$N = 2$ vacua in electrically gauged $N = 4$ supergravities

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Abstract

In this thesis we study $N = 2$ vacua in gauged $N = 4$ supergravity theories in four-dimensional spacetime. Using the embedding tensor formalism that describes general consistent magnetic gaugings of an ungauged $N = 4$ matter-coupled supergravity theory in a symplectic frame with $SO(1,1) \times SO(6,n)$ off-shell symmetry we formulate necessary conditions for partial supersymmetry breaking and find that the Killing spinor equations can be solved for the embedding tensor components. Subsequently, we show that the classification of theories that allow for vacua with partial supersymmetry amounts to solving a system of purely algebraic quadratic equations. Then, we restrict ourselves to the class of purely electric gaugings and explicitly construct a class of consistent super-Higgs mechanisms and study its properties. In particular, we find that the spectrum fills complete $N = 2$ supermultiplets that are either massless or BPS. Furthermore, we demonstrate that (modulo an abelian Lie algebra) arbitrary unbroken gauge Lie algebras can be realized provided that the number of $N = 4$ vector multiplets is sufficiently large. Finally, we compute the relevant terms of the effective action below the scale of partial supersymmetry breaking and argue that the special Kähler manifold for the scalars of the $N = 2$ vector multiplets has to be in the unique series of special Kähler product manifolds.

Zusammenfassung

Gegenstand der vorliegenden Arbeit sind $N = 4$ Supergravitationstheorien in vierdimensionaler Raumzeit mit $N = 2$ Vakua. Um solche Theorien zu konstruieren, verwenden wir den Einbettungstensorformalismus, welcher allgemeine konsistente magnetische Eichungen einer a priori ungeeichten $N = 4$ Supergravitationstheorie mit Materiekopplungen in einem symplektischen Rahmen mit $SO(1,1) \times SO(6,n)$ off-shell Symmetrie beschreibt. Zunächst formulieren wir die notwendigen Bedingungen für die partielle Supersymmetriebrechung und stellen fest, dass die Killingspinorgleichungen nach den Einbettungstensorkomponenten aufgelöst werden können. Als nächstes zeigen wir, dass die Klassifikation der Theorien mit einem Vakuum, welches nur einen Teil der Supersymmetrie respektiert, durch die Lösungen eines Systems rein algebraischer, quadratischer Gleichungen gegeben ist. Dann beschränken wir uns auf rein elektrische Eichungen und konstruieren eine Klasse konsistenter sogenannter Superhiggsmechanismen und studieren deren Eigenschaften. Wir finden heraus, dass in solchen Theorien das Spektrum durch vollständige $N = 2$ Supermultipletts gegeben ist, welche entweder masselos oder BPS sind. Ferner legen wir dar, dass abgesehen von einer abelschen Lie-Algebra beliebige ungebrochene Eich-Lie-Algebren realisiert werden können, so denn die Anzahl der $N = 4$ Vektormultipletts genügend groß ist. Schlussendlich rechnen wir die relevanten Terme der effektiven Theorie unterhalb der Supersymmetriebrechungsskala aus und argumentieren, dass die spezielle Kählermannigfaltigkeit für die skalaren Felder aus den $N = 2$ Vektormultipletts in der einzigen Serie von Produktmannigfaltigkeiten, die gleichzeitig speziell-Kähler sind, liegen muss.

TO WHOM IT MAY CONCERN ;-)

Contents

Chapter 1

Introduction

1.1 Supersymmetry

Supersymmetry has been a key concept in particle physics since its theoretical discovery in the early eighties [1–4]. Since then it has almost always been omnipresent as a tool in model building of nature even though, as of this writing, there are still no indications for supersymmetric realizations in nature at higher energies [5]. On the other hand, owing to the fact that even the most powerful collision experiments such as the Large Hadron Collider can only probe a rather limited energy range — up to a few terra electronvolts — it has also not been possible to exclude supersymmetry at a higher energy scale. While in physics the final word on whether or not supersymmetry exists in nature can only be given by future experiments, it is worth mentioning that supersymmetry has become a subject of its own in mathematics owing to its rich mathematical structure. In what follows we will introduce the main idea of supersymmetry and its extension as well as outline its appealing features.

While supersymmetry can also be present in non-relativistic quantum systems [6], it is usually thought of as an additional symmetry of relativistic classical or quantum field theory. In the context of a local quantum field theory in a four-dimensional spacetime it has been shown in [7] that supersymmetry is the only possible symmetry extension (on top of spacetime and gauge symmetry). More precisely, the extended supersymmetry algebra with or without central charges is the only graded Lie algebra that generates symmetries of the S-matrix — the central object of interest in phenomenology — in a relativistically consistent way. The evasion of the Coleman-Mandula theorem [8] was possible by relaxing the symmetry Lie group to what is called a super Lie group whose super Lie algebra also includes fermionic generators. In fact, the N-extended super Lie algebra of Minkowski spacetime is an extension of the Lie algebras of the Poincaré group and the gauge symmetry that contains N additional generators Q^A ($A = 1, \dots, N$) each of which is a Majorana or Weyl spinor with respect to the Poincaré group [9]. Following the paradigm of quantum field theory a particle in a supersymmetric theory is associated to a field transforming under an irreducible representation of the supersymmetry algebra.[1] Such a representation is usually called supermultiplet in allusion to the fact that it can be decomposed in terms of irreducible representations of the Poincaré symmetry into fields both with integer spin and half-integer spin. These fields are then referred to as supersymmetry partners which all have the same mass owing to the fact that $P_\mu P^\mu$ (where P^μ generate translations in Minkowski spacetime) is a Casimir operator of the supersymmetry algebra.

From a phenomenological point of view supersymmetry is a prominent means (of many others) to generalize the standard model of particle physics which is an $SU(3) \times SU(2) \times U(1)$ gauge theory with a chiral spectrum in Minkowski spacetime [10–15]. While the standard model, being a renormalizable quantum field theory [16], is in accordance with collider experiments to an extraordinary accuracy [5], it is not a theory of gravity and therefore cannot be considered as a fundamental theory of nature. Instead, it is rather thought of as an effective theory that can be used to make predictions for experiments below a certain energy scale where gravity is negligible and no new physics is present, but has to be augmented at a higher scale. In fact, at the latest at the reduced Planck scale $M_P = (8\pi G_{\text{Newton}})^{-1/2} = 2.4 \times 10^{18}$ GeV gravity is expected to play a significant role and should enter the theory. However, in an effective theory the momentum cutoff that can be introduced to regularize a priori ill-defined, divergent loop integrals in correlation functions of the quantum fields has to be taken seriously in that it no longer just appears in an intermediate step of a renormalization scheme but rather is a physical scale. In the standard model this leads to the "hierarchy problem" (references [1] in the review [17]): The quantum correction to the Higgs self-energy and, thus, the Higgs squared mass is quadratic in the cutoff. In the absence of new physics the natural cutoff would be M_P and, as a result, the quadratic correction would be some 30 orders of magnitude higher than the experimental Higgs mass at around 125 GeV [18, 19] which is considered unnatural as it requires fine tuning of the bare mass. Furthermore, it is not only the Higgs mass that is highly sensitive to high-scale physics because even though the self-energies for the fermions and massive gauge bosons do not obtain quadratic corrections (due to the chiral gauge symmetry), their masses depend via the Higgs mechanism on the Higgs mass and therefore also inherit the ultraviolet sensitivity. Supersymmetry provides an elegant solution to the hierarchy problem in that it consistently eliminates the quadratic corrections of scalar squared masses. In fact, in an $N = 1$ supersymmetric standard model the couplings among the superpartners are such that the loop contributions to the quadratic divergence cancel at all orders of perturbation theory. This is a consequence of the mass degeneracy of superpartners in a supersymmetric theory and the absence of quadratically divergent corrections to the

[1] Strictly speaking, one usually requires irreducibility with respect to a (super)symmetry that is extended by the discrete CPT symmetry.

self-energy of the fermions.

On the other hand, superpartners of degenerate mass have not been found in nature and, hence, the challenging question in any supersymmetric model that aims to describe the real world is how to break supersymmetry while maintaining a solution to the hierarchy problem. In the (next-to) minimal supersymmetric standard model the Lagrangian of the supersymmetrized standard model is augmented by terms that explicitly break supersymmetry without reintroducing quadratic divergencies [20]. In particular, these so-called soft-breaking terms contain mass terms for the scalars and for the gaugini the latter of which are the supersymmetry partners of the vector bosons. Being renormalizable quantum field theories satisfying the criterion of naturalness these models are of phenomenological interest, in particular, as far as the search of supersymmetric partners is concerned. However, they do not explain how supersymmetry is broken. The most prominent approaches to this question rely on spontaneous supersymmetry breaking in a so-called hidden sector that only couples weakly to the ordinary sector by means of gravity or gauge interactions.

In theories with spontaneously broken supersymmetry (or hidden supersymmetry) the vacuum respects only a part of the supersymmetry of the Lagrangian. However, for global supersymmetry (i.e. one whose parameters are independent of Minkowski spacetime) the fermionic analogue [4, 21] of the Goldstone theorem [22] gives rise to massless fermionic fields — the Goldstini — which in the absence of a hidden sector are in conflict with experimental data. In contrast, breaking supersymmetry spontaneously can be realized in theories with local supersymmetry, where the supersymmetry parameters may depend on the spacetime. In fact, in analogy with the Higgs mechanism in which massless Goldstone modes give rise to the longitudinal polarization of massive vector bosons, the massless Goldstini provide the missing degrees of freedom of massive gravitini that formerly were part of a supermultiplet including the graviton [23–25]. Such a super-Higgs mechanism will be the central part of this thesis albeit in a less phenomenological, less realistic way given that we will be dealing with extended $N > 1$ supergravity theories that do not allow for a chiral spectrum.

Finally, before discussing locally-supersymmetric theories we mention another phenomenological motivation for global supersymmetry given in the realm of grand unified theories in which the standard model gauge group is embedded into a higher-dimensional gauge group. Here, supersymmetry plays an important role in constructing models with gauge coupling unification below the Planck scale. In fact, the most prominent example is the $SU(5)$ extension [26] of the (next-to) minimal supersymmetric standard model in which the three fundamental gauge couplings of $SU(3) \times SU(2) \times U(1)$, when normalized in terms of the $SU(5)$ coupling and evolved according to the renormalization group flow, tend to unify at the one-loop level into the one of $SU(5)$ [27, 28].

1.2 Supergravity

A locally-supersymmetric theory, which was first constructed by [29, 30], necessarily includes gravity and is thus referred to as supergravity. On a conceptual level this can be best understood in terms of the superspace formalism ([9, 31] for $N = 1$ in four spacetime dimensions) where locally-supersymmetric Lagrangians are constructed in terms of superfields (subject to certain constraints) that depend on a supermanifold the construction of which is based on an extension of the spacetime coordinates by spinorial coordinates. Then local supersymmetry amounts to invariance under general such coordinate transformations and thus naturally includes general relativity. In a less rigorous way, the necessity to include gravity also follows from the fact that the super Lie algebra of local supersymmetry involves the spacetime-dependent derivative operator ∂_μ. In fact, the anticommutation relation

$$\{Q^A, \bar{Q}_B\} \sim \delta^A{}_B \, \Gamma^\mu \partial_\mu \,, \tag{1.1}$$

where the Γ^μ furnish a matrix representation of the Clifford algebra, implies that the commutator of two supersymmetry transformations after being bosonized through local spinorial parameters $\epsilon_1(x), \epsilon_2(x)$ closes into a diffeomorphism of the spacetime manifold defined by the vector field $\bar{\epsilon}_1(x)\Gamma^\mu \epsilon_2(x)\partial_\mu$. As a consequence, in locally-supersymmetric theories general coordinate transformations necessarily are part of the bosonic symmetry group.

Of course, the inclusion of gravity is an exciting feature in view of potential theories that might unify all fundamental interactions of nature. However, a gravity theory has a dynamical metric (vielbein) and, hence, the background spacetime can no longer be chosen at will as it must be part of a solution of the equations of motion of all fields. This seems to be one of the difficulties why, as of this writing, a consistent quantum field theory of gravity in which fluctuations around the background field configuration are quantized has not been found. Furthermore, even when quantizing a weak gravitational field around a Minkowski background, it seems that the quantum field theory is not renormalizable. In fact, for fluctuations of mass dimension one, infinitely many interaction terms of negative mass dimensions arise in the Lagrangian and therefore the theory is not power-counting renormalizable. Moreover, it has been shown that divergencies in scattering amplitudes of four-dimensional pure gravity start to emerge at the two-loop level [32]. The question whether or not finitely many counterterms suffice to eliminate all divergencies at any loop level is hard to answer given that an analogous computation at the three loop level has not yet been carried out due to its complexity. However, in adding supersymmetry to a theory including gravity the ultraviolet behavior is ameliorated in that divergencies start to emerge at a higher loop level. In fact, in the case of maximal[2] supergravity, i.e. $N = 8$, in four dimensions there are even promising

[2]The attribute “maximal” refers to a theory with maximal supersymmetry but without massless fields

indications of perturbative finiteness [33–35] but a proof thereof is still lacking.

In quest of an ultraviolet complete quantum theory that consistently describes all fundamental interactions of nature, string theory emerged more than 40 years ago and is still considered as a promising approach that deserves to be studied in more depth. At the classical level it generalizes the notion of a point particle to one of a one-dimensional extended object to be referred to as (open or closed) string. Upon quantization of the classical oscillation modes of the string one finds an infinite tower of particles owing to the fact that in a higher-dimensional Minkowski spacetime background they transform in representations of the Lorentz symmetry. Their masses are proportional to the only dimensionful parameter — the string length l_s — of the theory. While all particles of positive mass squared decouple in the low energy theory at a scale $\ll l_s^{-1}$ there are finitely many massless degrees of freedom which in the case of the closed string in particular include the quantum excitations of a higher-dimensional graviton field. Being a quantum theory with a well-defined notion of an S-matrix, this sparked the hope that string theory might be the fundamental theory of nature. At the same time, it is a rather generic feature of string theory to include a tachyonic mode — a particle of negative mass squared — in the spectrum indicating that the background is unstable. However, stable string theories (type I, type IIa, type IIb, heterotic) have been constructed with higher-dimensional spacetime (as opposed to worldsheet) supersymmetry. In their construction GSO projections are used that project out degrees of freedom in such a way that the resulting spectra furnish complete irreducible representations of the supersymmetry algebra. In particular, all tachyons are eliminated while the graviton excitations remain in the spectra. In fact, owing to the fact that tachyonic representations of the respective supersymmetry algebras, in particular (1.1), do not exist, the tachyon can consistently be eliminated from the spectrum in that it no longer appears as virtual particles in string loops of scattering amplitudes. Another obvious virtue of such a superstring theory is the inclusion of spacetime fermions which are needed in order to eventually incorporate the standard model of particle physics. In superstring theories the critical dimension of the spacetime for which the Hilbert space does not have negative norm states is ten rather than 26 in the case of the bosonic string.

In the S-matrix approach [36, 37] an effective low energy field theory describing only the massless modes of a given superstring theory in ten dimensions can be constructed that reproduces the string scattering amplitude in the limit $p^2 l_s^2 \ll 1$ for external momenta p. The resulting action is a ten-dimensional supergravity theory with 32 (type IIa, type IIb) or 16 (type I, heterotic) real supercharges.[3] A vast landscape of vacua exists. The ones of particular phenomenological interest are those with a four-dimensional

of helicity 5/2 or higher.

[3] The number of real supercharges is the real dimension of the spinorial representation of the Lorentz symmetry of (higher-dimensional) Minkowski spacetime under which the supersymmetry generators transform.

spacetime and six compactified extra spatial dimensions. These are called string compactifications and an effective gravity field theory in four-dimensional spacetime can be obtained upon integrating out the extra dimensions. The construction of phenomenologically realistic vacua with an embedding of the standard model of particle physics is the subject of string phenomenology (e.g. in the heterotic string). However, here we will only comment on the supersymmetry of the effective four-dimensional theory. For (fluxless) backgrounds built from a product manifold of a four-dimensional maximally-symmetric spacetime and a six-dimensional compact manifold, it was found that the amount of four-dimensional supersymmetry depends on the geometry (here: holonomy) of the internal manifold [38]. In fact, in choosing appropriate internal manifolds one finds four-dimensional supergravity theories with $N = 8$, $N = 4$, $N = 2$, $N = 1$ or no supersymmetry at all.

While not surprising, it is worthwile to point out that N-extended supersymmetric theories are increasingly constrained the higher N is. For maximal supersymmetry $N = 8$ in four dimensions the smallest massless supermultiplet is already the supergravity multiplet with the graviton as its highest helicity component. Given that only one graviton field shall be present in the spectrum of the theory, the (70-dimensional) target manifold $E_{7,7}/(SU(8)/\mathbb{Z}_2)$ of the non-linear sigma model for the scalars is completely fixed [39, 40]. In contrast, in half-maximal supersymmetry $N = 4$, massless matter supermultiplets without a graviton exist. Hence, while the helicity-3/2 supermultiplet is of no "phenomenological" interest, the spectrum of an $N = 4$ supergravity may comprise in addition to the gravity multiplet a certain number $n \in \mathbb{N}$ of vector multiplets. For a given n, the scalar manifold is also completely determined by $N = 4$ supersymmetry [41–46]: It is the symmetric space

$$SL(2)/SO(2) \times SO(6,n)/SO(6)\times SO(n) \ , \tag{1.2}$$

where the first factor is the two-dimensional target manifold of the scalars in the gravity multiplet (the dilaton and axion) and the second factor refers to the one of the $6n$ scalars of the vector multiplets. On the other hand, $N = 2$ and $N = 1$ supergravity theories are less constrained in that the scalar manifolds are not completely fixed and not necessarily a homogeneous space while supersymmetry still constraints their geometry: For $N = 2$ supergravity, the scalar manifold of the vector multiplets has to be projective special-Kähler while the one of the hypermultiplets with highest helicity component 1/2 is required to be a quaternionic-Kähler manifold [47]. In theories with minimal supersymmetry $N = 1$, scalars only reside in chiral multiplets and their target manifold must be a Kähler manifold [9, 48].

While for $N < 4$ there are various ways to deform the supergravity theory, the only known deformations for $N = 4$ and $N = 8$ supergravity theories with a given spectrum are gaugings of a subgroup of the a priori global symmetry. In fact, as the spacetime is four-dimensional and thus the Hodge dual of a field strength two-form is again a two-form, even a subgroup of the electromagnetic duality[4] group embedded into a symplectic group can be consistently gauged. General such magnetic gaugings have been described by means of the embedding tensor formalism [51]. It is based on the principle of introducing a gauge-covariant derivative that involves a certain gauge connection. Schematically, one has

$$\partial_\mu \to \partial_\mu + \Theta_{\text{electric}}\, A_\mu + \Theta_{\text{magnetic}}\, A_\mu^{\text{dual}}\,, \tag{1.3}$$

where Θ_{electric} and Θ_{magnetic} are the embedding tensors that select a subset of the vector bosons and their dual magnetic gauge fields and, in doing so, define an embedding of the gauge group into the global on-shell symmetry group. Since compatibility with supersymmetry of the action requires that non-trivial gaugings in particular contribute to the supersymmetry transformations of the fermionic fields and also generate a scalar potential, the vacuum structure of the otherwise ungauged theory can be significantly changed. In particular, new vacua may arise that respect only some of the original supersymmetries. Hence, in gauging extended supergravity theories it may be possible to spontaneously break N-extended supersymmetry to $N' < N$ supersymmetry. Such breakings are referred to as partial supersymmetry breakings. It is the topic of this thesis to analyze the vacuum structure of gauged $N = 4$ supergravities [41–45, 52–54] with respect to the preserved supersymmetry. In particular, we will embark on a classification of consistent electrically-gauged $N = 4$ theories with $N = 2$ vacua. From now on we will always think of the supergravity theory as a classical field theory of its own, regardless of a possible ultraviolet quantum completion.

1.3 Partial supersymmetry breaking

Partial supersymmetry has been investigated since the early nineties [55–57]. In the first place, for phenomenological reasons, the possibility of an $N = 2$ supersymmetric theory that is spontaneously broken to $N = 1$ was studied and in the original theories it was found to be impossible [55, 56]. However, some ten years later deformations of the theories to circumvent the no-go theorem were found for both globally and locally supersymmetric $N = 2$ theories. In fact, in global $N = 2$ theories partial breaking turned out to be possible in the presence of electric and magnetic Fayet-Iliopoulos terms that are not aligned [58, 59]. Not much later, in supergravity, simple examples with partial breaking have been constructed by formulating the problem in a symplectic frame in which no prepotential exists for the special geometry associated to the vector multiplets [60–62]. After many more examples were found, a systematic analysis in $N = 2$ supergravity with general matter content was carried out recently [63–65] using the embedding tensor formalism [51].

[4]The electromagnetic dualities [47, 49, 50] are so-called on-shell symmetries in that they are symmetries of the combined set of equations of motion and Bianchi identities for the field strengths.

As to $N = 4$, it was not long after the construction of the electrically gauged supergravity theory coupled to a given number of vector multiplets [43, 44] that partial supersymmetry breaking was investigated [66, 67]. There it was found that the simplest class of backgrounds of electrically gauged supergravity theories must necessarily be Minkowski (cosmological constant $\Lambda = 0$) provided that at least one supersymmetry is preserved. Furthermore, they constructed examples of electrically gauged $N = 4$ supergravities that are spontaneously broken to $N = 2$ and $N = 1$, and showed that $N = 3$ vacua do not exist in this class of gaugings. In order to also allow for anti-de Sitter backgrounds, more general deformations were introduced via a set of $SU(1,1)$[5] phases — the so-called de Roo-Wagemans angles between the semi-simple factors of the gauge group [45, 68]. In the embedding tensor language, non-trivial such de Roo-Wagemans angles correspond to particular non-vanishing embedding tensor components that define proper (i.e. non-electric) magnetic gaugings [53, 54]. Using these additional deformations it was found that vacua with negative cosmological constant $\Lambda < 0$ and any supersymmetry $1 < N < 4$ exist. Many years later, more examples of partial supersymmetry breaking in $N = 4$ were discussed some of which are based on more general magnetically gauged $N = 4$ supergravities [69, 70]. Furthermore, their relation to string theory compactifications has been studied in some detail (see, for example, [71–75] and references therein).

Similarly to the analogous problem for partial breaking of $N = 2$ in [63], it was the original aim of this thesis to classify all the gaugings of the $N = 4$ supergravity with $N = 2$ vacua using the embedding tensor formalism [51]. However, while implicitly the crucial terms of the general magnetically-gauged Lagrangian of $N = 4$ supergravity are known in terms of the embedding tensors, supersymmetry and the closure of the gauge Lie algebra demand that the embedding tensors satisfy a set of algebraic linear and quadratic constraints the latter of which unfortunately turn out to be hard to be solved in full generality. We therefore mainly concentrate on the class of purely electric gaugings and solve the constraint equations as much as possible. While we are not able to fully solve them, we demonstrate that many consistent solutions exist. This enables us to explicitly check the super-Higgs mechanism. This thesis is based on the publication [76].

After this introduction we will review $N = 4$ supergravity theory in section 2. Therein, we will give the matter-coupled ungauged $N = 4$ supergravity in a symplectic frame with $SO(1,1) \times SO(6,n)$ off-shell symmetry. Within this frame we will discuss the electromagnetic duality and will give the precise embedding of the isometry group $SL(2) \times SO(6,n)$ of the scalar manifold (1.2) into the symplectic group $Sp(2(6+n), \mathbb{R})$. Then we discuss the embedding tensor formalism which yields consistent magnetically gauged supergravity theories. Apart from the bosonic Lagrangian (up to topological terms), we also review the supersymmetry variations of the fermions and fermion bilin-

[5] The scalar manifold associated to the scalars of the gravity multiplet is $SL(2)/SO(2) \cong SU(1,1)/U(1)$ the two of which are isomorphic to each other, see appendix B.1.

ear terms in the Lagrangian that are needed in order to discuss partial supersymmetry breaking. While eventually we restrict ourselves to purely electric gaugings, we close section 2 after elucidating the importance of magnetically gauged theories. In section 3, we will discuss a simple but important class of background solutions of $N = 4$ gauged supergravity. Subsequently, in order to construct a super-Higgs mechanism that describes the spontaneous partial breaking of supersymmetry, we study the supersymmetry transformations of the background and analyze the resulting Killing spinor equations. We find that the analysis can drastically be simplified by exploiting the symmetry of the theory: In fact, we argue that, first, it is without loss of generality that we can choose the vacuum to be at the origin of (1.2) and, secondly, that one can always bring the gravitini mass matrix to diagonal shape. Partial supersymmetry requires solutions of a linear system of equations in terms of the embedding tensor components, while in order for the gaugings to be consistent quadratic constraints on the embedding tensor components have to be satisfied. For simplicity, we therefore restrict ourselves to purely electric gaugings. In passing we will first demonstrate that non-trivial $N = 4$ vacua can be constructed. Then we focus on the construction of $N = 2$ vacua. To this end, we prepare ourselves by discussing the representation theory of $N = 2$ global supersymmetry and try to solve the quadratic constraints as much as possible. Some more technical details are given in appendix D. While being unable to fully solve all constraints we give all solutions for $n \leq 6$ and special solutions where $n \in \mathbb{N}$ is arbitrary. Eventually, in section 4 we will discuss some aspects of the $N = 2$ low-energy effective theory. For the solutions found in section 3, we compute all mass terms in an $N = 2$ vacuum and find that all fields fill complete $N = 2$ supermultiplets some of which turn out to be massive. In particular, we will analyze the Gravity/Goldstini sector in which the two gravitini associated to the broken supersymmetry directions as well as their superpartners become massive. Then we will show that at the $N = 2$ vacuum all massive fields have to fill out $N = 2$ BPS multiplets. Furthermore, we demonstrate that many different gauge Lie algebras can be realized that preserve the vacuum. Finally, we compute the terms of the $N = 2$ supersymmetric effective action that may potentially be used to verify that the geometry of the scalar field space is consistent with $N = 2$ supersymmetry.

Chapter 2

Gauged $N = 4$ supergravity in $D = 4$

In this chapter we will review $N = 4$ gauged supergravity in a four-dimensional spacetime. To a large extent it is based on [52–54]. Starting with the ungauged theory, we will discuss the electromagnetic duality [47, 49, 50] and present the crucial aspects of the embedding tensor formalism [51] which describes a general class of magnetic gaugings of the supergravity theory. Apart from the discrete parameter n that labels the number of $N = 4$ vector multiplets in the ungauged theory, these are the only known deformations of $N = 4$ supergravity. Finally, we focus on purely electric gaugings.

2.1 Ungauged theory

The simplest version of an $N = 4$ supersymmetric theory in four-dimensional spacetime that couples an $N = 4$ gravity multiplet to a number $n \in \mathbb{N}$ of $N = 4$ vector multiplets all of which are generically massless[1] is the so-called ungauged $N = 4$ supergravity. According to the representation theory of the supersymmetry algebra, the physical degrees of freedom of the gravity multiplet are the spacetime metric $g_{\mu\nu}$ of signature $(-, +, +, +)$ also known as the graviton, four helicity-3/2 fermions ψ^i_μ to be referred to as gravitini, $(i = 1, \ldots, 4)$, six vector bosons A^m_μ, $(m = 1, \ldots, 6)$, four helicity-1/2 fermions χ^i and two real scalars. Labeling the vector multiplets by $a = 1, \ldots, n$, the particle content of n vector multiplets consists of n vector bosons A^a_μ, $(4n)$ helicity-1/2 fermions λ^{ai} and $(6n)$ real scalars. Note that $N = 4$ helicity-3/2 multiplets are not part of the spectrum as the presence of helicity-3/2 fermions other than the superpartners of the graviton is

[1]Strictly speaking, the notion of mass depends on the background. In the ungauged theory, however, the absence of a scalar potential implies that for a large class of backgrounds all the fields are massless. In general, this will change upon gauging the theory, see section 2.4.

phenomenologically undesirable. While the theory is locally supersymmetric, it is ungauged in that the vector bosons $A_\mu^M = (A_\mu^m, A_\mu^a)$ are abelian and all fields are uncharged with respect to the gauge group $U(1)^{6+n}$. This is tantamount to having a Lagrangian in which vector potentials A_μ^M only appear in terms of their gauge-invariant field strengths. These are defined by

$$H_{\mu\nu}{}^M = 2\,\partial_{[\mu} A_{\nu]}{}^M \ . \tag{2.1}$$

For future use the spinor conventions for the fermions are summarized in appendix C.1.

The bosonic part of the Lagrangian of the ungauged theory is given by [54]

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bos.}} &= \tfrac{1}{2}R - \tfrac{1}{4}\,\text{Im}(\tau) M_{MN} H_{\mu\nu}{}^M H^{\mu\nu N} + \tfrac{1}{8}\,\text{Re}(\tau)\,\eta_{MN}\,\epsilon^{\mu\nu\rho\sigma} H_{\mu\nu}{}^M H_{\rho\sigma}{}^N \\ &\quad + \tfrac{1}{16}(\partial_\mu M_{MN})(\partial^\mu M^{MN}) - \tfrac{1}{4\,\text{Im}(\tau)^2}(\partial_\mu \tau)(\partial^\mu \tau^*) \ , \end{aligned} \tag{2.2}$$

where R is the Ricci-scalar of the spacetime metric $g_{\mu\nu}$, $e = \sqrt{|\det g|}$ and $\epsilon^{\mu\nu\rho\sigma}$ is totally antisymmetric with $\epsilon^{0123} = e^{-1}$. Furthermore, η_{MN} is the $SO(6, n)$-invariant metric

$$\eta = (\eta_{MN}) = (\eta^{MN}) = \text{diag}(\underbrace{-1, \ldots, -1}_{6 \text{ times}}, \underbrace{1, \ldots, 1}_{n \text{ times}}) \ . \tag{2.3}$$

The two real degrees of freedom of the scalars associated to the gravity multiplet are described by $\tau \in \mathbb{C}$ with $\text{Im}\,\tau > 0$ and their kinetic term is given in terms of a non-linear sigma model on the homogeneous space

$$SU(1,1)/U(1) \cong SL(2)/SO(2) \cong \{\tau \in \mathbb{C} \,|\, \text{Im}\,\tau > 0\} \ , \tag{2.4}$$

see appendix B.1 for details. The target manifold of the scalars of the vector multiplets is the homogeneous space

$$SO(6, n)/[SO(6) \times SO(n)] \ , \tag{2.5}$$

which is a $(6n)$-dimensional manifold. Its (left) cosets defined by identifications with respect to the right action of $SO(6) \times SO(n)$ on $SO(6, n)$ can be represented by $M(x) = (M_{MN})(x) = \mathcal{V}(x)\mathcal{V}^T(x)$ where $\mathcal{V}(x) \in SO(6, n)$. The index calculus for the coset spaces (2.4) and (2.5) is discussed in appendix B.2.

The full Lagrangian is gauge-invariant with respect to the gauge group $U(1)^{6+n}$, which infinitesimally acts on the vector bosons as

$$\delta_\Lambda A_\mu^M = \partial_\mu \Lambda^M \tag{2.6}$$

where the $\Lambda^M(x)$ are the local gauge parameters. All other fields transform trivially under this symmetry (due to the absence of charges). Furthermore, the full Lagrangian has a global $SO(6, n)$ symmetry which acts from the left on (2.5), i.e. on the scalars of the vector multiplets as

$$\mathcal{V}(x) \to g\,\mathcal{V}(x) \ , \tag{2.7}$$

and on the vector bosons as

$$A_\mu^M(x) \to g^M{}_N\, A_\mu^N(x)\ , \tag{2.8}$$

for $g^{-1} = (g^M{}_N) \in SO(6,n)$. All other fields transform trivially under this global symmetry. It is in this sense that capital indices $M, N, \ldots$ are referred to as $SO(6,n)$-indices. They can be raised/lowered using the $SO(6,n)$-invariant metric given in (2.3). Furthermore, the full Lagrangian has a local

$$H = U(1) \times SU(4) \times SO(n) \tag{2.9}$$

invariance that acts non-trivially both on the fermions and the scalars while the remaining fields are left invariant. The first two factors amount to the $U(4)$ R-symmetry of the supersymmetry algebra. As $SU(4)$ is the double cover of $SO(6)$, an element of $SU(4)$ unambiguously defines an element in $SO(6)$ and, hence, together with $SO(n)$ amounts to redefining the coset representative $\mathcal{V}(x) \in SO(6,n)$ (by means of right multiplication). Similarly, $U(1) \cong SO(2)$ is the maximal, compact subgroup of $SU(1,1) \cong SL(2)$ and reparametrizes the cosets therein by right multiplication. The representations of the fields with respect to this symmetry are given in table 2.1. By virtue of this symmetry, indices $i, j, \ldots \in \{1, \ldots, 4\}$ are fundamental $SU(4)$ indices, while the ones $a, b, \ldots \in \{1, \ldots, n\}$ refer to the fundamental representation of $SO(n)$. Note that this local symmetry is not a gauge symmetry since the gauge connection is composite and not a physical degree of freedom.

field	$SO(6,n)$	$SU(4) \times SO(n)$	$U(1)$ charges
$g_{\mu\nu}$	$\mathbf{1}$	$(\mathbf{1},\mathbf{1})$	0
ψ_μ^i	$\mathbf{1}$	$(\mathbf{4},\mathbf{1})$	$-1/2$
A_μ^M	$\square$	$(\mathbf{1},\mathbf{1})$	0
χ^i	$\mathbf{1}$	$(\mathbf{4},\mathbf{1})$	$3/2$
λ^{ai}	$\mathbf{1}$	$(\mathbf{4},\mathbf{n})$	$1/2$
$\mathcal{V} = \mathcal{V}_{SO(6,n)}$	$\mathcal{V} \to g\,\mathcal{V}$	$\mathcal{V} \to \mathcal{V}\,h(x)$	0
$\mathcal{V}_\alpha$	$\mathbf{1}$	$(\mathbf{1},\mathbf{1})$	1

Table 2.1: Representations of the fields with respect to the global symmetry $SO(6,n)$ and the local symmetry $U(1) \times SU(4) \times SO(n)$. Here $g \in SO(6,n)$ and $h(x) \in SO(6) \times SO(n)$, i.e. in particular, matter scalar representatives $\mathcal{V}$ are charged with respect to $SO(6) \sim SU(4)$. The components $\mathcal{V}_\alpha$ for $\alpha = 1,2$ form an $SL(2)$ "vielbein" which is explained in appendix B.2.

Note that in the ungauged theory a scalar potential is absent. Furthermore, the supersymmetry transformations on the fermionic fields for a maximally symmetric background are given by

$$\delta_\epsilon \psi_\mu^i = 2 D_\mu \epsilon^i, \qquad \delta_\epsilon \chi^i = 0, \qquad \delta_\epsilon {\lambda_a}^i = 0\,, \tag{2.10}$$

for all free indices and supersymmetry parameters ϵ^i that are Weyl spinors each of which forms the right-handed spinor part of a Dirac spinor. The differential operator D_μ contains the spin connection and in a maximally symmetric background simplifies to (C.12). As a consequence, as we will discuss in great detail in section 3.1, maximally symmetric backgrounds of the ungauged theory with pointlike scalar background configurations are necessarily $N = 4$ Minkowski vacua. Moreover, in the absence of a scalar potential all the scalars are moduli fields and the target manifolds (2.4) and (2.5) are therefore also referred to as moduli space of the ungauged $N = 4$ supergravity theory. At each such vacum the $N = 4$ spectrum consists of the massless $N = 4$ gravity multiplet and n massless $N = 4$ vector multiplets.

In addition to the symmetries already discussed, ungauged $N = 4$ supergravity also has a global $SO(1,1)$ symmetry. Parametrizing an orthochronous (a non-orthochronous) $SO(1,1)$ element[2]

$$\begin{pmatrix} \pm\cosh\theta & \sinh\theta \\ \sinh\theta & \pm\cosh\theta \end{pmatrix} \in SO(1,1) \subset SL(2) \tag{2.11}$$

in terms of the rapidity $\theta \in \mathbb{R}$, its action on the scalars τ and the field strengths $H^M_{\mu\nu}$ is given by

$$\begin{aligned} \tau \mapsto \tau' &= (\exp(\pm\theta))^{-2}\,\tau\,, \\ H^M_{\mu\nu} \mapsto H'^M_{\mu\nu} &= (\pm\exp(\pm\theta))\,H^M_{\mu\nu}\,, \end{aligned} \tag{2.12}$$

while all other fields transform trivially. These actions are obviously representations of $SO(1,1)$ and the Lagrangian, in particular its bosonic part in (2.2), is invariant. As we will see in the next subsection the global symmetries associated to $SO(1,1)$ and $SO(6,n)$, respectively, are particular cases of the electromagnetic duality.

2.2 Electromagnetic duality

Being a four-dimensional theory including $(6+n)$ vector bosons, ungauged $N = 4$ has an electromagnetic duality group which is the isometry group

$$G = SL(2) \times SO(6,n) \tag{2.13}$$

of the scalar manifold given as a product of the coset spaces (2.4) and (2.5). While in general such duality transformations are not symmetries of the Lagrangian, they leave the combined set of equations of motion and Bianchi identities invariant, i.e. the duality transformation of a solution gives another solution of the same set of equations. Sometimes such duality transformations are referred to as on-shell symmetries.

[2]Orthochronous/non-orthochronous elements of the Lorentzgroup $SO(1,1)$ in two dimensions preserve/flip the first component (the "time component") of the $SO(1,1)$ vector, respectively, see (2.12).

In passing we mention that the existence of the $SL(2)$ symmetry in the ungauged $N = 4$ supergravity theory that is only realized on-shell was found in toroidal reduction of ungauged $N = (1, 0)$ supergravity (16 real supercharges) in ten dimensions with a given number $m \in \mathbb{N}$ of $N = (1, 0)$ vector multiplets [41, 54, 77]. There, the gravity multiplet contains the Kalb-Ramond field which is a two form and gives rise to another two form in four dimensions. The latter can be dualized at the level of the equations of motion and Bianchi identities to a scalar. Note, however, that this reduction necessarily gives rise to $n = (6 + m)$ $N = 4$ vector multiplets.

The terms that involve the field strengths in the bosonic Lagrangian (2.2) are

$$- \tfrac{1}{4}\,\mathrm{Re}(N_{MN}) H^{M\mu\nu} H^{N}_{\mu\nu} + \tfrac{1}{8}\,\mathrm{Im}(N_{MN}) \epsilon^{\mu\nu\rho\sigma} H^{M}_{\mu\nu} H^{N}_{\rho\sigma}\,, \tag{2.14}$$

where

$$N = (N_{MN}) = \mathrm{Im}\,\tau\, M + i\,\mathrm{Re}\,\tau\,\eta \in \mathrm{Mat}(\mathbb{C}, (6+n)\times(6+n)) \tag{2.15}$$

depends on the scalar fields. The magnetically dual field strengths $G^{\mu\nu}_{M}$ are defined in terms of functional derivatives of the full action S:

$$G^{\mu\nu}_{M} = e^{-1}\epsilon^{\mu\nu\rho\sigma}\frac{\delta S}{H^{M\rho\sigma}}\,. \tag{2.16}$$

Using the decomposition of the two-forms $H^{M}_{\mu\nu}$ (and their magnetic duals) into their self-dual (anti-self-dual) components, e.g.

$$H^{M(\pm)}_{\mu\nu} = \tfrac{1}{2}(H^{M}_{\mu\nu} \pm \tilde{H}^{M}_{\mu\nu}), \qquad \left(H^{M(\pm)}_{\mu\nu}\right)^{*} = H^{M(\mp)}_{\mu\nu} \tag{2.17}$$

where $\tilde{H}^{M\mu\nu} = -\frac{1}{2}i\epsilon^{\mu\nu\rho\sigma}H^{M}_{\rho\sigma}$ is the Hodge dual of $H^{M}_{\mu\nu}$, the Bianchi identities for the field strengths read[3]

$$\partial^{\mu}\,\mathrm{Im}\,H^{M(\pm)}_{\mu\nu} = 0\,, \tag{2.18}$$

and the equations of motion for the field strengths are

$$\partial^{\mu}\,\mathrm{Im}\,G^{(\pm)}_{M\mu\nu} = 0\,. \tag{2.19}$$

An electromagnetic duality is a real transformation[4]

$$\begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{(\pm)}_{M\mu\nu} \end{pmatrix} \mapsto \begin{pmatrix} H'^{M(\pm)}_{\mu\nu} \\ G'^{(\pm)}_{M\mu\nu} \end{pmatrix} = S \begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{(\pm)}_{M\mu\nu} \end{pmatrix}\,, \tag{2.20}$$

for a symplectic matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2(n+6), \mathbb{R})\,, \qquad S^{T}\Omega S = \Omega\,,\ \text{with}\ \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\,, \tag{2.21}$$

[3] These are the integrability conditions $dH^{M} = 0$ needed in order to locally express the field strengths two-forms $H^{M} = dA^{M}$ in terms of vector potential one-forms A^{M}.

[4] In a slight abuse of notation column vectors are just denoted by their components with respect to the index M.

provided that at the same time an isometry acts on the scalars in precisely such a way that N transforms as[5]

$$N \mapsto N' = -i(C + iDN)(A + iBN)^{-1} . \tag{2.22}$$

In fact, the new field strengths and dual field strengths still satisfy the Bianchi identities (2.18) and the equations of motion (2.19) and the relation (2.16) is preserved. Furthermore, it is owing to the isometry that the equations of motion for the scalar fields are also satisfied by the new field strengths and transformed scalars. On the other hand, the Lagrangian is in general not invariant under the duality.

The embedding of the isometries (2.13) of the scalar manifold into

$$\begin{aligned} Sp(2(6+n),\mathbb{R}) &\cong USp(6+n, 6+n) \\ &= Sp(2(6+n),\mathbb{C}) \cap U(6+n, 6+n) \end{aligned} \tag{2.23}$$

maps the maximal compact subgroup of (2.13) into the maximal compact subgroup $U(6+n)$ of $USp(6+n, 6+n)$ [78]. For $SO(6, n)$ we find the explicit maps

$$\begin{array}{ccccc} SO(6,n) & \hookrightarrow & Sp(2(6+n),\mathbb{R}) & \xrightarrow{\sim} & USp(6+n,6+n) \\ g & \mapsto & \begin{pmatrix} g^{-1} & 0 \\ 0 & g^T \end{pmatrix} & \mapsto & \frac{1}{2}\begin{pmatrix} g^{-1}+g^T & g^{-1}-g^T \\ g^{-1}-g^T & g^{-1}+g^T \end{pmatrix} , \end{array} \tag{2.24}$$

while for $SL(2)$ the embedding turns out to be

$$\begin{array}{ccccc} SL(2) & \hookrightarrow & Sp(2(6+n),\mathbb{R}) & \xrightarrow{\sim} & USp(6+n,6+n) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} s_a & s_b\,\eta \\ s_c\,\eta & s_d \end{pmatrix} & \mapsto & \frac{1}{2}\begin{pmatrix} a+d+i(b-c)\eta & b+c-i(d-a)\eta \\ b+c+i(d-a)\eta & a+d-i(b-c)\eta \end{pmatrix} , \end{array} \tag{2.25}$$

where η is the $SO(6, n)$-invariant tensor defined in (2.3) and

$$\begin{pmatrix} s_a \\ s_b \\ s_c \\ s_d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} . \tag{2.26}$$

In fact, upon restricting to the maximal compact subgroups the embedded matrices (2.24) and (2.25) are block-diagonal and can be identified with unitary matrices. Furthermore, note that the two embeddings commute as required by the direct product (2.13). For the sake of completeness we state the isomorphism between $Sp(2N, \mathbb{R})$ and $USp(N, N)$:

$$Sp(2N,\mathbb{R}) = C^{-1}\, USp(N,N)\, C, \qquad C = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix} \in \mathrm{Mat}(\mathbb{C}, (2N)\times(2N)) . \tag{2.27}$$

[5] In the fermionic part also terms exist that are linear in the field strength. These couplings must also have a similiar transformation induced by the isometries of the scalar manifold [51].

The embedding of $SO(6,n)$ gives rise to the global $SO(6,n)$ symmetry discussed in subsection 2.1, see (2.8). In fact, from (2.7) it is clear that isometries of (2.5) precisely generate the transformed N' as in (2.22). As to $SL(2)$ one finds that $SL(2)$ acts as Möbius transformations on τ. In fact, for an $SL(2)$ element as in (2.25) one finds another $SL(2)$ element

$$\begin{pmatrix} s_d & -s_c \\ -s_b & s_a \end{pmatrix} \in SL(2) \tag{2.28}$$

which parametrizes the transformation of τ as

$$\tau \mapsto \tau' = \frac{s_d\,\tau - s_c}{-s_b\,\tau + s_a}\,. \tag{2.29}$$

This is a (non-linear) representation of $SL(2)$ that precisely generates the transformation of N given in (2.22). Note that embedding an element (2.11) of $SO(1,1) \subset SL(2)$ yields (2.12).

Raising the $SO(6,n)$ index of $G_{M\mu\nu}$ with η the transformation (2.24) is covariant,

$$\begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{M(\pm)}_{\mu\nu} \end{pmatrix} \mapsto \begin{pmatrix} H'^{M(\pm)}_{\mu\nu} \\ G'^{M(\pm)}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} g^{-1} & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{M(\pm)}_{\mu\nu} \end{pmatrix}. \tag{2.30}$$

Similarly, in terms of $SO(6,n)$ covariant field strengths $H^M_{\mu\nu}$ and their magnetic duals $G^M_{\mu\nu}$ an $SL(2)$ transformation given by (2.20) and (2.25) reads

$$\begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{M(\pm)}_{\mu\nu} \end{pmatrix} \mapsto \begin{pmatrix} H'^{M(\pm)}_{\mu\nu} \\ G'^{M(\pm)}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} s_a & s_b \\ s_c & s_d \end{pmatrix} \begin{pmatrix} H^{M(\pm)}_{\mu\nu} \\ G^{M(\pm)}_{\mu\nu} \end{pmatrix}, \tag{2.31}$$

and, hence, they also constitute a vector in the fundamental representation of $SL(2)$,

$$H^{M\alpha}_{\mu\nu} = \begin{pmatrix} H^{M+}_{\mu\nu} \\ H^{M-}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} H^{M}_{\mu\nu} \\ G^{M}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} H^{M(+)}_{\mu\nu} + H^{M(-)}_{\mu\nu} \\ G^{M(+)}_{\mu\nu} + G^{M(-)}_{\mu\nu} \end{pmatrix}, \tag{2.32}$$

where $\alpha = (+,-)$ is an $SL(2)$ index — not to be confused with the $(\pm)$ that label self-dual (anti-self-dual) combinations of the two forms, see (2.17). As a result, the two-form $H^{M\alpha}$ transforms in the bifundamental representation $(2,\Box)$ with respect to the global on-shell symmetry (2.13).

In view of the electromagnetic duality the H^{M+} are referred to as electric field strengths whereas H^{M-} are the magnetic dual field strengths. In giving the (bosonic part of the) Lagrangian, (2.2), which only contains the electric field strengths, a symplectic frame has implicitly been chosen. This is the $SO(1,1) \times SO(6,n)$ covariant electric frame in which $SO(1,1) \times SO(6,n)$ is a global symmetry of the Lagrangian, whereas the entire global $SL(2)$ symmetry is only realized on-shell. All other symplectic frames are obtained by symplectic rotations to be explained at the end of section 2.5. In general, their off-shell symmetries — the symmetries of the Lagrangian — may differ. However, in any symplectic frame the equations of motion for the electric field strengths (2.19) are at the same time the Bianchi identities for the magnetic fields strengths and, hence, the integrability conditions required to locally introduce magnetic vector potentials A_μ^{M-} by means of

$$H_{\mu\nu}^{M-} = 2\,\partial_{[\mu} A_{\nu]}^{M-}\,. \tag{2.33}$$

Of course, (2.1) locally defines the electric vector potentials $A_\mu^M = A_\mu^{M+}$. Also, $A_\mu^{M\alpha}$ transforms in the bifundamental representation $(2,\Box)$ of (2.13).

2.3 Embedding tensor formalism

The only known deformations of the ungauged $N = 4$ matter coupled supergravity theory consist in consistently gauging a subgroup of the on-shell global symmetry (2.13), i.e. in promoting it to a local/gauge symmetry of the Lagrangian. These are deformations in that the deformed action shall still be invariant with respect to deformed $N = 4$ supersymmetry transformations. On the other hand, the global on-shell symmetry of the ungauged theory in general no longer persists in the gauged version (but may still be partly realized). The gauging relies on the idea of minimally coupling a selection of the scalar fields to the vector potentials, in general, both to electric and magnetic vector potentials. Consistency of such general gaugings requires sophisticated amendments of couplings in order not to upset the counting of degrees of freedom and furthermore puts constraints on the possible gaugings (closure of the gauge Lie algebra, mutual locality). In the embedding tensor formalism such general gaugings can be described in terms of the covariant embedding tensor (with respect to the original on-shell symmetry) [51, 79]. The following discussion summarizes the relevant aspects of the embedding tensor formalism applied to $N = 4$ matter coupled supergravity [52–54].

The gauging is parametrized in terms of infinitesimal gauge transformations. One therefore considers the Lie algebra of the isometry Lie group (2.13). In the fundamental representations[6] of the real Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{so}(6,n)$, respectively, the components of generators (the basis) can be written as

$$(t_{\alpha\beta})_\gamma{}^\delta = \delta^\delta_{(\alpha}\,\epsilon_{\beta)\gamma}\,, \qquad (t_{MN})_P{}^Q = \delta^Q_{[M}\,\eta_{N]P}\,, \tag{2.34}$$

where $\epsilon_{\alpha\beta}$ is the $SL(2)$-invariant antisymmetric tensor with $\epsilon_{-+} = 1$. As before, lower case Greek letters $\alpha, \beta, \ldots$ denote fundamental $SL(2)$ indices, while $M, N, \ldots$ are vector indices associated to $SO(6,n)$. In view of electric and magnetic vector potentials $A_\mu^{M\alpha}$ that after the gauging are to transform in the adjoint representation of the gauge Lie

[6] This is the one of importance since the action of (2.13) on the fields can be entirely expressed using only fundamental representations. To this end, the $SL(2)/SO(2)$ vielbein is used (rather than the coordinate τ), see appendix B.2 for details.

algebra, one parametrizes the most general generators $X_{M\alpha}$ of the gauge Lie algebra as

$$X_{M\alpha} = \Theta_{M\alpha}{}^{\lambda\sigma}\,(t_{\lambda\sigma}\otimes \mathbb{1}) + \Theta_{M\alpha}{}^{RS}\,(\mathbb{1}\otimes t_{RS}) \tag{2.35}$$

in terms of the real embedding tensor components $\Theta_{M\alpha}{}^{\lambda\sigma}, \Theta_{M\alpha}{}^{RS} \in \mathbb{R}$. In fact, the embedding tensor defines an embedding of the gauge Lie algebra into the Lie algebra of the isometry Lie group. In components (2.35) reads

$$X_{M\alpha N\beta}{}^{P\gamma} = \Theta_{M\alpha}{}^{\lambda\sigma}\,\delta^{\gamma}_{(\lambda}\,\epsilon_{\sigma)\beta}\,\delta^{P}_{N} + \Theta_{M\alpha}{}^{RS}\,\delta^{P}_{[R}\,\eta_{S]N}\,\delta^{\gamma}_{\beta}\,. \tag{2.36}$$

According to the gauging prescription the partial derivative acting on scalar vielbeins is replaced by a gauge covariant derivative given by

$$\begin{aligned} D_\mu =&\partial_\mu - g\,A_\mu^{M\alpha}X_{M\alpha} \\ &\partial_\mu - g\,A_\mu^{M\alpha}\,\Theta_{M\alpha}{}^{\lambda\sigma}\,(t_{\lambda\sigma}\otimes\mathbb{1}) - g\,A_\mu^{M\alpha}\,\Theta_{M\alpha}{}^{RS}(\mathbb{1}\otimes t_{RS})\,, \end{aligned} \tag{2.37}$$

where $g\in\mathbb{R}$ is a redundant but useful overall gauge coupling constant.

The embedding tensor components can be decomposed as

$$\begin{aligned} \Theta_{M\alpha}{}^{\beta\gamma} &= \tilde{\xi}_{M\delta}\,\epsilon^{\delta(\beta}\delta^{\gamma)}_{\alpha} + \hat{\Theta}^{(\beta\gamma)}_{M\alpha}\,, \\ \Theta_{M\alpha}{}^{NP} &= f_{\alpha[MRS]}\,\eta^{RN}\eta^{SP} + \delta^{[N}_{M}\,\xi^{P]}_{\alpha} + \eta_{MQ}\left(\hat{\Theta}^{Q[NP]}_{\alpha} + \hat{\Theta}^{[N|Q|P]}_{\alpha}\right)\,, \end{aligned} \tag{2.38}$$

with the following tracelessness conditions,

$$\begin{aligned} \hat{\Theta}^{(\beta\alpha)}_{M\alpha} &= 0\,, \\ \eta_{MQ}\left(\hat{\Theta}^{Q[NM]}_{\alpha} + \hat{\Theta}^{[N|Q|M]}_{\alpha}\right) &= 0\,. \end{aligned} \tag{2.39}$$

In fact, these are the tensor product decompositions

$$\begin{aligned} (\mathbf{2},\square)\otimes(\mathbf{3},\cdot) &= (\mathbf{2},\square)\oplus(\mathbf{4},\square)\,, \\ (\mathbf{2},\square)\otimes\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) &= \left(\mathbf{2},\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\oplus(\mathbf{2},\square)\oplus\left(\mathbf{2},\begin{smallmatrix}\square\square\\\square\;\;\end{smallmatrix}\right)\,, \end{aligned} \tag{2.40}$$

in terms of irreducible representations of the Lie algebra $\mathfrak{sl}(2)\oplus\mathfrak{so}(6,n)$. Here, $(\mathbf{2},\square)$ denotes the bifundamental representation, while $(\mathbf{3},\cdot)\equiv(\mathbf{3},\mathbf{1})$ and $\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$ denote the adjoint representation of $\mathfrak{sl}(2)$ and $\mathfrak{so}(6,n)$, respectively.

A consistent gauging requires that the embedding tensor components satisfy linear and quadratic constraints. The linear conditions read

$$X_{(\{M\alpha\}\{N\beta\}}{}^{Q\delta}\,\Omega^{(c)}_{\{P\gamma\})Q\delta} = 0\,, \tag{2.41}$$

where

$$\Omega^{(c)} = \Lambda\Omega\Lambda\,, \qquad \Lambda = \begin{pmatrix}\mathbb{1} & 0\\ 0 & \eta\end{pmatrix}\,, \tag{2.42}$$

defines the symplectic bilinear form in the $SO(6,n)$-covariant formulation.[7] As a result, the only potentially non-zero components of the embedding tensor are $f_{\alpha[MNP]}$ and $\xi_{M\alpha} = \tilde{\xi}_{M\alpha}$, and thus (2.38) simplifies to

$$\Theta_{M\alpha}{}^{NP} = f_{\alpha M}{}^{NP} + \tfrac{1}{2}\,\delta_M^{[N}\,\xi_\alpha^{P]} \qquad \Theta_{M\alpha}{}^{\beta\gamma} = \tfrac{1}{2}\,\xi_{\delta M}\,\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)}\,. \tag{2.43}$$

While in the general embedding tensor formalism the linear constraints (2.41) are not derived from first principles but rather postulated ad hoc, it is known to be a necessary condition for a supersymmetric action in $N=4$ supergravity (and $N=8$ which originally motivated this postulate).

On the other hand, the quadratic constraints

$$[X_{M\alpha}, X_{N\beta}] = -X_{M\alpha N\beta}{}^{P\gamma} X_{P\gamma}\,, \tag{2.44}$$

encode the closure of the gauge Lie algebra where the $X_{[\{M\alpha\}\{N\beta\}]}{}^{P\gamma}$ are the structure constants of the gauge Lie algebra. Note, however, that in general the $X_{(\{M\alpha\}\{N\beta\})}{}^{P\gamma}$ only vanish upon contracting them with $X_{P\gamma}$, cf. (2.36). In terms of the embedding tensor components, (2.44) yields

$$\begin{aligned} \xi_\alpha^M \xi_{\beta M} &= 0\,,\\ \xi^P_{(\alpha}\, f_{\beta)PMN} &= 0\,,\\ 3 f_{\alpha R[MN}\, f_{\beta PQ]}{}^R + 2\xi_{(\alpha[M}\, f_{\beta)NPQ]} &= 0\,,\\ \epsilon^{\alpha\beta}\left(\xi_\alpha^P\, f_{\beta PMN} + \xi_{\alpha M}\,\xi_{\beta N}\right) &= 0\,,\\ \epsilon^{\alpha\beta}\left(f_{\alpha MNR}\, f_{\beta PQ}{}^R - \xi_\alpha^R\, f_{\beta R[M[P}\,\eta_{Q]N]} - \xi_{\alpha[M}\, f_{N]PQ\beta} + \xi_{\alpha[P}\, f_{Q]MN\beta}\right) &= 0\,. \end{aligned} \tag{2.45}$$

It is important to note that in $N=4$ gauged supergravity it is due to both the linear and the quadratic constraints, (2.41) and (2.44), respectively, that mutual locality is guaranteed. We will come back to discuss this shortly.

All constraint equations are tensor equations with respect to the on-shell symmetry. As a consequence, while the embedding tensor components have been introduced as real numbers subject to (2.41) and (2.44), transforming them according to their index structure with respect to the on-shell symmetry gives another solution of the constraints. In fact, the full power of the embedding tensor formalism comes into effect after promoting the embedding tensor components to components of a tensor/spurion that transforms under the on-shell symmetry (2.13) of the ungauged theory. In doing so, (2.13) remains an on-shell symmetry in the gauged supergravity. However, fixing the embedding tensor explicitly breaks the original on-shell symmetry.

The gauge transformations act non-trivially on the scalars and the vector bosons, as well as on auxiliary[8] two-forms $B_{\mu\nu}^{MN} = B_{\mu\nu}^{[MN]}$ and $B_{\mu\nu}^{\alpha\beta} = B_{\mu\nu}^{(\alpha\beta)}$ that are needed for

[7] A symplectic transformation S as defined in (2.20) is equivalently described by $S^{(c)} = \Lambda^{-1} S \Lambda$ acting on $SO(6,n)$-covariant column vectors $(H^M, G^M)^T$ and satisfying $S^{(c)T}\Omega^{(c)}S^{(c)} = \Omega^{(c)}$.

[8] They do not have kinetic terms.

consistency, see [51]. On the other hand, all the fermions are inert since they transform trivially with respect to the on-shell symmetry. The infinitesimal gauge transformations on scalars are defined in terms of their action on the scalar vielbeins

$$\mathcal{V} = (\mathcal{V}_{SL(2)}, \mathcal{V}_{SO(6,n)})(x) \in SL(2) \times SO(6,n)\,, \tag{2.46}$$

as

$$\delta\mathcal{V} = g\,\Lambda^{M\alpha} X_{M\alpha}\mathcal{V}\,, \tag{2.47}$$

where $\Lambda^{M\alpha}(x)$ are the local gauge parameters. As to the vector bosons, gauge transformations are defined as

$$\delta A_\mu^{M\alpha} = \partial_\mu \Lambda^{M\alpha} + g\,A_\mu^{N\beta} X_{N\beta P\gamma}{}^{M\alpha}\Lambda^{P\gamma} + \ldots \tag{2.48}$$

where the ... denote further terms given in terms of the local gauge parameters $\Xi_\mu^{[MN]}(x)$ and $\Xi_\mu^{(\alpha\beta)}(x)$ associated to the auxiliary two-forms. In the course of this thesis these are not relevant (see [52–54] for details). Note that up to inhomogeneities the vectors $A_\mu^{M\alpha}$ transform in the adjoint representation of the gauge Lie algebra spanned by the (2.35). In fact, using (2.44) one has

$$\left[g\,\Lambda^{P\gamma}X_{P\gamma}, A_\mu^{N\beta}X_{N\beta}\right] = g\,A_\mu^{N\beta}X_{N\beta P\gamma}{}^{M\alpha}\Lambda^{P\gamma}\,X_{M\alpha}\,. \tag{2.49}$$

Upon exponentiation of the gauge Lie algebra spanned by non-vanishing $X_{M\alpha}$ a connected, simply connected component of the identity element of the non-abelian gauge Lie group can be constructed. This is a Lie subgroup of the isometry group (2.13). It is remarkable that the Lagrangian of the gauged theory is gauge-invariant even if a part of the on-shell symmetry has been gauged that had not been a symmetry of the ungauged Lagrangian. In general, as in the case of the ungauged theory, there may furthermore be an abelian gauge symmetry $U(1)^p$ to some power $p \in \mathbb{N}$.

Finally, note that the quadratic constraints (2.44) are equivalent to saying that the embedding tensor is gauge-invariant as required for objets to be interpreted as charges.

2.4 Gauged theory

We will now summarize the relevant terms of the gauged Lagrangian as given in [52–54]. Apart from topological terms for the vector and the auxiliary two-form fields the bosonic Lagrangian reads

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bos.}} = {}& \tfrac{1}{2}R + \tfrac{1}{16}(D_\mu M_{MN})(D^\mu M^{MN}) + \tfrac{1}{8}(D_\mu M_{\alpha\beta})(D^\mu M^{\alpha\beta}) \\ & - \tfrac{1}{4}\,\mathrm{Im}(\tau) M_{MN} H_{\mu\nu}{}^{M+} H^{\mu\nu N+} + \tfrac{1}{8}\,\mathrm{Re}(\tau)\eta_{MN}\epsilon^{\mu\nu\rho\sigma} H_{\mu\nu}{}^{M+} H_{\rho\sigma}{}^{N+} \\ & - V\,, \end{aligned} \tag{2.50}$$

where the scalar potential is given by

$$V = \tfrac{g^2}{16}\Big(f_{\alpha MNP}\, f_{\beta QRS}\, M^{\alpha\beta}\,\big[\tfrac{1}{3}M^{MQ}M^{NR}M^{PS} + (\tfrac{2}{3}\,\eta^{MQ} - M^{MQ})\,\eta^{NR}\,\eta^{PS}\big] - \tfrac{4}{9} f_{\alpha MNP}\, f_{\beta QRS}\,\epsilon^{\alpha\beta}M^{MNPQRS} + 3\,\xi^M_\alpha\,\xi^N_\beta\, M^{\alpha\beta} M_{MN}\Big)\,. \qquad (2.51)$$

Instead of using the coordinate τ the kinetic terms for the scalars on $SL(2)/SO(2)$ are expressed in terms of $(M_{\alpha\beta}) = \mathcal{V}_{SL(2)}\mathcal{V}^T_{SL(2)}$, see appendix B.2. Then, the covariant derivatives in (2.50) are explicitly given by

$$\begin{aligned} D_\mu M_{\alpha\beta} &= \partial_\mu M_{\alpha\beta} + g\, A_\mu^{M\gamma}\xi_{(\alpha M}\, M_{\beta)\gamma} - g\, A_\mu^{M\delta}\,\xi_{\epsilon M}\,\epsilon_{\delta(\alpha}\epsilon^{\epsilon\gamma}M_{\beta)\gamma}\,,\\ D_\mu M_{MN} &= \partial_\mu M_{MN} + 2g\, A_\mu^{P\alpha}\,\Theta_{P\alpha(M}{}^Q M_{N)Q}\,. \end{aligned} \qquad (2.52)$$

The deformed electric field strengths are

$$\begin{aligned} H_{\mu\nu}{}^{M+} &= 2\,\partial_{[\mu}A_{\nu]}^{M+} - g\,\Big(f_{\alpha NP}{}^M - \xi_{\alpha[N}\,\eta^{M]}{}_P - \tfrac{3}{2}\xi_{\alpha P}\,\delta^M_N\Big)\,A^{N\alpha}_{[\mu}A^{P+}_{\nu]}\\ &\quad + \tfrac{g}{2}\,\Theta^M{}_{-NP}\,B^{NP}_{\mu\nu} + \tfrac{g}{2}\,\xi^M_+\,B^{++}_{\mu\nu} + \tfrac{g}{2}\,\xi^M_-\,B^{+-}_{\mu\nu}\,. \end{aligned} \qquad (2.53)$$

Moreover,

$$M_{MNPQRS} = \epsilon_{mnpqrs}\mathcal{V}_M{}^m\,\mathcal{V}_N{}^n\,\mathcal{V}_P{}^p\,\mathcal{V}_Q{}^q\,\mathcal{V}_R{}^r\,\mathcal{V}_S{}^s\,, \qquad (2.54)$$

is given in terms of the totally antisymmetric tensor ϵ_{mnpqrs}.

In view of discussing the supersymmetry of the vacuum we list the fermion bilinear terms that include the gravitini and give the deformed supersymmetry transformations of the fermionic fields. One has[9]

$$\begin{aligned} e^{-1}\mathcal{L}_{3/2} \;=\;& \tfrac{2}{3}\,A_1^{ij}\,(\psi^i_\mu)^*\bar{\sigma}^{\mu\nu}\epsilon\,(\psi^j_\nu)^* + \text{h.c.}\\ &+\tfrac{1}{3}\,A_2^{ij}(\psi^i_\mu)^*\sigma^\mu\epsilon\,(\chi^j)^* - A_{2\,ai}{}^j\,\psi^i_\mu\,\epsilon\,\bar{\sigma}^\mu\epsilon\,(\lambda^{aj})^* + \text{h.c.}\,, \end{aligned} \qquad (2.55)$$

where the so-called fermion shift-matrices are given in terms of the scalar vielbeins and the embedding tensor components,

$$\begin{aligned} A_1^{ij} &= \epsilon^{\alpha\beta}(\mathcal{V}_\alpha)^*\,\mathcal{V}^M{}_{[kl]}\,\mathcal{V}_N{}^{[ik]}\,\mathcal{V}_P{}^{[jl]}\,f_{\beta M}{}^{NP}\,,\\ A_2^{ij} &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\,\mathcal{V}^M{}_{[kl]}\,\mathcal{V}_N{}^{[ik]}\,\mathcal{V}_P{}^{[jl]}\,f_{\beta M}{}^{NP} + \tfrac{3}{2}\,\epsilon^{\alpha\beta}\mathcal{V}_\alpha\,\mathcal{V}_M{}^{[ij]}\xi^M_\beta\,,\\ A_{2ai}{}^j &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\,\mathcal{V}^M{}_a\,\mathcal{V}^N{}_{[ik]}\,\mathcal{V}_P{}^{[jk]}\,f_{\beta MN}{}^P - \tfrac{1}{4}\,\delta^j_i\,\epsilon^{\alpha\beta}\mathcal{V}_\alpha\,\mathcal{V}^M_a\,\xi_{\beta M}\,, \end{aligned} \qquad (2.56)$$

which are tensors with respect to the local composite symmetry $U(1)\times SU(4)\times SO(n)$, discussed in section 2.1, which is still a symmetry of the gauged theory. In a consistent gauging, the shift-matrices are related to the scalar potential (2.51) by what is sometimes called the generalized Ward identity,

$$\tfrac{1}{3}\,A_1^{ik}\,(A_1^{jk})^* - \tfrac{1}{9}\,A_2^{ik}\,(A_2^{jk})^* - \tfrac{1}{2}\,A_{2aj}{}^k\,(A_{2ai}{}^k)^* \;=\; -\,\tfrac{1}{4}\,\delta^i_j\,V\,. \qquad (2.57)$$

[9]In Appendix C.1 we will give our spinor conventions and relate the Weyl spinors used here to Dirac spinors which are used frequently in the literature. Also note that in (2.55) we removed factors of i in the mixed terms of gravitini and spin-1/2 fermions given in [54].

The deformed supersymmetry transformations of the fermionic fields are also given in terms of the fermion shift-matrices,

$$\begin{aligned} \delta_\epsilon \psi_\mu^i &= 2D_\mu \epsilon^i + \tfrac{2}{3}\, A_1^{ij} \bar{\sigma}_\mu \epsilon\, (\epsilon^j)^* + \dots\,, \\ \delta_\epsilon \chi^i &= \tfrac{4}{3} i\, A_2^{ji} \epsilon\, (\epsilon^j)^* + \dots\,, \\ \delta_\epsilon {\lambda_a}^i &= 2i\, A_{2aj}^i\, \epsilon^j + \dots\,. \end{aligned} \tag{2.58}$$

Here, the supersymmetry parameter ϵ^i is a Weyl spinor that forms the right-handed spinor part of a Dirac spinor. Moreover, the ... denote terms that vanish in a maximally-symmetric background. In such a background the only relevant connection term in the covariant derivative $D_\mu \epsilon^i$ acting on ϵ^i is the torsion-free spin connection given explicitly in (C.12).

2.5 Electric gaugings

In the general case both electric and magnetic vector bosons $A_\mu^{M\alpha}$ are minimally coupled to scalars, cf. (2.52). Similarly, also the deformed field strengths, e.g. in (2.53), contain couplings to all the $A_\mu^{M\alpha}$. In the chosen symplectic frame it is, however, possible to only consider purely electric gaugings where the magnetic vector bosons A_μ^{M-} never appear in the Lagrangian. These are given by the restrictions

$$X_{M-} = 0\,, \tag{2.59}$$

which is equivalent to

$$f_{MNP} \equiv f_{+MNP} \in \mathbb{R}\,, \qquad f_{-MNP} = 0\,, \qquad \xi_{\alpha M} = 0\,. \tag{2.60}$$

For such gaugings the quadratic constraints simplify drastically to

$$f_{+R[MN}\, {f_{+PQ]}}^R = 0\,. \tag{2.61}$$

The scalar potential then reads

$$V = \tfrac{1}{16} \tfrac{1}{\operatorname{Im}\tau} f_{MNP}\, f_{QRS}\, \left[\tfrac{1}{3} M^{MQ} M^{NR} M^{PS} + \left(\tfrac{2}{3}\eta^{MQ} - M^{MQ}\right) \eta^{NR} \eta^{PS}\right]\,. \tag{2.62}$$

The covariant derivatives acting on coset parametrizations are

$$\begin{aligned} D_\mu M_{\alpha\beta} &= \partial_\mu M_{\alpha\beta}\,, \\ D_\mu M_{MN} &= \partial_\mu M_{MN} + 2g\, A_\mu^P\, {f_{P(M}}^Q M_{N)Q}\,, \end{aligned} \tag{2.63}$$

while the field strengths are as usual,

$$H_{\mu\nu}^{M+} \equiv H_{\mu\nu}^M = 2\,\partial_{[\mu} A_{\nu]}^M - g\, {f_{NP}}^M A_{[\mu}^N\, A_{\nu]}^P\,. \tag{2.64}$$

Moreover, the gauge transformations (2.48) simplify to

$$\delta A_\mu^M = \partial_\mu \Lambda^M - f_{PQ}{}^M A_\mu^P \Lambda^Q \,. \tag{2.65}$$

As a consequence, the $f_{MN}{}^P$ are the structure constants of the gauge Lie algebra and the quadratic constraint in (2.61) is but the Jacobi identity. Note, however, that not all gauge Lie algebras can occur since the $f_{MNP} = f_{MN}{}^L \eta_{LP}$ have to be completely antisymmetric. Here, the occurrence of the $SO(6,n)$ invariant metric η_{MN} puts constraints on the possible Lie algebras that can be gauged [43, 54].[10] In the course of this thesis we will not initially specify the gauge group, as partial supersymmetry breaking will put additional constraints on the deformation parameters f_{MNP}.

Note that (2.60) also eliminates the auxiliary two-forms from the Lagrangian. For future use we also give the fermion bilinear terms that upon evaluating at the background will give rise to masses of the spin-1/2 fermions. They read [43]

$$\begin{aligned} e^{-1}\mathcal{L}_{1/2} \;\; = \;\; & -A_{2ai}{}^j \chi^i (\lambda^{aj})^* + \text{h.c.} \\ & + \tfrac{1}{3} A_2^{ij} (\lambda^{ai})^* \epsilon\, (\lambda_a{}^j)^* + A_{ab}{}^{ij} (\lambda^{ai})^* \epsilon\, (\lambda^{bj})^* + \text{h.c.} \,, \end{aligned} \tag{2.66}$$

and are given in terms of the shift-matrices and another set of $SU(4) \times SO(n)$ tensors

$$A_{ab}{}^{ij} = \mathcal{V}_- \, f_{MN}{}^P \, \mathcal{V}^M{}_a \, \mathcal{V}^N{}_b \, \mathcal{V}_P{}^{[ij]} \,. \tag{2.67}$$

While in this thesis we will be mainly dealing with purely electric gaugings, it is important to note that even in the general case of magnetic gaugings no additional degrees of freedom are introduced. In fact, the field strengths of the magnetic vector bosons and of the auxiliary two-forms B^{MN} and $B^{\alpha\beta}$ are determined by their equations of motion. As a result one finds only $(n+6)$ physical (i.e. propagating) vector bosons out of the $2(n+6)$ vector bosons $A_\mu^{M\alpha}$ [51], as in the case of purely electric gaugings. On the other hand, the reason to consider magnetically gauged theories in general is due to the existence of many different ungauged theories each of which can be gauged individually and, in general, give different gauged theories. In fact, the ungauged $N = 4$ supergravity theory discussed in section 2.1 has been given in a symplectic frame in which $SO(1,1) \times SO(6,n)$ is a global symmetry of the Lagrangian. However, upon a general symplectic rotation $S \in Sp(2(n+6), \mathbb{R})$ — not necessarily an embedded isometry given by (2.24) or (2.25) — that rotates $H_{\mu\nu}^{M(\pm)}$ as in (2.20) and acts on N as in (2.22) (as well as on unspecified couplings in the fermionic sector of the theory) another Lagrangian of the ungauged $N = 4$ supergravity is obtained with a possibly different global off-shell symmetry (e.g. $SL(2) \times GL(6) \subset SL(2) \times SO(6, 6+n')$ for $n = 6 + n'$ [79]). While the on-shell symmetry is always $SL(2) \times SO(6,n)$, theories related by a

[10] In contrast, for a semisimple Lie algebras with structure constants $f_{ab}{}^c$ the Killing form κ_{ab} is non-degenerate and can therefore be used to raise/lower indices. Then $f_{abc} = f_{ab}{}^d \kappa_{cd}$ would be automatically completely antisymmetric.

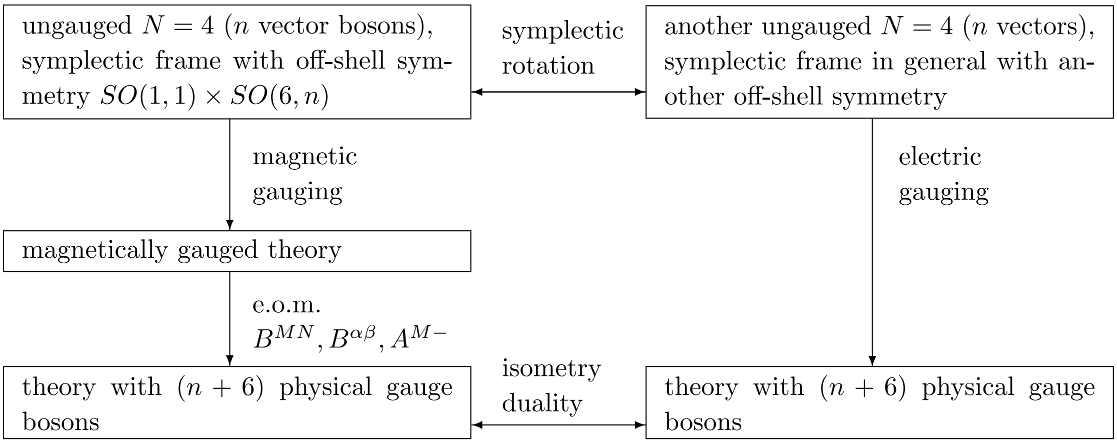

Figure 2.1: Embedding tensor formalism: Given a symplectic frame with a magnetically gauged $N = 4$ supergravity theory, there exists another symplectic frame with an equivalent $N = 4$ supergravity that is purely electrically gauged. Also, electrically gauged $N = 4$ theories in any symplectic frame can always equivalently be described by magnetically gauged $N = 4$ supergravity in a particular chosen symplectic frame.

symplectic rotation are in general not equivalent to each other. Note that the choice of a symplectic frame corresponds to choosing an embedding of the on-shell symmetry into the symplectic duality group. It is now by means of the embedding tensor formalism that electric gaugings in all possible symplectic frames can equivalently be described by consistent general magnetic gaugings in just one symplectic frame. In fact, after solving the equations of motion of the auxiliary two-forms and the magnetic vector potentials in the magnetically gauged theory, the equivalence is given in terms of isometry duality transformations [51, 52]. This is graphically depicted in figure 2.1. In order for this to work so-called mutual locality conditions are required. These are additional quadratic constraints on the embedding tensor which however in the case of $N = 4$ supergravity follow from the constraints on the embedding tensor given in (2.41) and (2.44) [51].

Chapter 3

$N = 2$ vacua in gauged $N = 4$ supergravities

In this chapter we construct gauged $N = 4$ supergravities in four dimensions with $N = 2$ vacua. In the standard particle physics terminology such theories are said to spontaneously break $N = 4$ supergravity to $N = 2$ and therefore give rise to what is called a super-Higgs mechanism. The construction of such theories consists of two steps: First we solve the system of Killing spinor equations/inequalities which give necessary conditions on the embedding tensor components. Being essentially a linear problem it can be solved for the general class of magnetic gaugings. In contrast, in a second step, consistency of the gauging requires solving the quadratic constraints in (2.45). The classification of all such theories with $N = 2$ vacua would require a complete solution of this system of algebraic quadratic equations. However, as its complexity scales badly in terms of the free parameter $n \in \mathbb{N}$ (the number of $N = 4$ vector multiplets) it seems hard to solve it in full generality. We therefore mainly concentrate on the case of purely electric gaugings. While even in this restricted class of gaugings we cannot solve all the consistency equations, we simplify the problem as much as possible and give some explicit solutions. To a large extent the following discussion has also been published in [76].

3.1 Killing spinor equations

A vacuum of an $N = 4$ supergravity theory — viewed as a classical field theory — is a background field configuration that satisfies the equations of motion/Bianchi identities and a stability condition[1]. The simplest class of such vacua — the one of interest in

[1] For instance, in Minkowski or AdS spacetime one demands that the background configuration minimizes the energy functional.

our construction — has a metric $g_{\mu\nu}$ of signature $(-+++)$ on a maximally symmetric spacetime and a constant (i.e. spacetime-independent) scalar field configuration

$$(\mathcal{V}_{SL(2)}, \mathcal{V}_{SO(6,n)}) \in SL(2) \times SO(6,n)\,, \tag{3.1}$$

while fermionic and vector background fields as well as the two-form background field strengths vanish. In fact, one only requires the bosonic Lagrangian — up to topological terms given in (2.50) — to check that the only non-trivial equations of motion evaluated at this background field configuration are the ones arising from varying the metric and the scalars. The former are but the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, R + V|_{\text{bg}}\, g_{\mu\nu} = 0\,, \tag{3.2}$$

with cosmological constant $V|_{\text{bg}}$ given by the scalar potential in (2.51) evaluated at the background. As an immediate consequence, in such a background the spacetime has to be Einstein with

$$R_{\mu\nu} = V|_{\text{bg}}\, g_{\mu\nu}, \qquad R = 4\, V|_{\text{bg}}\,. \tag{3.3}$$

While there also exists non-maximally symmetric solutions to (3.2) such as for instance the (AdS-)Schwarzschild solution [47], we will restrict ourselves to the maximally symmetric ones: de Sitter (dS) ($V|_{\text{bg}} > 0$), Minkowski ($V|_{\text{bg}} = 0$) and Anti-de Sitter (AdS) ($V|_{\text{bg}} < 0$). On the other hand, the equation of motions of the scalar fields written in terms of fluctuations[2] $\vec{\phi}$ around the point (3.1), when evaluated at the background, yield the criticality condition

$$\left.\frac{\partial V}{\partial \vec{\phi}}\right|_{\text{bg}} = 0 \tag{3.4}$$

for the scalar potential. Furthermore, in a maximally symmetric spacetime the stability conditions require that the mass squared parameters of all the fields satisfy certain bounds [80]: For $V|_{\text{bg}} \leq 0$ scalar mass parameters have to obey the Breitenlohner-Freedman bound $m^2 \geq 3V|_{\text{bg}}/4$ [81, 82] whereas the ones for the gravitino necessarily satisfy

$$m^2 \geq -\tfrac{1}{3} V|_{\text{bg}}\,. \tag{3.5}$$

In contrast, the bound for both scalar and gravitino mass parameters in dS background is $m^2 \geq 0$.

Supersymmetry acts on the background field configuration as in (2.58) and, being a symmetry of the action, maps it to another equivalent background field configuration of the theory. However, some supersymmetry generators may leave the vacuum invariant. In fact, the number of linearly independent supersymmetry generators that respect

[2] More precisely, the components of $\vec{\phi}$ are coordinates of a chart that maps $\vec{\phi} = 0$ to the point (3.1) of the scalar manifold.

the vacuum define the amount of unbroken supersymmetry. Schematically, for bosonic background fields B and fermionic ones $F = 0$ one has

$$\begin{aligned} \delta_\epsilon B &\sim \epsilon F = 0\,, \\ \delta_\epsilon F &\sim \begin{cases} D\epsilon\, +\, \epsilon B & \text{for } F = \text{gravitino} \\ \quad \epsilon B & \text{for } F \neq \text{gravitino} \end{cases} \end{aligned} \tag{3.6}$$

for a supersymmetry transformation associated to a right-handed spinor ϵ and a (bosonic) covariant derivative D including the spin connection. For maximally symmetric backgrounds it is therefore only the variations of the fermions that can be non-trivial. Hence, in $N = 4$ supergravity, unbroken supersymmetries labeled by $\epsilon = (\epsilon^i)$ require

$$\delta_\epsilon \psi^i = 0\ , \qquad \delta_\epsilon \chi^i = 0\ , \qquad \delta_\epsilon \lambda^{ai} = 0 \tag{3.7}$$

for all $i = 1, \ldots, 4$ and $a = 1, \ldots, n$, while for the broken ones at least one of the equations in (3.7) must be violated. Evaluating (2.58) at the background we can rewrite (3.7) as

$$\delta_\epsilon \psi^i_\mu = 2D_\mu \epsilon^i - \tfrac{2}{3} A_1^{ij} \Gamma_\mu \epsilon_j = 0\ , \tag{3.8a}$$

$$\delta_\epsilon \chi^i = -\tfrac{4}{3} i A_2^{ji} \epsilon_j = 0\ , \tag{3.8b}$$

$$\delta_\epsilon \lambda^{ai} = 2i A_{2aj}{}^i \epsilon^j = 0\,. \tag{3.8c}$$

As before, (A_1^{ij}),(A_2^{ij}) and $(A_{2aj}{}^i)$ for all a are the shift-matrices defined in (2.56), all of which are evaluated at the background (3.1). It is important to note that it is due to the presence of the covariant derivative that setting the variation of the gravitini to zero gives rise to an integrability condition [38, 47],

$$[D_\mu, D_\nu]\epsilon^i = \tfrac{2}{9} A_1^{ij} (A_1^{jk})^* \Gamma_{\nu\mu} \epsilon^k\,, \tag{3.9}$$

which constrains the geometry of the spacetime. In fact, the existence of at least one unbroken supersymmetry excludes the possibility of having a dS background. While we give the detailed calculation in appendix C.2, here we only state the result:

$$\left(R\, \delta^i_k + \tfrac{16}{3} A_1^{ij} (A_1^{jk})^* \right) \epsilon^k = 0\,. \tag{3.10}$$

As $A_1 = (A_1^{ij})$ is symmetric, one finds that

$$R = -\tfrac{16}{3} |a_i|^2 \leq 0\,, \tag{3.11}$$

where $|a_i|^2$ are the eigenvalues of $A_1 A_1^\dagger$ associated to unbroken supersymmetry directions i. Obviously, all such eigenvalues $|a_i|^2 \equiv |a|^2 = -\tfrac{3}{4} V|_{\text{bg}}$ are degenerate. In mathematics, the equations (3.8a) are referred to as Killing spinor equations — usually, written as an eigenspinor equation of the Dirac operator (using Majorana spinors) —

and any $(\epsilon^i(x))$ satisfying (3.8a) is called a Killing spinor of the spacetime[3]. On the other hand, in the physics literature [54, 83] it has become customary to name all the equations (3.8a) - (3.8c) Killing spinor equations. We will stick to the latter nomenclature.

We will construct gauged supergravity theories with a vacuum that shall be preserved by at least one supersymmetry, i.e. we want at least one right-handed spinor ϵ that satisfies the Killing spinor equations[4]. It then follows from the representation theory of the preserved supersymmetry that stability of the vacuum is guaranteed. In particular, for a Minkowski background this means that the scalar background configuration (3.1) must be a minimum (potentially with flat directions) of the scalar potential which is why (3.4) will indeed be satisfied. In what follows we will argue that for any given but non-vanishing number of preserved supersymmetry the system of Killing spinor equations (3.8a) - (3.8c) and the corresponding inequalities evaluated at an arbitrary scalar background (3.1) can be solved for the embedding tensor components by using the on-shell symmetry to go, without loss of generality, to the origin of the scalar manifold, c.f. [80], and, secondly, by using the residual symmetry to diagonalize the gravitini mass matrix at the origin. Note that owing to the fact that we want to construct a theory with certain vacuum structure our approach is different from the "analytic" one where you start with a given supergravity theory, i.e. pick a specific gauging, and then analyze the vacuum structure by e.g. looking for solutions of the Killing spinor equations. In contrast, our "synthetic" approach follows [63, 64] (albeit in a rather different language): we first specify the vacuum structure in choosing the amount of preserved supersymmetry (at least $N = 1$) and then use the Killing spinor equations to solve for the embedding tensor components.

3.1.1 Going to the origin of the scalar manifold

As already mentioned in section 2.3,

$$SL(2) \times SO(6, n) \tag{3.12}$$

is an on-shell symmetry of any gauged $N = 4$ supergravity theory provided that the embedding tensor components $f_{\alpha MNP}$ and $\xi_{\alpha M}$ transform as in (2.40) under (3.12) as opposed to being kept constant. In particular, this means that had we been able to construct a gauged supergravity theory with an arbitrary background configuration (3.1), we would always find a group element in (3.12) that, using (2.7) and the analogous transformation for the $SL(2)$ vielbein, maps (3.1) to the origin

$$(\mathbb{1}_2, \mathbb{1}_{6+n}) \in SL(2) \times SO(6, n)\,, \tag{3.13}$$

where $\mathbb{1}_k \in \mathrm{Mat}_{k,k}$ denotes the k-dimensional unit matrix. While this comes at the cost of redefining the vector bosons, the scalar fields and the embedding tensor components,

[3] Killing vectors of the spacetime can then be constructed from the Killing spinors.
[4] In Minkowski and AdS space such Killing spinors exist [47].

we would end up with an equivalent description of the gauged $N = 4$ supergravity theory we started with. It is therefore without loss of generality that we may choose the origin (3.13) as the scalar vacuum configuration. As we will see, this is of help because the shift-matrices in (2.56) evaluated at (3.13) end up being disentangled with respect to certain components of the embedding tensor components. The residual on-shell symmetry in a theory with vacuum (3.13), also called the isotropy symmetry, is a combination of $SO(2) \times SO(6) \times SO(n) \subset G$ and global (i.e. spacetime-independent) $U(1) \times SU(4) \times SO(n) \subset H$ composite gauge symmetries such that their compositions leave the origin (3.13) invariant, c.f. [79].

3.1.1.1 A-matrices at the origin of $SO(6, n)$

We now evaluate the fermion shift matrices in (2.56) as well as — for future use — the A-matrices ($A_{ab}{}^{ij}$) given in (2.67) at the origin of $SO(6, n)$. In a rather compact form, one finds

$$
\begin{aligned}
A_1^{ij} &= \tfrac{1}{8}(\mathcal{V}_-)^* \left([G_m]_{ik}\right)^* [G_n]_{kl} \left([G_p]_{lj}\right)^* (f_{+mnp} + i f_{-mnp}) \;, \\
A_2^{ij} &= \tfrac{1}{8}\mathcal{V}_- \left([G_m]_{ik}\right)^* [G_n]_{kl} \left([G_p]_{lj}\right)^* (f_{+mnp} + i f_{-mnp}) \\
&\quad - \tfrac{3}{4}\mathcal{V}_- \left([G_m]_{ij}\right)^* (\xi_{+m} - i\xi_{-m}) \;, \\
A_{2ai}{}^{j} &= -\tfrac{1}{4}\mathcal{V}_- \left[[G_m]_{ik} \left([G_n]_{kj}\right)^* (f_{+amn} - i f_{-amn}) + \delta_i{}^j (\xi_{+a} - i\xi_{-a}) \right] \;, \\
A_{ab}{}^{ij} &= -\tfrac{1}{2}\mathcal{V}_- \, [G_m]^{ij} f_{+abm} + \ldots \;,
\end{aligned}
\tag{3.14}
$$

where G_m are the 't Hooft matrices, which we review in appendix B.2. The ... in (3.14) denote possible further terms that may appear in general magnetically gauged theories but are not explicitly given in [54]. At the critical point (3.13) one has $\mathcal{V}_- = 1$ but later, when discussing purely electric gaugings we allow for non-trivial $\mathcal{V}_- > 0$.

It is convenient to express the result for A_1^{ij} and A_2^{ij} as in [80] in terms of

$$
f^{(\pm)}_{\alpha mnp} := f_{\alpha mnp} \pm \tfrac{1}{3!}\epsilon_{\alpha\beta}\epsilon_{mnpqrs} f_{\beta qrs}
\tag{3.15}
$$

where ϵ_{mnpqrs} and $\epsilon_{\alpha\beta}$ are completely antisymmetric with $\epsilon_{123456} = 1$ and $\epsilon_{-+} = 1$. The components of the symmetric matrix (A_1^{ij}) depend on 20 real parameters $f^{(+)}_{\alpha mnp}$:

$$\begin{aligned}
A_1^{11} &= \tfrac{3}{4}\left[(-f^{(+)}_{-123} + f^{(+)}_{-156} - f^{(+)}_{-246} + f^{(+)}_{-345}) + i(f^{(+)}_{+123} - f^{(+)}_{+156} + f^{(+)}_{+246} - f^{(+)}_{+345})\right] \\
A_1^{22} &= \tfrac{3}{4}\left[(-f^{(+)}_{-123} + f^{(+)}_{-156} + f^{(+)}_{-246} - f^{(+)}_{-345}) + i(f^{(+)}_{+123} - f^{(+)}_{+156} - f^{(+)}_{+246} + f^{(+)}_{+345})\right] \\
A_1^{33} &= \tfrac{3}{4}\left[(-f^{(+)}_{-123} - f^{(+)}_{-156} - f^{(+)}_{-246} - f^{(+)}_{-345}) + i(f^{(+)}_{+123} + f^{(+)}_{+156} + f^{(+)}_{+246} + f^{(+)}_{+345})\right] \\
A_1^{44} &= \tfrac{3}{4}\left[(-f^{(+)}_{-123} - f^{(+)}_{-156} + f^{(+)}_{-246} + f^{(+)}_{-345}) + i(f^{(+)}_{+123} + f^{(+)}_{+156} - f^{(+)}_{+246} - f^{(+)}_{+345})\right] \\
A_1^{12} &= \tfrac{3}{4}\left[(-f^{(+)}_{+125} - f^{(+)}_{+136}) + i(-f^{(+)}_{-125} - f^{(+)}_{-136})\right] \\
A_1^{34} &= \tfrac{3}{4}\left[(f^{(+)}_{+125} - f^{(+)}_{+136}) + i(f^{(+)}_{-125} - f^{(+)}_{-136})\right] \\
A_1^{13} &= \tfrac{3}{4}\left[(f^{(+)}_{+124} - f^{(+)}_{+236}) + i(f^{(+)}_{-124} - f^{(+)}_{-236})\right] \\
A_1^{24} &= \tfrac{3}{4}\left[(f^{(+)}_{+124} + f^{(+)}_{+236}) + i(f^{(+)}_{-124} + f^{(+)}_{-236})\right] \\
A_1^{14} &= \tfrac{3}{4}\left[(f^{(+)}_{+134} + f^{(+)}_{+235}) + i(f^{(+)}_{-134} + f^{(+)}_{-235})\right] \\
A_1^{23} &= \tfrac{3}{4}\left[(-f^{(+)}_{+134} + f^{(+)}_{+235}) + i(-f^{(+)}_{-134} + f^{(+)}_{-235})\right]
\end{aligned} \tag{3.16}$$

It is apparent that any symmetric complex 4×4 matrix can be written in this form. Writing

$$A_2^{ij} = (A_2^{ij})|_{\xi=0} + (A_2^{ij})|_{f=0}\,, \tag{3.17}$$

the result for $(A_2^{ij})|_{\xi=0}$ for $\xi_{\alpha M} = 0$ is obtained from (A_1^{ij}) after substituting

$$\begin{aligned}
f^{(\pm)}_{-mnp} &\rightarrow -f^{(\mp)}_{-mnp} \\
f^{(\pm)}_{+mnp} &\rightarrow f^{(\mp)}_{+mnp}.
\end{aligned} \tag{3.18}$$

Thus, for $\xi = 0$, A_2 depends on another 20 real parameters $f^{(-)}_{\alpha mnp}$. For nonvanishing ξ the additional antisymmetric contribution reads

$$(A_2^{ij})|_{f=0} = -\tfrac{3}{4}\begin{pmatrix} 0 & -i\xi^1 - \xi^4 & -i\xi^2 - \xi^5 & -i\xi^3 - \xi^6 \\ * & 0 & i\xi^3 - \xi^6 & -i\xi^2 + \xi^5 \\ * & * & 0 & i\xi^1 - \xi^4 \\ * & * & * & 0 \end{pmatrix} \tag{3.19}$$

where $\xi^m := \xi^m_+ - i\xi^m_-$. As to $(A_{2ai}{}^j)$ for all $a = 1, \ldots, n$, we write

$$A_{2ai}{}^j = A_{2ai}{}^j|_{\xi=0} + A_{2ai}{}^j|_{f=0}\,, \tag{3.20}$$

and the components of $(A_{2ai}{}^j)|_{\xi=0}$ read

$$\begin{aligned}
A_{2a1}{}^1|_{\xi=0} &= -\tfrac{1}{2}i(f_{a14}+f_{a25}+f_{a36})\\
A_{2a2}{}^2|_{\xi=0} &= -\tfrac{1}{2}i(f_{a14}-f_{a25}-f_{a36})\\
A_{2a3}{}^3|_{\xi=0} &= -\tfrac{1}{2}i(-f_{a14}+f_{a25}-f_{a36})\\
A_{2a4}{}^4|_{\xi=0} &= -\tfrac{1}{2}i(-f_{a14}-f_{a25}+f_{a36})\\
A_{2a1}{}^2|_{\xi=0} &= -\tfrac{1}{2}[(f_{a23}-f_{a56})+i(f_{a26}-f_{a35})]\\
A_{2a3}{}^4|_{\xi=0} &= -\tfrac{1}{2}[(-f_{a23}-f_{a56})+i(-f_{a26}-f_{a35})]\\
A_{2a1}{}^3|_{\xi=0} &= -\tfrac{1}{2}[(-f_{a13}+f_{a46})+i(-f_{a16}+f_{a34})]\\
A_{2a2}{}^4|_{\xi=0} &= -\tfrac{1}{2}[(-f_{a13}-f_{a46})+i(f_{a16}+f_{a34})]\\
A_{2a1}{}^4|_{\xi=0} &= -\tfrac{1}{2}[(f_{a12}-f_{a45})+i(f_{a15}-f_{a24})]\\
A_{2a2}{}^3|_{\xi=0} &= -\tfrac{1}{2}[(-f_{a12}-f_{a45})+i(f_{a15}+f_{a24})]
\end{aligned} \tag{3.21}$$

where $f_{amn} := f_{+amn} - if_{-amn}$. Moreover,

$$A_{2a2}{}^1|_{\xi=0} = -\tfrac{1}{2}[-(f_{a23}-f_{a56})+i(f_{a26}-f_{a35})] \tag{3.22}$$

etc. where the first of the two summands always gets an extra minus sign. For non-vanishing ξ there is a diagonal contribution

$$A_{2ai}{}^j|_{f=0} = -\tfrac{1}{4}\,\delta_i{}^j(\xi_{+a}-i\xi_{-a}). \tag{3.23}$$

We conclude that A_1 depends only on $f^{(+)}_{\alpha mnp}$ while A_2 is given in terms of both $f^{(-)}_{\alpha mnp}$ and $\xi_{\alpha m}$; A_{2a} is built from $f_{\alpha amn}$ and $\xi_{\alpha a}$. Note that at the origin embedding tensor components $f_{\alpha abm}$ and $f_{\alpha abc}$ do not appear in the fermion shift matrices (and therefore also not in the Killing spinor equations).

Finally, we give the explicit result for the antisymmetric A-matrices $(A_{ab}{}^{ij})$ for all a, b at the origin of the scalar manifold whose electric part is entirely given in terms of the components f_{+abm},

$$(A_{ab}{}^{ij}) = \frac{1}{2}\begin{pmatrix} 0 & if_{+ab1}+f_{+ab4} & if_{+ab2}+f_{+ab5} & if_{+ab3}+f_{+ab6}\\ -* & 0 & -if_{+ab3}+f_{+ab6} & if_{+ab2}-f_{+ab5}\\ -* & -* & 0 & -if_{+ab1}+f_{+ab4}\\ -* & -* & -* & 0 \end{pmatrix} + \ldots \tag{3.24}$$

for all a, b.

3.1.2 Diagonalizing the gravitino mass matrix

As in all supergravities, c.f. [57], the fermion shift matrix A_1 evaluated at the background configuration in the supersymmetry transformation of the gravitini is also the gravitino mass matrix:

$$e^{-1}\mathcal{L}_{m_{3/2}} = \tfrac{2}{3}\,A_1^{ij}\,(\psi^i_\mu)^*\bar{\sigma}^{\mu\nu}\epsilon\,(\psi^j_\nu)^* + \text{h.c.}\ . \tag{3.25}$$

As a consequence of what is called the Autonne decomposition [84], an arbitrary symmetric complex matrix (A_1^{ij}) can be "diagonalized" by means of an $SU(4)$ transformation in that one can always find an $S \in SU(4)$ such that

$$S(A_1^{ij})S^T = \mathrm{diag}(|a_1|e^{i\phi}, |a_2|, |a_3|, |a_4|)\,, \tag{3.26}$$

with $|a_1| \leq \ldots \leq |a_4|$. Note, however, that diagonalizing a non-diagonal matrix (A_1^{ij}) at the origin transforms also the matrices (A_2^{ij}) and $({A_{2aj}}^i)$, and affects the vacuum by an $SO(6) \subset H$ rotation moving it away from the critical point (3.28). Of course, the scalar vacuum always remains in the same coset of G/H. We now think of such an H transformation as acting globally and apply its inverse as a G transformation on the vacuum, the embedding tensors, and the vector bosons. In doing so, one returns to the origin of $SO(6,n)$ and at the same time has a diagonal gravitino mass matrix. Moreover, one now knows the A-matrices in terms of the transformed $f_{\alpha MNP}$ and $\xi_{\alpha M}$. We therefore may assume that, without loss of generality, (A_1^{ij}) is of the form (3.26) and the A-matrices are explicitly given as in (3.16) - (3.23). Inspecting (3.25) and the kinetic terms in (2.50) we see that the gravitini mass parameters are given by $2/3 \cdot |a_1|, \ldots, 2/3 \cdot |a_4|$.

We want to stress that diagonalizing the gravitino mass matrix also defines the notion of unbroken or broken supersymmetry directions labeled by $i = 1, \ldots, 4$. In fact, according to the Killing spinor equations, more precisely due to (3.11), one requires

$$|a_i| = \sqrt{-\tfrac{3}{4}V|_{\mathrm{bg}}} \tag{3.27}$$

for any unbroken supersymmetry direction labeled by i. Note that the phase ϕ in (3.26) can be absorbed into a redefinition of the first preserved spinor ϵ. As a result, for each unbroken supersymmetry there is a gravitino with mass parameter $m = \sqrt{-\tfrac{1}{3}V|_{\mathrm{bg}}}$ saturating the stability bound (3.5) of the maximally-symmetric spacetime. In contrast, for broken directions i it is necessary that diagonal entries $|a_i|$ differ from the ones given in (3.27). Furthermore, for each unbroken i one needs a zero row in matrices (A_2^{ij}) and $({A_{2ai}}^j)$ for all a. These conditions give rise to a set of equations and inequalities that we wish to solve for the embedding tensor components. While the latter only appear linearly in the shift matrices, the scalar potential V, given in (2.51), is quadratic in the $f_{\alpha MNP}$ and $\xi_{\alpha M}$. We will solve this problem in the following way: First we solve the linear problem for a given background value $V \leq 0$ of the scalar potential. In fact, it is apparent from the explicit form of the shift matrices given in (3.16) - (3.23) that the embedding tensor components can be chosen in such a way that the Killing spinor equations (and their inequalities) are fulfilled at the critical point (3.28) for any number of preserved supersymmetries and a given background value $V \leq 0$. In a second step, we (try to) solve the quadratic constraints (2.45) required by the consistency of the gauged supergravity (closure of gauge Lie algebra, super-Higgs mechanism, stability. . .). Having done so, the generalized Ward identity (2.57) holds [54] which implies that our $V \leq 0$

is in fact the background value of (2.51). It is in this spirit that the solution of the Killing spinor equations/inequalities is a necessary step that as soon as the quadratic constraints have been solved becomes sufficient. Note that it is by means of (2.57) that the Killing spinor equations (3.8a) already imply the ones for the spin-1/2 fermions (3.8b) and (3.8c)[5] which means that in principle we need not demand zero rows in (A_2^{ij}) and $(A_{2ai}{}^j)$ since this will follow from a solution of the quadratic constraints. However, solving the constraints turns out to be difficult and introducing zero rows into these shift matrices is a useful measure to simplify computations.

3.1.3 Electric gaugings

While within the general class of magnetic gaugings the procedure of going to the origin is always convenient, there is one inconvenience concerning the restriction to only electric gaugings. If we were to start with a purely electrically gauged $N = 4$ supergravity with an arbitrary scalar vacuum (3.1), we would in general require a non-trivial $SL(2)$ transformation in order to go to the origin (3.13) of the scalar manifold. However, such an $SL(2)$ transformation is a proper electric-magnetic duality and, thus, in going to the origin (3.13) we would leave the class of electric gaugings. This can be seen from the embedding tensors: We start in a symplectic frame in which purely electric gaugings are the ones with non-trivial $f_{+MNP} \in \mathbb{R}$ but with $f_{-MNP} = 0$ and $\xi_{\alpha M} = 0$. Upon a non-trivial $SL(2)$ transformation the f_{+MNP} would be rotated also into the f_{-MNP} which therefore would give rise to magnetic gaugings in the given symplectic frame. This is not a problem because the consistency constraints (2.45), being in particular $SL(2)$ tensor equations, are no harder to solve. On the other hand, we do not gain much in terms of simplicity in also going to the origin of $SL(2)$. In fact, it is the origin of $SO(6,n)$ that disentangles the embedding tensor components. Hence, when discussing purely electric gaugings we only apply an $SO(6,n)$ transformation to transform without loss of generality the scalar vacuum (3.1) to

$$(\mathcal{V}_{SL(2)}, \mathbb{1}_{6+n}) \in SL(2) \times SO(6,n)\,, \tag{3.28}$$

and, in doing so, we preserve the class of electric gaugings. Using the additional local symmetry $U(1) \sim SO(2)$ of the Lagrangian which acts as in table 2.1 both on gravity scalar representatives $\mathcal{V}_\alpha$ and on fermions, we can always bring the scalar vielbein to a form such that $\mathcal{V}_- = 1/\sqrt{\operatorname{Im}\tau} > 0$ (see appendix B.2 for details). This comes at the cost of redefining the fermion fields but simplifies the A-matrices in (2.56) and (2.67), in that $\mathcal{V}_- > 0$ becomes an overall scaling factor (to be added in equations (3.16), (3.19),(3.21),(3.22), and (3.23)) while $\mathcal{V}_+ \in \mathbb{C}$ never appears. Note that at a critical point $(\mathbb{1}_2, \mathbb{1}_{6+n})$ one would have $\mathcal{V}_- = 1$.

[5] This is always true for extended supergravity theories, see [57].

In theories with purely electric gaugings and using the gauge $\mathcal{V}_- > 0$, the symmetric shift-matrices (A_1^{ij}) and (A_2^{ij}) are identical. As a result, the Killing spinor equations (3.8a) and (3.8b) immediately imply $V = 0$, i.e. any vacuum of an electrically gauged $N = 4$ supergravity theory that contains at least one preserved supersymmetry is necessarily Minkowski as opposed to AdS. This result is in accordance with [67] in which equal $SU(1,1)$ phases are equivalent to having only electric gaugings [54]. Moreover, the Killing spinor equations (3.8a) - (3.8c) evaluated at the background do not depend on $\mathcal{V}_-$, i.e. the analysis of partial supersymmetry breaking does not depend on the critical point $\mathcal{V}_{SL(2)} \in SL(2)$ in the gravity scalar manifold. Furthermore, as can be seen from the generalized Ward identity (2.57) or the explicit form of the scalar potential given in (2.62), the scalar potential only depends on $\mathcal{V}_{SL(2)}$ by means of an overall scaling factor $(\mathcal{V}_-)^2$ and, hence, the complex coordinate τ parametrizing $SL(2)/SO(2)$ is a flat complex direction of the scalar potential. Upon inspecting mass terms, it turns out that a generic $\mathcal{V}_{SL(2)}$ leads to an overall scaling of all mass terms. This is, of course, a requirement of any preserved supersymmetry in Minkowski background. Note that it is only upon canonically normalizing the gauge kinetic terms that the mass terms for the vector bosons also scale appropriately.

In electrically gauged theories the Killing spinor equation (3.8a) for a preserved supersymmetry direction i simplifies to

$$\partial_\mu \epsilon^i = 0\,, \tag{3.29}$$

in a Minkowski background. Thus, the vacuum only preserves global supersymmetry. In other words, in the super-Higgs mechanism based on electric gaugings, $N = 4$ supergravity will be broken to global $N' < N$ supersymmetry.

Still restricting ourselves to the class of electric gaugings we will briefly consider vacua with $N = 4$ supersymmetry. Subsequently, we will focus on vacua with $N = 2$ supersymmetry. In studying them it will become apparent that $N = 3$ vacua in purely electric gaugings do not exist. In principle, one could also study $N = 1$ vacua.

3.1.3.1 $N = 4$ vacua

In order for the vacuum to respect $N = 4$ supersymmetry, i.e. there shall be four linearly independent Killing spinors ϵ^i satisfying (3.8a) - (3.8c), the shift-matrices $A_1 = A_2$ and $(A_{2ai}{}^j)$ for all a must all vanish. Using the explicit result of section 3.1.1.1 at the origin (3.28) this is only possible for $f_{+mnp} = 0$ and $f_{+amn} = 0$ for all indices. On the other hand, there are potentially non-trivial components f_{+mab} and f_{+abc}. While being unconstrained by the Killing spinor equations they have to satisfy the quadratic constraints (2.61), which in this case read

$$f_{abm}\, f_{acn} - f_{abn}\, f_{acm} = 0\,, \tag{3.30a}$$

$$f_{abc}\, f_{adm} + f_{abm}\, f_{acd} - f_{abd}\, f_{acm} = 0\,, \tag{3.30b}$$

$$f_{abc}\, f_{dea} + f_{abe}\, f_{cda} - f_{abd}\, f_{ace} = f_{mbc}\, f_{dem} + f_{mbe}\, f_{cdm} - f_{mbd}\, f_{mce}\,, \tag{3.30c}$$

for all free indices and where $f_{MNP} \equiv f_{+MNP}$. As a matter of fact, these equations are non-trivial and can be hard to solve for arbitrary $n \in \mathbb{N}$. In order to demonstrate that solutions exist, we will give a class of solutions as well as a special solution for $n = 3$:

First, we set $f_{mab} = 0$ which implies that all bosons and fermions are massless. In fact, the mass terms for all the bosons and fermions evaluated at the background (3.28) vanish. For the fermions this follows from (2.66). The vector bosons are massless because in this case the covariant derivative (2.63) evaluated at the vacuum reads

$$D_\mu M_{MN} = \partial_\mu M_{MN}\,, \tag{3.31}$$

and, hence, does not give rise to mass terms for the vector bosons. As to the scalars, one finds that the background is a critical point and that the curvature of the scalar potential (2.62) evaluated at the background vanishes. This is, of course, required by an $N = 4$ supersymmetric vacuum. Here, we will not give the computation as it is an special case of the computation for $N = 2$ vacua which we discuss in section 4.1. In fact, if $f_{abc} \in \mathbb{R}$ are the only potentially non-trivial components, then the scalar potential in (4.6) is zero up to cubic order in terms of the fluctuations (here: $e_a = f_a = g_a = 0$ and $O_{ab} = 0$). As a result, in this class of gaugings the $N = 4$ spectrum consists of one gravity multiplet and n vector multiplets. The potentially non-trivial quadratic constraints are

$$f_{abc}\, f_{dea} + f_{abe}\, f_{cda} - f_{abd}\, f_{ace} = 0\,, \tag{3.32}$$

which tally with the Jacobi identity in the adjoint representation of the compact form of a reductive[6] Lie algebra of rank n when expressed in an appropriate basis. Based on the classification of simple Lie algebras, many solutions to this equation are therefore well-understood. As can be seen from the gauge transformations of the vector bosons given in (2.65), the f_{abc} are the structure constants of the Lie algebra associated to the gauge group. It is apparent that (modulo $U(1)$ factors) any compact, reductive Lie group can be gauged for sufficiently large n.

Secondly, we set $n = 3$ because then (3.30c), being antisymmetric in $[bcde]$, is always satisfied. A very simple but physically non-trivial solution of (3.30a) - (3.30c) is then given by $f_{1ab} \neq 0$ while all other components vanish. It is an interesting solution in that it gives rise to two massive vector fields and their superpartners. Hence, in terms of representation theory of $N = 4$ the spectrum consists of seven massless $N = 4$ vector

[6] A Lie algebra is reductive if and only if it is the direct sum of a semisimple Lie algebra and an abelian Lie algebra.

multiplets and two $N = 4$ BPS vector multiplets. Note that this is the only possibility because a long massive $N = 4$ multiplet would already require a massive graviton. The unbroken gauge group that leaves the vacuum invariant turns out to be $U(1)^7$.

3.1.3.2 $N = 2$ vacua

Let us now turn to our main problem, which is to study spontaneous breaking of $N = 4$ to $N = 2$ supersymmetry. For unbroken $N = 2$ supersymmetry, one requires

$$(A_1^{ij}) = (A_2^{ij}) = \mathrm{diag}(0, 0, \mu_1, \mu_2) , \qquad A_{2a1}{}^i = A_{2a2}{}^i = 0 , \quad \forall i, a . \tag{3.33}$$

since the vacuum is necessarily Minkowski. Before we solve (3.33) it is useful to study the decomposition of $N = 4$ multiplets in terms of $N = 2$ multiplets. This is of interest as partial supersymmetry breaking requires massive gravitini to be embedded into massive supermultiplets of the preserved supersymmetry [85].

3.1.3.3 Representation theory of Minkowski $N = 2$ supersymmetry

In view of $N = 4$ gauged supergravities with $N = 2$ Minkowski vacua it is instructive to analyze how the degrees of freedom of the original $N = 4$ supermultiplets transform in terms of the $N = 2$ (sub)supersymmetry. In the following discussion we assume that the background is Minkowski which is guaranteed for purely electric gaugings. For an AdS background the discussion would have to be generalized owing to the fact that the supersymmetry algebra in AdS background differs from the one in Minkowski space [86]. Let us denote a multiplet of N-extended supersymmetry with mass m and highest spin/helicity s in Minkowski space by $M_{N,s,m}$. Using this terminology the $N = 4$ gravitational multiplet and the massless vector multiplet together with their component spectrum read [9]

$$\begin{aligned} N = 4 \text{ gravitational multiplet:} \qquad & M_{4,2,0} = \Big([2], 4[\tfrac{3}{2}], 6[1], 4[\tfrac{1}{2}], 2[0]\Big) , \\ N = 4 \text{ vector multiplet:} \qquad & M_{4,1,0} = \Big([1], 4[\tfrac{1}{2}], 6[0]\Big) , \end{aligned} \tag{3.34}$$

where $[s]$ denotes the spin/helicity of the component and the number in front is its multiplicity. Furthermore, the massless $N = 2$ multiplets are

$$\begin{aligned} N = 2 \text{ gravitational multiplet:} \qquad & M_{2,2,0} = \Big([2], 2[\tfrac{3}{2}], [1]\Big) , \\ N = 2 \text{ gravitino multiplet:} \qquad & M_{2,3/2,0} = \Big([\tfrac{3}{2}], 2[1], [\tfrac{1}{2}]\Big) , \\ N = 2 \text{ vector multiplet:} \qquad & M_{2,1,0} = \Big([1], 2[\tfrac{1}{2}], 2[0]\Big) , \\ N = 2 \text{ hypermultiplet:} \qquad & M_{2,1/2,0} = \Big(2[\tfrac{1}{2}], 4[0]\Big) , \end{aligned} \tag{3.35}$$

while the massive $N = 2$ multiplets read

$$\begin{aligned}
N = 2 \text{ massive gravitino multiplet:} \quad & M_{2,3/2,m\neq 0} = \Big([\tfrac{3}{2}], 4[1], 6[\tfrac{1}{2}], 4[0]\Big) \ , \\
N = 2 \text{ BPS gravitino multiplet:} \quad & M_{2,3/2,\mathrm{BPS}} = \Big(2[\tfrac{3}{2}], 4[1], 2[\tfrac{1}{2}]\Big) \ , \\
N = 2 \text{ massive vector multiplet:} \quad & M_{2,1,m\neq 0} = \Big([1], 4[\tfrac{1}{2}], 5[0]\Big) \ , \\
N = 2 \text{ BPS vector multiplet:} \quad & M_{2,1,\mathrm{BPS}} = \Big([1], 2[\tfrac{1}{2}], 1[0]\Big) \ , \\
N = 2 \text{ BPS hypermultiplet:} \quad & M_{2,1/2,\mathrm{BPS}} = \Big(2[\tfrac{1}{2}], 4[0]\Big) \ .
\end{aligned} \tag{3.36}$$

Note that there are two distinct $N = 2$ massive gravitino multiplets, the BPS gravitino multiplet $M_{2,3/2,\mathrm{BPS}}$ and the long massive gravitino multiplet $M_{2,3/2,m\neq 0}$. They differ in that only the BPS gravitino multiplet transforms under a central charge of the supersymmetry algebra in precisely the way that leads to multiplet shortening. BPS gravitini can only occur in pairs as each of them carries a non-vanishing BPS charge which by itself would not be CPT-invariant. This implies that $N = 4$ cannot be broken to $N = 3$ with a BPS gravitino multiplet.

The branching rules of the two $N = 4$ multiplets in terms of massless $N = 2$ multiplets are as follows

$$M_{4,2,0} = M_{2,2,0} + 2M_{2,3/2,0} + M_{2,1,0} \ , \qquad M_{4,1,0} = M_{2,1,0} + M_{2,1/2,0} \ , \tag{3.37}$$

from which we see that in breaking $N = 4 \to N = 2$ the gravity multiplet gives rise to a vector multiplet containing the dilaton and axion in the $N = 2$ spectrum.

As all degrees of freedom must be embedded into complete $N = 2$ multiplets, the two heavy gravitini must lie in massive $N = 2$ supermultiplets. As far as representation theory is concerned there are two options regarding the type of the gravitino multiplet(s). For the situation where the heavy $N = 2$ gravitini are in non-BPS multiplets one has

$$M_{4,2,0} + nM_{4,1,0} \to M_{2,2,0} + 2M_{2,3/2,m\neq 0} + n'_{\mathrm{v}} M_{2,1,m\neq 0} + n_{\mathrm{v}} M_{2,1,\cdot} + n_{\mathrm{h}} M_{2,1/2,\cdot} \ , \tag{3.38}$$

where n'_{v} counts long massive vector multiplets, n_{v} counts BPS vector multiplets and massless vector multiplets (as they have the same field content) and n_{h} counts BPS or massless hypermultiplets (as they also have the same field content). We use $\cdot$ to denote either massless or BPS multiplets. Inserting the spectrum (3.34)–(3.36) one finds the consistency conditions

$$n_{\mathrm{v}} = n - 3 - n'_{\mathrm{v}} \ , \qquad n_{\mathrm{h}} = n - 2 - n'_{\mathrm{v}} \ . \tag{3.39}$$

Thus in this case there have to be at least three $N = 4$ vector multiplets in the spectrum, i.e. $n \geq 3$. In this minimal case with also $n'_{\mathrm{v}} = 0$ there are, apart from the $N = 2$ gravitational multiplet and the two heavy gravitino multiplets, one massive or massless hypermultiplet after the symmetry breaking.

In case that the heavy $N = 2$ gravitini are contained in a BPS multiplet one has

$$M_{4,2,0} + nM_{4,1,0} \to M_{2,2,0} + M_{2,3/2,BPS} + n'_{\rm v} M_{2,1,m\neq 0} + n_{\rm v} M_{2,1,\cdot} + n_{\rm h} M_{2,1/2,\cdot} \ , \quad (3.40)$$

with the consistency conditions

$$n_{\rm v} = n + 1 - n'_{\rm v} \ , \qquad n_{\rm h} = n - 1 - n'_{\rm v} \ , \quad (3.41)$$

and thus there has to be at least one $N = 4$ vector multiplet in the spectrum, i.e. $n \geq 1$. In this minimal case with $n'_{\rm v} = 0$, one finds after the symmetry breaking the $N = 2$ gravitational multiplet, the BPS gravitino multiplet, and two massless/BPS vector multiplets. Note that according to equations (3.39) and (3.41) the case with two long massive gravitino multiplets $M_{2,3/2,m\neq 0}$, relative to the BPS case, yields one fewer hypermultiplet and four fewer vector multiplets in the spectrum.

3.1.3.4 The linear conditions

In this section we first solve the linear $N = 2$ conditions (3.33) and then embark on solving the quadratic constraints (2.61). While the linear equations can easily be solved, it seems hard to find the general solution for the quadratic constraints.

Let us first focus on the zero entries in $A_1 (= A_2)$. Using the explicit form given in section 3.1.1.1 one easily finds that only four of the f_{mnp} can be non-zero and they depend on only two parameters which we denote by c and d. More precisely one finds

$$f_{234} = f_{456} =: c \ , \qquad f_{126} = f_{135} =: d \ , \quad (3.42)$$

while all other f_{mnp} vanish. Moreover, A_1^{33} and A_1^{44} which are related to the gravitini mass parameters μ_1 and μ_2 introduced in (3.33) also depend on c and d via

$$A_1^{33} = -\tfrac{3}{2}\,\mathcal{V}_-(c+d) = \mu_1 > 0 \ , \qquad A_1^{44} = -\tfrac{3}{2}\,\mathcal{V}_-(c-d) = \mu_2 \geq \mu_1 \ , \quad (3.43)$$

where as pointed out before $\mu_2 \geq \mu_1$ is chosen without loss of generality. Let us now turn to the last set of equations in (3.33) and solve the system of linear equations for the shift matrices $(A_{2ai}{}^j)$. Using (3.21) and (3.22) the potentially non-trivial components of f_{amn} turn out to be

$$f_{a25} = -f_{a36} =: e_a \ , \qquad f_{a23} = f_{a56} =: f_a \ , \qquad f_{a26} = f_{a35} =: g_a \ , \quad (3.44)$$

while $f_{a1n} = f_{a4n} = 0$ for all a and n. Thus, for any a, the matrix $A_{2ai}{}^j$ is non-trivial only in its lower right block and given by

$$(A_{2ai}{}^j) = \begin{pmatrix} 0 & 0 \\ 0 & Z_a \end{pmatrix} , \qquad Z_a = f_a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i \begin{pmatrix} -e_a & g_a \\ g_a & e_a \end{pmatrix} \ . \quad (3.45)$$

This concludes our analysis of the linear equations arising from the Killing spinor equations (3.33). Let us now turn to the quadratic constraints.

3.2 Solutions of the quadratic conditions

In order for the gauging to be consistent with respect to supersymmetry and gauge invariance, we need to impose the quadratic constraints (2.61). However, in practice it is difficult to solve these equations in general. In order to solve them as much as possible we will make much use of their symmetry properties. In fact, (2.61) are $SO(6,n)$ tensor equations and it will prove crucial to exploit all the symmetries.

Let us first look at the equation $(M,N,P,Q) = (m,n,p,q) = (1,2,4,5)$ of the quadratic constraints (2.61) which, using (3.42), reads

$$c \cdot d = 0 \; . \tag{3.46}$$

Since $c = 0$ is inconsistent with the gauge choice of (3.43), we need to have $d = 0$ and $c < 0$. This in turn implies a first result, namely that the two heavy gravitini have to be degenerate in mass

$$m_{3/2} := \tfrac{2}{3}\, A_1^{33} = \tfrac{2}{3}\, A_1^{44} = -c\,\mathcal{V}_- \; , \tag{3.47}$$

as one expects when some fraction of supersymmetry is preserved in a Minkowski background. Let us also note that (3.46) immediately implies that in electrically gauged theories one can never break $N = 4$ to $N = 3$ since $A_1^{33} = 0$, $A_1^{44} \neq 0$ requires $c = -d \neq 0$, as was first shown in [67].

In order to proceed, it is necessary to make some simplifying assumptions. By inspection, one finds that for $g_a = 0$ the equations simplify considerably and therefore some of them can be solved. On the other hand, the $g_a \neq 0$ case is much more involved and solutions — should they exist — would have to be more sophisticated, as we point out in appendix D.1.2. In fact, there we show that $g_a \neq 0$ solutions do not exist for $n \leq 6$. In what follows we will therefore assume that $g_a = 0$, which also implies $e_a = 0$ due to the quadratic constraint for $(M,N,P,Q) = (b,n,p,q) = (b,2,4,6)$. This choice corresponds to turning-off certain components of the A-matrices and minimizes the coupling between gravitini and gaugini in the Lagrangian (2.55). Indeed, we shall see later that with this choice it is only the "first" $N = 4$ vector multiplet that contributes to the gravity/Goldstini sector. The fact that it is the components $g_a = f_{a26} = f_{a35} = 0$ and $e_a = f_{a25} = -f_{a36} = 0$ that allow for this simplification is due to our particular $SU(4)$ gauge choice for which gravitini remain massless (3.33), suitably translated into $SO(6)$ indices using the 't Hooft matrices (see (B.2)).

Let us now consider the quadratic constraint $(M,N,P,Q) = (m,n,p,q) = (2,3,5,6)$. Inserting (3.42) and (3.44) we find

$$\sum_a f_a^2 = c^2 > 0 \; , \tag{3.48}$$

i.e. at least one f_a must be different from zero. This implies, via (3.45), that $(A_{2ai}{}^j)$ has non-zero entries and from (2.55) and (2.66) we see that additional fermionic couplings have to be non-zero and related to the gravitino mass. As we will see in section 4.1, (3.48) is necessary for the super-Higgs mechanism and the appropriate couplings of the Goldstone fermions to the gravitinos. In order to simplify the analysis we use an $SO(n)$ transformation that leaves the origin invariant and choose $f_a = c\,\delta_{a7}$ which obviously solves (3.48). The quadratic constraints $(M, N, P, Q) = (b, n, p, q)$ then imply

$$f_{7bm} = 0 \ , \quad \forall\, b, m \ . \tag{3.49}$$

In appendix D.2 we list the remaining non-trivial quadratic constraints. A subset of them, (D.67a) - (D.67u), can be written in terms of the antisymmetric real $(n-1)\times(n-1)$ matrices

$$G_m = (f_{\tilde{b}\tilde{c}m}) \qquad \text{and} \qquad G_7 = (f_{\tilde{b}\tilde{c}7}) \qquad \text{with } \tilde{b}, \tilde{c} = 8, \ldots, 6 + n \ . \tag{3.50}$$

which satisfy

$$\begin{aligned} &[G_2, H_+] = -2c\, G_3 \ , \qquad [G_3, H_+] = +2c\, G_2 \ , \qquad [G_2, G_3] = cH_- \ , \\ &[G_5, H_+] = -2c\, G_6 \ , \qquad [G_6, H_+] = +2c\, G_5 \ , \qquad [G_5, G_6] = c\, H_- \ , \end{aligned} \tag{3.51}$$

where $H_\pm = G_4 \pm G_7$, while the remaining matrix commutators all vanish. As a consequence, (3.51) defines a Lie bracket on the 7-dimensional real vector space spanned by abstract elements $\{G_1, G_2, G_3, G_5, G_6, H_+, H_-\}$ and it can be checked that the Jacobi identities are satisfied.

Note that G_1 commutes with all other elements and thus we have a real 7-dimensional Lie algebra $\mathfrak{g}$ which decomposes into a sum of two ideals,

$$\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{s}, \tag{3.52}$$

spanned by G_1 and $\{G_2, G_3, G_5, G_6, H_+, H_-\}$, respectively. It can be further checked that $\mathfrak{s}$ is a solvable Lie algebra of dimension 6.[7] The problem of finding solutions to the quadratic constraints (D.67a) - (D.67u) is now equivalent to finding antisymmetric finite-dimensional representations of $\mathfrak{g}$. One obvious class of solutions is given by

$$G_2 = G_3 = G_5 = G_6 = H_- = 0 \tag{3.53}$$

and an arbitrary, antisymmetric H_+ that commutes with G_1. In this case one has $G_4 = G_7$. In appendix D.2.1 we will prove that no other solution exists. Our proof is based on Lie's theorem concerning complex representations of complex solvable Lie algebras.

[7] Recall that a Lie algebra $\mathfrak{g}$ is solvable if and only if the (upper) derived series of Lie algebras $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \ldots)$ terminates after finitely many steps.

The remaining equations (D.67a) to (D.68c) to be solved now simplify to

$$f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}4} = 0 \tag{3.54a}$$

$$f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{a}\tilde{d}1} + f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}\tilde{d}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}1} = 0 \tag{3.54b}$$

$$f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{a}\tilde{d}4} + f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}\tilde{d}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}4} = 0 \tag{3.54c}$$

$$f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{d}\tilde{e}\tilde{a}} + f_{\tilde{a}\tilde{b}\tilde{e}}\, f_{\tilde{c}\tilde{d}\tilde{a}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}\tilde{e}} = f_{1\tilde{b}\tilde{c}}\, f_{\tilde{d}\tilde{e}1} + f_{1\tilde{b}\tilde{e}}\, f_{\tilde{c}\tilde{d}1} - f_{1\tilde{b}\tilde{d}}\, f_{1\tilde{c}\tilde{e}}\,. \tag{3.54d}$$

Note that the gravitino mass parameter c has disappeared from the equations. Unfortunately, it is still hard to solve these equations in generality for any given integer n. Obviously, the minimal case of an $N = 2$ vacuum can be realized by either choosing $n = 1$ in which case embedding tensor components with indices $\tilde{a}, \tilde{b}, \ldots$ do not exist or setting the latter all to zero.

Let us first consider $G_1 = G_4 = 0$. In this case the only remaining non-trivial equation is

$$f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{d}\tilde{e}\tilde{a}} + f_{\tilde{a}\tilde{b}\tilde{e}}\, f_{\tilde{c}\tilde{d}\tilde{a}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}\tilde{e}} = 0\,, \tag{3.55}$$

which tallies with the Jacobi identity in the adjoint representation of the compact form of a reductive Lie algebra of rank $(n-1)$ when expressed in an appropriate basis. As already mentioned in section 3.1.3.1, based on the classification of simple Lie algebras, solutions to (3.55) are well-understood. As we will see in section 4.2, the components $f_{\tilde{a}\tilde{b}\tilde{c}}$ turn out to be the structure components of the Lie algebra that leaves invariant the vacuum of the theory and, hence, determine the unbroken gauge group up to abelian extensions.

Now we turn to the case of non-trivial G_1 and G_4. In appendix D.2.2 we will solve (3.54a), which in matrix notation reads

$$[G_1, G_4] = 0\,. \tag{3.56}$$

Here we will only explain the result. The solution of this $SO(n-1)$ tensor equation could be given in terms of $SO(n-1)$ representatives of an orbit of solutions. However, as it is also an $O(n-1)$ tensor equation, it is more convenient to give its solution in terms of $O(n-1)$ representatives, up to an additional simple reflection, so as to obtain this gauge by a $SO(n-1)$ rotation. Regardless of this subtlety our gauge choice proves useful in the following analysis. One finds that the most general solution consists of simultaneously block-diagonal G_1 and G_4 with blocks that square to a matrix proportional to the identity of the block. The explicit form of G_1 and G_4 in our gauge is given as follows: First of all, we have

$$G_1 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{3.57}$$

where $D = diag(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots)$ is a diagonal matrix with ordered positive eigenvalues $x_1 > x_2 > \ldots > 0$ and ε is the antisymmetric 2×2 matrix with $\varepsilon_{12} = 1$; the zeros in (3.57) denote zero matrices of appropriate dimensions. Then, we have

$$G_4 = \begin{pmatrix} A & 0 \\ 0 & (D' \otimes \varepsilon) \oplus 0 \end{pmatrix} , \tag{3.58}$$

where A is an antisymmetric matrix (of the same matrix dimensions as $D \otimes \varepsilon$) satisfying

$$[D \otimes \varepsilon, A] = 0 , \tag{3.59}$$

and D' is another invertible diagonal matrix. Furthermore, we show in appendix D.2.2 that both $D \otimes \varepsilon$ and A are block-diagonal. As a result, the four different types of blocks that can appear are listed in table 3.1.

G_1 block	G_4 block
$x_i \, \mathbb{1} \otimes \varepsilon$	$0 \cdot \mathbb{1} \otimes \mathbb{1}_2$
$x_i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}' \end{pmatrix} \otimes \varepsilon$	$\lvert y_{ij} \rvert \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}' \end{pmatrix} \otimes \varepsilon$
$x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon)$	$\lvert y_{ij} \rvert \, \mathbb{1} \otimes \left(\cos\phi_{ijk} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \varepsilon + \sin\phi_{ijk} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \right)$
$x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon)$	$D^{(ij0)} \otimes (\varepsilon \otimes \mathbb{1}_2)$

Table 3.1: The four different types of blocks appearing in the solution of $[D \otimes \varepsilon, A] = 0$. The label i refers to blocks in G_1 with eigenvalues $-x_i^2 \neq 0$ of $(G_1)^2$. Similarly, the label j is associated to subblocks in G_4 with eigenvalues $-y_{ij}^2 \neq 0$ of $(G_4)^2$. Moreover, $D^{(ij0)}$ is a diagonal matrix with eigenvalues $\pm y_{ij}$ and $\phi_{ijk} \in (0, \pi/2)$. Finally, k labels different possible angles ϕ_{ijk}.

We will now solve the tensor equation given in (3.54b). For a given G_1, these equations are linear in $f_{\tilde{a}\tilde{b}\tilde{c}}$ and can easily be solved for the latter in the gauge (D.72). Before we state the result, we introduce some index notation in that we distinguish $SO(n-1)$ indices $\tilde{a}, \tilde{b}, \ldots$ depending on whether or not they correspond to non-zero or zero blocks in G_1: Components of non-zero 2×2 blocks shall have subindices, e.g. $\tilde{a}_1 = 1, 2$, indicating the block they belong to. On the other hand, components associated to the zero block in G_1 shall be denoted by $\tilde{a}_0$. Furthermore, we introduce matrices

$$G^{(x_1)}_{\tilde{a}_0} = (G_{\tilde{a}_0 \tilde{b}_1 \tilde{c}_2}) = f_{\tilde{a}_0 \tilde{b}_1 \tilde{c}_2} , \tag{3.60}$$

where $\tilde{b}_1, \tilde{c}_2$ run over all indices associated to blocks with x_1 in G_1. The solution of (3.54b) is given in terms of three classes of potentially non-trivial components $f_{\tilde{a}\tilde{b}\tilde{c}}$. First,

$$f_{\tilde{a}_0 \tilde{b}_0 \tilde{c}_0} \in \mathbb{R} , \tag{3.61}$$

can be arbitrary; then one finds

$$G^{(x_1)}_{\tilde{a}_0} = S^{(x_1)} \otimes \varepsilon + A^{(x_1)} \otimes \mathbb{1}_2 , \tag{3.62}$$

for a symmetric matrix $S^{(x_1)}$ and an antisymmetric $A^{(x_1)}$; finally components $f_{\tilde{a}_1\tilde{b}_2\tilde{b}_3}$ are given in terms of two real numbers,

$$\begin{aligned}
f_{2_1 1_2 2_3} &= f_{1_1 1_2 1_3} \,, \\
f_{1_1 2_2 2_3} &= -f_{1_1 1_2 1_3} \,, \\
f_{2_1 2_2 1_3} &= f_{1_1 1_2 1_3} \,, \\
f_{2_1 1_2 1_3} &= -f_{2_1 2_2 2_3} \,, \\
f_{1_1 2_2 1_3} &= f_{2_1 2_2 2_3} \,, \\
f_{1_1 1_2 2 1_3} &= f_{2_1 2_2 2_3}
\end{aligned} \tag{3.63}$$

for $x_1 = x_2 + x_3$ ($x_1 \geq x_2 \geq x_3$) while they vanish for $x_1 \neq x_2 + x_3$.

Similarly, for a given G_4 the equations in (3.54c), being again linear in the $f_{\tilde{a}\tilde{b}\tilde{c}}$, can in principle be solved. On the other hand, owing to the aforementioned arbitrarily complicated block structure of G_4 it would be cumbersome to give the general solution. In contrast, (3.54d) are quadratic equations that we cannot solve in full generality. We will therefore proceed by discussing certain special solutions of them (still in the case $g_a = 0$).

3.2.1 Special solutions

We will discuss two special classes of solutions to the equations given in (3.54a) to (3.54d). First we will give all solutions in the case of $n \leq 6$, and secondly we construct special but physically non-trivial solutions that work for any $n \in \mathbb{N}$.

3.2.1.1 Solutions for $n \leq 6$

In appendix D.1.2 we show that for $n \leq 6$ consistency requires $g_a = 0$. As a consequence, the equations to be solved are precisely the ones in (3.54a) to (3.54d). As in (3.57), we will bring G_1 to the following gauge

$$G_1 = \left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \varepsilon \right] \oplus 0 \in \mathrm{Mat}_{5,5} \tag{3.64}$$

for $n = 6$ with $m_1, m_2 \in \mathbb{R}$, or to obvious truncations of (3.64) to matrices in $\mathrm{Mat}_{n-1,n-1}$ for $n \leq 5$. As discussed in appendix D.2.2, we distinguish between the following two cases: Given that matrices $(G_1)^2$ and $(G_4)^2$ have four nonzero degenerate eigenvalues each (which can only happen for $n \geq 5$), G_4 can be written as

$$G_4 = \pm n_1 \left[\cos\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \varepsilon + \sin\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \right] \oplus 0 \tag{3.65}$$

for $n = 6$ or its obvious truncation in the case of $n = 5$, while otherwise we can write

$$G_4 = \left[\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \otimes \varepsilon \right] \oplus 0 \tag{3.66}$$

for $n = 6$ or truncations thereof for $n \leq 5$. Here, we introduced $n_1, n_2 \in \mathbb{R}$ and $\phi \in [0, \pi/2]$. Note that the dimension of the matrices G_1 and G_4 being smaller than 6 does not allow for non-trivial deformation components of the kind given in (3.63). However, in general we will find components as in (3.61) that, as we will see, correspond to the structure constants of the unbroken gauge Lie algebra, as well as components as in (3.62) that in some cases for $n = 6$ are required to be non-trivial.

We state the result for $n \leq 5$ in terms of representatives of $SO(n-1)$ orbits in table 3.2. In anticipation of phenomenological aspects to be discussed in section 4, we also list some physical properties for the consistent solutions. Note that for $n \leq 4$ consistency is trivially given. Furthermore, in the case of $n = 5$ one cannot have $m_1, m_2 \neq 0$ which excludes solutions of the type (3.65).

n	non-trivial components	$N = 2$ multiplets	unbroken gauge group
1	no G_1, G_4	$2 \times M_{2,1,0}$	$U(1)^3$
2	$G_1 = G_4 = 0$	$3 \times M_{2,1,0}$, $1 \times M_{2,1/2,BPS}$ of mass $\lvert c\rvert$	$U(1)^{3+1}$
3	$G_1 = G_4 = 0$	$4 \times M_{2,1,0}$, $2 \times M_{2,1/2,BPS}$ of mass $\lvert c\rvert$	$U(1)^{3+2}$
	$m_1 \neq 0 \vee n_1 \neq 0$	$2 \times M_{2,1,0}$, $2 \times M_{2,1,BPS}$ of mass^2 $(m_1^2 + n_1^2)$, $1 \times M_{2,1/2,.}$ of mass^2 $m_1^2 + (\lvert c\rvert - n_1)^2$, $1 \times M_{2,1/2,BPS}$ of mass^2 $m_1^2 + (\lvert c\rvert + n_1)^2$	$U(1)^3$
4	$G_1 = G_4 = 0, g_{\tilde{1}\tilde{2}\tilde{3}} \neq 0$	...	$U(1)^3 \times SU(2)$
	$G_1 = G_4 = 0, g_{\tilde{1}\tilde{2}\tilde{3}} = 0$	...	$U(1)^{3+3}$
	$m_1 \in \mathbb{R}, n_1 \neq 0, g_{\tilde{1}\tilde{2}\tilde{3}} \in \mathbb{R}$	...	$U(1)^{3+1}$
5	$G_1 = G_4 = 0, g_{\tilde{1}\tilde{2}\tilde{3}} \neq 0$	...	$U(1)^{3+1} \times SU(2)$
	$G_1 = G_4 = 0, g_{\tilde{1}\tilde{2}\tilde{3}} = 0$	...	$U(1)^{3+4}$
	$G_1 = 0, n_1, n_2 \neq 0$	...	$U(1)^3$
	$m_1 \neq 0 \vee n_1 \neq 0, m_2, n_2 = 0$ and $g_{\tilde{1}\tilde{2}\tilde{3}} \in \mathbb{R}$	...	$U(1)^{3+2}$
	$m_1 \neq 0, m_2 = 0, n_2 \neq 0$	...	$U(1)^3$

Table 3.2: Consistent electric gaugings with $N = 2$ vacuum for $n \leq 5$. Explanations are given in section 3.2.1.1. We also always have the $N = 2$ gravity multiplet $M_{2,2,0}$ and the $N = 2$ BPS gravitino multiplet $M_{2,3/2,BPS}$ of mass $|c|$. For brevity for $n \geq 4$ we do not list the $N = 2$ spectrum (the ...). Note that here for convenience we set $\mathcal{V}_- = 1$.

The result for $n = 6$ is given in terms of $SO(5)$ gauge representatives in table 3.3.[8] We observe that consistent solutions may still have non-trivial deformation spaces.

G_1	G_4	solutions: non-trivial $f_{\tilde{a}\tilde{b}\tilde{c}}$, etc.	unbr. g. group
$m_1, m_2 = 0$	$n_1, n_2 = 0$	$f_{\tilde{1}\tilde{2}\tilde{3}} = 0$ $f_{\tilde{1}\tilde{2}\tilde{3}} \neq 0$	$U(1)^{3+5}$ $U(1)^{3+2} \times SU(2)$
$m_1, m_2 = 0$	$n_1 \neq 0,\ n_2 = 0$	$f_{\tilde{1}\tilde{2}\tilde{3}} \in \mathbb{R}$ $f_{\tilde{3}\tilde{4}\tilde{5}} \neq 0$	$U(1)^{3+3}$ $U(1)^3 \times SU(2)$
$m_1, m_2 = 0$	$0 \neq n_1^2 \neq n_2^2 \neq 0$	$f_{\tilde{1}\tilde{2}\tilde{5}} \in \mathbb{R}$	$U(1)^{3+1}$
$m_1, m_2 = 0$	$0 \neq n_1^2 = n_2^2 \neq 0$	$G_{\tilde{5}} = \begin{pmatrix} \frac{f^2_{\tilde{2}\tilde{4}\tilde{5}}}{f_{\tilde{3}\tilde{4}\tilde{5}}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon + \begin{pmatrix} 0 & f_{\tilde{2}\tilde{4}\tilde{5}} \\ -f_{\tilde{2}\tilde{4}\tilde{5}} & 0 \end{pmatrix} \otimes \mathbb{1}_2$ $f_{\tilde{1}\tilde{2}\tilde{5}} \in \mathbb{R}$	$U(1)^{3+1}$ $U(1)^{3+1}$
$m_1 \neq 0, m_2 = 0$	$n_1 \in \mathbb{R}, n_2 = 0$	$f_{\tilde{1}\tilde{2}\tilde{3}} \in \mathbb{R}$ $f_{\tilde{3}\tilde{4}\tilde{5}} \neq 0$	$U(1)^{3+3}$ $U(1)^3 \times SU(2)$
$m_1 \neq 0, m_2 = 0$	$n_1 \in \mathbb{R}, n_2 \neq 0$	$f_{\tilde{1}\tilde{2}\tilde{5}} \in \mathbb{R}$	$U(1)^{3+1}$
$0 \neq m_1^2 \neq m_2^2 \neq 0$	$n_1, n_2 \in \mathbb{R}$	$G_{\tilde{5}} = \begin{pmatrix} \frac{m_1 m_2}{f_{\tilde{3}\tilde{4}\tilde{5}}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$n_1, n_2 = 0$	$G_{\tilde{5}} = \begin{pmatrix} f_{\tilde{1}\tilde{2}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon + \begin{pmatrix} 0 & f_{\tilde{2}\tilde{4}\tilde{5}} \\ -f_{\tilde{2}\tilde{4}\tilde{5}} & 0 \end{pmatrix} \otimes \mathbb{1}_2$ with $m_1^2 = f_{\tilde{1}\tilde{2}\tilde{5}}\, f_{\tilde{3}\tilde{4}\tilde{5}} - f^2_{\tilde{2}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$n_1 \neq 0, n_2 = 0$	$G_{\tilde{5}} = \begin{pmatrix} f_{\tilde{1}\tilde{2}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon$ with $m_1^2 = f_{\tilde{1}\tilde{2}\tilde{5}}\, f_{\tilde{3}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$0 \neq n_1^2 \neq n_2^2 \neq 0$	$G_{\tilde{5}} = \begin{pmatrix} f_{\tilde{1}\tilde{2}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon$ with $m_1^2 = f_{\tilde{1}\tilde{2}\tilde{5}}\, f_{\tilde{3}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$0 \neq n_1^2 = n_2^2 \neq 0$ with $\sin\phi = 0$	$G_{\tilde{5}} = \begin{pmatrix} f_{\tilde{1}\tilde{2}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon$ with $m_1^2 = f_{\tilde{1}\tilde{2}\tilde{5}}\, f_{\tilde{3}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$0 \neq n_1^2 = n_2^2 \neq 0$ with $\cos\phi = 0$	$G_{\tilde{5}} = \begin{pmatrix} f_{\tilde{3}\tilde{4}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon + \begin{pmatrix} 0 & f_{\tilde{2}\tilde{4}\tilde{5}} \\ -f_{\tilde{2}\tilde{4}\tilde{5}} & 0 \end{pmatrix} \otimes \mathbb{1}_2$ with $m_1^2 = f^2_{\tilde{3}\tilde{4}\tilde{5}} - f^2_{\tilde{2}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$
$m_1 = m_2 \neq 0$	$0 \neq n_1^2 = n_2^2 \neq 0$ $\sin\phi, \cos\phi \neq 0$	$G_{\tilde{5}} =$ $\begin{pmatrix} 2\cot\phi f_{\tilde{2}\tilde{4}\tilde{5}} + f_{\tilde{3}\tilde{4}\tilde{5}} & 0 \\ 0 & f_{\tilde{3}\tilde{4}\tilde{5}} \end{pmatrix} \otimes \varepsilon + \begin{pmatrix} 0 & f_{\tilde{2}\tilde{4}\tilde{5}} \\ -f_{\tilde{2}\tilde{4}\tilde{5}} & 0 \end{pmatrix} \otimes \mathbb{1}_2$ with $m_1^2 = f^2_{\tilde{3}\tilde{4}\tilde{5}} - f^2_{\tilde{2}\tilde{4}\tilde{5}} + 2\cot\phi f_{\tilde{2}\tilde{4}\tilde{5}}\, f_{\tilde{3}\tilde{4}\tilde{5}}$	$U(1)^{3+1}$

Table 3.3: Consistent electric gaugings with $N = 2$ vacuum for $n = 6$. Explanations are given in section 3.2.1.1.

[8] There exist also solutions that are obtained from the ones given in table 3.3 by a reflection $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{1}_3 \in O(5)$.

3.2.1.2 Special solutions with $g_a = 0$ and $G_1 = 0$ for arbitrary $n \in \mathbb{N}$

A class of special solutions with $g_a = 0$ for arbitrary n is obtained by setting $G_1 = 0$ which drastically simplifies the equations (3.54a) to (3.54d). Similarly to the discussion for general G_1 in section 3.2, we can write G_4 as

$$G_4 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix} , \qquad (3.67)$$

where $D = diag(y_1, \dots, y_1, y_2, \dots, y_2, \dots)$ is a diagonal matrix with ordered positive eigenvalues $y_1 > y_2 > \dots > 0$. In doing so, the full solution to equation (3.54c) is analogous to the one given in terms of the (a priori) non-trivial components in (3.61), (3.62), and (3.63). For general such components, it is still hard to solve the last equations (3.54d). However, an interesting class of solutions is obtained after setting all but $f_{\tilde{a}_0\tilde{b}_0\tilde{c}_0}$ to zero since then (3.54d) is just the Jacobi identity (3.55) for the gauge Lie algebra with structure constants $f_{\tilde{a}_0\tilde{b}_0\tilde{c}_0} \in \mathbb{R}$. As stated above many non-trivial solutions to these equations are known, each of which corresponds to a compact reductive group G_{vac}. As we will see in section 4.2 in those cases the unbroken gauge group that leaves the vacuum invariant is

$$U(1)^3 \times G_{\text{vac}} \, . \qquad (3.68)$$

Finally, anticipating the discussion of mass terms, we list the $N = 2$ spectrum for such solutions in table 3.4.

block in G_4	mass	$N = 2$ multiplets
$\mathbb{0}_k$	0	$k \times M_{2,1,0}$
	$\lvert c\rvert$	$k \times M_{2,1/2,BPS}$
$y_i \otimes \varepsilon$	y_i	$2 \times M_{2,1,BPS}$
	$\big\lvert \lvert c\rvert - y_i \big\rvert$	$1 \times M_{2,1/2,.}$
	$\lvert c\rvert + y_i$	$1 \times M_{2,1/2,BPS}$

Table 3.4: $N = 2$ multiplets in the matter sector for the solutions in section 3.2.1.2. In the gravity sector one has the $N = 2$ gravity multiplet $M_{2,2,0}$, the $N = 2$ BPS gravitino multiplet $M_{2,3/2,BPS}$ of mass $|c|$, and two more $N = 2$ vector multiplets $M_{2,1,0}$. The consistency condition given in (3.41) is fulfilled with $n'_{\text{v}} = 0$, i.e. no non-BPS massive vector multiplets. Furthermore, note that for blocks with $y_i = |c|$ one obtains massless hypermultiplets. This is of interest because together with massless vector multiplets these give rise to a non-trivial geometry of the scalar manifold in the effective $N = 2$ theory.

Chapter 4

Aspects of the $N = 2$ low-energy effective theory

The low-energy effective theory of an $N = 2$ vacuum of $N = 4$ supergravity should be consistent with $N = 2$ supersymmetry. In the case of the $g_a = 0$ solutions in purely electrically gauged supergravities we will therefore show that the various fields can be consistently embedded into complete $N = 2$ multiplets that are either massless or BPS. We will then discuss the unbroken gauge group that preserves the vacuum and, finally, we will comment on the effective Lagrangian below the scale of partial supersymmetry breaking. Bearing in mind that we have not yet fully solved the quadratic constraint equations, we will start generally but then restrict ourselves to the solutions with $g_a = 0$. The following discussion has also been published in [76].

4.1 Mass terms in the gauged theory

The fermionic mass terms of the theory emerge from the fermion bilinears given in equations (2.55) and (2.66) after evaluating the A-matrices at the critical point (3.28). By construction, the gravitini mass matrix is diagonal and its two non-zero eigenvalues are given by (3.47). Masses for vector bosons arise from the gauge-covariant derivative acting on the scalar fields. At the same time, the mixed couplings of vector bosons and scalar fields single out the pseudo-Goldstone fields that provide the longitudinal polarization of massive vector bosons. In the case of electric gaugings the scalars in the gravity multiplet are neutral ($D_\mu M_{\alpha\beta} = \partial_\mu M_{\alpha\beta}$) and thus the pseudo-Goldstone fields can only arise from the scalars of the vector multiplets. Using (2.52) together with all the information about the f_{MNP} obtained in the previous section, the gauged kinetic term of those scalars yields

$$\begin{aligned}\frac{1}{16}(D_\mu M_{MN})\,(D^\mu M^{MN}) \;\;&=\;\; \frac{1}{16}(\partial_\mu M_{MN})\,(\partial^\mu M^{MN}) \\ &\quad -\frac{g^2}{2}\sum_{a=1}^{n}(e_a^2+f_a^2+g_a^2)\sum_{m\in\{2,3,5,6\}} A_\mu^m A^{\mu m} \\ &\quad -\frac{g^2}{2}\sum_{b,c=1}^{n} O_{bc}\, A_\mu^b A^{\mu c} + \dots\,, \end{aligned} \tag{4.1}$$

where we introduced a symmetric and positive semi-definite matrix $(O_{ab}) \in \mathrm{Mat}_{n,n}$ with components

$$O_{bc} \equiv \sum_{a=1}^{n}\sum_{m=1}^{6} f_{abm}\, f_{acm}. \tag{4.2}$$

The $\dots$ in (4.1) denote couplings of vectors and Goldstone bosons. Note that in (4.1) the terms mixing $A^{\mu m}$ and $A^{\mu b}$ are absent due to the quadratic constraints (b, m, n, p) for $m, n, p \in \{2, 3, 5, 6\}$.

Before reading off the masses of the vector bosons one has to canonically normalize their kinetic terms in (2.2). To this end, we redefine $A'^{\mu M} = \sqrt{\operatorname{Im}\tau}A^{\mu M}$, for a given background value τ, which amounts to scaling all mass terms in (4.1) by a factor of $1/\operatorname{Im}\tau$ as required by supersymmetry, cf. section 3.1.3. It is then apparent that only four gauge bosons $(A^{\mu 2}, A^{\mu 3}, A^{\mu 5}, A^{\mu 6})$ of the gravity multiplet become heavy and, due to (3.48), (D.11), their masses are degenerate and equal to the gravitino mass (3.47):

$$m^2_{A^2,A^3,A^5,A^6} = \mathcal{V}_-^2\sum_{a=1}^{n}(e_a^2+f_a^2+g_a^2) = c^2\,\mathcal{V}_-^2 = (m_{3/2})^2\,. \tag{4.3}$$

Thus, an $N = 2$ vacuum with two non-BPS gravitino multiplets would require at least four vector multiplets (i.e. $n \geq 4$), as in this case eight massive vector bosons are contained in the two gravitino multiplets (3.36). Eventually, the symmetric mass matrix (O_{ab}) will be diagonalized by means of an $SO(n)$ transformation and being positive semi-definite it will give rise to well-defined mass terms. Note that for the solutions discussed in section 3.2.1 we always have $g_a = e_a = 0$ and $G_2 = G_3 = G_5 = G_6 = 0$ and the above expressions are much simpler.

In order to analyze the potential (2.62) in a neighborhood of the origin of the scalar manifold, we employ the following chart

$$\begin{aligned}\mathbb{R}^{6n} \supset U \quad &\to \quad W \subset {}^{SO(6,n)}/_{SO(6)\times SO(n)}, \\ \phi^{ma} \quad &\mapsto \quad \exp\left(\sum_{m,a}\phi^{ma}[t_{ma}]\right) \equiv \mathcal{V}(\phi^{ma}) \equiv \mathcal{V}\,, \end{aligned} \tag{4.4}$$

where $[t_{ma}]_M{}^N = \delta_{[m}{}^N \eta_{a]M}$ are the non-compact generators of the coset space associated to the vector multiplets. We can then express the scalar kinetic term as

$$\frac{1}{16}\left(\partial_\mu M_{MN}\right)\left(\partial^\mu M^{MN}\right) = -\frac{1}{2}\left(\partial_\mu \phi^{ma}\right)\left(\partial^\mu \phi^{ma}\right) + \mathcal{O}((\partial\phi)^2\phi^2)\ . \tag{4.5}$$

As this kinetic term is canonically normalized, we can identify the coordinates ϕ^{ma} with the scalar degrees of freedom. Geometrically, these can be interpreted as fluctuations in $SO(6,n)/[SO(6)\times SO(n)]$ around the critical point (3.28). Computing the scalar potential (2.62) it turns out that in the case of electric gaugings the two scalars of the gravity multiplet remain massless. Therefore, in an infinitesimal neighborhood of the origin where higher-order interactions are negligible, the scalar manifold of the gravity multiplet remains unaffected and thus can be ignored in what follows. Up to cubic terms, one finds:

$$\begin{aligned}\mathcal{L}_{pot} = -\frac{\mathcal{V}_-^2}{2}\Bigg[&\sum_c (e_c^2+f_c^2+g_c^2)\sum_a \sum_{m\in\{2,3,5,6\}} (\phi^{ma})^2 + \sum_{b,c} O_{bc}\sum_{m=1}^{6}\phi^{mb}\phi^{mc}\\ &+\sum_{a,b}\sum_{l,k=1}^{6}\Bigg(\sum_c f_{abc}f_{lkc} + \sum_{m=1}^{6} f_{abm}f_{lkm}\\ &\qquad + \sum_c f_{akc}f_{lbc} + \sum_{m=1}^{6} f_{akm}f_{lbm}\Bigg)\phi^{la}\phi^{kb}\Bigg]\\ &+\mathcal{O}(\phi^3)\ .\end{aligned} \tag{4.6}$$

Note that the absence of linear terms in (4.6) is a necessary condition for metastability. Furthermore, the fact that the cosmological constant vanishes is due to the quadratic constraint (D.11), as we have seen earlier. Recall that $\mathcal{V}_- = 1/\sqrt{\operatorname{Im}\tau}$ where τ is a coordinate on $SL(2)/SO(2)$ and, in fact, a modulus field.

Now that we know all mass terms we can check the super-Higgs mechanism that is required by partial supersymmetry breaking. First, we will consider the gravity/Goldstini sector, and secondly, we will discuss the matter sector. As a result, we will also show that the vacuum solutions are metastable, as required by the preserved $N=2$ supersymmetry. We will restrict ourselves to the case $g_a = 0$, which as we have seen in section 3.2 implies $e_a = 0$ and $G_2 = G_3 = G_5 = G_6 = 0$. For such solutions the potential simplifies to

$$\begin{aligned}\mathcal{L}_{pot} = -\frac{\mathcal{V}_-^2}{2}\Bigg[&c^2\sum_{m\in\{2,3,5,6\}}\phi^{m\tilde a}\phi^{m\tilde a} + \sum_{m\in\{2,3,5,6\}} O_{\tilde a\tilde b}\phi^{m\tilde a}\phi^{m\tilde b} + 4c\, f_{\tilde a\tilde b 4}\,(\phi^{2\tilde a}\phi^{3\tilde b}+\phi^{5\tilde a}\phi^{6\tilde b})\\ &+ f_{\tilde c\tilde a 4}\, f_{\tilde c\tilde b 4}\,\phi^{1\tilde a}\phi^{1\tilde b} + f_{\tilde c\tilde a 1}\, f_{\tilde c\tilde b 1}\,\phi^{4\tilde a}\phi^{4\tilde b} - 2 f_{1\tilde a\tilde c}\, f_{4\tilde b\tilde c}\,\phi^{4\tilde a}\phi^{1\tilde b}\Bigg]\\ &+\mathcal{O}(\phi^3),\end{aligned} \tag{4.7}$$

where as before we denote the potentially non-trivial embedding tensor components by $f_{\tilde{a}\tilde{b}m}$ for $SO(n-1)$ indices $\tilde{a}, \tilde{b}$, etc.

4.1.1 Gravity/Goldstini sector

In the gauge where $f_a = c\,\delta_{a7}$ it is only the "first" $N = 4$ vector multiplet that contributes to the gravity/Goldstini sector. After canonically normalizing the kinetic terms of the fermions by means of the field redefinition $\chi'^i = \frac{1}{\sqrt{2}}\chi^i$ we find that the fermionic mass terms in this sector read[1]

$$\begin{aligned} c\big[&\psi^3_\mu\,\epsilon\,\sigma^{\mu\nu}\psi^3_\nu + \tfrac{1}{2}\sqrt{2}\,\bar{\eta}^{(3)}\,\sigma^\mu\,\psi^3_\mu \\ &+ \psi^4_\mu\,\epsilon\,\sigma^{\mu\nu}\psi^4_\nu + \tfrac{1}{2}\sqrt{2}\,\bar{\eta}^{(4)}\,\sigma^\mu\,\psi^4_\mu \\ &- \sqrt{2}\,\chi'^3(\lambda^{74})^* - \tfrac{1}{2}(\lambda^{74})^*\epsilon\,(\lambda^{74})^* \\ &+ \sqrt{2}\,\chi'^4(\lambda^{73})^* - \tfrac{1}{2}(\lambda^{74})^*\epsilon\,(\lambda^{74})^*\big] + \text{h.c.}\ , \end{aligned} \tag{4.8}$$

where the would-be Goldstino combinations eaten by the massive gravitini are

$$\begin{aligned} \bar{\eta}^{(3)} &= \bar{\eta}^{(3)\dot{A}} = \epsilon^{\dot{A}\dot{B}}\chi'^3_{\dot{B}} + \sqrt{2}(\lambda^{74A})^*, \quad \dot{A}, \dot{B} = 1, 2 \\ \bar{\eta}^{(4)} &= \bar{\eta}^{(4)\dot{A}} = \epsilon^{\dot{A}\dot{B}}\chi'^4_{\dot{B}} - \sqrt{2}(\lambda^{73A})^*\,. \end{aligned} \tag{4.9}$$

The mass terms for the spin-1/2 fermions $\chi'_1, \chi'_2, \lambda^7_1, \lambda^7_2$ are absent in (4.8) and thus these fermions are massless. As in [9], mixed terms involving both a gravitino and a spin-1/2 fermion can be removed by means of the following gravitino shifts

$$\begin{aligned} \tilde{\psi}^3_\mu &= \tilde{\psi}^{3A}_\mu = \psi^{3A}_\mu + \tfrac{\sqrt{2}}{6}\bar{\sigma}^{A\dot{B}}_\mu\bar{\eta}^{(3)}_{\dot{B}} + \mathcal{O}(\partial\eta^{(3)}), \\ \tilde{\psi}^4_\mu &= \tilde{\psi}^{4A}_\mu = \psi^{4A}_\mu + \tfrac{\sqrt{2}}{6}\bar{\sigma}^{A\dot{B}}_\mu\bar{\eta}^{(4)}_{\dot{B}} + \mathcal{O}(\partial\eta^{(4)})\ , \end{aligned} \tag{4.10}$$

yielding additional contributions to the mass matrix of the spin-1/2 fermions. As a result, their mass terms read

$$\begin{aligned} \tfrac{c}{2}\Bigg[&\Big((\lambda^{74A})^*, \epsilon^{\dot{A}\dot{B}}\chi'^3_{\dot{B}}\Big)\,\epsilon_{\dot{A}\dot{C}}\,M^{(-)}\begin{pmatrix}(\lambda^{74C})^* \\ \epsilon^{\dot{C}\dot{D}}\chi'^3_{\dot{D}}\end{pmatrix} \\ &+ \Big((\lambda^{73A})^*, \epsilon^{\dot{A}\dot{B}}\chi'^4_{\dot{B}}\Big)\,\epsilon_{\dot{A}\dot{C}}\,M^{(+)}\begin{pmatrix}(\lambda^{73C})^* \\ \epsilon^{\dot{C}\dot{D}}\chi'^4_{\dot{D}}\end{pmatrix}\Bigg] + \text{h.c.}\ , \end{aligned} \tag{4.11}$$

where the mass matrices $M^{(\pm)}$ are given by

$$M^{(\pm)} = \tfrac{1}{3}\begin{pmatrix} 1 & \pm\sqrt{2} \\ \pm\sqrt{2} & 2 \end{pmatrix}\ , \tag{4.12}$$

[1] From now on we will drop the overall scaling factor of $\mathcal{V}_-^2$.

and both have eigenvalues 0 and 1. In fact, the two zero eigenvalues give rise to two massless helicity-1/2 fermions to be identified as the would-be Goldstini associated to the broken supersymmetry. On the other hand, one finds two spin-1/2 fermions of mass $|c|$ that together with the two massive gravitini fit into the $N = 2$ BPS gravitino multiplet.

As to the bosons in this sector, (4.1) shows that the only massive vectors are A^2_μ, A^3_μ, A^5_μ,A^6_μ while the massless ones are A^1_μ, A^4_μ, A^7_μ. The four massive vectors belong to the $N = 2$ BPS gravitino multiplet as we shall show in 4.1.3. Finally, all eight scalars of this sector are massless, as can be seen from (4.7), four of which are to be interpreted as the would-be Goldstone bosons. In an infinitesimal neighborhood around the critical point these fluctuations are described by $\phi^{27}, \phi^{37}, \phi^{57}, \phi^{67}$.

To conclude, we have shown that the fields in the massive BPS gravitino multiplet all have the same mass, consistent with $N = 2$ supersymmetry. Furthermore, in the gravity/Goldstini sector the $N = 2$ gravity multiplet and the massive $N = 2$ BPS gravitino multiplet are accompanied by two massless $N = 2$ vector multiplets, which are the remnants of the minimal $N = 4$ multiplets required for spontaneous partial supersymmetry breaking to $N = 2$.

$N = 2$ multiplets	mass squared
$M_{2,2,0}$ gravity	0
$M_{2,3/2,BPS}$ BPS gravitino	c^2
$2 \times M_{2,1,0}$ vector	0

Table 4.1: Gravity/Goldstini sector of the $N = 2$ spectrum.

4.1.2 Matter sector

The mass squared matrix for vector bosons $A^{\mu\tilde{a}}$ defined in (4.2) now reads

$$O = -G_1^2 - G_4^2\,, \tag{4.13}$$

which according to the discussion in section 3.2 is already diagonal. For each block in G_1 and G_4 with degenerate eigenvalues

$$(G_1^{(ij)})^2 = -x^2\,\mathbb{1}_l, \qquad (G_4^{(ij)})^2 = -y^2\,\mathbb{1}_l\,, \tag{4.14}$$

where $x, y \in \mathbb{R}$, one finds l vectors of mass squared $x^2 + y^2$.

Using the explicit expression given for the A-matrices in (3.24) the mass terms (2.66) for the fermions $\lambda^{1\tilde{a}}, \lambda^{2\tilde{a}}$ are given by

$$\tfrac{1}{2}\left((\lambda^{\tilde{a}1})^*, (\lambda^{\tilde{a}2})^*\right) \epsilon\, U \begin{pmatrix} (\lambda^{\tilde{b}1})^* \\ \lambda^{\tilde{b}2})^* \end{pmatrix} + \text{h.c.}\,, \tag{4.15}$$

with

$$U = \begin{pmatrix} 0 & iG_1 + G_4 \\ -iG_1 - G_4 & 0 \end{pmatrix} . \tag{4.16}$$

Thus, their mass squared matrix

$$UU^\dagger = \begin{pmatrix} O & 0 \\ 0 & O \end{pmatrix} = \begin{pmatrix} -G_1^2 - G_4^2 & 0 \\ 0 & -G_1^2 - G_4^2 \end{pmatrix} , \tag{4.17}$$

is also diagonal by virtue of the quadratic constraints (D.71). Similarly, the mass terms for $\lambda^{3\tilde{a}}, \lambda^{4\tilde{a}}$ in (2.66) are given by

$$\tfrac{1}{2} \left((\lambda^{\tilde{a}3})^*, (\lambda^{\tilde{a}4})^* \right) \epsilon \, V \begin{pmatrix} (\lambda^{\tilde{b}3})^* \\ \lambda^{\tilde{b}4})^* \end{pmatrix} + \text{h.c.} , \tag{4.18}$$

where

$$V = \begin{pmatrix} -c & -iG_1 + G_4 \\ iG_1 - G_4 & -c \end{pmatrix} . \tag{4.19}$$

The corresponding mass squared matrix reads

$$VV^\dagger = \begin{pmatrix} c^2 - G_1^2 - G_4^2 & -2c\, G_4 \\ 2c\, G_4 & c^2 - G_1^2 - G_4^2 \end{pmatrix} . \tag{4.20}$$

As in (4.14), it can be shown that for each block in G_1 and G_4 the eigenvalues are

$$x^2 + (|c| \pm |y|)^2 , \tag{4.21}$$

with degeneracy l each.

We can read off the mass terms for the scalar fields $\phi^{1\tilde{a}}, \phi^{4\tilde{a}}$ directly from (4.7),

$$-\tfrac{1}{2} \left(\phi^{1\tilde{a}}, \phi^{4\tilde{a}} \right) Z \begin{pmatrix} \phi^{1\tilde{b}} \\ \phi^{4\tilde{b}} \end{pmatrix} , \tag{4.22}$$

where

$$Z = \begin{pmatrix} -G_4^2 & G_4 G_1 \\ G_1 G_4 & -G_1^2 \end{pmatrix} . \tag{4.23}$$

Obviously, for the trivial block in G_1 and G_4 with $x, y = 0$ one obtains $(2l)$ massless scalars. On the other hand, for each block with $x \neq 0$ or $y \neq 0$, $l = 2l'$ has to be even and the eigenvalues of Z turn out to have $(2l')$-fold degenerate eigenvalues

$$0 , \qquad (x^2 + y^2). \tag{4.24}$$

The zero eigenvalue set precisely corresponds to the would-be Goldstone modes eaten by the $(2l')$ vector bosons that become massive. Finally, the mass terms for the remaining scalars $\phi^{2\tilde{a}}, \phi^{3\tilde{a}}, \phi^{5\tilde{a}}, \phi^{6\tilde{a}}$ turn out to be

$$-\tfrac{1}{2}\left(\phi^{3\tilde{a}},\phi^{2\tilde{a}}\right)\begin{pmatrix} c^2-G_1^2-G_4^2 & -2c\,G_4 \\ 2c\,G_4 & c^2-G_1^2-G_4^2\end{pmatrix}\begin{pmatrix}\phi^{3\tilde{b}}\\ \phi^{2\tilde{b}}\end{pmatrix},$$
$$-\tfrac{1}{2}\left(\phi^{6\tilde{a}},\phi^{5\tilde{a}}\right)\begin{pmatrix} c^2-G_1^2-G_4^2 & -2c\,G_4 \\ 2c\,G_4 & c^2-G_1^2-G_4^2\end{pmatrix}\begin{pmatrix}\phi^{6\tilde{b}}\\ \phi^{5\tilde{b}}\end{pmatrix}, \tag{4.25}$$

where the mass squared matrices are precisely $VV^\dagger$. As a result, one has $(2l)$ scalars for each mass in (4.21). It is then clear that all masses-squared are positive and therefore metastability is guaranteed, as required for a supersymmetric theory with Minkowski background. Furthermore, one finds that all degrees of freedom in the matter sector fit into complete $N=2$ supermultiplets. The resulting $N=2$ spectrum is summarized in table 4.2. Note that blocks in G_1 and G_4 with $x=0$ and $|y|=|c|$ give rise to massless $N=2$ hypermultiplets.

block	$N=2$ multiplets	mass squared
$G_1^{(ij)}=G_4^{(ij)}=0\cdot\mathbb{1}_l$	$(l)\times M_{2,1,0}$ massless vector	0
	$(l)\times M_{2,1/2,BPS}$ BPS hyper	c^2
$(G_1^{(ij)})^2=-x^2\,\mathbb{1}_{2l'}$,	$(2l')\times M_{2,1,BPS}$ BPS vector	(x^2+y^2)
$(G_4^{(ij)})^2=-y^2\,\mathbb{1}_{2l'}$	$(l')\times M_{2,1/2,BPS}$ BPS hyper	$x^2+(\lvert c\rvert+\lvert y\rvert)^2$
with $x\neq 0$ or $y\neq 0$	$(l')\times M_{2,1/2,.}$ (BPS) hyper	$x^2+(\lvert c\rvert-\lvert y\rvert)^2$

Table 4.2: Matter sector of the $N=2$ spectrum. The matrices $G_1, G_4 \in \mathrm{Mat}_{n-1,n-1}$ are simultaneously block-diagonal with non-trivial blocks of the type given in table 3.1 or zero blocks.

4.1.3 BPS multiplets

So far in the discussion of mass terms we have only shown that all fields fit into complete $N=2$ multiplets. In particular, according to our assignments in tables 4.1 and 4.2 all massive fields lie in BPS representations. In the generic case where the masses of the various $N=2$ superfields are all different, the above assignments are obviously correct. However, in the case of mass degeneracies between various short $N=2$ superfields one should exclude the case where short multiplets combine in order to form long multiplets. In fact, in what follows we will show that in the case of $g_a=0$ all massive fields have to be in BPS representations and that no long $N=2$ multiplet can occur in this super-Higgs mechanism. To this end we will study the crucial parts of the supersymmetry transformations of the bosonic fields that we take from [43].[2] It suffices to analyze the supersymmetry transformations of the massive bosons.

[2]While our proof is somewhat indirect, it does not require the supersymmetry transformations of the fermions which are not fully given in [43].

We first consider the massive vectors $A_\mu^2, A_\mu^3, A_\mu^5, A_\mu^6$ in the gravity/Goldstini sector. Evaluating their supersymmetry transformations at the origin (3.28) of $SO(6,n)$ one finds

$$\delta_\epsilon A_\mu^m \sim [G_m]_{ij}(\epsilon^i \epsilon\, \psi_\mu^j + \epsilon^i \epsilon \bar{\sigma}_\mu \chi^j) + \text{h.c.} \tag{4.26}$$

for $m = 2, 3, 5, 6$. Moreover, as in (2.58), $\epsilon^i = q^i \eta$ contains the $SU(4)$ vector q^i and $[G_m]_{ij}$ denote the 't Hooft matrices given in (B.37). In our gauge, cf. (3.33), the unbroken supersymmetry directions are given by linear combinations of q^1 and q^2 (or ϵ^1 and ϵ^2). As a result, for $m = 2, 3, 5, 6$ the massive vectors A_μ^m transform into the fermions $\psi_\mu^3, \psi_\mu^4, \chi^3, \chi^4$. While massive scalars are not present in the gravity/Goldstini sector, we will now inspect the transformations of the four Goldstone bosons that provide the longitudinal polarization of the massive vector bosons. In an infinitesimal neighborhood of the origin these fluctuations are described by the scalars $\phi^{27}, \phi^{37}, \phi^{57}, \phi^{67}$. Using the explicit chart (4.4) of $SO(6,n)$ one finds

$$\delta_\epsilon \mathcal{V}_m{}^a = \delta_\epsilon \phi^{ma} + \mathcal{O}(\phi\, \delta\phi)\,, \tag{4.27}$$

which when evaluated at the origin can again be expressed in terms of the 't Hooft matrices as

$$\delta_\epsilon \phi^{ma} \sim [G_m]_{ij}\, \epsilon^i \epsilon\, \lambda^{aj} + \text{h.c.}\,. \tag{4.28}$$

In particular, we find that the Goldstone bosons $\phi^{27}, \phi^{37}, \phi^{57}, \phi^{67}$ transform under $N = 2$ into fermions $\lambda^{73}, \lambda^{74}$. As a result, the massive bosons of the gravity/Goldstini sector transform into the massive fermions of the same sector. Note that the gravitino shifts in (4.10) also only involves the aforementioned fermions.

Next, we will analyze the supersymmetry transformations of the bosonic fields in the matter sector. The supersymmetry transformations of the massive vectors $A_\mu^{\hat{a}}$ evaluated at the origin are given by[3]

$$\delta_\epsilon A_\mu^{\hat{a}} \sim \epsilon^i \epsilon \bar{\sigma}_\mu \epsilon\, (\lambda^{\hat{a}i})^* + \text{h.c.}\,. \tag{4.29}$$

As a consequence, restricting the transformations to $N = 2$ one finds that each massive vector boson $A_\mu^{\hat{a}}$ rotates into the gaugini $\lambda^{\hat{a}1}$ and $\lambda^{\hat{a}2}$ but not into $\lambda^{\hat{a}3}$ and $\lambda^{\hat{a}4}$. Furthermore, as we discussed below (4.22), the associated Goldstone bosons are accompanied by massive scalars. Infinitesimally, all of them are described by linear combinations of the scalar fields $\phi^{1\hat{a}}$ and $\phi^{4\hat{a}}$. Their transformations can be read off from (4.28). Owing to the fact that $[G_m]_{ij}$ for $m = 1$ or $m = 4$ is block-diagonal, one finds that under $N = 2$ supersymmetry transformations the scalars $\phi^{1\hat{a}}$ and $\phi^{4\hat{a}}$ only rotate into fermions $\lambda^{\hat{a}1}$ and $\lambda^{\hat{a}2}$. In particular, this also shows that neither the would-be Goldstone combinations nor the massive scalars in (4.22) transform into $\lambda^{\hat{a}3}$ and $\lambda^{\hat{a}4}$. Furthermore, it is worth mentioning that neither $A_\mu^{\hat{a}}$ nor the massive scalars in (4.22) transform into

[3] As in section 4.2 indices $\hat{a}, \hat{b}, \ldots$ denote $SO(n-1)$ indices $\tilde{a}, \tilde{b}, \ldots$ associated to massive vector bosons, i.e. to non-trivial blocks in either G_1 or G_4.

the spin-1/2 fermions in the gravity/Goldstini sector given in (4.11), let alone into the massive gravitini. Finally, the only remaining potentially massive bosons are the scalars $\phi^{2\hat{a}}, \phi^{3\hat{a}}, \phi^{5\hat{a}}, \phi^{6\hat{a}}$ in (4.25). As can again be seen from (4.28), they only transform into fermions $\lambda^{\hat{a}3}, \lambda^{\hat{a}4}$ and never into $\lambda^{\hat{a}1}, \lambda^{\hat{a}2}$, let alone into fermions of the gravity/Goldstini sector.

We can now conclude that all massive $N = 2$ supermultiplets have to be BPS multiplets. The argument goes as follows: We found that the massive fields in the gravity/Goldstini sector and the massive fields in the matter sector are not related by supersymmetry transformations acting on the bosonic fields. This implies that the massive fields in the gravity/Goldstini sector have to lie in a BPS gravitino multiplet as massive long gravitino multiplets can never be decomposed into two non-trivial sets of bosons and fermions such that within each set the bosons only mix into the fermions, respectively. This follows from the construction of supermultiplets as representations of the Clifford algebra. Furthermore, by the same token, the remaining massive vector bosons have to be in $N = 2$ BPS vector multiplets.

4.2 Unbroken gauge group

We shall now investigate the unbroken gauge group at the $N = 2$ critical point, i.e. the group (or rather its Lie algebra) that leaves the scalar vacuum configuration for consistent electric gaugings with $g_a = 0$ invariant. First, we note that the critical point in $SL(2)/SO(2)$ is not affected by gauge transformations. However, on the scalar matter fields a generic gauge transformations parametrized by a local gauge parameter $\Lambda^P(x)$ acts as

$$M_{MN} \to M_{MN} + 2\,\Lambda^P f_{P(M}{}^Q M_{N)Q}\,, \tag{4.30}$$

and, in particular, the coset representative of the origin of $SO(6,n)/[SO(6) \times SO(n)]$ transforms as

$$\mathbb{1}_{MN} \to \mathbb{1}_{MN} + 2\,\Lambda^P (f_{PM}{}^N + f_{PN}{}^M)\,. \tag{4.31}$$

In demanding invariance of the origin under (4.31), the gauge parameters are restricted to the ones with $\Lambda^m = 0$ for $m = 2, 3, 5, 6$, and $\Lambda^{\tilde{a}} = 0$ for each massive vector boson $A^{\tilde{a}}_\mu$, the latter of which requires a non-zero block in G_1 or G_4. The gauge transformations of vector fields are given in (2.65) Using our knowledge of certain embedding tensor components in the case of $g_a = 0$ one can compute the gauge transformation for the massless vector bosons, which in this section we will denote as $A^{\mu\bar{a}}$ so as to distinguish them from massive vectors $A^{\mu\hat{a}}$. While we dropped the ˜ above indices, $\bar{a}$ and $\hat{a}$ are still understood as $SO(n-1)$ indices. One finds

$$\begin{aligned}
\delta A^1_\mu &= \partial_\mu \Lambda^1 \,, \\
\delta A^4_\mu &= \partial_\mu \Lambda^4 \,, \\
\delta A^7_\mu &= \partial_\mu \Lambda^7 \,, \\
\delta A^{\bar a}_\mu &= \partial_\mu \Lambda^{\bar a} - \, f_{\bar a \bar b \bar c}\, A^{\bar b}_\mu \, \Lambda^{\bar c} \,.
\end{aligned} \tag{4.32}$$

Note that in the last line of (4.32) we made use of $f_{\bar a \bar b \hat c} = 0$, which we learned from the quadratic constraints $(\tilde b, \bar c, \bar d, 1)$ and $(\tilde b, \bar c, \bar d, 4)$. The transformations (4.32) imply that we can interpret the three fields $A^1_\mu, A^4_\mu, A^7_\mu$ as the vector bosons of a gauge group $U(1)^3$. On the other hand, the embedding tensor components $f_{\bar a \bar b \bar c}$ amount to the structure constants of the gauge Lie algebra associated to the massless vector bosons $A^{\mu \bar a}$. In fact, as already pointed out in the simple case of (3.55), the quadratic constraints for $(\bar b, \bar c, \bar d, \bar e)$ are simply the Jacobi identity

$$f_{\bar a \bar b \bar c}\, f_{\bar d \bar e \bar a} + f_{\bar a \bar b \bar e}\, f_{\bar c \bar d \bar a} - f_{\bar a \bar b \bar d}\, f_{\bar a \bar c \bar e} = 0 \ , \tag{4.33}$$

that gives rise to a gauge Lie group G_{vac}. Its dimension equals the number of massless vector bosons ($\leq n-1$). If n is sufficiently large, any compact reductive Lie group can be chosen in order to satisfy (4.33). As a result, the full unbroken gauge symmetry is

$$U(1)^3 \times G_{\text{vac}}. \tag{4.34}$$

On the other hand, it is important to note that there is an additional set of constraints on the components $f_{\bar a \bar b \bar c}$ coming from the quadratic equations for $(\bar b, \bar c, \hat d, \hat e)$:

$$f_{\bar a \bar b \bar c} f_{\bar a \hat d \hat e} + f_{\hat a \bar b \hat e} f_{\bar c \hat d \hat a} - f_{\hat a \bar b \hat d} f_{\hat a \bar c \hat e} = 0 \ . \tag{4.35}$$

As we have seen in section 3.2.1, it is not always possible to set all $f_{\bar a \hat b \hat c}$ (i.e. the components given in (3.62)) to zero such that (4.35) is trivially satisfied. However, we have already shown in section 3.2.1.2 that consistent examples exist for any given compact reductive Lie group G_{vac}.

4.3 Scalar manifold in the effective theory

Below the scale of supersymmetry breaking $m_{3/2}$ we may integrate out heavy particles and, in doing so, arrive at an $N = 2$ supersymmetric effective action. We are particularly interested in the geometry of the scalar manifold of this effective action. As before, we will consider the case of electric gaugings with $g_a = 0$. In the limit where momenta $p \ll m_{3/2}$ can be neglected, the equations of motion for the massive vectors are purely algebraic and can be solved for the massive vector bosons since their mass terms are automatically diagonal, as we discussed in section 4.1.2. One finds

$$A^n_\mu = -\tfrac{1}{2c^2} \sum_{m\in\{2,3,5,6\}} (\partial_\mu M_{m7})\, f_{7nm}\,,$$
$$A^{\hat b}_\mu = -\tfrac{1}{2m^2_{(\hat b)}} \sum_{m\in\{1,4\}} (\partial_\mu M_{m\tilde a})\, f_{\tilde a\hat b m} \tag{4.36}$$

for each $n \in \{2,3,5,6\}$ and massive vectors with index $\hat b$. When inserted back into the Lagrangian and using our knowledge about certain embedding tensor components, the scalar kinetic term yields[4]

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \tfrac{1}{16}\Bigg[& 2\sum_{m\in\{2,3,5,6\}} (\partial_\mu M_{m\tilde a})\left(\partial^\mu M^{m\tilde a}\right) + 2\sum_{m\in\{1,4\}} (\partial_\mu M_{m7})\left(\partial^\mu M^{m7}\right) \\ & + 2\sum_{m\in\{1,4\}} (\partial_\mu M_{m\tilde a})\left(\partial^\mu M^{m\tilde a}\right) + \sum_{\hat b} (m_{(\hat b)})^{-2} \sum_{m,n\in\{1,4\}} (\partial_\mu M_{m\hat a})(\partial^\mu M_{n\hat c})\, f_{\hat a\hat b m} f_{\hat c\hat b n} \\ & + (\partial_\mu M_{mn})\,(\partial^\mu M^{mn}) + (\partial_\mu M_{ab})\left(\partial^\mu M^{ab}\right)\Bigg]\,. \end{aligned} \tag{4.37}$$

Using the chart (4.4) one finds

$$\begin{aligned} & -\tfrac12 \sum_{m\in\{2,3,5,6\}} \left(\partial_\mu\phi^{m\tilde a}\right)\left(\partial^\mu\phi^{m\tilde a}\right) - \tfrac12 \sum_{m\in\{1,4\}} \left(\partial_\mu\phi^{m7}\right)\left(\partial^\mu\phi^{m7}\right) \\ & -\tfrac12 \sum_{m\in\{1,4\}} \left(\partial_\mu\phi^{m\bar a}\right)\left(\partial^\mu\phi^{m\bar a}\right) \\ & -\tfrac12 \left(\partial_\mu\phi^{1\hat a},\, \partial_\mu\phi^{4\hat a}\right)\left((O^{(\text{massive})})^{-2} Z^{(\text{massive})}\right)_{\hat a\hat b} \begin{pmatrix}\partial^\mu\phi^{1\hat b}\\ \partial^\mu\phi^{4\hat b}\end{pmatrix} + \mathcal{O}\left((\partial\phi)^2\phi^2\right)\,, \end{aligned} \tag{4.38}$$

where $O^{(\text{massive})}$ is the truncation of (4.13) to an invertible matrix obtained after deleting all its zero rows and columns, and similarly, $Z^{(\text{massive})}$ is the analogous truncation of the mass matrix Z defined in (4.23). Note that kinetic terms for the Goldstone modes ϕ^{m7} for $m = 2,3,5,6$ are absent in (4.38) as these scalars have been eaten by the massive vector bosons $A^{\mu m}$ for $m = 2,3,5,6$. Moreover, the same diagonalization scheme of section 4.1.2 also diagonalizes the kinetic terms of the scalars $\phi^{1\hat a}$ and $\phi^{4\hat a}$ associated to massive vectors with indices $\hat a$. As before, the zero eigenvalues of $Z^{(\text{massive})}$ ensure that the kinetic terms of the Goldstone modes in the matter sector vanish (again the Goldstone modes are eaten by the vector bosons $A^{\mu\hat a}$ that acquire mass). On the other hand, its nonzero eigenvalues are such that the remaining kinetic terms are canonically normalized, which justifies the mass assignment in section 4.1.2.

Let us now summarize the dynamical degrees of freedom in an infinitesimal neighborhood of the origin. The scalars $\phi^{m\tilde a}$ for $m = 2,3,5,6$ are part of $N = 2$ (BPS)

[4] Repeated indices are summed over their full index range unless otherwise specified by explicit summation symbols.

hypermultiplets, while ϕ^{17} and ϕ^{47} and the two scalars of $SL(2)/SO(2)$ lie in the two massless $N = 2$ multiplets that descend from the gravity/Goldstini sector. The scalars $\phi^{1\bar{a}}, \phi^{4\bar{a}}$ form $N = 2$ massless vector multiplets, while the non-Goldstone modes of the $\phi^{1\hat{a}}, \phi^{4\hat{a}}$ belong to $N = 2$ BPS vector multiplets. Note, however, that in the effective theory below the scale of partial supersymmetry breaking $m_{3/2}$, all scalars (and their supersymmetry partners) with masses larger than $m_{3/2}$ should also be integrated out.

As the scalars of $SL(2)/SO(2)$, described by τ, are moduli that lie in a massless $N = 2$ vector multiplet, the $SL(2)/SO(2)$ factor of the $N = 4$ scalar manifold descends without change to the scalar field space of the massless $N = 2$ vector multiplets in the low-energy theory. If the number of these vector multiplets is $(k+1)$, we conjecture that the vector multiplet field space of the $N = 2$ low-energy theory is the following product of coset spaces,

$$^{SL(2)}/_{SO(2)} \times {}^{SO(2,k)}/_{SO(2)\times SO(k)}\,, \tag{4.39}$$

which is known to be the only series of special Kähler product manifolds including a factor of $SL(2)/SO(2)$ [87]. Moreover, since we only analyze the potential to quadratic order, we can only infer that the moduli space is a submanifold of (4.39). To see this explicitly, one should reconstruct the metric of the scalar manifold order by order (due to the power expansion of the exponential map in (4.4)). As we saw in section 3.2.1.2, it is also possible to have light or massless hypermultiplets, in which case $N = 2$ supersymmetry requires the field space to be quaternionic Kähler. It may be that the coordinates (4.4) on $SO(6,n)/[SO(6)\times SO(n)]$ that we are using here are not well suited to checking the expected geometry explicitly.

Chapter 5

Conclusions and Outlook

In this thesis we studied $N = 2$ vacua in gauged $N = 4$ supergravity theories in four-dimensional spacetime. Having reviewed ungauged, matter-coupled $N = 4$ supergravity in a symplectic frame with $SO(1,1) \times SO(6,n)$ symmetry, we considered general magnetic gaugings to be described by the embedding tensor formalism. In such theories we first specified a class of maximally-symmetric backgrounds where the only non-vanishing fields are the metric and a spacetime-independent point configuration in the scalar manifold. In view of spontaneous partial supersymmetry breaking we studied the supersymmetry transformations of such a background and analyzed the resulting Killing spinor equations. In doing so, we first discussed the integrability conditions for Killing spinors and rederived the well-known result that supersymmetric vacua of the aforementioned kind are either Minkowski or anti-de Sitter vacua (as opposed to being de Sitter). Then we formulated necessary conditions for partial supersymmetry breaking that — being essentially a system of linear equations — could be solved for the embedding tensor components. On the other hand, the sufficient conditions on the deformation parameters are the quadratic constraints that ensure the consistency of the gauging. These equations are algebraic, quadratic tensor equations with respect to the isometry Lie group $SL(2)\times SO(6,n)$ of the scalar manifold and therefore scale badly with the free parameter n that counts the number of $N = 4$ vector multiplets in the spectrum of the ungauged $N = 4$ supergravity theory. For simplicity, we therefore restricted ourselves to the case of purely electric gaugings and solved the quadratic constraints as much as possible. While it was not difficult to solve all the constraints for $n \leq 6$, it was only after setting certain embedding tensor components to zero ($g_a \equiv f_{a26} = 0$) that a large set of physically interesting solutions with arbitrary $n \in \mathbb{N}$ could be constructed. Intuitively, setting the aforementioned components to zero amounts to a simplification of the gravity/Goldstini sector in that only one $N = 4$ vector multiplet suffices to provide the degrees of freedom in order for the two massive gravitini and their superpartners to become massive as is required by an $N = 2$ vacuum. It is noteworthy that in an intermediate step of the construction of such consistent solutions a subsystem of the quadratic equations, which probably cannot be solved by elementary means, could completely be solved using a corollary of Lie's theorem after recognizing it as a representation theoretical problem of a solvable Lie algebra. On the other hand, even in the case of purely electric gaugings it has not been possible to fully classify all deformations that allow for an $N = 2$ vacuum. While obviously this is due to the lack of a universal algorithm to solve a system of algebraic equations, it can already be seen from the special solutions we found that solutions may potentially be increasingly sophisticated. In accordance with the literature we found that electric gaugings in the specified symplectic frame only allow for Minkowski vacua and can never spontaneously break $N = 4$ to $N = 3$. Furthermore note that $N = 4$ supergravity is broken to global $N = 2$ supersymmetry.

For the explicit consistent solutions with $g_a = 0$ we studied the super-Higgs mechanisms and found that its consistency heavily relies on the consistency of the gauging. To this end, we computed the mass terms of all the fields and checked that the latter can be embedded into complete $N = 2$ supermultiplets as required by $N = 2$ supersymmetry. We found that massive supermultiplets necessarily have to be BPS, i.e. such fields are charged with respect to a central charge of the supersymmetry algebra in precisely the way that gives rise to multiplet shortening. Here one may wonder whether this is yet another consequence of setting $g_a = 0$. In particular, we found that stability is guaranteed. Next, we defined the notion of the unbroken gauge Lie algebra and demonstrated that up to an abelian Lie algebra many different sensible gauge Lie algebras that preserve the vacuum can be realized as long as the number n of $N = 4$ vector multiplets is sufficiently large. Finally, we computed the relevant terms of the effective $N = 2$ supersymmetric action below the scale of partial supersymmetry breaking. Using a theorem of special-Kählerity we inferred that the target manifold of the scalars of the $N = 2$ vector multiplets lies in the unique series of special-Kähler product manifolds. On the other hand, in the coordinates we were using to describe an open neighborhood of the origin this claim could not explicitly be verified.

As stressed at the end of section 2 in principle it would be important to also consider magnetic gaugings as these equivalently describe electric gaugings of many different a priori ungauged $N = 4$ supergravity theories. Unfortunately, then the system of quadratic constraints becomes even more complicated. Furthermore, it could also be interesting to study $N = 1$ vacua (or $N = 3$ vacua using magnetic gaugings). Finally, we would like to mention that the procedure that led to the construction of vacua with partial supersymmetry breaking may in principle be transferred to an analogous analysis in maximal $N = 8$ supergravity. There the scalar field space is fixed to be 70-dimensional homogeneous space $E_{7,7}/(SU(8)/\mathbb{Z}_2)$ and, hence, in the absence of an arbitrary parameter $n \in \mathbb{N}$, the quadratic constraints cannot become arbitarily complicated. On the other hand, the very nature of exceptional simple Lie algebras will still sufficiently complicate the search for consistent gaugings.

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Appendix A

Conventions

The spacetime metric $g_{\mu\nu}$ used in this paper has signature $(-,+,+,+)$ and the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\lambda}$ is defined with $\epsilon^{0123} = e^{-1}$, $\epsilon_{0123} = -e = -\sqrt{|\det g|}$.

We use the following indices:

indices	group
$\alpha, \beta, \gamma, \ldots \in \{-,+\}$	$SL(2)$
$M, N, P, \ldots \in \{1, \ldots, 6+n\}$	$SO(6,n)$
$m, n, p, \ldots \in \{1, \ldots, 6\}$	$SO(6)$
$i, j, k, \ldots \in \{1, \ldots, 4\}$	$SU(4)$
$a, b, c, \ldots \in \{1, \ldots, n\}$	$SO(n)$
$\tilde{a}, \tilde{b}, \tilde{c}, \ldots \in \{1, \ldots, n-1\}$	$SO(n-1) \subset SO(n)$

All indices other than the ones of $SU(4)$ transform under the fundamental representation of the given groups. In the case of $SU(4)$ upper/lower indices transform under the $\mathbf{4}$ ($\bar{\mathbf{4}}$), respectively. Upon complex conjugation such upper and lower indices are interchanged, e.g. $(X_i{}^j)^* = X^i{}_j$.

In addition, for the $g_a = 0$ solutions discussed in sections 4.2 and 4.3 it is sometimes convenient to use $SO(n-1) \subset SO(n)$ indices, $\bar{a}\ldots$ and $\hat{a}\ldots$ which are associated to massless vectors $A^{\mu\bar{a}}$ and massive vectors $A^{\mu\hat{a}}$, respectively.

Appendix B

Non-linear sigma model

B.1 $SU(1,1)/U(1) \cong SL(2)/SO(2)$

Here we review different formulations of the non-linear sigma model on

$$SU(1,1)/U(1) \cong SL(2)/SO(2)\,. \tag{B.1}$$

In doing so, it will also become apparent that the manifold (B.1) is diffeomorphic to the upper half of the complex plane. It is sometimes called the Poincaré plane [47]. The following discussion is close to the one in [78].

In the lifted formulation of non-linear sigma models on homogeneous spaces G/H [88] (which is used in order to couple scalars to fermions as required by supersymmetry) a coset field $\mathcal{V}'$ on four-dimensional Minkowski space $\mathbb{M}_{1,3}$ that encodes the scalar fields,

$$\mathcal{V}' : \mathbb{M}_{1,3} \to G/H\,, \tag{B.2}$$

is lifted by a section in the H-principal bundle $G \to G/H$ to a function

$$\mathcal{V} : \mathbb{M}_{1,3} \to G\,. \tag{B.3}$$

This amounts to choosing coset representatives for each spacetime point x. For matrix representations of the group G the pull-back of the Maurer-Cartan form with respect to $\mathcal{V}$ is

$$\mathcal{V}^{-1}(x)\,\partial_\mu \mathcal{V}(x) \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \tag{B.4}$$

for all $x \in \mathbb{M}_{1,3}$. Here, $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of the Lie groups G and H, respectively. Moreover, $\mathfrak{h} \perp \mathfrak{h}^\perp$ with respect to a given inner product κ on $\mathfrak{g}$. Let $\mathbb{P}$ denote the projection

$$\mathbb{P} : \mathfrak{g} \to \mathfrak{h}^\perp \tag{B.5}$$

with $\mathbb{P}^2 = \mathbb{P}$. The scalar kinetic term can then be written as

$$e^{-1}\mathcal{L}_{\text{scalar kinetic}} \propto \kappa(\mathbb{P}\mathcal{V}^{-1}\partial_\mu \mathcal{V},\, \mathbb{P}\mathcal{V}^{-1}\partial^\mu \mathcal{V})\,. \tag{B.6}$$

An arbitrary matrix $\mathcal{V} \in SU(1,1)$ can be written as

$$\mathcal{V} = \begin{pmatrix} \phi & \psi^* \\ \psi & \phi^* \end{pmatrix} , \tag{B.7}$$

with complex number $\phi, \psi \in \mathbb{C}$ satisfying

$$|\phi|^2 - |\psi|^2 = 1 . \tag{B.8}$$

Its real Lie algebra $\mathfrak{su}(1,1)$ can be decomposed as

$$\mathfrak{su}(1,1) = \mathfrak{u}(1) \oplus \mathfrak{u}(1)^\perp , \tag{B.9}$$

where in the fundamental representation[1]

$$\begin{aligned} \mathfrak{u}(1) &= \mathbb{R} i\sigma_3 , \\ \mathfrak{u}(1)^\perp &= \mathbb{R}\sigma_1 + \mathbb{R}\sigma_2 \end{aligned} \tag{B.10}$$

are given in terms of the ordinary Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Let $\mathbb{P}$ be the projection

$$\mathbb{P} : \mathfrak{su}(1,1) \to \mathfrak{u}(1)^\perp \tag{B.11}$$

with $\mathbb{P}^2 = \mathbb{P}$. The basis vectors $\sigma_1, \sigma_2, i\sigma_3$ are orthogonal with respect to the inner product

$$\begin{aligned} \mathfrak{su}(1,1) \times \mathfrak{su}(1,1) &\to \mathbb{R} \\ (X,Y) &\mapsto \mathrm{Tr}(XY) \end{aligned} \tag{B.12}$$

on $\mathfrak{su}(1,1)$. Then the scalar kinetic term of the non-linear sigma model on $SU(1,1)/U(1)$ is given by

$$\begin{aligned} -\tfrac{1}{2}\mathrm{Tr}\left(\mathbb{P}\mathcal{V}^{-1}(\partial_\mu\mathcal{V})\mathcal{V}^{-1}(\partial^\mu\mathcal{V})\right) =& \phi^*\psi\,\partial_\mu\psi^*\partial^\mu\phi + \psi^*\phi\,\partial_\mu\phi^*\partial^\mu\psi \\ &- |\phi|^2\partial_\mu\psi^*\partial^\mu\psi - |\psi|^2\partial_\mu\phi^*\partial^\mu\phi . \end{aligned} \tag{B.13}$$

Without loss of generality (by means of a $U(1)$ transformation) we can choose coordinates for representatives of $SU(1,1)/U(1)$,

$$\phi = \frac{1}{\sqrt{1-r^2}}, \qquad \psi = \frac{re^{i\chi}}{\sqrt{1-r^2}} , \tag{B.14}$$

with $r \in [0,1), \chi \in [0,2\pi)$ and thus (B.13) can be written as

$$-\frac{1}{(1-r^2)^2}\left(\partial_\mu r\partial^\mu r + r^2\partial_\mu\chi\partial^\mu\chi\right) . \tag{B.15}$$

[1]Here $\mathbb{R}v$ denotes the $\mathbb{R}$-vector space generated by a vector v, i.e. the $\mathbb{R}$-span of v.

Furthermore, writing

$$z = re^{-i\chi}\,, \qquad \tau = i\frac{1-z}{1+z} \tag{B.16}$$

one can express (B.15) in terms of a complex variable $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ as

$$-\tfrac{1}{4}\frac{\partial_\mu \tau \partial^\mu \tau^*}{(\operatorname{Im}\tau)^2}\,. \tag{B.17}$$

This shows that $SU(1,1)/U(1)$ is diffeomorphic to the upper half plane and furthermore is a complex manifold. Moreover, the metric is hermitian and its associated Kähler form is closed. As a matter of fact, $SU(1,1)/U(1)$ is a Kähler manifold and the Kähler potential can be chosen to be

$$K = \tfrac{1}{2}\log \operatorname{Im}\tau\,. \tag{B.18}$$

After yet another change of coordinates,

$$\tau = \sigma + ie^{-\phi}\,, \tag{B.19}$$

the scalar kinetic term (B.17) reads

$$-\tfrac{1}{4}\left(\partial_\mu\phi\,\partial^\mu\phi + e^{2\phi}\partial_\mu\sigma\,\partial^\mu\sigma\right)\,, \tag{B.20}$$

for $\sigma, \phi \in \mathbb{R}$. This is precisely the $SL(2)/SO(2)$ description given as follows: The Lie algebra of $SL(2,\mathbb{R}) = SL(2)$ decomposes as

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2,\mathbb{R}) \oplus \mathfrak{so}(2,\mathbb{R})^\perp \tag{B.21}$$

where in the fundamental representation

$$\begin{aligned} \mathfrak{so}(2,\mathbb{R}) &= -\mathbb{R}i\sigma_2\,,\\ \mathfrak{so}(2,\mathbb{R})^\perp &= \mathbb{R}\sigma_3 + \mathbb{R}\sigma_1\,. \end{aligned} \tag{B.22}$$

Again, let $\mathbb{P}$ denote the projection

$$\mathbb{P} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(2,\mathbb{R})^\perp \tag{B.23}$$

with $\mathbb{P}^2 = \mathbb{P}$ and the inner product on $\mathfrak{sl}(2,\mathbb{R})$ is

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) &\to \mathbb{R}\\ (X,Y) &\mapsto \operatorname{Tr}(XY)\,. \end{aligned} \tag{B.24}$$

Then, for spacetime-dependent $SL(2)/SO(2)$ representatives (in the triangular gauge)

$$\mathcal{V} = \exp\left(\sigma\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\exp\left(\phi\begin{pmatrix}-1/2 & 0\\ 0 & 1/2\end{pmatrix}\right) = \begin{pmatrix}e^{-\phi/2} & \sigma e^{\phi/2}\\ 0 & e^{\phi/2}\end{pmatrix} \in SL(2) \tag{B.25}$$

the scalar kinetic term is given by

$$-\tfrac{1}{2}\mathrm{Tr}\left(\mathbb{P}\mathcal{V}^{-1}(\partial_\mu\mathcal{V})\mathcal{V}^{-1}(\partial^\mu\mathcal{V})\right) \tag{B.26}$$

which indeed coincides with (B.20).

The natural $SL(2)$ action on $SL(2)/SO(2)$ is an isometry that leaves the metric invariant. In fact, in the description using the upper half plane $SL(2)$ acts as Möbius transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{B.27}$$

where $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,, \tag{B.28}$$

under which the scalar kinetic term in (B.17) is left invariant. In another description that makes this $SL(2)$-invariance manifest, one introduces a spacetime-dependent matrix $M = \mathcal{V}\mathcal{V}^T$ with $\mathcal{V}$ given in (B.25). The scalar kinetic terms (B.20) can then be written as

$$\tfrac{1}{8}\mathrm{Tr}\left((\partial_\mu M)(\partial^\mu M^{-1})\right)\,. \tag{B.29}$$

We will return to this in section B.2.

For the sake of completeness we state that the isomorphism between $SL(2)$ and $SU(1,1)$ is given by the following similarity transformation,

$$D^{-1}\,SL(2,\mathbb{R})\,D = SU(1,1), \qquad D = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\,, \tag{B.30}$$

which also maps the maximal compact subgroups $SO(2) \subset SL(2)$ and $U(1) \subset SU(1,1)$ onto each other.

B.2 Coset space representatives

The coset space ${}^{SO(6,n)}/_{SO(6)\times SO(n)}$ is represented by a matrix $\mathcal{V} = (\mathcal{V}_M{}^N) \in SO(6,n)$. Raising/lowering $SO(6,n)$ indices is defined via the $SO(6,n)$ invariant metric

$$\eta = (\eta_{MN}) = (\eta^{MN}) = \mathrm{diag}(\underbrace{-\ldots-}_{6\text{ times}},\underbrace{+\ldots+}_{n\text{ times}})\,,$$

and $\mathcal{V}^{-1T} = (\mathcal{V}^M{}_N)$. $\mathcal{V}$ transforms as

$$\mathcal{V} \to g\,\mathcal{V}\,h(x)\,, \tag{B.31}$$

which in terms of indices reads

$$\begin{aligned} \mathcal{V}_M{}^N &\to g_M{}^P\,\mathcal{V}_P{}^Q\,h(x)_Q{}^N\,, \\ \mathcal{V}^M{}_N &\to g^M{}_P\,\mathcal{V}^P{}_Q\,h(x)^Q{}_N\,, \end{aligned} \tag{B.32}$$

where $g = (g_M{}^P) \in SO(6,n)$ and a spacetime dependent $h(x) = (h(x)_Q{}^N) \in SO(6) \times SO(n)$ and $g^M{}_P$ and $h(x)^Q{}_N$ are obtained from the former via lowering/raising indices. It is apparent that global $SO(6,n)$ acts only on the first index of $\mathcal{V}_M{}^N$ and $\mathcal{V}^M{}_N$ while local $SO(6) \times SO(n)$ acts only on the second. The bosonic part of the Lagrangian can be conveniently expressed in terms of a symmetric positive definite matrix

$$M = (M_{MN}) := \mathcal{V}\mathcal{V}^T \,, \tag{B.33}$$

which transforms as a tensor of $SO(6,n)$, i.e.

$$M_{MN} \to g_M{}^Q g_N{}^R M_{QR} \,, \tag{B.34}$$

and is manifestly invariant under local $SO(6) \times SO(n)$ transformations. One also has $M^{-1} = (M^{MN})$ transforming as

$$M^{MN} \to g^M{}_Q g^N{}_R M^{QR}. \tag{B.35}$$

In describing the supergravity theory index calculus seems to be indispensable because $SO(6,n)$ indices associated to $SO(6) \times SO(n)$ need to be decomposed into those of $SO(6)$ and $SO(n)$, of which the $SO(6)$ indices are to be transferred to indices of the universal cover $SU(4)$ in order to describe the coupling of scalar representatives to fermions. The relation between these indices is due to the fact that in terms of representations of their common complex Lie algebra one has $(\mathbf{4} \otimes \mathbf{4})_{\text{antisymmetric}} \cong \mathbf{6}$. As in the appendix of [80], we therefore associate to every vector index m of $SO(6)$ a pair of anti-symmetric $SU(4)$ indices $[ij]$ in the following way

$$\phi_{ij} = \tfrac{1}{2} \sum_{m=1}^{6} \phi_m \, [G_m]_{ij} \,, \qquad \phi^{ij} = -\tfrac{1}{2} \sum_{m=1}^{6} \phi_m \, [G_m]^{ij} \,, \tag{B.36}$$

where ϕ_m shall be a generic $SO(6)$ vector and the G's are the 't Hooft matrices

$$[G_1]_{ij} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} , \quad [G_2]_{ij} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} ,$$

$$[G_3]_{ij} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} , \quad [G_4]_{ij} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} ,$$

$$[G_5]_{ij} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \quad [G_6]_{ij} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{B.37}$$

Furthermore, for every $m = 1, \ldots, 6$ one defines

$$[G_m]^{ij} = -\tfrac{1}{2}\epsilon^{ijkl}\,[G_m]_{kl} = -([G_m]_{ij})^*\,, \tag{B.38}$$

so as to obtain $(\phi_{ij})^* = \phi^{ij}$.

At the origin of $SO(6,n)$, cf. (3.28), one finds $\mathcal{V} = \mathcal{V}^{-1T} = \mathbb{1}$ which in components reads

$$\begin{aligned} \mathcal{V}_m{}^n = \mathcal{V}^n{}_m = \delta^n_m, \qquad & \mathcal{V}_m{}^a = \mathcal{V}^m{}_a = 0, \\ \mathcal{V}_a{}^b = \mathcal{V}^b{}_a = \delta^b_a, \qquad & \mathcal{V}_a{}^m = \mathcal{V}^m{}_a = 0. \end{aligned} \tag{B.39}$$

In terms of $SU(4)$ indices one now has

$$\mathcal{V}_M{}^{ij} = \begin{cases} \frac{1}{2}[G_m]^{ij}, & \text{if } M = m \\ 0, & \text{if } M = a \end{cases}, \qquad \mathcal{V}^M{}_{ij} = \begin{cases} \frac{1}{2}[G_m]_{ij}, & \text{if } M = m \\ 0, & \text{if } M = a \end{cases}. \tag{B.40}$$

As to ${}^{SL(2)}/_{SO(2)}$, a generic representative would be $\mathcal{V} = \mathcal{V}_{SL(2)} = (\mathcal{V}_\alpha{}^\beta) \in SL(2)$. Raising/lowering indices is defined via the antisymmetric matrix $\epsilon = (\epsilon_{\alpha\beta}) = (\epsilon^{\alpha\beta})$ with $\epsilon^{21} = \epsilon^{-+} = 1$[2] in such a way that

$$(\mathcal{V}^\alpha{}_\beta) = (\epsilon^{\alpha\gamma}\mathcal{V}_\gamma{}^\delta\epsilon_{\delta\beta}) = \epsilon\mathcal{V}\epsilon = -\mathcal{V}^{-1T}. \tag{B.41}$$

As before, transformations in terms of indices are

$$\begin{aligned} \mathcal{V} = (\mathcal{V}_\alpha{}^\beta) \quad &\to \quad g\mathcal{V}h(x) = (g_\alpha{}^\gamma\mathcal{V}_\gamma{}^\delta h(x)_\delta{}^\beta)\,, \\ -\mathcal{V}^{-1T} = (\mathcal{V}^\alpha{}_\beta) \quad &\to \quad (g^\alpha{}_\gamma\mathcal{V}^\gamma{}_\delta h(x)^\delta{}_\beta)\,, \end{aligned} \tag{B.42}$$

and the bosonic Lagrangian can be written in terms of the symmetric positive definite matrix

$$M := \mathcal{V}\mathcal{V}^T = (M_{\alpha\beta})\,, \tag{B.43}$$

that can be expressed in terms of $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ as

$$(M_{\alpha\beta}) = \frac{1}{\operatorname{Im}(\tau)}\begin{pmatrix} |\tau|^2 & \operatorname{Re}(\tau) \\ \operatorname{Re}(\tau) & 1 \end{pmatrix}. \tag{B.44}$$

This follows from (B.25) and (B.19). Its inverse is

$$M^{-1} = (M^{\alpha\beta}) = \frac{1}{\operatorname{Im}(\tau)}\begin{pmatrix} 1 & -\operatorname{Re}(\tau) \\ -\operatorname{Re}(\tau) & |\tau|^2 \end{pmatrix}, \tag{B.45}$$

and transforms accordingly. The fermionic sector of the supergravity theory requires a different representation of cosets, namely, in terms of $(\mathcal{V}_\alpha) \in \mathbb{C}^2$ such that

$$M_{\alpha\beta} = \operatorname{Re}(\mathcal{V}_\alpha(\mathcal{V}_\beta)^*)\,. \tag{B.46}$$

[2] The identification of indices $1 \equiv +, 2 \equiv -$ is due to (2.32).

For (B.44) one can always find appropriate $\mathcal{V}_\alpha$. Letting them transform as vectors under global $SL(2) = SL(2, \mathbb{R})$ gives the right transformation for $M_{\alpha\beta}$. For a given τ as above, $\mathcal{V}_\alpha$ is unique up to local $U(1)$ transformations

$$\mathcal{V}_\alpha \to e^{i\phi(x)}\mathcal{V}_\alpha \tag{B.47}$$

for arbitrary $\phi(x) \in \mathbb{R}$ (and up to a sign ambiguity[3]). As fermions also transform under this $U(1)$, they couple to coset representatives $\mathcal{V}_\alpha$. At the origin $\mathcal{V} = \mathbb{1}$ and thus in an appropriate gauge one finds $(\mathcal{V}_\alpha) = (i, 1)^T$, i.e. $\mathcal{V}_- = 1$.

[3]Fixing the gauge such that $\mathbb{R} \ni \mathcal{V}_- > 0$, one finds a sign ambiguity in the imaginary part of $\mathcal{V}_+$ as is apparent from $M_{\alpha\beta} = (\mathrm{Re}\,\mathcal{V}_\alpha)(\mathrm{Re}\,\mathcal{V}_\beta) + (\mathrm{Im}\,\mathcal{V}_\alpha)(\mathrm{Im}\,\mathcal{V}_\beta)$.

Appendix C

Spinors

C.1 Weyl & Dirac spinor conventions

While we find it more convenient to work with Weyl spinors, the fermionic terms in the literature [43, 54] are given in terms of Dirac spinors. Based on the conventions given in [54] we express Dirac spinors in terms of Weyl spinors. In what follows we will first summarize their conventions and then express fermionic terms using Weyl spinors.

The metric $(\eta_{\mu\nu})$ has signature $(-,+,+,+)$. The γ-matrices Γ_μ satisfying

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \tag{C.1}$$

are (chirally) represented by

$$\Gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma_\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}. \tag{C.2}$$

where

$$\sigma_\mu = (\mathbb{1}, \vec{\sigma}) = \bar{\sigma}^\mu, \qquad \sigma^\mu = \eta^{\mu\nu}\sigma_\nu = (-\mathbb{1}, \vec{\sigma}) = \bar{\sigma}_\mu\,,$$

and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is built from the usual σ-matrices. One then has

$$\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{C.3}$$

and

$$\begin{gathered} (\Gamma_\mu)^\dagger = \eta^{\mu\nu}\Gamma_\nu = \Gamma_0\Gamma_\mu\Gamma_0, \qquad (\Gamma^\mu)^\dagger = (\eta^{\mu\nu}\Gamma_\nu)^\dagger = \Gamma_0\Gamma^\mu\Gamma_0, \\ \Gamma_0^\dagger = -\Gamma_0, \qquad (\Gamma^{\mu\nu})^\dagger = \tfrac{1}{2}[\Gamma^\mu, \Gamma^\nu]^\dagger = -\Gamma_0\Gamma^{\nu\mu}\Gamma_0. \end{gathered} \tag{C.4}$$

In particular,

$$\Gamma^{\mu\nu} = 2\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{C.5}$$

where

$$\sigma^{\mu\nu} = \tfrac{1}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu), \qquad \bar{\sigma}^{\mu\nu} = \tfrac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu). \tag{C.6}$$

Using the charge conjugation matrix

$$B = i\Gamma_5\Gamma_2 = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \qquad \text{with} \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.7}$$

one defines for a generic Dirac spinor ϕ^i transforming in the $\mathbf{4}$ of $SU(4)$

$$\phi_i = B(\phi^i)^*, \tag{C.8}$$

which transforms again as Dirac spinor, but now in the complex conjugate representation $\bar{\mathbf{4}}$ of $SU(4)$. For a chiral spinor with $\Gamma_5\phi^i = \pm\phi^i$, one finds $\Gamma_5\phi_i = \mp\phi_i$, i.e. charge conjugation also flips the chirality of chiral spinors. Furthermore, one defines

$$\bar{\phi}_i = (\phi^i)^\dagger \Gamma_0, \qquad \bar{\phi}^i = (\bar{\phi}_i)^* B. \tag{C.9}$$

The fermionic spectrum of $N = 4$ supergravity in $D = 4$ with a gravity multiplet and n vector multiplets consists of Dirac spinors ψ^i_μ, λ^{ai}, χ^i that have the following chirality:

$$\begin{aligned}
\psi^i_\mu &= \begin{pmatrix} (\psi^i_\mu)^A \\ 0 \end{pmatrix} & \qquad \Gamma_5\psi^i_\mu &= \psi^i_\mu, \\
\lambda^{ai} &= \begin{pmatrix} (\lambda^{ai})^A \\ 0 \end{pmatrix} & \qquad \Gamma_5\lambda^{ai} &= \lambda^{ai}, \\
\chi^i &= \begin{pmatrix} 0 \\ (\chi^i)_{\dot{A}} \end{pmatrix} & \qquad \Gamma_5\chi^i &= -\chi^i.
\end{aligned} \tag{C.10}$$

Note that we have not introduced new symbols for Weyl spinors but the latter are recognized in the van der Waerden notation by undotted $(A, \ldots)$ and dotted indices $(\dot{A}, \ldots)$ transforming with respect to the two different $SU(2)$ groups of the Lorentz group. We can now express all the fermionic mass terms in terms of Weyl spinors

$$\begin{aligned}
\bar{\psi}_{\mu i}\Gamma^{\mu\nu}\psi_{\nu j} + \text{h.c.} &= 2(\psi^i_\mu)^* \bar{\sigma}^{\mu\nu}\epsilon\,(\psi^j_\nu)^* - 2(\psi^j_\nu)\,\epsilon\,\sigma^{\nu\mu}(\psi^i_\mu), \\
\bar{\psi}_{\mu i}\Gamma^\mu\chi_j + \text{h.c.} &= -(\psi^i_\mu)^*\sigma^\mu\epsilon\,(\chi^j)^* + (\chi^j)\,\epsilon\,\sigma^\mu(\psi^i_\mu), \\
\bar{\psi}^i_\mu\Gamma^\mu\lambda^a_j + \text{h.c.} &= -(\psi^i_\mu)\,\epsilon\,\bar{\sigma}^\mu\epsilon\,(\lambda^{aj})^* - (\lambda^{aj})\,\epsilon\,\bar{\sigma}^\mu\epsilon\,(\psi^i_\mu)^*, \\
\bar{\lambda}^a_i\lambda^b_j + \text{h.c.} &= (\lambda^{ai})^*\epsilon\,(\lambda^{bj})^* - (\lambda^{bj})\,\epsilon\,(\lambda^{ai}), \\
\bar{\chi}^i\lambda^a_j + \text{h.c.} &= (\chi^i)(\lambda^{aj})^* + (\lambda^{aj})(\chi^i)^*,
\end{aligned} \tag{C.11}$$

where on the right hand side we suppressed all dotted/undotted spinor indices. Note that bilinear terms made from $\bar{\chi}^i\chi^j$ are absent in gauged $N = 4$ supergravity, as no such term exists that is invariant under $U(1) \subset H$ and linear in the embedding tensor components. In our conventions all ϵ-tensors with upper/lower, dotted/undotted indices are numerically identical and given by the one in (C.7).

C.2 Integrability condition of the Killing spinor equations

Here we compute the integrability condition (3.9) of a Killing spinor satisfying (3.8a). In the background chosen in section 3.1 the covariant derivative acting on a right-handed spinor ϵ^i reads

$$D_\mu \epsilon^i = \partial_\mu \epsilon^i + \tfrac{1}{4}\omega_{\mu ab}\Gamma^{ab}\,\epsilon^i\,, \tag{C.12}$$

where $\omega_{\mu ab}$ is the torsion-free (Levi-Civita) spin connection. The indices $a, b, \ldots$ are local Lorentz indices while the ones $\mu, \nu, \ldots$ denote coordinate indices of the spacetime. Moreover, the index $i = 1, \ldots, 4$ labels the supersymmetries of $N = 4$ supersymmetry. Using the γ-matrix identity

$$[\Gamma^{cd}, \Gamma^{ab}] = -2(\eta^{ca}\Gamma^{db} - \eta^{da}\Gamma^{cb} - \eta^{cb}\Gamma^{da} + \eta^{db}\Gamma^{ca})\,, \tag{C.13}$$

one finds

$$\begin{aligned}[][D_\mu, D_\nu]\epsilon^i &= \tfrac{1}{4}\left(\partial_\mu\,\omega_{\nu ab} + \omega_{\mu ac}\,\omega_\nu{}^c{}_b - (\mu \leftrightarrow \nu)\right)\Gamma^{ab}\epsilon^i \\ &= \tfrac{1}{4}R_{\mu\nu ab}\,\Gamma^{ab}\epsilon^i\,,\end{aligned} \tag{C.14}$$

where $R_{\mu\nu ab}$ is the Riemann curvature tensor of the spacetime. On the other hand, assuming ϵ^i was a Killing spinor satisfying (3.8a) one finds

$$[D_\mu, D_\nu]\epsilon^i = \tfrac{2}{9}A_1^{ij}(A_1^{jk})^*\Gamma_{\nu\mu}\epsilon^k\,, \tag{C.15}$$

where we made use of

$$\Gamma_{[\nu}B\Gamma_{\mu]}B = \Gamma_{\nu\mu}\,. \tag{C.16}$$

As a result, due to the interchange symmetry $R_{cdab} = R_{abcd}$ one finds

$$\tfrac{1}{16}R_{cdab}\,\{\Gamma^{cd}, \Gamma^{ab}\}\epsilon^i = -\tfrac{1}{9}A_1^{ij}(A_1^{jk})^*\,\Gamma^{\mu\nu}\Gamma_{\mu\nu}\epsilon^k \tag{C.17}$$

which can be simplified using

$$\{\Gamma^{cd}, \Gamma^{ab}\} = 2\,\Gamma^{cdab} - 2(\eta^{ca}\eta^{db} - \eta^{da}\eta^{cb}), \qquad \Gamma^{\mu\nu}\Gamma_{\mu\nu} = -12\,. \tag{C.18}$$

In fact, the contribution from Γ^{cdab} cancels in the contraction with the Riemann tensor by means of the (first) Bianchi identity $R_{c[dab]} = 0$. Finally, the result is

$$\left(R\,\delta^i_k + \tfrac{16}{3}A_1^{ij}(A_1^{jk})^*\right)\epsilon^k = 0\,, \tag{C.19}$$

with Ricci tensor $R_{ab} = R^c{}_{acb}$ and Ricci scalar $R = R_{ab}\,\eta^{ab}$.

Appendix D

Partial solution of the quadratic constraints

D.1 Discussing constraint equations for $g_a \neq 0$

The quadratic constraints for electric gaugings in the case of $g_a \neq 0$ are hard to solve. In fact, so far we have not found any example of a consistent solution with $g_a \neq 0$. Here we will discuss the following two aspects: First, we will show that an electrically gauged $N = 4$ theory with $N = 2$ vacuum requires $f_a \neq 0$; secondly, we will give some details on a lengthy but elementary calculation that shows that $g_a \neq 0$ solutions, if at all, exist only in $n > 6$. These two aspects illustrate that $g_a \neq 0$ consistent solutions would have to be rather sophisticated. As in section 3.2 we label the quadratic constraints given in (2.61) by the quadruple (M, N, P, Q) of $SO(6, n)$-indices.

D.1.1 $N = 2$ vacua require $f_a \neq 0$

We will prove this claim by contradiction; we therefore assume $f_a = 0$. The constraint equations to be used in this proof are

$$(2,3,5,6) \qquad \vec{e}^2 + \vec{g}^2 = c^2 \neq 0\,, \tag{D.1}$$

$$(b,2,4,5) \qquad F_4\,\vec{e} = 2c\,\vec{g}\,, \tag{D.2}$$

$$(b,2,4,6) \qquad F_4\,\vec{g} = -2c\,\vec{e}\,, \tag{D.3}$$

$$(b,2,3,5) \qquad F_2\,\vec{g} = F_3\,\vec{e}\,, \tag{D.4}$$

$$(b,2,3,6) \qquad F_3\,\vec{g} = -F_2\,\vec{e}\,, \tag{D.5}$$

$$(b,c,2,3) \qquad ([G_2, G_3])_{bc} = c(F_4)_{bc} + 2(e_c\,g_b - e_b\,g_c)\,, \tag{D.6}$$

where for better legibility we use a matrix notation with $SO(n)$ vectors $\vec{e}, \vec{g}$ and matrices $(F_m)_{ab} = f_{mab}$. It is obvious from (D.1), (D.2), (D.3) that both $\vec{e}$ and $\vec{g}$ must be nonzero because an $N = 2$ vacuum requires $c \neq 0$. Thus, without loss of generality, using first an $SO(n)$ transformation and subsequently a transformation of the residual $SO(n-1)$ symmetry[1], one can write

$$\vec{e} = \begin{pmatrix} e \\ 0 \\ \vec{0} \end{pmatrix}, \qquad \vec{g} = \begin{pmatrix} g' \\ g \\ \vec{0} \end{pmatrix}, \tag{D.7}$$

with $e \neq 0, g, g' \in \mathbb{R}$. Then equations (D.2), (D.3) show that $g' = 0, g = \sigma e$ with $\sigma = \pm 1$ and

$$F_4 = \begin{pmatrix} \begin{matrix} 0 & -2c\sigma \\ 2c\sigma & 0 \end{matrix} & 0 \\ 0 & \tilde{F}_4 \end{pmatrix}, \tag{D.8}$$

where $\tilde{F}_4 \in \mathrm{Mat}_{n-2,n-2}$. Furthermore, (D.4) and (D.5) imply

$$F_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \vec{v} & \vec{w} & \tilde{F}_2 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \sigma\vec{w} & -\sigma\vec{v} & \tilde{F}_3 \end{pmatrix}, \tag{D.9}$$

with $\vec{v}, \vec{w} \in \mathrm{Mat}_{n-2,1}$ and antisymmetric matrices $\tilde{F}_2, \tilde{F}_3 \in \mathrm{Mat}_{n-2,n-2}$. As a consequence, (D.1) and (D.6) yield

$$\sigma[F_2, F_3]_{78} = -3c^2 = \vec{v}^2 + \vec{w}^2 \geq 0\,, \tag{D.10}$$

which contradicts $c \neq 0$. Hence, $\vec{f}$ cannot vanish in consistent solutions with $N = 2$ vacuum. This ends the proof.

[1] We assume that n is large enough.

D.1.2 $g_a \neq 0$ solutions do not exist in $n \leq 6$

First we will concentrate on the subset of non-trivial quadratic constraints in (2.61) that can easily be solved:

$$
\begin{aligned}
&(2,3,5,6) && \vec{e}^2 + \vec{f}^2 + \vec{g}^2 = c^2 \neq 0\,, && \text{(D.11)}\\
&(b,1,2,3) && F_1\,\vec{f} = 0\,, && \text{(D.12)}\\
&(b,1,2,5) && F_1\,\vec{e} = 0\,, && \text{(D.13)}\\
&(b,1,2,6) && F_1\,\vec{g} = 0\,, && \text{(D.14)}\\
&(b,2,3,4) && F_4\,\vec{f} = 0\,, && \text{(D.15)}\\
&(b,2,4,5) && F_4\,\vec{e} = 2c\,\vec{g}\,, && \text{(D.16)}\\
&(b,2,4,6) && F_4\,\vec{g} = -2c\,\vec{e}\,, && \text{(D.17)}\\
&(b,2,3,5) && F_3\,\vec{e} - F_5\,\vec{f} - F_2\,\vec{g} = 0\,, && \text{(D.18)}\\
&(b,2,3,6) && F_2\,\vec{e} - F_6\,\vec{f} + F_3\,\vec{g} = 0\,, && \text{(D.19)}\\
&(b,2,5,6) && F_6\,\vec{e} + F_2\,\vec{f} - F_5\,\vec{g} = 0\,, && \text{(D.20)}\\
&(b,3,5,6) && F_5\,\vec{e} + F_3\,\vec{f} + F_6\,\vec{g} = 0\,. && \text{(D.21)}
\end{aligned}
$$

Here we use the same matrix notation as in section D.1.1. Having shown that $\vec{f} = 0$ is impossible, without loss of generality we write it as

$$\vec{f} = \begin{pmatrix} f \\ \vec{0} \end{pmatrix}\,, \tag{D.22}$$

with $f \neq 0$ and due to (D.12) and (D.15) find

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \qquad F_4 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}\,, \tag{D.23}$$

with certain matrices $* \in \mathrm{Mat}_{n-1,n-1}$. Unlike in section 3.2 here we consider the case where $\vec{g} \neq 0$. Analogously to the discussion in section D.1.1, one can, without loss of generality and using (D.16) and (D.17), write

$$\vec{g} = \begin{pmatrix} 0 \\ \sigma e \\ 0 \\ \vec{0} \end{pmatrix}, \qquad \vec{e} = \begin{pmatrix} 0 \\ 0 \\ e \\ \vec{0} \end{pmatrix}\,, \tag{D.24}$$

with $e \neq 0$ and $\sigma = \pm 1$ to find

$$F_1 = \mathbb{0}_{3,3} \oplus \tilde{F}_1, \qquad F_4 = \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 2\sigma c & 0 \\ 0 & -2\sigma c & 0 & \\ & 0 & & \tilde{F}_4 \end{pmatrix}\,, \tag{D.25}$$

with matrices $\tilde{F}_1, \tilde{F}_4 \in \mathrm{Mat}_{n-3,n-3}$. Furthermore, equations (D.18) to (D.21) are solved by

$$F_2 = \begin{pmatrix} & & \mathbb{0}_{3,3} & * \\ \vec{a} & \vec{b} & -\sigma\vec{d} + f/e\vec{c}' & \tilde{F}_2 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} & & \mathbb{0}_{3,3} & * \\ \vec{c} & \vec{d} & \sigma\vec{b} + f/e\vec{a}' & \tilde{F}_3 \end{pmatrix},$$
$$F_5 = \begin{pmatrix} & & \mathbb{0}_{3,3} & * \\ \vec{a}' & \vec{b}' & -\sigma\vec{d}' - f/e\vec{c} & \tilde{F}_5 \end{pmatrix}, \qquad F_6 = \begin{pmatrix} & & \mathbb{0}_{3,3} & * \\ \vec{c}' & \vec{d}' & \sigma\vec{b}' - f/e\vec{a} & \tilde{F}_6 \end{pmatrix}, \qquad \text{(D.26)}$$

with $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{a}', \vec{b}', \vec{c}', \vec{d}' \in \mathrm{Mat}_{1,n-3}$ and antisymmetric $\tilde{F}_2, \tilde{F}_3, \tilde{F}_5, \tilde{F}_6 \in \mathrm{Mat}_{n-3,n-3}$.

There remains a large number of non-trivial quadratic constraints which we do not know how to fully solve. Here, we list only those that are useful in our argument:

$$(b, c, 1, m) \qquad [F_1, F_m] = 0\,, \qquad \text{(D.27)}$$
$$(b, c, 2, 4) \qquad [F_2, F_4] = -c\,F_3\,, \qquad \text{(D.28)}$$
$$(b, c, 3, 4) \qquad [F_3, F_4] = c\,F_2\,, \qquad \text{(D.29)}$$
$$(b, c, 4, 5) \qquad [F_5, F_4] = -c\,F_6\,, \qquad \text{(D.30)}$$
$$(b, c, 4, 6) \qquad [F_6, F_4] = c\,F_5\,, \qquad \text{(D.31)}$$
$$(b, c, 2, 3) \qquad ([F_2, F_3])_{bc} = c(F_4)_{bc} - f(F_7)_{bc} + 2(e_c\,g_b - e_b\,g_c)\,, \qquad \text{(D.32)}$$
$$(b, c, 5, 6) \qquad ([F_5, F_6])_{bc} = c(F_4)_{bc} - f(F_7)_{bc} + 2(e_c\,g_b - e_b\,g_c)\,, \qquad \text{(D.33)}$$
$$(b, c, 2, 6) \qquad ([F_2, F_6])_{bc} = -g(F_8)_{bc} + 2(e_b\,f_c - e_c\,f_b)\,, \qquad \text{(D.34)}$$
$$(b, c, 3, 5) \qquad ([F_3, F_5])_{bc} = -g(F_8)_{bc} + 2(e_b\,f_c - e_c\,f_b)\,, \qquad \text{(D.35)}$$
$$(b, c, 2, 5) \qquad ([F_2, F_5])_{bc} = -e(F_9)_{bc} - 2(f_c\,g_b - f_b\,g_c)\,, \qquad \text{(D.36)}$$
$$(b, c, 3, 6) \qquad ([F_3, F_6])_{bc} = e(F_9)_{bc} + 2(f_c\,g_b - f_b\,g_c)\,. \qquad \text{(D.37)}$$

Here, $(F_7)_{ab} = f_{7ab}$, etc. for the first three $SO(n)$ indices denoted by $7, 8, 9$. Using (D.25) and (D.26), equations (D.28) to (D.31) are equivalent to:

$$\tilde{F}_4\,\vec{a} = c\,\vec{c}\,, \tag{D.38}$$
$$\tilde{F}_4\,\vec{b} = 3c\,\vec{d} - 2c\sigma\tfrac{f}{e}\,\vec{c'}\,, \tag{D.39}$$
$$\tilde{F}_4\,\vec{c} = -c\,\vec{a}\,, \tag{D.40}$$
$$\tilde{F}_4\,\vec{d} = -3c\,\vec{b} - 2c\sigma\tfrac{f}{e}\,\vec{a'}\,, \tag{D.41}$$
$$\tilde{F}_4\,\vec{a'} = c\,\vec{c'}\,, \tag{D.42}$$
$$\tilde{F}_4\,\vec{b'} = 3c\,\vec{d'} + 2c\sigma\tfrac{f}{e}\,\vec{c}\,, \tag{D.43}$$
$$\tilde{F}_4\,\vec{c'} = -c\,\vec{a'}\,, \tag{D.44}$$
$$\tilde{F}_4\,\vec{d'} = -3c\,\vec{b'} + 2c\sigma\tfrac{f}{e}\,\vec{a}\,, \tag{D.45}$$
$$[\tilde{F}_2, \tilde{F}_4] = -c\,\tilde{F}_3\,, \tag{D.46}$$
$$[\tilde{F}_3, \tilde{F}_4] = c\,\tilde{F}_2\,, \tag{D.47}$$
$$[\tilde{F}_5, \tilde{F}_4] = -c\,\tilde{F}_6\,, \tag{D.48}$$
$$[\tilde{F}_6, \tilde{F}_4] = c\,\tilde{F}_5\,. \tag{D.49}$$

While tedious, it is possible to find the general solution to equations (D.38) to (D.45). Rather than discussing this in detail we will content ourselves with showing that consistent solutions require as a necessary condition that $\vec{a}, \vec{b}$, etc. be at least nonzero column vectors of dimension 4. This then immediately allows us to prove the claim of this section (obviously due to (D.26)). To this end, we solve equations (D.32) to (D.37) for F_7, F_8, F_9, respectively, and invoke the antisymmetry of f_{abc}. This gives rise to another set of quadratic constraints. The ones of interest for this argument are

$$\vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{c}\,, \tag{D.50}$$

$$\vec{a}' \cdot \vec{d}' = \vec{b}' \cdot \vec{c}'\,, \tag{D.51}$$

$$\vec{a} \cdot \vec{d}' = \vec{b} \cdot \vec{c}'\,, \tag{D.52}$$

$$\vec{c} \cdot \vec{b}' = \vec{d} \cdot \vec{a}'\,, \tag{D.53}$$

$$\sigma(\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d}) = \tfrac{f}{e}(\vec{c} \cdot \vec{c}' - \vec{a} \cdot \vec{a}')\,, \tag{D.54}$$

$$\sigma(\vec{a}' \cdot \vec{b}' + \vec{c}' \cdot \vec{d}') = -\tfrac{f}{e}(\vec{c} \cdot \vec{c}' - \vec{a} \cdot \vec{a}')\,, \tag{D.55}$$

$$\sigma(\vec{b} \cdot \vec{b}' + \vec{d} \cdot \vec{d}') = \tfrac{f}{e}(\vec{a} \cdot \vec{b} + \vec{c}' \cdot \vec{d}')\,, \tag{D.56}$$

$$\sigma(\vec{b} \cdot \vec{b}' + \vec{d} \cdot \vec{d}') = -\tfrac{f}{e}(\vec{a}' \cdot \vec{b}' + \vec{c} \cdot \vec{d})\,, \tag{D.57}$$

$$\sigma(\vec{d} \cdot \vec{a}' - \vec{a} \cdot \vec{d}') = \tfrac{f}{e}(\vec{a} \cdot \vec{c} + \vec{a}' \cdot \vec{c}')\,, \tag{D.58}$$

$$\sigma(\vec{b} \cdot \vec{c}' - \vec{c} \cdot \vec{b}') = -\tfrac{f}{e}(\vec{a} \cdot \vec{c} + \vec{a}' \cdot \vec{c}')\,, \tag{D.59}$$

$$\sigma(\vec{d} \cdot \vec{b}' - \vec{b} \cdot \vec{d}') = \tfrac{f}{e}(\vec{b} \cdot \vec{c} + \vec{b}' \cdot \vec{c}')\,, \tag{D.60}$$

$$\sigma(\vec{b} \cdot \vec{d}' - \vec{d} \cdot \vec{b}') = -\tfrac{f}{e}(\vec{a} \cdot \vec{d} + \vec{a}' \cdot \vec{d}')\,, \tag{D.61}$$

$$\vec{a} \cdot \vec{b}' - \vec{b} \cdot \vec{a}' = \vec{d} \cdot \vec{c}' - \vec{c} \cdot \vec{d}'\,, \tag{D.62}$$

$$\sigma(\vec{b}^2 + \vec{d}^2) + \tfrac{f}{e}(\vec{b} \cdot \vec{a}' - \vec{d} \cdot \vec{c}') = \sigma(\vec{b}'^2 + \vec{d}'^2) - \tfrac{f}{e}(\vec{a} \cdot \vec{b}' - \vec{c} \cdot \vec{d}')\,, \tag{D.63}$$

$$\sigma(\vec{a} \cdot \vec{b}' + \vec{d} \cdot \vec{c}') - \tfrac{f}{e}(\vec{a}^2 + \vec{c}'^2) = -\sigma(\vec{b} \cdot \vec{a}' + \vec{c} \cdot \vec{d}') - \tfrac{f}{e}(\vec{a}'^2 + \vec{c}^2)\,, \tag{D.64}$$

$$\sigma e(6e^2 + \vec{b}^2 + \vec{d}^2) = f\left(-\vec{a} \cdot \vec{b}' - \vec{b} \cdot \vec{a}' + \tfrac{f}{e}\sigma(\vec{a}^2 + \vec{c}'^2)\right)\,, \tag{D.65}$$

$$\sigma e(6e^2 + \vec{b}^2 + \vec{d}^2) = f(\vec{a} \cdot \vec{b}' + \vec{d} \cdot \vec{c}' - 2\vec{b} \cdot \vec{a}')\,, \tag{D.66}$$

where for the last two equations we also used (D.11) and (D.24). Those two equations imply that not all $\vec{a}, \vec{b}, \ldots$ can vanish because by assumption $e \neq 0$. Furthermore, one finds that solutions satisfying (D.38) to (D.45) subject to the additional constraints (D.50) to (D.66) necessarily require nonzero column vectors $\vec{a}, \vec{b}$, etc. of dimension at least 4. Since $\vec{a}, \vec{b}, \ldots \in \mathrm{Mat}_{1,n-3}$ we conclude that $g_a \neq 0$ solutions do not exist in $n \leq 6$.

D.2 Discussing constraint equations for $g_a = 0$

Here we list the quadratic constraint equations that are not trivially satisfied, c.f. section 3.2. In what follows the quadruple (M, N, P, Q) in the first column refers to the free indices in (2.61):

$$(\tilde{b}, \tilde{c}, 1, 2) \qquad f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}2} = 0\,, \tag{D.67a}$$

$$(\tilde{b}, \tilde{c}, 1, 3) \qquad f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}3} = 0\,, \tag{D.67b}$$

$$(\tilde{b}, \tilde{c}, 1, 4) \qquad f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}4} = 0\,, \tag{D.67c}$$

$$(\tilde{b},\tilde{c},1,5) \qquad f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}5} = 0\,, \tag{D.67d}$$
$$(\tilde{b},\tilde{c},1,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}1} - f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}6} = 0\,, \tag{D.67e}$$
$$(\tilde{b},\tilde{c},2,5) \qquad f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}2} - f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}5} = 0\,, \tag{D.67f}$$
$$(\tilde{b},\tilde{c},2,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}2} - f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}6} = 0\,, \tag{D.67g}$$
$$(\tilde{b},\tilde{c},3,5) \qquad f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}3} - f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}5} = 0\,, \tag{D.67h}$$
$$(\tilde{b},\tilde{c},3,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}3} - f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}6} = 0\,, \tag{D.67i}$$
$$(\tilde{b},\tilde{c},3,4) \qquad f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}3} - f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}4} = c\, f_{\tilde{b}\tilde{c}2}\,, \tag{D.67j}$$
$$(\tilde{b},\tilde{c},2,4) \qquad f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}2} - f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}4} = -c\, f_{\tilde{b}\tilde{c}3}\,, \tag{D.67k}$$
$$(\tilde{b},\tilde{c},4,5) \qquad f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}4} - f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}5} = c\, f_{\tilde{b}\tilde{c}6}\,, \tag{D.67l}$$
$$(\tilde{b},\tilde{c},4,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}4} - f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}6} = -c\, f_{\tilde{b}\tilde{c}5}\,, \tag{D.67m}$$
$$(\tilde{b},\tilde{c},2,3) \qquad f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}2} - f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}3} = c\,(f_{\tilde{b}\tilde{c}4} - f_{7\tilde{b}\tilde{c}})\,, \tag{D.67n}$$
$$(\tilde{b},\tilde{c},5,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}5} - f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}6} = c\,(f_{\tilde{b}\tilde{c}4} - f_{7\tilde{b}\tilde{c}})\,, \tag{D.67o}$$
$$(\tilde{b},\tilde{c},7,1) \qquad f_{\tilde{a}\tilde{b}1}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}1} = 0\,, \tag{D.67p}$$
$$(\tilde{b},\tilde{c},7,4) \qquad f_{\tilde{a}\tilde{b}4}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}4} = 0\,, \tag{D.67q}$$
$$(\tilde{b},\tilde{c},7,2) \qquad f_{\tilde{a}\tilde{b}2}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}2} = c\, f_{\tilde{b}\tilde{c}3}\,, \tag{D.67r}$$
$$(\tilde{b},\tilde{c},7,3) \qquad f_{\tilde{a}\tilde{b}3}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}3} = -c\, f_{\tilde{b}\tilde{c}2}\,, \tag{D.67s}$$
$$(\tilde{b},\tilde{c},7,5) \qquad f_{\tilde{a}\tilde{b}5}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}5} = c\, f_{\tilde{b}\tilde{c}6}\,, \tag{D.67t}$$
$$(\tilde{b},\tilde{c},7,6) \qquad f_{\tilde{a}\tilde{b}6}\, f_{\tilde{a}\tilde{c}7} - f_{\tilde{a}\tilde{b}7}\, f_{\tilde{a}\tilde{c}6} = -c\, f_{\tilde{b}\tilde{c}5}\,, \tag{D.67u}$$

$$(\tilde{b},\tilde{c},\tilde{d},m) \qquad 0 = f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{a}\tilde{d}m} + f_{\tilde{a}\tilde{b}m}\, f_{\tilde{a}\tilde{c}\tilde{d}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}m}\,, \tag{D.68a}$$
$$(\tilde{b},\tilde{c},\tilde{d},7) \qquad 0 = f_{\tilde{a}\tilde{b}\tilde{c}}\, f_{\tilde{d}7\tilde{a}} + f_{\tilde{a}\tilde{b}7}\, f_{\tilde{c}\tilde{d}\tilde{a}} - f_{\tilde{a}\tilde{b}\tilde{d}}\, f_{\tilde{a}\tilde{c}7}\,, \tag{D.68b}$$
$$(\tilde{b},\tilde{c},\tilde{d},\tilde{e}) \qquad f_{a\tilde{b}\tilde{c}}\, f_{\tilde{d}\tilde{e}a} + f_{a\tilde{b}\tilde{e}}\, f_{\tilde{c}\tilde{d}a} - f_{a\tilde{b}\tilde{d}}\, f_{a\tilde{c}\tilde{e}} = f_{r\tilde{b}\tilde{c}}\, f_{\tilde{d}\tilde{e}r} + f_{r\tilde{b}\tilde{e}}\, f_{\tilde{c}\tilde{d}r} - f_{r\tilde{b}\tilde{d}}\, f_{r\tilde{c}\tilde{e}}\,. \tag{D.68c}$$

D.2.1 The most general solution to equations (3.51)

Here we will prove the claim that the most general solution of equations (3.51) is given by (3.53) and an arbitrary, antisymmetric H_+ that commutes with G_1. In fact, it suffices to consider the Lie subalgebra $\mathfrak{s}' \subset \mathfrak{g}$ spanned by $\{G_2, G_3, H_+, H_-\}$ which is also solvable. Its non-vanishing Lie brackets are

$$\begin{aligned} &[G_2, H_+] = -2c\, G_3, \qquad && [G_2, G_3] = c\, H_-, \\ &[G_3, H_+] = +2c\, G_2\,. && \end{aligned} \tag{D.69}$$

We shall prove the following theorem:
Theorem: The most general solution to system (D.69) consists of solutions with

$$G_2 = G_3 = H_- = 0, \qquad H_+ = -H_+^T \text{ arbitrary.} \tag{D.70}$$

Our proof requires two elementary lemmata about matrices and a corollary of Lie's theorem concerning finite-dimensional representations of complex, solvable Lie algebras.

Lemma: An antisymmetric matrix $A \in \mathrm{Mat}(\mathbb{R}, m \times m)$ is nilpotent if and only if $A = 0$.
Proof: Being antisymmetric A can be brought to diagonal form $PAP^{-1} = diag(\lambda_1, \ldots, \lambda_m)$ with a $P \in GL(\mathbb{C}, m \times m)$ and $\lambda_i \in i\mathbb{R}$. As $PA^nP^{-1} = (PAP^{-1})^n$ for all $n \in \mathbb{N}$, nilpotency is basis-independent. It is then obvious that,

$$(PAP^{-1})^n = diag(\lambda_1^n, \ldots, \lambda_m^n) \; ,$$

is nilpotent iff $\lambda_i = 0 \, \forall i$ which implies $A = 0$. The converse is trivial.

Lemma: Given matrices $A_1, \ldots, A_k \in \mathrm{Mat}(\mathbb{C}, m \times m)$ for $k \in \mathbb{N}$. For simultaneously triangularizable matrices $A_1, \ldots, A_k$ the commutator $[A_i, A_j]$ is nilpotent for all $i, j = 1, \ldots, k$.
Proof: The commutator of two upper triangular matrices is strictly upper triangular and, hence, nilpotent.

Corollary of Lie's theorem[2]: Let $\mathfrak{g}$ be a complex, solvable Lie algebra and (V, ρ) a finite-dimensional representation of $\mathfrak{g}$. Then there exists a basis of V such that all elements of $\mathfrak{g}$ are represented as upper triangular matrices.
Proof: Lecture script by W. Soergel [89].

In order to be able to apply this corollary we need to complexify our real Lie algebra (D.69).

Lemma: Given a real Lie algebra $\mathfrak{g}$ and a finite-dimensional real representation (V, ρ) of $\mathfrak{g}$. Then one finds a finite-dimensional representation $(V_\mathbb{C}, \rho_\mathbb{C})$ of the complexified Lie algebra $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ (with $\mathbb{C}$-linear extension of the Lie bracket) defined by $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ and

$$\rho_\mathbb{C}(X + iY) := \rho(X) + i\rho(Y),$$

for all $X, Y \in \mathfrak{g}$.
Proof: $\mathbb{C}$-linearity of $\rho_\mathbb{C}$ is obvious and so is the proof of

$$\rho_\mathbb{C}([X + iY, U + iV]) = [\rho_\mathbb{C}(X + iY), \rho_\mathbb{C}(U + iV)]$$

for all $X, Y, U, V \in \mathfrak{g}$. As a result, $(V_\mathbb{C}, \rho_\mathbb{C})$ is a finite-dimensional representation of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$.

[2] Given a complex, solvable Lie algebra, then all its finite-dimensional irreducible representations are one-dimensional.

Now we can prove the theorem:

Proof of the theorem: Assume that there exists a solution of (D.69) *with an antisymmetric* $G_2 \neq 0 \in Mat(\mathbb{R}, m \times m)$*. Any such solution would be a finite-dimensional real representation* $(\mathbb{R}^m, \rho)$ *of our real solvable Lie algebra* $\mathfrak{s}'$*. In this proof such a solution will be denoted by* $\rho(G_2), \rho(G_3), \rho(H_-), \rho(H_+)$ *with* $\rho(G_2) \neq 0$ *by assumption, while* $G_2, G_3, H_-, H_+ \in \mathfrak{s}'$ *shall refer to the abstract elements of the Lie algebra. We denote the induced representation of the complexified Lie algebra* $\mathfrak{s}'_{\mathbb{C}}$ *as* $(\mathbb{C}^m, \rho_{\mathbb{C}})$*. Since also* $\mathfrak{s}'_{\mathbb{C}}$ *is solvable, we apply the corollary and find that* $\rho_{\mathbb{C}}(G_2), \rho_{\mathbb{C}}(G_3), \rho_{\mathbb{C}}(H_-), \rho_{\mathbb{C}}(H_+) \in Mat(\mathbb{C}, m \times m)$ *are simultaneously triangularizable. Then, according to the second lemma we find that*

$$[\rho_{\mathbb{C}}(G_3), \rho_{\mathbb{C}}(H_+)] = 2c\, \rho_{\mathbb{C}}(G_2)$$

is nilpotent. As $c \neq 0$ *one finds* $\rho_{\mathbb{C}}(G_2) = \rho(G_2)$ *is nilpotent. However, being antisymmetric* $\rho(G_2)$ *must be zero by the first lemma which is in contradiction with* $\rho(G_2) \neq 0$*. We therefore conclude that* $\rho(G_2) = 0$ *which, by means of the Lie algebra* (D.69)*, immediately implies* $\rho(G_3) = \rho(H_-) = 0$*. As a result, solutions* (D.70) *are already the most general solutions to* (D.69)*. This ends the proof.*

D.2.2 Solving $[G_1, G_4] = 0$

We will now solve (3.54a), which in matrix notation reads

$$[G_1, G_4] = 0\,. \tag{D.71}$$

It is by means of an $O(n-1)$ transformation that, without loss of generality, any G_1 can be written in block-diagonal form as

$$G_1 = (D \otimes \varepsilon) \oplus 0 = \begin{pmatrix} D \otimes \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \tag{D.72}$$

where $D = diag(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots)$ is a diagonal matrix with ordered positive eigenvalues $x_1 > x_2 > \ldots > 0$ and ε is the antisymmetric 2×2 matrix with $\varepsilon_{12} = 1$; the zeros in (D.72) denote zero matrices of appropriate dimensions. Note that, in general, this gauge can only be obtained by also using reflections (in addition to rotations). While strictly speaking we are only allowed to use $SO(n-1) \subset G$ rotations, the quadratic constraints (3.54a) - (3.54d) are also $O(n-1)$ tensor equations. We may therefore also use reflections to arrive, as an intermediate step, at the gauge (D.72) — which simplifies the subsequent analysis — as long as, in the end, we return to only using rotations, in that we apply another reflection that flips two directions but preserves the block structure (e.g. $x_i \to -x_i$ for one 2×2 block). Since $D \otimes \varepsilon$ is invertible, (D.71) implies (also using another gauge choice for the lower right block)

$$G_4 = \begin{pmatrix} A & 0 \\ 0 & (D' \otimes \varepsilon) \oplus 0 \end{pmatrix}, \tag{D.73}$$

where A is an antisymmetric matrix (of the same matrix dimensions as $D\otimes\varepsilon$) satisfying

$$[D\otimes\varepsilon,\, A] = 0 \tag{D.74}$$

and D' is another invertible diagonal matrix. In order to solve (D.74) we note that any even-dimensional antisymmetric A can be written as

$$A = S\otimes\varepsilon + A_1\otimes\mathbb{1} + A_2\otimes\sigma_1 + A_3\otimes\sigma_3\,, \tag{D.75}$$

where S is symmetric, A_1, A_2, A_3 are antisymmetric, and σ_1, σ_3 are the usual Pauli matrices. Now (D.74) implies[3]

$$[D, S] = 0, \qquad [D, A_1] = 0, \qquad \{D, A_2\} = 0, \qquad \{D, A_3\} = 0\,, \tag{D.76}$$

which in the reflection gauge (D.72) implies $A_2 = A_3 = 0$ and $S_{ij} = (A_1)_{ij} = 0$ for all i, j with $x_i \neq x_j$. As a result, we obtain

$$A = S\otimes\varepsilon + A_1\otimes\mathbb{1}\,, \tag{D.77}$$

where now S and A_1 are block-diagonal with blocks associated to degenerate x_i in D. We will now refine the block-structure in G_4. To this end, we will use the residual symmetry of the blocks in G_1 and G_4 to bring each G_4 block associated to some x_i to the form

$$(\text{ith block in } G_4) = (diag(y_{i1}, \ldots, y_{i1}, y_{i2}, \ldots, y_{i2}, \ldots)\otimes\varepsilon)\oplus 0\,, \tag{D.78}$$

with $y_{i1} > y_{i2} > \ldots > 0$. While this, of course, temporarily spoils the gauge (D.72), it is by means of (D.71) that we find, using the same argument as before, that the ith block in G_1 has a subblock structure with blocks associated to degenerate y_{ij} or zero in the ith G_4 block. Now we apply symmetries that respect these subblocks to bring G_1 back to our gauge (D.72) and at the same time maintain the subblock structure in G_4. Then, repeating the argument that lead to (D.77), we know that the subblock associated to x_i in G_1 and y_{ij} in G_4 is given by

$$((i,j) \text{ block in } G_4) = S^{(ij)}\otimes\varepsilon + A_1^{(ij)}\otimes\mathbb{1}\,, \tag{D.79}$$

where

$$\left(S^{(ij)}\otimes\varepsilon + A_1^{(ij)}\otimes\mathbb{1}\right)^2 = -(y_{ij})^2\,\mathbb{1}\otimes\mathbb{1}. \tag{D.80}$$

The (i,j) block in G_1 is $x_i\,\mathbb{1}\otimes\varepsilon$ and is thus invariant under orthogonal transformations that only act on the first tensor product factor. Such transformations can be used to bring $S^{(ij)}$ to diagonal form

$$D^{(ij)} = diag(d_{ij1}, \ldots, d_{ij1}, -d_{ij1}, \ldots, -d_{ij1}, \ldots)\oplus 0\,, \tag{D.81}$$

[3] $\{.,.\}$ denotes the anticommutator.

where $d_{ijk} > 0$ and the dimensions of positive and negative eigenvalues can in general be different. In doing so, (D.80) gives rise to the following system of equations

$$(A_1^{(ij)})^2 + (y_{ij})^2 = (D^{(ij)})^2, \qquad \{D^{(ij)}, A_1^{(ij)}\} = 0. \tag{D.82}$$

The second equation gives

$$(A_1^{(ij)})_{kl} = 0 \qquad \vee \qquad (D^{(ij)})_{kk} = -(D^{(ij)})_{ll} \tag{D.83}$$

and, hence, the $D^{(ij)}$ and $A_1^{(ij)}$ have the following block-diagonal form

$$D^{(ij)} = \begin{pmatrix} d_{ij1}\mathbb{1} & 0 & \cdots & 0 \\ 0 & -d_{ij1}\mathbb{1}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad A_1^{(ij)} = \begin{pmatrix} 0 & F^{(ij1)} & \cdots & 0 \\ -F^{(ij1)^T} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F^{(ij0)} \end{pmatrix}, \tag{D.84}$$

where $F^{(ijk)}$ are rectangular matrices and $F^{(ij0)}$ are antisymmetric square matrices subject to the following conditions (from (D.82)):

$$d_{ijk}^2 + \begin{pmatrix} F^{(ijk)}F^{(ijk)^T} & 0 \\ 0 & F^{(ijk)^T}F^{(ijk)} \end{pmatrix} = y_{ij}^2, \qquad (F^{(ij0)})^2 = -y_{ij}^2. \tag{D.85}$$

Without loss of generality we can use the residual symmetry to bring each $F^{(ij0)}$ into diagonal form

$$D^{(ij0)} \otimes \varepsilon\,, \tag{D.86}$$

where the eigenvalues of $D^{(ij0)}$ must be $\pm y_{ij}$ in order to satisfy (D.85). In particular, $F^{(ij0)}$ must have even dimension. As to the $F^{(ijk)}$, (D.85) implies that

$$F^{(ijk)}F^{(ijk)^T} = \xi_{ijk}\,\mathbb{1}\,, \tag{D.87a}$$

$$F^{(ijk)^T}F^{(ijk)} = \xi_{ijk}\mathbb{1}' \tag{D.87b}$$

for some non-negative number ξ_{ijk}. In the case where $\xi_{ijk} = 0$ one finds $F^{(ijk)} = 0$, and (D.85) implies $d_{ijk} = y_{ij}$. On the other hand, for $\xi_{ijk} > 0$, (D.87a), (D.87b), respectively, shows that the rows/columns of $1/\sqrt{\xi_{ijk}}F^{(ijk)}$ are orthonormal which, however, is only possible if $F^{(ijk)}$ is a square matrix. In this case, $1/\sqrt{\xi_{ijk}}F^{(ijk)}$ is an orthogonal matrix that without loss of generality can be orthogonally transformed to the unit element: In fact, the (i,j,k) block in $D^{(ij)}$

$$\begin{pmatrix} d_{ijk}\mathbb{1} & 0 \\ 0 & -d_{ijk}\mathbb{1} \end{pmatrix} \tag{D.88}$$

is invariant under an orthogonal transformation

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \tag{D.89}$$

that at the same time acts on the (i,j,k) block in $A^{(ij)}$ as

$$\begin{pmatrix} 0 & F^{(ijk)} \\ -F^{(ijk)T} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} T^T & 0 \\ 0 & S^T \end{pmatrix} \begin{pmatrix} 0 & F^{(ijk)} \\ -F^{(ijk)T} & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$$
$$= \begin{pmatrix} 0 & T^T F^{(ijk)} S \\ -(T^T F^{(ijk)} S)^T & 0 \end{pmatrix} . \tag{D.90}$$

Choosing $S = \mathbb{1}, T = 1/\sqrt{\xi_{ijk}} F^{(ijk)}$ one obtains

$$F^{(ijk)} = \sqrt{\xi_{ijk}} \mathbb{1} \, . \tag{D.91}$$

The condition (D.85) finally reads

$$d_{ijk}^2 + \xi_{ijk} = y_{ij}^2 \tag{D.92}$$

and, hence,

$$d_{ijk} = |y_{ij}| \cos\phi_{ijk}, \qquad \sqrt{\xi_{ijk}} = |y_{ij}| \sin\phi_{ijk} \tag{D.93}$$

for some angle $\phi_{ijk} \in (0, \pi/2)$. To conclude, we have the following block types,

$$\begin{aligned} G_1^{(ijk)} &= x_i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \otimes \varepsilon, \\ G_4^{(ijk)} &= |y_{ij}| \left(\cos\phi_{ijk} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \otimes \varepsilon + \sin\phi_{ijk} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \otimes \mathbb{1}_2 \right) \end{aligned} \tag{D.94}$$

for $\phi_{ijk} \in (0, \pi/2)$, while blocks with $F^{(ijk)} = 0$ read

$$\begin{aligned} G_1^{(ijk)} &= x_i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}' \end{pmatrix} \otimes \varepsilon, \\ G_4^{(ijk)} &= |y_{ij}| \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}' \end{pmatrix} \otimes \varepsilon. \end{aligned} \tag{D.95}$$

Finally, zero blocks in $D^{(ij)}$ give rise to the following blocks:

$$\begin{aligned} G_1^{(ij0)} &= x_i (\mathbb{1} \otimes \mathbb{1}_2) \otimes \varepsilon, \\ G_4^{(ij0)} &= (D^{(ij0)} \otimes \varepsilon) \otimes \mathbb{1}_2 \, . \end{aligned} \tag{D.96}$$

Using appropriate orthogonal transformations it is possible to write (D.94) as

$$\begin{aligned} G_1^{(ijk)} &= x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon), \\ G_4^{(ijk)} &= |y_{ij}| \, \mathbb{1} \otimes \left(\cos\phi_{ijk} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \varepsilon + \sin\phi_{ijk} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \right), \end{aligned} \tag{D.97}$$

and, similarly, we transform (D.96) to[4]

$$\begin{aligned} G_1^{(ij0)} &= x_i \, \mathbb{1} \otimes (\mathbb{1}_2 \otimes \varepsilon), \\ G_4^{(ij0)} &= D^{(ij0)} \otimes (\varepsilon \otimes \mathbb{1}_2). \end{aligned} \tag{D.98}$$

Note that both (D.97) and (D.98) are block-diagonal matrices with non-trivial 4×4 blocks. From these blocks and using (D.73) we can construct the full solution of (D.71) for the gauge choice outlined above. As mentioned already, in the end one may have to apply another reflection so that this gauge can be obtained from generic matrices G_1 and G_4 only by rotations, rather than reflections.

[4]Note that (D.98) yields (D.97) for $\phi_{ijk} = \pi/2$ provided that $D^{(ij0)}$ has only positive eigenvalues. But the latter need not be the case in general.

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