{f}(i\sigma) &= \int_0^\infty dr r^{i\sigma-1} f(r) \\ &= \int_{-\infty}^\infty dx e^{i\sigma x} f(e^x). \end{aligned}$$

Let f be positive definite and a good function. Then, by lemma 6, $|\tilde{f}|$ has a maximum at $\sigma = 0$ and

$$\tilde{f}(i\sigma) = e^{-\frac{b}{2}\sigma^2 + ia\sigma} + \dots$$

if f is normalized appropriately. The coefficients are

$$a = -i\frac{\partial}{\partial\sigma}f(i\sigma)\Big|_{\sigma=0} = \frac{\partial}{\partial s}f(s)\Big|_{s=0} = \int \frac{dr}{r}\ln r f r \tag{3.38}$$

$$b = -i\frac{\partial^2}{\partial\sigma^2}f(i\sigma)\Big|_{\sigma=0} = \frac{\partial^2}{\partial s^2}f(s)\Big|_{s=0} = \int \frac{dr}{r}\ln^2 r f r. \tag{3.39}$$

Theorem 10 *Consider the following convolution product on $\mathbb{R}_+$ SU(2)*

$$e^{-V^n} = c_0^{*N+1} * c_5^{*N}$$

with $N = \frac{1}{2}(3^n - 1)$, $c_i = c_i^0$, $c_0 = e^{-V}$ and $c_5 = |\det\Phi|^2 e^{-V}$. If we choose the constant in V so that $c_0 * c_5$ is a normalized measure on $\mathcal{H}$ and $\int d\phi e^{-V(\phi)} = \mathcal{N}$, then we get the following flow of the Higgs potential for large n

$$e^{-V^n} \sim \frac{\mathcal{N}}{(2\pi B)^{\frac{1}{2}}}\exp\big[-\frac{1}{2B}\ln^2(re^A)\big].$$

with

$$a' = \int \frac{dr}{r}\ln r a_0(r), \quad a = \int \frac{dr}{r}\ln r(a_0 * a_5)(r),$$

$$b' = \int \frac{dr}{r}\ln^2 r a_0(r), \quad b = \int \frac{dr}{r}\ln^2 r(a_0 * a_5)(r)$$

$$B = b' + Nb \quad \textit{and} \quad A = a' + Na.$$

PROOF: According to eq.(3.38)

$$\tilde{c}_0(i\sigma) = e^{-\frac{b'}{2}\sigma^2 + i\sigma a'}$$
$$\widetilde{(c_0 * c_5)}(i\sigma) = e^{-\frac{b}{2}\sigma^2 + i\sigma a}$$

It follows

$$\begin{aligned}
e^{-V^n(r)} &= f^N(r) = \int_{-\infty}^{\infty} d\sigma r^{i\sigma} \tilde{f}^N(i\sigma) = \int_{-\infty}^{\infty} d\sigma r^{i\sigma} \tilde{c}_0(i\sigma) \big(\widetilde{c_0 * c_5}\big)^N(i\sigma) \\
&= \int_{-\infty}^{\infty} d\sigma r^{i\sigma} e^{-\frac{b'}{2}\sigma^2 + i\sigma a'} \left(e^{-\frac{b}{2}\sigma^2 + i\sigma a}\right)^N = \int_{-\infty}^{\infty} d\sigma r^{i\sigma} e^{-\frac{B}{2}\sigma^2 + i\sigma A} \\
&\overset{x=\ln r}{=} \int_{-\infty}^{\infty} d\sigma e^{i\sigma x} e^{-\frac{B}{2}\sigma^2 + i\sigma A} = \int_{-\infty}^{\infty} d\sigma e^{-\frac{B}{2}\sigma^2 + i\sigma(A+x)} \\
&= \frac{1}{(2\pi B)^{\frac{1}{2}}} e^{-\frac{1}{2B}(x+A)^2} = \frac{1}{(2\pi B)^{\frac{1}{2}}} e^{-\frac{1}{2B}\ln^2(re^A)} \qquad \textbf{q.e.d.}
\end{aligned}$$

The other coefficients in the effective Boltzmannian can be treated in the same way, *provided* the coefficients which enter into the recursion relation are either positive or negative definite. This is true in all cases except for G = SU(2), H = GL(2,$\mathbb{C}$).

A and B are of order $\frac{1}{2}(3^n + 1)$. Their explicit value is determined by the starting Higgs potential V and is obtained by applying eqs.(3.38) and (3.39). A can be transformed to $\mathcal{O}(1)$ by a wave function renormalization. It is equivalent to modifying the block spin definition to

$$\phi^{n+1}(z, z+3a_n) = Z^{1/2}\phi^n(z, z+a_n)\epsilon\phi^n(z+2a_n, z+a_n)^T\epsilon^T\phi^n(z+2a_n, z+3a_n) \quad (3.40)$$

with a suitable n-independent constant $Z^{1/2}$. For $A = 0$: $V(ru) = V(\phi), \phi \mapsto Z^{\frac{1}{2}}\phi$ and rescaling is happening with every renormalization group step. There is a fixpoint for $A = 0$ and a flat Higgs potential.

The group H = GL(2,$\mathbb{C}$)

Here the aspects of the two previous cases are combined. The representations are labeled by $\Xi = (l, s = i\sigma, \rho)$ and the transform is

$$\tilde{f}(\Xi) = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{dr}{r} e^{il\theta} r^{i\sigma} \sinh^2\eta d\eta f(\zeta d(\eta)) \frac{2\sin\frac{1}{2}\rho\eta}{\rho\sinh\eta}, \quad (3.41)$$

which has its maximum at $l = 0$, $s = 0$ and $\rho = 0$. The dominance of $l = 0$ is the reason for the asymptotic behavior of $\exp(-V^n(\zeta d(\eta)))$ being independent of the phase ζ, $\phi = \zeta a$ $a \in$ SL(2,$\mathbb{C}$).

Corollary 5 *With A and B of order $N + 1 = \frac{1}{2}(3^n + 1)$ as defined in theorem 10 and $B' = Nb$ for $N = 3^n$ as in eq.(3.37) we get the following flow of the Higgs potential for* GL(2, $\mathbb{C}$)

$$e^{-V^n(\zeta d(\eta))} \sim \frac{\mathcal{N}}{(2\pi B')^{\frac{3}{2}}(2\pi B)^{\frac{1}{2}}} \exp\Big[-\frac{1}{8B'}\eta^2 - \frac{1}{2B}\ln^2(re^A)\Big]. \quad (3.42)$$

Again the constant A grows like N, and due to this fact A can be transformed to $\mathcal{O}(1)$ by a wave function renormalization. The exponential tends to a constant in the limit of large n (as for $\mathbb{R}_+$SU(2,$\mathbb{C}$)), when A stays bounded.

Remark: In the special case $\mathrm{G} = \mathrm{SU}(2), \mathrm{H} = \mathrm{GL}(2,\mathbb{C})$ we are not able to compute the asymptotic behavior of some coefficients in the fermionic action. The reason is that some of the coefficients are not measures or negatives of measures. This case is special in that G = SU(2) is not the maximal compact subgroup of H.

The group $\mathbb{R}_+$SL(2,$\mathbb{C}$) is treated in the same way, but here the quantum number l is absent. In this case SU(2) is the maximal compact subgroup.

Model with gauge group U(2)

Due to the fact that for H = GL(2,$\mathbb{C}$), SU(2) is not the maximal compact subgroup we wish to consider also the maximal compact subgroup G = U(2) of GL(2,$\mathbb{C}$). Locally, U(2) = SU(2) $\times$ U(1).
The results can be extended to the model with gauge group U(2) and H = GL(2,**C**). To do so, we have to investigate the consequences of an additional local U(1)-invariance. Under a U(1)-transformation,

$$\begin{aligned} \phi(z,z+1) &\mapsto e^{-i\vartheta(z)}\phi(z,z+1)e^{i\vartheta(z+1)}, \\ \psi(z) &\mapsto e^{-i\vartheta(z)}\psi(z)\ , \\ \bar\psi(z+1) &\mapsto e^{i\vartheta(z+1)}\bar\psi(z+1)\ . \end{aligned}$$

U(1)-biinvariance implies that the Higgs-potential $V(\zeta A)$ is independent of the phase of ζ.

The block spin (3.26) is not U(1)-covariant, and must be changed by a factor which involves the phase of $\det\phi^n$ to

$$\phi^{n+1}(z,z+3a_n) = \\ e^{i\arg\,\det\phi^n(z+a_n,z+2a_n)}\phi^n(z,z+a_n)\epsilon\phi^n(z+2a_n,z+a_n)^T\epsilon^T\phi^n(z+2a_n,z+3a_n)\ .$$

A parallel transporter that is also covariant under U(1)-gauge transformations leads to a modification of the factor $\det\phi$ that may appear in the coefficients

$$[\det\phi(z,z+1)] \mapsto e^{-i\arg\,\det\phi^n(z,z+1)}[\det\phi(z,z+1)].$$

It is appropriate to change also the definition of the four fermion interaction term k^n that was formerly given as $k^n(z,z+a_n) = \frac{1}{4}\bar\psi\epsilon\bar\psi(z)\times\psi\epsilon\psi(z+a_n)$ to

$$k^n(z,z+a_n) = \frac{1}{4}e^{i\arg\,\det\phi^n(z,z+a_n)}\bar\psi\epsilon\bar\psi(z)\times\psi\epsilon\psi(z+a_n).$$

When these changes are made, the master integration formulas (3.24) remain valid as they stand. Therefore, the recursion relations theorem 5 for the coefficients c_j^n remain literally valid, and also corollary 4 remains valid with the understanding that harmonic expansion of U(2)-bicovariant functions on GL(2,$\mathbb{C}$) enters.

The initial values (3.33) change as a consequence of the changed definition of k. In the formulas (3.33), $\det\phi$ and $\det\phi^*$ have to be replaced by $|\det\phi|$.

As a consequence, given that $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$, all the quantities $c_j^0[\phi]d\phi$ are normalizable measures, or negatives of normalizable measure. It follows from this in the same way as for the other models with this property that the model's RG-flow moves towards an infrared fixpoint. The asymptotic behavior of the Higgs potential is the same as in the G = SU(2) - H = GL(2,$\mathbb{C}$) model.

The behavior of the flow of the Higgs potential

To summarize the results of the calculations, we have seen explicitly that for gauge group SU(2) and holonomy group $\mathbb{R}_+$ SU(2) the Higgs potential becomes flat in the limit of large numbers of RG-steps n. It tends to a definite function for holonomy group SL(2,$\mathbb{C}$) with gauge group SU(2) and for holonomy group GL(2, $\mathbb{C}$) with gauge group U(2), as well as for gauge group SU(2) and holonomy group $\mathbb{R}_+$ SL(2, $\mathbb{C}$). In all cases there is a flow towards an infrared (IR) fixpoint, that is described by this functions.

There is an ultraviolet (UV-)fixpoint as well. It is a pure gauge theory with gauge group G. In other words, the parallel transporters are unitary $\phi(z, z+1) \in \mathrm{G} = \mathrm{SU}(2)$ or U(2). On this fixpoint the exponentiated Higgs potential $e^{-V(\phi)} = \delta_{\mathrm{G}}(\Phi)$ is a function whose support is concentrated on the gauge group G. The δ_{G}-function is defined by the formula

$$\int d\phi f(\phi)\delta_{\mathrm{G}}(\phi) = \int_{\mathrm{G}} du f(u), \quad \text{for} \quad u \in \mathrm{G}.$$

This model is equivalent to a free fermion theory because unitary parallel transporters can be transformed away in one dimension. But still the one-dimensional model shows the feature where if we are arbitrarily close to this fixpoint of pure gauge theory there is a flow away towards an IR-fixpoint after a large number of RG-steps. In our one-dimensional model this means only that our Higgs potential at the starting point has a nearly infinitely deep minimum and the curvature of that minimum decreases upon lowering the cutoff until we end up at the IR-fixpoint. Our one-dimensional model does not allow us to start a nontrivial flow, starting with a pure gauge theory, because the pure gauge theory is a fixpoint, we cannot leave. We need fluctuations to start the flow. Think of fluctuations in the length, e.g. if we consider holonomy group $\mathbb{R}_+$ SU(2), where the parallel transporter ϕ can be decomposed into $\phi = ru$, $u \in \mathrm{SU}(2)$, with length r; r can be chosen arbitrarily close to 1. This problem does not occur in more than one dimension. This will be explained in the following section.

Unfortunately, a typical double well Higgs potential will not come out of a pure gauge theory just by performing a RG-flow in one dimension. We hope and expect that the situation will change in more dimensions, provided there are sufficiently many fermions in the theory. One could think of an effective four-dimensional theory of the bilayered membrane type and suppose that a formal continuum limit of our discrete action has the form

$$S(\eta,\bar{\psi},\psi,\phi)=\int d^4z\big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\mathcal{V}(\eta)+\bar{\psi}(i\gamma^\mu\partial_\mu+g\begin{pmatrix}-\phi & 0\\ 0 & \phi\end{pmatrix})\psi\big) \qquad (3.43)$$

with $\mu=0,1,2,3$ and two components of ψ or $\bar{\psi}$. For this calculation we just consider nonunitary parallel transporters of the form $\phi=u_1d(\eta)u_2$ with a symmetric Higgs potential function $\mathcal{V}(\eta)=\mathcal{V}(-\eta)$ and coupling constant g. Perturbation calculations suggest that the integration of fermions pushes the Higgs potential towards a Mexican hat shape, while the integration of the bosons pushes it towards single well shape. This is because the contributions to renormalization of mass terms $m^2\phi\phi^*$ have opposite sign [30]. Suppose now that the extra dimension has finite extent d. Then the RG-flow will be stopped after a finite number of steps, when d equals one block lattice spacing. Assuming that the final Higgs potential has Mexican hat shape, this will lead to mass generation as described in section 4.

3.4 Beyond the one-dimensional model

In this section a qualitative discussion of the model in more dimensions will be presented. It gives another good reason to investigate nonunitary parallel transporters. We have seen that we are able to compute the RG-flow in one dimension exactly. Unfortunately the one-dimensional model does not show the feature that we could start with an ordinary gauge theory and obtain a flow towards a Higgs model with a double well Higgs potential, which would lead to spontaneous symmetry breaking. We now want to discuss what we can learn from our simple model about what has to be added if we want to involve the conventional four dimensions.

To illustrate the possible generalization we keep the formulation with Weyl spinors and assume a suitable discretization of the Weyl term.

$$\langle\partial_\mu\psi(z+a_\mu),\epsilon\sigma_\mu C\psi(z)\rangle=\langle\partial_\mu\psi(z+a_\mu),\epsilon\sigma_\mu\bar{\psi}(z)\rangle,$$

where μ enumerates the space-time direction, σ_μ are the Pauli matrices and C is the operator of complex conjugation. The block spin construction depends on these matrices and thus on the directions, but can be performed in the same way as in our one-dimensional example. A nontrivial problem appears in the order of the integrations. Again, we integrate out the fermions at two neighbored sites to obtain a Boltzmannian with a term of zeroeth order in the Fermi fields. But we have several choices to choose these two sites, and of course we have to sum over different paths as it is illustrated in figure 3.3. We have to keep in mind that locality should be preserved, what is a nontrivial question concerning the order of integration.

For simplicity we think of one extra direction, perpendicular to the one considered so far. Figure 3.2 illustrates the integration of fermions and bosons. Let us think again of gauge group SU(2) and holonomy group $\mathbb{R}_+$ SU(2) with the decomposition into a unitary part

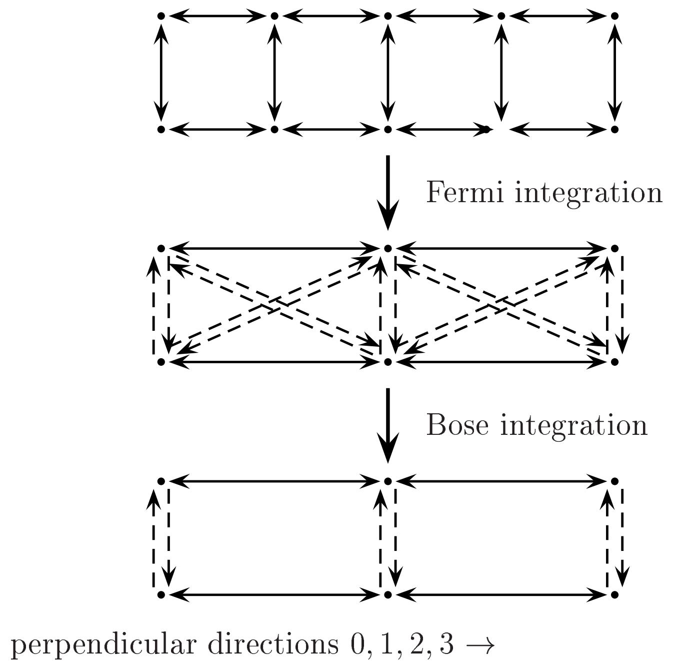


Figure 3.2: Performing a RG-step with an additional perpendicular direction

$u \in \mathrm{SU}(2)$ and a selfadjoint part that we already called length $r \in \mathbb{R}_+$. The initial fluctuations of this length needed in one dimension to start the flow away from the pure gauge theory UV-fixpoint towards the IR-fixpoint is not any longer necessary in two dimensions. Here a coarsening that involves unitary parallel transporters from both the extra and the perpendicular direction will produce block spins ϕ^1 in the linear span of the gauge group SU(2). The block spin is still gauge covariant. The mechanism is explained in figure 3.2. Therefore, we get the fluctuation length by a single RG-step. This is so because the sum of elements of SU(2) is not an element of SU(2) any more, see figure 3.3.

After such a RG-step we could of course integrate out the length to obtain again pure gauge theory. It may be possible or not to reduce to ordinary gauge theory by integrating out the length at any scale. But this depends on whether the resulting effective action stays local after this integration. We could be forced into a Higgs model if this locality is not preserved. This will happen if the mass that is determined by the curvature at the minima of the Higgs potential falls below the cutoff scale. We expect that there will be a domain of scales, where the effective theory needs a Higgs field for its locality.

The question of the order of integrating out fermions at lattice sites leads to the suggestion that it may be possible to obtain loop terms after some RG-steps. Figure 3.4 illustrates this. The dashed lines come up after integrating out the fermions at the sites that are labeled with 1. In the second step the fermions at 2 are integrated out. One obtains parallel transporters with target and source at the same point.

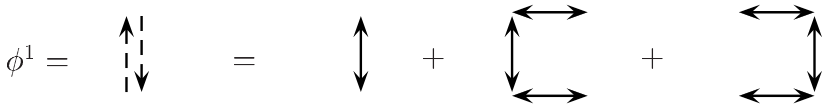


Figure 3.3: After performing the Fermi integration the block spin will be in the linear span.

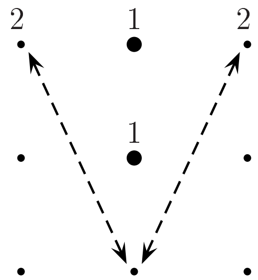


Figure 3.4: Possibility of loop terms.

Such a loop block spin involves the parallel transporter from the left to the right site (both labeled by 2), or vice versa. This must be obtained by some latter RG-transformations maybe in a further dimension. Admitting a triangular lattice loop-terms would appear by a single RG-step.

Some nonunitary link variables $\phi(z, z + a_{0\mu})$ have already been regarded by G. Mack in the framework of dielectric lattice gauge theories [37]. Here $\phi(z, z + a_{0\mu})$ takes its value in the linear space $\mathcal{G}$, which consists of real linear combinations of matrices in the gauge group G and can be polar decomposed via $\phi(z, z + a_{0\mu}) = u(z, z + a_{0\mu})\sigma_\mu(z)$ with an ordinary unitary gauge field $u(z, z + a_{0\mu}) \in$ G and a function of a dielectric field ε_μ. Here the integration is performed by involving the Lebesgue measure $d\phi$ on $\mathcal{G}$. It is possible to use the Haar measure on the group of invertible elements of $\mathcal{G}$ instead.

Chapter 4

Group theory of flavor mass splitting

We start again with the important biinvariance property of the Higgs potential. As in the last chapter we consider a gauge theory with nonunitary parallel transporters. Again we admit parallel transporters that are elements of a noncompact holonomy group H. It will be explained how a splitting of the masses of fermions of different flavors can arise through spontaneous symmetry breaking. The group theoretic preliminaries lead to patterns of symmetry breaking. To present the main feature we give a simple example that is compatible with our one-dimensional model of section 3.3.

4.1 Group theoretic preliminaries

The group theoretical basics which are important in explaining symmetry breaking in our models are introduced here. It should be mentioned that we do not use the conventional notations, in order to avoid confusion later on. It is assumed throughout that G (the unitary gauge group) is the maximal compact subgroup of H (the holonomy group). In the group theoretic literature, our H is called G, G is called K, and the content of theorem 11 below is known as the KAK-decomposition.

Some standard facts about noncompact semi simple Lie groups are recalled here.
Let H be a semisimple Lie group with Lie algebra $\mathfrak{h}$ over $\mathbb{R}$. The adjoint action of H on $\mathfrak{h}$ is denoted by Ad. For matrix groups, $\mathrm{Ad}(g)X = gXg^{-1}$. $\mathfrak{h}$ possesses a Cartan decomposition,

$$\mathfrak{h} = \mathfrak{g} + \mathfrak{p},$$

with an involutive automorphism σ of $\mathfrak{h}$, that is $\sigma^2 = \mathrm{id}$, such that $\sigma X = X$ for $X \in \mathfrak{g}$ and $\sigma X = -X$ for $X \in \mathfrak{p}$, and the Killing form of $\mathfrak{h}$ is negative definite on $\mathfrak{g}$ and positive definite on $\mathfrak{p}$. $\mathfrak{g}$ is a subalgebra, and if H has a finite center, then its maximal compact subgroup G has Lie algebra $\mathfrak{g}$ [47]. It follows that $\mathrm{Ad}(g)X \in \mathfrak{p}$ for all $X \in \mathfrak{p}, g \in$ G. Let $\mathfrak{a}$ be a maximal abelian Lie algebra in $\mathfrak{p}$, and let A be the corresponding subgroup of H. We call A the maximal noncompact abelian subgroup of H. $\mathfrak{a}$ is unique up to conjugation with elements of G. Corresponding with the Cartan decomposition of $\mathfrak{h}$, the elements h

of H admit a decomposition

$$h = \exp(X)g, \quad g \in \mathrm{G}, \quad X \in \mathfrak{p}. \tag{4.1}$$

In applications, we encounter reductive Lie groups $\mathrm{H} = \mathrm{Z}\ \mathrm{H}_s$, where H_s is a semi simple Lie group and Z a Lie group contained in the center of H. It will factor as $\mathrm{Z} = \mathrm{Z}_c\ \mathrm{Z}_n$ into a compact group Z_c and a noncompact group Z_n. If G is the maximally compact subgroup of H then $\mathrm{Z}_c \subset \mathrm{G}$. In our application, Z_n will be $\mathbb{R}_*$ or $\mathbb{R}_+$.
We will make use of the following theorem.

Theorem 11 [28]
Let $\mathrm{H} = \mathrm{ZH}_s$ *be a reductive Lie group with semi simple factor* H_s, G *the maximal compact subgroup of* H, A *the maximal noncompact abelian subgroup of* H_s *as introduced above, and* Z_n *the noncompact factor of* Z.

i) Every element h *of* H *admits a decomposition*

$$h = \rho g_1 a g_2, \quad g_1, g_2 \in \mathrm{G}, a \in \mathrm{A}, \rho \in \mathrm{Z}_n$$

ii) Every irreducible representation τ *of* H *of finite dimension* r *admits a scalar product, such that* $\tau(\eta)$ *are selfadjoint for* $\eta \in \mathfrak{a}$. *Therefore they can be simultaneously diagonalized, hence*

$$\tau(a) = \tau(\exp(\eta)) = \begin{pmatrix} e^{\eta_1} & & 0 \\ & \ddots & \\ 0 & & e^{\eta_r} \end{pmatrix}.$$

iii) If H *is a group of* $n \times n$*-matrices which is its own irreducible representation then the decomposition* $h = \rho g_1 a g_2$ *with diagonal matrices* a *holds for a group* H' *of* $n \times n$*-matrices which is isomorphic to* H.

Let us now recall the definition of the Weyl group W(A) of the pair (H_s,A):
Let M be the centralizer of $\mathfrak{a}$ in G that is $\mathrm{M} = \{g \in \mathrm{G}, \mathrm{Ad}(g)|_{\mathfrak{a}} = \mathbb{I}\}$.
Let M^* be the normalizer of $\mathfrak{a}$ in G that is $\mathrm{M}^* = \{g \in \mathrm{G}, \mathrm{Ad}(g)\mathfrak{a} \subset \mathfrak{a}\}$.
Then the *Weyl group* is defined as

$$\mathrm{W(A)} = \mathrm{M}^*/\mathrm{M}.$$

It is a finite group and the map $\mathrm{W(A)} \mapsto \mathrm{GL}(\mathfrak{a})$ given by $w(X) = \mathrm{Ad}(m^*)X$ for $X \in \mathfrak{a}$, $m^* \in w \in \mathrm{M}^*/\mathrm{M}$ is injective.

Corollary 6 (Weyl group as symmetry group of biinvariant functions)
Suppose that the function V *on* H *possesses the* G*-biinvariance property* $V(g_1 h g_2) = V(h)$ *for all* $g_1, g_2 \in \mathrm{G}$, $h \in \mathrm{H}$. *Then there exists a function* $\mathcal{V}$ *on* $\mathrm{Z}_n \times \mathfrak{a}$ *such that*

$$V(h) = V(\rho g_1 \exp(\eta) g_2) = \mathcal{V}(\rho, \eta) \quad \textit{for all } g_1, g_2 \in \mathrm{G}, \eta \in \mathfrak{a}, \rho \in \mathrm{Z}_n$$

and

$$\mathcal{V}(\rho, w(\eta)) = \mathcal{V}(\rho, \eta) \quad \textit{for all } w \in \mathrm{W(A)}.$$

PROOF: The first part is immediate from theorem 11. Let $m \in \mathrm{M}^*$ and $w = m^* \bmod \mathrm{M}$. It follows from the biinvariance property that $V(\rho m \exp(\eta) m^{-1}) = V(\rho \exp(\eta)) = \mathcal{V}(\rho, \eta)$. On the other hand $V(\rho m \exp(\eta) m^{-1}) = V(\rho \exp(w(\eta))) = \mathcal{V}(\rho, w(\eta))$. **q.e.d.**

Remark: One could include the Lie algebra of Z_n in $\mathfrak{a}$, thereby avoiding the factors of ρ. This is not convenient for later applications, however, because the factors ρ will play a special role. Moreover, the special role of the point $\eta = 0$ would get lost.

4.2 Symmetry breaking patterns

Based on the group theoretical background we discuss the possibilities of spontaneous symmetry breaking. A simple example will demonstrate the main feature.

We define U to be the centralizer of $\hat{\eta}$ in G that is it consists of $g \in \mathrm{G}$ with $\mathrm{Ad}(g)\hat{\eta} = \hat{\eta}$. It will be called the *unbroken symmetry group*. Let $\mathrm{W}_{\hat{\eta}}$ be the subgroup of elements $w \in \mathrm{W(A)}$ which leave $\hat{\eta}$ invariant, $w(\hat{\eta}) = \hat{\eta}$. It holds

$$\mathrm{W}_{\hat{\eta}} = (\mathrm{U} \cap \mathrm{M}^*)/\mathrm{M}. \tag{4.2}$$

Let us call an element $\hat{\eta}$ of $\mathfrak{a}$ *generic* if the centralizer U of $\hat{\eta}$ in G is M, that is if $\mathrm{Ad}(g)\hat{\eta} = \hat{\eta}$ for $g \in \mathrm{G}$ implies $\mathrm{Ad}(g)\eta = \eta$ for all $\eta \in \mathfrak{a}$. Nongeneric $\hat{\eta}$ form lower dimensional subalgebras of $\mathfrak{a}$. The point $\hat{\eta} = 0$ is nongeneric.

Corollary 7 (Possibility of complete symmetry breaking)

i) If $\mathfrak{a}$ is a completely noncompact Cartan subalgebra of $\mathfrak{h}$ then $\mathfrak{m}$, the Lie algebra of M*, is trivial.* M *is a discrete group.*

ii) If $\hat{\eta}$ is generic, then $\mathrm{U} = \mathrm{M}$ *and therefore $\mathrm{W}_{\hat{\eta}}$ is trivial.*

PROOF: $\mathfrak{m}$ is, of course, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. If $\mathfrak{a}$ is a Cartan subalgebra, then no element of $\mathfrak{g}$ can be in its centralizer, because otherwise the Cartan subalgebra could be enlarged. Therefore $\mathfrak{m}$ is trivial. ii) is the definition of generic $\hat{\eta}$ and because of (4.2) $\mathrm{W}_{\hat{\eta}}$ is trivial. **q.e.d.**

Since M has in general several connected components, it will be a nontrivial discrete group if $\mathfrak{m}$ is trivial.

Example to corollary 7

Let H = SL(2,$\mathbb{R}$). It possesses a completely noncompact Cartan subgroup consisting of matrices $a = \exp(\eta) = \begin{pmatrix} e^{\eta_1} & 0 \\ 0 & e^{-\eta_1} \end{pmatrix}$. The Weyl group has two elements $(1, w)$ and $w(\eta) = -\eta$. Since $\mathfrak{a}$ is one-dimensional, its only lower dimensional subalgebra is 0. Thus if $\hat{\eta} \neq 0$ is a minimum of $\mathcal{V}$, then the Lie algebra $\mathfrak{u}$ of the unbroken symmetry group U is trivial, and the continuous symmetry is broken completely.

4.2.1 Physical application

In the following, as in the last chapter we assumed that our Higgs fields are parallel transporters along links in extra directions. We denote them by ϕ. They are elements of a typically noncompact holonomy group H, $\phi(x, y) \in$ H and transform under gauge transformation according to

$$\phi(x, y) \mapsto u(x)\phi(x, y)u(y)^{-1},$$

where $u(x)$ and $u(y)$ are elements of the gauge group G $\subset$ H. Only the unitary gauge group G is a local symmetry. As already discussed in the previous chapter, matter fields form representation spaces of the holonomy group H and lattice gauge fields, on which we focus in this work, take their values in this holonomy group. In section 3.2.3 we saw that local G-biinvariance leads to the essential biinvariance property of the Higgs potential $V(\phi)$

$$V(\phi) = V(u_1\phi u_2), \quad \text{for all } u_1, u_2 \in \text{ G}, \phi \in \text{ H}.$$

Moreover, it follows from Corollary 6 that the Higgs potential can be written in the form

$$V(\phi) = V(\rho g_1 e^{\eta} g_2) = \mathcal{V}(\rho, \eta),$$

with $\eta \in \mathfrak{a}$ and $\rho \in \mathrm{Z}_n$, and $\mathcal{V}$ is invariant under elements w of the Weyl group W(A)

$$\mathcal{V}(\rho, w(\eta)) = \mathcal{V}(\rho, \eta), \quad \text{for } w \in \text{W(A)}.$$

This symmetry under the discrete group W = W(A) arises from local G-biinvariance. The minima $\hat{\eta}$ of $\mathcal{V}$ form orbits under W. Therefore we may distinguish between two major cases.

i) The W-symmetry is spontaneously broken, either completely or partially.
This is the case if the orbit(s) contain(s) more than one point. Then minima $\hat{\eta}$ are situated on a nontrivial orbit of the Weyl group W. Let us assume that there is only one such orbit. The symmetry is completely broken if $w(\hat{\eta}) \neq \hat{\eta}$ implies that w is the identity. Otherwise there is an unbroken subgroup $\mathrm{W}_{\hat{\eta}}$ of W(A) which consists of $w \in$ W(A) such that $w(\hat{\eta}) = \hat{\eta}$. In the real world fermions of different flavor show the effect of mass splitting, thus the W-symmetry must be spontaneously broken in this application.

i) The W-symmetry is not spontaneously broken.
In this trivial case we have only one minimum and thus $w(\hat{\eta}) = \hat{\eta}$ for all $w \in$ W(A). In particular this happens when the only minimum occurs at $\hat{\eta} = 0$. In our one-dimensional model examined in the previous chapter we had only one minimum and thus no symmetry breaking. (In the standard model fermions of different color have the same mass.)

When spontaneous symmetry breaking occurs, an exponential mass hierarchy occurs naturally. This is suggestive in view of how the big quotients of the masses in the standard model appear. To bring out the principles as clearly as possible, let us look at a simplified model.

4.2.2 An example for flavor mass splitting

Let us suppose we would get a Higgs potential $\mathcal{V}(\eta)$ with nontrivial minima $\hat{\eta} \neq 0$ in our one-dimensional model described in the previous chapter with gauge group SU(2) and noncompact holonomy group H = SL(2,$\mathbb{C}$). The Lie algebra $\mathfrak{a}$ is one-dimensional and its elements may therefore be parametrized by real numbers $\eta \in \mathbb{R}$. The Weyl group W(A) has one nontrivial element w which takes $\eta \mapsto -\eta$. Therefore, $\phi \in$ SL(2,$\mathbb{C}$) may be decomposed as $\phi = u_1 d(\eta) u_2$ with $u_1, u_2 \in$ SU(2), $\eta \in \mathbb{R}$,

$$d(\eta) = \begin{pmatrix} e^{-\frac{\eta}{2}} & 0 \\ 0 & e^{\frac{\eta}{2}} \end{pmatrix} \text{ and } d(w(\eta)) = \begin{pmatrix} e^{\frac{\eta}{2}} & 0 \\ 0 & e^{-\frac{\eta}{2}} \end{pmatrix}.$$

Assuming that the orbit of the minima under the discrete group W $= \{1, w\}$ is nontrivial, the minima of $\mathcal{V}(\eta)$ are at $\pm\hat{\eta} \neq 0$. Then at the minima of $V(\phi)$, $\phi = u_1 d(\hat{\eta}) u_2$ and the interaction term with a real coefficient $\mathfrak{z}$ becomes

$$\mathfrak{z}\overline{\psi}\phi\psi = \sum_{\alpha=1}^{2} \overline{\psi}'_\alpha m_\alpha \psi'_\alpha \tag{4.3}$$

with $\overline{\psi}' = \overline{\psi} u_1$, $\psi' = u_2 \psi$, and masses

$$m_1 = \mathfrak{z} e^{-\hat{\eta}/2} \ , \quad m_2 = \mathfrak{z} e^{\hat{\eta}/2} \ .$$

We see that we have fermions of two flavors with different masses.

4.3 Adjoint vs. nonadjoint Higgs field

A nonunitary paralleltransporter admits a polar decomposition into a unitary and a selfadjoint part. The unitary part transforms like a conventional lattice gauge field. While the selfadjoint part transforms like a Higgs field, one might expect that the Higgs field transforms according to the adjoint representation of G. We will show that this does not need to be the case. When it is the case, the Cartan subgroup of G will always remain unbroken. But there also exist groups H such that the G-symmetry can be completely broken.

To see this, let us examine consequences of the biinvariance property of the Higgs potential V from a different point of view which makes closer contact with conventional wisdom.

It is known fact [47] which follows from the Cartan decomposition (4.1) that elements $\Phi \in$ H can be decomposed as $\Phi = \rho e^X g, g \in$ G, $X \in \mathfrak{p}$ and $\rho \in$ Z$_n$. Let us decompose the paralleltransporter in the extra direction and let $\hat{\mu}$ be the lattice vector in this extra direction.

$$\phi(x, y = x + \hat{\mu}) = \rho e^{\Xi_\mu(x)} U(x, y)$$

with $U(x,y) \in \mathrm{G}$, $\Xi_\mu(x) \in \mathfrak{p}$ and $\rho \in \mathrm{Z}_n$. Gauge transformations act according to

$$\begin{aligned} U(x,y) &\mapsto u(x)U(x,y)u(y)^{-1} \\ \Xi_\mu(x) &\mapsto u(x)\Xi_\mu(x)u(x)^{-1}. \end{aligned}$$

The function $\mathcal{V}$ on $\mathfrak{a}$ may be extended to a function on $\mathfrak{p} \supset \mathfrak{a}$ which is invariant under the action $X \to \mathrm{Ad}(g)X$ of G on $\mathfrak{p}$. Let us tentatively identify Ξ_μ as a Higgs field and $U(x,y)$ as a conventional lattice gauge field. Using the biinvariance property, the Higgs potential in the action can be written as

$$V(\phi(x, x+\hat{\mu})) = \mathcal{V}(\rho, \Xi_\mu(x)).$$

Suppose $\hat{\Xi}_\mu$ is the minimum of $\mathcal{V}$, which determines the vacuum expectation value of the Higgs field Ξ_μ. Then the unbroken subgroup U$'$ of G consisits of those $g \in \mathrm{G}$, which obey

$$g\hat{\Xi}_\mu g^{-1} = \hat{\Xi}_\mu.$$

In other words, U$'$ is the centralizer of $\hat{\Xi}_\mu$ in G. Since elements of $\mathfrak{p}$ are conjugate in G to elements of $\mathfrak{a}$, there is a $g_1 \in \mathrm{G}$ such that

$$\hat{\Xi}_\mu = g_1 \hat{\eta} g_1^{-1}$$

for some $\hat{\eta} \in \mathfrak{a}$, which is unique up to the action of the Weyl group. It follows that $\mathrm{U}' = g_1 \mathrm{U} g_1^{-1}$, where U is the unbroken symmetry group defined in section 4.2. Therefore we have:

Proposition 12 *Retaining the notation introduced above, the unbroken symmetry group* U$'$ *is isomorphic to the group* U *defined in section 4.2.*

In any case, the Higgs field takes its values in $\mathfrak{p}$ which is a linear representation space for G via the tranformation $X \to \mathrm{Ad}(g)X, X \in \mathfrak{p}, g \in \mathrm{G}$. It depends on the group H whether this representation is isomorphic to the adjoint representation of G or not.

It is true if $\mathfrak{h} = \mathrm{Lie\ H}$ can be interpreted as a complex Lie algebra of half the dimension, because in this case $\mathfrak{p} = i\mathfrak{g}$. The groups H = SL(n,$\mathbb{C}$) belong to this class. However, it is not true in general. For H = SL(n,$\mathbb{R}$) $\mathfrak{g}$ consists of antisymmetric real $n \times n$-matrices; it has dimension $n(n-1)/2$. In contrast, $\mathfrak{p}$ consists of traceless symmetric real matrices; it has dimension $n(n+1)/2 - 1$. These representation spaces have different dimensions and they are therefore not isomorphic for $n > 1$.

Examples

- The Lorentz group H = SL(2,$\mathbb{C}$), with its maximal compact subgroup G = SU(2) consists of rotations and $\mathfrak{g}$ transforms according to the three-dimensional adjoint representation of G, while $\mathfrak{p}$ consists of infinitesimal boosts. They are also specified by 3-vectors.

- The maximal compact subgroup of H = SL(2,$\mathbb{R}$) is G = U(1) whose irreducible representations are labelled by $l = 0, \pm 1, \pm 2, ...$ The adjoint representation of U(1) is the trivial representation $l = 0$, whereas $\mathfrak{p}$ carries a two-dimensional real representation of U(1) which decomposes into $-1 \oplus 1$ in the complex domain.

From the point of view of symmetry breaking, complex Lie algebras $\mathfrak{h}$ and normal real forms of complex Lie algebras are extreme opposites. The unbroken subgroup U of G is the stability group of the minima of $\mathcal{V}$, regarded as a function on $\mathfrak{p}$. If $\mathfrak{g}$ is a complex Lie algebra, then U always includes the Cartan subgroup of G. In contrast, if $\mathfrak{g}$ is a normal form then $\mathfrak{a}$ is a completely noncompact Cartan algebra and corollary 7 tells us that the symmetry may be completely broken.

4.4 Classification of the possible unbroken subgroups

The following result gives us the possible remaining symmetry groups U. It involves the notion of a parabolic subgroup of H. Any noncompact semi simple Lie group H with finite center admits an Iwasawa decomposition $\mathrm{H} = \mathrm{GAN}^+$, where N^+ has nilpotent Lie algebra $\mathfrak{n}^+$. If M is the centralizer of A in G as before, then $\mathrm{P} = \mathrm{MAN}^+$ is known as the minimal prabolic subgroup. Any subgroup of H, which contains P or a subgroup conjugate to it is called a parabolic subgroup. Parabolic subgroups are classified in the mathematical literature [48].

Proposition 13 (G. Mack, M. de Riese [44])
Let in addition to the above mentioned assumptions H *be a simple Lie group. For any* $\eta \in \mathfrak{a}$*, the centralizer of* η *in* G *is* $\mathrm{M}_\eta = \mathrm{P}_\Theta \cap \mathrm{G}$*, where* P_Θ *is some parabolic subgroup of* H *if* $\eta \neq 0$ *and* $\mathrm{M}_0 = \mathrm{G}$*. Traversing through* $\mathfrak{a}$ *all conjugacy classes of parabolic subgroups are obtained in this way. In other words* $\{\mathrm{M}_\eta : \eta \in \mathfrak{a}, \eta \neq 0\}$ *is the set of all parabolic subgroups of* H *intersected with* G*, plus* G *itself.*

If the minimum of the Higgs potential $\mathcal{V}$ takes its value at $\hat{\eta}$, then the unbroken symmetry group U is $\mathrm{M}_{\hat{\eta}}$ as classified in proposition 13, because U was defined as a centralizer in section 4.2.

Proof: (The definitions and notations of [48] will be used.)
Fix $\eta \in \mathfrak{a}$. M_η is a Lie subgroup of G with identity component M'_η and Lie algebra $\mathfrak{m}_\eta$. Let $\mathfrak{a}_\eta$ be the centralizer of $\mathfrak{m}_\eta$ in $\mathfrak{a}$ or, alternatively, the fixed-point set of $\mathrm{Ad}(m)$ for all $m \in \mathrm{M}_\eta$. Let $\mathfrak{a}^\perp$ be the orthogonal complement of $\mathfrak{a}_\eta$ in $\mathfrak{a}$ with respect to the Killing form.

Let Σ^+ be the set of positive restricted roots and $\Sigma^\pm_\eta$ the set of positive resp. negative restricted roots λ with $\lambda|_{\mathfrak{a}_\eta} = 0$. Define $\mathfrak{n}^\pm_\eta := \sum_{\lambda \in \Sigma^\pm_\eta} \mathfrak{g}_\lambda$ and as usual $\mathfrak{n}^+ := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda$ is the root space to the restricted root λ.

With $\mathfrak{m}' := \mathfrak{m} \oplus \mathfrak{a}^\perp \oplus \mathfrak{n}^+_\eta \oplus \mathfrak{n}^-_\eta$ the centralizer of $\mathfrak{a}_\eta$ in $\mathfrak{h}$ obviously is $\mathfrak{m}' \oplus \mathfrak{a}_\eta$. Therefore it holds on the one hand $\mathfrak{m}_\eta = \mathfrak{m}' \cap \mathfrak{g}$ and on the other $\mathfrak{p}_\eta := \mathfrak{m}' \oplus \mathfrak{a}_\eta + \mathfrak{n}^+$ is a parabolic subalgebra of $\mathfrak{h}$, because it contains the minimal parabolic subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$.

Let A, N^+, N_η^+ and N_η^- be the subgroups of H corresponding to $\mathfrak{a}$, $\mathfrak{n}^+$, $\mathfrak{n}_\eta^+$ and $\mathfrak{n}_\eta^-$, and let L_η be the centralizer of $\mathfrak{a}_\eta$ in H. L_η contains M, A, N_η^+ and N_η^- as subgroups. Now, $P_\eta := L_\eta\, N^+$ is a parabolic subgroup, because it contains the minimal parabolic subgroup MAN^+. The compact subgroup of P_η lies in L_η, because the Killing form vanishes on $\mathfrak{n}^+$, and therefore is equal to M_η.

In order to show that all conjugacy classes of parabolic subgroups are obtained in this way note that according to [48] all these classes are obtained by choosing subsets Θ of the set of simple restricted roots and taking the space generated by the duals of the elements of Θ in $\mathfrak{a}$ as $\mathfrak{a}^\perp$. Let $\mathfrak{a}_\eta$ be the orthogonal complement of $\mathfrak{a}^\perp$. Now choose η to lie in $\mathfrak{a}_\eta$ in such a way that $\lambda(\eta) \neq 0$ for all roots not in the root system spanned by Θ. The procedure above leads now to $P_\eta = P_\Theta$. **q.e.d.**

The tables on the following pages list the Lie algebras of the corresponding Lie groups. Table 4.1 assigns the Lie algebra $\mathfrak{m}$ of the centralizer of $\mathfrak{a}$ in G to the Lie algebra $\mathfrak{h}$ of a given simple Lie group. For some examples table 4.4 lists the Lie algebras of all possible unbroken symmetry groups of simple groups H with Lie algebra $\mathfrak{h}$ according to proposition 13. $\mathfrak{m}$ is the Lie algebra of the minimal and $\mathfrak{g}$ the Lie algebra of the maximal possible unbroken symmetry group. The intermediate Lie algebras $\mathfrak{m}_{\hat{\eta}}$ are listed. Information about the latter are obtained by using the software package [43] of M. de Riese that can also help to treat other simple Lie groups.

$\mathfrak{h}$	$\mathfrak{m}$	$\mathfrak{a}$	$\mathfrak{g}$	
$\mathfrak{sl}(l+1,\mathbb{R})$	trivial	$\mathbb{R}^l$	$\mathfrak{so}(l+1)$	
$\mathfrak{sl}(l+1,\mathbb{C})$	$\mathfrak{u}(1)^l$	$\mathbb{R}^l$	$\mathfrak{su}(l+1)$	
$\mathfrak{su}^*(l+1)$	$\bigoplus_{\frac{l+1}{2}} \mathfrak{su}(2)$	$\mathbb{R}^{\frac{l-1}{2}}$	$\mathfrak{sp}(\frac{1}{2}(l+1))$	$l-1 \in 2\mathbb{N}$ $l \geq 3$
$\mathfrak{su}(m,n)$	$\mathfrak{su}(m-n) \oplus \bigoplus_n \mathfrak{u}(1)$	$\mathbb{R}^n$	$\mathfrak{su}(m) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$	$m \geq 2, n \geq 1,$ $m > n$
$\mathfrak{so}(m,n)$	$\mathfrak{so}(m-n)$	$\mathbb{R}^n$	$\mathfrak{so}(m) \oplus \mathfrak{so}(n)$	$m \geq n$
$\mathfrak{sp}(l,\mathbb{R})$	trivial	$\mathbb{R}^l$	$\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	
$\mathfrak{sp}(l,\mathbb{C})$	$\mathfrak{u}(1)^l$	$\mathbb{R}^l$	$\mathfrak{sp}(l)$	
$\mathfrak{sp}(m,n)$	$\mathfrak{sp}(m-n) \oplus \bigoplus_n \mathfrak{su}(2)$	$\mathbb{R}^n$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(n)$	$m \geq n \geq 1$
$\mathfrak{so}^*(2l)$	$\bigoplus_{\frac{l}{2}} \mathfrak{su}(2)$	$\mathbb{R}^{\frac{l}{2}}$	$\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	$l \in 2\mathbb{N}$ $l \geq 4$
$\mathfrak{so}^*(2l)$	$\bigoplus_{\frac{l-1}{2}} \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathbb{R}^{\frac{l-1}{2}}$	$\mathfrak{su}(l) \oplus \mathfrak{u}(1)$	$l-1 \in 2\mathbb{N}$ $l \geq 4$
$\mathfrak{e}_{6(6)}$	trivial	$\mathbb{R}^6$	$\mathfrak{sp}(4)$	
$\mathfrak{e}_{6(2)}$	$\mathfrak{u}(1) \oplus \mathfrak{u}(1)$	$\mathbb{R}^4$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(4) \oplus \mathfrak{u}(1)$	$\mathbb{R}^2$	$\mathfrak{so}(10) \oplus \mathfrak{u}(1)$	
$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}(8)$	$\mathbb{R}^2$	$\mathfrak{f}_4$	
$\mathfrak{e}_{7(7)}$	trivial	$\mathbb{R}^7$	$\mathfrak{su}(8)$	
$\mathfrak{e}_{7(-5)}$	$\bigoplus_3 \mathfrak{su}(2)$	$\mathbb{R}^4$	$\mathfrak{su}(12) \oplus \mathfrak{su}(2)$	
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(8)$	$\mathbb{R}^3$	$\mathfrak{e}_6 \oplus \mathfrak{u}(1)$	
$\mathfrak{e}_{8(8)}$	trivial	$\mathbb{R}^8$	$\mathfrak{so}(16)$	
$\mathfrak{e}_{8(-26)}$	$\mathfrak{so}(8)$	$\mathbb{R}^4$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	
$\mathfrak{f}_{4(4)}$	trivial	$\mathbb{R}^4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	
$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(7)$	$\mathbb{R}$	$\mathfrak{so}(9)$	
$\mathfrak{g}_{2(2)}$	trivial	$\mathbb{R}^2$	$\mathfrak{su}(2) \times \mathfrak{su}(2)$	

Table 4.1: Lie algebra of the centralizer M and the maximal compact subgroup G of H

$\mathfrak{h}$	$\mathfrak{m}$	$\mathfrak{g}$	$\mathfrak{m}_{\hat\eta}$ for $\hat\eta \neq 0$ and $\hat\eta$ nongeneric
$\mathfrak{su}(m,3)$	$\mathfrak{su}(m-3)\oplus 3\mathfrak{u}(1)$	$\mathfrak{su}(m)\oplus\mathfrak{su}(3)\oplus\mathfrak{u}(1)$	$\mathfrak{su}(m-3)\oplus\mathfrak{su}(2)\oplus 2\mathfrak{u}(1)$ $\mathfrak{su}(m-2)\oplus\mathfrak{su}(3)\oplus 2\mathfrak{u}(1)$ $\mathfrak{su}(m-2)\oplus 3\mathfrak{u}(1)$ $\mathfrak{su}(m-2)\oplus\mathfrak{su}(2)\oplus 2\mathfrak{u}(1)$ $\mathfrak{su}(m-1)\oplus\mathfrak{su}(2)\oplus 2\mathfrak{u}(1)$
$\mathfrak{so}(m,2)$	$\mathfrak{so}(m-2)$	$\mathfrak{so}(m)\oplus\mathfrak{u}(1)$	$\mathfrak{so}(m-2)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-1)$
$\mathfrak{so}(m,3)$	$\mathfrak{so}(m-3)$	$\mathfrak{so}(m)\oplus\mathfrak{su}(3)$	$\mathfrak{so}(m-3)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus\mathfrak{so}(3)$ $\mathfrak{so}(m-2)$ $\mathfrak{so}(m-2)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-1)\oplus\mathfrak{u}(1)$
$\mathfrak{so}(m,4)$	$\mathfrak{so}(m-4)$	$\mathfrak{so}(m)\oplus\mathfrak{su}(4)$	$\mathfrak{so}(m-4)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-4)\oplus\mathfrak{so}(3)$ $\mathfrak{so}(m-4)\oplus\mathfrak{so}(4)$ $\mathfrak{so}(m-3)$ $\mathfrak{so}(m-3)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus 2\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus 2\mathfrak{so}(3)$ $\mathfrak{so}(m-2)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-2)\oplus 2\mathfrak{u}(1)$ $\mathfrak{so}(m-1)\oplus 2\mathfrak{so}(3)$
$\mathfrak{so}(m,5)$	$\mathfrak{so}(m-5)$	$\mathfrak{so}(m)\oplus\mathfrak{su}(5)$	$\mathfrak{so}(m-5)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-5)\oplus 2\mathfrak{u}(1)$ $\mathfrak{so}(m-5)\oplus\mathfrak{so}(3)$ $\mathfrak{so}(m-5)\oplus\mathfrak{so}(4)$ $\mathfrak{so}(m-5)\oplus\mathfrak{so}(5)$ $\mathfrak{so}(m-5)\oplus\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-4)$ $\mathfrak{so}(m-4)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-4)\oplus 2\mathfrak{u}(1)$ $\mathfrak{so}(m-4)\oplus\mathfrak{so}(3)$ $\mathfrak{so}(m-3)$ $\mathfrak{so}(m-3)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus 2\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ $\mathfrak{so}(m-3)\oplus\mathfrak{so}(5)$ $\mathfrak{so}(m-3)\oplus\mathfrak{so}(5)\oplus\mathfrak{u}(1)$

$\mathfrak{h}$	$\mathfrak{m}$	$\mathfrak{g}$	$\mathfrak{m}_{\hat{\eta}}$ for $\hat{\eta} \neq 0$ and $\hat{\eta}$ nongeneric
$\mathfrak{su}^*(6)$	$3\mathfrak{su}(2)$	$\mathfrak{sp}(3)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$
$\mathfrak{su}^*(8)$	$4\mathfrak{su}(2)$	$\mathfrak{sp}(4)$	$2\mathfrak{sp}(2)$ $\mathfrak{sp}(2) \oplus \mathfrak{sp}(3)$ $3\mathfrak{sp}(2)$
$\mathfrak{su}(m,2)$	$\mathfrak{su}(m-2) \oplus 2\mathfrak{u}(1)$	$\mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(m-2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $\mathfrak{su}(m-2) \oplus 3\mathfrak{u}(1)$ $\mathfrak{su}(m-1) \oplus 2\mathfrak{u}(1)$
$\mathfrak{so}^*(8)$	$2\mathfrak{su}(2)$	$\mathfrak{su}(4) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $\mathfrak{sp}(2)$
$\mathfrak{so}^*(12)$	$3\mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{u}(1)$	$3\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ $\mathfrak{sp}(3)$
$\mathfrak{so}^*(16)$	$4\mathfrak{su}(2)$	$\mathfrak{su}(8) \oplus \mathfrak{u}(1)$	$4\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $2\mathfrak{su}(3)$ $2\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ $\mathfrak{su}(5)$ $\mathfrak{su}(5) \oplus \mathfrak{u}(1)$ $\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ $2\mathfrak{sp}(2)$ $\mathfrak{sp}(2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$ $\mathfrak{sp}(3) \oplus \mathfrak{u}(1)$ $\mathfrak{sp}(4)$
$\mathfrak{e}_{8(-26)}$	$\mathfrak{so}(8)$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(7) \oplus \mathfrak{so}(5)$ $\mathfrak{so}(8) \oplus \mathfrak{u}(1)$ $\mathfrak{so}(9)$ $\mathfrak{so}(9) \oplus \mathfrak{u}(1)$ $\mathfrak{so}(9) \oplus \mathfrak{so}(3)$ $\mathfrak{so}(9) \oplus \mathfrak{so}(5) \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)$ $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ $\mathfrak{so}(11) \oplus \mathfrak{so}(3)$ $\mathfrak{so}(13) \oplus \mathfrak{u}(1)$

Table 4.2: List of Lie algebras that are candidates for unbroken symmetries. $2\mathfrak{u}(1)$ stands for $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$, $2\mathfrak{su}(1)$ for $\mathfrak{su}(1) \oplus \mathfrak{su}(1)$, and so on.

Chapter 5

Origin of quark masses and Cabibbo-Kobayashi-Maskawa matrix

Here a gauge theoretic model, which retains the standard model as effective theory is introduced. This chapter is based on joint work with G. Mack and T. Prüstel [31]. Within this model the appearance of the quark masses of the standard model as well as the principle of the computation of the Cabibbo-Kobayashi-Maskawa matrix is explained. The basic idea is the bilayered membrane, which is introduced in section 3.2.2. The model lives on a five-dimensional space-time with two four-dimensional boundaries R and L, where the right- and left-handed quarks and leptons live, a small distance d apart. Quarks of the same weak hypercharge but belonging to different generations are assumed to be in the same irreducible representation of the holonomy group of the bulk. Their masses are determined by the minima of $V(\phi)$. Assuming multiple minima of $\mathcal{V}$, flavor mass splitting will arise in the manner discussed in the last chapter. We imagine that the model originates from a continuum theory which is a conventional gauge theory with a certain gauge group in the bulk, with an UV-cutoff if needed, and gauge groups as specified below on the boundaries. As a consequence of renormalization group (RG-)transformations, an effective lattice theory with nonunitary parallel transporters is supposed to be obtained, with a distance of one lattice spacing between the two boundaries, and with a nontrivial Higgs potential V which depends on parallel transporters ϕ across the bulk, with values $\phi \in$ H. Since the RG-transformation must preserve locality, it can be performed to a good accuracy in a theory with ∞ extent in the fifth direction. As a result, the Higgs potential V will not depend on the physics on the boundaries.

5.1 Description of the model

The model lives on a five-dimensional space-time, called bulk, with two four-dimensional boundaries R and L. As in the bilayered membrane the right-handed isospin singlet quarks $q_R^U = (u_R, c_R, t_R)$ and $q_R^D = (d_R, s_R, b_R)$, and also right-handed leptons l_R^e and l_R^μ are situated on the boundary R. On the boundary L we have left-handed quarks with isospin doublets $q_L = (d_L, u_L, s_L, c_L, b_L, t_L)$ and left-handed leptons l_L. In the bulk there may be Dirac fermions, presumably as many as there are chiral fermions on the boundaries. The assignment of the fermions to boundaries is compatible with the action of the Pati-Salam subgroup $SU(4)_C \times SU(2)_L \times SU(2)_R$ of the GUT group SO(10), but not with the action of SO(10) itself. One may speculate that at scales shorter than the GUT scale, SO(10) is an unbroken local symmetry of the bulk and light fermions migrate to the boundary when $SU(2)_L \times SU(2)_R$ breaks in the bulk, surviving temporarily on the boundary. Our model is valid at larger scales, where $SU(2)_R$ is broken on the boundary R. Migration of the modes from bulk to branes has been discussed in the literature [19].

5.1.1 Possible assignment of the gauge groups

Here we propose an assignment of gauge groups and holonomy groups to the boundaries and the bulk. The choice is based on the wish to obtain the standard model as effective theory.

The standard model consists of six quarks. They are put together in three families $(u,d), (c,s)$ and (t,b). Left-handed quarks transform according to the fundamental representation of $SU(2) \times U(1) \times SU(3)_C$

$$q_L = \begin{pmatrix} q^D \\ q^U \end{pmatrix}_L \in V_L = \tilde{V}_L \times \mathbb{C}^3 \times \mathbb{C}^3,$$

with a two-dimensional representation space $\tilde{V}_L$. Fields q^D and q^U are elements of a vectorspace $\mathbb{C}^3$, which is spanned by the Eigenstates of flavor. The second $\mathbb{C}^3$-factor accommodates color. Right-handed quarks transform according to a trivial representation of $SU(2) \times U(1)$. We have singlets q_R^U and q_R^D, that are put together into a vector of two components

$$q_R = \begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix} \in V_R^D \oplus V_R^U = (\tilde{V}_R^D \oplus \tilde{V}_R^U) \times \mathbb{C}^3 \times \mathbb{C}^3.$$

Again, the two $\mathbb{C}^3$-factors accommodate flavor and color. $\tilde{V}_R^D$ and $\tilde{V}_R^U$ are one-dimensional representation spaces. Figure 5.1 summarizes the asignments.

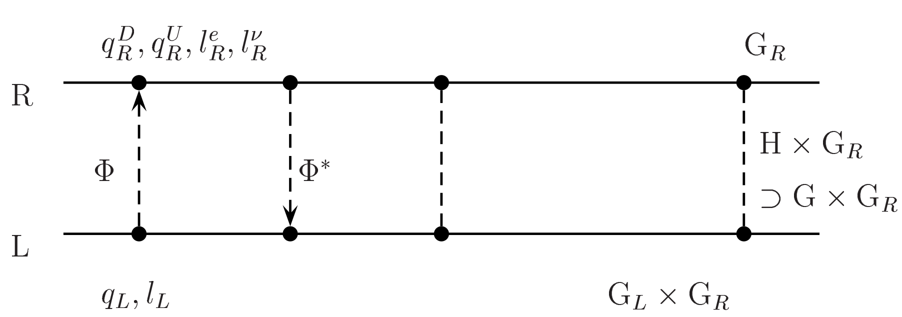


Figure 5.1: Assignment of the gauge and holonomy groups

- G_R is the right-handed unitary gauge group, with $\mathrm{G}_R = \mathrm{U}(1) \times \mathrm{SU}(3)_C$, weak hypercharge and color, in accordance to the standard model. We have two one-dimensional representation spaces $\tilde{V}_R^D$ and $\tilde{V}_R^U$ and thus two isospin singlets $q_R^U = (u_R, c_R, t_R) \in V_R^U$ and $q_R^D = (d_R, s_R, b_R) \in V_R^D$.

- $\mathrm{G}_L \times \mathrm{G}_R$ is the left-handed unitary gauge group, $\mathrm{G}_L \times \mathrm{G}_R = \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SU}(3)_C$, with a 2-dimensional representation space $\tilde{V}_L$ and a isospin doublet $q_L = (d_L, u_L, s_L, c_L, b_L, t_L) \in V_L$.

- $\mathrm{G} \times \mathrm{G}_R$ is the unitary gauge group of the bulk. A possible choice would be $\mathrm{G} = \mathrm{SO}(3) \times \mathrm{SU}(3)$.

- $\mathrm{H} \times \mathrm{G}_R$ is the bulk's holonomy group. The bulk's unitary gauge group should be a subgroup of $\mathrm{H} \times \mathrm{G}_R$ for reasons that we have already mentioned in section 3.4. We postulate that G is the maximal compact subgroup G. In our proposed model we choose $\mathrm{H} = \mathbb{R}_* \ \mathrm{H}^D \times \mathbb{R}_* \ \mathrm{H}^U$, with $\mathbb{R}_* = \mathbb{R}\backslash\{0\}$.
  $\mathrm{H}^D = \mathrm{SL}(3,\mathbb{R})$, $\mathrm{H}^U = \mathrm{SL}(3,\mathbb{C})$ or vice versa. At least one of them has to be a group of complex matrices otherwise we could not have CP-violation. It is proposed to choose two different groups because otherwise it would be difficult to explain the different mass splitting in the U and D family at this energy scale.

The photon of the standard model is massless and interacts equally with right-and left-handed quarks, furthermore it mediates interactions between them. Therefore we need an abelian gauge field that propagates in the bulk and separates the quarks. For this reason all unitary gauge groups have the U(1) in common. In the following we disregard color. That means we set $V_L = \tilde{V}_L \times \mathbb{C}^3$ and $V_R = \tilde{V}_R \times \mathbb{C}^3$.

We denote the parallel transporter across the bulk from the R-boundary to the L-boundary by $\Phi \in \mathrm{H} \times \mathrm{G}_R = \mathbb{R}_* \ \mathrm{H}^D \times \mathbb{R}_* \ \mathrm{H}^U \times \mathrm{G}_R$. Such parallel transporters have the form

$$\Phi = \begin{pmatrix} \rho^D \phi^D u_R^D & 0 \\ 0 & \rho^U \phi^U u_R^U \end{pmatrix} : V_L \to V_R^D \oplus V_R^U. \tag{5.1}$$

According to (5.11) $u_R^D = e^{-\frac{i\vartheta}{3}}$ and $u_R^U = e^{\frac{2i\vartheta}{3}}$. We could also say that $\rho^D \phi^D$ is an element of the linear span of H^D and $\rho^U \phi^U$ of H^U, as $\rho^D, \rho^U \in \mathbb{R}_*$ are real numbers. As we have seen in section 3.4, we expect to obtain nonunitary parallel transporters by performing RG-steps. The parallel transporter Φ of the above form (5.1) is supposed to be such a nonunitary one that appears in the effective theory, that is described here. Elements of the unitary gauge group G can be written as

$$\phi = \begin{pmatrix} \phi^D & 0 \\ 0 & \phi^U \end{pmatrix}.$$

with unitary $\phi^D \in \mathrm{G}^D$, $\phi^U \in \mathrm{G}^U$.

5.2 Mass splitting in the model

The essentials of chapter 4 are now applied to our realistic model.

Flavor

In general ϕ^D and ϕ^U are elements of the noncompact groups H^D and H^U. The maximal noncompact abelian Lie algebras $\mathfrak{a}$ associated with the groups H^D and H^U are two-dimensional and its elements may be parameterized by triples $\underline{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ subject to the condition $\sum \eta_i = 0$. Accordingly, the diagonal matrices $\exp(\underline{\eta}) = d(\underline{\eta})$ have the form

$$d(\underline{\eta}^D) = \begin{pmatrix} e^{\eta_1^D} & 0 & 0 \\ 0 & e^{\eta_2^D} & 0 \\ 0 & 0 & e^{\eta_3^D} \end{pmatrix} \text{ and } d(\underline{\eta}^U) = \begin{pmatrix} e^{\eta_1^U} & 0 & 0 \\ 0 & e^{\eta_2^U} & 0 \\ 0 & 0 & e^{\eta_3^U} \end{pmatrix}.$$

The groups H^D and H^U are semi simple. Factors ρ^D and ρ^U have been pulled out already. According to eq.(3.6) of section 3.2.3, their elements ϕ^D and ϕ^U can be parameterized as

$$\begin{aligned} \phi^D &= A_R^{D*} d(\underline{\eta}^D) A_L^D \\ \phi^U &= A_R^{U*} d(\underline{\eta}^U) A_L^U \end{aligned} \tag{5.2}$$

where $A_R^D, A_R^D \in \mathrm{G}^D$ and $A_R^U, A_R^U \in \mathrm{G}^U$. The Higgs potential $V(\Phi)$ fulfills the biinvariance property

$$V(\Phi) = V\begin{pmatrix} \rho^D \phi^D u_R^D & 0 \\ 0 & \rho^U \phi^U u_R^U \end{pmatrix} = \mathcal{V}(\rho^D, \rho^U; \underline{\eta}^D, \underline{\eta}^U).$$

The Weyl groups $\mathrm{W}(\mathrm{A}^D)$ and $\mathrm{W}(\mathrm{A}^U)$ are isomorphic to the group of permutations π of three objects. To every permutation π of $\{1, 2, 3\}$ there exist a $w_\pi^D \in \mathrm{G}^D$ and a $w_\pi^U \in \mathrm{G}^U$ such that

$$\begin{aligned} w_\pi^D d(\underline{\eta}^D) w_\pi^{D*} &= d(\pi(\underline{\eta}^D)) = d(\eta^D_{\pi(1)}, \eta^D_{\pi(2)}, \eta^D_{\pi(3)}) \\ w_\pi^U d(\underline{\eta}^U) w_\pi^{U*} &= d(\pi(\underline{\eta}^U)) = d(\eta^U_{\pi(1)}, \eta^U_{\pi(2)}, \eta^U_{\pi(3)}). \end{aligned}$$

The pairs of elements (w_π^D, w_π^U) form the Weyl group $\mathrm{W}(\mathrm{A})$ of the product of the subgroup $\mathrm{A} = \mathrm{A}^D \times \mathrm{A}^U$. Again it follows from the G^D and G^U biinvariance property that the Higgs potential has the discrete symmetry

$$\mathcal{V}(\rho^D, \rho^U, \pi(\underline{\eta}^D), \pi(\underline{\eta}^U)) = \mathcal{V}(\rho^D, \rho^U, \underline{\eta}^D, \underline{\eta}^U).$$

If the symmetry of $\mathrm{W}(\mathrm{A})$ is spontaneously broken, then the quark masses will all come out

$$\begin{aligned} m_d &= \hat{\rho}^D e^{\hat{\eta}_1^D}, & m_s &= \hat{\rho}^D e^{\hat{\eta}_2^D}, & m_b &= \hat{\rho}^D e^{\hat{\eta}_3^D}, \\ m_u &= \hat{\rho}^U e^{\hat{\eta}_1^U}, & m_c &= \hat{\rho}^U e^{\hat{\eta}_2^U}, & m_t &= \hat{\rho}^U e^{\hat{\eta}_3^U}, \end{aligned} \tag{5.3}$$

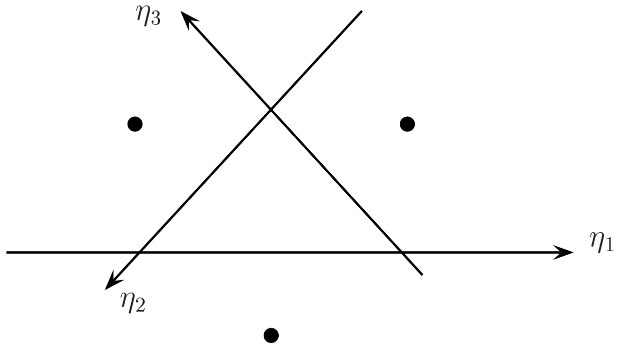


Figure 5.2: Example for $V(\Phi) = V(d(\eta_1, \eta_2, \eta_3))$ with three minima $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ (black points) that fulfills $V(d(\eta_1, \eta_2, \eta_3)) = V(d(\eta_{\pi(1)}, \eta_{\pi(2)}, \eta_{\pi(3)}))$. Each minimum $(\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3)$ is part of a triplet $(\hat{\eta}_{\pi(1)}, \hat{\eta}_{\pi(2)}, \hat{\eta}_{\pi(3)})$. Two of the three η_i can be chosen independently.

where the minimum of $\mathcal{V}(\rho^D, \rho^U; \eta^D, \eta^U)$ is at $(\hat{\rho}^D, \hat{\rho}^U, \hat{\eta}^D, \hat{\eta}^U)$. Therefore, quark masses in our realistic model are determined by the minima of the Higgs potential. If the W(A)-symmetry would not be broken spontaneously we would have no mass splitting. $\underline{\hat{\eta}} = 0$ is the only point which is invariant under W(A). The fermions (d, s, b) would therefore retain equal mass in the absence of spontaneous W(A)-symmetry breaking, and similarly for (u, c, t).

Color

We disregarded color, but nevertheless we have a brief look at it to emphasize the difference between flavor and color in our model. We could also admit nonunitary parallel transporters χ across the bulk in a possibly noncompact color holonomy group H_C. χ would appear in the nonunitary parallel transporter as a factor of it and could be also parameterized via $\chi = r u_1 d(\eta_C) u_2$, $r \in \mathbb{R}_*, u_1, u_2 \in \mathrm{SU}(3)_C$. If the symmetry is to be broken down, the minimum of the Higgs potential V, considered as a function of χ, must be at $\eta_C = 0$. In the standard model we do not have splitting of the masses of fermions of different color. The W(A)-symmetry is not spontaneously broken for color, which means that the orbits of the minima of the Higgs potential V under the Weyl group W(A) are trivial for color, but nontrivial for flavor.

5.3 Explicit symmetry breaking and the CKM-matrix

In section 4.2.2 we explained the essentials of flavor mass splitting at the example of a one-dimensional model, where only the extra dimension is retained. The generalization to five dimensions involves a subtle point which is connected with the appearance of a nontrivial Cabibbo-Kobayashi-Maskawa matrix. This will now be explained.

The action will include terms

$$\sum_x \bar{q}_R(x)\Phi(x)q_L(x). \tag{5.4}$$

They will produce masses, as in ordinary Higgs models. $\Phi(x) = \Phi(x, R; x, L)$ is the nonunitary parallel transporter along the bulk between the sites (x, L) on the L-boundary and (x, R) on the R-boundary. They are specified by a 4-dimensional coordinate x, and $\Phi(x)$ has the form (5.1) and depends on x. At discrete minima of the Higgs potential

$$\begin{aligned} \phi^D(x) &= A_R^{D*}(x) d(\underline{\hat{\eta}}^D) A_L^D(x) \\ \phi^U(x) &= A_U^{D*}(x) d(\underline{\hat{\eta}}^U) A_L^U(x). \end{aligned} \tag{5.5}$$

The Plaquette term in the action, to be discussed below, produces a kinetic term for ϕ^D, ϕ^U. Because the minima $\underline{\hat{\eta}}$ of the Higgs potential are discrete by hypothesis, an x-dependence of $\underline{\hat{\eta}}^D$ or $\underline{\hat{\eta}}^U$ is strongly disfavored by this kinetic term. But $A_R^D(x)$, $A_R^U(x)$, $A_L^D(x)$, $A_L^U(x)$ have arbitrary x-dependence to begin with.

G_R is a local symmetry on the boundaries. If G^D and G^U were also exact local symmetries on the boundary, and therefore of the whole theory, the factors $A_L^D(x)$, $A_R^U(x)$, $A_L^D(x)$, $A_L^U(x)$ could be transformed to unity by a local gauge transformation. But then there will be no nontrivial Cabibbo-Kobayashi-Maskawa matrix $C = A_L^D A_L^{U*}$. Therefore we postulated that there is explicit symmetry breaking on the boundaries as specified above. Then the ground state of the theory will single out special values of A_L^D and A_L^U. How they could be determined will be discussed below. Assuming that 4-dimensional translation invariance is not broken spontaneously, these values will not be x-dependent. Assuming x-independent A_L^D, A_R^U, A_L^D, A_L^U in eq.(5.5) and neglecting fluctuations of ρ^D, ρ^U away from their values $\hat{\rho}^D$ and $\hat{\rho}^U$ at the minima of the Higgs potential, the parallel transporter Φ takes the form

$$\Phi = \begin{pmatrix} A_R^{D*} m_D A_L^D u_R^D & 0 \\ 0 & A_R^{U*} m_U A_L^U u_R^U \end{pmatrix}$$

$$\text{with } m_D = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \text{ and } m_U = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

with quark masses m_d, m_s, m_b and m_u, m_c, m_t given by eq.(5.3). The mass term (5.4) then takes the form

$$\bar{q}_R \Phi q_L = \bar{q}_R^{D'} m_D q_L^{D'} + \bar{q}_R^{U'} m_U q_L^{U'} \tag{5.6}$$

with $\bar{q}_R^{D'} = \bar{q}_R^D A_R^{D*} u_R^D$, $\bar{q}_R^{U'} = \bar{q}_R^U A_R^{U*} u_R^U$, $q_L^{D'} = A_L^D q_L^D$ and $q_L^{U'} = A_L^U q_L^U$. In the model at hand, the constant factors A_R^D and A_R^U can be transformed away because the parallel transporters $u_R \in \mathrm{G}_R$ on the R-boundary commute with elements of G^U, G^D, so that actually $q_R^{D'} = q_R^D$ and $q_R^{U'} = q_R^U$. This feature is familiar from the standard model.

5.3.1 Diagonal subgroup and Cabibbo-Kobayashi-Maskawa matrix

The factors $A_L^D(x) \in \mathrm{G}^D$, $A_L^U(x) \in \mathrm{G}^U$ as well as $A_R^D(x) \in \mathrm{G}^D$, $A_R^U(x) \in \mathrm{G}^U$ are part of the trans bulk parallel transporters and are therefore dynamical fields of our model. G_R

is a local symmetry of bulk and both boundaries.

It is a salient feature of our model that the groups $G = G^D \times G^U$ and G_L act on the same 6-dimensional representation space $\tilde{V}_L \times \mathbb{C}^3$, where $\mathbb{C}^3$ is the flavor space, but their action does not commute. As a result, $G^D \times G^U$ is neither a local nor a global symmetry of the full theory. As global symmetry it is broken on the L-boundary.

Consider the diagonal subgroup $G_{\text{diag}} = G^D \cap G^U$. There are global symmetries

$$A_L^D(x) \mapsto hA_L^D(x), \qquad A_L^U(x) \mapsto hA_L^U(x), \qquad h \in G_{\text{diag}} \tag{5.7}$$
$$A_R^D(x) \mapsto h'A_R^D(x), \qquad A_R^U(x) \mapsto h''A_R^U(x), \qquad h' \in G^D, h'' \in G^U. \tag{5.8}$$

The first symmetry obtains because h commutes with the parallel transporters $u_L(x, x+\hat{\mu}) \in G_L$ on the L-boundary. To see this explicitly, examine the following parallel transporter which appears in the Plaquette term across the bulk.

$$\begin{aligned}
\Phi u_L \Phi^* &= \begin{pmatrix} m_D A_L^D & 0 \\ 0 & m_U A_L^U \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} A_L^{D*} m_D & 0 \\ 0 & A_L^{U*} m_U \end{pmatrix} \\
&= \begin{pmatrix} m_D & 0 \\ 0 & m_U \end{pmatrix} \begin{pmatrix} A_L^D u_{11} A_L^{D*} & A_L^D u_{12} A_L^{U*} \\ A_L^D u_{21} A_L^{U*} & A_L^U u_{22} A_L^{U*} \end{pmatrix} \begin{pmatrix} m_D & 0 \\ 0 & m_U \end{pmatrix} \\
&= \begin{pmatrix} m_D & 0 \\ 0 & m_U \end{pmatrix} \begin{pmatrix} u_{11} & A_L^D u_{12} A_L^{U*} \\ A_L^D u_{21} A_L^{U*} & u_{22} \end{pmatrix} \begin{pmatrix} m_D & 0 \\ 0 & m_U \end{pmatrix}.
\end{aligned} \tag{5.9}$$

u_{ij} are complex numbers. (5.7) is not a local symmetry because the corresponding gauge field on the L-boundary is missing. One can see the non-invariance under local transformations by writing arguments $x, x+\hat{\mu}$ for all the entries in eq.(5.9).

The symmetry (5.8) obtains because $h' \in G^U$, G^D commutes with parallel transporters $u_R(x, x+\mu) \in G_R$ on the R-boundary. Again, it is not a local symmetry as the corresponding G-gauge fields on the R-boundary are missing.

Let us introduce the Cabibbo-Kobayashi-Maskawa field

$$C(x) = A_L^D(x) A_L^{U*}(x).$$

It is invariant under the unbroken diagonal subgroup $G_{\text{diag}} = G^D \cap G^U$. To sum up, the groups G_L and G (and thus the gauge group of the left-handed boundary and the gauge group of the bulk) do not commute. On the L-boundary the local G-invariance is thus broken down to the diagonal subgroup G_{diag}.

Let $W(A_L^D(x), A_L^U(x), A_R^D(x), A_R^U(x))$ be the effective action that is obtained by integrating out the quark fields and gauge fields that are associated only with the boundary gauge groups G_R and G_L, and the fluctuations of η^D, η^U and of ρ^D, ρ^U away from the minimum of the Higgs potential V. The quark masses determined be minimization of V enter into the effective action W as parameters. Assuming that 4-dimensional translation symmetry is not spontaneously broken, the minimum of W will be assumed at constant values of $A_L^D, A_L^U, A_R^D, A_R^U$. The effective action W, restricted to constant fields, is invariant under the transformations (5.7) and (5.8). Therefore, it depends only on the Cabibbo-Kobayashi-Maskawa matrix

$$C = A_L^D A_L^{U*}$$

and its minimum determines C.
The Cabibbo-Kobayashi-Maskawa matrix (CKM) could be transformed away if it were not for the G-symmetry breaking by the L-boundary.

5.4 The full action

We will study terms appearing in the action and its invariant properties. The fermions on R are denoted by q_R, those on L by q_L. For now we disregard the leptonic parts of the action.

Weak hypercharge and Higgs doublet

The assignment of the weak hypercharge is as follows [23]

l_L	l_R	q_L	q_R^U	q_R^D	φ	$\tilde{\varphi}$
$-\frac{1}{2}$	-1	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$.

$\hat{\varphi}(x)$ is the Higgs doublet of length 1 and

$$\hat{\varphi}(x) = \begin{pmatrix} \hat{\varphi}_+(x) \\ \hat{\varphi}_0(x) \end{pmatrix}, \qquad \tilde{\hat{\varphi}}(x) = \begin{pmatrix} \bar{\hat{\varphi}}_0(x) \\ -\bar{\hat{\varphi}}_+(x) \end{pmatrix}. \tag{5.10}$$

It transforms under $v \in SU(2)$ according to $\hat{\varphi} \mapsto \hat{\varphi} v^*$, $\tilde{\hat{\varphi}} \mapsto \tilde{\hat{\varphi}} v^*$. The Higgs doublet φ will not appear in the action to start with. We will describe later how it is constructed from other fields.

The elements $u_R = e^{\frac{i\vartheta}{6}}$ of the unitary gauge group G_R of the R-boundary can be parametrized by an angle ϑ. It acts on the right-handed quarks according to

$$\begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix} \mapsto t_R(u_R) \begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\vartheta}{3}} & 0 \\ 0 & e^{\frac{2i\vartheta}{3}} \end{pmatrix} \begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix}, \tag{5.11}$$

on left-handed quarks according to

$$q_L \mapsto t_L(u_R) q_L = e^{\frac{i\vartheta}{6}} q_L \tag{5.12}$$

and on the Higgs doublets according to

$$\varphi \mapsto t_\varphi(u_R)\varphi = e^{\frac{i\vartheta}{2}}\varphi \tag{5.13}$$

$$\tilde{\varphi} \mapsto t_\varphi(u_R)^*\tilde{\varphi} = e^{-\frac{i\vartheta}{2}}\tilde{\varphi}. \tag{5.14}$$

Ignoring color the unitary gauge group of L is $\mathrm{G}_L\times$ G_R = SU(2) × U(1). G_L acts on left-handed quarks as in the standard model and consists of elements

$$u_L = \begin{pmatrix} u_{11}\mathbb{I} & u_{12}\mathbb{I} \\ u_{21}\mathbb{I} & u_{22}\mathbb{I} \end{pmatrix}$$

where $\mathbb{I}$ is the 3×3 unit matrix. It acts on the flavor space $\mathbb{C}^3$.

The action

The lattice action consists of the standard terms on the boundary and the terms across the bulk. We neglect to write Plaquette terms within the boundaries.

$$\begin{aligned} S = &\sum_x \Big(\bar{q}_R(x)\Phi(x)q_L(x) \\ &+\sum_\mu \left(\bar{q}_L(x)u_{L\mu}(x,L)u_R(x,L)q_L(x+\hat{\mu}) + \bar{q}_R(x)u_{R\mu}(x,R)q_R(x+\mu)\right) \\ &+\big(\operatorname{tr} g^{-2}[\Phi^*(x)t_R(u_{R\mu}(x,R))\Phi(x+\hat{\mu})t_L(u_{R\mu}(x,L))^*u_{L\mu}(x)^* - \Phi^*(x)\Phi(x)] \\ &+V(\Phi)\big)\Big) + \text{h.c.} + \text{Plaquette terms on the boundaries,} \end{aligned} \tag{5.15}$$

with $q_L(x) = q_L(x,L)$, $q_R(x) = q_R(x,R)$, $\Phi(x) = \Phi(x,R;x,L)$, $u_{R\mu}(x,R) = u_R(x,R;x+\hat{\mu},R)$ and $t_L(u_{R\mu}(x,L))^*u_{L\mu}(x)^* = u_L(x+\hat{\mu},L;x,L)t_L(u_R(x+\hat{\mu},L;x,L))$. $\hat{\mu}$ is the unit vector in μ-direction for the conventional four-dimensional space-time on the boundaries with $\mu = 0,...3$. The Plaquette term across the bulk involves

$$\Phi^*(x)t_R(u_{R\mu}(x,R))\Phi(x+\hat{\mu})t_L(u_{R\mu}(x,L))^*u_{L\mu}(x)^*.$$

If we get to effective theories, the Φ has to be included. Because the Φ-field appears also in $\bar{q}_R\Phi q_L$, we are not able to rescale ϕ^D, ϕ^U so that coupling constants multiplying them in the Plaquette term appear. Thus, there are two different coupling constants that remain in the Plaquette term. They are taken into account in the combined coupling constant g with

$$g^{-2} = \begin{pmatrix} g_D^{-2}\mathbb{I} & 0 \\ 0 & g_U^{-2}\mathbb{I} \end{pmatrix}. \tag{5.16}$$

Finally,

$$\sum_\mu \left(\bar{q}_L(x)u_{L\mu}(x,L)u_R(x,L)q_L(x+\hat{\mu}) + \bar{q}_R(x)u_{R\mu}(x,R)q_R(x+\hat{\mu})\right)$$

is the standard part of the lattice action. $\bar{q}_R(x)u_{R\mu}(x,R)q_R(x+\mu)$ is invariant under G_R-gauge transformations and $\bar{q}_L(x)u_{L\mu}(x,L)u_R(x,L)q_L(x+\hat{\mu})$ is invariant under $\mathrm{G}_L\times$ G_R-gauge transformations. This term in the action is not invariant under local gauge transformations of the bulks gauge group G.

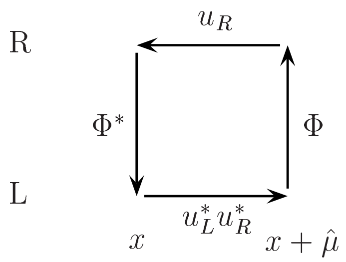


Figure 5.3: Plaquette with lattice spacing $a_\mu = 1$

5.5 Standard model Higgs doublet

It would appear that the G_L-invariance on boundary L is broken also by the interaction with the bulk. It turns out that this is the customary Higgs mechanism. The standard model's Higgs doublet φ emerges when one passes to a G_L-covariant description. This will now be explained.

We refer to the action (5.15). We introduce an interface $g(x) \in \mathrm{SU}(2)$ between bulk and L-boundary (bulk $\to$ L) and fix it at 1 initially. Thus we have to replace $\Phi(x)$ by $\Phi(x)g^*(x)$. The action is now invariant under SU(2)-gauge transformations $v(x) \in \mathrm{SU}(2)$ which act on q_L and $u_{L\mu}$ as usual, and which take $g(x) \to g(x)v(x)^*$. To every $g \in \mathrm{SU}(2)$ there is a complex 2-vector $\hat{\varphi}$ of unit length such that $g = L[\hat{\varphi}]$. The Plaquette term P_μ becomes

$$\begin{aligned} P_\mu = &\sum_\mu \big(\operatorname{tr} g^{-2}[\Phi^*(x) t_R(u_{R\mu}(x,R))\Phi(x+\hat{\mu}) L[\hat{\varphi}(x+\hat{\mu})]^* t_L(u_{R\mu}(x,L))^* u_{L\mu}(x)^* L[\hat{\varphi}(x)] \\ &- \Phi^*(x)\Phi(x)]\big). \end{aligned}$$

$L[\hat{\varphi}(x)] \in \mathrm{SU}(2)$ is uniquely determined by the requirement

$$\hat{\varphi} = L[\hat{\varphi}] \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.17}$$

Explicitly

$$L[\hat{\varphi}] = \begin{pmatrix} \bar{\hat{\varphi}}_0 & \hat{\varphi}_+ \\ -\bar{\hat{\varphi}}_+ & \hat{\varphi}_0 \end{pmatrix}, \quad \text{if } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_+ \\ \hat{\varphi}_0 \end{pmatrix}. \tag{5.18}$$

We will retain this formula for $\varphi = \rho\hat{\varphi}$ for arbitrary length.

We now define $\tilde{\Phi}$ by taking out the u_R-factor $\Phi = \mathrm{diag}(u_R^D, u_R^U)\tilde{\Phi} = u_R\tilde{\Phi}$ and get

$$P_\mu = \sum_\mu \big(\operatorname{tr} g^{-2}[\tilde{\Phi}^*(x) t_R(u_R(x)^* u_{R\mu}(x,R) u_R(x+\hat{\mu}))\tilde{\Phi}(x+\hat{\mu}) \tag{5.19}$$

$$\times L[\hat{\varphi}(x+\hat{\mu})]^* t_L(u_{R\mu}(x,L))^* u_{L\mu}(x)^* L[\hat{\varphi}(x)] \tag{5.20}$$

$$- \Phi^*(x)\Phi(x)]\big).$$

It follows from the defining property (5.17) that $L[v\varphi] = vL[\varphi], v \in \mathrm{SU}(2)$. Under gauge transformation in U(1)

$$u_R \mapsto u_R w, \qquad u_{R\mu} \mapsto u_{R\mu} w^* \qquad \text{and} \qquad \varphi \mapsto t_\varphi(w)\varphi,$$

$w \in \mathrm{U}(1)$. The weak hypercharge acts according to eqs.(5.11),(5.12) and (5.13).

Corollary 8 *With $L[\hat\varphi]$ defined in eq.(5.18) and the actions of the weak hypercharges every $w \in \mathrm{U}(1)$ fulfills the gauge covariance condition*

$$t_L(w)L[\hat\varphi] = L[t_\varphi(w)^*\hat\varphi]t_R(w).$$

Proof: This follows directly from the requirement of the SU(2)-covariance of terms like $u_{L\mu}t_L(u_{R\mu})L[\hat\varphi]t_R(u_R)$ appearing in the Plaquette term. Further, direct calculation gives

$$\begin{aligned} t_L(w)L[\hat\varphi] &= e^{\frac{i\vartheta}{6}} \begin{pmatrix} \bar{\hat\varphi}_0 & \hat\varphi_+ \\ -\bar{\hat\varphi}_+ & \hat\varphi_0 \end{pmatrix} = \begin{pmatrix} e^{\frac{i\vartheta}{2}}\bar{\hat\varphi}_0 & e^{\frac{-i\vartheta}{2}}\hat\varphi_+ \\ -e^{\frac{i\vartheta}{2}}\bar{\hat\varphi}_+ & e^{\frac{-i\vartheta}{2}}\hat\varphi_0 \end{pmatrix} \begin{pmatrix} e^{\frac{-i\vartheta}{3}} & 0 \\ 0 & e^{\frac{2i\vartheta}{3}} \end{pmatrix} \\ &= L[t_\varphi(w)^*\hat\varphi]t_R(w). \ \textbf{q.e.d.} \end{aligned}$$

Theorem 12 *Define*

$$W_\mu(x) := L[\hat\varphi(x+\hat\mu)]u_{L\mu}(x)^*L[t_\varphi(u_{R\mu}(x,L))\hat\varphi(x)]^* = \begin{pmatrix} W_\mu^{DD} & W_\mu^{DU} \\ W_\mu^{UD} & W_\mu^{UU} \end{pmatrix},$$

and $t^{-\frac{1}{3}}(u_R(\partial p)) = e^{-\frac{i\vartheta}{3}}, t^{\frac{2}{3}}(u_R(\partial p) = e^{\frac{2i\vartheta}{3}}$ *and let*

$$u_R(\partial p) = u_{R\mu}(x,L)^*u_R(x)^*u_{R\mu}(x,R)u_R(x+\hat\mu)$$

be the product of the U(1)*-factors around a Plaquette. The Plaquette ∂p depends on μ. g_D and g_U are the coupling constants and*

$$\tilde\Phi = \begin{pmatrix} \tilde\Phi^D & 0 \\ 0 & \tilde\Phi^U \end{pmatrix}.$$

The Plaquette term becomes

$$\begin{aligned} P_\mu = \sum_\mu \Big[& g_D^{-2}\big(\operatorname{tr}[\tilde\Phi^{D*}(x)t^{-\frac{1}{3}}(u_R(\partial p))\tilde\Phi^D(x+\hat\mu)W_\mu^{DD}(x) - \tilde\Phi^{D*}(x)\tilde\Phi^D(x)]\big) \\ & + g_U^{-2}\big(\operatorname{tr}[\tilde\Phi^{U*}(x)t^{\frac{2}{3}}(u_R(\partial p))\tilde\Phi^U(x+\hat\mu)W_\mu^{UU}(x) - \tilde\Phi^{U*}(x)\tilde\Phi^U(x)]\big)\Big], \end{aligned}$$

where the trace is 3-dimensional, when we disregard color.

Proof: Using the covariance condition the Plaquette (5.19) becomes

$$\begin{aligned} P_\mu = \sum_\mu & \big(\operatorname{tr} g^{-2}[t_R(u_{R\mu}(x,L)^*)\tilde\Phi^*(x)t_R(u_R(x)^*u_{R\mu}(x,R)u_R(x+\hat\mu))\tilde\Phi(x+\hat\mu) \\ & \times L[\hat\varphi(x+\hat\mu)]u_{L\mu}(x)^*L[t_\varphi(u_{R\mu}(x,L))\hat\varphi(x)]^* \\ & - \Phi^*(x)\Phi(x)]\big) \end{aligned}$$

with the gauge invariant term $W_\mu(x) = L[\hat{\varphi}(x+\hat{\mu})]u_{L\mu}(x)^* L[t_\varphi(u_{R\mu}(x,L))\hat{\varphi}(x)]^*$. Using

$$t_R(u) = \begin{pmatrix} t^{-\frac{1}{3}}(u) & 0 \\ 0 & t^{\frac{2}{3}}(u) \end{pmatrix}, t_\varphi(u) = t^{-\frac{1}{2}}(u)\mathbb{I}$$

and canceling U(1)-factors in the $\Phi^*\Phi$-term, we get

$$\begin{aligned} P_\mu = \sum_\mu \big[& g_D^{-2}\big(\operatorname{tr}[t^{-\frac{1}{3}}(u_{R\mu}(x,L)^*)\tilde{\Phi}^{D*}(x)t^{-\frac{1}{3}}(u_R(x)^* u_{R\mu}(x,R)u_R(x+\hat{\mu}))\tilde{\Phi}^D(x+\hat{\mu})W_\mu^{DD}(x) \\ & \quad - \tilde{\Phi}^{D*}(x)\tilde{\Phi}^D(x)]\big) \\ & + g_U^{-2}\big(\operatorname{tr}[t^{\frac{2}{3}}(u_{R\mu}(x,L)^*)\tilde{\Phi}^{U*}(x)t^{\frac{2}{3}}(u_R(x)^* u_{R\mu}(x,R)u_R(x+\hat{\mu}))\tilde{\Phi}^U(x+\hat{\mu})W_\mu^{UU}(x) \\ & \quad - \tilde{\Phi}^{U*}(x)\tilde{\Phi}^U(x)]\big)\big]. \end{aligned}$$

$t^{-\frac{1}{3}}(u)$ and $t^{\frac{2}{3}}(u)$ are factors that commute with the other matrices. **q.e.d.**

Theorem 13 *We assume that the dependence of x in the Higgs fields $\tilde{\Phi}^D(x)$ and $\tilde{\Phi}^U(x)$ comes only from the x-dependence of the factors $\rho^D(x)$ and $\rho^U(x)$*

$$\begin{aligned} \tilde{\Phi}^D(x) &= \rho^D(x)A_R^D d(\eta^D)A_L^D \\ \tilde{\Phi}^U(x) &= \rho^U(x)A_R^U d(\eta^U)A_L^U. \end{aligned}$$

The Plaquette term can then be written as follows

$$\begin{aligned} P_\mu = \sum_\mu \big\{ & g_D^{-2} \operatorname{tr}\big[d(\eta^D)^2\big(\rho^D(x)[t^{-\frac{1}{3}}(u_R(\partial p))\rho^D(x+\hat{\mu}) - \rho^D(x)] \\ & \quad + \rho^D(x)[W_\mu^{DD}(x)\rho^D(x+\hat{\mu}) - \rho^D(x)]\big)\big] \\ & + g_U^{-2} \operatorname{tr}\big[d(\eta^U)^2\big(\rho^U(x)[t^{\frac{2}{3}}(u_R(\partial p))\rho^U(x+\hat{\mu}) - \rho^U(x)] \\ & \quad + \rho^U(x)[W_\mu^{UU}(x)\rho^U(x+\hat{\mu}) - \rho^U(x)]\big)\big]\big\} \end{aligned}$$

if $t^{-\frac{1}{3}}$ and $t^{\frac{2}{3}}$ defined as in theorem 12.

Proof: We insert

$$\begin{aligned} t(u_R(\partial p)) &= 1 + [t(u_R(\partial p)) - 1] \text{ and} \\ \tilde{\Phi}^D(x)W_\mu^{DD}(x)\tilde{\Phi}^{D*}(x) &= \tilde{\Phi}^D(x)\tilde{\Phi}^{D*}(x) + [\tilde{\Phi}^D(x)W_\mu^{DD}(x)\tilde{\Phi}^{D*}(x+\hat{\mu}) - \tilde{\Phi}^D(x)\tilde{\Phi}^{D*}(x+\hat{\mu})]. \end{aligned}$$

The Plaquette term is

$$\begin{aligned} P_\mu = \sum_\mu \big\{ & g_D^{-2} \operatorname{tr}\big(\tilde{\Phi}^{D*}(x)[t^{-\frac{1}{3}}(u_R(\partial p))\tilde{\Phi}^D(x+\hat{\mu}) - \tilde{\Phi}^D(x)] \\ & \quad + [\tilde{\Phi}^D(x)W_\mu^{DD}(x)\tilde{\Phi}^{D*}(x+\hat{\mu}) - \tilde{\Phi}^D(x)\tilde{\Phi}^{D*}(x)]\big) \\ & + g_U^{-2} \operatorname{tr}\big(\tilde{\Phi}^{U*}(x)[t^{\frac{2}{3}}(u_R(\partial p))\tilde{\Phi}^U(x+\hat{\mu}) - \tilde{\Phi}^U(x)] \\ & \quad + [\tilde{\Phi}^U(x)W_\mu^{UU}(x)\tilde{\Phi}^{U*}(x+\hat{\mu}) - \tilde{\Phi}^U(x)\tilde{\Phi}^{U*}(x)]\big)\big\} \end{aligned}$$

because $A_{L,R}^{D*} = (A_{L,R}^D)^{-1}$ **q.e.d.**

5.6 Effective 4-dimensional theory and the spontaneously broken extra local U(1)-symmetry

We retain the assumption that we may neglect fluctuations of η^D, η^U away from the minima of the Higgs potential as well as fluctuations of the fields $A_L^D(x)$ etc. away from the constants which determine the CKM-matrix. We wish to show that the action reduces to a discretization of the standard model, except that there is an extra spontaneously broken U(1)-symmetry associated with an extra U(1)-lattice gauge field $u'_\mu(x)$ and a complex Higgs field. When we re-institute color, it will be seen that there is in addition an octet of axigluons. They appear because $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_L$ is broken down to the diagonal subgroup.

Let us first discuss the U(1)-symmetries. From a 4-dimensional point of view the original model has two U(1)-groups of gauge transformations $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$. They come from the possibility of performing gauge transformations at (x, L) in the L-boundary and at (x, R) in the R-boundary independently. As we shall see there will be spontaneous breaking to the diagonal U(1)-subgroup. The previous rewritings eq.(5.6) of the Yukawa term $\bar{q}_L \Phi q_R$ as mass terms, the standard model rewriting of the Plaquette terms in the boundaries, which brings in the CKM-matrix, and the formula of theorem 13 for the Plaquette term brought the action to the form of the action of a 4-dimensional theory except for one feature: The U(1)-parallel transporters $t_R(u_R(x))$ and $t_R(u_R(x+\mu))$ across the bulk appear as factors in $t_R(u(\partial p))$ in the formula for the Plaquette. Moreover, the doublet $\hat{\varphi}$ has unit length, in contrast with the standard model's Higgs field.

To remedy this we make a change of variables and identify the Higgs fields and the extra lattice gauge field.
In place of the real non-vanishing factors $\rho^D(x)$ and $\rho^U(x)$, introduce $\rho(x)$ and $\sigma(x)$ by

$$\rho^D(x) = \sigma(x)\rho(x), \qquad \rho^U(x) = \sigma(x)^2\rho(x) \tag{5.21}$$

and define the Higgs doublet φ of fluctuating length by

$$\varphi(x) = \rho(x)\hat{\varphi}(x), \text{ with } L[\varphi] = \rho L[\hat{\varphi}],$$

an extra complex Higgs field $\xi(x)$ by

$$\xi(x) = \sigma(x)u_R(x),$$

and a new lattice gauge field

$$u'_\mu(x) = u_{R\mu}(x, L)u_{R\mu}(x, R)^* \tag{5.22}$$

besides $u_{R\mu}(x, L)$. It couples to the difference of the $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ charges. Under gauge transformations $(v_L, v_R) \in \mathrm{U}(1)_L \times \mathrm{U}(1)_R$

$$\xi \mapsto v_R \xi v_L^{-1}$$

and

$$u'_\mu(x) \mapsto v_L(x)u_{R\mu}(x,L)v_L(x+\hat{\mu})^{-1}v_R(x+\hat{\mu})^* u_{R\mu}(x,R)^* v_R^{-1*}(x).$$

This shows that ξ and u'_μ are invariant under the diagonal subgroup.

Now we are ready to complete the rewriting of the Plaquette term in the form of a lattice action of a 4-dimensional theory (theorem 13). First we note that the second and fourth term involve φ because

$$\rho^D(x)W_\mu^{DD}\rho^D(x+\hat{\mu}) = \big(L[\varphi(x+\hat{\mu})]u_{L\mu}(x)^* L[t_\varphi(u_{R\mu}(x,L))\varphi(x)]^*\big)^{DD}\sigma(x)\sigma(x+\hat{\mu})$$

and similarly for the fourth term. Secondly,

$$\rho^D(x)[t^{-\frac{1}{3}}(u_R(\partial p))\rho^D(x+\hat{\mu}) - \rho^D(x)] = \rho(x)\rho(x+\hat{\mu})\xi(x)t^{-\frac{1}{3}}(u'_\mu(x))\xi^*(x) - |\rho(x)|^2\xi(x)\xi^*(x)$$

and similarly

$$\rho^U(x)[t^{\frac{2}{3}}(u_R(\partial p))\rho^U(x+\hat{\mu}) - \rho^U(x)] = \rho(x)\rho(x+\hat{\mu})\xi^{*2}(x)t^{\frac{2}{3}}(u'_\mu(x))\xi^2(x) - |\rho(x)|^2\xi^2(x)\xi^{*2}(x).$$

Neglecting higher order terms $\rho(x)\rho(x+\hat{\mu}) \approx \rho(x)^2 = |\varphi(x)|^2$. The last two terms are the U(1)-covariant kinetic term for the ξ-field on a lattice. It might appear that there are two different kinetic terms for the φ-field involving W^{DD} and W^{UU}. But they are proportional because

$$W^{UU} = \overline{W}^{DD}.$$

It is understood that every term in the action comes together with its hermitean conjugate.

Massive vector meson

The kinetic term of the gauge field of weak hypercharge leads to a massive vector meson M.
The mass term of the vector meson comes out of

$$g_D^{-2}\,\mathrm{tr}\,\big(d(\eta^D)^2\rho^D(x)[t^{-\frac{1}{3}}(u_R(\partial p))\rho^D(x+\mu) - \rho^D(x)] +$$
$$g_U^{-2}\,\mathrm{tr}\,\big(d(\eta^U)^2\rho^U(x)[t^{\frac{2}{3}}(u_R(\partial p))\rho^U(x+\mu) - \rho^U(x)]\big),$$

replacing $\eta^U, \eta^D, \rho^U, \rho^D$ by their values $\hat{\eta}^U, \hat{\eta}^D, \hat{\rho}^U, \hat{\rho}^D$ at the minimum of the Higgs potential and by inserting the quark masses. To extract the mass, we have to recall the correspondence between the terms in the action in the discretized theory and in the continuum. If φ is some Higgs field, and U_μ are lattice gauge fields corresponding to vector potentials $A_\mu = ig'\mathbb{A}_\mu$ in mathematical and physical normalization, respectively, then on a lattice of lattice spacing a

$$\begin{aligned} S_{\mathrm{kin}} &= \frac{1}{2a^2}\sum_{\pm\mu}\big(\varphi^*(x)\varphi(x) - \varphi^*(x)U_\mu^*(x)\varphi(x+\mu)\big) \\ &= \frac{1}{2a^2}\sum_{\pm\mu}\big(\rho(x)^2 - \rho(x)W_\mu(x)^*\rho(x+\mu)\big) \\ &= \frac{1}{2}\sum_{\pm\mu}\big(-\varphi^* D_\mu\varphi + \frac{1}{2}\varphi^* D_\mu^2\varphi\big), \end{aligned}$$

with $D_\mu = \frac{1}{a}(\partial_\mu + A_\mu) = \frac{1}{a}(\partial_\mu + ig'\mathbb{A}_\mu)$. The last expression is obtained by inserting

$$U_\mu(x) = 1 - aA_\mu(x) + \frac{a^2}{2}A_\mu^2(x) + ... \text{ and}$$
$$\varphi(x+\mu) = \varphi(x) + a\partial_\mu\varphi(x) + \frac{a^2}{2}\partial_\mu^2\varphi(x) + ... \quad .$$

The relevant term for the mass generation is $\frac{1}{2}\varphi^* A_\mu^2 \varphi$, yielding a mass squared $g'^2\langle\varphi^*, \varphi\rangle$. Since our lattice spacing is d it follows for the mass

$$m_M^2 = d^2\big(\frac{g_D^{-2}}{9}(m_d^2 + m_s^2 + m_b^2) + \frac{4g_U^{-2}}{9}(m_u^2 + m_c^2 + m_t^2)\big)g'^2 \tag{5.23}$$

Axigluons

So far we neglected to write the lattice gauge fields associated with color. When we take them into account we see that there is, besides the massive vector meson associated with weak hypercharge, an octet of massive axigluons. They arise for the same reason [36]: There are two independent $\mathrm{SU}(3)_C$-gauge groups associated with the L- and R-boundary respectively, and this (SU(3) × SU(3))-symmetry is broken down to SU(3) by the Higgs mechanism induced by the nontrivial minimum of V.

One can write formulae in SU(3)-covariant form. Writing for all the gauge fields G_R

$$u_R = u_R^W u_R^C, \qquad u_R^W \in \ \mathrm{U}(1), u_R^C \in \mathrm{SU}(3).$$

One sets

$$\xi(x) = \sigma(x)u_R^W u_R^C$$
$$\xi(x)^{*2}(x) = \sigma(x)^2\bar{u}_R^{W2}\bar{u}_R^C \wedge \bar{u}_R^C = \sigma(x)^2\bar{u}_R^{W2}u_R^C$$

where $\bar{u}_R^C \wedge \bar{u}_R^C$ is the antisymmetric tensor product of two factors in the 3^* representation of SU(3).

For simplicity we go to the gauge when the cross bulk SU(3) parallel transporters are set to 1. Then the above formulae remain valid "mutatis mutandis". In particular, we may retain eq.(5.22) for the lattice gauge field u'_μ of the massive vector mesons. But now

$$u'_\mu \in \ \mathrm{U}(1) \ \times \ \mathrm{SU}(3).$$

The formula for the axigluon masses is the same as in eq.(5.23) except that g' is to be replaced by the strong gauge coupling constant, and factors $\frac{1}{9}$ and $\frac{4}{9}$ are to be replaced by $\mathrm{tr}\left(\frac{\lambda_a}{2}\right)^2 = \frac{1}{2}$, where λ_a are the Gell-Mann matrices.

Chiral color is not anomaly free in four dimensions for the standard model's fermion content. On the other hand, the five-dimensional continuum theory with a single color SU(3) from which we supposedly started is anomaly free by hypothesis. This raises the question whether there are not further fermionic remnants in the effective theory. The following simple system of exotic fermions is known to suffice [13] to cancel anomalies:

$$(\mathbf{6}^*, \mathbf{1}, -\frac{1}{3}) \oplus (\mathbf{1}, \mathbf{6}, -\frac{1}{3})$$

that is a fundamental fermion transforming according to the six-dimensional representation of color SU(3) with conventional charge and a dichromatic fermion multiplet with no electric charge.

Fermionic terms on R

Of course, one also has to reformulate the conventional fermionic term on the R-boundary

$$\bar{q}_R(x)u_R(x, x+\mu)q_R(x+\mu).$$

We imagine that this term is to be pulled down on the 4-dimensional L-boundary with the u_R-factor that appears in the nonunitary Higgs field

$$\bar{q}_R(x)u_R(x, R; x, L)^* u_R(x, x+\mu)u_R(x+\mu, R; x+\mu, L)q_R(x+\mu).$$

5.6.1 The photon and its extra modes

As we already mentioned, the massless photon interacts equally with right- and left-handed quarks and mediates interactions between them. This is the reason why in our possible assignment of section 5.1.1 the gauge groups in the bulk and on the boundary have the right-handed abelian gauge group $\mathrm{G}_R = \mathrm{U}(1)$ in common. This choice leads to the fact that we do not have to insert an interface term between R-boundary and bulk in this special model. The vector potential should fulfill Neumann boundary conditions on R and L. (If it fulfilled Dirichlet boundary conditions it would vanish on the boundary.) Then the lowest mode is independent of the fifth direction. But there are additional modes forming Kaluza-Klein towers.

The photon in the bulk thus may have extra modes (for example massless scalars). Think of polarization in the fifth direction. We can demand that the field strength tensor $F_{\mu 5} = \partial_\mu A_5 - \partial_5 A_\mu$ vanishes on the boundary. If A_μ is independent of the fifth direction, then $F_{\mu 5} = 0$ implies that $A_5 = \text{const.}$ on the boundary. This eliminates massless modes, as we will see in the following theorem.

Theorem 14 *Suppose we impose boundary conditions*

$$\begin{aligned} \partial_5 A_\mu = 0 \ &\textit{on the boundary.} \\ \textit{and we gauge to } A_5 = 0 \ &\textit{in the bulk.} \end{aligned} \tag{5.24}$$

The extent of the fifth direction is d. Then the homogenous Maxwell equations admit massless solutions A^0_μ which are x_5-independent and satisfy 4-dimensional Maxwell equations, plus massive modes with masses $m_n = \frac{n\pi}{d}$, with $n = 1, 2, 3...$

Proof: Assuming Minkowski metric the wave equation is

$$\partial_M \partial^M A_N - \partial_N \partial^M A_M = 0$$

with $M, N = 0, 1, 2, 3, 5$. With eq.(5.24) one obtains

$$\partial_5 \partial^\mu A_\mu = 0 \text{ and} \tag{5.25}$$
$$(\Box - \partial_5^2) A_\mu - \partial_\mu \partial^\nu A_\nu = 0. \tag{5.26}$$

Because of the Neumann boundary conditions and an extent of the bulk from $z_5 = 0$ to $z_5 = d$, we write

$$A_\mu(\mathbf{x}, x^5) = A_\mu^0(\mathbf{x}) + \sum_{n=1}^{\infty} A_\mu^n(\mathbf{x}) \cos(\frac{n\pi x^5}{d}).$$

Consider first the x_5-independent modes A_μ^0. Equation (5.26) reduces to

$$\Box A_\mu^0 - \partial_\mu \partial^\nu A_\nu^0 = 0.$$

By an x_5-independent gauge transformation we may enforce $\partial^\nu A_\nu^0 = 0$, and we obtain the wave equation for massless modes. For the other modes eq.(5.26) leads to

$$(\Box + (\frac{n\pi}{d})^2) A_\mu^n - \partial_\mu \partial^\nu A_\nu^n = 0.$$

This is the well known wave equation for a massive vector field. It implies

$$(\Box + (\frac{n\pi}{d})^2) A_\mu^n = 0.$$

and $\partial^\nu A_\nu^n = 0$. **q.e.d.**

This is interpreted as Kaluza-Klein tower, we find a tower of massive vector mesons in the bulk in our model.

5.6.2 The two different bulk's gauge groups

In our model we introduced the bulk's holonomy group H = $\mathbb{R}_*$ SL(3,$\mathbb{C}$) $\times$ $\mathbb{R}_*$ SL(3,$\mathbb{R}$) $\times$ U(1). According to sections 4.3 and 4.4, H^D = SL(3,$\mathbb{R}$) and H^U = SL(3,$\mathbb{C}$) belong to different classes of spontaneous symmetry breaking. In case of SL(3,$\mathbb{R}$) the Higgs part does not transform according to the adjoint representation of its maximal compact subgroup SO(3) (that is, three-dimensional). But the Higgs part of SL(3,$\mathbb{C}$) with its 8-dimensional maximal compact subgroup SU(3) does so. The W-symmetry for SL(3,$\mathbb{R}$) is broken completely. In case of SL(3,$\mathbb{C}$) there remains an unbroken U(1) $\times$ U(1). Nevertheless, we obtain three minima in the three-dimensional representation.

Consequences for the CKM-matrix

The decomposition as it is used in eqs.(5.1) and (5.2) of the nonunitary parallel transporter allows us to perform a change of variables

$$\begin{aligned} \bar{q}_R &\mapsto \bar{q}'_R = \bar{q}_R A^*_R u_R = \begin{pmatrix} \bar{q}^D_R & \bar{q}^U_R \end{pmatrix} \begin{pmatrix} A^{D*}_R & 0 \\ 0 & A^{U*}_R \end{pmatrix} \begin{pmatrix} u^D_R & 0 \\ 0 & u^U_R \end{pmatrix} \\ q_L &\mapsto q'_L = A_L q_L = \begin{pmatrix} A^D_L & 0 \\ 0 & A^U_L \end{pmatrix} \begin{pmatrix} q^D_L \\ q^U_L \end{pmatrix}, \end{aligned}$$

so that $\bar{q}_R \Phi q_L \mapsto \bar{q}'_R \hat{\rho} d(\eta) q'_L$ if we neglect fluctuations of ρ. The SU(2) gauge term is

$$\bar{q}_L \gamma^\mu (\partial_\mu - A_\mu) q_L = \bar{q}_L \gamma^\mu \partial_\mu q_L + J^a_\mu A^\mu_a,$$

where A_μ is the vector potential, a the index of the isospin and J^a_μ the current.

$$J^a_\mu = \frac{1}{2} \bar{q}_L \gamma_\mu \tau^a q_L = \bar{q}'_L A_L \gamma_\mu \tau^a A^*_L q_L,$$

where τ^a are 6×6-matrices that involve the Pauli-matrices. The CKM-matrix $C = A^D_L A^{U*}_L$ appears in the charged current

$$J^{\mathrm{ch}}_\mu = \frac{1}{2} \bar{q}'_L A^D_L \gamma_\mu A^{U*}_L q'_L = \frac{1}{2} \bar{q}'_L C \gamma_\mu q'_L$$

when we change to mass eigenstates. Our $\mathrm{H}^U = \mathrm{SL}(2,\mathbb{C})$ involves an unbroken subgroup $\mathrm{U}(1) \times \mathrm{U}(1) \propto \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)) \subset \mathrm{SU}(3)$, that is spanned by $e^{i\lambda_3\vartheta_3}$, $e^{i\lambda_8\vartheta_8}$ and $e^{-i(\lambda_3\vartheta_3+\lambda_8\vartheta_8)}$. The Lie algebra is spanned by the Gell-Mann matrices λ_3 and λ_8. We perform a gauge transformation

$$\begin{aligned} &\bar{q}'_R \mapsto \bar{q}'_R h \\ &q'_L \mapsto h^* q'_L, \quad h \in \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)), \end{aligned}$$

that describes a local symmetry of the Higgs term in the bulk but not on the boundary. But this transformation leads to a transformation of the CKM-matrix

$$C = h A^D_L A^{U*}_L.$$

The unbroken symmetry is part of the global symmetry, which is used to transform away four phases in the CKM-matrix. Usually the CKM-matrix is only dependent of four instead of eight parameters. In our model this global symmetry originates in the local symmetry of the unbroken subgroup. This is, again, an argument for the model proposed here.

5.7 Further comments on the model

Leptons

In the presence of leptons one needs extra nonunitary parallel transporters.

A parallel transporter $\phi^l \in \mathrm{SL}(3,\mathbb{C})$ that enters into the mass term of charged leptons $\propto \bar{e}_R \phi^l l_L$ is needed to give mass to them. An economical choice would be

$$\phi^l = \phi^D.$$

Another parallel transporter ϕ^ν would be needed to give mass to neutrinos. Most likely one would also need a Majorana mass term for right-handed neutrinos in order to invoke the seesaw mechanism. We refrain from speculating what ϕ^ν might be.

Chapter 6

Summary and outlook

Locality and gauge covariance of an effective theory are the main issues discussed in this thesis. In order to solve linear discretized differential equations via multigrid methods it has been shown how effective problem operators with good locality properties can be constructed in a general framework without the need to specify a Hilbert space a priori. By means of a twogrid, several schemes have been introduced that involve the solution of a high and low frequency problem. We investigated applicability to problems that are not translation invariant. The convergence rate of the so-called compatible relaxation has been computed. The basics are applicable to a variety of problems, without the need to first inspect the problem (or fix the domains that are represented on the coarser scale) and find a suitable mechanism afterwards. Without fixing moving frames in advance the blocks spins constructed here are gauge covariant and, of course, they are strictly local. The effective problem operator involves not necessarily unitary parallel transporters.

This motivates the study of renormalization group transformations with respect to the appearance of nonunitary parallel transporters. The features are explained in a one-dimensional model and can be generalized. Here the renormalization group flow of the Higgs potential and its asymptotic behavior was computed exactly. The ordinary gauge theory is a fixpoint and the starting point of the RG-flow. Our fundamental hypothesis is that Higgs fields appear in higher dimensional field theories. At the level of effective theories they are nonunitary parallel transporters along links of the lattice in extra directions and they assume values in a typically noncompact holonomy group. One might consider integrating out the nonunitary part. At some scale the wish to retain locality is expected to forbid a possible integration of nonunitary parts. Gauge invariant functions like Higgs potentials that depend on such nonunitary parallel transporters fulfill the essential biinvariance property. This is important when explaining mechanisms of spontaneous symmetry breaking. Perturbation theoretical considerations would suggest that fermions generate negative contributions to the squared mass leading to spontaneous symmetry breaking.

Nonunitary parallel transporters define a noncompact holonomy group H. Patterns of spontaneous symmetry breaking have been discussed by applying group theoretical tools. One has to investigate the form of the Weyl group of the pair (H_s,A), where H_s is the semi simple factor of the whole reductive Lie group H. The Weyl-symmetry may be broken completely, partially broken or not broken at all. For the case of partial symmetry breaking a list of possible candidates for unbroken subgroups has been compiled. A five-dimensional model is presented. The particles live on two four-dimensional branes a small distance d apart. We have shown how the Cabibbo-Kobayashi-Maskawa matrix is computable in principle due to explicit symmetry breaking by the boundary on which left-handed particles live. For the given assignments of gauge and holonomy groups, the features of the resulting four-dimensional model are discussed. The main features of the standard model has been obtained. Quark masses can be determined by minimization of the Higgs potential. It is shown that the model predicts the existence of extra massive vector mesons.

It is a dream to compute these quark masses and the CKM-matrix explicitly. One would also like to include the leptonic sector and understand better the comprehension of color in the proposed model. One may wish to consider more complicated models where higher dimensional spaces intersect with boundaries, as well as models with other assignments of gauge groups. Furthermore, one is interested in a generalized mechanism of how Higgs fields appear out of gauge fields, including approaches with orbifolding [39]. It would be important to generalize the classification of possible unbroken symmetry groups to general Higgs fields which arise out of gauge fields. The basics about gauge covariance outlined in chapter 2 may lead to the idea to consider other interesting mechanisms on the way to get to effective theories from the GUT to the Planck scale. Obviously, preserving gauge covariance does not imply that the gauge groups on different scales need to be the same. Studying renormalization group flows with respect to this idea in detail would be an interesting topic. In order to apply a mechanism based on the local practical interpolation operators, which appear in the linear problem, one has to renormalize the effective operators. The question of a generalization of the standard renormalization theory for problems that are not translation invariant has not yet been answered. Concerning our method of the solution of linear equations, one would like to demonstrate that it works for some known difficult problems, including the Helmholtz equation and other problems coming from hyperbolic equations.

Appendix A

Generalized Neumann and tight Dirichlet boundary conditions

Suppose a division of the set of sites of Λ^0 into disjoint nonempty subsets x is given. The edges $\langle zw \rangle$ of x are all edges in Λ^0 with $z \in x$ and $w \in x$. The union X of these grids is also a grid with several disconnected pieces. The boundary ∂X of X consists of links $\langle z, w \rangle$ with z, w in different blocks. Given the boundary ∂X, we define an operator L^N with Neumann boundary conditions for arbitrary positive (or positive semidefinite) L as specified. The definition depends on a choice of scalar products which make fibres into Hilbert spaces. They may come from a positive diagonal operator D. We define L^N such that

$$L^N \leq L.$$

Further we define tight Dirichlet boundary conditions. An operator with such boundary conditions is denoted by L^{tD}, with

$$L^{tD} \geq L.$$

In case of the ordinary Laplacian it differs from the standard definition of $-\triangle^D$ in that it obeys

$$(-\triangle)^{tD} u = -\triangle u$$

for scalar functions u obeying $u(z) = -u(w)$ for edges $\langle zw \rangle \in \partial X$. If a function with this property is thought to be obtained by smooth interpolation from a function on the continuum, then this function has zeroes "in the middle" of the boundary edges, rather than at their sources.

Given scalar products in fibres $\mathbb{V}_z$, we may introduce orthonormal bases $\{e^z_j\}$ and dual bases $\{e^j_z\}$ in $\mathbb{V}_z$ with defining property

$$\langle e^j_z, e^w_k \rangle = \delta^j_k \delta^w_z$$

and translate operators into matrices M and vice versa. The adjoint matrix is denoted M^+, viz $M^+_{ij} = \bar{M}_{ji}$. Using this we define positive semidefinite operators

$$^wP_z e^z_i = \sum_j e^j_z (M^+M)^{\frac{1}{2}}_{ji} \quad \text{if} \quad L_{wz} e^z_i = \sum_j e^j_w M_{ji} \tag{A.1}$$

or $^wP_z = (L^+_{wz} L_{wz})^{\frac{1}{2}}$ for short, to be read as an equation for matrices. If the pair w, z is not an edge, we set $^wP_z = 0$.

Definition 3 *If L is positive semidefinite, an operator with generalized Neumann boundary conditions on ∂X is defined by*

$$L^N_{zw} = \begin{cases} 0 & \textit{for } \langle zw \rangle \in \partial X \\ L_{zw} & \textit{for } z \neq w \textit{ and } \langle zw \rangle \notin \partial X \\ L_{zz} - \epsilon \sum_{z':\langle z'z \rangle \in \partial X} {}^{z'}P_z & \textit{for } z = w. \end{cases}$$

where $\langle zw \rangle$ stands for the edge between z and w and $\epsilon = 1$. For positive L an operator with tight Dirichlet boundary conditions L^{tD} is defined by the same formula with $\epsilon = -1$. The definition is extended to negative definite L by $L^N = -(-L^N)$ and $L^{tD} = -(-L^{tD})$., but now $\epsilon = -1$ for Neumann boundary conditions and $\epsilon = 1$ for tight Dirichlet boundary conditions.

For $L = -\triangle$ the definition of L^N agrees with the standard definition of $-\triangle^N$.

The definition of an operator L^N is nonlinear in L. It depends on the scalar product. There is no guarantee that it yields an optimal choice of an operator L^N with the desired properties. It can happen that one may add positive diagonal boundary terms while still maintaining the inequality $L^N \leq L$.

Lemma 7 *Let M be any not necessarily square matrix, and let M^+ be its adjoint, i.e. $M^+_{ij} = \bar{M}_{ji}$. Then the block matrix*

$$\mathcal{M} = \begin{pmatrix} (MM^+)^{\frac{1}{2}} & M \\ M^+ & (M^+M)^{\frac{1}{2}} \end{pmatrix}$$

is positive semidefinite.

Proof: The matrix M admits a singular value decomposition $M = UDV^+$, where D is a positive semidefinite diagonal matrix, $U^+U = 1$ and $V^+V = 1$. It follows that

$$(MM^+)^{\frac{1}{2}} = UDU^+, (M^+M)^{\frac{1}{2}} = VDV^+,$$

since this obeys $(MM^+)^{\frac{1}{2}}(MM^+)^{\frac{1}{2}} = UD^2U^+ = MM^+$ and similarly for M^+M. Let $\varphi = (v, w)^T$. One computes $\langle \varphi, \varphi \rangle = \langle D(v' + w'), v' + w' \rangle \geq 0$ with $v' = U^+v, w' = V^+w$. Therefore $\mathcal{M}$ is positive semidefinite. **q.e.d.**

Proposition 14 *Let D be the diagonal part of L and $P : \mathcal{V} \mapsto \mathcal{V}'$ the diagonal operator with entries*

$$P_{zz} = \sum_w {}^wP_z.$$

If L is positive semidefinite, and L^{tD} and L^N are defined as in definition 3, then

$$D + P \geq L^{tD} \geq L \geq L^N \geq D - P$$

Proof of theorem 14: For $\epsilon = \pm 1$ we define

$$\mathcal{L}^{\epsilon}_{zw} = \begin{pmatrix} {}^wP_z & \epsilon L_{zw} \\ \epsilon L_{wz} & {}^zP_w \end{pmatrix} : \mathcal{V}_z \oplus \mathcal{V}_w \mapsto \mathcal{V}'_z \oplus \mathcal{V}'_w.$$

Using the matrix representations (A.1) of P and L, we deduce from lemma 7 that $\mathcal{L}^{\epsilon}_{zw}$ is positive semidefinite both for $\epsilon = 1$ and $\epsilon = -1$. For arbitrary sets B of edges, define the operator ${}^{\epsilon}L^B : \mathcal{V} \mapsto \mathcal{V}'$ by

$$\langle {}^{\epsilon}L^B u, v\rangle = \frac{1}{2} \sum_{\langle zw\rangle \notin B} \langle \mathcal{L}^{\epsilon}_{zw}(u_z \oplus u_w), v_z \oplus v_w\rangle.$$

By positive semidefiniteness of $\mathcal{L}^{\epsilon}_{zw}$, every term in the sum is positive semidefinite. From this we deduce monotonicity in B: If $B \supset C$ then ${}^{\epsilon}L^B \leq {}^{\epsilon}L^C$. An easy calculation shows that

$$\begin{aligned} {}^1L^{\partial X} &= P + L^N - D \\ {}^{\epsilon}L^{\emptyset} &= P + \epsilon(L - D) \\ {}^{-1}L^{\partial X} &= P - L^{tD} + D. \end{aligned}$$

Therefore monotonicity of ${}^{\epsilon}L^B$ in B implies $L^{tD} \geq L Ł^N$ and positivity of ${}^{\epsilon}L^{\partial X}$ implies $L^N \geq D - P$ and $D + P \geq L^{tD}$. **q.e.d.**

Appendix B

Balanced Neumann Neumann method

Here the balanced Neumann Neumann method for the solution of the discretized Laplace equation

$$-\triangle u = f$$

is described. (u is a real function on the grid Ω with Dirichlet boundary conditions on the boundary $\partial\Omega$.)

This is a method that uses blocks which overlap in boundaries, and exactly orthogonal high and low frequency spaces $\mathcal{V}_0$ and $\mathcal{V}_1$. The blocks are labeled by x. The total grid Ω is divided into blocks Ω_x, which overlap in boundary points. The set of boundary points of Ω_x is denoted by $\partial\Omega_x$, and

$$\Gamma = (\bigcup_x \partial\Omega_x) \backslash \partial\Omega$$

is the union of all boundaries, excluding the boundary of Ω. The Laplacian $\triangle^{N,x}$ on Ω_x with Neumann boundary conditions on $\partial\Omega_x$ is well defined, positive semidefinite, and possesses a zero mode which is constant on Ω_x. These zero modes will span the low frequency space $\mathcal{V}^1$. In our language this means that an energy cutoff ϵ is chosen such that there is exactly one eigenmode of the problem operator with Neumann boundary conditions below ϵ, and the restriction operator is constructed from these zero modes.

The interpolation is done with the help of the notion of an harmonic extension. A function u on Ω is called *harmonic* if

$$\triangle u = 0.$$

Given a function v on the boundaries of $\cup_x \partial\Omega_x$, which vanishes on $\partial\Omega$, the harmonic extension u of v, denoted by

$$u = \mathcal{H}v$$

is a function u on Ω, which is harmonic and obeys $u(z) = v(z)$ for $z \in \cup_x \partial\Omega_x$. A point $z \in \cup_x \partial\Omega_x$ may be in the boundary of several blocks Ω_x. To handle this, a partition of unity is used. It employs functions ν_x on $\cup_x \partial\Omega_x$, which are defined by

$$\nu_x^{-1}(z) = \text{ number of blocks whose boundary contains } z.$$

It follows that $\sum_x \nu_x(z) = 1$ for $z \in \Gamma$. ν_x will appear multiplying functions v on boundaries.

In the following, we shall first describe the high and low frequency spaces $\mathcal{V}_0$ and $\mathcal{V}_1$, and subsequently the preconditioners to solve the high frequency problem. It is customary in application of the balanced Neumann Neumann method to make very simple coarse grid corrections corresponding to the poor low frequency preconditioner $(D^1)^{-1}$.

B.1 High and low frequency spaces

On starts from the following decomposition of $\mathcal{V}$ into two parts

$$\mathcal{V} = \tilde{\mathcal{V}} + \sum_x \mathcal{W}_x,$$

where $\mathcal{W}_x$ consists of functions with support on Ω_x, which vanish on $\partial\Omega_x$ and $\tilde{\mathcal{V}}$ consists of harmonic extensions $\mathcal{H}(v)$ of functions v on $\cup_x \partial\Omega_x$ which vanish on $\partial\Omega$. $\sum_x \mathcal{W}_x$ will become part of the high frequency space $\mathcal{V}_0$ while the low frequency space $\mathcal{V}_1$ is contained in $\tilde{\mathcal{V}}$. It will be specified below.

$$\mathcal{V}_0 = \tilde{\mathcal{V}}_1^\perp + \sum_x \mathcal{W}_x \tag{B.1}$$

where $\tilde{\mathcal{V}}_1^\perp$ is the orthogonal complement of $\mathcal{V}_1$ in $\tilde{\mathcal{V}}$ with respect to the scalar product $a(\cdot,\cdot)$ furnished by the problem operator $-\triangle$, viz. $a(\cdot,\cdot) = \vec{\nabla} u \vec{\nabla} v$. The elements u of $\tilde{\mathcal{V}}_1^\perp$ are called *balanced*. They fulfill

$$a(u', u) = 0 \text{ for all } u' \in \mathcal{V}_1.$$

Their use gives the name to the method. $\mathcal{V}^1$ consists of real functions $x \mapsto U(x) \in \mathbb{R}$ and $\mathcal{V}_1$ consists of interpolates $\mathcal{J}U$, which are constructed as follows. Regard $U(x)$ as a function on $\cup_y \partial\Omega_y$, which is constant on $\partial\Omega_x$ and vanishes elsewhere. Set

$$\mathcal{J}U = \mathcal{H}(\nu_x U) \in \mathcal{V}_1.$$

Note the locality properties of the interpolation. Suppose that $v \neq 0$ on $\partial\Omega_x$ while it vanishes on the rest of Γ. Then its harmonic extention is only non-vanishing on Ω_x and on its neighbors, i.e. on those blocks Ω_y whose boundaries share a point with $\partial\Omega_x$. Given $\mathcal{V}_1$, there will be an orthogonal projector P_1

$$P_1 : \tilde{\mathcal{V}} \mapsto \mathcal{V}_1$$

and $\tilde{\mathcal{V}}_1^\perp = (I - P_1)\tilde{\mathcal{V}}$. This identifies $\tilde{\mathcal{V}}_1^\perp$ and $\tilde{\mathcal{V}}/\mathcal{V}_1$, $(I - P_1) : \tilde{\mathcal{V}} \mapsto \tilde{\mathcal{V}}_1^\perp$.

B.2 Iterative solution of the high frequency problem

The solution of the high frequency problem has two components, one in $\sum \mathcal{W}_x$ and one in $\tilde{\mathcal{V}}_1^{\perp}$. The determination of the first component is a local problem, because the blocks are decoupled. This is easy to solve by iteration. The component in $\tilde{\mathcal{V}}_1^{\perp}$ will be found by Schwarz iterations, which involve the operator $T : \tilde{\mathcal{V}}_1^{\perp} \mapsto \tilde{\mathcal{V}}_1^{\perp}$. T will be constructed as a sum of contributions from blocks

$$T = (I - P_1) \sum T_x (I - P_1) : \tilde{\mathcal{V}}_1^{\perp} \mapsto \tilde{\mathcal{V}}_1^{\perp}$$

with T_x as follows. For selected blocks x, let $\mathcal{V}_x \subset \tilde{\mathcal{V}}$ be the space of harmonic extensions u of functions v on $\cup_y \partial\Omega_y$, which vanish on the complement of $\partial\Omega_x$. Introduce a scalar product $\tilde{a}_x$ on $\mathcal{V}_x / \mathcal{V}_x \cap \mathcal{V}_1$

$$\tilde{a}_x(u_1, u_2) = \langle -\triangle^{N.x} u_1, u_2 \rangle = \langle -\triangle^{N.x} \mathcal{H}(\nu_x v_1), \mathcal{H}(\nu_x v_2) \rangle.$$

Remember that $\tilde{\mathcal{V}}_1^{\perp} \approx \tilde{\mathcal{V}} / \mathcal{V}_1$ consists of balanced functions. Finally one specifies the operator

$$T_x : \tilde{\mathcal{V}} \mapsto \mathcal{V}_x / \mathcal{V}_1 \cap \mathcal{V}_x$$

by the defining equation

$$\tilde{a}_x(T_x u, [u']) = a(u, u'), \quad [u'] \in \mathcal{V}_x / \mathcal{V}_1 \cap \mathcal{V}_x.$$

The right hand side is well defined, i.e. independent of the choice of representative u' of the equivalence class $[u']$.

Remark that the balancing condition on $u \in \tilde{\mathcal{V}}_1^{\perp}$ permits to rewrite the scalar product as

$$\tilde{a}_x(u_1, u_2) = \langle [-\triangle^{N,x} + \kappa \mathcal{C}^* D^1 \mathcal{C}] u_1, u_2 \rangle$$

for any κ, D^1 using the aforementioned restriction operator $\mathcal{C}$, which is constructed from the zero modes. As is illustrated in the main text, this possibility is the key to the proof of lower bounds, which guarantee low condition number and thereby fast convergence of the high frequency iteration. A proof that is in this spirit is found in [49].

Appendix C

Harmonic analysis

Let $f(\phi)$ be a SU(2)-biinvariant function on the holonomy group H. We expand $f(\phi)$ in spherical functions $P^{\Xi}(\Phi)$ that are associated with the unitary representation Ξ of the holonomy group H. The expansion reads

$$f[\phi] = \int d\Xi \tilde{f}(\Xi) P^{\Xi}(\phi). \tag{C.1}$$

$d\Xi$ is the restriction of the Plancherel measure on H to representations that contribute to the expansion of SU(2)-biinvariant functions. The transform is

$$\tilde{f}(\Xi) = \int_{\mathrm{H}} d\phi f(\phi) P^{\Xi}(\phi^{-1}). \tag{C.2}$$

We will call $\tilde{f}(\Xi)$ Fourier coefficients. The spherical functions are special cases of representation functions of H that appear in the decomposition of SU(2)-biinvariant functions.

C.1 The groups $\mathbb{R}_+$ and $\mathbb{R}_+$SU(2)

Consider the multiplicative group $\mathbb{R}_+$, $r \in \mathbb{R}_+$ and the one-dimensional representations $P^s(r) \equiv T^s(r) = r^{-s}$ labeled by $s \in i\mathbb{R}$, where T^s stands for the representation operator. They act on (1-dimensional) Hilbert spaces $\mathcal{H}^s$. The unitary representations with $T^s(r^{-1}) = T^s(r)^*$ are then labeled by $s \in i\mathbb{R}$ due to $r^{-s} = \overline{r^s}$.
Thus we get

$$f(r) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} T^s(r) \tilde{f}(s) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} r^{-s} \tilde{f}(s),$$

where $\frac{ds}{i}$ is the Plancherel measure and

$$\tilde{f}(s) = \int_0^{\infty} dr r^{-1} T^s(r^{-1}) f(r) = \int_0^{\infty} dr r^{s-1} f(r).$$

Remark:

- Integration with the left-invariant Haar measure on the multiplicative group $H = \mathbb{R}_+$ is given by $\int r^{-1} dr f(r)$.

- For H $= \mathbb{R}_+$ SU(2) with $\phi = ru$, $r \in \mathbb{R}_+$, $u \in \mathrm{SU}(2)$ we require the left-invariant Haar measure, which obeys

$$\int d\phi f(\phi_1 \phi) = \int d\phi f(\phi).$$

  We make the ansatz $\int d\phi f(\phi_1\phi) = \int r^k dr du f(r_1 u_1 r u)$. This measure is left-invariant for $k = -1$ and yields

$$d\phi = d(ru) = r^{-1} dr du = (d\ln r)(du).$$

We can label the representations of H $= \mathbb{R}_+\mathrm{SU}(2)$ by $\chi = (s, l)$

$$T^{sl}(\rho u) = r^{-s} T^l(u) : \mathcal{H}^\chi \mapsto \mathcal{H}^\chi,$$

where $\mathcal{H}^\chi$ is the $(2l+1)$-dimensional representation space of SU(2) with $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Again the unitary representations are labeled by imaginary s. Since the representations of SU(2) are unitary, $T^{sl}(ru)^{-1} = T^{sl}(ru)^*$. The harmonic analysis leads to

$$\begin{aligned} f(ru) &= \mathrm{tr} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_l \tilde{f}_l(s) T^{sl}(ru) \\ &= \mathrm{tr} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_l \tilde{f}_l(s) r^{-s} T^l(u), \\ \tilde{f}_l(s) &= \int_0^\infty du dr T^l(u^{-1}) r^{s-1} f(ru). \end{aligned}$$

$\tilde{f}_l(s)$ is a linear operator in the $(2l+1)$-dimensional representation space of SU(2). We now apply this to functions $f(\phi)$, which are biinvariant in the sense that $f(u_1 \phi u_2) = f(\phi)$ for $u_1, u_2 \in \mathrm{SU}(2)$ and $\phi \in$ H. For H $= \mathbb{R}_+\mathrm{SU}(2)$ SU(2)-biinvariance implies that $f(ru)$ is independent of $u \in \mathrm{SU}(2)$. This leads to the simplification

$$f(ru) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \tilde{f}(s) r^{-s}, \tag{C.3}$$

$$\tilde{f}(s) = \int_0^\infty dr r^{s-1} f(ru) \tag{C.4}$$

independent of u.

C.2 The quantum-mechanical Lorentz group SL(2,ℂ)

Every element a of the noncompact group H = SL(2,ℂ) can be parametrized as

$$a = u_1 d(\eta) u_2, u_1, u_2 \in \mathrm{SU}(2), d(\eta) = \begin{pmatrix} e^{-\frac{\eta}{2}} & 0 \\ 0 & e^{\frac{\eta}{2}} \end{pmatrix}. \tag{C.5}$$

The group SL(2,ℂ) is a twofold covering of the Lorentz group SO(3,1), whose representations are labeled by $\chi = (l, \rho)$ with $l = 0, \pm 1, \pm 2, ...$ The representations of SL(2,ℂ) are labeled in the same way $\chi = (l, \rho)$, but with $l = 0, \pm\frac{1}{2}, \pm 1, ...$ The representation operators $T^\chi(a)$ act on Hilbert spaces $\mathcal{H}^\chi$ and the expansion of a square integrable function $f(a)$ on SL(2,ℂ) with the Plancherel measure $d\chi$ on the group is

$$f(a) = \int d\chi \quad \mathrm{tr}_\chi \left(\tilde{f}(\chi) T^\chi(a)\right) \tag{C.6}$$

$$= \sum_l \int_{-\infty}^{\infty} (l^2 + \rho^2) \tfrac{d\rho}{8\pi} \quad \mathrm{tr}_\chi \left(\tilde{f}(\chi) T^\chi(a)\right), \tag{C.7}$$

where tr_χ is the trace over $\mathcal{H}^\chi$ and $\tilde{f}(\chi)$ are operators in $\mathcal{H}^\chi$. They are obtained of

$$\tilde{f}(\chi) = \int_{\mathrm{SL}(2,\mathbb{C})} da f(a) T^\chi(a^{-1}). \tag{C.8}$$

da is the *Haar measure on* $\mathrm{SL}(2,\mathbb{C})$.

Proposition 15 *If we decompose* $a \in \mathrm{SL}(2,\mathbb{C})$ *into a product of two* $\mathrm{SU}(2)$*-matrices and a diagonal matrix* $d(\eta)$

$$a = u_1 d(\eta) u_2$$

as in eq.(C.5) the left- and right-invariant measure on $\mathrm{SL}(2,\mathbb{C})$, *normalized as in [42] is given by*

$$da = \frac{1}{16} \sinh^2(\eta) d\eta du_1 du_2.$$

PROOF: In the parametrization

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

the Haar measure is given by

$$da = \left(\frac{i}{2}\right)^3 |\alpha|^{-2} d\alpha d\bar{\alpha} d\beta d\bar{\beta} d\gamma d\bar{\gamma}$$

[42] and, thus, depends on six real parameters $\alpha = q_1 + iq_2$, $\beta = q_3 + iq_4$ and $\gamma = q_5 + iq_6$. If we parametrize

$$u_i = u_z(\varphi_i) u_y(\theta_i) u_z(\psi_i), \quad (i = 1, 2, \varphi_2 = 0)$$

$$u_z(\varphi) = \begin{pmatrix} e^{-\frac{i\varphi}{2}} & 0 \\ 0 & e^{\frac{i\varphi}{2}} \end{pmatrix}, \quad u_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

one gets the Haar measure depending on the parameters $\eta, \varphi_1, \theta_1, \theta_2, \psi_1, \psi_2$ by computing the functional determinant J of the transformation

$$\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma} \to \eta, \varphi_1, \theta_1, \theta_2, \psi_1, \psi_2$$

and multiplication with $\left(\frac{i}{2}\right)^3 |\alpha|^{-2}$. Evaluating everything using MATHEMATICA yields

$$da = \frac{1}{16} \sinh^2(\eta) \sin(\theta_1) \sin(\theta_1) d\eta d\varphi_1 d\psi_1 d\psi_2,$$

where $du_1 = \sin\theta_1 d\varphi_1 d\psi_1$ and $du_2 = \sin\theta_2 d\psi_2$. **q.e.d.**

The set of functions

$$\begin{aligned} \psi^l_m(u) &= (2k+1)^{\frac{1}{2}} T^k_{l,m}(u) \\ k &= |l|, \quad k = 0, 1, 2, ..., \quad -k \le m \le k \end{aligned} \tag{C.9}$$

forms a canonical basis of $\mathcal{H}^\chi$, where $T^l_{m,m'}$ is the matrix element of a representation of SU(2). For matrix elements in the canonical basis we write

$$T^\chi_{kmk'm'} = (\psi^k_m, T^\chi(a)\psi^{k'}_{m'}) = \int du \overline{\psi^k_m(u)} T^\chi(a) \psi^{k'}_{m'}(u),$$

where du denotes the normalized Haar measure on SU(2). Because of decomposition (C.5) we can write

$$T^\chi_{km_1k'm_2}(a) = \sum_q T^k_{m_1m}(u_1) T^\chi_{kmk'm}(d(\eta)) T^k_{m_2m}(u_2)$$

with

$$T^l_{mm'}(a) = \left[\frac{(l+m)!(l-m)!}{(l+m')!(l-m')!}\right]^{\frac{1}{2}} \sum_n \binom{l+m'}{n} \binom{l-a_2}{l+m-n} a^n_{11} a^{l+m-n}_{12} a^{l+m'-n}_{21} a^{n-m-m'}_{22}$$

For an element $u \in \mathrm{SU}(2)$ we have

$$T^\chi_{kmk'm'}(u) = \delta_{kk'} T^k_{mm'}(u)$$

[42]. Again there is a simplification when this is applied to functions $f(a)$ with the biinvariance property $f(u_1 a u_2) = f(a)$ for $u_1, u_2 \in \mathrm{SU}(2)$. $\mathcal{H}^\chi$ decomposes into irreducible representations of SU(2). As seen from the canonical basis (C.9) the $(2k+1)$-dimensional representation occurs if $k \ge |l|, k - l \in \mathbb{Z}$. Thus, in particular, the trivial representation appears only for $l = 0$. The canonical basis (C.9) $\{\psi^k_m\}$ in $\mathcal{H}^l$ involves vectors which transform according to the $(2k+1)$-dimensional representation of SU(2) $(m = -k, .., k)$. $m = 0$ if $k = 0$ and one gets

$$P^\rho(a) = (\psi^0_0, T^{(0,\rho)}(a)\psi^0_0) = \frac{2\sin(\frac{1}{2}\eta\rho)}{\rho \sinh \eta}. \tag{C.10}$$

Thus, the expansion simplifies to

$$\begin{aligned} f(a) \equiv f(\eta) = & \int_{-\infty}^{\infty} \tfrac{\rho^2 d\rho}{16\pi} \tilde{f}(\rho) \quad (\psi_0^0, T^{(0,\rho)}(a)\psi_0^0) \\ = & \int_{-\infty}^{\infty} \tfrac{\rho^2 d\rho}{16\pi} \tilde{f}(\rho) \quad \frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta}, \end{aligned} \tag{C.11}$$

where $(\psi_m^k, \tilde{f}(\chi)\psi_{m'}^{k'}) = \delta_{l0}\delta_{k0}\delta_{k'0}\tilde{f}(\rho)$ and

$$\tilde{f}(\rho) = \int \sinh^2(\eta) d\eta f(\eta) \frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta} = \tilde{f}(-\rho). \tag{C.12}$$

Lemma 8 *The inversion formula (C.11) and in particular the normalization factor* $\frac{1}{8\pi}$ *can be obtained by using the orthogonality relation of the* sin*-function*

$$\int_{-\infty}^{\infty} d\rho \sin\frac{1}{2}\eta\rho \sin\frac{1}{2}\eta'\rho = 2\pi[\delta(\eta-\eta') - \delta(\eta-\eta')] \tag{C.13}$$

.

PROOF: First, we verify eq.(C.13). With $\sin\frac{1}{2}\eta\rho = \frac{1}{2i}(e^{i\eta\rho/2} - e^{-i\eta\rho/2})$ we get

$$\begin{aligned} \rightarrow \quad & \int_{-\infty}^{\infty} d\rho \sin(\frac{1}{2}\eta\rho)\sin(\frac{1}{2}\eta'\rho) \\ = \; & -\frac{1}{4}\int d\rho\big(e^{i(\eta+\eta')\rho/2} + e^{-i(\eta+\eta')\rho/2}\big) - \big(e^{i(\eta-\eta')\rho/2} + e^{-i(\eta-\eta')\rho/2}\big) \\ = \; & -\frac{1}{4}\big(2\delta(\frac{\eta+\eta'}{2}) - 2\delta(\frac{\eta-\eta'}{2})\big) \\ = \; & 2\pi[\delta(\eta-\eta') - \delta(\eta-\eta')]. \end{aligned}$$

Inserting eq.(C.12) in $\mathcal{N}\int d\rho\rho^2\tilde{f}(\rho)\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta}$ leads to the normalization factor in the inversion formula.

$$\begin{aligned} f(\eta) &= \mathcal{N}\int d\rho\rho^2\big(\frac{2}{\rho}\int \sinh\eta' d\eta' f(\eta')\sin(\frac{1}{2}\eta'\rho)\big)\frac{2\sin(\frac{1}{2}\eta\rho)}{\rho\sinh\eta} \\ &= \mathcal{N}\int d\rho \int \frac{\sinh\eta'}{\sinh\eta} f(\eta') 4\sin(\frac{1}{2}\eta\rho)\sin(\frac{1}{2}\eta'\rho) \\ &= 8\pi\mathcal{N}[\delta(\eta-\eta') - \delta(\eta-\eta')]f(\eta')\frac{\sinh\eta'}{\sinh\eta} \\ &= 8\mathcal{N}\pi[f(\eta) + f(-\eta)] = 16\mathcal{N}\pi f(\eta). \qquad \textbf{q.e.d.} \end{aligned} \tag{C.14}$$

Lemma 9 (Convolution) *Under harmonic analysis on* SL(2, $\mathbb{C}$)*, convolution goes into products. Thus, if the square integrable functions* f_i *admit the expansion*

$$f_i(a) = \int d\chi tr_\chi \left(\tilde{f}_i(\chi)T^\chi(a)\right)$$

for $i = 1...n$*, then*

$$\int da_1...da_n f_1(a_1)...f_N(a_n)\delta(a_1...a_n a^{-1}) = \int d\chi tr_\chi \left(\tilde{f}_1(\chi)...\tilde{f}_n(\chi)T^\chi(a)\right).$$

Herein, $d\chi$ *is the Plancherel measure on* $\mathrm{SL}(2,\mathbb{C})$*, and* $T^{\chi}(a)$ *are the representation operators for representations of the unitary principal series.*

This is well known, and generalizes to the other groups H.

C.3 The groups GL(2,$\mathbb{C}$) and $\mathbb{R}_+$SL(2,$\mathbb{C}$)

Every element $\phi \in$GL(2,$\mathbb{C}$), which has eight real parameters, can be parametrized as

$$\phi = \zeta a, \zeta \in \mathbb{C} \setminus \{0\}, a \in \mathrm{SL}(2,\mathbb{C}). \tag{C.15}$$

Note that the pair $(-\zeta, -a)$ determines the same element as (ζ, a) and that SU(2)-biinvariant functions f obey $f(-\phi) = f(-\zeta a) = f(\zeta a) = f(\phi)$ because $(-\mathbf{1}) \in \mathrm{SU}(2)$. Here we combine harmonic analysis on SL(2,$\mathbb{C}$) with harmonic analysis on the multiplicative group $\mathbb{C}\setminus\{0\}$. With $\zeta = re^{i\varphi}$, $\varphi \in [0, 2\pi]$ and $r = |\zeta|$ considered as an element of the dilatation group we get

$$f(\phi = \zeta a) = \underbrace{\sum_{l\in\mathbb{Z}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \int d\chi}_{\int d\Xi}\, e^{il\varphi} r^s \mathrm{tr}\big(\tilde{f}^{ls}(\chi) T^{\chi}(a)\big).$$

where $d\chi$ is regarded as the Plancherel measure of the representations of SL(2,$\mathbb{C}$) and $r^s e^{il\varphi}$ is the one dimensional representation (l, s) of $\mathbb{C} \setminus \{0\}$. Thus the representations of GL(2, $\mathbb{C}$) are labeled by $\Xi = (l, s, \chi)$. According to our SU(2)-biinvariance property we can replace χ by ρ as in section C.2. Again $T^{\chi}(a)$ is the representation operator of SL(2,$\mathbb{C}$) because we can write for the spherical function of GL(2,$\mathbb{C}$)

$$P^{\Xi}(\phi = re^{i\varphi} a) = e^{il\varphi} r^s P^{\rho}(a).$$

The transformation formulae for H = GL(2, $\mathbb{C}$) are

$$f(\phi = \zeta a) = \sum_{l\in\mathcal{Z}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho^2 d\rho}{\pi} \tilde{f}^{ls}(\rho) e^{il\varphi} r^s P^{\rho}(a),$$
$$\tilde{f}^{ls}(\rho) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-il\varphi} r^{-s-1} \int da f(\zeta a) P^{\rho}(a).$$

As for SL(2,$\mathbb{C}$), we have $P^{\rho}(a) = \frac{2\sin(\frac{1}{2}\eta\rho)}{\rho \sinh \eta}$ and we can gauge away the SU(2)-part.

The case H = $\mathbb{R}_+$SL(2,$\mathbb{C}$) is a direct derivative of H = $\mathbb{R}_+$SU(2) and H = SL(2,$\mathcal{C}$) and thus a simplification of H = GL(2, $\mathbb{C}$). For $\phi = ra \in \mathbb{R}_+\mathrm{SL}(2,\mathbb{C})$, $r \in \mathbb{R}_+, a \in$SL(2,$\mathbb{C}$) we get

$$f(ra) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} r^s \int_{-\infty}^{\infty} \frac{\rho^2 d\rho}{\pi} \tilde{f}^{s}(\rho) P^{\rho}(a)$$
$$\tilde{f}^{s}(\rho) = \int_0^{\infty} dr r^{-s-1} \int da P^{\rho}(a).$$

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