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The Finsler spacetime framework: backgrounds for physics beyond metric geometry

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Abstract

The fundamental structure on which physics is described is the geometric spacetime background provided by a four dimensional manifold equipped with a Lorentzian metric. Most importantly the spacetime manifold does not only provide the stage for physical field theories but its geometry encodes causality, observers and their measurements and gravity simultaneously. This threefold role of the Lorentzian metric geometry of spacetime is one of the key insides of general relativity.

During this thesis we extend the background geometry for physics from the metric framework of general relativity to our Finsler spacetime framework and ensure that the threefold role of the geometry of spacetime in physics is not changed. The geometry of Finsler spacetimes is determined by a function on the tangent bundle and includes metric geometry. In contrast to the standard formulation of Finsler geometry our Finsler spacetime framework overcomes the differentiability and existence problems of the geometric objects in earlier attempts to use Finsler geometry as an extension of Lorentzian metric geometry. The development of our non-metric geometric framework which encodes causality is one central achievement of this thesis. On the basis of our well-defined Finsler spacetime geometry we are able to derive dynamics for the non-metric Finslerian geometry of spacetime from an action principle, obtained from the Einstein–Hilbert action, for the first time. We can complete the dynamics to a non-metric description of gravity by coupling matter fields, also formulated via an action principle, to the geometry of our Finsler spacetimes. We prove that the combined dynamics of the fields and the geometry are consistent with general relativity. Furthermore we demonstrate how to define observers and their measurements solely through the non-metric spacetime geometry. Physical consequence derived on the basis of our Finsler spacetime are: a possible solution to the fly-by anomaly in the solar system; the possible dependence of the speed of light on the relative motion between the observer and the light ray; modified dispersion relation and possible propagation of particle modes faster than light and the propagation of light on Finsler null-geodesics.

Our Finsler spacetime framework is the first extension of the framework of general relativity based on non-metric Finslerian geometry which provides causality, observers and their measurements and gravity from a Finsler geometric spacetime structure and yields a viable background on which action based physical field theories can be defined.

Zusammenfassung

Eine fundamentale Erkenntniss aus der Einsteinschen allgemeinen Relativitätstheorie ist, dass die Raumzeit nicht nur die Bühne der Physik ist auf der physikalische Felder wechselwirken, sondern, dass die Geometrie der Raumzeit zugleich die kausale Struktur, die Beobachter, deren Messungen sowie die Gravitation beschreibt. Die Raumzeit selbst ist hierbei eine vierdimensionale Mannigfaltigkeit mit Lorentzscher Metrik, welche die Geometrie definiert.

Wir werden in dieser Arbeit einen erweiterten Raumzeitbegriff entwickeln, der statt auf einer Metrik auf einer Tangentialbündelfunktion basiert, die ein Längenmaß für Kurven definiert. Das Besondere an unserem neuen Zugang ist, dass sich die Rolle der Raumzeitgeometrie in der Physik dabei im Vergleich zur allgemeinen Relativitätstheorie nicht ändert. Die Grundlage für dieses Projekt legen wir mit unserer Erweiterung der Finslerschen Geometrie, die problemlos metrische Geometrie mit Lorentzsignatur verallgemeinert und insbesondere eine kausale Struktur vorgibt. Die so konstruierten Finslerraumzeiten ermöglichen uns die Einstein–Hilbert Wirkung der allgemeinen Relativitätstheorie aus einem völlig neuen Blickwinkel zu betrachten und diese umzuschreiben und zu verallgemeinern. Die so erhaltene Wirkung definiert nun die Dynamik der Tangentialbündelfunktion, die die Geometrie der Finslerraumzeiten bestimmt. Um diese Dynamik mit der Beschreibung der Gravitation in Zusammenhang zu bringen, koppeln wir Materiefelder über Wirkungsintegrale, die die zugehörigen Feldtheorien definieren an die nicht-metrische Geometrie. Das Kopplungsprinzip ist so konzipiert, dass die kombinierte Dynamik der Materiefelder und der Geometrie konsistent mit den Einsteinschen Feldgleichungen ist. Die jetzt noch fehlenden Beobachter auf Finslerschen Raumzeiten werden mit Hilfe eines Vierbeins eingeführt, das einzig und allein durch die Tangentialbündelfunktion bestimmt ist, welche die Geometrie definiert.

Direkte Konsequenzen der nicht-metrischen Finslerschen Raumzeitgeometrie, die wir besprechen sind: Eine mögliche Erklärung der fly-by Anomalie im Sonnensystem; eine mögliche Abhängigkeit der Lichtgeschwindigkeit vom Bewegunszustand des Beobachters; modifizierte Dispersionsrelationen und die prinzipielle Möglichkeit, dass es Teilchen gibt, die sich schneller als Licht bewegen und die Tatsache, dass sich Licht auf Finslergeodäten bewegt.

Unsere Finslerraumzeiten sind die erste Verallgemeinerung von Lorentzschen metrischen Mannigfaltigkeiten auf der Basis von Finslergeometrie, die Beobachter, deren Messungen sowie die Gravitation beinhaltet.

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Introduction

The notion of time as a fourth dimension existed before Einstein formulated special and general relativity with the help of Minkowski and Großmann. As Goenner nicely summarises [1] the appearance of a fourth "time" dimension goes back to the 18th century and appeared from there on as idea in physics, philosophy and literature.

Nowadays the geometry of spacetime plays a fundamental role in the modern understanding of physics. It consistently provides the stage on which physics takes place, the causal structure of events, the description of observers and their measurements and a description of one of the fundamental interactions, namely gravity. All these attributes of spacetime as unified entity arise from Einstein's theory of general relativity where the geometry of spacetime is derived from a Lorentzian metric, exactly in order to have all the properties just mentioned. One can say that general relativity simultaneously geometrises causality, observers and gravity.

Spacetime itself provides positions and time labels for all physical events caused by the interaction of physical systems built from physical fields or particles, mathematically modelled by a four dimensional manifold. The causal structure on this set is a relation between the events which determines which event causes another event, i.e., a time ordering. This time order is the same for all possible observers and physical fields who may observe or interact with the events. It is realised by equipping the spacetime manifold with a metric of Lorentzian signature and by modelling observers as timelike curves. Moreover observers are able to measure distances, lengths and physical fields what is realised by an orthonormal frame of the metric that represents the observers time and space directions at the position of the observer. Finally gravitational interaction is described by the curvature of the spacetime defined through the Lorentzian metric, which itself is determined by the physical fields on spactime through the Einstein equations. In this way the causal structure, the description of observers and their measurements and the description of gravity is tied crucially to one single object in general relativity: the Lorentzian metric of the four dimensional spacetime.

The geometry of spacetime in extensions of general relativity

Since general relativity was developed there exist suggestions how to modify and extend the theory, driven either by the dream of one unified geometric picture of the fundamental field theories in physics, beginning with the ideas of Kaluza and Klein [2, 3], or the need to explain observations which cannot be explained within general relativity such as for example the flat rotational curves of galaxies [4] and the accelerated expansion of the universe [5, 6].

But for whatever reason one modifies general relativity, and with it the geometry of spacetime, one has to ensure that in the modified framework the spacetime still provides all the features just discussed. Moreover, one has to discuss carefully which of the properties, causal structure, observers, description of gravity is related to which objects in the modified theory. We like to stress that especially the observer model which is employed is crucial for the viability tests of a theory since it relates observations and theoretical predictions. These issues are usually barely discussed. We will now have a brief look at the role of spacetime in some classes of modifications of general relativity.

A mild modification of general relativity from the viewpoint of the properties of spacetime are so called $f(R)$ theories [7]. In this framework the only thing changed with respect to general relativity are the dynamics which determine the metric of spacetime. It is not always clear whether the new dynamics are well-behaved and determine Lorentzian spacetime geometries which make sense from physical point of view. However there is still only one single object which describes the causal structure, observers and gravity and that is still the Lorentzian metric of spacetime.

The role of spacetime is changed more severely in extensions of general relativity which aim for the unification of fundamental physical field theories like Kaluza-Klein theories, supergravity and string theory [8]. In these theories the dimension of spacetime is increased and the higher dimensional metric spacetime geometry is determined from higher dimensional general relativity. Here the geometry of the higher dimensional spacetime shall describe all physical fields, among them a four dimensional metric describing gravity. Here the role of spacetime geometry goes far further compared to general relativity. It does not only describe gravity but all interactions and matter fields. The causal structure, observers and gravity are still related to one object, the resulting four dimensional Lorentzian metric obtained from the framework. But the spacetime is higher dimensional which has to be connected to the four dimensional spacetime we observe.

Theories adding additional fields to the metric for the description of gravity like a scalar, a vector, a scalar and a vector, or another two tensor, examples of such theories can be found in [9, 10], perform a more drastic modification of the role of spacetime compared to general relativity. In these theories the gravitational field is described not only by a Lorentzian metric, but in addition by other spacetime fields, whereas the geometry of the spacetime is still determined only by the Lorentzian metric as are observers and their measurements. Thus the geometry of spacetime which determines the causal structure is no longer identical to the description of gravity and, the other way around, the fields which describe gravity do not correspond to the field which encodes observers and their measurements. Here a disentanglement or decoupling between the description of gravity and the properties of spacetime took place, while the entanglement or coupling of these properties into the spacetime geometry is a key feature in general relativity.

During the discussion we have just seen that many modifications and extensions of general relativity violate at least one of its key ingredients: the geometrisation of causality, observers and gravity at the same time. If gravity is described by more fields than the metric or if spacetime has more than four dimensions why shouldn't this influence the causal structure or the measurements of observers? In general relativity this question does not appear since there is only one object in the theory, the metric of the four dimensional spacetime.

The question we investigate throughout this thesis is: Is it possible to extend the framework of general relativity such that the simultaneous geometrisation of causality, observers and gravity is maintained, the dimension of spacetime stays four and one may use more fields than just the metric or no metric at all to describe gravity? The answer to this question is positive and we will present an extended spacetime geometry framework throughout this thesis.

Aim and structure of this thesis

In this thesis we will construct and present an extension of the geometry of spacetime and general relativity which keeps the causal structure, observers and their measurements and the description of gravity consistently encoded into the geometry. It will be possible to use more fields than just a metric or no metric and just other tensor fields to describe gravity, but still all the fields which describe gravity determine the geometry of spacetime and vice versa. Moreover also observers and their measurements are based on all of the geometry of spacetime. Furthermore it will be possible to apply our new framework to unify the dynamics of several fields without going to spacetimes of higher dimension. We will develop a spacetime geometry based on Finsler geometry. The one single object which determines the geometry of spacetime will be a function on the tangent bundle instead of a metric on the manifold.

Certainly we are not the first ones who address extensions of the framework of general relativity with Finsler geometry. This is why we need to discuss the previous approaches, their ideas and their shortcomings, and our arguments which lead us to our new ideas how to overcome these shortcomings in the first part of this thesis. It contains in the chapters 1 to 4 the necessary mathematical preliminaries, the standard textbook formulation of Finsler geometry, its application in physics and our preliminary work on Finsler geometry based gravity theories. All this then lays the foundation for our development of our new Finsler spacetime framework in part II.

The mathematical foundation of Finsler geometry is laid in chapter 1 where we discuss fibre bundles and the tangent bundle as special instance of a fibre bundle. We summarise these well known mathematical frameworks and point out how the geometry of the tangent bundle is related to Lorentzian metric geometry used in general relativity. In chapter 2 we discuss Finsler geometry as it can be found in the literature. It is a well known generalisation of Riemannian geometry. It is studied and developed since it was created by Finsler in his thesis 1918 [11]. Instead of on a metric the geometry of a manifold is based on a more general length measure for curves on the manifold. During our review of Finsler geometry we will work out in detail why it is a suitable generalization only for Riemannian geometry and not for geometries based on a metric with indefinite signature. This issue is a major drawback when one wants to apply Finsler geometry in physics and one of our motivations to extend the formulation of Finsler geometry to our Finsler spacetime framework. In chapter 3 we collect applications of Finsler geometry in physics which inflict on an extension of Finsler geometry to a framework which generalises Lorentzian metric geometry. It has been realised that Finsler geometry is the appropriate tool to study the propagation of waves in all sorts of media, or more generally the ray approximation of solutions of partial differential equations. Moreover there are attempts to use Finsler geometry as a phenomenological tool to explain astronomical observations and as the fundamental geometry of spacetime. In our survey through the applications of Finsler geometry in physics we stress the mathematical problems which arise from the use of the standard Finsler geometry framework. Thus the need to put the applications of Finsler geometry in physics on a more solid mathematical footing is one more motivation for the development of the Finsler spacetime framework. First preliminary steps into that direction are summarised in chapter 4. It contains the results of the diploma theses [12] and [13] and concludes part I with the open questions left after these specific attempts to use Finsler geometry as an extended description of gravity.

In part II of this thesis we present our complete Finsler spacetime framework in all details. Mathematically speaking it is an extension of Finsler geometry such that it is a suitable generalisation of Lorentzian metric geometry and not only a generalisation of Riemannian geometry. From the physics point of view it provides a non-metric Finslerian spacetime geometry which is capable to encode the causal structure, observers and their measurements and the description of gravity in one fundamental object at the same time. It extends the framework of general relativity preserving the role of spacetime in the sense that it still geometrises causality, observers and gravity simultaneously. Since a Finsler geometric extension of Lorentzian spacetimes is also interesting from the purely mathematical point of view we begin part II of this thesis with our precise mathematical definition of Finsler spacetimes and derive their geometric properties in chapter 5. We will explain how our Finsler spacetimes overcome the problems of the standard formulation of Finsler geometry and discuss that they indeed are the suitable Finslerian extension of Lorentzian metric spacetimes. Most importantly for physics we explain how our Finsler spacetime geometry encodes causality. Moreover we present explicit examples of Finsler spacetime where the geometry is based on two metrics or a vector field and a metric. To be an extension of general relativity Finsler spacetimes must admit dynamics which determine their geometry. In chapter 6 we will derive such dynamics from our Finsler spacetime version of the Einstein–Hilbert action. The derived dynamics can be used from two viewpoints: they can be seen as generalised Einstein equations determining gravitational dynamics or they can potentially be seen as unified dynamics a la Kaluza and Klein for all the fields encoded into the Finslerian spacetime geometry. During this thesis we focus on the interpretation of Finsler spacetimes and their dynamics as extension of general relativity and comment on the latter option in the outlook. In chapter 7 we present a physically interesting Finsler spacetime which is a perturbative solution of the dynamics beyond metric geometry.It can be interpreted as Finsler refinement of the linearised Schwarzschild solution of general relativity and addresses the fly-by anomaly in the solar system. Afterwards in chapter 8 we complete the threefold geometrisation role of spacetime by the introduction of observers and their measurements on our Finsler spacetimes. They are completely defined through the geometry. As an example we discuss how the non-metric geometry changes the observers measurement of the velocity of another object compared to the same measurement on Lorentzian metric spacetimes. We especially compare the measurement of the speed of light in the two situations. In chapter 9 we equip Finsler spacetimes with action based physical field theories. We develop a coupling principle such that the field theories source the dynamics of the geometry and that in case Finsler spacetimes are identical to Lorentzian metric spacetimes all dynamics, the ones for the geometry an the ones for the other physical fields, are identical to those of general relativity. On the example of a scalar field theory and a theory of electrodynamics we present features of field theories on Finsler spacetime such as the propagation of modes faster than a specific speed of light and the propagation of light along null geodesics. This chapter concludes the demonstration that

Finsler spacetimes are viable non-metric geometric backgrounds for physics.

The technical details for the proofs of theorems as well as the explicit coordinate expressions of certain vector fields can be found in the appendices A of this thesis.

Our results on the Finsler spacetime framework presented in part II of this thesis have been published in several journal articles. A preliminary definition of Finsler spacetimes and the theory of electrodynamics on Finsler spacetime can be found in [14]. The scalar field theory and a first definition of observers is published in [15], and the dynamics of Finsler spacetime, the refined linearised Schwarzschild solution and the transformations between observers on Finsler spacetimes are presented in [16].

Part I.

Finsler geometry: Mathematical formulation and application in physics

This first part of the thesis contains the necessary preliminaries and the ideas that lead to the development of the Finsler spacetime formalism presented in part II. The mathematical language of fibre bundles and the standard formulation of Finsler geometry is introduced, applications of Finsler geometry in physics are presented and the Einstein–Hilbert action is reinterpreted in the context of general non-metric Finsler geometry. During every step of the discussion we will point out why an extension of the standard Finsler geometry framework is needed in order to apply Finsler geometry to describe geometric backgrounds for physics.

In chapter 1 the mathematical tools on which Finsler geometry, and so Finsler spacetimes, are built will be introduced. The main goal of this mathematical guide is to introduce the concept and the consequences of connections on the tangent bundle of a manifold. To achieve this goal in an efficient way some facts about locally trivial fibre bundles in general are recalled and then applied to the tangent bundle.

Finsler geometry, as it can be found in the literature, is reviewed in chapter 2. It relies on the mathematical objects introduced in the foregoing chapter. Moreover limitations of the framework are discussed. It will become clear that Finsler geometry works without any problems as long as it is considered as generalization of Riemannian geometry based on a metric with definite signature, but that it runs into problems when considered as generalization of geometry based on an indefinite metric. The latter issue is examined with a focus on the applications of Finsler geometry in physics.

In chapter 3 different applications of Finsler geometry in physics are collected. This includes the emergence of Finsler geometry as particle limit of field theories or ray approximation of wave equations, attempts to use Finsler geometry as fundamental geometry of spacetime including dynamics and in phenomenological approaches which use the freedom of Finsler geometry to fit observational data

A new perspective on dynamics of Finsler spaces and a discussion of open questions which arise in the application of Finsler geometry as spacetime geometry is presented in chapter 4. The Einstein–Hilbert action is carefully investigated from the viewpoint of Finsler geometry. It is straightforward to see that it is just a special metric geometric version of an action determining dynamics for for Finsler spaces. This part is then concluded by a discussion of questions arising when one studies the application of Finsler geometry in physics. It is explained why the existing framework needs to be extended to be used as fundamental spacetime geometry and what the extension has to provide an order to be used as non-metric spacetime geometry. The study of these questions then will directly lead to our development of Finsler spacetimes in the second part of this thesis.

1. A mathematical guide to Finsler geometry

Finsler geometry describes the geometry of a manifold M through tensor fields which live on the tangent bundle TM of the manifold and not on the manifold itself. Rather then considering only the points of M, the tangent bundle consists of the points and all directions of M. In this chapter we will make this statement mathematical precise and set up the mathematical stage used throughout this thesis. The mathematical structure offered by the tangent bundle will be crucial for the review of standard Finsler spaces in the following chapter, and especially for our extension of this framework to physical Finsler spacetimes and the analysis of physics on Finsler spacetimes in part II of this thesis.

Here we first introduce all necessary facts about general fibre bundles in section 1.1 in order to give a most systematic description of the tangent bundle and its properties in section 1.2. Already here we will get a first idea how Riemannian, and in general, Finsler geometry is related to connections on the tangent bundle; the details how the tangent bundle is the natural stage for Finsler geometry are presented in the next chapter.

1.1. Locally trivial fibre bundles

The tangent bundle itself is, as suggested by the name, a special instant of the mathematical structure called fibre bundle. Here we introduce the important features of fibre bundles which, when studied on the tangent bundle, provide the framework for Finsler geometry.

We begin with the definition, introduce a special set of coordinates to describe locally trivial smooth fibre bundles, its tensor spaces and their natural split in subsection 1.1.1, before we introduce the notion of a connection in subsection 1.1.2 and the corresponding curvature in subsection 1.1.3. The basic facts of locally trivial smooth fibre bundles can be found for example in the books [17, 18, 19] or the compendium [20].

1.1.1. Definition and vertical tangent spaces

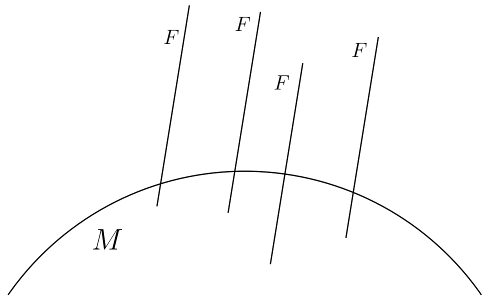
Before we can study tensors on the tangent bundle we need to introduce the canonical vertical tangent space which exists naturally on every locally trivial smooth fibre bundle. We begin with the definition of locally trivial smooth fibre bundles:

Definition 1.1. ***Locally trivial smooth fibre bundles***
The object $(P \xrightarrow{\pi} M, F)$, consisting of smooth manifolds P, called the total space, M, called the base space, F, called the fibre, and a projection map π such that $\pi^{-1}(p) = F, \forall\, p \in M$, is called a locally trivial smooth fibre bundle, if, and only if, there exists an open cover $\{U_i\}_{i\in I}$ of M such that for every $U_i \subset M$ exists a diffeomorphism $\Psi_i : \pi^{-1}(U_i) \to U_i \times F$, called trivialization, satisfying $\pi \circ \psi_i^{-1} = pr_1$, where pr_1 is the projection onto the first factor of ψ_i.

This definition can be made more general by considering P, M and F to be topological spaces instead of smooth manifolds, but the latter case suffices for this thesis. In the following we will suppress the terms smooth and locally trivial in locally trivial smooth fibre bundle for the sake of readability. The definition above is summarised in the commutative diagram in figure 1.1(a), and leads to the picture of a fibre bundle as a base space M with a fibre F attached to each point p in M, displayed in figure 1.1(b).

$$\begin{tikzcd}
P \supset \pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[d, "\pi"'] & U_i \times F \subset M \times F \arrow[dl, "pr_1"] \\
M \supset U_i &
\end{tikzcd}$$

(a) Defining structures of a fibre bundle.



(b) Visualization of a fibre bundle.

Figure 1.1. Fibre bundles.

The manifolds M and F admit local coordinates; for open sets $U \subset M$ and $V \subset F$ we have

$$\Xi : U \to \mathbb{R}^{\dim(M)};\ \Xi = (x^1, \ldots, x^{\dim(M)}) \quad \xi : V \to \mathbb{R}^{\dim(F)};\ \xi = (y^1, \ldots, y^{\dim(F)})\,. \tag{1.1}$$

These coordinates induce local coordinates of P via the trivialization in the following way. Consider $u \in P$, then there exists an open set $U \subset M$ such that $u \in \pi^{-1}(U)$ and $\psi_U(u) = (\psi_1(u), \psi_2(u)) = (\pi(u), \psi_2(u))$. Now let Ξ_U be a coordinate map on U and ξ_V be a coordinate map on $V \subset F$ containing $\psi_2(u)$. Therefore induced coordinates in a neighbourhood of $u \in P$ are given by

$$\Phi = (\Xi_U, \xi_V) \circ \psi_U \tag{1.2}$$

We may write $\Phi(u) = (\Xi_U(\pi(u)), \xi_V(\psi_2(u))) = (x^1, \ldots, x^{\dim(M)}, y^1, \ldots, y^{\dim(F)})$. The coordinates just obtained turn out to be very suitable to study tensors and tensor fields on P. For the rest of this section we will use the index convention that Latin indices $a, b, c, \ldots$ run from 1 to $\dim(M)$, while Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to $\dim(F)$.

The induced coordinates give rise to a coordinate basis of the tangent- and cotangent spaces T_uP and T_u^*P around a point $u \in P$, respectively given by

$$\Big\{\frac{\partial}{\partial x^a} = \partial_a\Big\} \cup \Big\{\frac{\partial}{\partial y^\alpha} = \bar{\partial}_\alpha\Big\} \text{ and } \{\mathrm{d}x^a\} \cup \{\mathrm{d}y^\alpha\}\,. \tag{1.3}$$

Hence a vector field X on P evaluated at $u \in P$ can be expressed as

$$X_{|u} = X^a(u)\partial_{a|u} + \bar{X}^\alpha(u)\bar{\partial}_{\alpha|u}\,. \tag{1.4}$$

A similar decomposition into base space basis and fibre basis can be done for every tensor field on P.

From the definition of the projection π of a fibre bundle it is clear that it induces a map from the tangent spaces of the total space to the tangent spaces of the base space which acts in the following way

$$d\pi_{|u} : T_uP \to T_{\pi(u)}M;\ d\pi_{|u}(X_u) = X^a(u)\partial_{a|\pi(u)}\,. \tag{1.5}$$

The kernel $\ker(d\pi_{|u})$ is exactly that part of the tangent space T_uP spanned by the induced basis elements $\{\bar{\partial}_\alpha\}$ which come from the fibre F, while the image of $d\pi_{|u}$ is a vector tangent to the base space at $\pi(u)$ expressed in the coordinate basis from the base space. Observe the difference between $\partial_{a|u} \in T_uP$ while $\partial_{a|\pi(u)} \in T_{\pi(u)}M$. We see that by its kernel, the projection map π singles out the part of the tangent space T_uP which is tangent to the fibre, and since $d\pi$ has constant rank all over P the following definition singles out a subspace of T_uP of dimension $\dim(F)$ for all $u \in P$:

Definition 1.2. ***Vertical tangent spaces***
Let $(P \xrightarrow{\pi} M, F)$ be a fibre bundle. The vertical tangent space $V_uP \subset T_uP$ at u is defined as $V_uP = \ker(d\pi_u)$. It is spanned by the induced coordinate basis elements $\{\bar{\partial}_\alpha\}$ which are induced by the fibre coordinates.

Having identified the vertical space as the subspace of T_uP which is tangent to the fibre F in a natural way, it remains to fix a complement which can be interpreted to be tangent to the base space M. This will be done next by introducing a so called connection on P and it will turn out that there is some freedom in the choice.

1.1.2. Connections and horizontal tangent spaces

Establishing the notion of a connection, or connection one-form, on P will allow us to decompose the tangent spaces T_uP into two subspaces: the canonical vertical space V_uP tangent to the fibre F presented in the previous subsection, and the horizontal space H_uP tangent to the base space M defined by the connection. The connection will be the central object which defines the geometry of a manifold in chapter 2 and in part II.

Definition 1.3. ***Connection one-forms***
Let $(P \xrightarrow{\pi} M, F)$ be a fibre bundle. A connection map is a projection map $\omega_u : T_uP \to V_uP$ which projects T_uP onto the vertical tangent space V_uP for all $u \in P$, so that $\omega_u \circ \omega_u = \omega_u$. We call $\omega_{(\cdot)} : P \to V_{(\cdot)}P; u \mapsto \omega_u$, or short ω, the connection one-form.

In an induced coordinate basis, see equation (1.3), ω can be expressed as [12]

$$\omega_u = \left(\mathrm{d}y^\alpha_{|u} + N^\alpha{}_b(u)\mathrm{d}x^b_{|u}\right) \otimes \bar{\partial}_{\alpha|u}\,, \tag{1.6}$$

where the $N^\alpha{}_a(u)$ are called connection coefficients and define the connection one-form of choice. Since ω is a projection onto the vertical tangent spaces it is possible to write the tangent space at $u \in P$ as $T_uP = V_uP \oplus \ker(\omega_u)$. Alternatively to a connection one-form on can define a connection:

Definition 1.4. ***Connection***
Let $(P \xrightarrow{\pi} M, F)$ be a fibre bundle. A connection is a map H which associates to each $u \in P$ a subspace $H(u) = H_uP \subset T_uP$ such that $T_uP = H_uP \oplus V_uP$, where V_uP is the vertical tangent space.

In other words a connection is a distribution, a smooth map which associates to each pint of a manifold a subspace of the tangent bundle. Starting from a connection one-form or from a connection is equivalent by setting $H_uP = \ker(\omega_u)$. In both cases we obtain the complement of V_uP we were looking for either as $\ker(\omega_u)$ or as $H(u)$. Therefore we define:

Definition 1.5. *The horizontal tangent space*
Let $(P \xrightarrow{\pi} M, F)$ be a fibre bundle with connection one-form ω. The horizontal tangent space $H_uP \subset T_uP$ at u is defined as $H_uP = \ker(\omega_u)$.

Observe that the horizontal spaces depend on the choice of the connection coefficients $N^\alpha{}_a(u)$. In contrast to the vertical spaces V_uP which are already determined by the definition of the fibre bundle, further input is needed to define the horizontal spaces H_uP. A short calculation reveals that, expressed in induced coordinate basis, the following horizontal basis spans H_uP

$$\ker \omega_u = \mathrm{span}\left(\delta_{a|u}\right);\ \delta_{a|u} = \partial_{a|u} - N^\alpha{}_a(u)\bar{\partial}_{\alpha|u}\,. \tag{1.7}$$

A sketch of this horizontal and vertical split of the tangent space to a fibre bundle is depicted in figure 1.2. From the definition of the horizontal tangent space one immediately realises that $\dim(H_uP) = \dim(T_{\pi(u)}M)$ and we recall that $d\pi$ is an isomorphism between H_uP and $T_{\pi(u)}M$. It allows to identify the vectors in the horizontal spaces with vectors tangent to the base space. A closer look reveals that in case of non-vanishing connection coefficients the horizontal space is not really embedded tangent to the base space manifold into the bundle tangent space, but tilted. This tilt, which may vary from point to point in the bundle and depends on the choice of coordinates, is caused by the mixing between manifold induced and fibre induced coordinate basis. It is governed by the connection coefficients. The deeper meaning of this mixture and the coordinate independent consequences will become clear in the next section when we discuss the curvature of a connection.

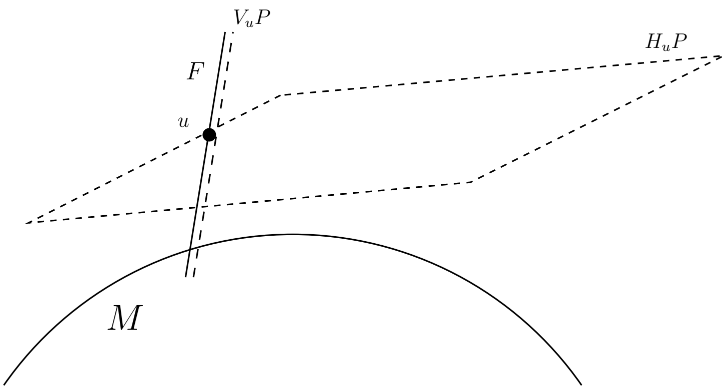


Figure 1.2. Dotted: The horizontal and the vertical tangent space at a point u in P

The split of the tangent space of the fibre bundle into horizontal and vertical tangent space induces a split of the cotangent space of the fibre bundle in a similar fashion. The dual to the horizontal and vertical tangent spaces, expressed in induced coordinates are given by

$$H^*_uP = \mathrm{span}(\mathrm{d}x^a{}_{|u}) \text{ and } V^*_uP = \mathrm{span}(\delta y^\alpha{}_{|u});\ \delta y^a{}_{|u} = \mathrm{d}y^\alpha{}_{|u} + N^\alpha{}_a \mathrm{d}x^a{}_{|u}\,. \tag{1.8}$$

As a remark see that the vertical dual annihilates the horizontal tangent space

$$\delta y^\alpha(\delta_b) = (dy^\alpha + N^\alpha{}_q dx^q)(\partial_b - N^\beta{}_b\bar{\partial}_\beta) = -N^\beta{}_b\delta^\alpha_\beta + N^\alpha_q\delta^q_b = 0\,. \tag{1.9}$$

1.1.3. Curvature and integrability

The split of the tangent spaces of P into the horizontal and vertical subspaces gives rise to the question whether these subspaces belong to integral manifolds which are submanifolds of P. The answer to this question is given by the definition of the curvature of a connection with the help of the Frobenius theorem which we cite here without proof; for a proof see for example [21].

Theorem 1.1. ***Frobenius Theorem***
Let M be a smooth manifold and D be a distribution; hence a mapping which associates to each $p \in M$ a subspace $D_p \subset T_pM$. Let $dim(D_p) < \dim(M)$ and let $X_i, i = 1, \ldots, \dim(D_p)$ be vector fields such that $\{X_i(p)\}$ form a basis of D_p. The subspaces D_p of T_pM are tangent spaces to a $\dim(D_p)$ dimensional submanifold N of M if and only if the commutator of the X_i satisfies $[X_i, X_j](p) \in D_p, \; \forall\, i, j$ and for all $p \in M$. In case N exists we call D integrable.

The vertical tangent spaces define a distribution V which maps each $u \in P$ to V_uP. They are spanned by $\{\bar{\partial}_{\alpha|u}\}$ which are evaluations of the vector fields $\{\bar{\partial}_\alpha\}$ which commute obviously. Hence $[\bar{\partial}_\alpha, \bar{\partial}_\beta](u) = 0 \in T_uP \; \forall\, \alpha, \beta$. By the Frobenius theorem we conclude that there exists a submanifold of P to which the V_uP are the tangent spaces; this submanifold is the fibre F.

For the horizontal tangent spaces the situation is more complicated. They also define a distribution by virtue of definition 1.4 which is spanned by the vector fields δ_a at each $u \in P$. In order to analyse whether this distribution is integrable we calculate the commutator of the basis vector fields

$$[\delta_a, \delta_b] = (\delta_b N^\alpha{}_a - \delta_a N^\alpha{}_b)\bar{\partial}_\alpha\,. \tag{1.10}$$

Hence in general we cannot conclude that the distribution defined by the horizontal tangent spaces is integrable. This failure of integrability is measured by a tensor, displayed above and leads to the definition of the curvature of a connection.

Definition 1.6. ***Curvature***
Let $(P \xrightarrow{\pi} M, F)$ be a fibre bundle and ω a connection one-form on P with connection coefficients $N^\alpha{}_a(u)$. The curvature tensor R of the connection one-form ω is defined by the components

$$R^\alpha{}_{ab} = [\delta_a, \delta_b]^\alpha\,. \tag{1.11}$$

It is obvious that a horizontal distribution is integrable, if, and only if, $R^\alpha{}_{ab} = 0$. When we discuss the tangent bundle in the next section we will see that this rather abstractly defined curvature tensor is straightforwardly connected with the curvature tensor for a covariant derivative on the base manifold M.

1.2. Tangent bundle geometry

The tangent bundle of a manifold is a special fibre bundle which is naturally constructed for every smooth n-dimensional manifold M, which then, in bundle language, is the base space. Despite the structures described for general fibre bundles the tangent bundle admits further important features due to its emergence from the base space manifold. Here we recall the most important ones: in subsection 1.2.1 the existence of purely base space induced coordinates and the existence of distinguished tensor fields on the bundle which behave like tensor fields on the base space are presented; in subsection 1.2.2 special lifts from the base space to the bundle are described and in subsection 1.2.3 a suitable covariant derivative for such tensor fields which yields a covariant expression for the autoparallels of a connection is discussed.

The framework of Finsler geometry naturally leads to a description of the geometry of the manifold in terms of a connection on the tangent bundle. We will see here that one can understand Riemannian geometry from the same point of view, namely that the geometric objects are special tensors on the tangent bundle derived from a unique connection on the tangent bundle. Moreover in this section we get an idea how Finsler geometry generalises the notion of geometry with respect to Riemannian geometry, before we review the details in the next chapter 2. The key differences are the properties of the unique connection one-form.

1.2.1. Manifold induced coordinates and distinguished tensor fields

We recall the definition of the tangent bundle, express it in so called manifold induced coordinates and introduce distinguished tensor fields.

Definition 1.7. ***The tangent bundle***
Let M be a n-dimensional differentiable manifold. The tangent bundle TM is the union of all tangent spaces T_pM of M

$$TM = \bigcup_{p \in M} T_pM \,. \tag{1.12}$$

An element $u \in TM$ is a vector $X_{|p}$ on M located in some tangent space T_pM. The bundle structure is as follows: The total space is the set TM with natural projection $\pi : TM \to M$ which acts as $\pi(u) = \pi(X_p) = p$; the fibres $F = \pi^{-1}(p)$ are given by the tangent spaces $T_pM \approx \mathbb{R}^n$ and so the tangent bundle is the fibre bundle $(TM \xrightarrow{\pi} M, \mathbb{R}^n)$. Note that the fibres of the tangent bundle have the same dimension as the base space and so the indices of the fibre coordinates have the same range as the indices labelling the coordinates of the manifold.

On an open subset $U \subset M$ we consider coordinates $\Xi = (x^1, \dots, x^n)$; at a point $p \in M$ they induce the standard coordinate basis $\{\partial_a\}$ of T_pM, hence every vector $Z_p \in T_pM$ can be written as $Z = y^a \partial_{a|p}$. We can identify T_pM with $\mathbb{R}^n$ when we identify the coordinate basis ∂_a with the canonical basis $e_a = (e^1_a, \dots, e^n_a)$ with $e^j_a = 0$ for $j \neq a$ and $e^a_a = 1$ of $\mathbb{R}^n$. In this way we introduced manifold induced coordinates $\{y^a\}$ on T_pM. The coordinates from the manifold and the induced coordinates on the tangent spaces together induce coordinates on TM. The coordinate representation of a vector $u = X_p = X^a \partial_{a|p}$ is then given by $u = (\Xi(p), y^1, \dots, y^n) = (x, y)$. Hence solely from coordinates of the manifold M, which is the base space of the bundle, we introduced coordinates on the whole tangent bundle. From now on we will consider TM in the manifold induced coordinates, and refer to the y coordinates as tangent space or fibre coordinates. A feature of these coordinates is that a coordinate change on the manifold $x \to \tilde{x}(x)$ induces a coordinate change on the bundle

$$(x^a, y^b) \to (\tilde{x}^a(x), \tilde{y}^b(x, y)); \ \tilde{y}^b(x, y) = \partial_q \tilde{x}^b(x) y^q \,. \tag{1.13}$$

From the general discussion of fibre bundles we can easily deduce the change of the bases of the bundle tangent spaces T_uTM under manifold induced coordinate transformations. This will lead to the discovery of tensor fields on TM which behave under such transformations like tensor fields on M. It turns out that the induced coordinate basis, equation (1.3), of the tangent spaces of TM transforms as

$$\begin{aligned}(\partial_a, \bar{\partial}_b) \to (\tilde{\partial}_a, \tilde{\bar{\partial}}_b) &= (\tilde{\partial}_a x^q \partial_q + \tilde{\partial}_a y^p \bar{\partial}_p, \tilde{\partial}_b x^p \bar{\partial}_p) &(1.14)\\ (\mathrm{d}x^a, \mathrm{d}y^b) \to (d\tilde{x}^a, d\tilde{y}^b) &= (\partial_q \tilde{x}^a \mathrm{d}x^q, \partial_q \tilde{y}^b \mathrm{d}x^q + \partial_p \tilde{x}^b \mathrm{d}y^p)\,. &(1.15)\end{aligned}$$

The $\bar{\partial}_a$ and dx^a transform as if they where tensor fields on the base manifold, the ∂_a and δy^a do not. However it is possible to find a complete basis of the T_uTM which has the transformation behaviour of tensors on the manifold under manifold induced coordinate transformations. To find this basis we consider a connection one-form on TM , definition 1.3, defined through its connection coefficients ${N^a}_b$. It is a $(1,1)$-tensor on TM and takes the form

$$\omega = (dy^a + {N^a}_b dx^b) \otimes \bar{\partial}_a\,, \tag{1.16}$$

in the induced coordinates. From this expression one deduces the transformation behaviour of the connection coefficients to be, see also [12, 22],

$${N^a}_b(x,y) \to {\tilde{N}^a}_b(\tilde{x},\tilde{y}) = {N^p}_q \partial_p \tilde{x}^a \tilde{\partial}_b x^q + \tilde{\partial}_b y^m \partial_m \tilde{x}^a\,. \tag{1.17}$$

Studying now the transformation behaviour of the horizontal and vertical basis of T_uTM and T^*_uTM , introduced in the previous section, under manifold induced coordinate transformations we find

$$\begin{aligned}(\delta_a, \bar{\partial}_b) &\to (\tilde{\delta}_a, \tilde{\bar{\partial}}_b) = (\tilde{\partial}_a x^q \delta_q, \tilde{\partial}_b x^p \bar{\partial}_p) &(1.18)\\ (\mathrm{d}x^a, \delta y^b) &\to (d\tilde{x}^a, \delta\tilde{y}^b) = (\partial_q \tilde{x}^a \mathrm{d}x^q, \partial_p \tilde{x}^b \delta y^p)\,. &(1.19)\end{aligned}$$

Indeed this horizontal-vertical basis is the basis we were looking for. Under manifold induced coordinate transformations it behaves like basis vector-, respectively covector fields on the base manifold. As a remark we point out that the transformations respect the horizontal-vertical split of the tangent bundle, they cause no mixing. Tensor fields on TM which have the property to transform under manifold induced coordinate transformations as if they were tensor fields on the manifold, and which define tensor fields on M in case their components are not dependent on the tangent space coordinates y, are called distinguished or short d-tensors.

Definition 1.8. ***d-tensors***
Let $X^i;\ i = 1,\ldots,s$ be vector fields on TM, $\Omega^j;\ j = 1,\ldots,r$ be covector fields on TM, q_j be projectors on the horizontal or vertical cotangent bundle and p_i be projectors on the horizontal or vertical tangent bundle of TM. An (r,s)-tensor field T on TM is called d-tensor field (distinguished tensor field), if, and only if,

$$T(\Omega^1,\ldots,\Omega^r,X^1,\ldots,X^s) = T(q_1(\Omega^1),\ldots,q_r(\Omega^r),p_1(X^1),\ldots,p_s(X^s))\,. \tag{1.20}$$

They can always be expressed in the horizontal-vertical basis as

$$T = {T^{a_1a_2\ldots a_r}}_{b_1b_2\ldots b_s}\, \delta_{a_1}\otimes..\otimes\delta_{a_i}\otimes\bar{\partial}_{a_i+1}\otimes..\otimes\bar{\partial}_{a_j}\otimes \mathrm{d}x^{b_1}\otimes..\otimes\mathrm{d}x^{b_k}\otimes\delta y^{b_k+1}\otimes..\otimes\delta y^{b_l}\,. \tag{1.21}$$

How many horizontal or vertical basis elements appear depends on the type of tensor and on the number of horizontal or vertical projectors. When we describe field theories on Finsler spacetimes in chapter 9 it will turn out that physical fields are in general d-tensors on TM. A mathematical example for a d-tensor is the curvature of a connection, equation (1.11),

$$R = R^a{}_{bc}\mathrm{d}x^b \wedge \mathrm{d}x^c \otimes \bar{\partial}_a\,. \tag{1.22}$$

This example is also suitable to demonstrate how, under certain circumstances, d-tensors define tensors on the base manifold and moreover how the curvature of a connection on the tangent bundle is related to the Riemann curvature tensor. Consider a connection one-form ω on TM with connection coefficients of the form $N^a{}_b(x,y) = \Gamma^a{}_{bc}(x)y^c$, it follows by direct calculation

$$R^a{}_{bc} = [\delta_b,\delta_c]^a = -(\partial_b\Gamma^a{}_{pc} - \partial_c\Gamma^a{}_{pb} + \Gamma^a{}_{bs}\Gamma^s{}_{pc} - \Gamma^a{}_{cs}\Gamma^s{}_{pb})y^p = -R^a{}_{pbc}(x)y^p\,. \tag{1.23}$$

Therefore the d-tensor field R with components $R^a{}_{bc}(x,y)$ on TM defines a tensor field R^M with components $\bar{\partial}_p R^a{}_{bc}(x)$ on M. In this way the curvature of a general connection on TM is nothing but a generalization of the Riemann curvature tensor.

Here we recalled a point of view on the Riemann curvature tensor which is not so common: it measures the integrability of the horizontal distribution defined by a connection one-form on the tangent bundle with connection coefficients $\Gamma^a{}_{bc}(x)y^c$, linear in y. The $\Gamma^a{}_{bc}$ may come from the definition of a covariant derivative, or a notion of parallel transport, on M, so that the path dependence of parallel transport between points on M is equivalent to the non-integrability of a linear connection on TM. We realise here that the choice of the connection is the key point to connect the curvature of a connection on the manifold to Riemannian geometry, and we will see in chapter 2 that in this choice lies the difference to Finsler geometry. Riemannian geometry is connected to a unique connection one-form with connection coefficients linear in y, while Finsler geometry leads to unique connection coefficients not linear, but only homogeneous in the tangent space coordinates, and so more general than Riemann geometry.

While it is clear for the linear connection case how to construct a covariant derivative, this is not so well known for the general case. Before we discuss a covariant derivative for d-tensors based on a general connection on TM, we need to introduce the lifts of vector and general tensor fields from the manifold to the tangent bundle. These will also be important when we study symmetries of Finsler spacetimes in section 5.4.

1.2.2. Tensor field lifts from the manifold to the tangent bundle

With the definition of d-tensors we found tensors on the tangent bundle which behave like tensors on the manifold, with the only difference that their components depended on the full tangent bundle coordinates (x,y) and not only on the manifold coordinates (x). The other way around it is possible to obtain tensors on the tangent bundle from tensors on the base manifold. We will introduce three type of lifts, the vertical and the horizontal lift of tensor fields, depending on the connection on the tangent bundle, and the complete lift for vector fields which is independent of the connection. The first two lifts are needed to define how the covariant derivative induced by a general connection acts on tensor fields which live on the manifold in subsection 1.2.3. The third lift generates diffeomorphisms on the tangent bundle induced by diffeomorphisms on the manifold generated by the original vector fields. This fact will play a major role when we study symmetries of Finsler spacetimes in section 5.4.

Definition 1.9. ***Vertical lifts***
Let ω be a connection on TM, T be an (r,s)-tensor field on a smooth manifold M and let $X^i;\ i=1,\dots,s$ be vector fields on TM, $\Omega^j;\ j=1,\dots,r$ be covector fields on TM. Moreover let P and Q be the projectors on the vertical tangent respectively cotangent spaces combined with their identification with the tangent space to the manifold

$$\begin{aligned} P(X^j(u)) &= P(X^{ja}(u)\delta_{a|u} + \bar{X}^{jb}(u)\bar{\partial}_{b|u}) = \bar{X}^{jb}(u)\partial_{b|\pi(u)} && (1.24)\\ Q(\Omega^j(u)) &= Q(\Omega^j_a(u)dx^a_{|u} + \bar{\Omega}^j_b(u)\delta y^b_{|u}) = \bar{\Omega}^j_b(u)dx^b_{|\pi(u)} && (1.25)\end{aligned}$$

The vertical lift of T to a tensor field T^v on TM is defined as

$$T^v(u) = T(\pi(u))\,, \tag{1.26}$$

so that

$$T^v(u)(\Omega^1,\dots,\Omega^r,X^1,\dots,X^s) = T(\pi(u))(Q(\Omega^1(u)),\dots,Q(\Omega^r(u)),P(X^1(u)),\dots,P(X^s(u)))\,. \tag{1.27}$$

In the horizontal-vertical basis induced by manifold induced coordinates of TM, and with $T^{a_1\dots a_r}{}_{b_1\dots b_s}(p)$ being the components of the manifold tensor field T in the canonical coordinate basis, this reads

$$T^v(u) = T^{a_1\dots a_r}{}_{b_1\dots b_s}(\pi(u))\bar{\partial}_{a_1}\otimes\dots\otimes\bar{\partial}_{a_r}\otimes\delta y^{b_1}\otimes\dots\otimes\delta y^{b_s}\,. \tag{1.28}$$

The horizontal lift of a tensor field on the manifold to a tensor field on the tangent bundle is defined in the same way, just replacing the vertical projectors with horizontal projectors.

Definition 1.10. ***Horizontal lifts***
Let ω be a connection on TM, T be an (r,s)-tensor field on a smooth manifold M and let $X^i; i=1,\dots,s$ be vector fields on TM, $\Omega^j; j=1,\dots,r$ be covector fields on TM. Moreover let P and Q be the projectors on the horizontal tangent respectively cotangent spaces.

$$\begin{aligned} P(X^j(u)) &= P(X^{ja}(u)\delta_{a|u} + \bar{X}^{ja}(u)\bar{\partial}_{a|u}) = X^{jb}(u)\partial_{b|\pi(u)} && (1.29)\\ Q(\Omega^j(u)) &= Q(\Omega^j_a(u)dx^a_{|u} + \bar{\Omega}^j_b(u)\delta y^b_{|u}) = \Omega^j_a(u)dx^a_{|\pi(u)} && (1.30)\end{aligned}$$

The horizontal lift of T a tensor field T^H on TM is defined as

$$T^H(u) = T(\pi(u))\,, \tag{1.31}$$

so that

$$T^H(u)(\Omega^1,\dots,\Omega^r,X^1,\dots,X^s) = T(\pi(u))(Q(\Omega^1(u)),\dots,Q(\Omega^r(u)),P(X^1(u)),\dots,P(X^s(u)))\,. \tag{1.32}$$

In the horizontal-vertical basis the horizontal lift can be expressed as

$$T^h(u) = T^{a_1\dots a_r}{}_{b_1\dots b_s}(\pi(u))\delta_{a_1}\otimes\dots\otimes\delta_{a_r}\otimes dx^{b_1}\otimes\dots\otimes \mathrm{d}x^{b_s}\,. \tag{1.33}$$

Finally we define the complete lift of a vector field on M to a vector field on TM.

Definition 1.11. ***Complete lift***
Let X be a vector field on M, expression in local coordinates by $X_{|x} = X^a(x)\partial_{a|x}$. The complete lift of X to a vector field X^C on the tangent bundle is defined as

$$X^C_{|(x,y)} = X^a(x)\partial_{a|(x,y)} + (y^p\partial_p X^a(x))\bar{\partial}_{a|(x,y)}\,. \tag{1.34}$$

Next we introduce a covariant derivative which is defined by a connection on the tangent bundle. The horizontal and vertical lifts introduced in this section make it possible that this covariant derivative on TM defines a covariant derivative on M.

1.2.3. Connection induced covariant derivatives and autoparallels

Based on the connection coefficients of a connection one-form ω on the tangent bundle it is possible to define a covariant derivative for (r,s)-d-tensor fields. It is a tensor derivative, obeys the Leibniz rule and defines parallel transport along horizontal curves in TM. The latter is important since autoparallels of the connection on TM fall in that class of curves, as we will see during this subsection. In chapter 2 it will turn out that in the framework of Finsler geometry there exists a unique connection such that its autoparallels are the geodesics of Finsler spaces. In part II we will see that those facts also hold for our Finsler spacetimes and that the dynamical covariant derivative allows a precise control on writing equations of motion for physical fields covariantly. Here we shortly review the basic facts about this so called dynamical covariant derivative and about autoparallels of a connection, further details can be found in [12, 22].

Definition 1.12. ***Dynamical covariant derivative***
Let $N^a{}_b(x,y)$ be the connection coefficients of a connection one-form ω on the tangent bundle TM of a n-dimensional differentiable manifold M in manifold induced coordinates. Moreover let $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,y)$ be the components of a (r,s)-d-tensor field T. The dynamical covariant derivative ∇ maps the (r,s)-d-tensor field T on the (r,s)-d-tensor fields ∇T with components

$$\begin{aligned}
&\nabla T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,y) \\
=\;& y^q\delta_q T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,y) \\
+\;& N^{a_1}{}_m(x,y)T^{ma_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,y) + \cdots + N^{a_r}{}_m(x,y)T^{a_1a_2\ldots m}{}_{b_1b_2\ldots b_s}(x,y) \\
-\;& N^m{}_{b_1}(x,y)T^{a_1a_2\ldots a_r}{}_{mb_2\ldots b_s}(x,y) - \cdots - N^m{}_{b_s}(x,y)T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots m}(x,y)\,.
\end{aligned} \tag{1.35}$$

Introducing the abbreviation $S = y^a\delta_a$ the action of the dynamical covariant derivative on functions can be written as $\nabla f(x,y) = S(f)(x,y) = y^q\delta_q f(x,y)$. Observe that this covariant derivative in general has no special properties with respect to the y coordinates; it will have in Finsler geometry where we choose a specific unique connection. As aside we remark that for (r,s)-tensor fields K on M, their vertical respectively horizontal lifts together with the dynamical covariant derivative define a covariant derivative with respect to a vector field X on the manifold by setting

$$\nabla^M_X K^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x) = \nabla K^{h\,a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,X) = \nabla K^{v\,a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}(x,X)\,. \tag{1.36}$$

In general this covariant derivative is not linear with respect to the direction of differentiation X. But similar as in the case for the curvature of a general connection on TM, equation (1.11), we recover the standard linear covariant derivative on the manifold $\tilde{\nabla}^M_X$ when the connection has coefficients $\Gamma^a{}_{bc}(x)y^c$ linear in the fibre coordinates, and when the (r,s)-d-tensor field components are independent of the y. Then,

$$\begin{aligned}\nabla T^{a_1a_2\dots a_r}{}_{b_1b_2\dots b_s}(x,y) &= y^q\big(\partial_q T^{a_1a_2\dots a_r}{}_{b_1b_2\dots b_s}(x)\\ &+ \Gamma^{a_1}{}_{mq}(x)T^{ma_2\dots a_r}{}_{b_1b_2\dots b_s}(x)+\dots+\Gamma^{a_r}{}_{mq}(x)T^{a_1a_2\dots m}{}_{b_1b_2\dots b_s}(x)\\ &- \Gamma^{m}{}_{b_1q}(x)T^{a_1a_2\dots a_r}{}_{mb_2\dots b_s}(x)-\dots-\Gamma^{m}{}_{b_sq}(x)T^{a_1a_2\dots a_r}{}_{b_1b_2\dots m}(x)\,.\\ &= \tilde{\nabla}^M_y T^{a_1a_2\dots a_r}{}_{b_1b_2\dots b_s}(x)\,.\end{aligned} \tag{1.37}$$

When we study Finsler geometry it will turn out that we deal with a unique homogeneous connection, i.e., $N^a{}_b(x,\lambda y)=\lambda N^a{}_b(x,y)$, for which the linear case is also only a special case, as it is here in the discussion of general connections.

Autoparallels of a connection on the tangent bundle are special curves on the base manifold. On Finsler spaces and Finsler spacetimes we will find a unique connection such that its autoparallels are identical to the geodesics of the manifold.

Definition 1.13. ***Autoparallels of a connection***
Let $\gamma:\mathbb{R}\to M$ be a smooth curve on M. Consider its natural lift to the tangent bundle $\tilde{\gamma}=(\gamma,\dot{\gamma})$, expressed in manifold induced coordinates. The curve γ is an autoparallel of the connection ω, if, and only if, there exists a parametrisation of γ such that its natural lift is a horizontal curve, i.e., if the tangent of $\dot{\tilde{\gamma}}\in T_{\tilde{\gamma}}TM$ is a horizontal vector field. In coordinates, this yields the autoparallel equation

$$\ddot{\gamma}^a+N^a{}_b(\gamma,\dot{\gamma})\dot{\gamma}^b=0\,. \tag{1.38}$$

Using the horizontal lift, see equation (1.33), $\dot{\gamma}^H=\dot{\gamma}^a\delta_a\in H_{\tilde{\gamma}}TM$ of the tangent $\dot{\gamma}$ of a curve γ on M, the autoparallel equation (1.38) can be written with help of the dynamical covariant derivative

$$\nabla\dot{\gamma}^H(\gamma,\dot{\gamma})=0\,. \tag{1.39}$$

The dynamical covariant derivative of a connection generalises the usual covariant derivative on the manifold when one considers the geometry of a manifold described by a general connection on the tangent bundle. It reduces to the standard covariant derivative on the manifold in case of a linear connection and features a special class of curves, its autoparallels, which characterize the geometry. In the next chapter we will see how one finds a unique connection such that its autoparallels are the geodesics of a general length measure for curves.

With the mathematical structure discussed during this chapter we are now in the position to describe Finsler spaces; manifolds equipped with a not necessarily metric length measure which determines the geometry.

2. A review of Finsler spaces

Finsler geometry is a well known subject in mathematics for nearly a century. It goes back to the thesis "Über Kurven und Flächen in allgemeinen Räumen" by P. Finsler from 1918 [11] in which he analysis the geometry of embedded curves and surfaces based on a general length measure on the background manifold.

During this chapter we will review the basic facts about the standard approach to Finsler geometry where all details can be found in textbooks like [22, 23]. Finsler geometry works similar to metric geometry with the main difference that the geometry of the manifold M is determined by a general length measure, the so called Finsler function F, and and not by a metric g. We will define Finsler spaces in section 2.1 which generalise Riemannian manifolds. Next we introduce the canonical tensors given on a Finsler space in section 2.2 before we introduce the unique connection and corresponding curvature of a Finsler space as well as the associated linear covariant derivatives in section 2.3. In section 2.4 it will turn out that the unique non-linear connection is such that its autoparallels are identical to the geodesics of the Finsler length measure and that its curvature governs the geodesic deviation on a Finsler space.

At the different stages of our review we will point out where the difficulties appear, which arise when one tries to use Finsler geometry to generalise semi-Riemannian geometry. It will become clear that one either excludes a huge class of interesting examples of Finsler geometries or that one immediately runs into ill-definedness of the geometry of the manifold along the appearing null-directions of the geometry. For the application of Finsler geometry in physics this is a serious issue since one is interested in a generalization of Lorentzian geometry, geometry based on a metric with signature $(-,+,+,+)$, such that causality is encoded into the background geometry and the null-directions are interpreted as the directions along which light respectively massless particles propagate. We will present how to overcome this limitation of the standard Finsler geometry framework by introducing our Finsler spacetimes in part II of this thesis. Finsler spacetimes turn out to be very general well-defined extension of Lorentzian metric manifolds and are able to serve as viable non-metric geometric background for physics.

In the next chapter 3 we will collect applications of Finsler geometry in physics in general and we discuss previous attempts to use Finsler geometry as geometry of spacetime.

2.1. Finsler functions

We give the definition of Finsler spaces which is the starting point of our extension of the framework in part II. Since from here on homogeneity properties of functions will play a special role in all derivations a short reminder of the Euler Theorem on homogeneous functions is presented.

In Finsler geometry one equips a smooth manifold with a general length measure instead of a metric. This length measure is defined through a tangent bundle function F, called Finsler function, such that the length of a curve $\gamma : \tau \mapsto \gamma(\tau)$ on the manifold in consideration can be represented by the parametrization invariant integral

$$S[\gamma] = \int \mathrm{d}\tau \; F(\gamma, \dot{\gamma}) \,. \tag{2.1}$$

The requirement to be parametrization invariant leads to the conclusion that F has to be homogeneous of degree one with respect to its second argument. Based on this length integral it is possible to describe the geometry of a manifold purely by tensors derived from the Finsler function.

Definition 2.1. ***Finsler function***
Let M be an n-dimensional manifold and TM its tangent bundle. A continuous real function $F : TM \to \mathbb{R}$ is called Finsler function if it satisfies:

- *F is smooth on the tangent bundle without the zero section $\widetilde{TM} = TM \setminus \{0\}$;*
- *F is homogeneous of degree one with respect to the fibre coordinates of TM*

$$F(x, \lambda y) = \lambda F(x, y) \,; \tag{2.2}$$

- *the Hessian g^F_{ab} of F^2 with respect to the tangent space coordinates has constant rank and is non-degenerate on $\widetilde{TM}$*

$$g^F_{ab} = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2 \,. \tag{2.3}$$

A Finsler function then leads immediately to the definition of Finsler spaces:

Definition 2.2. ***Finsler space***
An n-dimensional manifold M equipped with a Finsler function F is called a Finsler space (M, F).

The definition of the Finsler function ensures the well-definedness of the geometric objects introduced in the next subsection 2.2. An obvious example for a Finsler spacetime is a Finsler function induced by a symmetric $(0, n)$-tensor G on the manifold M

$$F(x, y) = \left(|G_{a_a \dots a_n}(x) y^{a_1} \dots y^{a_n}|\right)^{\frac{1}{n}} \,. \tag{2.4}$$

For $n = 2$ we obtain a Finsler space (M, F) which is identical to a metric manifold (M, g). The Hessian of F^2 with respect to the fibre coordinates, which is of major importance in what follows, then is equal to the metric g.

Already here we see that the example above does not yield a well-defined Finsler space for $n > 2$, if the null structure

$$N = \{(x, y) \in TM | F(x, y) = 0\} \,, \tag{2.5}$$

is non-triviel i.e., $N \neq \{(x, 0) \in TM\}$. Then F^2 is no longer smooth on $\widetilde{TM}$ but only on $\widetilde{TM} \setminus N$. But exactly such Finsler spaces one would consider as non metric generalizations of semi-Riemannian geometry because of the existence of a non trivial null structure. Now there are basically two options: Either one has to introduce further restrictions in the definition of the Finsler function (definition 2.1) to include these cases or one excludes them. Both is done in the literature [24, 25]. In the next section we will see that the non-differentiability of Finsler functions of this type with non trivial null structure leads to the non-existence of the geometric objects of the space along the null structure. Excluding a huge class of interesting examples or not being able to describe the geometry of a manifold everywhere, especially along its null structure, both is not satisfactory. The way out of this problem and to use Finsler geometry also as generalization of semi-Riemannian geometry is at the heart of our construction of Finsler spacetimes in part II.

Since the objects we will derive from the Finsler function in the next section inherit homogeneity properties, Euler's Theorem on homogeneous function will play an important role in future calculations.

Theorem 2.1. ***Euler's theorem on homogeneous functions***
Let $f : V \to \mathbb{R}$ be a homogeneous differentiable function of degree r, i.e., $f(\lambda x) = \lambda^r f(x)$, from some vector space V into the real numbers. The following holds

$$x^a \partial_a f(x) = r f(x)\,. \tag{2.6}$$

A proof can be found for example in [23]. For a function $H : TM \to \mathbb{R}$ which is homogeneous of degree r with respect to the tangent space coordinate y this implies $y^a \bar{\partial}_a H(x, y) = rH(x, y)$. The tensor components which we introduce next to describe the geometry of a manifolds will all satisfies such homogeneity properties.

2.2. Canonical geometric tensors

Here we introduce the geometric tensors which are derived from the Finsler function. Objects derived solely from the definition of Finsler spaces without further assumptions are the so called Finsler one-form, the Finsler metric, the Cartan tensor and the canonical Cartan one- and two-forms. They define d-tensor fields (definition 1.8) on $\widetilde{TM}$. At the end of this section we comment on their connection to metric Finsler spaces on which $F = \sqrt{|g_{ab}(x) y^a y^b|}$.

We list the objects directly determined by the definition of Finsler spaces according to the number of derivatives of F^2 with respect to the tangent space coordinates y.

Definition 2.3. ***Finsler one-form***
Let (M, F) be a Finsler space. The first derivative of F^2 with respect to the tangent space coordinates y defines the components p_a of a one-homogeneous d-one-form field

$$p_a(x, y) = \frac{1}{2} \bar{\partial}_a F^2(x, y)\,. \tag{2.7}$$

Observe that due to Euler's theorem on homogeneous function, equation (2.1), $y^a p_a = F^2$. From the viewpoint of physics the p_a are similar to canonical momenta.

Definition 2.4. ***Finsler metric***
Let (M, F) be a Finsler space. The Hessian of F^2 with respect to the tangent space coordinates y defines the components g^F_{ab} of a zero-homogeneous symmetric $(0,2)$-d-tensor field

$$g^F_{ab}(x, y) = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2(x, y)\,. \tag{2.8}$$

Again by Euler's theorem we have $y^a y^b g^F_{ab} = y^a p_a = F^2$. Observe that the Finsler metric is not a metric on the base manifold neither a metric on the tangent bundle. It can be seen as inner product between only horizontal, only vertical or between both types of vector fields, depending on which kind of $(0,2)$-d-tensor field is constructed from the coefficients in equation (2.8)

$$g^{F1} = g^F_{ab}(x,y)\mathrm{d}x^a \otimes \mathrm{d}x^b,\ g^{F2} = g^F_{ab}(x,y)\delta y^a \otimes \delta y^b,\ g^{F3} = g^F_{ab}(x,y)\delta y^a \otimes \mathrm{d}x^b\,. \tag{2.9}$$

In what follows we omit the index labelling the d-tensor type of g^F, it should always be clear from the context what is meant.

Definition 2.5. ***Cartan tensor***
Let (M,F) be a Finsler space. The third derivative of F^2 with respect to the tangent space coordinates y defines the components C_{abc} of a minus one-homogeneous symmetric $(0,3)$-d-tensor field

$$C_{abc}(x,y) = \frac{1}{4}\bar{\partial}_a\bar{\partial}_b\bar{\partial}_c F^2(x,y)\,. \tag{2.10}$$

For the Cartan tensor the Euler theorem yields by direct calculation $y^a C_{abc} = 0$. The Cartan tensor measures the deviation from metric geometry on a Finsler space in the sense that if $C_{abc}(x,y) = 0$ everywhere on $\widetilde{TM}$ the Finsler space is a metric manifold with metric components in a coordinate basis $g_{ab}(x) = g^F_{ab}(x)$, [22]. From the Finsler one-form components one defines:

Definition 2.6. ***Cartan one-form***
Let (M,F) be a Finsler space. The Finsler one-form components define the following one-form called Cartan one-form Θ

$$\Theta = p_a(x,y)\mathrm{d}x^a\,. \tag{2.11}$$

The Cartan two-form is derived from the Cartan one-form in the obvious way:

Definition 2.7. ***Cartan two-form***
Let (M,F) be a Finsler space. From the Cartan one-form one defines the Cartan two-form Ω, which we express in the horizontal-vertical basis (see equation (1.8))

$$\Omega = \mathrm{d}\Theta = \frac{1}{2}\delta_a\bar{\partial}_b F^2 \mathrm{d}x^a \wedge \mathrm{d}x^b + g^F_{ab}\delta y^a \wedge \mathrm{d}x^b\,. \tag{2.12}$$

The Cartan two-form will play a crucial role in finding a unique non-linear connection on a Finsler space determined by the Finsler function.

In the case of a Finsler function $F = \sqrt{|g_{ab}(x)y^a y^b|}$, which is induced by a metric the geometric objects introduced reduce to known tensor fields from Riemannian geometry: the Finsler one-form becomes the standard covector p associated to a vector y by the metric g; the Finsler metric $g^F(x,y)$ is identical to the metric $g(x)$ on the manifold which induces the Finsler function and the Cartan tensor C_{abc} vanishes.

We emphasize that all objects defined here depend on the fact that F^2 is differentiable. Since this is not guaranteed for Finsler functions with non trivial null structure precisely along the null structure, see the example in equation (2.4), the d-tensors cease to exist there. In the next subsection we will see that the non-linear connection which defines the geometry of a Finsler space is built from the tensors introduced here, which consequently also would not be defined where F^2 is not differentiable. As a consequence it would in particular not be possible to study null geodesics, see section 2.4. On our Finsler spacetimes in part II this problem does not occur.

2.3. Connections

The basic object which defines the geometry of a manifold, like the curvature and parallel transport, is a connection. In metric geometry we know that there is a unique distinguished covariant derivative: the Levi-Civita covariant derivative solely defined by a metric g on the manifold through the Christoffel symbols

$$\hat{\Gamma}^a{}_{bc}(x) = \frac{1}{2} g^{aq}(\partial_b g_{qc} + \partial_c g_{qb} - \partial_q g_{bc})\,. \tag{2.13}$$

In virtue of equation (1.37) they induce a linear connection on the tangent bundle with coefficients $N^a{}_b(x,y) = \hat{\Gamma}^a{}_{bc}y^c$, called Levi-Civita connection. In Finsler geometry there is a unique distinguished non-linear connection (definition 1.4) with coefficients $N^a{}_b(x,y)$ which are only homogeneous and not linear in their y dependence, called Cartan non-linear connection. It defines parallel transport on the manifold due to the induced covariant derivative (see definition 1.12 and the discussion below) and the curvature of the manifold. Despite this unique non-linear connection there exist several linear covariant derivatives on the tangent bundle, seen as manifold itself, derived from the Finsler function. These linear covariant derivatives respect the horizontal-vertical structure induced by the non-linear connection in the sense that the covariant derivative of a horizontal, respectively vertical, (co-)vector field is again horizontal, respectively vertical. We will discuss the Cartan and Berwald linear covariant derivative in detail, since they will appear naturally in several calculations. The others appearing in the literature will be mentioned briefly for completeness. Having introduced the unique non-linear connection and the linear covariant derivatives we will investigate their relation to the Levi-Civita covariant derivative in metric geometry.

2.3.1. Cartan non-linear connection

We introduce the fundamental object which defines the geometry of Finsler spaces, the unique Cartan non-linear connection.

Definition 2.8. ***Cartan non-linear connection one-form***
Let (M,F) be a Finsler space. The Cartan non-linear connection one-form on the tangent bundle is defined by the connection coefficients

$$N^a{}_b(x,y) = \Gamma^a{}_{bc}(x,y)y^c - g^{F\,aq}(x,y)C_{qbc}(x,y)\Gamma^c{}_{pq}(x,y)y^p y^q\,. \tag{2.14}$$

where $g^{F\,ab}$ is the inverse of the Finsler metric and $\Gamma^a{}_{bc}(x,y) = g^{F\,aq}(\partial_b g^F_{qc} + \partial_c g^F_{qb} - \partial_q g^F_{bc})$.

Important properties of the connection, which can be proven by a direct calculation, are the one-homogeneity with respect to the tangent space coordinates $N^a{}_b(x,\lambda y) = \lambda N^a{}_b(x,y)$ and the fact that the Finsler function F is horizontally constant $\delta_a F = 0$, where δ_a denotes the horizontal derivative respectively the horizontal basis introduced in equation (1.7).

Theorem 2.2. ***Uniqueness of the Cartan non-linear connection***
Let $N^a{}_b(x,y)$ be the connection coefficients of the Cartan non-linear connection, ∇ be the corresponding dynamical covariant derivative (definition 1.12) and let p be the projector onto the horizontal bundle, the union of all horizontal tangent spaces (definition 1.5) defined by the connection and let Ω be the Cartan two-form. The Cartan non-linear connection is the unique connection satisfying

$$\nabla g^F_{ab}(x,y) = 0,\ \Omega(p(X), p(Y)) = 0 \forall\ X, Y \in TTM\,. \tag{2.15}$$

A proof of this theorem can be found in [22]. The basic idea is that the first condition determines the symmetric part $N_{ab} + N_{ba}$ and the second the anti symmetric part $N_{ab} - N_{ba}$, where the first index of $N^a{}_b$ was raised and lowered with the Finsler metric. The conditions in equation (2.15) which uniquely determine the $N^a{}_b$ given in equation (2.14) are generalizations of the metric compatibility and torsion freeness conditions which uniquely define the Levi-Civita covariant derivative in case of metric geometry. A more compact form of the non-linear connection coefficients is given by

$$N^a{}_b = \frac{1}{2}\bar{\partial}_b\big(\Gamma^a{}_{ij}y^i y^j\big)\,. \tag{2.16}$$

The non-linear curvature of the Cartan non-linear connection is by definition (1.6),

$$R^a{}_{bc} = \delta_c N^a{}_b - \delta_b N^a{}_c\,. \tag{2.17}$$

As discussed in chapter 1, a connection splits the tangent bundle of the tangent bundle into horizontal and vertical part; so does the Cartan non-linear connection. We now introduce the different linear covariant derivatives on the tangent bundle which respect this split.

2.3.2. Linear connections

Regarding the tangent bundle as a manifold in its own right, one can associate linear covariant derivatives to this manifold. In Finsler geometry there exist special covariant derivatives on TM which are compatible with the structure induced by the Cartan non-linear connection. This can be understood in the following way. Consider the tangent bundle in induced coordinates (x,y) with corresponding horizontal-vertical basis of its tangent bundle TTM: $\{E_A\} = \{\delta_a, \bar{\partial}_b\}$ with $A = 1,\ldots,2n$, the first n basis elements denote the horizontal basis, the indices from $n+1$ to $2n$ the vertical basis, small indices only run from 1 to n. A general linear covariant derivative on the tangent bundle ∇^{TM} is defined by its coefficients Ξ, which can be expressed in the horizontal-vertical basis

$$\nabla^{TM}_{E_A} E_B = \Xi^C{}_{AB} E_C = \hat{\Xi}^c{}_{AB}\delta_c + \bar{\Xi}^c{}_{AB}\bar{\partial}_c\,. \tag{2.18}$$

As for every linear covariant derivative one obtains the associated linear curvature $(1,3)$-tensor of the tangent bundle by

$$R^{TM}(E_A, E_B)E_C = \nabla^{TM}_{E_A}\nabla^{TM}_{E_B}E_C - \nabla^{TM}_{E_B}\nabla^{TM}_{E_A}E_C - \nabla^{TM}_{[E_A,E_B]}E_C\,. \tag{2.19}$$

We say the covariant derivative respects the horizontal-vertical split in case

$$\hat{\Xi}^c{}_{AB} = \begin{cases} V^c{}_{ab} & \text{for } A,B = 0,\dots,n \text{ both horizontal} \\ W^c{}_{ab} & \text{for } A = n+1,\dots,2n \text{ vertical and } B = 0,\dots,n \text{ horizontal} \\ 0 & \text{else} \end{cases} \tag{2.20}$$

$$\bar{\Xi}^c{}_{AB} = \begin{cases} V^c{}_{ab} & \text{for } A = 0,\dots,n \text{ horizontal and } B = n,\dots,2n \text{ vertical} \\ W^c{}_{ab} & \text{for } A,B = n+1,\dots,2n \text{ both vertical} \\ 0 & \text{else}\,. \end{cases} \tag{2.21}$$

Only then is the covariant derivative of a vertical basis element again vertical, and the covariant derivative of a horizontal basis element again horizontal. Moreover, the covariant derivative with respect to a horizontal respectively vertical direction then comes always with the same coefficient V respectively W. Accordingly the components of the curvature of the covariant derivatives respecting the horizontal-vertical split decay into three sets in the following way

$$\begin{aligned} R^{TM}(\delta_a,\delta_b)\delta_c &= R^q{}_{cab}\delta_q, \quad R^{TM}(\delta_a,\delta_b)\bar{\partial}_c = R^q{}_{cab}\bar{\partial}_c, \\ R^{TM}(\bar{\partial}_a,\delta_b)\delta_c &= P^q{}_{cab}\delta_c, \quad R^{TM}(\bar{\partial}_a,\delta_b)\bar{\partial}_c = P^q{}_{cab}\bar{\partial}_c, \\ R^{TM}(\bar{\partial}_a,\bar{\partial}_b)\delta_c &= S^q{}_{cab}\delta_c, \quad R^{TM}(\bar{\partial}_a,\bar{\partial}_b)\bar{\partial}_c = S^q{}_{cab}\bar{\partial}_c\,. \end{aligned} \tag{2.22}$$

The difference between the linear covariant derivatives appearing in the literature for Finsler geometry lies in the choice of the coefficients V and W. For us the two most interesting covariant derivatives are the Cartan and the Berwald linear covariant derivatives.

Definition 2.9. ***Cartan linear covariant derivative***
The Cartan linear covariant derivative ∇^{CL} is a covariant derivative on the tangent bundle of a Finsler space which respects the horizontal and vertical split induced by the Cartan non-linear connection (definition 2.8) in the sense of equations (2.20) and (2.21). It is defined by the coefficients

$$V^c{}_{ab} = \Gamma^{\delta c}{}_{ab} = \frac{1}{2}g^{Fcq}\big(\delta_a g^F_{bq} + \delta_b g^F_{aq} - \delta_q g^F_{ab}\big), \quad W^c{}_{ab} = g^{Fcq}C_{abq}\,, \tag{2.23}$$

where C_{abq} is the Cartan tensor (see definition 2.5) and we call Γ^δ the δ-Christoffel symbols.

The Cartan linear covariant derivative has the property that it leaves the Finsler metric horizontally as well as vertically covariant constant, i.e.

$$\nabla^{CL}_{\delta_a} g^F_{ab} = 0, \quad \nabla^{CL}_{\bar{\partial}_a} g^F_{ab} = 0\,, \tag{2.24}$$

no matter which kind of d-tensor (see equation (2.9)) the components g^F_{ab} define. The horizontal part of the curvature $R(\delta_a,\delta_b)(\cdot)$ of the Cartan linear covariant derivative is given by

$$R^{CLq}{}_{cab} = \delta_a\Gamma^{\delta q}{}_{cb} - \delta_b\Gamma^{\delta q}{}_{ca} + \Gamma^{\delta q}{}_{ma}\Gamma^{\delta m}{}_{cb} - \Gamma^{\delta q}{}_{mb}\Gamma^{\delta m}{}_{ca} - C^q{}_{cm}R^m{}_{ab}\,. \tag{2.25}$$

It is linked to the non-linear curvature $R^a{}_{bc}$ of the Cartan non-linear connection, see [22], through

$$R^q{}_{ab} = -R^{CLq}{}_{cab}y^c\,. \tag{2.26}$$

The other covariant derivative of interest is defined as follows:

Definition 2.10. ***Berwald linear covariant derivative***
The Berwald linear covariant derivative ∇^B is a covariant derivative on the tangent bundle of a Finsler space which respects the horizontal and vertical split induced by the Cartan non-linear connection (definition 2.8) in the sense of equations (2.20) and (2.21). It is defined by the coefficients

$$V^c{}_{ab} = \bar{\partial}_a N^c{}_b, \quad W^c{}_{ab} = 0\,, \tag{2.27}$$

It does not leave the metric covariant constant and its horizontal part of the curvature $R(\delta_a, \delta_b)(\cdot)$ can be expressed with the help of the non-linear connection coefficients

$$R^{Bq}{}_{cab} = \delta_a(\bar{\partial}_b N^q{}_c) - \delta_b(\bar{\partial}_a N^q{}_c) + (\bar{\partial}_a N^q{}_m)(\bar{\partial}_b N^m{}_c) - (\bar{\partial}_b N^q{}_m)(\bar{\partial}_a N^m{}_c)\,. \tag{2.28}$$

It can be derived from the curvature of the Cartan non-linear connection by

$$\bar{\partial}_c R^q{}_{ab} = -R^{Bq}{}_{cab}\,. \tag{2.29}$$

During upcoming calculations these two linear covariant derivatives and their horizontal curvatures will naturally appear and will be used as abbreviations. For completeness we list the other linear covariant derivatives discussed in the literature in the context of Finsler spaces by giving their coefficients V and W:

Name	$V^a{}_{bc}$	$W^a{}_{bc}$
Chern-Rund covariant derivative ∇^{CR}	$\Gamma^{\delta a}{}_{bc}$	0
Hashiguchi covariant derivative ∇^H	$\partial_a N^b{}_c$	$g^{Fcq} C_{abq}$

(2.30)

In case the Finsler space is induced by a metric $F = \sqrt{g_{ab}(x) y^a y^b}$ the different covariant derivatives and the non-linear connection contain the same information. For the non-linear connection the conditions in equation (2.15) become the usual metric compatibility and torsion freeness conditions of a covariant derivative on the manifold, the connection coefficients become $N^a{}_b(x,y) = \hat{\Gamma}^a{}_{bc} y^c$ constructed from the usual Christoffel symbols $\hat{\Gamma}^a{}_{bc}$, and the dynamical covariant derivative defines the standard Levi-Civita covariant derivative on the manifold in the sense of equation (1.37). As presented in equation (1.23) the curvature for a non-linear connection with coefficients $N^a{}_b(x,y) = \hat{\Gamma}^a{}_{bc} y^c$ becomes the standard Riemann curvature tensor. The different linear covariant derivatives all become equal since their coefficients V all become $V^a{}_{bc} = \hat{\Gamma}^a{}_{bc}$ and the coefficients W all vanish, since the Cartan tensor vanishes. The only non-vanishing component of their curvature is $R^a{}_{bcd}$ which becomes identical to the usual Riemann curvature tensor of the base manifold. Hence in the metric Finsler space case all geometric information is encoded in the usual Christoffel symbols, while in the general Finsler case one has the Cartan non-linear connection as fundamental object and additional linear covariant derivatives on the tangent bundle containing further information.

The non-linear connection as well as the linear covariant derivatives discussed here are well-defined as long as F^2 is differentiable. But the non-linear connection coefficients $N^a{}_b$ as well as the coefficients $V^a{}_{bc}$ and $W^a{}_{bc}$ are no longer well-defined when F^2 fails to be differentiable. This is the same problem as discussed for the other geometric objects previously. Again it becomes clear that Finsler geometry in the Finsler space formulation is not a suitable non metric extension for semi-Riemannian geometry. Our Finsler spacetimes presented in part II of this thesis as extension of Finsler spaces are.

In the next section we will demonstrate explicitly that it would be impossible to study null-geodesics on Finsler geometries built from Finsler functions with non trivial null structure based on the definition of Finsler spaces. We stress again that from the viewpoint of physics this is a severe problem, since null geodesics are interpreted as trajectories along which light propagates.

2.4. Finsler geodesics and geodesic deviation

The geometry of an n-dimensional manifold influences the behaviour of embedded lower dimensional objects; the most prominent ones being embedded curves. Here we will discuss curves which extremise the distance between two points of a Finsler space, leading to the notion of Finsler geodesics and how the curvature of the Finsler space influences the distance of nearby geodesics. The latter leads to the Finsler space geodesic deviation equation. Moreover we will see explicitly why the Finsler geometry framework in the literature is not sufficient to study Finsler functions with non trivial null structure as one would like to do in physics. In part II we will discuss that Finsler geodesic model the trajectories of test bodies and that the geodesic deviation is interpreted as relative gravitational acceleration between them. Here we discuss these concepts mathematically in the context of Finsler spaces.

The heart of Finsler geometry is the Finsler length measure for curves $\gamma : \tau \mapsto \gamma(\tau)$ on the manifold displayed in equation (2.1). There exist, as in metric geometry, distinguished curves which extremise this integral. By the virtue of variational calculus these curves are the solutions of the Euler-Lagrange equations

$$\frac{d}{d\tau}\bar{\partial}_a F(\gamma,\dot{\gamma}) - \partial_a F(\gamma,\dot{\gamma}) = 0\,. \tag{2.31}$$

Definition 2.11. *Finsler geodesics*
Let (M,F) be a Finsler space. A curve γ on M is called Finsler geodesic in case it extremises the Finsler length integral defined in equation (2.1), i.e., it solves equation (2.31).

Consequently, to be a Finsler geodesic, a curve γ on M has to satisfy equation (2.31) which can be written nicely with the help of the Cartan non-linear connection coefficients, respectively with the dynamical covariant derivative, as

$$g^F_{ab}(\ddot{\gamma}^b + N^b{}_q(\gamma,\dot{\gamma})\dot{\gamma}^q) = g^F_{ab}\nabla\dot{\gamma}^H(\gamma,\dot{\gamma}) = \partial_a F(\gamma,\dot{\gamma})\frac{d}{d\tau}F(\gamma,\dot{\gamma})\,, \tag{2.32}$$

where $\dot{\gamma}^H$ denotes the horizontal lift (definition 1.10) of $\dot{\gamma}$. A proof can be found in [12, 22]. The factor on the right hand side of the geodesic equation (2.32) is due to the fact that we did not fix the parametrization of the curve so far. Since the Finsler length integral is reparametrization-invariant we may consider γ in any parametrization we like. The most common one is the so called arclength parametrization fixed by the fact that $F(\gamma,\dot{\gamma}) = 1\;\forall\,\tau$. In arclength parametrization the geodesic equation becomes

$$g^F_{ab}(\ddot{\gamma}^b + N^b{}_q(\gamma,\dot{\gamma})\dot{\gamma}^q) = g^F_{ab}(\nabla\dot{\gamma}^H)^b(\gamma,\dot{\gamma}) = 0\,, \tag{2.33}$$

and by comparison with definition 1.13 we see that arclength parametrized Finsler geodesics are autoparallels of the Cartan non-linear connection. In consequence, Finsler geodesics are curves whose natural lift to the tangent bundle $\tilde{\gamma} = (\gamma, \dot{\gamma})$ has a purely horizontal tangent $\dot{\tilde{\gamma}} = \dot{\gamma}^a \delta_a$. From the discussions about the metric limit and the problems arising for Finsler geometries based on a Finsler function with non trivial-null structure the following two remarks arise: first it is easy to see that in case of a metric Finsler space the Finsler geodesics become the geodesics of metric geometry and the geodesic equation is the metric geodesic equation expressed in terms of the Levi-Civita covariant derivative; second it also becomes clear that for a Finsler function with non trivial null structure of the form in equation (2.4) the geodesic equation cannot be studied for curves γ for which $F(\gamma, \dot{\gamma})$ vanishes. The Finsler metric as well as the non-linear connection coefficients are not well-defined. For physics this is a severe problem, since we want to interpret such curves as trajectories of light.

Considering nearby geodesics or, equivalently autoparallels, being described by a smooth variation Γ, i.e., a smooth surface

$$\Gamma : \mathbb{R} \times \mathbb{R} \to M, \tag{2.34}$$

allows us to discuss their deviation induced by the geometry of the manifold. Let $\Gamma(\tau, s)$ for fixed s be an arclength parametrized geodesic with tangent $\partial_\tau \Gamma(\tau, s)$. The derivative of Γ with respect to s is called deviation vector field $V = \partial_s \Gamma(\tau, s)$. The length of its integral curves measure the distance between the geodesic $\Gamma(\tau, s_0)$ and $\Gamma(\tau, s_1)$ for given τ, the change of this distance to first order in the parameter s is given by V. Studying the condition that the curves $\Gamma(\tau, s_0)$ and $\Gamma(\tau, s_1)$ shall be nearby geodesics, hence considering a small difference $s_0 - s_1 = \epsilon$, one finds that V must satisfy, see again [12, 22] for a detailed derivation,

$$\nabla\nabla V^a(\tau, s_0) + R^a{}_{bc} \dot{x}^b V^c(\tau, s_0) = 0\,. \tag{2.35}$$

The tensor $R^a{}_{bc}$ is the curvature tensor of the Cartan non-linear connection and we used $\dot{x}^b = \partial_\tau \Gamma(\tau, s_0)^b$ as abbreviation of the tangent along the geodesic $\Gamma(\tau, s_0)$. Equation (2.35) is called geodesic deviation equation. Physically the geodesic deviation is interpreted as the relative gravitational acceleration of nearby test particles in a given gravitational field.

This concludes our description of the Finsler geometry framework how it commonly can be found in the literature. In the next section we give a broad overview where Finsler geometry appears in physics.

3. Finsler geometry in physics

The applications of Finsler geometry in physics fall basically into two subjects. On the one hand it appears as an effective geometric description of point particle mechanics, point particle limits of field theories, like ray theory in media, and as a geometric description of fluid mechanics. Examples of this kind are collected in section 3.1. On the other hand there are attempts to use Finsler geometry as the geometry of spacetime which describes gravity, as we will see in section 3.2. We will mention the two most prominent Finsler length measures in this context, which include not only a metric, but also a vector field as building ingredients; we will encounter some work where Finsler geometry is used as a phenomenological tool to describe dark matter, dark energy, as well as quantum gravity effects and we discuss approaches to find field equations determining Finsler geometries dynamically.

The applications of Finsler geometry as spacetime geometry give rise to a number of questions concerning the equations used to determine the geometry of spacetime, the existence of the geometric objects appearing, the description of observers and the coupling of matter fields. We will summarise these questions in section 4.2 of the next chapter. In part II of this thesis the Finsler spacetime framework is developed as an answer to these issues.

3.1. Geometry of ray and particle limits

From the physics point of view the Finsler length measure in equation (2.1) is nothing but an action for a point particle and the Finsler geodesic equations (2.31) are the corresponding equations of motion. One simply uses the Lagrange function of the system in consideration as Finsler function F. For the free particle metric geometry usually suffice, but when one considers the point particle limits of general field theories or the analysis of the ray limit of the propagation of waves in media, one naturally encounters Hamiltonians and respectively Lagrangians which lead to a Finsler geometric description of these particle, or ray, limits. Here we present some concrete examples.

The most prominent example of a field theory leading to light ray propagation along Finsler geodesics are the equations of motion of linear electrodynamics in media. In its most general form it is described by so called area metric electrodynamics, see [26] and references therein for all details. There the relation between the electromagnetic field strength tensor F and the induction H is given by a general $(4,0)$ tensor field $\chi^{[ab][cd]}$, anti-symmetric in each pairs of indices but symmetric in the pairs, called area metric:

$$H^{ab} = \chi^{abcd} F_{cd}\,, \tag{3.1}$$

The area metric encodes the inductive properties of the medium in consideration. The field equations for H and F are given by $\mathrm{d}F = 0$ and $\mathrm{d}H = 0$, where the principal symbol of the latter equation, which describes the ray approximation of the field propagation, yields the so called Fresnel polynomial $P[\chi](k)$ which is of degree four in k and is interpreted as Hamiltonian describing the propagation of light rays [27]. After a Legendre transformation one obtains the corresponding non-metric Lagrangian, and it turns out that the light ray approximation of area metric electrodynamics yields propagation of light along Finsler geodesics. This is just one example where a non-metric field theory yields propagation of its ray or point particle approximation along Finsler geodesics. This principle can be extended to a wide class of causally well behaved field theories as discussed in [28]. In these frameworks, the Finsler geometry emerging through the point particle limit is not necessarily considered to be the geometry of spacetime, so the problem of non-existing geometric objects caused by possible non-differentiability of the Finsler function does not appear.

In a similar spirit but a complete different topic Finsler geometry appears in the description of seismic rays, respectively in the geometric optics approximation of waves in media, see for example [29]. There the starting point is to consider the Cauchy equation which describes the displacement $u(t,x)$ of a continuum

$$\rho(x)\frac{d^2u^j(t,x)}{dt^2} = \partial_i\sigma^{ij}(x)\,. \tag{3.2}$$

The indices i,j run only from 1 to 3 since one describes the propagation of waves through a spatial medium in absolute non-relativistic time. The quantity $\rho(x)$ is the mass density and σ^{ij} denotes the stress tensor of the medium. As in the case of general field theories described above, on studies the solvability condition of this equation and obtains a Hamiltonian $H(p)$ which may be a higher then second order polynomial in the momenta, depending on the properties of the stress tensor. Via Legendre transformation this Hamiltonian leads to a Lagrangian which describes the propagation of rays through the medium along Finsler geodesics. In [30] the objects appearing in such Finsler geometries are related directly to slowness and ray-velocity surfaces coming from the properties of the medium. More concretely in [31] sound waves in media are studied with the help of different techniques among which one finds the Finsler geometric description. This analysis of the Cauchy equation works in a fixed 3+1 split of spacetime with a positive definite spatial Finsler structure, therefore usually no problems with the differentiability of the Finsler function appear.

Since for the description of rays from general Hamiltonians Finsler geometry has been found to be a useful tool, Fermats principle has been in discussed in this context, i.e., that the path along which particles, respectively, rays, travel are geodesics of the Finsler length measure. For the propagation of light on Finsler geometries this analysis was done by Perlick in [32] and for the propagation of rays in anisotropic inhomogeneous media by Cerveny in [33].

3.2. Spacetime geometry

In contrast to the appearance of Finsler geometry in the study of Hamiltonians and Lagrangians derived from partial differential equations describing physical systems, there exist several attempts to consider Finsler geometry as the geometry of spacetime, on a fundamental and on a phenomenological level.

3.2.1. Anisotropic geometry and electrodynamics

The two best known Finsler length measures applied in this context are the so called Randers and Bogoslovsky length elements. They contain a one-form A, respectively a vector field V in addition to a metric g.

The Randers length element is given by

$$F_R = \sqrt{|g_{ab}(x)y^a y^b|} + A_a(x)y^a \,. \tag{3.3}$$

It was introduced in 1941 by G. Randers [34] on a four dimensional spacetime in order to study the inclusion of a preferred time direction into the geometry of spacetime. Finsler spaces with a Randers length element are called Randers spaces nowadays. The geodesics of Randers spaces are solutions to the Lorentz Force equations, hence can be seen as paths of charged particles in an external electromagnetic field described by the vector potential A. Therefore one can view Randers spaces as an approach to understand both electromagnetism and gravity, as being described by an anisotropic geometry, where the vector potential A induces the anisotropy. Randers suggests dynamics for such a length element by mapping it to a five dimensional metric spacetime and employing five dimensional Einstein equations. The Randers length measure is often considered as a simple example of a Finsler length measure which is not metric.

Another famous anisotropic spacetime model was introduced by Bogoslovsky 1977 in [35]. Studying the most general transformations which do not change the massless wave equation he deduces deformed Lorentz transformations which leave the following length measure invariant

$$F_B = (|V_a y^a|)^r (\sqrt{|\eta_{ab}y^a y^b|})^{1-r},\ \eta_{ab}V^a V^b = V^a V_a = 0\,. \tag{3.4}$$

It contains a null-vector V of the Minkowski metric η which introduces an anisotropy into the geometry. To obtain dynamics for such a length measure Bogoslowsky suggested to replace η by a general Lorentzian metric g determined by the Einstein equations and to treat the vector field as additional matter field [36]. The term very special relativity was introduced by Cohen and Glashow [37] where they study field theories which are invariant under the subgroup of the Poincare group formed by similarly deformed Lorentz transformations. This group theoretical approach was then connected to the Finsler length element of Bogoslovsky by Gibbons, Gomis and Pope in [38]. As a continuation of this idea there exist attempts to construct a general very special relativity and apply it to cosmology for example by changing the Minkowski metric η in Bogoslovskys length element to a Friedmann-Robertson Walker metric by Stavrinos, Kouretsis and Stathakopoulos [39].

Here we like to point out that in these two examples, the Randers and the Bogoslovsky length measure, the geometric tensors of Finsler geometry, introduced in section 2.2, do not exist for all $y \in TM \setminus \{0\}$. As mentioned many times before during the review of Finsler spaces in the previous section the problem is the failure of differentiability of the length measures on certain subsets of $TM \setminus \{0\}$. For F_B this subset includes the null structure.

We will revisit these anisotropic length measures when we discuss examples of Finsler spacetimes in section 5.5. It will turn out that we can lift the Bogoslovsky length measure to an equivalent much smoother one. Anyway this lifted length measure as well as the Randers length measure will still not have the desired smoothness properties. Nevertheless these length measures will inspire us to built an anisotropic length measure from a metric and a one-form, respectively a vector field, which posses all properties to be viable for physics.

3.2.2. Phenomenology of quantum gravity, string theory and dark matter

The possible violation of Lorentz invariance on a spacetime geometry level, not emergent as particle limit of field theories or from physics inside media, is one field in theoretical physics where Finsler geometry is used in certain different topics. Lorentz invariance violations manifesting themselves by modifications of the usual metric dispersion relation are assumed to be generic feature of phenomenological quantum gravity models. In [40] Girelli and co-authors connect these modified dispersion relations to Finsler functions which describe the trajectories of particles whose momentum obeys the modified dispersion relation. This is mathematically similar to the application from the analysis about rays or particles limits from field theories, but differently motivated. In connection to String Theory and Leptogenesis Finsler geometry appears in [41] by Mavromatos and Sarkar where the geometry derived from a string background leads to different non-metric dispersion relations for fermions and antifermions. Moreover in Lorentz invariance violating extensions of the Standard model Finsler length measures appear again as particle Lagrangians as discussed for example by Kostelecky in [42].

Not from Lorentz invariance violation but motivated from mismatch of observational data with predictions of general relativity, like the observations of dark matter and dark energy, the freedom of Finsler geometry is used to fit these data. In [43] Chang and Li develop a Finsler length measure and dynamical equations such that the result coincides with the so called MOdifed Newton Dynamics. This work is then modified and further developed to explain the dark matter observations of the bullet cluster [44].

These phenomenological models do not do not discuss the issue of non-differentiability in detail neither give a precise definition of observers and their measurements, which is important for the comparison with experimental observations.

3.2.3. Dynamics for Finsler geometries

For a proper application of Finsler geometry to be the geometry of spacetime field equations are required which determine the geometry dynamically. In the literature different equations are discussed.

The method used in the application of Bogoslovskys length measure to cosmology [39] is to apply the osculating formalism of Asanov [24] to derive the field equations. In the osculating formalism one replaces the independent tangent bundle coordinates (x, y) on which the geometrical objects of a Finsler space depends by $(x, y(x))$, where $y(x)$ is a vector field on the base manifold which has to be determined or to be given a priori. The fairly complicated gravitational equations derived from the osculating curvature scalar by Asanov, based on the curvature of the Cartan linear connection $R^q{}_{cab}(x, y)$ (see equation 2.25),

$$\mathcal{R}(x) = R(x, y(x)) = g^{Fcb}(x, y(x))R^q{}_{cqb}(x, y(x)) \tag{3.5}$$

reduces in the framework of [39] after some approximations to equations formally identical to the Einstein equations where instead of the Lorentzian metric $g_{ab}(x)$ the osculating Finsler metric $g^F_{ab}(x, y(x))$ is used.

Another way to arrive at equations formally similar to the Einstein equations, but living on the tangent bundle, see [22], is to use the Cartan linear connection introduced in equation (2.24). One proceeds by constructing the tangent bundle curvature tensor of the Cartan linear connection and the corresponding Ricci tensor and Ricci scalar of the manifold TM. These are then combined to the Einstein Tensor and equated with an energy momentum tensor on the tangent bundle to yield Einstein equations on TM. Using the notation from equation (2.22) for the different curvature components the equations decay into

$$R^q{}_{iqj} - \frac{1}{2}(g^{F\,ab} R^q{}_{aqb} + g^{F\,ab} S^q{}_{aqb}) g_{ij} \quad = \quad kT_{ij} \tag{3.6}$$

$$S^q{}_{iqj} - \frac{1}{2}(g^{F\,ab} R^q{}_{aqb} + g^{F\,ab} S^q{}_{aqb}) g_{ij} \quad = \quad kT_{\bar{i}\bar{j}}, \text{ and } P^q{}_{iqj} = kT_{i\bar{j}}\,, \tag{3.7}$$

where k is some gravitational constant, T_{ij} are the horizontal-horizontal, $T_{i\bar{j}}$ are the horizontal-vertical and $T_{\bar{i}\bar{j}}$ are the vertical-vertical components of some energy momentum tensor T on TM.

A third, yet different, dynamical equation for Finsler geometries is derived from the geodesic deviation equation (2.35) by Rutz [45]. Here the reasoning follows Piranis argument comparing the deviation of neighbouring trajectories in Newtonian gravity and in general relativity outside of matter. Doing the same on the level of general relativity and Finsler geometry, namely comparing the geodesic deviation equation of both frameworks, it is concluded that outside matter the dynamical equation in terms of the non-linear curvature $R^a{}_{bc}$ is

$$R^a{}_{ab} y^b = 0\,. \tag{3.8}$$

So there is a variety of suggestions of dynamical equations determining a Finslerian geometry. All of the different dynamical equations presented here make use of the geometric objects derived from a Finsler function. The Cartan non-linear connection and the curvature of the Cartan non-linear or linear connections appear explicitly. In the spirit of the discussion during the previous chapter this means that one has to be very careful for which Finsler length measures and on which subset of $TM \setminus \{0\}$ the equations are valid. Moreover, in all these approaches a discussion of a suitable matter coupling is missing. For the first two equations it is not discussed what the energy momentum tensor means in the Finslerian geometry context and the last equation is anyway considered to be valid only in the vacuum case.

We have seen that Finsler geometry is a useful tool in the analysis of the behaviour of solutions of partial differential equations on the one hand, and that it is a hopeful candidate to refine our understanding of the geometry of spacetime, hence gravity, on the other hand. What is missing, especially for the application as spacetime geometry, is a rigorous general framework circumventing the mathematical difficulties of indefinite Finsler length measures, rigorously derived dynamics which determine the Finsler geometry of spacetime from its matter content, and therefore a discussion of matter fields on a Finsler geometric background as well as the description of observers. Our Finsler spacetime framework in part II of this thesis is provides these missing ingredients.

Before we present the Finsler spacetime framework, we study the Einstein–Hilbert action from the viewpoint of Finsler geometry. The insight of this study in the next chapter is that the Einstein–Hilbert action $S_{EH}[g]$, determining the dynamics a metric g is just a special case of a more general action $S_{FS}[F]$ determining the dynamics for a, in general non-metric Finsler, function F. This result triggered the systematic analysis of questions, among them the open issues mentioned above, which lead to the development of Finsler spacetimes.

4. From Finsler spaces to Finsler spacetimes

In chapter 2 we have seen that the geometry of a manifold can be described much more generally by Finsler geometry then by metric geometry. This works without any problems as long as one works in the Finsler spaces framework as generalization of Riemannian geometry, but is not straightforward when one wants to generalise semi-Riemannian or, especially for the application in physics, Lorentzian geometry. In physics Finsler geometry is used for different purposes mentioned in the previous chapter 3. But, for its applications as spacetime geometry it is unsatisfactory that so far no framework has been developed which treats the mathematical issues in indefinite Finsler geometry on the one hand and on the other hand provides a complete treatment of the use of Finsler geometry as description of gravity. This should include dynamics, matter coupling and the description of observers. As a preliminary step towards this goal a toy model has been developed in the diploma theses [12] and [13]. Dynamics for Finsler spaces were found directly from the Einstein Hilbert–action, which from the Finsler geometric point of view is just a special case of an action which determines dynamics for general non-metric Finslerian geometries.

In section 4.1 we will illustrate the reinterpretation of the Einstein–Hilbert action as an action for a metric Finsler function. The associated dynamics turn out to be equivalent to the Einstein equations which determine a metric of a Riemannian manifold. Moreover we show that the Einstein Hilbert–action in its Finsler space interpretation is just a special case of more general action determining dynamics of general non-metric Finsler spaces. In section 4.2 we will summarise the questions concerning the application of Finsler geometry as the geometry of spacetime.

These will motivate the detailed study of the questions and issues which arise when one wants to use Finsler geometry as geometry of spacetime which lead to the development of our Finsler spacetime formalism presented in part II.

4.1. The Einstein–Hilbert action from a Finslerian viewpoint

Before we will rewrite the Einstein–Hilbert action for Finsler spaces we briefly review its origin and its importance in physics.

The most successful understanding of gravity for over a hundred years now is provided by Einstein's theory of general relativity [46]. It tells us that gravity is described by the curvature of a four dimensional Lorentzian spacetime sourced by the energy-momentum content existing on this geometrical background where physics takes place. The dynamical equations determining this interplay between matter and geometry are the Einstein equations

$$R_{ab} - \frac{1}{2}R g_{ab} = \frac{8\pi G}{c^4} T_{ab}\,, \tag{4.1}$$

where R_{ab} and R are the Ricci tensor respectively the Ricci scalar, g_{ab} is the spacetime metric and T_{ab} is the energy-momentum tensor. These equations are most effectively derived from an action composed of the Einstein–Hilbert action containing the Ricci-Scalar as Lagrangian and a matter action containing the matter Lagrangian L_M of choice

$$S[g, \phi_i] = \int_M \mathrm{d}^4x \sqrt{|\det g|} \left(\frac{c^4}{16\pi G} R + L_M[g, \phi_i] \right) . \tag{4.2}$$

Variation with respect to the spacetime metric yields the Einstein equations, as it is long known and can be found in textbooks like [21]. Recall that the form of the action and the Einstein equations is independent of the signature of the metric. In physics one considers a metric with Lorentzian signature $(-, +, +, +)$ but one would produce the same equations for a metric with any signature, if one would consider the action in equation (4.2) on any metric manifold.

This is important, since due to the difficulties of indefinite Finsler geometry, we want to consider the Einstein–Hilbert action during this section as an action on a four dimensional Riemannian manifold (M, g), i.e. on a manifold with metric of definite signature $(+, +, +, +)$. Equivalently, in the language of Finsler spaces, one can say we consider the Einstein–Hilbert action on a Finsler space $(M, F = \sqrt{g_{ab}y^a y^b})$. For the proofs of the results presented during this section and further details we refer to [12].

The insight we demonstrate now is that the Einstein–Hilbert action on a metric Finsler space can be understood as an action for the Finsler function F on the tangent bundle TM, not merely as an action for the metric g on the manifold M. At each point of M we consider the sphere S_p in the tangent space T_pM defined by

$$S_p = \left\{ y \in T_pM | \sqrt{g_{|p}(y, y)} = 1 \right\} . \tag{4.3}$$

It is now just a short calculation to rewrite the Einstein–Hilbert action as an action on the Sphere-Bundle Σ, which is the subset of the tangent bundle obtained by the union over all points of the manifold over the Spheres $S_p \subset T_pM$. By introducing suitable spherical coordinates $(x^a, \theta^\alpha); a = 0, \dots, 3; \alpha = 1, \dots 3$ on the seven dimensional manifold Σ, derived from the induced coordinates (x^a, y^b) of the tangent bundle one equates

$$S_{EH}[g] = \int_M \mathrm{d}^4x \sqrt{\det g} R = \int_\Sigma \mathrm{d}^4x \mathrm{d}^3\theta \sqrt{\det g}\, \sqrt{\det h} \frac{4}{\mathrm{Vol}(S_p)}\, R_{ab} y^a(\theta) y^b(\theta) , \tag{4.4}$$

where h is the pull-back of the scalar product, induced by the metric g, from T_pM to S_p. Observe that the quantity $R_{ab}y^a y^b$ is, up to a sign, nothing but the metric geometry version of the contracted non-linear curvature tensor of the Cartan non-linear connection $R^a{}_{ab}y^b$, see equation (2.17). This insight leads to the following reinterpretation of the Einstein–Hilbert action as an action for metric Finsler spaces S_{FS}, where the Finsler function F is considered as fundamental geometric field instead of the metric and we omitted the volume pre-factor

$$S_{FS}[F] = \int_\Sigma \mathrm{d}^4x \mathrm{d}^3\theta (\sqrt{\det g}\, \sqrt{\det h}\, R_{ab} y^a y^b)_{|\Sigma} . \tag{4.5}$$

Adding the matter field actions, lifted trivially to the sphere bundle to this action yields the Finsler space formulation of general relativity

$$S[F, \phi_i] = \int_\Sigma \mathrm{d}^4x \mathrm{d}^3\theta \Big(\sqrt{\det g}\, \sqrt{\det h} \Big(\frac{c^4}{4\pi G} R_{ab} y^a y^b + L_M[g, \phi_i] \Big) \Big)_{|\Sigma} . \tag{4.6}$$

Every metric appearing in the actions is considered as second derivative of F^2 with respect to the tangent space coordinates y. The matter field action is added trivially since it does not depend on the y coordinates and one could perform the integral to recover the standard action for the matter fields on the metric manifold. One of the central results of [12] is that the variation of the action S in equation (4.6) with respect to F yields a scalar field equation to determine F, which is equivalent the Einstein equations (4.1).

Next we seek to construct an action integral which generalises the action S_{FS} from the metric Finsler spaces to general Finsler spaces. Therefore a general notion of the sphere bundle Σ, integration on Σ with a suitable measure and a suitable Lagrangian must be found. The latter is the easiest task since there is a canonical candidate, the curvature scalar built from the contracted non-linear curvature of the Cartan non-linear connection

$$R^F = R^a{}_{ab}(x,y)y^b\,. \tag{4.7}$$

It does not add further derivatives of F to the action and reduces, up to a sign, to the case discussed so far in this section in the metric limit. Moreover this curvature tensor causes the geodesic deviation on Finsler spaces, as shown in equation (2.35). It is the same term which was considered by Rutz as vacuum equation for her Finsler theory of gravity, as discussed in section 3.2.3. We now sketch the extension of the action S_{FS}. All mathematical details will be discussed when we improve and extend this procedure to obtain dynamics for Finsler spacetimes in part II.

The general Finsler space sphere bundle is the set $\Sigma = \{(x,y) \in TM | F(x,y) = 1\}$. A volume form on this set is obtained by introducing the so called Sasaki type metric G on TM which respect the horizontal-vertical split of the Cartan non-linear connection in the sense that horizontal and vertical vectors are orthogonal to each other. Moreover this metric is covariantly constant with respect to the Cartan linear and the dynamical covariant derivative. This immediately leads to the fact that G must be built from the Finsler metric g^F of the Finsler space in the following way

$$G = g^F_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b + g^F_{ab}\delta y^a \otimes \delta y^b\,. \tag{4.8}$$

Its pull-back G^* to Σ yields the desired volume element. The extended action S_{FSE} from which dynamics for Finsler spaces can be derived by variation with respect to F is

$$S_{FSE}[F] = \int_\Sigma \mathrm{d}^4x\mathrm{d}^3\theta(\sqrt{\det G^*}\; R^F)_{|\Sigma}\,. \tag{4.9}$$

Observe that in contrast to the action for metric Finsler space here all building blocks of the action depend on all coordinates (x,y) respectively after their restriction to Σ only on (x,θ). A first variation of this action without reaching finally covariant equations of motion was attempted in [12] and a further reinterpretation as an action for the Finsler metric g^F was investigated in [13]. These were the first mathematically rigorous attempts to derive dynamical vacuum field equations for a Finsler geometry from an action principle.

Having arrived at this stage we see that the Einstein–Hilbert action is just a special case of a much more general framework. We will now conclude the first part of this thesis with a summary of questions concerning the application of Finsler geometry as spacetime geometry.

4.2. Reasons to generalise Finsler spaces

The construction of an action principle for dynamics of Finsler spaces and the insight about the connection to general relativity lead to questions which need to be answered in order to use Finsler geometry as spacetime geometry. In previous such attempts collected in section 3.2, these questions have not been answered satisfactorily. The urge to clarify these questions will lead us to the development of Finsler spacetimes.

Perhaps the most important question for the application of Finsler geometry in physics is:

- How do we implement the concept of causality into a Finslerian spacetime geometry?

In other words can we extend the Finsler geometry framework such that it extends Lorentzian metric geometry, without excluding a huge class of examples or not being able to describe null geodesics, respectively the geometry of the Finsler space along its null directions? As mentioned before, the importance for physics lies in the interpretation of null geodesics as the curves along which massless particles propagate.

The extended dynamics from the toy model Finsler spaces from the previous section give a hint on how it becomes possible to find a non-metric extension of general relativity. In order to understand if this is really the case and what the consequences are we need to address:

- What is the form of the extended Einstein–Hilbert action on proper non-metric extensions of Lorentzian metric spacetimes?
- What is the dynamical equation derived from the extended Einstein–Hilbert action?
- What do the solutions teach us about gravity?

The next open issue is the description of observers and their measurements. In the context of Lorentzian geometry and general relativity the description of observers and their measurements is tied to orthogonal frames of the Lorentzian spacetime metric. A frame is interpreted as the normalized space and time directions of a specific observer who is linked to other observers by Lorentz transformations. But if there is no spacetime metric but only a Finsler length measure:

- How are physical observers defined in a Finsler geometry setting?
- What kind of symmetry transformation relates different observers?
- What are the generic observational consequences of a non-metric spacetime geometry?

The answers to these observer and observation-related questions are the key to make predictions which can be compared with experimental data and be confirmed or falsified. Without a precise observer model predictions are hardly possible.

In order to perform a measurement there must be something for an observer to measure, like matter fields. We discussed an action as candidate for a theory of gravity which generates dynamics for Finsler spaces in the previous section; what is missing are matter fields and their dynamics.

- How can matter field theories be lifted to a Finsler spacetime geometry framework?

- How do the fields couple to the non-metric spacetime geometry?

This lifting and coupling of matter field theories to non-metric geometry has to be done in such a way that in case the Finsler geometry is a metric geometry one recovers general relativity and the usual matter field theories on Lorentzian spacetimes.

As we have seen on Finsler spaces the geometric tensors depend not only on the coordinates of the spacetime manifold but also on its tangent directions. To get a real understanding of this dependences it is necessary to explain:

- What is the interpretation of the fibre coordinate dependence of the geometrical tensors?

We will see during the development of the Finsler spacetime formalism that the answer to this question is closely related to the issue about observers and their measurements.

The questions collected here prepare the logical next steps in the program to obtain a mathematical framework for Finsler geometry to extend Lorentzian metric geometry and to apply it as the geometry of spacetime.

Answers to these questions are given by the Finsler spacetime formalism which we will now present in the second part of this thesis. Finsler spacetimes provide a clear notion of causality which is encoded into the geometry, a precise definition of observers and their measurements, field theories coupled to the geometry and gravitational dynamics which determine the geometry of spacetime from its matter field content. The latter is constructed in such a way that, in case the Finsler geometry is identical to metric geometry, one recovers all the standard field theories known from general relativity. In this sense Finsler spacetimes become viable non-metric geometric backgrounds for physics.

Part II.

The Finsler spacetimes framework

This second part is the heart of this thesis. It contains the construction of our newly developed Finsler spacetime framework. We will demonstrate in every detail that Finsler spacetimes provide a geometric background for physics which encodes causality, observers and their measurements and gravity simultaneously based on Finsler geometry, i.e., based on a general length measure instead of on a metric. It will become clear that Finsler spacetimes are capable to address unexplained gravitational phenomena, for example the fly-by anomaly in the solar system, and we discuss how the non-metric geometry influences the behaviour of fields theories living on this background. Moreover our Finsler spacetime framework gives the answers to the open questions discussed in section 4.2.

The foundation of our Finsler spacetime framework makes the precise definition of Finsler spacetimes in chapter 5. They are manifolds equipped with a smooth homogeneous function on the tangent bundle from which the whole geometry of the manifold is derived. We discuss the physical necessity of the requirements in our definition, we will see that they guarantee the existence of a causal structure and we will derive the important geometric objects. The similarities and crucial differences between our and the existing Finsler space framework we reviewed in chapter 2 will be worked out. Finsler spacetimes extend the existing framework similarly as indefinite metric geometry extends Riemannian geometry. To get a feeling for our new framework we present non trivial examples of Finsler spacetimes at the end of the chapter.

In order to encode gravity into the geometry of Finsler spacetime dynamics are needed which determine the geometry. Extending the ideas from section 4.1 to Finsler spacetimes, where we had a look at the Einstein–Hilbert action from the viewpoint of Finsler geometry, yield such dynamics in chapter 6. They are derived from the Finsler spacetime version of the Einstein–Hilbert action which can be seen as an action for a homogeneous smooth function on the tangent bundle. Moreover we prove that the dynamics obtained from the extended Einstein–Hilbert action are consistent with the Einstein equations.

In chapter 7 we then derive a first order non-metric solution of the Finsler spacetime dynamics. We linearise the previously derived equation which governs the dynamics of Finsler spacetimes around metric geometry and find a spherical symmetric solution close to Minkowski spacetime. This solution can be interpreted as refinement of the linearised Schwarzschild solution known from general relativity. We discuss how this solution addresses the fly-by anomaly in the solar system and thus is an interesting non-metric geometry for physics.

Having obtained dynamics for Finsler spacetimes we present how observers and their measurements are modelled by the geometry in chapter 8, without using a metric on spacetime. Observers will be timelike worldlines equipped with a special orthonormal frame. It is determined by the geometry-defining tangent bundle function of Finsler spacetimes and interpreted as the observers unit time and unit space directions. Measurements of observers are values of physical fields at the observers tangent bundle position evaluated with respect to the observers frame.

In chapter 9 we show how Finsler spacetimes can be equipped with matter fields observers can measure and we discuss a general coupling principle for physical fields described by differential forms. The coupling causes that the fields source the dynamics of Finsler spacetime. We then use the coupling principle to study the scalar field and electrodynamics on Finsler spacetimes.

We remark that most of the results presented in this second part of the thesis have been published in the articles [14], [15] and [16].

5. Finsler spacetime geometry

Finsler spacetime geometry is a version of Finslerian geometry which we will especially develop for the application as spacetime geometry in physics. It will become clear that it is capable to encode causality, observers and their measurements and gravity consistently as the Lorentzian metric spacetime geometry in general relativity. Introducing Finsler spacetimes during this chapter we discuss here how causality is encoded into the geometry.

The mathematical foundations on which Finsler spacetime geometry will be formulated are given by the geometry of the tangent bundle summarised in chapter 1. Finsler spacetime geometry extends Finsler spaces, i.e., geometries with Finsler metrics of definite signature, to Finsler spacetimes, i.e., geometries with associated metrics of indefinite signature. The framework is a very general non-metric Finslerian generalization of Lorentzian metric geometry, just like Finsler space geometry is the non-metric Finslerian generalization of Riemannian geometry which we reviewed in chapter 2.

We will begin this chapter by presenting the definition of Finsler spacetimes in section 5.1. We will comment on the motivation for the definition from physics, especially on the existence of a causal structure obtained from the geometry. Moreover we discuss the relation of Finsler spacetimes to Finsler spaces. In section 5.2 we discuss the geodesic equation, the Cartan non-linear connection, its curvature and the Berwald and Cartan linear covariant derivative on Finsler spacetimes as well as their relation to their counterparts on Finsler spaces. The Cartan non-linear connection is the central object which defines the geometry of Finsler spacetimes. Later we will equip Finsler spacetimes with internal dynamics and physical field theories which evolve according to certain well-defined field equations. These are defined and derived from action integrals, as it is most common in physics. The necessary techniques are developed in section 5.3 where we will see how Finsler spacetimes can be equipped with action integrals so that it becomes possible to derive field equations through variation. Afterwards we will implement the notion of symmetries of the spacetime geometry in section 5.4. Certain symmetries are associated to different physical situations and heavily simplify the task of solving field equations. At the end of this chapter in section 5.5 we will demonstrate that Lorentzian metric spacetimes are indeed Finsler spacetimes, we will present a bimetric example, with a more complex causal structure, in detail and return to the Randers and Bogoslovsky length measure which were introduced earlier as examples of the application of Finsler geometry in physics.

Our formulation of Finsler spacetime geometry here lays the foundation for the development of a complete framework which enables us to use Finsler geometry as the dynamical geometry of spacetime. In the next chapter we will make a first attempt to interpret Finsler spacetimes from the viewpoint of physics and equip them with canonical dynamics derived from the Einstein–Hilbert action.

5.1. Finsler spacetimes

The foundation on which the Finsler spacetime framework is based is the definition of Finsler spacetimes. This definition is designed so that it meets several requirements from physics. For example it guarantees a precise notion of timelike directions and the existence of the geometry of the manifold along null directions. Finsler spacetimes can be seen as a very general non-metric generalization of Lorentzian metric manifolds. In this section we will present our definition of Finsler spacetimes and comment on the necessity of the different requirements in the definition. As our most important result we will present theorem 5.1 on the existence of a convex cone of timelike vectors in the tangent spaces of Finsler spacetimes. Moreover, we will discuss the relationship between the different geometric objects on Finsler spacetimes to the corresponding geometric objects known from Finsler spaces. It will become clear that Finsler spacetimes are a mathematical construction which include manifolds with length measure $F(x,y) = (G_{a_1\dots a_n}(x)y^{a_1}\dots y^{a_n})^{1/n}$ with non trivial null structure. The description of such length measures was not possible in the language of Finsler spaces, see chapter 2.

5.1.1. Definition

The definition of Finsler spacetimes yields a manifold whose geometry is solely determined by a smooth homogeneous function on the tangent bundle. In contrast to Finsler spaces this function must not be one-homogeneous with respect to the tangent space coordinates y. The one-homogeneous length measure for curves, which is the defining object on Finsler spaces, is then derived from the fundamental geometry-defining function.

Definition 5.1. ***Finsler spacetime***
A Finsler spacetime, denoted by the tripel (M, L, F), is a four dimensional, connected, Hausdorff, paracompact, smooth manifold M equipped with a continuous fundamental geometry function $L : TM \to \mathbb{R}$ on the tangent bundle which has the following properties:

(i) L is smooth on the tangent bundle without the zero section $TM \setminus \{0\}$;

(ii) L is positively homogeneous of degree $r \geq 2$ with respect to the fibre coordinates of TM,

$$L(x, \lambda y) = \lambda^r L(x,y) \quad \forall\, \lambda > 0\,; \tag{5.1}$$

(iii) L is reversible in the sense

$$|L(x,-y)| = |L(x,y)|\,; \tag{5.2}$$

(iv) the Hessian g^L_{ab} of L with respect to the fibre coordinates, the L metric, is non-degenerate nearly everywhere on TM and especially nearly everywhere along the null structure of the spacetime $N_L = \{(x,y) \in TM \,|\, L(x,y) = 0\}$

$$g^L_{ab}(x,y) = \frac{1}{2}\bar\partial_a\bar\partial_b L\,. \tag{5.3}$$

I.e., g^L_{ab} is non-degenerate on $TM \setminus A$ with A for a measure zero subset A so that $B = N \cap N_L$ is a lower dimensional subset of N_L.

(v) the unit timelike condition holds, i.e., for all $x \in M$ the set

$$\Omega_x = \Big\{ y \in T_xM \,\Big|\, |L(x,y)| = 1\,,\ g^L_{ab}(x,y) \textit{ has signature } (\epsilon, -\epsilon, -\epsilon, -\epsilon)\,,\ \epsilon = \frac{|L(x,y)|}{L(x,y)} \Big\} \tag{5.4}$$

contains a non-empty closed connected component $S_x \subset \Omega_x \subset T_xM$.

The function F, defining the length measure S for curves γ on M

$$S[\gamma] = \int \mathrm{d}\tau F(\gamma, \dot\gamma)\,, \tag{5.5}$$

and its corresponding Hessian g^F associated to L are derived objects given by

$$F(x,y) = |L(x,y)|^{1/r}, \quad g^F_{ab} = \frac{1}{2}\bar\partial_a\bar\partial_b F^2\,. \tag{5.6}$$

This function F nearly defines a Finsler function according to definition 2.1 except that F may not be smooth on all of $TM \setminus \{0\}$ and that the Finsler metric g^F may be degenerate on $TM \setminus A$, where A is the set on which the Hessian g^L degenerates. This becomes clear from the relations between g^F and g^L displayed in (equation 5.15) below. For $L = g_{ab}(x)y^ay^b$, Finsler spacetimes become Lorentzian metric spacetimes; the details are derived section 5.5.

Already here we see how the Finsler spacetime formulation of Finsler geometry extends the Finsler space formulation. Examples which have differentiability issues along a non trivial null structure, and thus can not be discussed on the basis of the Finsler space definition from chapter 2, can now be studied on the basis of the fundamental geometry function L:

$$L(x,y) = G_{a_1\dots a_n}(x)y^{a_1}\dots y^{a_n} \Rightarrow F(x,y) = \left(|G_{a_a\dots a_n}(x)y^{a_1}\dots y^{a_n}|\right)^{\frac{1}{n}}\,. \tag{5.7}$$

Since L is smooth on $TM \setminus \{0\}$, so especially on N_L, no differentiability problems appear, in contrast to the discussion in section 2.1. The geometry of our Finsler spacetimes will be formulated in terms of derivatives acting on L as we will see throughout the remaining part of this chapter. With this it is now possible to study Finsler geometries with non trivial null structure.

A much more restrictive definition of indefinite Finsler spaces was given by Beem in [25], where he uses a two-homogeneous fundamental geometry function L and demands moreover that the Hessian g^L_{ab} is non-degenerate everywhere on $TM \setminus \{0\}$. We will see in section 5.5 that the definition of Beem excludes already simple bimetric geometries like the one with $L = (g_{ab}(x)y^ay^b)(h_{mn}(x)y^ay^b)$, where g and h are Lorentzian metrics, while they are included in our definition. The concept of deriving a Finsler function from another function has been used in the context of Lagrange spaces which define certain differential equations, so called sprays, by Antonelli in [47]. But this concept has so far never been brought into contact with indefinite Finsler geometry; neither has it been used to construct a spacetime geometry in physics.

We will now discuss properties of Finsler spacetimes emerging directly from the definition and comment on their importance for physics. Special attention is given to requirement (v), since it yields a theorem that ensures the existence of a convex cone of timelike vectors, just like the cone present in Lorentzian metric geometry.

5.1.2. Properties of Finsler spacetimes and causality

The various requirements on the fundamental geometry function L in the definition 5.1 of Finsler spacetimes are motivated from physics in the following way.

Requirement (i) and (iv), the smoothness of L and the regularity conditions on g^L ensure the existence of the geometric objects on $TM \setminus A$ especially along nearly the whole null structure $N_L \setminus B$ of spacetime. This guarantees the existence of null-geodesics in nearly all directions which can be interpreted as trajectories of massless particles. We allow for a subset $B \subset N_L$ along which the L metric may degenerate to include bimetric examples with intersecting light cones. Such causal structures appear in the description of electrodynamics in general non-disspative linear optical media [26]. Observe that even though on Finsler spacetimes there still may appear directions in the null structure of spacetime along which geodesics and curvature are not well-defined we still achieved a huge advantage compared to earlier approaches based on the definition of Finsler spaces. On the latter the geometric objects and geodesics were not well-defined on the complete null structure, here this is only the case for the measure zero set B. In section 5.5 we discuss Finsler spacetimes with B being the empty set and with B being non-empty. Neither of them could be discussed on the basis of Finsler spaces.

Requirement (ii) and (iii) together with the derived Finsler function in equation (5.6) ensures that the length integral for curves (5.5) is independent of the parametrization and orientation of the curve, hence indeed a geometric quantity. Later we interpret the length of an observers worldline as the observers measurement of time, hence it is ensured that time is a geometric quantity.

Requirements (iv) and (v) ensure the existence of a set of unit timelike vectors S_x which rescale to a convex cone of timelike vectors. This property of Finsler spacetimes is of major importance since we interpret the timelike vectors as tangents to observer worldlines. We formulate the following theorem:

Theorem 5.1. ***Cone of timelike vectors***
Each tangent space T_xM of a Finsler spacetime (M, L, F) contains an open convex cone

$$C_x = \bigcup_{\lambda>0} \lambda S_x = \bigcup_{\lambda>0} \{\lambda u \,|\, u \in S_x\} \,. \tag{5.8}$$

Proof of Theorem 5.1.
The techniques for this proof are adapted from Beem [25]. To begin note that the shell of unit timelike vectors S_x is a three dimensional closed submanifold of $T_xM \cong \mathbb{R}^4$. We now proceed in three steps. First we will determine the normal curvatures of S_x at some point $y_0 \in S_x$. These are defined [48] as $\kappa_n(z) = \sum_a \ddot{\gamma}^a(0) n^a$ for curves $\tau \mapsto \gamma(\tau)$ in S_x with normalized tangent vectors $\sum_a \dot{\gamma}^a \dot{\gamma}^a = 1$ and initial conditions $\gamma^a(0) = y_0^a$ and $\dot{\gamma}^a(0) = z^a$. The initial tangent z is tangent to S_x in y_0, i.e., it satisfies

$$0 = \bar{\partial}_a |L|(x, y_0) z^a = \frac{2|L(x, y_0)|}{(r-1)L(x, y_0)} g^L_{ab}(x, y_0) z^a y_0^b \,. \tag{5.9}$$

The unit normal is given by

$$n^a = \frac{1}{N(x, y_0)} \bar{\partial}_a |L|(x, y_0) = \frac{|L(x, y_0)| g^L_{ab}(x, y_0) y_0^b}{L(x, y_0) \left(\sum_c g^L_{cp}(x, y_0) g^L_{cq}(x, y_0) y_0^p y_0^q\right)^{1/2}} \,, \tag{5.10}$$

with $N(x,y_0)=2(r-1)^{-1}(\sum_c g^L_{cp}(x,y_0)g^L_{cq}(x,y_0)y_0^p y_0^q)^{1/2}$ being the norm of $\bar\partial_a|L|(x,y_0)$. Since the curves $\gamma(\tau)$ lie in S_x where $|L|=1$, we may obtain a useful relation for $\ddot\gamma(0)$ by differentiating $|L(x,\gamma(\tau))|=1$; we find that $g^L_{ab}(x,y_0)y_0^b\ddot\gamma^a(0)=-(r-1)g^L_{ab}(x,y_0)z^az^b$. Combining these results we find the normal curvatures

$$\kappa_n(z)=-\frac{(r-1)|L(x,y_0)|}{N(x,y_0)L(x,y_0)}g^L_{ab}(x,y_0)z^az^b=-\epsilon\frac{(r-1)}{N(x,y_0)}g^L_{ab}(x,y_0)z^az^b\,,\text{ with }\epsilon=\frac{L}{|L|}. \tag{5.11}$$

Second we will show that all these normal curvatures are positive. Note that the homogeneity of L implies $g^L_{ab}(x,y_0)y_0^ay_0^b=\frac{1}{2}r(r-1)L(x,y_0)$ so that y_0 is $g^L(x,y_0)$-timelike, i.e. $sign(g^L_{ab}(x,y_0)y_0^ay_0^b)=\epsilon$. Due to (5.9) we know that z and y_0 are $g^L(x,y_0)$-orthogonal, hence z must be $g^L(x,y_0)$-spacelike, i.e., $sign(g^L_{ab}(x,y_0)z^az^b)=-\epsilon$. This immediately confirms the result $\kappa_n(z)>0$.

Finally we will show the convexity of the set C_x defined in the theorem. Positivity of the normal curvatures implies positivity of the principal curvatures of S_x and so the convexity of S_x itself as embedded surface. Therefore the interior of S_x together with S_x seen as the set $\tilde C^1_x=\bigcup_{\lambda\geq 1}\lambda S_x$ is closed, connected and convex with boundary S_x. Because of the homogeneity of L, we then conclude that for all $\mu>0$ the sets $\tilde C^\mu_x=\bigcup_{\lambda\geq\mu}\lambda S_x$ are closed, connected and convex. Hence,

$$\bigcup_{\mu>0}\tilde C^\mu_x=C_x \tag{5.12}$$

is an open convex cone, which concludes the proof. □

The boundary of C_x must be made of $y\in T_xM$ such that $L(x,y)=0$ or that $g^L(x,y)$ degenerates since C_x can be written as the set

$$C_x=\{y\in T_xM|\exists\lambda>0:|L(x,y)|=\lambda^r,\mathrm{sign}(g^L_{ab})=(\epsilon,-\epsilon,-\epsilon,-\epsilon),\epsilon=\frac{L}{|L|}\}\,. \tag{5.13}$$

The existence of a convex cone of timelike vectors on Finsler spacetimes generalises the known cone of timelike vectors from Lorentzian metric geometry and gives rise to a causal structure encoded into the spacetime geometry. Moreover we will see in section 9 when we discuss field theories on Finsler spacetimes that the Lorentzian signature of g^L is connected to the principal symbol of the field equations on the tangent bundle. In this way local causality is encoded into the non-metric geometry of Finsler spacetimes.

Having clarified the need for the various requirements in the definition 5.1 we will next compare the geometric objects derived as derivatives with respect to the y coordinate from the fundamental geometry function L with those derived from the Finsler function F known from Finsler spaces.

5.1.3. Derivatives of the fundamental geometry function

As on Finsler spaces one obtains canonical geometric objects on Finsler spacetimes from differentiation of the geometry-defining function L with respect to the tangent space coordinates y. These can then be expressed in terms of derivatives acting on the derived Finsler function and so be related to the objects known from Finsler spaces. The following formulae are directly derived from the definition $F=|L|^{1/r}$ and sorted by the number of derivatives acting. For the sake of readability we use both functions F and L where they appear.

Proposition 5.1. ***The canonical on form components***
Let p_a be the Finsler one-form (see definition 2.3) derived from the Finsler function of Finsler spacetimes. The following holds:

$$p^L_a = \frac{1}{2}\bar{\partial}_a L = \frac{rL}{2F^2}p_a \tag{5.14}$$

Moreover by Euler's Theorem on homogeneous functions (theorem 2.1) $y^a p^L_a = \frac{r}{2}L$.

The relation between the Finsler metric g^F and the L metric g^L is straightforwardly derived.

Proposition 5.2. ***The Finsler and the L metric***
Let g^F and g^L be the Hessians of the functions F respectively L with respect to the fibre coordinates y from definition 5.1. Everywhere where F is differentiable we have

$$g^F_{ab} = \frac{2F^2}{rL}\Big(g^L_{ab} + \frac{(2-r)}{2rL}\bar{\partial}_a L\bar{\partial}_b L\Big), \qquad g^{F\,ab} = \frac{rL}{2F^2}\Big(g^{L\,ab} - \frac{2(2-r)}{r(r-1)L}y^a y^b\Big), \tag{5.15}$$

$$\det g^F \;=\; \frac{16F^{(8-4r)}}{r^4(r-1)}\det g^L\,. \tag{5.16}$$

Since it will be lead to important simplifications during calculations throughout the next sections and chapters we display the consequences from the homogeneity properties of L on the contracted L metric.

Proposition 5.3. ***Homogeneity properties of the L metric***

$$y^a y^b g^L_{ab} = (r-1)p^L_a y^a = \frac{r(r-1)}{2}L\,. \tag{5.17}$$

From the precise relation between the L and F metrics we can derive the signature of one from the other. This will be of importance when we discuss observers on Finsler spacetime in section 8.2.

Theorem 5.2. ***Signature of the metrics***
On the set $TM\setminus(A\cup\{L=0\})$ the metric g^L is nondegenerate of signature $(-1_m, 1_p)$ for natural numbers m, p with $m+p=4$. Then the Finsler metric has the same signature $(-1_m, 1_p)$ on the connected components of TM where $L(x,y)>0$, and reversed signature $(-1_p, 1_m)$ where $L(x,y)<0$.

Proof of Theorem 5.2.
We proof the relation between the signature of a matrix A_{ab} and a matrix C_{ab} that differs from A_{ab} by a one-form B_a that is spacelike with respect to A

$$C_{ab} = A_{ab} - B_a B_b\,,\ \text{with } A_{ab} = \frac{2F^2}{rL}g^L_{ab},\ B_a = \sqrt{\frac{(r-2)F^2}{r^2L^2}}\bar{\partial}_a L\,. \tag{5.18}$$

Analysing the signature of C_{ab} in an orthonormal basis of A_{ab} and using the $SO(m,p)$ freedom in the choice of this basis proves the theorem. The details are worked out in appendix A.1. □

For unit timelike vectors $y \in S_x$ which are contained in $TM\setminus(A\cup\{L=0\})$ we have by definition that the metric $g^L_{(x,y)}$ has signature $(1,-1,-1,-1)$ for $L(x,y)>0$ and $(-1,1,1,1)$ for $L(x,y)<0$. Hence we conclude about the signature of g^F on S_x from the theorem above:

Corollary 5.1.
The Finsler metric $g^F_{(x,y)}$ evaluated on the unit timelike shell S_x, and hence on the cone of timelike vectors C_x, has always Lorentzian signature: $\mathrm{sign}(g^F_{|C_x}) = \mathrm{sign}(g^F_{|S_x}) = (1,-1,-1,-1)$.

Having collected the important relations between the second fibre coordinate derivatives of L and F, we now display the relation of the third derivative of L to the Cartan tensor from definition 2.5.

Proposition 5.4. ***The Cartan tensor***
Let C_{abc} be the Cartan tensor derived from the Finsler function of Finsler spacetimes. The following holds:

$$C^L_{abc} = \frac{1}{2}\bar{\partial}_c g^L_{ab} = \frac{rL}{2F^2}\Big(C_{abc} - \frac{r-2}{2F^4}p_a p_b p_c + \frac{3(r-2)}{F^2}g^F_{(ab}p_{c)}\Big)\,. \tag{5.19}$$

The Cartan one-form Θ^L is, equivalent as on Finsler spaces (definition 2.6), the one-form with coefficients p^L_a

$$\Theta^L = p^L_a \mathrm{d}x^a = \frac{1}{2}\bar{\partial}_a L\, \mathrm{d}x^a\,, \tag{5.20}$$

and the Cartan two-form Ω^L is its differential

$$\Omega^L = \mathrm{d}\Theta^L = \frac{1}{2}\delta_a\bar{\partial}_b L \mathrm{d}x^a \wedge \mathrm{d}x^b + g^L_{ab}\delta y^a \wedge \mathrm{d}x^b\,. \tag{5.21}$$

Observe that all the geometric objects defined through derivatives acting on L are well-defined on the null structure N_L of the spacetime. We stress again that even for the simple examples $L = G_{a_1\dots a_n}y^{a_1}\dots y^{a_n}$ this is not the case for the geometric objects derived through F. The same will be true for the geometry-defining connection of Finsler spacetimes.

5.2. Finsler geodesics, connection and curvature

As for Finsler spaces the geometry of Finsler spacetimes is based on a non-linear homogeneous connection on the tangent bundle, the concept discussed in chapter 1.2. This distinguished non-linear connection should be such that the geodesics of the spacetime are autoparallels. In order to achieve this goal we will first derive the geodesic equation of Finsler spacetimes expressed through the fundamental geometry function L and present how null geodesics can be derived and studied on Finsler spacetimes. We can then compare the geodesic equation to the autoparallel equation of non-linear connections to get a hint for the coefficients of the connection we are searching for. We will then define an appropriate non-linear connection and discuss its properties. The most important property is highlighted in theorem 5.4, where we prove that the non-linear connection coefficients can equally be expressed through our fundamental geometry function L and the derived Finsler function F. Afterwards we introduce the associated Berwald- and Cartan-linear covariant derivatives, their difference tensor, their curvatures and their relation to the non-linear curvature of the non-linear connection.

5.2.1. Geodesics

Geodesics are curves on Finsler spacetimes which extremise the length integral for curves γ displayed in equation (5.5). The Euler-Lagrange equations take the usual form

$$\frac{d}{d\tau}\bar{\partial}_a F(\gamma,\dot{\gamma}) - \partial_a F(\gamma,\dot{\gamma}) = 0\,, \tag{5.22}$$

but since F is not the fundamental object of our formalism we rewrite these equations in terms of the fundamental geometry function L

$$\frac{d}{d\tau}\Big(\frac{F}{r}\frac{\bar{\partial}_a L}{L}\Big) - \frac{F}{r}\frac{\partial_a L}{L} = 0 \Leftrightarrow \frac{rL}{F}\frac{d}{d\tau}\Big(\frac{F}{rL}\Big)\bar{\partial}_a L + \Big(\frac{d}{d\tau}\bar{\partial}_a L - \partial_a L\Big) = 0\,. \tag{5.23}$$

Expanding the derivative of $\bar{\partial}_a L$ with respect to the curve parameter τ and introducing the L metric yields

$$g^{Lab}\frac{L}{F}\frac{d}{d\tau}\Big(\frac{F}{L}\Big)\bar{\partial}_a L + \ddot{\gamma}^b + \frac{1}{2}g^{Lab}\Big(\dot{\gamma}^c\partial_c\bar{\partial}_a L - \partial_a L\Big) = 0\,. \tag{5.24}$$

Hence in arclength parametrization $F(\gamma,\dot{\gamma}) = 1 \Leftrightarrow L = \pm 1$ the geodesic equation is

$$\ddot{\gamma}^b + \frac{1}{2}g^{Lab}\Big(\dot{\gamma}^c\partial_c\bar{\partial}_a L - \partial_a L\Big) = 0\,. \tag{5.25}$$

Observe that equation (5.25) could be used for null curves on Finsler spacetimes, since all objects appearing are well-defined along null directions, but it was not derived for this case. Therefore, in order to discuss null geodesics, we have to employ a different method. Instead of using the length measure for curves as an action we may now study the constraint action

$$\tilde{S}[\gamma,\lambda] = \int \mathrm{d}\tau\Big(L(\gamma,\dot{\gamma}) + \lambda(\tau)[L(\gamma,\dot{\gamma}) - \kappa]\Big)\,. \tag{5.26}$$

Variation with respect to the Lagrange multiplier λ yields the constraint $L(\gamma,\dot{\gamma}) = \kappa$ with constant κ, variation with respect to the curve γ yields

$$\frac{d}{d\tau}((1+\lambda)\bar{\partial}_a L) - (1+\lambda)\partial_a L = 0 \Leftrightarrow \frac{\dot{\lambda}}{1+\lambda}\bar{\partial}_a L + \Big(\frac{d}{d\tau}\bar{\partial}_a L - \partial_a L\Big) = 0\,. \tag{5.27}$$

For any κ we now conclude from the constraint $L(\gamma,\dot{\gamma}) = \kappa$

$$\frac{d}{d\tau}L = \ddot{\gamma}^a\bar{\partial}_a L + \dot{\gamma}^a\partial_a L = 0\,,. \tag{5.28}$$

Combining this equation with contraction of the the second equation in (5.27) with $\dot{\gamma}^a$ and using the r-homogeneity of the fundamental geometry function L yields

$$\begin{aligned} &\dot{\gamma}^a\Big(\frac{\dot{\lambda}}{1+\lambda}\bar{\partial}_a L + \Big(\frac{d}{d\tau}\bar{\partial}_a L - \partial_a L\Big)\Big) \\ = \quad &\frac{\dot{\lambda}}{1+\lambda}rL + \frac{d}{dt}(rL) - \ddot{\gamma}^a\bar{\partial}_a - \dot{\gamma}^a\partial_a L = \frac{r\dot{\lambda}}{1+\lambda}\kappa = 0\,. \end{aligned} \tag{5.29}$$

For non-null curves, $\kappa \neq 0$, we now have to conclude that $\dot{\lambda} = 0$, so equation (5.27) becomes identical to the geodesic equation (5.23) in arclength parametrization. For null curves, $\kappa = 0$, we cannot conclude from (5.29) that $\dot{\lambda} = 0$. Nevertheless equation (5.27) is equivalent to the geodesic equation (5.23) also for null curves since we can change the parametrization $\tau \mapsto s(\tau)$ of the curve without altering the constraint $L(\gamma, \dot{\gamma}) = \kappa = 0$. Choosing $s(\tau)$ such that it solves

$$\frac{\ddot{s}}{\dot{s}}(r-1) + \frac{\dot{\lambda}(s(\tau))}{1+\lambda(s(\tau))} = 0\,, \tag{5.30}$$

transforms equation (5.27) in the geodesic equation (5.23) with respect to the parameter s.

After this discussion we conclude that curves satisfying equation (5.23), or equivalently (5.25), are geodesics of the Finsler spacetime. We like to point out one more time that only due to the differentiability of L along the null structure of the spacetime is it possible to discuss null geodesics. In terms of F, like on Finsler spaces, it is in general not possible to study the geodesic equation along the null structure since F usually fails to be differentiable precisely along the null structure.

In order to find the non-linear connection whose autoparallel equation (1.38) coincides with the geodesic equation (5.25) we compare the two and realise that the non-linear connection coefficients ${N^a}_b$ must satisfy

$${N^a}_b(x,y)y^b = \frac{1}{2}g^{Lab}(x,y)\Big(y^c\partial_c\bar{\partial}_a L(x,y) - \partial_a L(x,y)\Big)\,. \tag{5.31}$$

As an immediate consequence we realise that ${N^a}_b$ has to be homogeneous of degree one with respect to y. Next we will define a non-linear connection which satisfies the above equation, and this connection will even turn out to be unique.

5.2.2. Connections and curvature

We will show that on Finsler spacetimes (M, L, F) there exists a distinguished non-linear connection whose autoparallels are geodesics of the manifold. It turns out that this connection is the analogue to the Cartan non-linear connection on Finsler spaces , but with the important advantage that it is derived from the fundamental geometry function L and so well-defined on $N_L \setminus B$. Recall that B is the subset of the null structure N_L of spacetime where g^L may degenerate. We will discuss in detail the relation between the derivation of the connection from L and the derivation from F, and on the consequences for the associated linear covariant derivatives and the non-linear curvature.

Definition 5.2. ***Cartan non-linear connection on Finsler spacetimes***
Let (M, L, F) be a Finsler spacetime. The homogeneous non-linear connection on the tangent bundle defined through the connection coefficients

$${N^a}_b = \frac{1}{4}\bar{\partial}_b\Big[g^{Laq}\big(y^p\partial_p\bar{\partial}_q L - \partial_q L\big)\Big]\,, \tag{5.32}$$

is called Cartan non-linear connection.

Observe that everywhere where g^L is non-degenerate these connection coefficients are well-defined, especially nearly everywhere on the null structure of spacetime.

Theorem 5.3. ***Uniqueness of the Cartan non-linear connection on Finsler spacetimes***
Let (M, L, F) be a Finsler spacetime and $N^a{}_b$ be the connection coefficients of the Cartan non-linear connection of Finsler spacetimes. The connection is the unique one-homogeneous connection which satisfies

$$\nabla g^L_{ab} = 0 \text{ and } \Omega^L(p(X), p(Y)) = 0\ \forall\ X, Y \in TTM, \tag{5.33}$$

where p denotes the projector onto the horizontal bundle defined by the non-linear connection, Ω denotes the Cartan two-form (see equation 5.21) and ∇ the dynamical covariant derivative from definition 1.12.

In other words: The $N^a{}_b$ in equation (5.32) define the unique connection which leaves the L metric covariantly constant and fulfil the symmetry condition $\bar\partial_{[a}N^q{}_{b]} = 0$.

Proof of Theorem 5.3.
We derive the explicit form of $N^a{}_b$ from the conditions in the theorem. The ideas are similar to the ones which prove the analogue result on Finsler spaces formulated in theorem 2.2. The metric compatibility condition ∇g^L_{ab} determines the symmetric part while the Cartan two-form condition determines the anti-symmetric part of N_{ab}, where the index of $N^a{}_b$ is lowered with the L metric. The result is

$$\begin{aligned} N_{ab} + N_{ba} &= y^c\delta_c g^L_{ab} = y^c\partial_c g^L_{ab} - y^c N^q{}_c\bar\partial_q g^L_{ab} && (5.34)\\ N_{ab} - N_{ba} &= \frac{1}{2}\bar\partial_a\partial_b L - \frac{1}{2}\bar\partial_b\partial_a L\,. && (5.35)\end{aligned}$$

The combination of these equations, raising the first index with the L metric yields

$$\begin{aligned} N^a{}_b &= \frac{1}{2}g^{Lap}\Big(y^c\partial_c g^L_{ap} - y^c N^q{}_c\bar\partial_q g^L_{bp} + \frac{1}{2}\bar\partial_p\partial_b L - \frac{1}{2}\bar\partial_b\partial_p L\Big) && (5.36)\\ &= \frac{1}{4}\Big(g^{Lap}\bar\partial_b(y^c\partial_c\bar\partial_p L - \partial_p L) - 2g^{Lab}y^c N^q{}_c\bar\partial_q g^L_{bp}\Big) && (5.37)\\ \Rightarrow y^b N^a{}_b &= \frac{1}{2}g^{Laq}(y^c\partial_c\bar\partial_p L - \partial_p L)\,. && (5.38)\end{aligned}$$

The last conclusion is due to the homogeneity properties of L. Inserting equation (5.38) into equation (5.37) yields the desired result. □

As a remark we like to point out that, by direct calculation, one finds that the fundamental geometry function L, as well as the corresponding Finsler function $F(L)$, is horizontally constant. Moreover using this and the relation between the g^L and g^F we find that g^F is covariantly constant

$$\delta_a L = \partial_a L - N^q{}_a\bar\partial_q L = 0; \quad \nabla g^F_{ab} = 0\,. \tag{5.39}$$

Now after we collected the properties of the Cartan non-linear connection we consider its autoparallels. Recall from definition 1.13 that these are curves γ on the manifold which satisfy

$$\ddot\gamma^a + N^a{}_b(\gamma, \dot\gamma)\dot\gamma^b = 0. \tag{5.40}$$

For the connection coefficients of Cartan non-linear connection (5.32) this equation becomes

$$\ddot\gamma^a + \frac{1}{2}g^{Laq}\big(y^p\partial_p\bar\partial_q L - \partial_q L\big) = 0\,, \tag{5.41}$$

which is identical to the geodesic equation (5.25). The derivation only requires the Euler Theorem 2.1 due to the homogeneity properties of the terms appearing in the non-linear connection coefficients.

To name the connection identical to its counterpart on Finsler spaces is justified by the following important result:

Theorem 5.4. ***Connection coefficient identity***
Let (M, L, F) be a Finsler spacetime and $N^a{}_b$ be the connection coefficients of the Cartan non-linear connection of Finsler spacetimes (equation 5.32). Then wherever F is differentiable and g^F is non-degenerate

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\Big[g^{Faq}\big(y^p\partial_p\bar{\partial}_q F^2 - \partial_q F^2\big)\Big] = \Gamma^a{}_{bc}y^c - g^{Faq}(x,y)C_{qbc}\Gamma^c{}_{pq}y^p y^q\,, \tag{5.42}$$

where $\Gamma^a{}_{bc}(x,y) = g^{Faq}(\partial_b g^F_{qc} + \partial_c g^F_{qb} - \partial_q g^F_{bc})$ and C_{abc} is the Cartan tensor.

Proof of Theorem 5.4.
The first equality is obtained by expanding the connection coefficients in equation (5.32) by use of the relation $F = |L|^{1/r}$ and the propositions on the geometric objects obtained from y derivatives of L from section 5.1.3. Collecting all terms yields the desired result. For the second equality one simply performs the derivative and uses the definition of the Finsler metric in equation (5.6) and the Cartan tensor C_{abc} in equation (2.10). $\square$

A similar statement has been proven in [47] in the context of Lagrange spaces. The last expression in equation (5.42) precisely equals the connection coefficients of the Cartan non-linear connection on Finsler spaces introduced in equation (2.14). This proves that the non-linear connection on Finsler spacetimes is nothing but the generalisation of the Cartan non-linear connection on Finsler spaces. On Finsler spacetimes the coefficients can equally be expressed by derivatives acting on L respectively on the corresponding Finsler function F. In the first case the domain of definition is $TM \setminus A$ where g^L is non degenerate, while in the second one has to examine carefully where F is differentiable. Hence the Cartan non-linear connection coefficients on Finsler spacetimes expressed through L are the smooth continuation of the Cartan non-linear connection coefficients introduced in terms of F on Finsler spaces to the set where F is not differentiable. This nicely demonstrates the advantages of formulating Finslerian geometry in terms of our Finsler spacetimes: the differentiability problems of F are nicely removed.

Associated to the Cartan non-linear connection there are linear covariant derivatives respecting the horizontal-vertical structure induced by the connection, similar as in the situation on Finsler spaces discussed in section 2.3. The important ones for us are the Berwald- and the Cartan-linear covariant derivative. Their definition is nearly identical to the one on Finsler spaces except that they are defined through the fundamental geometry function L instead of the Finsler function F. It turns out that this only makes a difference in the Cartan linear derivative with respect to a vertical direction; for this case we can define two different derivatives. Only the second will be of relevance when we study observer transformations in section 8.2.2, but for completeness we present both. The first version of the Cartan linear covariant derivative

∇^{CL} is defined through the following rules

$$\nabla^{CL}_{\delta_a}\delta_b \;=\; \Gamma^{\delta q}{}_{ab}\delta_q;\; \nabla^{CL}_{\bar\partial_a}\delta_b = g^{Lqp}C^{L}{}_{pab}\delta_q \tag{5.43}$$
$$\nabla^{CL}_{\delta_a}\bar\partial_b \;=\; \Gamma^{\delta q}{}_{ab}\bar\partial_q;\; \nabla^{CL}_{\bar\partial_a}\bar\partial_b = g^{Lqp}C^{L}{}_{pab}\bar\partial_q\,. \tag{5.44}$$

where the δ-Christoffel symbols Γ^{δ} are defined by

$$\Gamma^{\delta\,a}{}_{bc} = \frac{1}{2}g^{F\,aq}\big(\delta_b g^F_{qc} + \delta_c g^F_{qb} - \delta_q g^F_{bc}\big) = \frac{1}{2}g^{L\,aq}\big(\delta_b g^L_{qc} + \delta_c g^L_{qb} - \delta_q g^L_{bc}\big)\,. \tag{5.45}$$

They are related to the non-linear connection coefficients by $\Gamma^{\delta a}{}_{bc}y^c = N^a{}_b$ and can be expressed through L or F. The proof uses the relation between L and the relations from section 5.1.3. The horizontal curvature component of the covariant derivative is given by

$$R^{CLq}{}_{cab} = \delta_b\Gamma^{\delta q}{}_{ac} - \delta_a\Gamma^{\delta q}{}_{cb} + \Gamma^{\delta q}{}_{pb}\Gamma^{\delta p}{}_{ac} - \Gamma^{\delta q}{}_{pa}\Gamma^{\delta p}{}_{cb} - g^{Lpq}C^L_{pmc}R^m{}_{ab}\,. \tag{5.46}$$

The second version ∇^{CL2} differs in the derivative with respect to the vertical direction where it is defined through $F(L)$ instead of L and can only be used wherever F is differentiable.

$$\nabla^{CL2}_{\bar\partial_a}\delta_b = g^{Fqp}C_{pab}\delta_q;\; \nabla^{CL2}_{\bar\partial_a}\bar\partial_b = g^{Fqp}C_{pab}\bar\partial_q\,, \tag{5.47}$$

which changes the horizontal components of the curvature to

$$R^{CL2q}{}_{cab} = \delta_b\Gamma^{\delta q}{}_{ac} - \delta_a\Gamma^{\delta q}{}_{cb} + \Gamma^{\delta q}{}_{pb}\Gamma^{\delta p}{}_{ac} - \Gamma^{\delta q}{}_{pa}\Gamma^{\delta p}{}_{cb} - g^{Fpq}C_{pmc}R^m{}_{ab}\,. \tag{5.48}$$

The Berwald linear covariant derivative ∇^B is defined through the following rules

$$\nabla^B_{\delta_a}\delta_b = \bar\partial_a N^q{}_b\delta_q;\quad \nabla^B_{\delta_a}\bar\partial_b = \bar\partial_a N^q{}_b\bar\partial_q;\quad \nabla^B_{\bar\partial_a}\delta_b = 0;\quad \nabla^B_{\bar\partial_a}\bar\partial_b = 0\,, \tag{5.49}$$

and its horizontal curvature components are

$$R^{Bq}{}_{cab} = \delta_b(\bar\partial_a N^q{}_c) - \delta_a(\bar\partial_c N^q{}_b) + (\bar\partial_p N^q{}_b)(\bar\partial_a N^p{}_c) - (\bar\partial_p N^q{}_a)(\bar\partial_c N^p{}_b)\,. \tag{5.50}$$

All three linear covariant derivatives turn out to be useful tools during upcoming calculations in the next sections and chapters, as well as the tensor which characterises their non trivial difference.

Theorem 5.5. *The S Tensor*
Let (M, L, F) be a Finsler spacetime. Wherever the L metric g^L is non-degenerate,

$$S^a{}_{bc} = \Gamma^{\delta\,a}{}_{bc} - \bar\partial_c N^a{}_b \tag{5.51}$$

defines a d-tensor field (see definition 1.8). The components $S^a{}_{bc}$ can be written with the help of the Finsler function F as

$$S^a{}_{bc} = -g^{Faq}\nabla C_{qbc} = -\nabla C^a{}_{bc} \tag{5.52}$$

The tensor S measures the difference between the Cartan- and the Berwald linear covariant derivative and vanishes in case the Finsler spacetime is metric, $L = g_{ab}(x)y^ay^b$.

Proof of Theorem 5.5.
That S is indeed a tensor is obvious from the fact that it is the difference between coefficients which define a covariant derivative. The non tensorial behaviour of $\Gamma^{\delta\,a}{}_{bc}$ and $\bar{\partial}_c N^a{}_b$ under induced coordinate transformations cancels in their difference. To prove equation (5.52) we use the relation $N^a{}_b = \Gamma^{\delta\,a}{}_{bc} y^c$ to equate

$$S^a{}_{bc} = -y^q \bar{\partial}_c \Gamma^{\delta\,a}{}_{bq}\,. \tag{5.53}$$

Expanding this equation with help of the definition of Γ^δ (equation (5.45)) in terms of F and using that g^F is covariantly constant (equation (5.39)) yields equation 5.52. From this representation of S it is clear that it vanishes in the metric limit $L = g_{ab}(x)y^a y^b$ and $F^2 = |g_{ab}(x)y^a y^b|$, since the Cartan tensor does so. $\square$

As a final remark on covariant derivatives observe that the linear covariant derivatives are related to the dynamical covariant derivative by

$$y^a \nabla^{CL/CL2/B}_{\delta_a} = \nabla\,, \tag{5.54}$$

and that the non-linear curvature of the Cartan non-linear connection can be expressed through the coefficients of the linear covariant derivatives as

$$\begin{aligned} R^a{}_{bc} &= [\delta_b, \delta_c]^a && (5.55)\\ &= -y^q\big(\delta_b \Gamma^{\delta a}{}_{qc} - \delta_c \Gamma^{\delta a}{}_{qb} + \Gamma^{\delta a}{}_{pb}\Gamma^{\delta p}{}_{qc} - \Gamma^{\delta a}{}_{pc}\Gamma^{\delta p}{}_{qb}\big) && (5.56)\\ &= -y^q\big(\delta_b(\bar{\partial}_q N^a{}_c) - \delta_c(\bar{\partial}_q N^a{}_b) + (\bar{\partial}_p N^a{}_b)(\bar{\partial}_q N^p{}_c) - (\bar{\partial}_p N^a{}_c)(\bar{\partial}_q N^p{}_b)\big). && (5.57) \end{aligned}$$

We succeeded in finding a distinguished non-linear connection whose autoparallels are arc-length parametrized geodesics of the Finsler spacetime. Moreover we could show that, due to the existence of the fundamental geometry function L, the non-linear connection, its curvature and its associated linear covariant derivatives are well-defined objects along all directions of spacetime except on the set A where the L metric degenerates. This is a huge advantage over the formulation of the geometry of a manifold in terms of the Finsler function F in case the length measure admits a non trivial null structure. Observe that the connections and the curvature tensors introduced here have the same metric limits for $L = g_{ab}(x)y^a y^b$ as the ones we discussed in context of Finsler spaces. The non-linear connection coefficients become $N^a{}_b(x,y) = \Gamma^a{}_{bc}y^c$, the coefficients of the covariant derivatives become $\Gamma^{\delta a}{}_{bc} = \Gamma^a{}_{bc}(x)$, $C^a{}_{bc} = 0$ and the non-linear curvature becomes $R^a{}_{bc}(x,y) = -R^a{}_{dbc}(x)y^d$, where the x dependent objects are the standard Christoffel symbols respectively the Riemann curvature tensor from metric geometry.

A distinguished connection, its curvature and associated covariant derivatives are the toolbox which describe the well-defined geometry of Finsler spacetimes. This geometric structure lays the foundation for the use of manifolds equipped with an in general non-metric fundamental geometry function L as geometric backgrounds for physics. But a geometric background for physics is only interesting if we can define physical theories which live on and determine this background. The most common way to do so is to introduce action integrals which define the dynamics of field theories. Therefore we will now discuss integration over homogeneous tangent bundle functions on Finsler spacetimes.

5.3. Integration and the unit tangent bundle

Classical physics is described by field theories on Lorentzian metric spacetimes. Since Finsler spacetimes present a natural generalization of metric spacetimes, we should be able to formulate field theories. We saw during the previous sections that all geometric objects on Finsler spacetimes are homogeneous tensor fields on the tangent bundle; the same will be true for physical fields in our construction. In this section we will develop the technology needed to write down well-defined action integrals, and to derive the corresponding equations of motion. As a first attempt we will consider integrals with homogeneous integrands over the whole tangent bundle. Studied in so called adapted coordinates, it will turn out that they cannot be well-defined. This observation leads us to the development of the unit tangent bundle where integrals over homogeneous tangent bundle functions can be defined. These integrals over the unit tangent bundle also resemble the situation on Finsler spaces in section 4.1 where integrals over the sphere bundle were discussed in connection to the Einstein–Hilbert action.

5.3.1. Tangent bundle integrals and adapted coordinates

The most obvious Ansatz for action integrals defining dynamics of tensors living on the tangent bundle of a Finsler spacetime (M, L, F) would be integrals of the form

$$\int_{TM} \mathrm{d}^4x\mathrm{d}^4y\, f(x,y)\,, \tag{5.58}$$

where f is some Lagrangian density of choice. All geometric objects on Finsler spacetimes and, as it will turn out, all physical fields which may appear in a Lagrangian density will have homogeneity properties with respect to the fibre coordinates y; and so will the Lagrangian density itself. Analysing integrals over the tangent bundle with homogeneous integrands f in adapted coordinates then reveals that the integral cannot be well-defined.

Instead of manifold induced coordinates $Z^A = (x^a, y^b);\ a, b = 0, \dots, 3;\ A = 0, \dots, 7$ on TM we consider now coordinates in which the homogeneity of functions is absorbed into one radial coordinate $\hat{Z}^B = (\hat{x}^a, u^\alpha, R);\ a = 0, \dots, 3; \alpha = 1, 2, 3;\ B = 0, \dots, 7$:

$$\hat{Z}(Z) = (\hat{x}^a(x,y), u^\alpha(x,y), R(x,y)) = (x^a, u^\alpha(x, y/F(x,y)), F(x,y))\,. \tag{5.59}$$

This is a well-defined coordinate transformation on $TM \setminus N_L$ where $L \neq 0$. The coordinate transformation matrices are given by

$$\frac{\partial \hat{Z}^A}{\partial Z^B} = \left[\begin{array}{c|c} \delta^a_b & 0 \\ \hline \partial_b u^\alpha & \bar{\partial}_b u^\alpha \\ \partial_b |L|^{1/r} & \bar{\partial}_b |L|^{1/r} \end{array}\right], \qquad \frac{\partial Z^A}{\partial \hat{Z}^B} = \left[\begin{array}{c|cc} \delta^a_b & 0 & 0 \\ \hline \hat{\partial}_b y^a & \partial_{u^\beta} y^a & \frac{y^a}{R} \end{array}\right]. \tag{5.60}$$

and satisfy the invertibility properties

$$\frac{\partial \hat{Z}^A}{\partial Z^C}\frac{\partial Z^C}{\partial \hat{Z}^B} = \left[\begin{array}{c|cc} \delta^a_b & 0 & \\ \hline \partial_b u^\alpha + \bar{\partial}_c u^\alpha \hat{\partial}_b y^c & \bar{\partial}_c u^\alpha \partial_{u^\beta} y^c & \bar{\partial}_c u^\alpha \frac{y^c}{R} \\ \partial_b |L|^{1/r} + \bar{\partial}_c |L|^{1/r} \hat{\partial}_b y^c & \bar{\partial}_c |L|^{1/r} \partial_{u^\beta} y^c & 1 \end{array}\right] = \left[\begin{array}{c|cc} \delta^a_b & 0 & \\ \hline 0 & \delta^\alpha_\beta & 0 \\ 0 & 0 & 1 \end{array}\right], \tag{5.61}$$

$$\frac{\partial Z^A}{\partial \hat{Z}^C}\frac{\partial \hat{Z}^C}{\partial Z^B} = \left[\begin{array}{c|c} \delta^a_b & 0 \\ \hline \hat{\partial}_b y^a + \partial_b u^\gamma \partial_{u^\gamma} y^a + \frac{y^a}{R}\partial_b |L|^{1/r} & \partial_{u^\gamma} y^a \bar{\partial}_b u^\gamma + \frac{y^a}{R}\bar{\partial}_b |L|^{1/r} \end{array}\right] = \left[\begin{array}{c|c} \delta^a_b & 0 \\ \hline 0 & \delta^a_b \end{array}\right]. \tag{5.62}$$

To see the problem with integrals like the one in equation (5.58) we look at the homogeneity property of f in the new coordinates. Let $f(x,y)$ be homogeneous of degree n with respect to y and let $\hat{\partial}_a$, ∂_α and ∂_R denote the derivatives with respect to $\hat{x}^a$, u^α respectively R, then, by the homogeneity properties of the $\hat{Z}$ coordinates and Euler's theorem 2.1,

$$\begin{aligned} y^a \bar{\partial}_a f(x,y) &= n f(x,y) && (5.63)\\ y^a \bar{\partial}_a f(\hat{x}(x,y), u(x, y/F(x,y)), R(x,y)) &= y^a(\bar{\partial}_a \hat{x}^q \hat{\partial}_q f + \bar{\partial}_a \hat{u}^\alpha \partial_\alpha f + \bar{\partial}_a R \partial_R f) && (5.64)\\ \Rightarrow R\partial_R f &= n f\,. && (5.65) \end{aligned}$$

Hence the homogeneity properties of a function f with respect to y are absorbed into homogeneity properties with respect to R, so $f(x,\lambda y) = f(\hat{x}, u, \lambda R) = \lambda^n f(x,u,R)$. Since R is just one coordinate we can rewrite the integral (5.58) to

$$\begin{aligned} f(\hat{x}, u^\alpha, R) &= R^n f(x, u^\alpha, 1) && (5.66)\\ \Rightarrow \int_{TM\setminus N_L} \mathrm{d}^4x \mathrm{d}^4y\; f(x,y) &= \int_{TM\setminus N_L} \mathrm{d}^4\hat{x}\mathrm{d}^3u\mathrm{d}R\; \det\left[\frac{\partial Z^A}{\partial \hat{Z}^B}\right] R^n f(\hat{x}, u, 1)\,, && (5.67) \end{aligned}$$

where N_L denotes the null structure of the Finsler spacetimes where L vanishes. No matter what properties $f(\hat{x},u,1)$ has, there always exist functions of homogeneity n for which the R integration of the above integral diverges. Even if for a special n the R integration may not be problematic since there may appear cancellations with the Jacobian from the coordinate transformation, we have to conclude that the integral is not well-defined for general homogeneous functions.

When it comes to the variation of an action integral constructed as an integral over TM like the one discussed here, the homogeneity divergence forbids us to derive and read off the field equations for the homogeneous object of interest. In general, a field ϕ on Finsler spacetimes is a tensor field with fixed homogeneity on the tangent bundle, as we will describe in detail in chapter 9. Therefore, the components of the field $\phi^{A\cdots}{}_{B\ldots}(x,y)$ are homogeneous of some degree n with respect to y, and the variation of an action integral $S[\phi]$ built on the basis of (5.58) gives

$$\begin{aligned} \delta S[\phi] &= \int_{TM} \mathrm{d}^4x\mathrm{d}^4y\,\delta\tilde{\mathcal{L}}[\phi](x,y) = \int_{TM} \mathrm{d}^4x\mathrm{d}^4y \left(\frac{\delta S[\phi]}{\delta\phi^{A\cdots}{}_{B\ldots}}\delta\phi^{A\cdots}{}_{B\ldots}\right)(x,y)\\ &= \int_{TM\setminus N_L} \mathrm{d}^4\hat{x}\mathrm{d}^3u\mathrm{d}R\;\det\left[\frac{\partial Z^A}{\partial\hat{Z}^B}\right] R^n \left(\frac{\delta S[\phi]}{\delta\phi^{A\cdots}{}_{B\ldots}}\delta\phi^{A\cdots}{}_{B\ldots}\right)(\hat{x},u,1) && (5.68) \end{aligned}$$

The integral over R is sensitive to the n-homogeneity of the integrand and diverges. Consequently we cannot require $\delta S[\phi]=0$ in order to read off equations of motion. Note that this problem cannot be cured by considering compactly supported $\delta\phi^{A\ldots}{}_{B\ldots}$; these simply do not exist because their homogeneity always leads to non-compact support along the fibre directions.

Anyway we are interested in integrals over general homogeneous functions and their variation with respect to homogeneous fields, in order to be as free as possible in the construction of field theory actions. The solution to this problem is to consider only the $\mathrm{d}^4\hat{x}\mathrm{d}^3u$ part of the integral

(5.68) and to omit the $\mathrm{d}R$ part. Technically clean this will be achieved by the introduction of the unit tangent bundle, on which the homogeneity of homogeneous tangent bundle functions is removed without the loss of information.

5.3.2. Action integrals on the unit tangent bundle

Since integrals over the tangent bundle over homogeneous functions diverge generically, we must consider an alternative formulation of action integrals. We construct them on the unit tangent bundle and show that they lead to well-defined field equations through variation.

In the adapted coordinates defined in equation (5.59) it is clear that every function $f(x,y)$ homogeneous of degree n with respect to y can be expressed through

$$R^n f(\hat{x},u,1) = R^n f_{|R=1}(x,u). \tag{5.69}$$

Hence knowing a homogeneous function on the tangent bundle is equivalent to know the function on the set $R=1$ and its homogeneity. We call the set where $R=1$ the unit tangent bundle. It generalises the sphere bundle of equation (4.3) from Finsler spaces to Finsler spacetimes

Definition 5.3. ***The unit tangent bundle***
Let (M,L,F) be a Finsler spacetime. The unit tangent bundle Σ is the set in TM on which the induced Finsler function F is unity

$$\Sigma = \{(x,y)\in TM | F(x,y)=1\}\,. \tag{5.70}$$

The restriction of a n-homogeneous function f on the tangent bundle to the unit tangent bundle is

$$f(x,y)_{|\Sigma} = f(\hat{x},u,1) = \frac{1}{R^n} f(\hat{x},u,R) = \frac{1}{F(x,y)^n} f(x,y)\,. \tag{5.71}$$

The other way around one can extend any function $h(\hat{x},u)$ on Σ to an n-homogeneous function $h(x,y)$ on the tangent bundle

$$h(x,y) = R(x,y)^n h(\hat{x}(x,y), u(x, y/F(x,y)))\,. \tag{5.72}$$

For us the following procedure is important. Starting from homogeneous objects on the tangent bundle, we will restrict them to the unit tangent bundle, where we write down an action which yields field equations on the unit tangent bundle for the restricted object. This equation can then be extended to the whole tangent bundle by suitable multiplication with $F(x,y)=R$ and determines the unrestricted object. Observe that zero-homogeneous objects on the tangent bundle are identical to their restriction. With this in mind we consider the following integrals for homogeneous f

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ (f(x,y))_{|\Sigma} = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ f(\hat{x},u,1)\,. \tag{5.73}$$

To formulate action integrals in terms of Lagrange functions instead of densities we now introduce a metric G on TM. Its pull-back G^* of G to Σ induces a canonical volume form. The requirements for the metric are that it shall be compatible with the geometric structure of the Finsler spacetime, i.e., respect the horizontal-vertical structure induced by the Cartan non-linear connection. Horizontal and vertical vectors shall be orthogonal and be treated equivalently, the metric shall be covariant constant with respect to the dynamical covariant derivative and it shall be homogeneous as tensor. This directly leads to two Finsler spacetime versions of the Sasaki-type metric which we already encountered during the generalisation of the Einstein–Hilbert action on Finsler spaces in equation (4.8):

Definition 5.4. ***Sasaki metric on Finsler spacetimes***
Let (M, L, F) be a Finsler spacetime and $\mathrm{d}x^a$ and δy^a be the local horizontal and vertical cotangent bundle basis on the tangent bundle. The following metrics on the tangent bundle are called F Sasaki metric G^F and L Sasaki metric G^L

$$\begin{aligned} G^F &= -g^F_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b - \frac{1}{F^2} g^F_{ab}\delta y^a \otimes \delta y^b && (5.74) \\ G^L &= g^L_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b + \frac{g^L_{ab}}{F^2}\delta y^a \otimes \delta y^b\,. && (5.75) \end{aligned}$$

They are 0 respectively $r-2$-homogeneous as tensor and identical in the metric limit $L = g_{ab}(x)y^a y^b$ for timelike ($g_{ab}(x)y^a y^b < 0$) directions.

When we discuss the coupling of fields to the geometry in chapter 9 we will work with the metric G^F. It will turn out that after deriving equations of motion on Σ and lifting them to TM the equations obtained from a coupling to G^F are preferable to the ones obtained with respect to G^L. The homogeneity of the metrics is counted tensorially by including the homogeneity of the basis elements. On the set N_L where L vanishes this leads to a degeneracy of G^L and G^F. This issue does not bother us, since we are interested in the pull-back of the two metrics to Σ. To calculate this pull-back we perform a coordinate transformation from manifold induced coordinates to adapted coordinates and restrict ourselves afterwards to the set $R = 1$.

Theorem 5.6. ***Sasaki metrics in adapted coordinates***
The metrics G^F and G^L from definition 5.4 can be expressed in the adapted coordinates $(\hat{x}, u, R)$ (see equation (5.59)) as

$$\begin{aligned} G^F &= -g^F_{ab}\,\mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b - \frac{1}{R^2} h^F_{\alpha\beta}\,\delta u^\alpha \otimes \delta u^\beta - \frac{1}{R^2}\,\mathrm{d}R \otimes \mathrm{d}R && (5.76) \\ G^L &= g^L_{ab}\mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b + \frac{h^L_{\alpha\beta}}{R^2}\delta u^\alpha \otimes \delta u^\beta + \frac{r(r-1)L}{2\,R^4}\mathrm{d}R \otimes \mathrm{d}R\,. && (5.77) \end{aligned}$$

where $h^{F\backslash L} = g^{F\backslash L}_{ab}\partial_\alpha y^a \partial_\beta y^b$ and $\delta u^\alpha = du^\alpha + (\bar{\partial}_b u^\alpha N^b{}_a - \partial_a u^\alpha)d\hat{x}^a = du^\alpha + N^\alpha{}_a d\hat{x}^a$. Recall that r is the degree of homogeneity of L.

Proof of Theorem 5.6.
The formulae are proven by expressing the horizontal-vertical cotangent basis from the induced coordinates with the help of the coordinate transformation matrices (5.60) in the new coordinates. For simplifications the invertibility properties from equations (5.61) and (5.62) are used. The detailed calculation is postponed to the appendix A.2 due to its length. □

Observe that the signature of $h^{F\backslash L}_{ab}$ can be determined by knowing the signature of $g^{F\backslash L}_{ab}$ and the sign of L, since the signature of $G^{F\backslash L}$ is always $(\mathrm{sign}(g^{F\backslash L}), \mathrm{sign}(g^{F\backslash L}))$. The pull-back to Σ can now easily be calculated by setting $\mathrm{d}R = 0$ and restricting the components

$$\begin{aligned} G^{F*} &= -g^F_{ab|\Sigma}\,\mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b - h^F_{\alpha\beta|\Sigma}\,\delta u^\alpha \otimes \delta u^\beta && (5.78) \\ G^{L*} &= g^L_{ab|\Sigma}\mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b + h^L_{\alpha\beta|\Sigma}\delta u^\alpha \otimes \delta u^\beta\,. && (5.79) \end{aligned}$$

The volume element induced by these metrics is expressed with the help of their determinants

$$\det G^{F*} = \det g^F_{|\Sigma} \det h^F_{|\Sigma}, \quad \det G^{L*} = \det g^L_{|\Sigma} \det h^L_{|\Sigma}\,. \tag{5.80}$$

Their difference lies only in a factor due to the relation between the L and F metric (equations (5.15) and (5.16)) and the invertibility properties (5.61) and (5.62):

$$\begin{aligned} \det g^F_{|\Sigma} &= (\det g^F)_{|\Sigma} = (\frac{16F^{(8-4r)}}{r^4(r-1)} \det g^L)_{|\Sigma} = \frac{16}{r^4(r-1)} \det g^L_{|\Sigma}, & (5.81)\\ h^F_{\alpha\beta|\Sigma} &= (g^F_{ab}\partial_\alpha y^a \partial_\beta y^b)_{|\Sigma} = \frac{2}{r} h^L_{\alpha\beta|\Sigma} \Rightarrow \det h^F_{|\Sigma} = \frac{8}{r^3} \det h^L_{|\Sigma}, & (5.82)\\ \Rightarrow \det G^{F*} &= \frac{128}{r^7(r-1)} \det G^{L*}\,. & (5.83) \end{aligned}$$

Note that for $r = 2$ the determinants coincide. Using the shorthand notation $g^{F\backslash L} = |\det g^{F\backslash L}_{ab}|$ and $h^{F\backslash L} = |\det h^{F\backslash L}_{\alpha\beta}|$, a well-defined integral over Σ of a homogeneous tangent bundle function f now reads

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^{F\backslash L} h^{F\backslash L}}_{|\Sigma}\, f(x,y)_{|\Sigma}\,. \tag{5.84}$$

It is invariant under manifold induced diffeomorphisms, respectively coordinate changes, by construction. The d^4x and $\sqrt{g^{F\backslash L}}$ terms form an invariant scalar, the d^3u and $\sqrt{h^{F\backslash L}}$ are separately invariant and f is a scalar function. Such integrals can now be used as action integrals for fields ϕ on TM replacing the divergent equation (5.68) by

$$\delta S[\phi] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \delta(\tilde{\mathcal{L}}[\phi](x,y))_{|\Sigma} = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u, \Big(\frac{\delta S[\phi]}{\delta\phi^{A\dots}{}_{B\dots}} \delta\phi^{A\dots}{}_{B\dots}\Big)_{|\Sigma}(\hat{x},u)\,. \tag{5.85}$$

One obtains the same equations from $\delta S[\phi] = 0$ no matter whether the volume element we use is based on the metric G^{F*} or G^{L*}, thanks to the relation (5.83).

By restricting the domain of integration from the tangent bundle TM to the unit tangent bundle Σ we achieved the construction of integrals which may serve as field theory actions with well-defined variation. That means it is possible to read off equations of motion by considering variations with compact support. The solutions of the equations of motions are extremal points of the action.

We complete this section by deriving important integration by parts formulae which are used extensively to derive field equations in the chapters 6 and 9.

5.3.3. Integration by parts

For tangent bundle functions $A^a(x,y)$ that are homogeneous of degree m the following formulae for integration by parts, expressed in terms of the L and F metric as well as the connection coefficients of the Cartan linear covariant derivative (equation (5.45)) and the S tensor (equation (5.51)), hold

$$\begin{aligned} \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^F h^F}_{|\Sigma}\big(\delta_a A^a\big)_{|\Sigma} &= -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^F h^F}_{|\Sigma}\Big[\big(\Gamma^{\delta\,p}{}_{pa} + S^p{}_{pa}\big)A^a\Big]_{|\Sigma}, & (5.86)\\ \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^L h^L}_{|\Sigma}\big(\delta_a A^a\big)_{|\Sigma} &= -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^L h^L}_{|\Sigma}\Big[\big(\Gamma^{\delta\,p}{}_{pa} + S^p{}_{pa}\big)A^a\Big]_{|\Sigma},\\ \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^F h^F}_{|\Sigma}\big(\bar\partial_a A^a\big)_{|\Sigma} &= -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^F h^F}_{|\Sigma}\Big[\big(g^{F\,pq}\bar\partial_a g^F_{pq} - (m+3)y^p g^F_{pa}\big)A^a\Big]_{|\Sigma},\\ \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^L h^L}_{|\Sigma}\big(\bar\partial_a A^a\big)_{|\Sigma} &= -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^L h^L}_{|\Sigma}\Big[\big(g^{L\,pq}\bar\partial_a g^L_{pq} - \frac{2(4r+m-5)}{r(r-1)L} y^p g^L_{pa}\big)A^a\Big]_{|\Sigma}, \end{aligned}$$

We omitted boundary terms since, in the context where we apply these formulae, during the variation of action integrals, the boundary terms always vanish. For covariant derivatives the following formulae arise

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\nabla^{CL}_{\delta_a}A^a\big)_{|\Sigma} = -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[S^p{}_{pa}A^a\Big]_{|\Sigma}\,, \tag{5.87}$$

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\nabla^{B}_{\delta_a}A^a\big)_{|\Sigma} = -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[2S^p{}_{pa}A^a\Big]_{|\Sigma}\,, \tag{5.88}$$

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\nabla A\big)_{|\Sigma} = 0\,. \tag{5.89}$$

The relations for the covariant derivatives are straightforwardly derived from the relations for the horizontal and vertical derivatives above. They in turn can be proven with the help of the coordinate transformation relations (5.60) and the invertibility relations (equations (5.61) and (5.62)). Explicit calculations can be found in the appendix A.3.

With the ability to define physical fields theories from action integrals that we gained during this section, we are one step further in using Finsler spacetimes as non-metric geometrical background in physics. As a further mathematical feature we now introduce symmetries of Finsler spacetimes.

5.4. Symmetries

Symmetries of the background geometry of spacetime are associated to different physical situations. Spherical symmetry is used in the description of the solar system, homogeneous and isotropic symmetry for cosmology and flat maximal symmetry in the absence of gravity. In these symmetric situations the task to solve the dynamical equations describing the evolution of a physical system simplify enormously. Here we introduce the concept of symmetries on Finsler spacetimes and discuss the consequences of spherical, homogeneous and isotropic and maximal symmetry on the fundamental geometry function.

5.4.1. Definition

On a symmetric Lorentzian manifold (M, g), the metric is invariant under certain diffeomorphisms; similarly we wish to define symmetries of a Finsler spacetime (M, L, F) as an invariance of the fundamental geometry function L. Consider a diffeomorphism generated by the vector field $X = \xi^a(x)\partial_a$; this acts as a coordinate change on local coordinates on M as $(x^a) \to (x^a + \xi^a)$, and on the induced coordinates on the tangent bundle TM defined in section 1.2 as $(x^a, y^a) \to (x^a + \xi^a, y^a + y^q\partial_q\xi^a)$. Hence the diffeomorphism on M induces a diffeomorphism on TM that is generated by the complete lift $X^C = \xi^a\partial_a + y^q\partial_q\xi^a\bar{\partial}_a$ of X (see definition 1.11). The idea of implementing symmetries via complete lifts in a Finsler geometry setting appears already in [49]; here we want to make this concept precise for Finsler spacetimes and next apply it concretely to maximal, cosmological and spherical symmetry:

Definition 5.5. *Symmetries of Finsler spacetimes*
A symmetry of a Finsler spacetime (M, L, F) is a diffeomorphism generated by a vector field Y over the tangent bundle TM so that $Y(L) = 0$ and Y is the complete lift X^C of a vector field X over M. A Finsler spacetime is called symmetric if it possesses at least one symmetry.

The following theorem summarises important properties of Finsler spacetime symmetries. The symmetry generators form a Lie algebra with the commutator of vector fields on TM, and they are isomorphic to a Lie algebra of vector fields on M which becomes the usual symmetry algebra of Lorentzian manifolds in the metric geometry limit. This not only shows that Definition 5.5 of symmetry is consistent with that of Lorentzian spacetimes, but also that the usual Killing vectors, e.g., those for spherical symmetry, can be used to study symmetries of Finsler spacetimes.

Theorem 5.7. *The symmetry algebra*
Let $\mathcal{S}$ be the set of symmetry-generating vector fields of a Finsler spacetime as defined in definition 5.5 above.

(i) $(\mathcal{S}, [\cdot, \cdot])$ is a Lie subalgebra of the set of vector fields over TM;

(ii) $(\mathcal{S}, [\cdot, \cdot])$ is isomorphic to the Lie subalgebra $(\pi_*(\mathcal{S}), [\cdot, \cdot])$ of the set of vector fields over M;

(iii) in the metric geometry limit, $(\pi_*(\mathcal{S}), [\cdot, \cdot])$ becomes the symmetry algebra of the emerging Lorentzian spacetime.

Proof of Theorem 5.7.
We prove each statement of the theorem separately:

(i) Let $Y \in \mathcal{S}$; then $Y(L) = 0$ and $(\pi_* Y)^C - Y = 0$. Both properties are linear, so that $\mathcal{S}$ is a vector subspace of the Lie algebra of all vector fields on TM. It remains to be proven that the commutator of two elements $Y_1, Y_2 \in \mathcal{S}$ closes in $\mathcal{S}$. It is clear that $[Y_1, Y_2](L) = 0$; to show that $(\pi_*[Y_1, Y_2])^C = [Y_1, Y_2]$, one uses that $Y_i = X_i^C$ for some vector fields X_i on M and that $[X_1^C, X_2^C] = [X_1, X_2]^C$. The latter is shown by expanding both sides and use of the definition of complete lifts of vector fields.

(ii) The inverse for π_* on $\pi_*(\mathcal{S})$ is given by the complete lift, hence $\mathcal{S}$ and $\pi_*(\mathcal{S})$ are isomorphic as vector spaces. The Lie algebra structure is preserved in both directions because of $[X_1^C, X_2^C] = [X_1, X_2]^C$, and hence also $\pi_*[Y_1, Y_2] = [\pi_* Y_1, \pi_* Y_2]$.

(iii) For $Y = X^C \in \mathcal{S}$, we have $\xi^a \partial_a L + y^q \partial_q \xi^a \bar{\partial}_a L = 0$. In the metric geometry limit $L(x, y) = g_{ab}(x) y^a y^b$, and hence $y^p y^q (\xi^a \partial_a g_{pq} + g_{ap} \partial_q \xi^a + g_{aq} \partial_p \xi^a) = y^p y^q \mathcal{L}_X g_{pq}(x) = 0$. Since the Lie-derivative of the metric g does not depend on the fibre coordinates of the tangent bundle we conclude $\mathcal{L}_X g_{pq}(x) = 0$. This is the condition that defines X as the symmetry generator of a metric spacetime. $\square$

The consequences of what is called spherical, cosmological and maximal symmetry will be investigated now.

5.4.2. Spherical symmetry

Consider a Finsler spacetime (M, L, F) and coordinates $(t, r, \theta, \phi, y^t, y^r, y^\theta, y^\phi)$ on its tangent bundle. Spherical symmetry is defined by the following three vector fields, that generate spatial rotations and form the algebra $\mathfrak{so}(3)$,

$$X_4 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi\,, \quad X_5 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi\,, \quad X_6 = \partial_\phi\,. \tag{5.90}$$

Their complete lifts are obtained by utilising definition 1.11: For $X = \xi^a\partial_a$ its complete lift is $X^C = \xi^a\partial_a + y^q\partial_q\xi^a\bar{\partial}_a$. Hence

$$\begin{aligned}
X_4^C &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi + y^\phi\cos\phi\bar{\partial}_\theta - \Big(y^\theta\frac{\cos\phi}{\sin^2\theta} + y^\phi\cot\theta\sin\phi\Big)\bar{\partial}_\phi\,, &(5.91)\\
X_5^C &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi + y^\phi\sin\phi\bar{\partial}_\theta - \Big(y^\theta\frac{\sin\phi}{\sin^2\theta} - y^\phi\cot\theta\cos\phi\Big)\bar{\partial}_\phi\,, &(5.92)\\
X_6^C &= \partial_\phi\,. &(5.93)
\end{aligned}$$

Applying the symmetry condition $X_6^C(L) = 0$ implies $\partial_\phi L = 0$, while using $X_4^C(L) = 0$ and $X_5^C(L) = 0$ to deduce $(\sin\phi X_4^C - \cos\phi X_5^C)(L) = 0$ and $(\cos\phi X_4^C + \sin\phi X_5^C)(L) = 0$ yields

$$\partial_\theta L = y^\phi\cot\theta\bar{\partial}_\phi L\,, \quad y^\phi\sin^2\theta\bar{\partial}_\theta L = y^\theta\bar{\partial}_\phi L\,. \tag{5.94}$$

In order to analyse the implications of these equations on L we introduce new coordinates

$$u(\theta) = \theta\,, \quad v(y^\theta) = y^\theta\,, \quad w(\theta, y^\theta, y^\phi)^2 = (y^\theta)^2 + \sin^2\theta(y^\phi)^2\,, \tag{5.95}$$

while keeping (t, y^t, r, y^r, ϕ). The associated transformation of the derivatives

$$\begin{aligned}
\partial_t &= \partial_t\,, \quad \partial_r = \partial_r\,, \quad \partial_\theta = \frac{w^2 - v^2}{w}\cot u\partial_w + \partial_u\,, \quad \partial_\phi = \partial_\phi\,, &(5.96)\\
\bar{\partial}_t &= \bar{\partial}_t\,, \quad \bar{\partial}_r = \bar{\partial}_r\,, \quad \bar{\partial}_\theta = \frac{v}{w}\partial_w + \partial_v\,, \quad \bar{\partial}_\phi = \sin u\frac{\sqrt{(w^2 - v^2)}}{w}\partial_w\,, &(5.97)
\end{aligned}$$

makes the equations (5.94) equivalent to the simple constraints $\partial_u L = 0$ and $\partial_v L = 0$.

Hence we conclude from the analysis of the symmetry conditions $X_i^C(L) = 0$ that the most general spherically symmetric Finsler spacetime is described by a fundamental function which is r-homogeneous in (y^t, y^r, w) and of the form

$$L(t, r, \theta, \phi, y^t, y^r, y^\theta, y^\phi) = L(t, r, y^t, y^r, w(\theta, y^\theta, y^\phi))\,, \tag{5.98}$$

where $w(\theta, y^\theta, y^\phi)$ is defined in (5.95).

5.4.3. Cosmological symmetry

After our discussion of the spherically symmetric case in full detail above, we will now present the results of a similar analysis for cosmologically Finsler spacetimes before we discuss maximal symmetry.

Cosmological symmetry describes an isotropic and homogeneous spacetime. This is a much more symmetric situation than in the spherically symmetric scenario, and is implemented by requiring the following six vector fields to be symmetry generators, see for example [50],

$$X_1 = \chi \sin\theta \cos\phi \partial_r + \frac{\chi}{r} \cos\theta \cos\phi \partial_\theta - \frac{\chi}{r} \frac{\sin\phi}{\sin\theta} \partial_\phi \,, \tag{5.99}$$
$$X_2 = \chi \sin\theta \sin\phi \partial_r + \frac{\chi}{r} \cos\theta \sin\phi \partial_\theta + \frac{\chi}{r} \frac{\cos\phi}{\sin\theta} \partial_\phi \,, \tag{5.100}$$
$$X_3 = \chi \cos\theta \partial_r - \frac{\chi}{r} \sin\theta \partial_\theta \,, \tag{5.101}$$
$$X_4 = \sin\phi \partial_\theta + \cot\theta \cos\phi \partial_\phi \,, \quad X_5 = -\cos\phi \partial_\theta + \cot\theta \sin\phi \partial_\phi \,, \quad X_6 = \partial_\phi \,, \tag{5.102}$$

where we write $\chi = \sqrt{1 - kr^2}$ and k is constant. The complete lifts of these vector fields are listed in appendix A.4. Applying the symmetry conditions $X_i^C(L) = 0$ to the fundamental function L and introducing the new coordinates

$$q(r) = r \,, \quad s(y^r) = y^r \,, \quad u(\theta) = \theta \,, \quad v(y^\theta) = y^\theta \,, \tag{5.103}$$
$$w_C(r, \theta, y^r, y^\theta, y^\phi)^2 = \frac{(y^r)^2}{1 - kr^2} + r^2 \big((y^\theta)^2 + \sin^2\theta (y^\phi)^2 \big), \tag{5.104}$$

while keeping (t, y^t) yields the following result: the homogeneous and isotropic fundamental function L is r-homogeneous in (y^t, w_C) and has the form

$$L(t, r, \theta, \phi, y^t, y^r, y^\theta, y^\phi) = L(t, y^t, w_C(r, \theta, y^r, y^\theta, y^\phi)) \,. \tag{5.105}$$

The constant k only appears in the expression for the coordinate w_C. The value of w_C can be understood as the metric length measure on a three dimensional manifold of constant curvature k. The same metric appears in the spatial part of the standard Robertson–Walker metric.

5.4.4. Maximal symmetry

A Finsler spacetime is called maximally symmetric if it admits the maximal number of symmetry vectors possible. At this point this statement is vague since it is not clear that there exist a maximal number of symmetry vectors, as is the case on metric manifolds, see for example [51]. To see that this really is the case consider the following argument. First we prove that the second derivative of a symmetry vector field at a point is completely determined by its first derivative and itself at that point; hence by applying derivatives to the second derivative the same holds for all higher derivatives. Second we conclude from the Taylor expansion of the symmetry vector field around a given point that there can maximally be be $n(n+1)/2$ independent symmetry vector fields in n dimensions, hence 10 for $n = 4$.

Having proven that there exists a maximal number of symmetry vector fields on Finsler spacetimes we analyse them and find that they have to be equal to the maximal symmetric Lorentzian manifolds.

Theorem 5.8. ***Second derivative of a symmetry vector field***
Let (M, L, F) be a Finsler spacetime and $X = \xi^a(x)\partial_a$ be a symmetry vector field as defined in definition 5.5. The second derivative of the horizontal lift $X^H = \xi^a \delta_a$ of a symmetry vector field, $\nabla^B_{\delta_a} \nabla^B_{\delta_b} \xi^c$, expressed through the Berwald linear covariant derivative ∇^B, can be expressed through the first derivative $\nabla^B_{\delta_a} \xi^c$ and the vector field ξ^a itself.

Proof of Theorem 5.8.
Since the fundamental geometry functions is horizontally constant, $\delta_a L = 0$, the symmetry condition $X^C(L) = 0$ can written as

$$\nabla(\xi^m \bar{\partial}_m L) = 0\,. \tag{5.106}$$

Performing two derivatives with respect to the y coordinates, introducing the Berwald linear covariant derivative (equation (5.49)) and the S-tensor defined in (5.51) we obtain from (5.106)

$$\nabla^B_{\delta_p}\xi_q + \nabla^B_{\delta_q}\xi_p + \bar{\partial}_m g^L_{pq}\nabla\xi^m - 2\xi^a S^r{}_{ap} g^L_{rq} = 0\,. \tag{5.107}$$

The curvature of the Berwald connection $R^{Bq}{}_{cab}$ satisfies the following cyclic summation identity

$$R^{Bq}{}_{cab} + R^{Bq}{}_{abc} + R^{Bq}{}_{bca} = 0\,, \tag{5.108}$$

and the commutator of the Berwald linear covariant derivative can be written as

$$\nabla^B_{\delta_a}\nabla^B_{\delta_b}\xi_c - \nabla^B_{\delta_b}\nabla^B_{\delta_a}\xi_c = R^{Bq}{}_{cab}\xi_q + R^q{}_{ab}\bar{\partial}_q g^L_{cd}\xi^d\,. \tag{5.109}$$

A cyclic summation of (5.109) together with (5.108) yields

$$\begin{aligned} &\nabla^B_{\delta_a}\nabla^B_{\delta_b}\xi_c - \nabla^B_{\delta_b}\nabla^B_{\delta_a}\xi_c + \nabla^B_{\delta_b}\nabla^B_{\delta_c}\xi_a - \nabla^B_{\delta_c}\nabla^B_{\delta_b}\xi_a + \nabla^B_{\delta_c}\nabla^B_{\delta_a}\xi_b - \nabla^B_{\delta_a}\nabla^B_{\delta_c}\xi_b \\ = \;& R^q{}_{ab}\bar{\partial}_q g^L_{cp}\xi^p + R^q{}_{bc}\bar{\partial}_q g^L_{ap}\xi^p + R^q{}_{ca}\bar{\partial}_q g^L_{bp}\xi^p\,. \end{aligned} \tag{5.110}$$

Using (5.107) on $\nabla^B_{\delta_b}\nabla^B_{\delta_c}\xi_a$, $\nabla^B_{\delta_c}\nabla^B_{\delta_a}\xi_b$ and $\nabla^B_{\delta_a}\nabla^B_{\delta_c}\xi_b$ to interchange the index on the second covariant derivative with the index on the vector field, and using (5.109) to recombine $\nabla^B_{\delta_a}\nabla^B_{\delta_b}\xi_c - \nabla^B_{\delta_b}\nabla^B_{\delta_a}\xi_c$ to curvature tensors yields for the second derivative of the symmetry vector field

$$\begin{aligned} 2\nabla^B_{\delta_c}\nabla^B_{\delta_b}\xi_a \;=\;& 2R^{Bq}{}_{cab}\xi_q + \xi^i(R^q{}_{ab}\bar{\partial}_q g^L_{ci} - R^q{}_{bc}\bar{\partial}_q g^L_{ai} - R^q{}_{ca}\bar{\partial}_q g^L_{bi}) \\ +\;& \nabla^B_{\delta_a}(\bar{\partial}_m g^L_{bc}\nabla\xi^m - 2\xi^d S^q{}_{bd} g^L_{cq}) \\ -\;& \nabla^B_{\delta_b}(\bar{\partial}_m g^L_{ac}\nabla\xi^m - 2\xi^d S^q{}_{ad} g^L_{cq}) \\ -\;& \nabla^B_{\delta_c}(\bar{\partial}_m g^L_{ab}\nabla\xi^m - 2\xi^d S^q{}_{ad} g^L_{bq})\,. \end{aligned} \tag{5.111}$$

On the right hand side of the equations are still contracted second derivatives acting on the symmetry vector field of the form $\nabla_{\delta_a}\nabla\xi_b$. To get rid of these terms we contract the above equation with y^b and y^c and obtain with the help of equation (5.107) and the facts that $S^q{}_{ab}y^b = 0$ as well as $S^q{}_{ab}g^L_{qi}y^i = 0$

$$y^b y^c \nabla^B_{\delta_c}\nabla^B_{\delta_b}\xi_a = \nabla\nabla\xi_a = R^q{}_{ab}y^b\xi_q\,. \tag{5.112}$$

Contracting (5.111) with y^b we obtain an expression for $\nabla^B_{\delta_a}\nabla\xi^m$ in terms of lower order derivatives of ξ

$$\begin{aligned} & 2y^b\nabla^B_{\delta_c}\nabla^B_{\delta_b}\xi_a = 2\nabla^B_{\delta_c}\nabla\xi_a \\ =\;& 2R^{Bq}{}_{cab}\xi_q y^b + y^b\xi^i(R^q{}_{ab}\bar{\partial}_q g^L_{ci} - R^q{}_{bc}\bar{\partial}_q g^L_{ai} - R^q{}_{ca}\bar{\partial}_q g^L_{bi}) \\ & +\nabla^B_{\delta_a}((r-2)\nabla\xi_c) - \nabla(\bar{\partial}_m g^L_{ac}\nabla\xi^m - 2\xi^d S^q{}_{ad} g^L_{cq}) - \nabla^B_{\delta_c}((r-2)\nabla\xi_a) \\ =\;& 2R^{Bq}{}_{cab}\xi_q y^b + y^b\xi^i(R^q{}_{ab}\bar{\partial}_q g^L_{ci} - R^q{}_{bc}\bar{\partial}_q g^L_{ai} - R^q{}_{ca}\bar{\partial}_q g^L_{bi}) + (r-2)(\nabla^B_{\delta_a}\nabla\xi_c - \nabla^B_{\delta_c}\nabla\xi_a) \\ & -\nabla(\bar{\partial}_m g^L_{ac})\nabla\xi^m - \bar{\partial}_m g^L_{ac}\nabla\nabla\xi^m + 2\nabla(\xi^d S^q{}_{ad} g^L_{cq})\,. \end{aligned} \tag{5.113}$$

In the last line of this equation the second derivative $\nabla\nabla\xi^m$ can be reduced to the symmetry vector field itself by use of the equation (5.112) and the remaining second derivative term $\nabla^B_{\delta_a}\nabla\xi_c - \nabla^B_{\delta_c}\nabla\xi_a$ combines again into a curvature expression. Contracting (5.107) with y^p yields $(r-1)\nabla\xi_q = \nabla^B_q(\xi_c y^c)$. Acting with another ∇^B_p on this expression, anti-symmetrising the free indices and using (5.109) yields

$$\nabla^B_{\delta_a}\nabla\xi_c - \nabla^B_{\delta_c}\nabla\xi_a = -R^q{}_{ac}\xi_q\,. \tag{5.114}$$

Thus we can replace all second derivative of ξ appearing on the right hand side of equation (5.111) with first and zeroth derivatives wit the help equations (5.112) and (5.113). □

Theorem 5.8 guarantees that wherever a second or higher derivative of the symmetry vector field appears, as for example in a Taylor expansion, they can be rewritten in terms of first derivatives and the original ξ^a. Hence symmetry vector fields around a point p are completely determined by the expressions ξ_a and $\nabla^B_{\delta_a}\xi_b$ at p of which there are $n+n^2$ in n dimensions. But due to the symmetry equation $X^C(L)=0$ these are not all independent, equation (5.107) can be rewritten as

$$L^{pq}{}_{(ab)}\nabla^B_p\xi_q = (\delta^q_a\delta^p_b + \delta^q_b\delta^p_q + y^p g^{Lqm}\bar\partial_m g^L_{ab})\nabla^B_p\xi_q = 2\xi^p S^q{}_{pa}g^L_{qb}\,, \tag{5.115}$$

which are $n(n+1)/2$ constraint equations on the n^2 components $\nabla^B_p\xi_q$. Thus there are only $n+n^2-n(n+1)/2 = n(n+1)/2$ independent ξ_a and $\nabla^B_{\delta_a}\xi_b$ which determine the symmetry vector fields completely. In four dimensions this makes exactly ten, as claimed above.

Using the notations from [51] to analyse the situation on Finsler spacetimes the ten symmetry vectors in four dimensions generating maximal symmetry are given by

$$X_\alpha = C(x)\alpha^c\partial_c\,,\quad X_\Omega = \Omega^a{}_b x^b\partial_a\,, \tag{5.116}$$

with $C(x)=\sqrt{1-KC_{pq}x^px^q}$, constant K, and constant 4×4 matrices C_{ab} and $\Omega^a{}_b$ satisfying $\Omega^q{}_bC_{qa} = -\Omega^q{}_aC_{qb}$. The four X_α are called quasi translations and the six X_Ω are called quasi rotations; their complete lifts are

$$X^C_\alpha = C(x)\alpha^c\partial_c - y^b\frac{KC_{bm}x^m}{C(x)}\alpha^c\bar\partial_c\,,\quad X^C_\Omega = \Omega^a{}_bx^b\partial_a + y^b\Omega^a{}_b\bar\partial_a\,. \tag{5.117}$$

Evaluating the symmetry conditions $X^C_\alpha(L)=0$ and $X^C_\Omega(L)=0$ on the fundamental function, and introducing new coordinates

$$u^a(x)=x^a\,,\quad v^\gamma(y)=y^\gamma\,,\quad w_M(x,y)^2 = C_{ab}y^ay^b + \frac{K}{C(x)^2}C_{ap}x^py^aC_{bq}x^qy^b = g_{ab}(x)y^ay^b\,, \tag{5.118}$$

where γ runs over any three indices in $\{0,1,2,3\}$, yields the following result: the maximally symmetric fundamental function L is r-homogeneous in w_M, and of the form

$$L(x,y) = L(w_M(x,y)) = A\,w_M(x,y)^n\,. \tag{5.119}$$

The final equality is obtained from Euler's theorem for homogeneous functions. Observe that the maximally symmetric fundamental geometry function is always a metric one. Hence all maximally symmetric Finsler spacetimes are the well known maximally symmetric Lorentzian spacetimes characterised by the curvature constant K.

The symmetries introduced here will play an important role when we study the dynamics of Finsler spacetimes in the next chapter. Spherical symmetry will give rise to a Finsler spacetime which can be interpreted as a non-metric Finslerian refinement of the Schwarzschild solution known from general relativity.

We conclude this chapter in which we presented the geometry of Finsler spacetimes by showing explicitly how Lorentzian metric spacetimes are Finsler spacetimes and examining non-metric examples.

5.5. Illustrative examples

After we introduced the mathematical details of the geometry of Finsler spacetimes (M, L, F) we are now in the position to discuss in detail some explicit examples. These illustrate the strength of our definition and the general theorems derived above. First we show explicitly that Lorentzian metric spacetimes are a special case of Finsler spacetimes. In particular we will exhibit how connection and curvature, and the causal structure of a Lorentzian metric fit into the more general scheme discussed above. The second example shows a more complicated causal structure with two different lightcones at each point. This Finsler spacetime goes beyond metric manifolds, but nevertheless has well-defined timelike cones and allows a full description of geometric objects and propagation along null directions. Afterwards we discuss anisotropic spacetimes, which contain a Lorentzian metric and in addition a vector or a one-form field. There will revisit the Randers and Bogoslowsky length measures and discuss them from the Finsler spacetime point of view.

5.5.1. Lorentzian metric spacetimes

Lorentzian manifolds $(M, \tilde{g})$ with metric $\tilde{g}$ of signature $(-,+,+,+)$ are a special type of Finsler spacetimes (M, L, F). They are described by the metric-induced function

$$L(x,y) = \tilde{g}_{ab}(x)y^a y^b \tag{5.120}$$

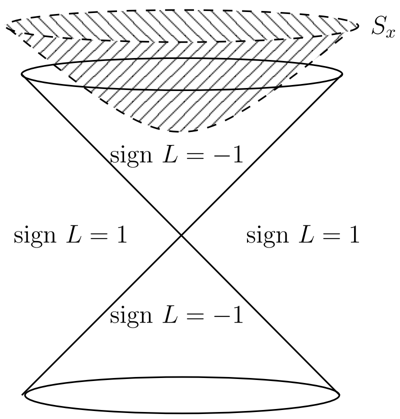
which is homogeneous of degree $r = 2$. Recalling the definition, $L(x, y)$ leads to the Finsler function $F(x, y) = |\tilde{g}_{ab}(x)y^a y^b|^{1/2}$ that is easily recognized as the Lorentzian metric spacetime length measure for curves.

Clearly L is smooth on TM and obeys the reversibility property. The metric $g^L_{ab}(x, y) = \tilde{g}_{ab}(x)$, and hence is non-degenerate on TM; so the measure zero set $A = \emptyset$ and so also is $B \subset N_L$. The signature of g^L is globally $(-,+,+,+)$, so the unit timelike condition tells us to consider the set

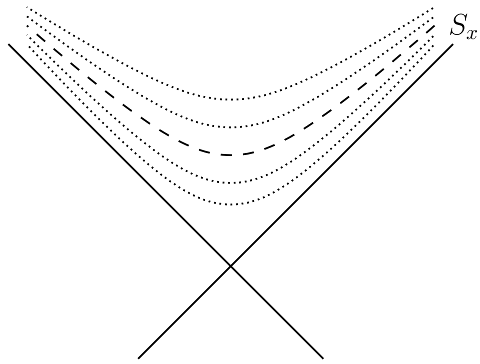
$$\Omega_x = \left\{ y \in T_xM \,\middle|\, \epsilon(x,y) = \frac{|L(x,y)|}{L(x,y)} = -1 \right\}. \tag{5.121}$$

This set has precisely two connected components, both of which are closed. We may call one of these S_x, as displayed in figure 5.1(a). From theorem 5.1 we learn that the shell of unit timelike vectors S_x can be rescaled to form an open convex cone C_x that contains all the usual timelike vectors of $\tilde{g}$ at a point $x \in M$, see figure 5.1(b).

The Finsler function F is non-differentiable on the null structure N_L; there the Finsler metric is not defined. Calculating the Finsler metric yields the results that $g^F_{ab}(x, y) = -\tilde{g}_{ab}(x)$ on the



(a) Null structure and unit timelike vectors on Lorentzian metric spacetimes.



(b) Rescaling the unit shell to form the cone of timelike vectors.

Figure 5.1. Causal structure on Lorentzian spacetimes.

$\tilde{g}$-timelike vectors and $g^F_{ab}(x,y) = +\tilde{g}_{ab}(x)$ on the $\tilde{g}$-spacelike vectors, hence the Finsler metric changes its signature. From the behaviour of the L and the F metric we see that the Sasaki type metrics G^L and G^F coincide on the timelike vectors.

The geometric objects derived from the fundamental geometry function take the familiar form for Lorentzian manifolds. The Cartan tensor C_{abc} and the tensor $S^a{}_{bc}$ that measure the departure from metricity vanish, because the Finsler metric does not depend on the fibre coordinates. Moreover, the coefficients of the linear covariant derivative simply become the δ-Christoffel symbols of the metric $\tilde{g}$, i.e., $\Gamma^{\delta\,a}{}_{bc}(x,y) = \Gamma^a{}_{bc}(x)$. The non-linear connection reduces to a linear connection with coefficients

$$N^a{}_b(x,y) = \Gamma^a{}_{bc}(x)y^c\,, \tag{5.122}$$

and according to (5.55) its curvature is given by the Riemann tensor of $\tilde{g}$ as $R^c{}_{ab}(x,y) = -y^d R^a{}_{dbc}(x)$. For Finsler spacetimes induced by Lorentzian metrics it is easy to see that connection and curvature are expressible in terms of $g^L_{ab}(x,y) = \tilde{g}_{ab}(x)$ and hence defined everywhere on TM.

5.5.2. Simple bimetric Finsler structures

A simple example of a Finsler spacetime (M,L,F) that goes beyond Lorentzian metric manifolds can be defined through two Lorentzian metrics h and k of signature $(-,+,+,+)$ for which the cone of h-timelike vectors is contained and centred in the cone of k-timelike vectors. We discuss this simple example here in detail and comment on the case when the two null cones of the metrics are tilted against each other and intersect. Such null structures are relevant as covariant descriptions for certain aspects of crystal optics [26] and can now be investigated with the help of our Finsler spacetime framework. It is worth noting that it was thought impossible to realise two signal cones consistently in Finsler geometry [52], but we will see that this is not a problem at all. Our example is based on the function

$$L = h_{ab}(x)y^a y^b k_{ab}(x)y^a y^b\,. \tag{5.123}$$

It is clear that L is homogeneous of degree $r = 4$, smooth on TM and obeys the reversibility condition. The corresponding Finsler function is defined as $F(x, y) = |h_{ab}(x)y^a y^b k_{cd}(x)y^c y^d|^{1/4}$.

The null structure N_L is the union of the null cones of the metrics h and k, and the metric $g^L_{ab}(x, y)$ turns out to be degenerate on a measure zero subset $A \neq \emptyset$ that forms an additional structure between the null surfaces without intersecting them; hence in this example $B = \emptyset$. The situation is displayed in figure 5.2.

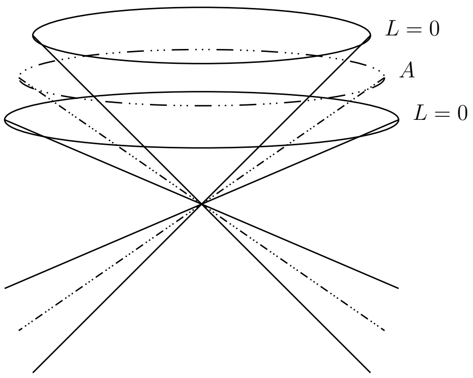
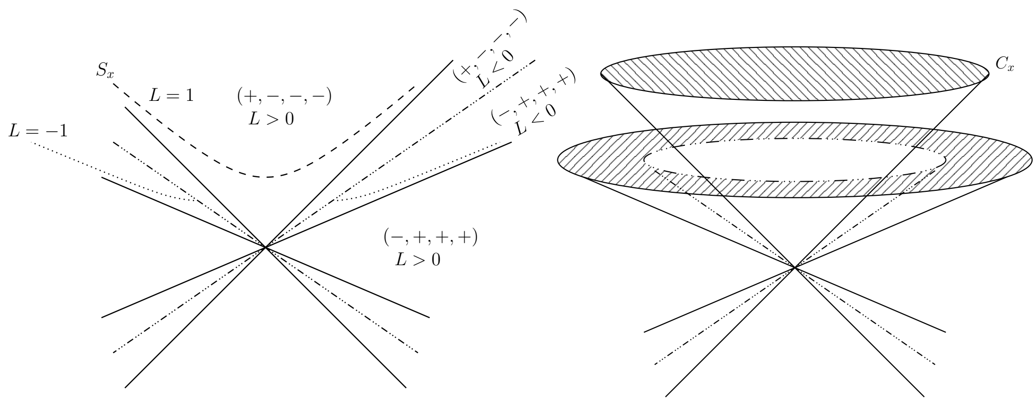


Figure 5.2. Null structure of the bimetric Finsler spacetime (solid) and degeneracy set A of g^L (dashed dotted).

Across A, the metric g^L changes its signature from $(+, -, -, -)$ to $(-, +, +, +)$. This change of signature would exclude this kind of length measures from Beem's indefinite Finsler spaces [25] as mentioned earlier. In order to analyse the unit timelike condition, we need to compare the signature of g^L with the sign of L. One finds four connected components of the set Ω_x. Two of these are closed, two are not; one of each is displayed in figure 5.3(a). Choosing one of the closed components to be the set S_x we can rescale it to form the complete convex cone C_x of timelike vectors at $x \in M$ according to theorem 5.1. The non-closed components will not give rise to a convex cone when rescaled in the same way, as can be seen in figure 5.3(b).



(a) Cut through two connected components of Ω_x; the inner set S_x is closed, the outer set is not.

(b) Rescaling S_x leads to a convex cone C_x; rescaling the outer set does not.

Figure 5.3. Causal structure on bimetric Finsler spacetimes.

As in the Lorentzian metric case the Finsler function F of this Finsler spacetime is not differentiable where $L = 0$. There the Finsler metric is not defined; it changes its signature across the null structure and is degenerate on A. But since on Finsler spacetimes all geometric objects are defined through the fundamental geometry function L they are well-defined on $TM \setminus A$, in particular on N_L. Hence this bimetric Finsler spacetime has a well-defined causal structure which is more general than that of Lorentzian metric manifolds but still admits all the necessary properties to be applicable in physics. So this bimetric Finsler spacetime is indeed a geometric background which generalises Lorentzian metric manifolds in a well-defined way.

We remark that an example where the set B is non-empty is given by a more complicated bimetric structure where the light cones of the two metrics h_{ab} and k_{ab} are tilted against each other and intersect. Along the intersection of the cones the L metric degenerates clearly, since both $h_{ab}y^a y^b$ and $k_{ab}y^a y^b$ vanish

$$g^L_{ab} = h_{ab}k_{cd}y^c y^d + k_{ab}h_{cd}y^c y^d + 2h_{ac}y^c k_{bd}y^d + 2h_{bc}y^c k_{ad}y^d\,. \tag{5.124}$$

As long as the intersection of the cones is lower dimensional than the null structure itself, one has control over the geometric objects nearly everywhere on the null structure; so we included these examples into our definition of Finsler spacetimes.

5.5.3. Anisotropic Finsler spacetimes

We call a Finsler spacetimes (M, L, F) anisotropic when the fundamental geometry function L is not only built from a Lorentzian metric $\tilde{g}(x)$ but also from a vector field $V(x)$ or a one-form $A(x)$ on the manifold. These special Finsler spacetimes we introduce here for the first time, they have not been considered in our journal articles. We show that special bimetric Finsler spacetimes can be obtained in this way, but first we discuss the length measures introduced by Randers and Bogoslovsky which we encountered when we discussed the application of Finsler geometry in physics in chapter 3. Recall the form of length measures

$$F_R(x,y) = \sqrt{|\tilde{g}_{ab}(x)y^a y^b|} + A_a(x)y^a, \quad F_B(y) = (|V_a y^a|)^q(\sqrt{|\eta_{ab}y^a y^b|})^{1-q}\,. \tag{5.125}$$

The Randers length measure F_R cannot be a Finsler spacetime since there does not exist an r-homogeneous smooth function L_R on $TM \setminus \{0\}$ such that $F_R = |L_R|^{1/r}$; the problem is caused by the term $\sqrt{\tilde{g}_{ab}y^a y^b}$. But since the non-differentiability of F_R is also not on the null structure of F_R but on the set set where $\tilde{g}_{ab}y^a y^b = 0$ one might try to refine the definition of Finsler spacetimes by relaxing the smoothness condition on L perhaps to include the Randers space into the framework. Anyway it is possible to study the geometry of a manifold equipped with the Randers length measure (M, F_R) with the tools from Finsler spacetimes everywhere where $\tilde{g}_{ab}y^a y^b \neq 0$.

Bogoslovsky's length measure was considered as a generalization of flat Minkowski spacetime. Consider the following generalization which is Bogoslovsky's length measure for $\tilde{g}_{ab} = \eta_{ab}$

$$\tilde{F}_B(x,y) = (|V_c(x)y^c|)^r(\sqrt{|\tilde{g}_{ab}(x)y^a y^b|})^{1-r}, \quad \tilde{g}_{ab}V^a V^b = V_b V^b = 0\,. \tag{5.126}$$

For this length measure there exists a smooth function L_B on $TM \setminus \{0\}$ so that $\tilde{F}_B = |L_B|^{(1-r)/2}$

$$L_B = (V_c y^c)^n \tilde{g}_{ab}y^a y^b, \quad \tilde{g}_{ab}V^a V^b = V_b V^b = 0, \quad n \in \mathbb{N},\ n \geq 0\,. \tag{5.127}$$

We introduced the parameter $n = 2r/(1-r)$ to give precise conditions which ensure the smoothness of L. They translate into conditions on the original parameter $r = n/(n+2)$, i.e., for a non-metric geometry r has to take a value between 1 and $1/3$. Despite having this nice properties L_B does not define a Finsler spacetime. Its corresponding L metric is too degenerate

$$g^{L_B}_{ab} = |y^c V_c|^n \Big(\tilde{g}_{ab} + \frac{n(n-1)}{2(y^c V_c)^2}\tilde{g}_{pq} y^p y^q V_a V_b + \frac{n}{y^c V_c}[y_a V_b + y_b V_a]\Big)\,. \tag{5.128}$$

It degenerates along the three dimensional hyperplane $A = \{(x,y) \in TM)|\tilde{g}_{ab}y^a V^b = 0\}$ tangent to the light cone of $\tilde{g}$. Thus the set $B = A \cap N_L = A$ where the geometry of (M, L_B, F_B) is not guaranteed to exist is not a lower dimensional subset of the null structure. It is possible to study the geometry of that manifold with the tools of Finsler spacetimes everywhere except on the rather large set B; remarkably on the complete null cone of $\tilde{g}$ except the one dimensional subspace spanned by the null vector V^a. On the basis of Finsler spaces, starting from F_B, not even this would be possible.

Positive examples of anisotropic Finsler spacetime are special bimetric Length measures. Consider a bimetric fundamental geometry function $L = (\tilde{g}_{ab}y^a y^b)(k_{cd}y^c y^d)$ with

$$k_{cd} = \tilde{g}_{cd} + Q\frac{V_a V_b}{\tilde{g}_{mn}V^m V^n}\,. \tag{5.129}$$

Depending on the parameter Q and the causal properties of the vector field V with respect to the metric $\tilde{g}$ the metric k is a Lorentzian or Riemannian metric, or degenerate. Let V for example be timelike with respect to the metric $\tilde{g}$, then k is degenerate for $Q = 1$; is Riemannian for $Q > 1$ and Lorentzian for $Q < 1$.

Choosing V indeed $\tilde{g}$ timelike and $Q < 1$ but $Q \neq 0$ the anisotropic length measure we just constructed realises the bimetric Finsler spacetime example with non-intersecting null cones discussed in the previous subsection.

Keeping V timelike but $Q > 2$ leads to an interesting anisotropic Finsler spacetime which is composed from a Lorentzian metric $\tilde{g}$ and a Riemannian metric k. The null structure of this spacetime is given only by the cone of the Lorentzian metric. The shell of unit timelike vectors S_x can identified by checking the signature condition on the L metric from the definition of Finsler spacetime. It lies in the interior of the null cone of the Lorentzian metric $\tilde{g}$ and is flattened compared to the shell of unit timelike vectors in the metric situation. The shell of unit spacelike vectors here is distorted compared to the corresponding shell in Lorentzian metric geometry. The rescaling of S_x yields the cone of timelike vectors C_x on this bimetric Finsler spacetime which is identical to the cone of timelike vectors on Lorentzian metric spacetimes. The requirement $Q > 2$ guarantees that the degeneracy set of the L metric A lies outside the cone of timelike vectors and does not intersect the null structure, i.e., $B = \emptyset$. We depicted the situation in figure 5.4.

The examples presented during this section demonstrate the existence of non trivial Finsler spacetimes which can be discussed as non-metric geometric backgrounds for physics in the Finsler spacetime framework. They admit a well-defined geometry on $TM \setminus A$, especially nearly on all of their null structure N_L; there may only be a lower dimensional subset $B \subset N_L$ where the geometry is not defined. The physical significance of the set A, where the L metric degenerates has to be discussed case by case. We like to point out that, due to the degeneracy of the L

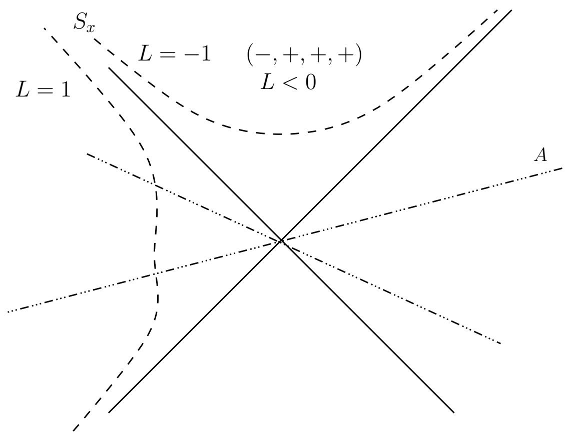


Figure 5.4. Null structure (solid), degeneracy set A of g^L (dashed dotted), one unit timelike shell S_x and one unit spacelike shell (dashed).

metric and the non differentiability of F^2 along the null structure, neither of the non-metric Finsler spacetime examples could be discussed as Finsler spaces nor as indefinite Finsler space based on the definition of Beem.

This finishes our presentation and discussion of Finsler spacetime geometry. We saw that Finsler spacetimes nicely generalise Lorentzian metric manifolds and admit a well-defined geometry. In the following we will focus on the application of Finsler spacetimes as geometric backgrounds for physics.

6. Dynamics of Finsler spacetimes

There exists a huge variety of Finsler spacetimes as interesting geometric backgrounds, some of them were just discussed as examples in the previous section. The question is what is their interpretation from a physical point of view?

The applications of Finsler geometry in physics discussed in chapter 3 suggest the following interpretations. The geometry of Finsler spacetimes can be used to describe characteristics of media, by encoding their properties into the fundamental geometry function, or it can be interpreted as the non-metric geometry of spacetime. We will focus on the latter during this thesis.

With our construction of Finsler spacetimes we have provided so far the solid mathematical geometric framework which can be used as the fundamental geometry of spacetime encoding causality. What is still missing are dynamics determining the geometry. During this chapter we derive such dynamics. In section 6.1 we first recall our results from our Finsler geometric view on the Einstein–Hilbert action from chapter 4 and then extend it to Finsler spacetimes. Afterwards, in section 6.2, we derive the dynamical equation which shall determine the geometry of a Finsler spacetime in the absence of additional non-geometric fields. It turns out that in the metric limit it is equivalent to the Einstein vacuum equations. At the end of this chapter in section 6.3 we discuss the consequence of the invariance of the generalised Einstein–Hilbert action under manifold induced diffeomorphisms.

Our construction of Finsler spacetimes and their dynamics aim for an extension of general relativity and will be further discussed in the following chapters. We will present first order solutions to the dynamics of Finsler spacetimes beyond metric geometry which can be interpreted as refinement of the linearised Schwarzschild solution and which is capable to address the fly-by anomaly in the solar system in chapter 7; we will introduce observers and their measurements modelled through the geometry of Finsler spacetimes in chapter 8 and we will equip Finsler spacetimes with action based physical field theories in chapter 9 and show how they source the dynamics of the geometry derived in this chapter.

Moreover the dynamics we derive here lay the foundation for yet another interpretation of Finsler spacetime geometry we like to mention here; namely to encode different physical fields, via a fundamental geometry function L constructed from the different fields, into one geometric picture. The spirit is similar to the ideas of Kaluza and Klein, supergravity and string theorie [8] or Randers [34], with the difference that four dimensional Finsler spacetime geometry is employed instead of higher dimensional metric geometry that must be dimensionally reduced. We will comment on this option further in the outlook of this thesis

6.1. Action principle

The key observation to find an action for the fundamental geometry function L of Finsler spacetimes is the fact that the Einstein–Hilbert action we usually use in general relativity can be seen as a very special case of a more general action which determines a general Finslerian geometry instead of a metric geometry. We briefly repeat the facts from chapter 4: On manifolds with definite metric $\hat{g}$ the Einstein Hilbert action can be written as an integral over all unit spheres S_p of the metric

$$S_{EH}[\hat{g}] = \int_M \mathrm{d}^4x\sqrt{\det\hat{g}}R = \int_M \mathrm{d}^4x\frac{4}{\mathrm{Vol}(S_p)}\int_{S_p}\mathrm{d}^3\theta\sqrt{\det\hat{g}}\;\sqrt{\det\hat{h}}\;R_{ab}y^a(\theta)y^b(\theta)\,. \tag{6.1}$$

Realising that the integrand in the last expression is, up to a sign, nothing but the metric limit of a contraction of the non-linear curvature tensor $R^F = R^a{}_{ab}y^b$ we generalised the Einstein–Hilbert action to an action for a general Finsler function

$$S_{FSE}[F] = \int_{\{F=1\}}\mathrm{d}^4x\mathrm{d}^3\theta(\sqrt{\det G^*}\;R^F)_{|\{F=1\}}\,. \tag{6.2}$$

This action was derived for Finsler functions with trivial null structure in section 4.1.

In the last chapter we have developed the notion of Finsler spacetimes which enable us to discuss Finsler geometries with non trivial null structure. As part of the discussion we obtained well-defined integrals of the unit tangent bundle in section 5.3 which enable us to formulate action integrals. Now in principle one could look at the Einstein–Hilbert action on Lorentzian spacetimes with metric $\tilde{g}$ and try to rewrite it in the same manner as we did for the Riemannian case; as integral over the set $\Sigma = \{\tilde{g}_{ab}(x)y^ay^b = \pm1\} \subset TM$. When we introduced the integration on Finsler spacetimes we called Σ the unit tangent bundle and we equipped it with coordinates $(\hat{x}^a, u^\alpha); a = 0,\dots,3; \alpha = 1,\dots,3$. The obvious problem here is that for fixed x the set $\Sigma_x = \{\tilde{g}_{ab}(x)y^ay^b = \pm1\}$ is non compact and so the integration we used in case of a positive definite metric to derive equation (6.1) is in general not well-defined. Hence we cannot derive the action for the dynamics of Finsler spacetimes from Lorentzian metric geometry in the strict sense as we did for Finsler spaces. Nevertheless, as explained in section 5.3, the variation of an action defined as an integral over Σ is well-defined and enables us to obtain field equations. Furthermore, it was proven in [12] that the variation of the action (6.2) with respect to F for the case $F = \sqrt{|\tilde{g}_{ab(x)y^ay^b}|}$ yields equations equivalent to the Einstein equations, independent of the signature of $\tilde{g}$.

Moreover, the Finsler curvature scalar on Finsler spacetimes $\mathcal{R} = R^a{}_{ab}y^b = R^F$, see equation (5.55), is the unique curvature scalar built solely from the non-linear curvature tensor and no different tensors depending on L; the only geometric structure that enters is the unique Cartan non-linear connection. No further derivatives on L are needed and none of the different linear covariant derivatives.

The derivation on positive definite Finsler spaces, the result that in the metric case the equations of motion obtained by variation are independent of the signature of the metric and the uniqueness argument just stated lead us to the following definition.

Definition 6.1. ***Finsler spacetime dynamics***
The canonical action which defines the dynamics of Finsler spacetimes is given by

$$S_L[L] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \sqrt{g^F h^F}_{|\Sigma}\, \mathcal{R}_{|\Sigma}\,. \tag{6.3}$$

The Lagrangian is given by the canonical curvature scalar $\mathcal{R} = R^a{}_{ab}y^b$ built form the curvature d-tensor $R^a{}_{bc}$ of the Cartan non-linear connection.

It is an integral over the unit tangent bundle in the form of equation (5.84) and so possesses a well-defined variation with respect to the dynamical variable L. For the volume element, which comes from the Sasaki metric introduced in definition 5.4, we choose g^F and h^F because of their convenient homogeneity properties. As discussed earlier, for the resulting field equations the choice between g^L and h^L, or g^F and h^F, makes no difference. We consider this action as the canonical generalisation of the Einstein–Hilbert action to Finsler spacetimes due to the line of argument above.

6.2. Field equation

We will now derive the equations of motion determining the fundamental geometry function L from our canonical generalisation of the Einstein–Hilbert to Finsler spacetimes. On the basis of the interpretations mentioned previously the field equation can be seen as a gravitational vacuum equation, extending the Einstein vacuum equations, or as an equation containing the dynamics of several fields, including the gravitational field, on a Lorentzian metric background.

The integrand of the action (6.3) is a homogeneous tangent bundle function restricted to the unit tangent bundle Σ. To perform the variation with respect to L for an m-homogeneous function $f(x,y)$ on TM restricted to Σ it is useful to realise that

$$\delta(f_{|\Sigma}) = (\delta f)_{|\Sigma} - \frac{m}{r} f_{|\Sigma}\frac{\delta L}{L}\,, \tag{6.4}$$

where r is the homogeneity of L. With the help of this formula and our results for integrations by parts in (5.86) we can derive the vacuum field equations in three steps. The first uses the variation formula above with $f(x,y) = \sqrt{g^F h^F}\; R^a{}_{ab}y^b$ and $m = 5$, which yields

$$\delta S_L[L] = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u \left[\delta\Big(\sqrt{g^F h^F}\;\mathcal{R}\Big) - \frac{5}{r}\sqrt{g^F h^F}\;\mathcal{R}\frac{\delta L}{L}\right]_{|\Sigma}\,. \tag{6.5}$$

The second step is the variation of the volume element which leads to

$$\delta S_L[L] = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u\sqrt{g^F h^F}_{|\Sigma}\left[\Big(g^{F\,pq}\delta g^F_{pq} - \frac{6}{r}\frac{\delta L}{L}\Big)\mathcal{R} + y^b\delta R^a{}_{ab}\right]_{|\Sigma}\,, \tag{6.6}$$

while in the third step we use the linear covariant derivatives (equations (5.43) and(5.49)) to gain the following identities

$$\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\left[\sqrt{g^F h^F}\; g^{F\,pq}\delta g^F_{pq}\mathcal{R}\right]_{|\Sigma} = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u\left[\sqrt{g^F h^F}\; g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R}\frac{\delta L}{rL}\right]_{|\Sigma}\,, \tag{6.7}$$

$$\int \mathrm{d}^4\hat{x}\mathrm{d}^3u \left[\sqrt{g^F h^F}\ y^b \delta R^a{}_{ab}\right]_{|\Sigma} = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u \left[\sqrt{g^F h^F}\ 2g^{F\,ab}\big(\nabla^{CL}_a S_b + S_a S_b + \bar{\partial}_a \nabla S_b\big)\frac{\delta L}{rL}\right]_{|\Sigma}. \tag{6.8}$$

We finally arrive at the following form of the variation of the general non-metric, Finslerian version of the Einstein–Hilbert action (6.3):

$$\delta S_L[L] = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u \sqrt{g^F h^F}_{|\Sigma} \left[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - 6\mathcal{R} + 2g^{F\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big)\right]_{|\Sigma}\frac{\delta L}{rL}. \tag{6.9}$$

For further details of this variation we refer the reader to appendix A.5.

Now we can read off the dynamical equation on Σ which determines L in the absence of other fields

$$\left[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - 6\mathcal{R} + 2g^{F\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big)\right]_{|\Sigma} = 0\,. \tag{6.10}$$

Observe that all terms in the bracket are zero-homogeneous on TM, except the second term $\mathcal{R}$ that is homogeneous of degree two. Since $(\mathcal{R})_{|\Sigma} = (\mathcal{R}/F^2)_{|\Sigma}$ we can replace the second term by $\mathcal{R}/F^2$ which is now also zero-homogeneous. Hence the equation can be lifted to TM in the form

$$g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - \frac{6}{F^2}\mathcal{R} + 2g^{F\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big) = 0\,. \tag{6.11}$$

It seems that this equation could be invalid on $\{L=0\} = \{F=0\}$ where F is not differentiable so that the Finsler metric g^F would not exist. However, this is not the case: the equation is valid also on nearly all the null structure. To see this, one expresses g^F through g^L with the help of formula (5.15) and multiplies by F^2

$$\frac{rL}{2}g^{L\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - \frac{2(2r-1)}{(r-1)}\mathcal{R} + rLg^{L\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big) = 0\,. \tag{6.12}$$

The resulting equation is well-defined wherever g^L is nondegenerate, and in particular on nearly all of the null structure. Note that equation (6.11) is invariant under the transformation $L \mapsto L^k$. For sure L and L^k lead to the same Finsler function F thus g^F is invariant under this transformation. The same is true for the curvature scalar $\mathcal{R}$ and the S tensor since they derived from the non-linear connection coefficients and the δ-Christoffel symbols, which are invariant according to equations (5.42) and (5.45).

In the metric limit $L = \tilde{g}_{ab}(x)y^a y^b$ the tensors in the field equation for L reduce to $\mathcal{R} = -y^a y^b R_{ab}$ and $S_a = 0$, where R_{ab} are the components of the Ricci tensor of the metric $\tilde{g}$. Accordingly, the field equation becomes

$$-2\frac{L}{|L|}R + \frac{6}{|L|}R_{ab}y^a y^b = 0 \tag{6.13}$$

which is equivalent to the Einstein vacuum equations $R_{ab} = 0$ by differentiating twice with respect to y. We conclude that a family of solutions of the Finsler spacetime dynamics (6.11) is induced by solutions $\tilde{g}_{ab}(x)$ of the vacuum Einstein equations via the fundamental functions $L_k = (\tilde{g}_{ab}(x)y^a y^b)^k$.

Our action based approach to derive dynamical equations for Finsler spacetimes is a huge improvement compared to the previous attempts discussed in section 3.2.3. We do not need an a priory vector field from the manifold, as in the osculating formalism, the equation is a scalar equation on the tangent bundle for a scalar field on the tangent bundle and not simply form equivalent to some known filed equations and the action based derivation enables us to couple matter fields to the geometry also via the an action principle, what guarantees that the geometry side of the equation obeys the same conservation equation as the matter source term automatically. The latter is for example not guaranteed by the approach of Rutz.

The field equation (6.11) for L we derived here from the non-metric Finsler spacetime version of the Einstein–Hilbert action is the first dynamical equation for a Finsler geometry derived from a well-defined action.

In chapter 9 we will couple further fields to the geometry of Finsler spacetimes via well-defined field theory actions. These additional fields will produce a source term of the field equation for L which we will derive in section 9.3.

6.3. Diffeomorphism invariance of the action

Invariances of an action yield conserved quantities. In metric geometry the invariance of the Einstein–Hilbert action under diffeomorphisms of the manifold guarantees the divergence freeness of the Einstein tensor and so ensures consistency with energy conservation, i.e., the divergence freeness of the energy momentum tensor. From the invariance of the Finsler spacetime Einstein–Hilbert action S_L (6.3) under manifold induced diffeomorphisms we obtain a generalised conservation equation which is satisfied by the scalar on the left hand side of the field equation (6.11).

As discussed for the symmetries of Finsler spacetimes in section 5.4 a diffeomorphism on M, induced by a vector field $X = \xi^a(x)\partial_a$, induces a diffeomorphism on TM by its complete lift $X^c = \xi^a\partial_a + y^q\partial_q\xi^a\bar{\partial}_a$. Here we do not assume X to be a symmetry vector field of the Finsler spacetime. The variation of the action S_L under this diffeomorphism is

$$\delta_X S_L[L] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\, \Big(\frac{\delta\mathcal{L}}{\delta L}\delta_X L\Big)_{|\Sigma} \tag{6.14}$$

with $\mathcal{L} = R^a{}_{ab}y^b$ and $\delta_X L = X^c(L) = \bar{\partial}_a L\nabla\xi^a$. Now integration by parts and the fact that $\bar{\partial}_a L$ is constant with respect to the dynamical covariant derivative ∇ yields

$$\delta_X S_L[L] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\sqrt{g^F g^F}_{|\Sigma}\Big[\nabla\Big(\frac{1}{\sqrt{g^F h^F}}\frac{\delta\mathcal{L}}{\delta L}\bar{\partial}_a L\Big)\Big]_{|\Sigma}\xi^a_{|\Sigma}\,. \tag{6.15}$$

The components ξ^a only depend on x and S_L is diffeomorphism invariant, i.e., $\delta_X S_L = 0$ for all X. Hence we obtain the integral conservation equation

$$\delta_X S_L[L] = 0 \Rightarrow \int_{\Sigma_x} \mathrm{d}^3u\sqrt{g^F g^F}_{|\Sigma_x}\Big[\nabla\Big(\frac{1}{\sqrt{g^F h^F}}\frac{\delta\mathcal{L}}{\delta L}\bar{\partial}_a L\Big)\Big]_{|\Sigma_x} = 0\,. \tag{6.16}$$

Using equation (6.9) this can be rewritten as

$$\int_{\Sigma_x} \mathrm{d}^3u\sqrt{g^F h^F}_{|\Sigma_x}\Big[\nabla\Big(\Big[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - 6\mathcal{R} + 2g^{F\,ab}(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b)\Big]\frac{\bar{\partial}_a L}{rL}\Big)\Big]_{|\Sigma_x} = 0\,. \tag{6.17}$$

To conclude from here that the integrand has to vanish is certainly to strong. To get an inside about the connection of the above conservation equation and the divergence freeness of the

Einstein tensor we are used to in general relativity, we study the metric limit

$$\begin{aligned} &\int_{\Sigma_x} \mathrm{d}^3u \sqrt{\tilde{g}\tilde{h}}_{|\Sigma_x} \Big[\frac{y^b y^c y^d}{|L|} \Big(-2\tilde{g}_{bd}\tilde{g}^{mn}\nabla_c R_{mn} + 6\nabla_c R_{bd} \Big) \frac{\tilde{g}_{ai} y^i}{L} \Big]_{|\Sigma_x} \\ = \ &\Big(-2\tilde{g}_{bd}\tilde{g}^{mn}\nabla_c R_{mn} + 6\nabla_c R_{bd} \Big) \tilde{g}_{ai} \int_{\Sigma_x} \mathrm{d}^3u \sqrt{gh}_{|\Sigma_x} \Big[y^b y^c y^d y^i \Big]_{|\Sigma_x} = 0\,. \end{aligned} \tag{6.18}$$

The action of the dynamical covariant derivative here becomes equal to the action of the Levi-Civita covariant derivative ∇_a. If $\tilde{g}$ would be a Riemannian metric, and so Σ_x would be compact, we could perform the integral above and would find in terms of the volume of the unit three sphere S^3_p, see appendix A.6 for the calculation,

$$\int_{\Sigma_x} \mathrm{d}^3u \sqrt{\tilde{g}\tilde{h}}_{|\Sigma_x} \Big[y^b y^c y^d y^i \Big]_{|\Sigma_x} = \sqrt{\tilde{g}} \frac{2}{\mathrm{Vol}(S^3_p)} \tilde{g}^{(bc}\tilde{g}^{di)}\,. \tag{6.19}$$

Together with equation (6.18) this would then yield the conservation of the Einstein tensor

$$\Big(-2\tilde{g}_{bd}\tilde{g}^{mn}\nabla_c R_{mn} + 6\nabla_c R_{bd} \Big) \tilde{g}_{ai} \frac{2}{\mathrm{Vol}(S^3_p)} \tilde{g}^{(bc}\tilde{g}^{di)} = \frac{8\tilde{g}_{as}}{\mathrm{Vol}(S^3_p)} \Big(\nabla_m R^{ms} - \frac{1}{2}\nabla_m \tilde{g}^{ms} R \Big) = 0\,. \tag{6.20}$$

But in the metric limit of Finsler spacetimes the metric g has Lorentzian signature and so Σ_x is non-compact and the evaluation of the above integral cannot be done so easily. A way to understand the integral in the Lorentzian and in the general Finsler spacetime case may be consider it as renormalised integral [53], but this requires further investigation.

7. First order non-metric solution of Finsler spacetime dynamics

To work as extension of general relativity the Finsler spacetime dynamics should have solutions L which are non-metric and they should be interesting from the point of view of physics. As a matter of fact the field equation (6.11) is in general hard to solve. Therefore in this chapter we will study Finsler spacetimes that describe mild deviations from Lorentzian geometry. In this situation, the complicated Finsler gravity field equation allows a simplified perturbative treatment. After a general discussion of the linearised field equation in section 7.1, we will employ what we learned about spacetime symmetries to present a spherically symmetric solution in section 7.2. This particular model turns out to be a refinement of the linearised Schwarzschild solution of general relativity, and we will show that it is capable of modelling unexplained effects in the solar system like the fly-by anomaly. With this solution we get a glimpse of what kind of non-metric geometries can be derived from the dynamics of Finsler spacetimes, and we see that these have interesting features from the physics point of view.

In order to construct a complete extension of general relativity it is necessary to define observers and to discuss how they perform measurements on Finsler spacetimes, what we will do in chapter 8. It is also necessary to introduce physical matter fields and their coupling to the Finsler spacetime geometry as well as to derive the resulting gravitational field equation including the matter fields as source, which will be done in chapter 9.

7.1. Finsler modifications of Lorentzian geometry

Recall that the fundamental functions $L = L_0$ and $L = (L_0)^k$ define the same geometry, and that this is respected by the field equation for L. Hence the following class of fundamental functions gives us good control over deviations from Lorentzian metric geometry,

$$L = \big(\tilde{g}_{ab}(x)y^a y^b\big)^k + H(x,y) = G(x,y) + H(x,y)\,. \tag{7.1}$$

Here, $H(x,y)$ is a $2k$-homogeneous function with respect to y that causes the Finsler modifications of the Lorentzian metric spacetime $(M,\tilde{g})$. The abbreviations $G(x,y) = \tilde{g}^k = (g_{ab}(x)y^a y^b)^k$ should not be confused with the Sasaki-type metric on TM.

The field equation for L, here interpreted as Finsler gravity vacuum equation, can be expressed in terms of L and g^L_{ab}, as derived in equation (6.12):

$$\frac{rL}{2}g^{L\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - \frac{2(2r-1)}{(r-1)}\mathcal{R} + rLg^{L\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big) = 0\,. \tag{7.2}$$

We now expand this equation to linear order in the modification $H(x,y)$, where $\tilde{g}_{ab}y^a y^b \neq 0$. In the following calculations we suppress all higher order terms. To simplify calculations later we expand G and derivatives acting on G in terms of $\tilde{g}_{ab}$. From the above equation we see that we need to expand the L metric, the curvature scalar $\mathcal{R}$ and the S-tensor; the latter two require the knowledge of the non-linear connection coefficients and the δ-Christoffel symbols. We display here the results of the expansion of the required objects to first order in H. We begin with the L metric and its inverse

$$g^L_{ab} = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b L = G_{ab} + H_{ab} = k\tilde{g}^{k-1}\Big(\tilde{g}_{ab} - \frac{2(k-1)}{\tilde{g}}y_a y_b\Big) + H_{ab} \tag{7.3}$$

$$g^{Lab} \approx G^{ab} - H_{ij}G^{ia}G^{jb} = \frac{1}{k\tilde{g}^{k-1}}\Big(\tilde{g}^{ab} - \frac{2(k-1)}{(2k-1)\tilde{g}}y^a y^b\Big) - H_{ij}G^{ia}G^{jb}\,, \tag{7.4}$$

where G_{ab} and H_{ab} denote the Hessians of G respectively H. Using these expansions the linearisation of the non-linear connection coefficients are

$$\begin{aligned} N^a{}_b &= \frac{1}{4}\bar{\partial}_b\Big(g^{Laq}(y^m\partial_m\bar{\partial}_q L - \partial_q L)\Big) \\ &\approx N^{0a}{}_b + N^{1a}{}_b \\ &= \Gamma^a{}_{bc}y^c + \frac{1}{4}\bar{\partial}_b\Big[\frac{1}{k\tilde{g}^{k-1}}g^{aq}(y^m\delta^{\mathbf{0}}_m\bar{\partial}_q H - \partial_q H) - \frac{2(k-1)}{k\tilde{g}^k}y^a y^m\delta^{\mathbf{0}}_m H\Big]\,, \end{aligned} \tag{7.5}$$

where the $\Gamma^a{}_{bc}$ are the usual Christoffel symbols of the metric $\tilde{g}_{ab}(x)$ and $\delta^{\mathbf{0}}_m = \partial_m - \Gamma^i{}_{mc}y^c\bar{\partial}_i$ is the horizontal derivative induced by the zeroth order of the non-linear connection coefficients. The extra upper index 0 or 1 on objects indicates their order in H. To express the curvature scalar $\mathcal{R} = R^a{}_{ab}y^b$ we define

$$\begin{aligned} T^a = \frac{1}{2}N^a{}_b y^b &\approx T^{0a} + T^{1a} \\ &= \frac{1}{2}\Gamma^a{}_{bc}y^c y^b + \frac{1}{4}\Big[\frac{1}{k\tilde{g}^{k-1}}g^{aq}(y^m\delta^{\mathbf{0}}_m\bar{\partial}_q H - \partial_q H) - \frac{2(k-1)}{k\tilde{g}^k}y^a y^m\delta^{\mathbf{0}}_m H\Big]. \end{aligned} \tag{7.6}$$

With help of T and the components of the Ricci tensor R_{ab} of the metric $\tilde{g}_{ab}(x)$ we can express the curvature scalar to first order in H in a compact form

$$\begin{aligned} \mathcal{R} = y^b\delta_b N^a{}_a - y^b\delta_a N^a{}_b &= y^b\partial_b\bar{\partial}_a T^a - 2\partial_a T^a + \bar{\partial}_a T^b\bar{\partial}_b T^a - 2T^a\bar{\partial}_a\bar{\partial}_b T^b \\ &\approx \mathcal{R}^0 + \mathcal{R}^1 \qquad (7.7) \\ &= -R_{ab}y^a y^b + y^b\partial_b\bar{\partial}_a T^{1a} - 2\partial_a T^{1a} \\ &+ 2\bar{\partial}_a T^{0m}\bar{\partial}_m T^{1a} - 2T^{0m}\bar{\partial}_m\bar{\partial}_a T^{1a} - 2T^{1m}\bar{\partial}_m\bar{\partial}_a T^{0a}\,. \qquad (7.8) \end{aligned}$$

The only S-tensor components appearing in equation (7.2) are $S_a = S^q{}_{qa}$; from its definition in equation (5.51) we calculate that it has no zeroth order term

$$\begin{aligned} S_a = -\frac{1}{2}y^m\bar{\partial}_a\Gamma^{\delta q}_{qm} &= -\frac{1}{4}y^m\bar{\partial}_a(g^{Lcd}\delta_m g^L_{cd}) \qquad (7.9) \\ &\approx S^1_a \\ &= -\frac{1}{2}y^m\bar{\partial}_a\Big[\frac{1}{k\tilde{g}^{k-1}}\delta^{\mathbf{0}}_m(\tilde{g}^{ab}H_{ab}) - \frac{2(k-1)(k+2)}{k\tilde{g}^k}\delta^{\mathbf{0}}_m H\Big]\,. \qquad (7.10) \end{aligned}$$

We now collect all terms to write equation (7.2) to first order in H

$$\begin{aligned} 0 \approx\ & \tilde{g}\tilde{g}^{ab}\bar{\partial}_a\bar{\partial}_b R^0 - 6R^0 \\ +\ & \big[\tilde{g}\tilde{g}^{ab}\bar{\partial}_a\bar{\partial}_b R^1 - 6R^1 - k\tilde{g}^k G^{ai}G^{bj}H_{ij}\bar{\partial}_a\bar{\partial}_b R^0 + kHG^{ab}\bar{\partial}_a\bar{\partial}_b R^0 \\ +\ & 2\tilde{g}\tilde{g}^{ab}(2\delta^{\mathbf{0}}_a S^1_b - 2\Gamma^q{}_{ab}S^1_q + y^m\delta^{\mathbf{0}}_m\bar{\partial}_a S^1_b - y^m\Gamma^q{}_{ma}\bar{\partial}_q S^1_b - y^m\Gamma^q{}_{mb}\bar{\partial}_a S^1_q)\big]\,. \end{aligned} \tag{7.11}$$

The zeroth order contribution in the first line is equivalent to the Einstein vacuum equations, as discussed in section 6.2. The first order terms in square brackets determine the Finsler modification of the metric background solution. We will now solve the linearised Finsler spacetime dynamics for a perturbation of Minkowski spacetime.

7.2. Refinements of the linearised Schwarzschild solution

We will now use our results on symmetries and on the linearisation of vacuum Finsler gravity around metric spacetimes to derive a particular model that refines the linearised Schwarzschild solution and can be used to study solar system physics.

Recall from section 5.4.2 that the dependence of the general spherically symmetric fundamental function in tangent bundle coordinates induced by (t, r, θ, ϕ) is restricted to the form $L(t, r, y^t, y^r, w(\theta, y^\theta, y^\phi))$ where $w^2 = (y^\theta)^2 + \sin^2\theta(y^\phi)^2$. We wish to study such a spherically symmetric fundamental function that describes a Finsler modification of Lorentzian geometry. For simplicity, we consider a bimetric four-homogeneous Finsler spacetime that perturbs the maximally symmetric vacuum solution of the dynamics (6.11). This background solution is given by Minkowski spacetime since for maximal symmetry we are back to metric geometry, as discussed in section 5.4.4. The model we like to discuss assumes the following form of L

$$L = \big(\eta_{ab}y^a y^b\big)^2 + \eta_{ab}y^a y^b h_{cd}y^c y^d = (\eta_{ab}y^a y^b)(\eta_{cd} + h_{cd})y^c y^d \tag{7.12}$$

with $h_{ab} = \mathrm{diag}(a(r), b(r), c(r)r^2, c(r)r^2\sin^2\theta)$. This Ansatz has the explicit form

$$L(r, y^t, y^r, w) = \big(-y^{t^2} + y^{r^2} + r^2w^2\big)\big([-1 + a(r)]y^{t^2} + [1 + b(r)]y^{r^2} + [1 + c(r)]r^2w^2\big). \tag{7.13}$$

Observe that the function $c(r)$ cannot be transformed away by defining a new radial coordinate. Although this could remove $c(r)$ from the metric in the right hand bracket, such a coordinate change would generate extra terms in the metric appearing in the left hand bracket. Therefore, the existence of the function $c(r)$ as a physical degree of freedom is a Finsler geometric effect that appears as a consequence of the bimetric spacetime structure assumed here.

We will now solve the linearised Finsler gravity equation (7.11) for $a(r)$, $b(r)$ and $c(r)$ with the Ansatz (7.13). Sorting the equation with respect to powers in y^t, y^r and w gives rise to three equations that have to be satisfied:

$$-2a' - ra'' = 0\,,\quad ra'' + 2b' - 4c' - 2rc'' = 0\,,\quad ra' + 2b + rb' - 2c - 4rc' - r^2c'' = 0\,. \tag{7.14}$$

The solution of these equations is

$$a(r) = -\frac{A_1}{r} + A_2,\quad b(r) = -\frac{A_1}{r} + \frac{A_3}{r^2},\quad c(r) = \frac{A_4}{r} - \frac{A_3}{r^2}\,. \tag{7.15}$$

To compare this non-metric spacetime to the linearised Schwarzschild solution of general relativity we study its geodesics. With help of the linearised expression for the non-linear connection coefficients in (7.5), the geodesic equation (5.40) $\ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b = 0$ can be calculated to first order in H. As usual in spherical symmetry, setting $\theta = \frac{\pi}{2}$ solves one of the four component equations immediately; the remaining equations are

$$0 = \ddot{t} - \frac{1}{2}\frac{A_1}{r^2}\,\dot{t}\,\dot{r} \tag{7.16}$$

$$0 = \ddot{r} - \frac{1}{4}\frac{A_1}{r^2}\,\dot{t}^2 + \frac{1}{4}\Big(\frac{A_1}{r^2} - 2\frac{A_3}{r^3}\Big)\,\dot{r}^2 + \Big(-r - \frac{A_1}{2} - \frac{A_4}{4} + \frac{1}{2}\frac{A_3}{r}\Big)\,\dot{\phi}^2 \tag{7.17}$$

$$0 = \ddot{\phi} + \frac{2}{r}\Big(1 - \frac{1}{4}\frac{A_4}{r} + \frac{1}{2}\frac{A_3}{r^2}\Big)\,\dot{\phi}\,\dot{r}\,. \tag{7.18}$$

From these equations we find two constants of motion

$$E = \dot{t}\,\Big(1 + \frac{1}{2}\frac{A_1}{r}\Big), \quad \ell = r^2\Big(1 + \frac{1}{2}\frac{A_4}{r} - \frac{1}{2}\frac{A_3}{r^2}\Big)\dot{\phi}\,. \tag{7.19}$$

These can be used to deduce the orbit equation from the affine normalisation condition that $F(x, \dot{x}) = 1$ along the Finsler geodesic; we write σ for the sign of the background length measure $\eta_{ab}\dot{x}^a\dot{x}^b = -\dot{t}^2 + \dot{r}^2 + r^2\dot{\phi}^2$ to obtain

$$\frac{1}{2}\dot{r}^2 = \frac{E^2}{2}\Big(1 - \frac{A_2}{2}\Big) + \frac{1}{2}\sigma\Big(1 + \frac{A_1}{2r}\Big) - \frac{\ell^2}{2r^2}\Big(1 + \frac{A_1}{2r} - \frac{A_4}{2r}\Big) + \frac{A_3}{4r^2}\Big(\sigma - E^2\Big). \tag{7.20}$$

The geodesic equations, the constants of motion and the orbit equation are well suited to compare the bimetric linearised Finsler solution with the linearised Schwarzschild solution. To see the differences to this solution of Einstein gravity we first note that A_2 can be absorbed into a redefinition of E, hence can be assumed to be zero. Second we introduce the Schwarzschild radius r_0 to redefine $A_1 = -2r_0(1 + a_1)$, $A_3 = 2\ell^2 a_3/(E^2 - \sigma)$ and $A_4 = 2r_0 a_4$ in terms of dimensionless small constants a_1, a_3 and a_4. Then the orbit equation becomes

$$\frac{1}{2}\dot{r}^2 = \frac{E^2}{2} + \frac{\sigma}{2} - \frac{\sigma r_0}{2r}(1 + a_1) - \frac{\ell^2}{2r^2}(1 + a_3) + \frac{r_0\ell^2}{2r^3}(1 + a_1 + a_4)\,. \tag{7.21}$$

In the special case $a_1 = a_3 = a_4 = 0$ this is precisely the orbit equation in the linearised Schwarzschild geometry, see [54]; the same limit also applies to the geodesic equations and the constants of motion.

The Finsler geometric refinements of the metric Schwarzschild geometry are encoded in the constants a_1, a_3 and a_4. These can in principle be fitted to data from solar system experiments. Indeed, there are certain observations that cannot be fully explained by the Schwarzschild solution [55], for instance, the fly-by anomaly [56]: for several spacecrafts it has been reported that swing-by manoeuvres lead to a small unexplained velocity increase. This corresponds to a change in the shape of the orbit of the spacecraft. Such a change can in principle be modelled by Finsler refinements; the perturbations a_1, a_3 and a_4 certainly provide possibilities to alter the wideness of the swing-by orbit as compared to that expected from Einstein gravity. This can be confirmed by simple numerical calculations, see figure 7.1.

We have seen that Finsler geometries exist that are extremely close to metric geometries. Our specific example of a spherically symmetric bimetric perturbation around Minkowski spacetime could be reinterpreted as a geometry close to the linearised Schwarzschild solution of Einstein gravity. The more complex causal structure, however, leads to additional constants that modify the geodesic equations and in particular the shape of test particle orbits. This could be a means to explain the fly-by anomaly in the solar system. We emphasize that this consequence already at first order perturbation theory gives a glimpse on the potential of Finsler gravity.

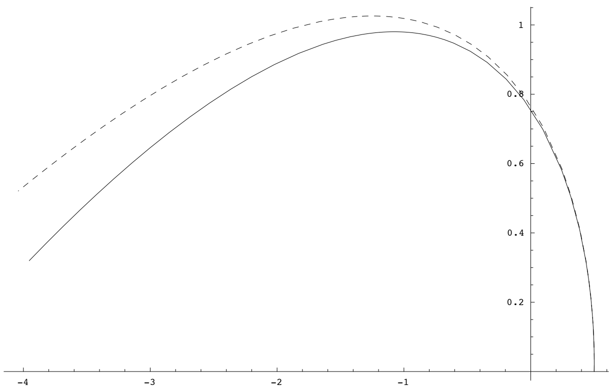


Figure 7.1. *Numerical fly-by solutions of the geodesic equations for linearised Schwarzschild geometry (dashed line) and the bimetric Finsler refinement (solid line) with* $a_1 = 0$*,* $a_3 \simeq 0.156$ *and* $a_4 = 0.1$*. The mass is centred at the origin an has Schwarzschild radius* $r_0 = 0.1$*. The initial conditions are* $r(0) = 0.5$*,* $\dot{r}(0) = 0.02$*,* $\phi(0) = 0$*,* $\dot{\phi}(0) = 1.1$ *and* $t(0) = 0$ *for both curves, and* $\dot{t}(0)$ *is calculated from the respective unit normalization condition* $F(x, \dot{x}) = 1$*.*

Further understanding of the dynamics and non-perturbative solutions are needed to study the full potential of Finsler spacetimes and their dynamics as extension of Lorentzian metric spacetimes and general relativity.

8. Physical observers, measurements and the length measure

In general relativity, the description of a physical observer and measurement procedures are tied to the metric geometry of spacetime. Thus, in order to interpret Finsler spacetimes as dynamical non-metric geometric backgrounds for physics extending Lorentzian spacetimes, and to make observable predictions, it is necessary to define a mathematical model of physical observers in terms of the available non-metric geometry. A precise description how observers measure time, spatial distances and physical fields has to be implemented.

We begin this chapter in section 8.1 with an investigation of the Finsler length measure. It turns out to be the action for point-like test particles as well as the geometric clock which determines the time an observer measures. After this discussion of the length measure we will describe observers and their measurements by the definition of a reference frame of the L, respectively the F metric, in section 8.2. We will interpret the frame vectors as the observers unit time and unit space directions. The components of physical fields are then measured and interpreted by an observer with respect to this local observer frame. A model of observers immediately poses the question of the relation between different observers and their observations. On Lorentzian metric spacetimes different observers are related by Lorentz transformations. On Finsler spacetimes it will turn out that these transformations still play a central role as the building block of a composite observer transformation with the structure of a groupoid. In section 8.3 we study as an example measurement an observer's measurement of the speed of light.

After our construction of an observer model here we will equip Finsler spacetimes with physical field theories, in addition to the geometry, in the next chapter, to complete our Finsler spacetime framework.

8.1. Interpreting the length measure

Our two guiding principles for the interpretation of the geometric length measure are the weak equivalence principle and the clock postulate.

The weak equivalence principle states that the trajectories of small test bodies, neither affected by gravitational tidal forces nor by forces other than gravity, are independent of their internal structure and composition [57]. Experimentally, this principle is confirmed with extremely high precision [58]; in gravity theory it has been implemented already by Newton who postulated that gravitational mass should equal inertial mass, and then by Einstein who formulated the motion of test bodies in terms of geodesics on Lorentzian spacetime where all test bodies couple to the geometry of spacetime in the same way. We will implement the weak equivalence principle on Finsler spacetimes by interpreting the induced Finsler length measure as the action for point particles. This will hold in particular for test particles that do not influence the structure of spacetime by their presence.

Point particles are idealized small objects propagating through spacetime along a curve γ. Here we consider point particles affected only by the Finslerian geometry of spacetime and not by any other fields. Their dynamics are defined through the action $S[\gamma]$ given by the length functional

$$S[\gamma] = \int \mathrm{d}\tau F(\gamma, \dot{\gamma})\,. \tag{8.1}$$

The resulting equation of motion from the variation of the action with respect to γ, is the geodesic equation on Finsler spacetimes derived in section 5.2. Expressed with the help of the Cartan non-linear connection coefficients $N^a{}_b$ defined in definition 5.2

$$\ddot{\gamma}^a + N^a{}_b(\gamma, \dot{\gamma})\dot{\gamma}^b + \dot{\gamma}^a \frac{2}{r-1} \frac{L}{F} \frac{d}{d\tau} \frac{F}{L} = 0\,. \tag{8.2}$$

We conclude that test particles propagate along the geodesics of the spacetime and so they can be used as probes of the geometry. There exist situations in which we can treat objects as test particles, to good approximation, and compare their trajectories with geodesics from different spacetime geometries. From this we can draw conclusions about how well certain spacetime geometry models describe a physical situation. One prominent situation of this type is the comparison of the trajectory of mercury in the solar system with the geodesics in metric Schwarzschild spacetime, or its post Newtonian approximation, as it can be found in textbooks like [54], respectively in the living reviews [57]. When we applied the Finsler spacetime framework to the solar system in chapter 7, we saw how the geodesics in a special spherically symmetric Finsler spacetime model address the fly-by anomaly appearing when spacecrafts gain velocity in swing-by manoeuvres around planets.

The second interpretation of the Finsler length measure comes from the clock postulate. It identifies the mathematical length of observer worldlines with the proper time an observer measures. Thus the length measure for timelike curves is a geometric definition of a clock. Observers propagate through spacetime along their worldlines. In order to ensure a fixed causal order of events for all observers, the wordlines have to be timelike curves, i.e., curves γ with tangents $\dot{\gamma}$ lying in the cone of timelike vectors defined in theorem 5.1. The proper time an observer measures between two events P and Q along the observer's path is then connected to the length of the wordline segment in the following way:

Postulate 8.1. ***Clock Postulate***

The proper time T_{PQ} an observer on a Finsler spacetime (M, L, F) measures along its timelike worldline γ, between two events $P = \gamma(\tau_0)$ and $Q = \gamma(\tau_1)$ along the wordline, is given by the length of the wordline segment between them

$$T_{PQ} = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau\; F(\gamma, \dot{\gamma})\,. \tag{8.3}$$

Freely falling observers, observers only subject to forces encoded in the geometry of spacetime, move on worldlines that extremise their proper time. They are propagating along Finsler geodesics and so are special instants of test particles.

From this definition of a finite time measure of observers we now continue to describe observers by the introduction of a local observer frame.

8.2. Physical observers

Physical observers, or objects performing measurements, measure more than just time intervals; they also measure spatial length and various physical fields. To achieve this, a model of observers requires four tangent vectors that build an orthonormal frame $\{e_\mu\}_{\mu=0}^3$, their time and space directions. Then measurable quantities are the components of physical fields with respect to this frame, evaluated at the observers tangent bundle position $(\gamma, \dot\gamma)$. To compare measurements of different observers it is necessary to communicate the results obtained by one observer to another. This communication is realised by a certain class of transformations between different observers; we will show that these transformations have the algebraic structure of a groupoid that generalises the usual Lorentz group in metric geometry. In the next section we will explicitly calculate the illustrative example of an observer's measurement of the speed of light.

8.2.1. Observer frames

To construct an observer frame e_μ at the observer's tangent bundle position $(\gamma, \dot\gamma)$ we first identify the time direction e_0 before we introduce three further spatial directions e_α which complete it into a frame. The frame will be constructed such that it lives in the horizontal tangent space of the tangent bundle $H_{(\gamma,\dot\gamma)}TM$ since this space is isomorphic to the tangent space of the spacetime manifold $T_\gamma M$ and thus can be identified with directions tangent to spacetime. Recall that the isomorphism is given by the differential of the projection map of the tangent bundle $\mathrm{d}\pi_{|(\gamma,\dot\gamma)}$, as discussed below equation (1.7). We will now discuss how to construct the frame and then summarise the procedure in a precise definition below.

As explained in the previous section observers move along spacetime curves $\gamma : \tau \mapsto \gamma(\tau)$ in M with timelike tangents. The parametrization can be chosen so that $\dot\gamma \in S_\gamma$ is unit timelike. According to the definition of Finsler spacetimes we now have $|L(\gamma, \dot\gamma)| = 1$ and the signature of $g^L_{ab}(\gamma, \dot\gamma)$ is Lorentzian $\mathrm{sign}(g^L) = (\epsilon, -\epsilon, -\epsilon, -\epsilon)$ with $\epsilon = L/|L|$. Then the clock postulate expressed through equation (8.3) tells us that $\dot\gamma$ must be interpreted as the local unit time direction and so be identified with e_0 of the observers frame. We may write the normalization condition $|L(\gamma, \dot\gamma)| = 1$ in the form $g^F_{(\gamma,\dot\gamma)}(e_0, e_0) = 1$ using the horizontal lift $e_0 = \dot\gamma^H$ of $\dot\gamma$ as defined in definition 1.10. This technical identification places the observer frame in the horizontal tangent bundle $H_{(\gamma,\dot\gamma)}TM$.

To identify the spatial three-space seen by an observer, we will complete e_0 to a four dimensional basis e_μ of $H_{(\gamma,\dot\gamma)}TM$. We determine the three horizontal vectors e_α with help of the Cartan one-form Θ^L (see equation (5.20)), respectively Θ (see equation (2.11)), of the Finsler spacetime by the condition

$$0 = \Theta^L_{(\gamma,\dot\gamma)}(e_\alpha) = \frac{1}{2}\bar\partial_a L_{(\gamma,\dot\gamma)}\mathrm{d}x^a(e_\alpha) = \frac{r}{4}\frac{L(\gamma,\dot\gamma)}{F(\gamma,\dot\gamma)^2}\bar\partial_a F^2_{(\gamma,\dot\gamma)}\mathrm{d}x^a(e_\alpha) \Leftrightarrow 0 = \Theta_{(\gamma,\dot\gamma)}(e_\alpha). \tag{8.4}$$

As long as $L \neq 0$ the above conditions, the one expressed in terms of L and the one expressed in terms of F, are equivalent. In terms of the metrics the conditions simply becomes an orthogoinality condition

$$g^L_{(\gamma,\dot\gamma)}(e_0, e_\alpha) = 0 \text{ and } g^F_{(\gamma,\dot\gamma)}(e_0, e_\alpha) = 0. \tag{8.5}$$

This construction is justified by the observation that a horizontal three-space is defined by a conormal horizontal one-form. The only linearly independent one-form available in terms of geometric data is the vertical form $\mathrm{d}L = \bar{\partial}_a L\, \delta y^a$. This can be mapped globally to the horizontal one-form $\widetilde{\mathrm{d}L} = \bar{\partial}_a L\, \mathrm{d}x^a$, which is proportional to the Cartan one-form. We remark that the e_α may depend less trivially on $\dot{\gamma} = \pi_* e_0$ than in Lorentzian geometry because in their defining equation $g^F_{(\gamma,\dot{\gamma})}(e_0, e_\alpha) = 0$ the time direction $e_0 = \dot{\gamma}^H$ appears not only as a vector argument but also in the argument of the components of the g^F.

So far we identified the three dimensional orthogonal, and therefore spatial, complement subspace $\mathrm{span}\langle e_\alpha \rangle$ to the time direction e_0. But the basis of this subspace is not yet normalised. To define a normalised frame we have the choice between the L respectively the F metric to normalise the vectors e_α. Due to the relation between both and the orthogonality condition of the vectors e_0 and e_α it turns out that the difference lies in a constant factor depending only on the homogeneity of L. Since observers' tangents lie in the shell S_γ, we know by theorem 5.2 that the signature of the g^F is $(1,-1,-1,-1)$. It is clear from the normalization $g^F_{(\gamma,\dot{\gamma})}(e_0,e_0) = 1$ of the unit time direction e_0 that it corresponds to the $+1$ direction. Hence the other three frame vectors e_α can be normalized to $g^F_{ab}(\gamma,\dot{\gamma}) e^a_\alpha e^b_\alpha = -\delta_{\alpha\beta}$. This fixes an observer frame on Finsler spacetimes. Observe that the normalization with respect to g^F is related to the normalization with respect to g^L by

$$g^F_{ab}(\gamma,\dot{\gamma}) e^a_\alpha e^b_\beta = \frac{2}{r}\frac{L}{|L|} g^L_{ab}(\gamma,\dot{\gamma}) e^a_\alpha e^b_\beta\,, \tag{8.6}$$

due to the orthogonality condition equation (8.4) and the normalization of the observers tangent. Observer frames lead to a simultaneous diagonalisation of g^L and g^F. We prefer the metric g^F for orthonormalisation over g^L, since g^F is invariant under $L \to L^k$ just like the Cartan non-linear connection. We now summarise our construction of observers' frames into a precise definition.

Definition 8.1. ***Observer frames***
Let (M,L,F) be a Finsler spacetime. Physical observers along worldlines $\tau \mapsto \gamma(\tau)$ in M are described by a frame basis $\{e_\mu\}_{\mu=0}^3$ of $H_{(\gamma,\dot{\gamma})}TM$ which

(i) has a timelike vector e_0 in the sense $\pi_* e_0 = \dot{\gamma} \in S_\gamma$; and

(ii) is g^F-orthogonal, $g^F_{(\gamma,\dot{\gamma})}(e_\mu, e_\nu) = -\eta_{\mu\nu} = diag(1,-1,-1,-1)$.

They measure the components of horizontal tensor fields over TM with respect to their frame at their tangent bundle position $\{e_\mu(\gamma,\dot{\gamma})\}_{\mu=0}^3$.

By equation (8.6) one can express the second condition in the definition in terms of the L metric

$$g^F_{(\gamma,\dot{\gamma})}(e_\mu,e_\nu) = -\eta_{\mu\nu} \Leftrightarrow g^L_{(\gamma,\dot{\gamma})}(e_0,e_0) = \frac{r}{2}\frac{L}{|L|}(r-1),\ g^L_{(\gamma,\dot{\gamma})}(e_\alpha,e_\beta) = -\frac{r}{2}\frac{L}{|L|}\delta_{\mu\nu}\,. \tag{8.7}$$

We emphasize again that the frame $\{e_\mu\}_{\mu=0}^3$ in $H_{(\gamma,\dot{\gamma})}TM$ can be identified one to one with a frame $\{\pi_* e_\mu\}_{\mu=0}^3$ in $T_\gamma M$, or reversely by the horizontal lift. We will now demonstrate which kind of transformations relate two observers and so extend Lorentz transformations.

8.2.2. generalised Lorentz transformations

We already stressed the importance that observers should be able to communicate their measurements. Consider two observers whose worldlines meet at a point $x \in M$. Since observers by definition 8.1 measure the components of horizontal tensor fields in their frame and at their tangent bundle position, we need to determine which transformation uniquely maps an observer frame $\{e_\mu\}_{\mu=0}^3$ in $H_{(x,y)}TM$ to a second observer frame $\{f_\mu\}_{\mu=0}^3$ in $H_{(x,z)}TM$. Their respective four-velocities, or time directions, $y = \pi_* e_0$ and $z = \pi_* f_0$ generically are different, so that the two observer frames are objects in tangent spaces to TM at different points. As a consequence, we will now demonstrate that the transformations between observers consist of two parts: the first is a transport of the frame $\{e_\mu\}_{\mu=0}^3$ from (x,y) to (x,z), the second will turn out to be a Lorentz transformation. From here on we suppress the label $\{\}_{\mu=0}^3$.

Theorem 8.1. ***Observer transformations***
Consider two observer frames $\{e_\mu\}$ in $H_{(x,y)}TM$ and $\{f_\mu\}$ in $H_{(x,z)}TM$ on a Finsler spacetime (M,L,F). If z is in a sufficiently small neighbourhood around $y \in T_xM$, then the following procedure defines a unique map $\{e_\mu\} \mapsto \{f_\mu\}$:

(i) Let $t \mapsto v(t)$ be a vertical autoparallel of the the Cartan linear covariant derivative ∇^{CL2} (see equation (5.47)) that connects $v(0) = (x,y)$ to $v(1) = (x,z)$; this satisfies $\pi_*\dot v = 0$ and $\nabla^{CL2}_{\dot v}\dot v = 0$. Determine a frame $\{\hat e_\mu(v(t))\}$ along $v(t)$ by parallel transport $\nabla^{CL2}_{\dot v}\hat e_\mu = 0$ with the initial condition $\hat e_\mu(v(0)) = e_\mu$.

(ii) Find the unique Lorentz transformation Λ so that $f_\mu = \Lambda^\nu{}_\mu \hat e_\nu(v(1))$.

We use the second version of the Cartan linear covariant derivative in the theorem to ensure invariance of the procedure under $L \to L^k$, since these define the same geometry. In the proof it will become clear that the use of the Cartan linear covariant derivative based on F is suitable in this case since $F(v) \neq 0$ and so F^2 is differentiable along v.

Proof of Theorem 8.1.
We first show that the curve v required in (i) exists. A general curve $\gamma : \tau \mapsto \gamma(\tau) = (x(\tau), y(\tau))$ on TM has a tangent $\dot\gamma = \dot x^a \delta_a + (\dot y^a + N^a{}_q \dot x^q)\bar\partial_a$; it is vertical, if, and only if, $\dot x = 0$. Hence the verticality condition $\pi_*\dot v = 0$ in the theorem implies $\dot v = \dot V^a \bar\partial_a$. The definition of the Cartan linear covariant derivative (5.47) then tells us that $\nabla^{CL2}_{\dot v}\dot v = 0$ is equivalent to solving

$$\ddot V^a + \frac{1}{2} g^{F\,ap} \bar\partial_p g^F_{bc}(v) \dot V^b \dot V^c = 0\,. \tag{8.8}$$

This has a unique solution connecting (x,y) to any point (x,z) for z in a sufficiently small neighbourhood around y in T_xM. Now let $\{\hat e_\mu(v(t))\}$ be the parallely transported vector fields $\nabla^{CL2}_{\dot v}\hat e_\mu = 0$ with $\hat e_\mu(v(0)) = e_\mu$. The properties of the Cartan linear covariant derivative ensure that the $\hat e_\mu$ are horizontal fields. Observe also that $\nabla^{CL2}_{\dot v}\big(g^F_v(\hat e_\mu, \hat e_\nu)\big) = 0$ along v since g^F is covariantly constant under ∇^{CL2}. It follows that

$$g^F_{v(t)}(\hat e_\mu(v(t)), \hat e_\nu(v(t))) = -\eta_{\mu\nu} \tag{8.9}$$

is independent of t, and holds in particular at the final point of the transport $v(1) = (x,z)$. We realise that $\{\hat e_\mu(v(1))\}$ is an orthonormal frame with respect to g^F in $H_{(x,z)}TM$ as well as $\{f_\mu\}$; hence they are related by a unique Lorentz transformation as stated in point (ii). □

The procedure described in theorem 8.1 provides a map between the frames of two observers at the same point of the manifold $x \in M$, but with different four-velocities $y, z \in S_x \subset T_xM$; we display the two parts of this procedure as $\Lambda \circ P_{y\to z}$, i.e., as parallel transport followed by Lorentz transformation, which is illustrated in figure 8.1. The combined maps transform observers uniquely into one another as long as the autoparallel v connecting the vertically different points in TM exists and is unique. This is certainly the case if (x, y) and (x, z) are sufficiently close to each other. Whether the geometric structure of a specific, or maybe all, Finsler spacetimes is such that unique transformations between all observers exist requires still further investigation. For specific Finsler spacetime types one can for example explicitly analyse the vertical geodesic deviation equation

$$\nabla^{CL2}_{\dot{v}}\nabla^{CL2}_{\dot{v}} z^a + z^m \dot{V}^b \dot{V}^c (C^{ai}{}_c C_{imb} - C^{ai}{}_m C_{icb}) = 0 \tag{8.10}$$

where v is a vertical geodesic and z is the deviation vector to another nearby vertical geodesic. From the analysis one obtains information about the uniqueness of the vertical geodesic connecting two such observers we are looking for. It is clear that in the metric limit ($C^a{}_{bc} = 0$) the distance between vertical geodesics emerging from one observer grows linearly since, $\ddot{z} = 0$, and do not shrink again. Hence there exists only one vertical geodesic which connects two points.

In the observer transformations on generic Finsler spacetimes there appears an additional ingredient that is not present on metric spacetimes. Before applying the Lorentz transformation to the frame, one has to perform a parallel transport in the vertical tangent space. In the metric limit the vertical covariant derivative becomes trivial (the Cartan tensor vanishes) so that the parallely transported frame does not change at all along the curve v. In this special case the transformation of an observer thus reduces to $\Lambda \circ \mathrm{id}_{y\to z}$ which is fully determined by a Lorentz transformation.

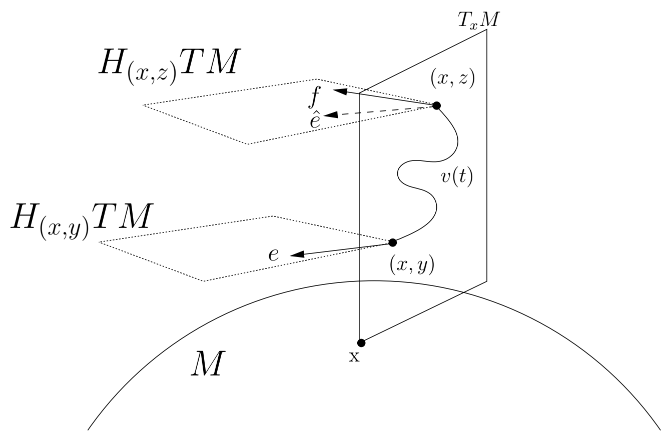


Figure 8.1. *Transformation between two observer frames: the frame $\{e_\mu\}$ in $H_{(x,y)}TM$ is first parallely transported to $\{\hat{e}_\mu\}$ in $H_{(x,z)}TM$, second Lorentz transformed into the final frame $\{f_\mu\}$.*

The observer transformations on Finsler spacetimes essentially have the algebraic structure of a groupoid that reduces to the Lorentz group in the metric limit. We first review the general definition of a groupoid and then show how this applies to our case.

Definition 8.2. ***Groupoid***
A groupoid $\mathcal{G}$ consists of a set of objects G_0 and a set of arrows G_1. Every arrow A is assigned a source $e = s(A)$ and a target $f = t(A)$ by the maps $s : G_1 \to G_0$ and $t : G_1 \to G_0$; one writes this as $A : e \to f$. For arrows A and B whose source and target match as $t(A) = s(B)$ there exists an associative multiplication $G_1 \times G_1 \to G_1, (A, B) \mapsto BA$ with

$$s(BA) = s(A)\,, \quad t(BA) = t(B)\,, \quad C(BA) = (CB)A\,. \tag{8.11}$$

A unit map $G_0 \to G_1, e \mapsto \mathbb{1}_e$ where $\mathbb{1}_e : e \to e$ exists so that

$$\mathbb{1}_{t(A)}A = A = A\,\mathbb{1}_{s(A)}\,. \tag{8.12}$$

For every arrow A exists an inverse arrow A^{-1} that satisfies

$$s(A^{-1}) = t(A)\,, \quad t(A^{-1}) = s(A)\,, \quad A^{-1}A = \mathbb{1}_{s(A)}\,, \quad AA^{-1} = \mathbb{1}_{t(A)}\,. \tag{8.13}$$

Groupoids are generalizations of groups. These can be expressed as groupoids with a single object in G_0; then the arrows correspond to group elements all of which can be multiplied since sources and targets always match. The multiplication is associative, the identity element and inverse elements exist.

Consider $G_0 = S_x \subset T_xM$ as the set of unit timelike vectors which contains the different four-velocities of observers at the point $x \in M$. Let the arrows in G_1 be the set of all maps between two observer frames at x which are defined by the procedure stated in theorem 8.1. In case the involved vertical autoparallels connect the four-velocities uniquely, the sets G_0 and G_1 define a groupoid: source and target of a map $A = \Lambda \circ P_{y\to z}$ between two frames $\{e_\mu \in H_{(x,y)}TM\}$ and $\{f_\mu \in H_{(x,z)}TM\}$ are simply given by $s(A) = y \in S_x$ and $t(A) = z \in S_x$; the multiplication BA is defined by applying the procedure of theorem 8.1 to construct the map between $s(A)$ and $t(B)$, which gives the properties (8.11); we choose the unit map $\mathbb{1}_y$ that provides (8.12) as $\mathbb{1}_y = \mathbb{1} \circ \mathrm{id}_{y\to y}$, i.e., as trivial parallel transport of the frame $\{e_\mu \in H_{(x,y)}TM\}$ with respect to the Cartan linear connection along the vertical autoparallel that stays at (x, y) followed by the identity Lorentz transformation. Finally, we define the inverse $A^{-1} = \Lambda^{-1} \circ P_{z\to y}$, where $P_{z\to y}$ denotes parallel transport backwards along the unique vertical autoparallel connecting (x, y) and (x, z) which is also used for $P_{y\to z}$; to check the properties (8.13), one simply shows that parallel transport of the frames and Lorentz transformation commute. Thus we have shown the following result:

Theorem 8.2. ***Observer transformations as groupoid***
On Finsler spacetimes (M, L, F) the transformations between observer frames at $x \in M$, $\{e_\mu \in H_{(x,y)}TM\}$, that are attached to points $(x, y) \in U_x \subset S_x$ define a groupoid $\mathcal{G}$ under the condition that any pair of points in U_x can be connected by a unique vertical autoparallel of the Cartan linear connection.

We already discussed that the transformations of observer frames reduce to the form $A = \Lambda \circ \mathrm{id}_{y\to z}$ in the limit of metric geometry. Hence the only information contained in the reduced groupoid $\tilde{\mathcal{G}}$ with $\tilde{G}_0 = S_x$ and $\tilde{G}_1 = \{\Lambda \circ \mathrm{id}_{y\to z}\}$ is given by the Lorentz transformations. In mathematically precise language this can be expressed as the equivalence of $\tilde{\mathcal{G}}$ to the Lorentz group seen as a groupoid $\mathcal{H}$ with a single object $H_0 = \{x\}$ and arrows $H_1 = \{\Lambda\}$. The functor $\varphi : \tilde{\mathcal{G}} \to \mathcal{H}$ establishing the equivalence can be defined by the projection $\varphi_0 = \pi : \tilde{G}_0 \to H_0$ and by $\varphi_1 : \tilde{G}_1 \to H_1, \Lambda \circ \mathrm{id}_{y\to z} \mapsto \Lambda$. Indeed, φ can be checked to be injective, full and essentially surjective, and so it makes $\tilde{\mathcal{G}}$ and $\mathcal{H}$ equivalent. See [59] for details on the required mathematical definitions.

By constructing observer transformations we have now obtained a complete description of observers and their measurements on Finsler spacetimes which enables us to make observable predictions. As an example we now consider the speed of light measured by an observer on Finsler spacetimes.

8.3. Measurement example: The speed of light

Definition 8.1 of the observer frame includes the statement that a physical observable is given by the components of a horizontal tensor field with respect to the observer's frame, evaluated at her position on the tangent bundle, i.e., at her position on the manifold and her four-velocity. The motivation for this is as follows. The geometry of Finsler spacetimes is formulated on the tangent bundle TM, and hence matter tensor fields coupling to this gravitational background must also be defined over TM. Not all such tensor fields can be interpreted as tensor fields from the perspective of the spacetime manifold M, only those which are purely horizontal since $H_{(x,y)}TM \simeq T_{\pi(x,y)}M = T_xM$. This interpretation requires that the tensor fields be horizontal; then they are multilinear maps built on the horizontal space $H_{(x,y)}TM$ and its dual. This interpretation will have major influence in the construction of field dynamics on Finsler spacetimes in chapter 9. Consider the example of a 2-form field Φ over TM. In horizontal-vertical basis it can be expanded as

$$\Phi = \Phi_{1\,ab}(x,y)\,\mathrm{d}x^a \wedge \mathrm{d}x^b + 2\,\Phi_{2\,ab}(x,y)\,\mathrm{d}x^a \wedge \delta y^b + \Phi_{3\,ab}(x,y)\,\delta y^a \wedge \delta y^b\,. \tag{8.14}$$

Only the purely horizontal part $\Phi_{1\,ab}(x,y)\,\mathrm{d}x^a \wedge \mathrm{d}x^b$ has a clear interpretation as a field along the manifold due to the isomorphism $d\pi$. Note that such horizontal tensor fields are automatically d-tensor fields (see definition 1.8) and have the same number of components as a tensor field of the same rank on M. The difference is that the components depend on the tangent bundle position. The measurement of a horizontal tensor field by an observer at the tangent bundle position $(\gamma,\dot\gamma)$ clearly requires an observer frame of $H_{(\gamma,\dot\gamma)}TM$ in order to read out the components.

We emphasize that the dependence of observables on the four-velocity of the observer is not surprising. Neither is it problematic, as long as observers can communicate their results. In general relativity, observables are the components of tensor fields over M with respect to the observer's frame in $T_\gamma M$; they clearly depend on $\dot\gamma$ which induces the splitting of $T_\gamma M$ into time and space directions. On Finsler spacetimes the dependence of observables on the observer's four-velocity is not only present in the time/space split of $H_{(\gamma,\dot\gamma)}TM$, but also in the argument of the tensor field components. The difference between the situation on a metric manifold and a Finsler spacetime is depicted in figure 8.2. We will continue the discussion of the dependence of the components of fields on tangent space directions in chapter 9 when we discuss field theories on Finsler spacetimes.

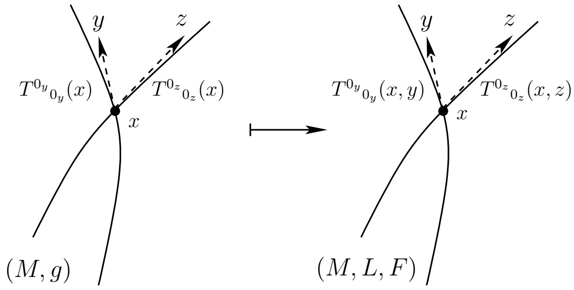


Figure 8.2. The measurement process on metric and Finsler spacetimes. The observers with tangents y and z measure their respective $(0,0)$ component $T^{0_y}_{0_y}$ and $T^{0_z}_{0_z}$ of the $(1,1)$-tensor T.

As a simple example we discuss the measurement of the spatial velocity of a point particle that moves on a worldline ρ with tangent $\dot\rho$. At the position of the observer, where $\rho = \gamma$, one can use the horizontal lift to map $\dot\rho$ from $T_\gamma M$ to $\dot\rho \in H_{(\gamma,\dot\gamma)}TM$. This lift, also denoted by $\dot\rho$ from here on and can now be expanded in the orthonormal frame of an observer as $\dot\rho = \dot\rho^0 e_0 + \vec{\dot\rho} = \dot\rho^0 e_0 + \dot\rho^\alpha e_\alpha$, where we recall that $e_0 = \dot\gamma$ is the observer's four velocity. The time $\dot\rho^0$ passes while the particle moves in spatial direction $\dot\rho^\alpha$, so the spatial velocity $\vec v$ and its square v^2 are

$$\vec v = \frac{\vec{\dot\rho}}{\dot\rho^0}, \quad v^2 = \frac{\delta_{\alpha\beta}\dot\rho^\alpha\dot\rho^\alpha}{(\dot\rho^0)^2} = -\frac{g^F_{(\gamma,\dot\gamma)}(\vec{\dot\rho},\vec{\dot\rho})}{g^F_{(\gamma,\dot\gamma)}(\dot\rho,\dot\gamma)^2}\,. \tag{8.15}$$

From this formula we may derive the speed of light seen by a given observer. For now we assume that light propagates on null worldlines ρ with $L(\rho,\dot\rho) = 0$. When we introduce a theory of electrodynamics on Finsler spacetimes in section 9.5, this will be confirmed. The null condition can equivalently be written as $L(\rho,\dot\rho^0 e_0 + \vec{\dot\rho}) = 0$, which we use to replace the Finsler metric in the formula for the velocity above by Taylor expanding around $\vec{\dot\rho} = 0$

$$0 = (\dot\rho^0)^r L(\rho,\dot\gamma) \quad + \quad (\dot\rho^0)^{r-1}\bar\partial_a L(\rho,\dot\gamma)\vec{\dot\rho}^{\,a} + (\dot\rho^0)^{r-2} g^L_{(\rho,\dot\gamma)}(\vec{\dot\rho},\vec{\dot\rho}) \tag{8.16}$$

$$+ \quad \sum_{k=3}^{\infty}\frac{(\dot\rho^0)^{r-k}}{k!}\bar\partial_{c_1}...\bar\partial_{c_k}L(\rho,\dot\gamma)\vec{\dot\rho}^{\,c_1}...\vec{\dot\rho}^{\,c_k}. \tag{8.17}$$

Multiplying by $(\dot\rho^0)^{-r+2}$, using the orthogonality of $\vec{\dot\rho}$ and $\dot\gamma$ and inserting the relation between g^L and g^F from equation (5.15) yields

$$0 = (\dot\rho^0)^2 L(\gamma,\dot\gamma) + \frac{rL(\gamma,\dot\gamma)}{2F^2(\gamma,\dot\gamma)} g^F_{(\gamma,\dot\gamma)}(\vec{\dot\rho},\vec{\dot\rho}) + \sum_{k=3}^{\infty}\frac{(\dot\rho^0)^{2-k}}{k!}\bar\partial_{c_1}...\bar\partial_{c_k}L(\gamma,\dot\gamma)\vec{\dot\rho}^{\,c_1}...\vec{\dot\rho}^{\,c_k}. \tag{8.18}$$

We immediately obtain an expression for the speed of light $c^2_{(\gamma,\dot\gamma)}(\vec{\dot\rho})$, i.e., the speed of light travelling in spatial direction $\vec{\dot\rho}$ and measured by the observer $(\gamma,\dot\gamma)$ by rearranging the above equation:

$$c^2_{(\gamma,\dot\gamma)}(\vec{\dot\rho}) = -\frac{g^F_{(\gamma,\dot\gamma)}(\vec{\dot\rho},\vec{\dot\rho})}{g^F_{(\gamma,\dot\gamma)}(\dot\rho,\dot\gamma)^2} = \frac{2}{r} + \frac{2}{r}\frac{1}{L(\gamma,\dot\gamma)}\sum_{k=3}^{\infty}\frac{(\dot\rho^0)^{-k}}{k!}\bar\partial_{c_1}...\bar\partial_{c_k}L(\gamma,\dot\gamma)\vec{\dot\rho}^{\,c_1}...\vec{\dot\rho}^{\,c_k}\,. \tag{8.19}$$

Observe that the infinite sum in the formulae above becomes a finite sum for polynomial L, as for example in our bimetric example defined in equation (5.123). The $\dot\rho^0$ are determined by solving the null condition $L(\gamma,\dot\rho^0\dot\gamma + \vec{\dot\rho}) = 0$; on a generic Finsler spacetime there can be more than one solution since the null structure can be very complicated. The formulae (8.15) and (8.19) enable us to compare experimental results on particle and light velocities with predictions on specific Finsler spacetime models. Take for example the maximal anisotropy of the speed of light an observer could detect. It is given by

$$\Delta c^2_{(\gamma,\dot\gamma)} = \max_{\dot{\vec\rho}} c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) - \min_{\dot{\vec\rho}} c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) \tag{8.20}$$

This quantity can be calculated and compared to the bounds on from anisotropy measurements. From equation (8.19) we see that the anisotropy depends on the higher than second order derivatives of the fundamental geometry function function; these do vanish on special Finsler spacetimes where the higher order derivative terms in the sum cancel each other and in the metric limit $L = g_{ab}y^a y^b,\ r = 2$ where we recover $\Delta c^2_{(\gamma,\dot\gamma)} = 0$ and thus $c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = 1$ independent of the observer and the spatial direction of the light ray.

This example of the measurement of the speed of light demonstrates how important it is to define observers and their measurements before one can compare predictions with experimental data. Measurements of our observers depend stronger on their motion relative to the system on which they perform measurements, but this does not cause an problems in the theoretical description of physics if there is a transformation which relates the measurements of different observers. Here these transformations are provided by the generalised Lorentz transformations introduced previously.

9. Matter fields on Finsler spacetimes

Physical phenomena are events on spacetime caused by the interaction of fields, which in principle can be measured by an observer. These fields are on the one hand the fundamental geometry function L we already encountered in chapter 5, which describes the geometry of spacetime itself, on the other hand there are additional non-geometric matter fields. When the geometric field L describes the gravitational interaction, additional matter fields are needed to describe all other interactions and matter. We briefly mentioned how such fields may look like when we discussed measurements of observers in section 8.3. The interpretation of Finsler spacetimes as generalisation of Lorentzian metric spacetimes which provide a geometrisation of causality, observers and gravity at the same time, requires that we introduce a scheme how such matter fields couple to the non-metric geometry and how they source the dynamics of the geometry.

The behaviour of the fields shall be described by field equations which determine their evolution and interaction and thus the events on spacetime. A fundamental ingredient is the coupling of the non-geometric fields to the geometry-defining one. On Lorentzian metric spacetimes this coupling is realised through the appearance of the spacetime metric in the matter field equations and the appearance of the matter fields through the energy-momentum tensor in the Einstein equations. On general Finsler spacetimes there is no spacetime metric and no energy momentum tensor in the sense of general relativity available. In this chapter we will construct such a coupling with the help of the geometric structure available and obtain classical field theories on Finsler spacetimes. Their quantisation has to be investigated but this remains an open question here and we comment on it in the outlook. We begin with a general discussion about viability conditions on the coupling of the matter fields to the geometry in section 9.1. Afterwards we present a specific coupling scheme in section 9.2 which meets all requirements discussed previously and we derive how the coupled fields appear as source term in the Finsler spacetime dynamics in section 9.3. Afterwards we study properties of the field theories obtained via the coupling principle. We consider the scalar field in section 9.4 where we derive and discuss the dispersion relation for momentum modes of the field. In section 9.5 we present a theory of electrodynamics on Finsler spacetimes which yields the propagation of light along Finsler null-geodesics.

9.1. Requirements on the coupling

The Finsler spacetime framework is a generalisation of the Lorentzian metric geometric background for physics. The non-geometric matter fields which can be used to describe physics on this background shall be modified as minimalistically as possible. We present now the arguments that lead to certain requirements on the coupling between matter fields and the geometry which should hold for any coupling one comes up with. In the next section we then present a specific coupling principle which meets all requirements discussed here.

The geometry of Finsler spacetimes is described completely by homogeneous d-tensor fields on the tangent bundle TM of the spacetime manifold M, as discussed in every detail in chapter 5. Recall from definition 1.8: a (r, s)-d-tensor field on TM differs from an (r, s)-tensor field on M mainly by the fact that its components depend in general not only on the points of the manifold but on the points of the tangent bundle; especially they do not differ in the amount of components.

In order to couple to a geometry described by tensors on TM, generic matter fields have also to be tensor fields on TM. To change them as minimalistically as necessary we want to keep the number of components identical to the fields known from field theories on metric spacetimes and we have to ensure tensorial transformation behaviour under manifold induced coordinate changes. This suggests to use d-tensors as physical fields on the tangent bundle. As a special case these would include tensor fields whose components only depend on the coordinates of the base manifold. Moreover horizontal d-tensors can always be expressed in the horizontal basis of the tangent bundle of the tangent bundle $\{\delta_a\}_{a=0}^3$ and its dual $\{\mathrm{d}x^a\}_{a=0}^3$. Since the map $d\pi_{(x,y)}$ of the tangent bundle is an isomorphism between $H_{(x,y)}TM$ and T_xM these purely horizontal d-tensors can be mapped to the tensor spaces of the manifold. Because of these arguments we require that the measurable part of a physical field shall be described by purely horizontal (r, s)-d-tensors on TM.

The geometry of Finsler spacetimes coincides with Lorentzian metric geometry in case the fundamental geometry function can be expressed through powers of Lorentzian metric length element $L_n = (g_{ab}(x)y^a y^b)^n$. In this case all geometric tensors can be identified with the geometric tensor of Lorentzian geometry and the same should hold for the matter fields. A requirement on the coupled field theory therefore is that in case the Finsler spacetime geometry is nothing but metric geometry the matter fields shall be identifiable with tensor fields on the manifold and that their field equations shall reduce to the ones known from the field theory on Lorentzian spacetimes.

The coupling of the fields to the geometry will involve the fundamental geometry function L and the field theory shall give a source term to the dynamical equation of the geometry derived in section 6.2. Similarly as stated above for the matter field dynamics, in the case of a metric Finsler spacetime this coupled equation shall be equivalent to the Einstein equations including the matter source term.

We derived the dynamics for the geometry of Finsler spacetimes from an action which we derived from the Einstein–Hilbert action. The most simple way to realise all requirements discussed is, to define matter field theories on Finsler spacetimes from an action integral on the unit tangent bundle, similar to what is usually done to define field theories on Lorentzian metric spacetimes.

To sum this discussion up we conclude that matter fields on Finsler spacetimes should be homogeneous horizontal (r, s)-d-tensor fields. In case the Finsler spacetime is a Lorentzian metric spacetime they should be identifiable with their counterparts on Lorentzian metric manifold and their dynamics, as well as the dynamics determining the geometry including the source term induced by the matter fields, should be equivalent to field equations known from general relativity.

We like to remark here again and in more detail that the dependence of physical fields on the direction of the manifold is nothing special, as we did when we discussed the measurements of observers in section 8.3. This dependence does especially not mean, that the field depends on the observer, only the measurement does. Every tensorial field on a Lorentzian metric spacetime yields a measurable physical number only when evaluated with respect to some observer frame at the position of the observer. On Finsler spacetimes exactly the same is the case, only that the measurable quantities may depend stronger on the on the observer's frame since the physical fields will be d-tensors on the tangent bundle. Nonetheless the physical field itself is for sure independent of any observer and evolves according to its field equation. For example every n-form Λ on M can be seen as $(n-1)$-form $\hat{\Lambda}$ on TM, with components linear in the tangent space coordinates via the identification

$$\begin{aligned}\Lambda = \Lambda_{a_1\ldots a_n}(x)\mathrm{d}x^{a1}\wedge\cdots\wedge\mathrm{d}x^{a_n}\mapsto\hat{\Lambda} &= \Lambda_{qa_1\ldots a_{n-1}}(x)y^q\mathrm{d}x^{a1}\wedge\cdots\wedge\mathrm{d}x^{a_{n-1}}\\ &= \hat{\Lambda}_{a_1\ldots a_{n-1}}(x,y)\mathrm{d}x^{a1}\wedge\cdots\wedge\mathrm{d}x^{a_{n-1}}\,. \qquad (9.1)\end{aligned}$$

Finsler spacetime fields now could be not linear but only homogeneous with respect to the tangent space coordinates. Such a lifting of fields would be analogous to what happens to the non-linear curvature tensor when one passes from linear connections on TM, respectively metric geometry to a one-homogeneous connection on TM respectively Finsler spacetime geometry, see the discussion below equation (5.55).

Having argued for requirements on a coupling principle and that it is nothing special to consider fields which depend on the coordinates of the tangent bundle we present a procedure that realises all the requirements and can be seen as a minimal coupling.

9.2. Minimal coupling

Matter field theories on a Lorentzian metric manifold are usually derived from an action integral. Here we will present a lifting procedure how one obtains consistent counterpart matter field theory actions on Finsler spacetimes. In the discussion we restrict our attention to p-form fields; for spinor fields further studies are required. After the discussion of the general procedure we derive the influence on the dynamics of Finsler spacetimes in the next section.

Consider an action $S_m[\tilde{g},\phi]$ for a physical p-form field ϕ on a Lorentzian spacetime $(M,\tilde{g})$,

$$\tilde{S}_m[\tilde{g},\phi] = \int_M \mathrm{d}^4x\ \sqrt{\tilde{g}}\ \mathcal{L}(\tilde{g},\phi,\mathrm{d}\phi)\,. \qquad (9.2)$$

The corresponding matter action on Finsler spacetime is obtained by lifting $\tilde{S}_m$ to the tangent bundle equipped with the Sasaki-type metric (TM, G^F), defined in definition 5.4, according to the following procedure:

(i) consider the Lagrangian density $\mathcal{L}(\ldots)$ of the standard theory on M as a contraction prescription that forms a scalar function from various tensorial objects;

(ii) replace the Lorentzian metric $\tilde{g}(x)$ in $\mathcal{L}(\ldots)$ by the Sasaki-type metric $G^F(x,y)$;

(iii) replace the p-form field $\phi(x)$ on M by a zero-homogeneous p-form field $\Phi(x,y)$ on TM;

(iv) introduce Lagrange multipliers λ for all not purely horizontal components of Φ;

(v) finally integrate over the unit tangent bundle Σ with the volume form given by the pull-back G^{F*} of the Sasaki-type metric.

We call this procedure a minimal coupling since we only exchanged the spacetime metric by the Sasaki-type metric on the tangent bundle, lifted fields on M to fields on TM, and kept the number of field components fixed. Especially no further dependence on the geometry of spacetime, except through the metric, is introduced. The result of this procedure is the Finsler spacetime field theory action

$$S_m[L,\Phi,\lambda] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u \Big[\sqrt{g^F h^F}\Big(\mathcal{L}(G^F,\Phi,\mathrm{d}\Phi) + \lambda(1-P^H)\Phi\Big)\Big]_{|\Sigma}\,. \tag{9.3}$$

In the example of the general two-form on TM

$$\Phi = \Phi_{1\,ab}(x,y)\,\mathrm{d}x^a\wedge\mathrm{d}x^b + 2\,\Phi_{2\,ab}(x,y)\,\mathrm{d}x^a\wedge\delta y^b + \Phi_{3\,ab}(x,y)\,\delta y^a\wedge\delta y^b\,, \tag{9.4}$$

the two-form Φ is zero-homogeneous as tensor if its component $\Phi_{1\,ab}(x,y), \Phi_{2\,ab}(x,y)$ and $\Phi_{3\,ab}(x,y)$ are $0,-1$ respectively -2-homogeneous. The projection P^H projects to the purely horizontal part of the 2-form

$$P^H\Phi = \Phi_{1\,ab}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\,. \tag{9.5}$$

The Lagrange multiplier guarantees that the physical on-shell degrees of freedom of Φ are precisely those. They have a clear physical interpretation as tensor fields from the base manifold perspective, as discussed in section 8.3 in context of the measurement of an observer and in the previous section from the field theory point of view.

The matter field equations obtained by extremising the action with respect to the p-form field Φ and the Lagrange multiplier λ can be studied most easily if expressed in components with respect to the horizontal/vertical basis. The calculation is performed in detail in appendix A.7. We display the results with the convention that barred indices $\bar{a},\bar{b},\dots$ denote vertical components, unbarred indices $a,b,\dots$ now denote horizontal components, and capital indices $A,B,\dots$ both horizontal and vertical components. Variation with respect to the Lagrange multiplier yields the constraints

$$\Phi_{\bar{a}_1\dots\bar{a}_i a_{i+1}\dots a_p} = 0\,,\quad \forall\, i = 1\dots p\,. \tag{9.6}$$

Variation for the purely horizontal components of Φ gives

$$\frac{\partial\mathcal{L}}{\partial\Phi_{a_1\dots a_p}} - (p+1)(\nabla^{CL}_q + S_q)\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{qa_1\dots a_p})} - (\bar{\partial}_{\bar{q}} + g^{F\,mn}\bar{\partial}_{\bar{q}}g^F_{mn} - 4g^F_{\bar{q}q}y^q)\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{\bar{q}a_1\dots a_p})} = 0 \tag{9.7}$$

which determines the evolution of the physical field components, while variation with respect to the remaining components produces

$$\begin{aligned}&\lambda^{\bar{a}_1 A_2\dots A_p}\\ =\;& -\frac{\partial\mathcal{L}}{\partial\Phi_{\bar{a}_1A_2\dots A_p}} + (p+1)(\nabla^{CL}_q + S_q)\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{q\bar{a}_1A_2\dots A_p})} + \frac{p(p+1)}{2}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{PQA_2\dots A_p})}\gamma^{\bar{a}_1}{}_{PQ}\\ &+(\bar{\partial}_{\bar{q}} + g^{F\,mn}\bar{\partial}_{\bar{q}}g^F_{mn} - 4g^F_{\bar{q}q}y^q)\frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{\bar{q}\bar{a}_1A_2\dots A_p})}\end{aligned} \tag{9.8}$$

which fixes the components of the Lagrange multiplier. The $\gamma^{\bar{a}}{}_{PQ}$ are the commutator coefficients of the horizontal/vertical basis.

Our coupling principle is consistent with the metric limit, i.e., the equations of motion obtained from the Finsler spacetime action reduce to the equations of motion on Lorentzian spacetime in the case $L = \tilde{g}_{ab}(x)y^a y^b$ and $\Phi_{A_1\ldots A_p}(x,y) = \phi_{A_1\ldots A_p}(x)$. Then we have the geometric identity $S_a = 0$; moreover

$$\mathrm{d}\Phi_{a_1\ldots a_{p+1}} = (p+1)\partial_{[a_1}\phi_{a_2\ldots a_{p+1}]} \tag{9.9}$$

by using the constraints (9.6) and the fact that the horizontal derivative acts as a partial derivative on the y-independent p-form components. Finally,

$$\frac{\partial \mathcal{L}}{\partial(\mathrm{d}\Phi_{\bar{q}a_1\ldots a_p})} = 0 \tag{9.10}$$

because, as a consequence of our coupling principle where the Sasaki-type metric is block-diagonal in the horizontal/vertical basis, the vertical index of $\mathrm{d}\Phi_{\bar{q}a_1\ldots a_p}$ must appear in the Lagrangian $\mathcal{L}(G,\Phi,\mathrm{d}\Phi)$ contracted via g^F into either a vertical derivative or into components of Φ with at least one vertical index. In the metric limit, vertical derivatives give zero, while the constraints (9.6) guarantee that all components of Φ with at least one vertical index vanish. Combining these observations shows that equation (9.7) reduces to

$$\frac{\partial \mathcal{L}}{\partial \Phi_{a_1\ldots a_p}} - (p+1)\nabla^{CL}_q \frac{\partial \mathcal{L}}{\partial(\mathrm{d}\Phi_{qa_1\ldots a_p})} = 0\,, \tag{9.11}$$

where ∇^{CL} now operates in the same way as the Levi-Civita connection of the metric g. Again, as a consequence of our minimal coupling principle with the block-diagonal form of the Sasaki-type metric in the horizontal/vertical basis, we can conclude in the metric limit that

$$\frac{\partial \mathcal{L}(G,\Phi,\mathrm{d}\Phi)}{\partial \Phi_{a_1\ldots a_p}} = \frac{\partial \mathcal{L}(\tilde{g},\phi,\mathrm{d}\phi)}{\partial \phi_{a_1\ldots a_p}}\,, \quad \frac{\partial \mathcal{L}(G,\Phi,\mathrm{d}\Phi)}{\partial(\mathrm{d}\Phi_{qa_1\ldots a_p})} = \frac{\partial \mathcal{L}(\tilde{g},\phi,\mathrm{d}\phi)}{\partial(\mathrm{d}\phi_{qa_1\ldots a_p})} \tag{9.12}$$

so that (9.11) becomes equivalent to the standard p-form field equation of motion on metric spacetime.

Note that the coupling can be easily extended to the case of interacting form fields of any degree with metric spacetime action

$$\tilde{S}_m[\tilde{g},\phi_1,\phi_2,\ldots] = \int_M \mathrm{d}^4x\,\sqrt{\tilde{g}}\,\mathcal{L}(\tilde{g},\phi_1,\mathrm{d}\phi_1,\phi_2,\mathrm{d}\phi_2,\ldots)\,. \tag{9.13}$$

The minimal coupling procedure then leads to the action

$$\begin{aligned} &S_m[L,\Phi_1,\lambda_1,\Phi_2,\lambda_2,\ldots] \\ = \quad &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\Big[\sqrt{g^F h^F}\Big(\mathcal{L}(G^F,\Phi_1,\mathrm{d}\Phi_1,\Phi_2,\mathrm{d}\Phi_2,\ldots) + \sum_I \lambda_I(1-P^H)\Phi_I\Big)\Big]_{|\Sigma}\,. \end{aligned} \tag{9.14}$$

The equations of motion for each field ϕ_I have the same form as in the single field case, and the metric limit leads to the standard field equations by arguments that proceed in a completely analogous way as before.

Varying the matter field action with respect to the fundamental geometry function L yields a scalar source term of the Finsler spacetime dynamics (6.11), we call energy-momentum scalar T, since it formally replaces the energy-nomentum tensor from general relativity.

During the next section we demonstrate further features resulting from the coupling principle. We demonstrate that the Finsler spacetime dynamics including this source term are consistent with the Einstein-equations and we discuss the resulting scalar field theory and electrodynamics; in these explicit examples we derive observable consequences.

With the coupling principle presented here we so demonstrate that it is very well possible to obtain field theories on non-metric geometric backgrounds.

Nevertheless we like to remark that this does for sure not mean that there are no other ways to couple fields consistently to Finsler spacetimes. One may for example construct a coupling principle directly for extended p-form fields on M as homogeneous $(p-1)$-form fields on TM as outlined in equation (9.1). The advantage of these fields would be that they are immediately horizontal, the disadvantage that it is not clear which Lagrangians one should use to define the field theory. A rewriting of the known field theories on Lorentzian metric spacetimes into this language would give first insights how such field theories could look like.

9.3. Source of the geometric dynamics of spacetime

We are now in the position to study the interplay between the matter actions S_m introduced in (9.3) and the pure Finsler geometry action S_L in (6.3). Their sum provides a complete description of gravitational dynamics and classical matter fields on Finsler spacetimes:

$$\begin{aligned} & S[L,\Phi,\lambda] \\ = \; & \kappa^{-1}S_L[L]+S_m[L,\Phi,\lambda] \\ = \; & \kappa^{-1}\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\Big[\sqrt{g^Fh^F}\;\mathcal{R}\Big]_{|\Sigma}+\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\Big[\sqrt{g^Fh^F}\;\big(\mathcal{L}(G,\Phi,\mathrm{d}\Phi)+\lambda(1-P^H)\Phi\big)\Big]_{|\Sigma}. \end{aligned} \tag{9.15}$$

As usual, the matter field equations following from this are the same as for the pure matter action. The gravitational field equations are obtained by variation with respect to the fundamental geometry function L. The variation of S_m with respect to L is

$$\delta S_m=\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\Big(\frac{\delta S_m}{\delta L}\delta L\Big)_{|\Sigma}=\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\Big(\sqrt{g^Fh^F}\frac{rL}{\sqrt{g^Fh^F}}\frac{\delta S_m}{\delta L}\Big)_{|\Sigma}\frac{\delta L}{rL}\,, \tag{9.16}$$

and leads us to the definition of the energy momentum scalar $T_{|\Sigma}$ on the unit tangent bundle as

$$T_{|\Sigma}=\Big(\frac{rL}{\sqrt{g^Fh^F}}\frac{\delta S_m}{\delta L}\Big)_{|\Sigma}. \tag{9.17}$$

With this definition the complete gravitational field equations on Finsler spacetime including energy-momentum sources becomes

$$\Big[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R}-6\frac{\mathcal{R}}{F^2}+2g^{F\,ab}\big(\nabla^B_a S_b+\bar{\partial}_a\nabla S_b\big)\Big]_{|\Sigma}=-\kappa T_{|\Sigma}\,. \tag{9.18}$$

As in the vacuum case with $T_{|\Sigma}=0$, these equations can be lifted to TM. The terms in the bracket on the left hand side are all zero-homogeneous and can be lifted trivially. The terms in T without the restriction, on the right hand side of the equation, can in principle have different homogeneities; to lift these one simply multiplies each term by the appropriate power of F in order to make them zero-homogeneous. This is the same procedure applied in section 6.2 to the gravity side.

The gravitational constant κ will now be determined so that the gravitational field equation on Finsler spacetimes becomes equivalent to the Einstein equations in the metric limit. Variation with respect to L of the concrete form of the matter action in (9.15) and performing the metric limit, i.e., $g^F{}_{ab}(x,y) = -\tilde{g}_{ab}(x)$ for observers and $\Phi_{A_1\ldots A_p}(x,y) = \phi_{A_1\ldots A_p}(x)$, the gravity equation (9.18) becomes

$$2\tilde{g}^{ab}R_{ab} + 6\frac{R_{ab}y^a y^b}{|\tilde{g}_{pq}y^p y^q|} = -\kappa\Big(4\mathcal{L} - 4\tilde{g}_{ab}\frac{\partial\mathcal{L}}{\partial\tilde{g}_{ab}} - 24\frac{y_a y_b}{|\tilde{g}_{pq}y^p y^q|}\frac{\partial\mathcal{L}}{\partial\tilde{g}_{ab}}\Big)\,. \tag{9.19}$$

The detailed calculation of this result is involved and can be found again in appendix A.7. Introducing the standard energy momentum tensor of p-form fields on Lorentzian metric spacetimes $\tilde{T}^{ab} = \tilde{g}^{ab}\mathcal{L} + 2\frac{\partial\mathcal{L}}{\partial g_{ab}}$ and its trace $\tilde{T} = \tilde{T}^{ab}\tilde{g}_{ab} = 4\mathcal{L} + 2\tilde{g}_{ab}\frac{\partial\mathcal{L}}{\partial\tilde{g}_{ab}}$ we can rewrite the equation above as

$$2R - 6\frac{R_{ab}y^a y^b}{\tilde{g}_{pq}y^p y^q} = -\kappa\Big(-2\tilde{T} + 12\frac{\tilde{T}^{ab}y_a y_b}{\tilde{g}_{pq}y^p y^q}\Big), \tag{9.20}$$

if evaluated at $\tilde{g}$-timelike observer four-velocities y. Now we take a second derivative with respect to y, contract with $\tilde{g}^{-1}$, reinsert the result, and conclude

$$\Big(R_{ab} - \frac{1}{2}\tilde{g}_{ab}R\Big)y^a y^b = 2\kappa\,\tilde{T}_{ab}y^a y^b\,. \tag{9.21}$$

Since there is no y-dependence beyond the explicit one, a second derivative with respect to y yields the Einstein equations, if we choose the gravitational constant $\kappa = \frac{4\pi G}{c^4}$.

The gravity equation on Finsler spacetime including the coupling to matter therefore is

$$\Big[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b\mathcal{R} - 6\mathcal{R} + 2g^{F\,ab}\big(\nabla^B_a S_b + \bar{\partial}_a\nabla S_b\big)\Big]_{|\Sigma} = \frac{4\pi\mathcal{G}}{c^4}T_{|\Sigma}\,. \tag{9.22}$$

Observe that this field equation including the matter part is invariant under $L \to L^k$ by construction of the coupling principle, as the vacuum equation is. This leads to the interesting conclusion that every solution $\tilde{g}_{ab}(x)$ of the Einstein equations induces a family L_k of solutions of the Finsler gravity solution with $L_k = (\tilde{g}_{ab}(x)y^a y^b)^k$.

With the presentation of the matter coupled equation which determines the geometry of Finsler spacetimes as the Einstein equations determine the geometry of Lorentzian metric spacetimes we provide a complete framework which is capable to extend general relativity consistently. Even though a lot of questions still remain to be answered in the future, like for example the initial value problem of the above equation (9.22), we demonstrated that it is very well possible to study spacetime geometries beyond metric geometry and to equip them with consistent dynamics.

We proceed now to demonstrate explicitly how our coupling principle works. We construct the matter field theories for the scalar field and the electromagnetic potential on Finsler spacetimes.

9.4. The scalar field

The minimal coupling procedure for matter fields to Finsler spacetime we introduced can be applied immediately, for instance to the scalar field. We derive the equations of motion on a general Finsler spacetime and demonstrate their consistency with the Klein-Gordon equation in the metric limit. Then we specialise to a bimetric flat Finsler spacetime which is close to Minkowski spacetime to derive the dispersion relations of modes of the field with momentum P. It will turn out that, depending on the energy of the mode, its velocity may be larger than the velocity of light. We will see that the dispersion relation we derive in the end approaches modified dispersion relations discussed in the physics literature [60] from a purely geometric point of view. The modification is caused by the non-metric geometry of Finsler spacetimes.

9.4.1. The action and equations of motion

The Finsler spacetime action for massive scalar field $\phi(x,y)$ according to the minimal coupling principle discussed above is

$$\begin{aligned} S[L,\phi] &= \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\sqrt{G^{F*}}\,\mathcal{L}[G^F,\phi,\mathrm{d}\phi]_{|\Sigma} \\ &= -\frac{1}{2}\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\sqrt{G^{F*}}\Big[G^{F\,AB}\partial_A\phi\partial_B\phi+m^2\phi^2\Big]_{|\Sigma}\,, \end{aligned} \tag{9.23}$$

where the capital indices A,B label the eight induced coordinates (x^a,y^b) on TM and G^{F*} is the pull back of the Sasaki-metric G^F to Σ. The equations of motion for ϕ are obtained by variation. We first expand the action in the horizontal/vertical basis $\{\delta_a,\,\bar\partial_a\}$ of TTM and obtain

$$S[L,\phi]=\frac{1}{2}\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\sqrt{G^{F*}}\,\Big[g^{F\,ab}\delta_a\phi\delta_b\phi+g^{F\,ab}\bar\partial_a\phi\bar\partial_b\phi-m^2\phi^2\Big]_{|\Sigma}\,. \tag{9.24}$$

To find $\delta S[\phi]$ we use the integration by parts formulae (5.86) and we find the equations of motion

$$\Big[-g^{F\,ab}\Big(\delta_a\delta_b-\Gamma^{\delta\,p}{}_{ab}\delta_p+\bar\partial_a\bar\partial_b+S^p{}_{pa}\delta_b\Big)\phi-m^2\phi\Big]_{|\Sigma}=0\,, \tag{9.25}$$

which can be expressed in terms of the coordinates $(\hat{x}^a,u^\alpha)$ of Σ, see section 5.3.2 for the definitions, as

$$\begin{aligned} 0 &= -g^{F\,ab}\Big[\hat\delta_a\hat\delta_b+\bar\partial_a u^\alpha\bar\partial_b u^\beta\partial_\alpha\partial_\beta-\Gamma^{\delta p}{}_{ab}\hat\delta_p+S^p{}_{pa}\hat\delta_b+\bar\partial_a u^\beta\partial_\beta\bar\partial_b u^\alpha\partial_\alpha\Big]_{|\Sigma}\phi-m^2\phi \\ &= G^{F*\,\hat{M}\hat{N}}\partial_{\hat{M}}\partial_{\hat{N}}\phi-g^{F\,ab}\Big[-\Gamma^{\delta p}{}_{ab}\hat\delta_p+S^p{}_{pa}\hat\delta_b+\bar\partial_a\bar\partial_b u^\alpha\partial_\alpha\Big]_{|\Sigma}\phi-m^2\phi\,. \end{aligned} \tag{9.26}$$

The upper case hatted indices denote the coordinates of Σ. In metric geometry $L=\tilde{g}_{ab}(x)y^ay^b$ and $\phi(x,y)=\tilde\phi(x)$ the field equation reduces to the standard Klein-Gordon equation expressed with help of the Levi-Civita covariant derivative ∇_a

$$(\tilde{g}^{ab}\nabla_b\nabla_a-m^2)\tilde\phi=0\,. \tag{9.27}$$

We remark that the principal symbol of the equation of motion (9.26) is governed by the pull-back of the Sasaki metric to the unit tangent bundle. In case the Sasaki metric has Lorentzian signature everywhere on TM the standard theorems on hyperbolic partial differential equations can be applied and ensure a well posed initial value problem [61].

We know that the Sasaki-Metric has Lorentzian signature on the set of all timelike vectors on TM; the union over all cones C_x of timelike vectors. This can easily be seen from the explicit form of the Sasaki metric on TM in adapted coordinates (equations (5.76) and (5.77)), and from the fact that, by definition 5.1, $g^{L\setminus F}$ has Lorentzian signature on this set. So in general our definition of Finsler spacetimes does not guarantee that the Sasaki-Metric has Lorentzian signature everywhere. How this effects the propagation of solutions of the equations of motion obtained from our minimal coupling principle is so far not clear and has to be investigated carefully in the future.

We now study the scalar field on a special flat Finsler spacetime close to Minkowski spacetime.

9.4.2. On flat bimetric Finsler spacetimes

Our aim is to analyse the dispersion relation fulfilled by momentum modes of the scalar field on a a flat bimetric Finsler spacetime close to Minkowski spacetime. We will compare the velocity of a mode of the scalar field with momentum P with the velocity of light and we will comment on the relation of our result with modified dispersion relations discussed in the literature.

We already encountered bimetric geometries as prototype examples for Finsler spacetimes in section 5.5.2 and when we discussed the refined linearised Schwarzschild geometry in section 7.2. For a flat bimetric geometry close to Minkowski spacetime we use global Cartesian coordinates in which the fundamental geometry function takes the form

$$L(y) = \eta_{ab}y^a y^b k_{cd}y^c y^d = \eta_{ab}y^a y^b(\eta_{cd} + h_{cd})y^c y^d = \eta(y,y)^2 + \eta(y,y)h(y,y), \tag{9.28}$$

where h_{ab} are components of a symmetric $(0,2)$ tensor such that $k_{ab} = \eta_{ab} + h_{ab}$ is a Lorentzian metric. To simplify the discussion we furthermore assume that the null-cone of the metric $\eta + h$ is the outer light cone and so the light cone of η is the inner cone representing the boundary velocity of observers. The cone of timelike vectors C_x is given by the cone of timelike vectors of η. The situation for a general bimetric Finsler spacetime is depicted in the figures 5.3a and 5.3b.

In the global Cartesian coordinate system we consider, the non-linear connection coefficients $N^a{}_b$ and the δ-Christoffel symbols vanish and so does the S-tensor. We expand the scalar field ϕ into modes of momentum $P = P_a d\hat{x}^a + P_\alpha \delta u^\alpha = ((P_a + \hat{N}^\alpha{}_a P_\alpha)d\hat{x}^a + P_\alpha du^\alpha)$, where P_a denote the horizontal and P_α the vertical components of the momentum covector. The derivative operators $\hat{\delta}_a$ and ∂_α act as multiplication operators with iP_a respectively iP_α and hence the equation of motion (9.26) becomes the dispersion relation

$$0 = -g^{F\,ab}\Big[- P_a P_b - \bar{\partial}_a u^\alpha \bar{\partial}_b u^\beta P_\alpha P_\beta + i\bar{\partial}_a\bar{\partial}_b u^\alpha P_\alpha\Big]_{|\Sigma} - m^2\,. \tag{9.29}$$

The corresponding direction of propagation X of the field mode with momentum P is given by the map

$$X = \frac{1}{m}G^{F*\,-1}(P,\cdot)\,. \tag{9.30}$$

As discussed in section 8.2, observers measure the horizontal modes; so we interpret the pure horizontal modes, $P_\alpha = 0$, as particle excitations of the field. The resulting dispersion relation for any observer on worldline γ with timelike tangent $\dot\gamma \in C_\gamma$ is

$$-g^{Fab}(\gamma,\dot\gamma)P_a P_b = -m^2\,. \tag{9.31}$$

In each observers frame $\{e_0 = \dot\gamma, e_\alpha\}$, introduced in definition 8.1, the dispersion relation takes the standard form

$$\eta^{\mu\nu} P_\mu P_\nu = -m^2\,. \tag{9.32}$$

Thus for each observer the dispersion relation of the scalar field in Minkowski spacetime is realised and we identify the components of the momentum P in the frame of an observer with the energy $P_0 = E$ and with the spatial momentum (P_1, P_2, P_3) of the mode seen by an observer. Since the pull-back of the Sasaki-metric G^{F*} is block-diagonal in the horizontal-vertical basis the corresponding direction of propagation is now given by

$$X^\mu = \frac{1}{m}\eta^{\mu\nu} P_\nu\,. \tag{9.33}$$

We now compare the velocity of the particle with momentum P moving in spatial direction $\dot{\vec\rho}$ with the velocity of light which is the boundary velocity of observers measured by an observer. Recall from equations (8.15) and (8.19) that the velocity of a general particle moving in spatial direction $\dot{\vec\rho} = \dot{\vec\rho}^\alpha e_\alpha$ measured by an observer on a worldline γ is given by

$$v^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = \frac{\delta_{\alpha\beta}\dot\rho^\alpha\dot\rho^\alpha}{(\dot\rho^0)^2} = -\frac{g^F_{(\gamma,\dot\gamma)}(\dot{\vec\rho},\dot{\vec\rho})}{g^F_{(\gamma,\dot\gamma)}(\dot\rho,\dot\gamma)^2}\,, \tag{9.34}$$

and that the speed of light measured by the observer is

$$c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = \frac{2}{r} + \frac{2}{r}\frac{1}{L(\gamma,\dot\gamma)}\sum_{k=3}^{\infty}\frac{(\dot\rho^0)^{-k}}{k!}\bar\partial_{c_1}...\bar\partial_{c_k}L(\gamma,\dot\gamma)\dot{\vec\rho}^{\,c_1}...\dot{\vec\rho}^{\,c_k}\,. \tag{9.35}$$

We will calculate their difference to first order in h on the Finsler spacetime in consideration (9.28) and comment on the relation of the result to modified dispersion relation in the literature. On the Finsler spacetime we are studying the infinite sum terminates at $k = 4$

$$c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = \frac{2}{r}\big(1+2(\dot\rho^0)^{-3}[2\eta(\dot{\vec\rho},\dot\gamma)\eta(\dot{\vec\rho},\dot{\vec\rho})+h(\dot{\vec\rho},\dot\gamma)\eta(\dot{\vec\rho},\dot{\vec\rho})+\eta(\dot{\vec\rho},\dot\gamma)h(\dot{\vec\rho},\dot{\vec\rho})]+(\dot\rho^0)^{-4}L(\gamma,\dot{\vec\rho})\big)\,. \tag{9.36}$$

To proceed we have to find $\dot\rho^0$ from the condition $L(\rho,\dot\rho) = 0$; since we are interested in the boundary velocity of observers this reduces to study $\eta(\dot\rho,\dot\rho) = 0$. In an observer's frame we have

$$(\dot\rho^0)^2\eta(\dot\gamma,\dot\gamma) + 2\dot\rho^0\eta(\dot\gamma,\dot{\vec\rho}) + \eta(\dot{\vec\rho},\dot{\vec\rho}) = 0\,. \tag{9.37}$$

In order to find a solution for $\dot\rho^0$, and to further analyse the expression for the speed of light, we expand the normalization of the observers worldline $L(\gamma,\dot\gamma) = 1$ and the orthogonality of $\dot\gamma$ and $\dot{\vec\rho}$ at the observers position $g^L_{(\gamma,\dot\gamma)}(\dot\gamma,\dot{\vec\rho}) = 0$ to first order in h:

$$\begin{aligned} & L(\gamma,\dot\gamma) = 1 \\ \Leftrightarrow\quad & \eta(\dot\gamma,\dot\gamma) = \frac{1}{2}\big(-h(\dot\gamma,\dot\gamma) - \sqrt{4+h(\dot\gamma,\dot\gamma)^2}\,\big) = -1 - \frac{1}{2}h(\dot\gamma,\dot\gamma) + \mathcal{O}(h^2) \end{aligned} \tag{9.38}$$

$$\begin{aligned} & g^L_{(\gamma,\dot\gamma)}(\dot\gamma,\dot{\vec\rho}) = 0 \\ \Leftrightarrow\quad & \eta(\dot\gamma,\dot{\vec\rho}) = -\frac{\eta(\dot\gamma,\dot\gamma)}{2\eta(\dot\gamma,\dot\gamma)+h(\dot\gamma,\dot\gamma)}h(\dot\gamma,\dot{\vec\rho}) = -\frac{1}{2}h(\dot\gamma,\dot{\vec\rho}) + \mathcal{O}(h^2)\,. \end{aligned} \tag{9.39}$$

In equation 9.38 we choose the negative root since $\dot\gamma$ is supposed to be timelike, hence it has to be timelike with respect to η. Using these relations in the equation for the speed of light, equation (9.35) yields

$$c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = \frac{1}{2}\Big(1 + (\dot\rho^0)^{-4}\big[\eta(\dot{\vec\rho},\dot{\vec\rho})^2 + \eta(\dot{\vec\rho},\dot{\vec\rho})h(\dot{\vec\rho},\dot{\vec\rho})\big]\Big) + \mathcal{O}(h^2)\,. \tag{9.40}$$

The solutions for $\dot\rho^0$ up to first order in h is obtained from equation (9.37)

$$\dot\rho^0 = \sqrt{\eta(\dot{\vec\rho},\dot{\vec\rho})}\left(\pm 1 - \frac{h(\dot{\vec\rho},\dot\gamma)}{2\sqrt{\eta(\dot{\vec\rho},\dot{\vec\rho})}} \mp \frac{1}{4}h(\dot\gamma,\dot\gamma)\right) + \mathcal{O}(h^2)\,. \tag{9.41}$$

Considering light propagating into the future, hence having a positive $\dot\rho^0$ component, the expression for the speed of light finally becomes

$$\begin{aligned} c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) &= 1 + \frac{1}{2}\frac{1}{\eta(\dot{\vec\rho},\dot{\vec\rho})}\Big(h(\dot{\vec\rho},\dot{\vec\rho}) + \eta(\dot{\vec\rho},\dot{\vec\rho})h(\dot\gamma,\dot\gamma) + 2\sqrt{\eta(\dot{\vec\rho},\dot{\vec\rho})}\,h(\dot\gamma,\dot{\vec\rho})\Big) + \mathcal{O}(h^2) \\ &= 1 + \frac{1}{2}\frac{1}{\eta(\dot{\vec\rho},\dot{\vec\rho})}h(\dot\rho,\dot\rho) + \mathcal{O}(h^2) = 1 - \frac{1}{2}\frac{1}{\eta(\dot{\vec\rho},\dot{\vec\rho})}|h(\dot\rho,\dot\rho)| + \mathcal{O}(h^2)\,, \end{aligned} \tag{9.42}$$

where we used the expansion $\dot\rho = \dot\rho^0\dot\gamma + \dot{\vec\rho}$ of the null vector to derive the second equality. Observe that $h(\dot\rho,\dot\rho)$ is negative, since we assumed that the null vectors of the metric $\eta+h$ form the outer null-cone and $\dot\rho$ is a null vector on the inner null-cone, the one of η. The argument goes as follows: Assume X is a null-vector of η, then, since the null cone of η is the inner null cone, X is a timelike vector of the metric $\eta+h$. Hence $\eta(X,X)+h(X,X)<0$, but since $\eta(X,X)=0$ this implies $h(X,X)<0$. Moreover it can be seen from equation (9.37) that $\eta(\dot{\vec\rho},\dot{\vec\rho})$ is positive to zeroth order on h since $\eta(\dot\gamma,\dot\gamma)$ is negative and $\eta(\dot{\vec\rho},\dot\gamma)$ does not contribute. Thus the Finslerian correction term to the speed of light is always negative under the assumptions we made during this section.

Employing the dispersion relation (9.32) and the identification with the corresponding tangent directions the velocity of a particle mode with momentum P can be written as

$$v^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = \frac{P^\alpha P^\beta\delta_{\alpha\beta}}{E^2} = 1 - \frac{m^2}{E^2} \tag{9.43}$$

and so the difference to the speed of light, depending on the energy and the mass of the particle mode measured by an observer on a worldline γ, is

$$v^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) - c^2_{(\gamma,\dot\gamma)}(\dot{\vec\rho}) = -\frac{m^2}{E^2} + \frac{1}{2}\frac{|h(\dot\rho,\dot\rho)|}{\eta(\dot{\vec\rho},\dot{\vec\rho})} + \mathcal{O}(h^2)\,, \tag{9.44}$$

We see that particle modes with large mass and low energy will always be slower than the 'slow' speed of light, but we also see that as soon as the mass is sufficiently small and the energy sufficiently high, they can be superluminal. The exact ratio of the particles mass and energy necessary to become superluminal here depends on the spatial motion of the particle $\dot{\vec\rho}$ relative to an observer on a worldline γ and corresponding time direction $\dot\gamma$.

Velocity dispersion relations that deviate from the one on Lorentzian metric spacetime have been studied in physics in the context of quantum spacetimes [62] and in Lorentz symmetry violating extensions of the Standard Model, for example the one by Coleman and Glashow [63]. Here superluminal particle modes occur due to the non-metric geometry of the spacetime background. Thus our analysis of the scalar field on Finsler spacetime provides a geometric origin of modified dispersion relations. This may indicate that the field theories which led to modified dispersion relations in the literature may be more suitable described as field theories living on a Finsler spacetime background instead of as field theories on a Lorentzian metric background.

During the discussion of the scalar field, we did not need to use Lagrange multipliers to constrain the field to have only horizontal components. This feature of our coupling principle will now become visible when we discuss electrodynamics on Finsler spacetimes.

9.5. Finsler Electrodynamics

Our key objective in this section is to proof that the propagation of light on Finsler spacetimes indeed takes place on Finsler null geodesics when we apply our minimal coupling principle to the theory of electrodynamics.

Before we are able to analyse the propagation of light we have to lift the theory and to derive the equation of motion. The difficulty here is to ensure that the one-form potential A and the field strength tensor F are purely horizontal on shell. Having solved this problem we analyse the equations of motion and find that the singularities of the solutions propagate along Finsler null geodesics, a fact which we interpret as propagation of light.

9.5.1. Action and field equations

In the standard formulation of electrodynamics, the action is a functional $\tilde{S}_m[g, A]$ of a one-form potential A, but the classical physical field is $F = dA$, which is not encoded in the action. Equivalently one can use an interacting action of the form $\tilde{S}_m[g, A, F]$ which provides the complete set of Maxwell equations $F = dA$ and $d \star_g F = 0$ by variation. Our minimal coupling principle tells us to introduce Lagrange multiplier to restrict all fields to be purely horizontal; using the first action then only kills the vertical components of the lift of A, but does not guarantee that the lift of F is purely horizontal, the second action ensures horizontal of both fields A and $F = dA$.

Explicitly the action for classical electrodynamics on a Lorentzian metric spacetime $(M, \tilde{g})$ involving a one-form $\tilde{A}$ and a field strength two-form $\tilde{F}$ separately is

$$\tilde{S}_m[g, A, F] = -\frac{1}{2}\int_M \mathrm{d}^4x\sqrt{\tilde{g}}\; \tilde{g}^{ab}\tilde{g}^{cd}\tilde{F}_{ac}\Big((d_M\tilde{A})_{bd} - \frac{1}{2}\tilde{F}_{bd}\Big)\,. \tag{9.45}$$

The equations of motion are obtained from this action by variation with respect to $\tilde{A}$ and $\tilde{F}$:

$$\nabla^{\tilde{g}}_b\tilde{F}^{ba} = 0\,,\quad \tilde{F}_{ab} = \partial_a\tilde{A}_b - \partial_b\tilde{A}_a\,, \tag{9.46}$$

where $\nabla^{\tilde{g}}$ denotes the Levi-Civita connection of the spacetime metric. The advantage of this formulation over the standard action $-\frac{1}{4}\int \mathrm{d}^4x\sqrt{\tilde{g}}\;\tilde{F}^{ab}\tilde{F}_{ab}$ lies in the fact that the relation $\tilde{F} = d\tilde{A}$, that $\tilde{A}$ is a gauge potential, does not need to be imposed by hand.

We now apply the minimal coupling principle to the action (9.45). Using the horizontal/vertical basis, the fields $\tilde{A}$ and $\tilde{F}$ are lifted to

$$A = A_a(x,y)\mathrm{d}x^a + A_{\bar{a}}(x,y)\delta y^a\,,\quad F = \frac{1}{2}F_{ab}(x,y)\mathrm{d}x^a\wedge\mathrm{d}x^b + F_{\bar{a}b}\delta y^a\wedge\mathrm{d}x^b + \frac{1}{2}F_{\bar{a}\bar{b}}(x,y)\delta y^a\wedge\delta y^b\,. \tag{9.47}$$

The forms A and F are required to be zero-homogeneous in the fibre coordinates; this implies the homogeneities zero for A_a, F_{ab}, minus one for $A_{\bar{a}}, F_{\bar{a}b}$ and minus two for $F_{\bar{a}\bar{b}}$ with respect to the y coordinates. The corresponding action on Finsler spacetimes now becomes

$$\begin{aligned} S[L,A,F] \;=\; \int \mathrm{d}^4x\mathrm{d}^3u\sqrt{G^{F*}}\Big[&-\frac{1}{2}G^{F\;AB}G^{F\;CD}F_{AC}\Big((d_{TM}A)_{BD} - \frac{1}{2}F_{BD}\Big) \\ &+\lambda^{\bar{a}}A_{\bar{a}} + \lambda^{\bar{a}b}F_{\bar{a}b} + \lambda^{\bar{a}\bar{b}}F_{\bar{a}\bar{b}}\Big]_{|\Sigma}\,, \end{aligned} \tag{9.48}$$

where the induced coordinates $(Z^A) = (x^a, y^a)$ and the corresponding partial derivatives are used before restricting to Σ. One may not be confused by the multiple meanings of F here. It should be clear from the context where F labels the field strength tensor, the Finsler function or is used as label for the objects derived from the Finsler function. Observe the appearance of the Lagrange multipliers $\lambda^{\bar{a}}$, $\lambda^{\bar{a}b}$ and $\lambda^{\bar{a}\bar{b}}$ that kill the non-horizontal parts of A and F on-shell.

The variation of the generalised action with respect to A, F and the Lagrange multipliers is technically straightforward; the calculation uses the Berwald bases and requires the integration by parts identities (5.86). Using the immediate constraints

$$A_{\bar{a}} = 0\,, \quad F_{\bar{a}b} = 0\,, \quad F_{\bar{a}\bar{b}} = 0\,, \tag{9.49}$$

we thus find the field equations, expressed in terms of the horizontal Cartan linear covariant derivative,

$$\begin{aligned} F_{ab} &= \delta_a A_b - \delta_b A_a = \nabla^{CL}_a A_b - \nabla^{CL}_b A_a\,, && (9.50)\\ 0 &= g^{F\,ab} g^{F\,cd} \left(\nabla^{CL}_a F_{bd} - S^p{}_{pa} F_{bd}\right), && (9.51)\\ \lambda^{\bar{a}b} &= -g^{F ap} g^{F bq} \bar{\partial}_p A_q\,, \quad \lambda^{\bar{a}} = \frac{1}{2} F^{pq} R^a{}_{pq}\,, \quad \lambda^{\bar{a}\bar{b}} = 0\,. && (9.52) \end{aligned}$$

The field strength F whose components are interpreted as electric and magnetic fields is gauge-invariant under the transformations

$$A_a \mapsto A_a + B_a\,, \quad \delta_{[a} B_{b]} = 0\,. \tag{9.53}$$

In general these may change the solution for the Lagrange multiplier $\lambda^{\bar{a}b}$, but this has no physical relevance. Observe that the transformation $B_b = \delta_b \phi(x) = \partial_b \phi(x)$, which is the gauge freedom of the original theory on metric spacetimes, still is a gauge transformation of the lifted theory and also leaves the Lagrange multiplier $\lambda^{\bar{a}b}$ invariant.

As it should be from our general considerations about the minimal coupling principle the field equations reduce to the standard Maxwell equations (9.46) in case the Finsler spacetime is induced by a Lorentzian metric and the fields only depend on the coordinates of the manifold M but not on the fibre coordinates of TM.

Having the equations of motion of electrodynamics on Finsler spacetimes at hand we are in the position to study the propagation of light.

9.5.2. Propagation of light

Any theory of electrodynamics determines the motion of light through the corresponding system of partial differential equations. Light trajectories are obtained in the geometric optical limit by studying the propagation of singularities of the electromagnetic fields. Since our field equations on Finsler spacetime are formulated over the tangent bundle, also the resulting singularity propagation will follow curves $\tau \mapsto (x(\tau), y(\tau))$ on the tangent bundle TM. Of these only the natural lifts $\tau \mapsto (x(\tau), \dot{x}(\tau))$ that arise from curves $\tau \mapsto x(\tau)$ on the manifold M have an immediate interpretation as light trajectories. We will demonstrate the strong result that in the proposed extended electrodynamics (9.48) all light trajectories are Finsler null geodesics.

In the following analysis we regard the components of the one-form A as the fundamental variables. We insert (9.50) into (9.51) to obtain the following system of linear second order partial differential equations

$$0 = g^{F\,a[b} g^{F\,d]c} \left(\nabla^{CL}_a \nabla^{CL}_b A_d - S^p{}_{pa} \nabla^{CL}_b A_d\right). \tag{9.54}$$

A solution of this system for A_a determines solutions for F_{ab} and $\lambda^{\bar{a}b}$ according to our field equations (9.50)–(9.52). Following standard methods for partial differential equations we now extract the principal symbol from the equations above. For this purpose we use the gauge condition $g^{F\,ab}\nabla^{CL}_a A_b = 0$ which generalises the usual Lorentz gauge. Then the terms of highest derivative order can be written in the form

$$\begin{aligned} 0 &= g^{F\,ab}\left(\partial_a\partial_b - 2N^p{}_a\partial_b\bar{\partial}_p + N^p{}_a N^q{}_b\bar{\partial}_p\bar{\partial}_q\right)A_c + \dots \\ &= \delta^p_c P^{AB}\partial_A\partial_B A_p + \dots \end{aligned} \tag{9.55}$$

where the dots represent terms with less than two derivatives acting on the A_a. Following Dencker [64] we need to check if the principal symbol

$$P^p_c(x,y,k,\bar{k}) = \delta^p_c\frac{1}{2}P^{AB}k_Ak_B = \delta^p_c\frac{1}{2}g^{F\,ab}k^H_a k^H_b\,, \quad k^H_a = k_a - N^p{}_a\bar{k}_p\,, \tag{9.56}$$

is of real principal type. This is indeed the case since there exists $\tilde{P}^c_q = \delta^c_q$ such that $\tilde{P}^c_q P^p_c = Q\delta^p_q$ with $Q = \frac{1}{2}g^{F\,ab}k^H_a k^H_b$. Moreover Q is real and has a Hamiltonian vector field X_Q which is non-vanishing and not radial in case $Q = 0$, as can be seen form the explicit form X_Q below. Thus the singularities of the field A propagate along the projection to TM of the integral curves of the Hamiltonian vector field X_Q that lie in the surface $Q = 0$

$$X_Q = \partial_{k_a}Q\partial_a + \partial_{\bar{k}_a}Q\bar{\partial}_a - \partial_a Q\partial_{k_a} - \bar{\partial}_a Q\partial_{\bar{k}_a}\,. \tag{9.57}$$

The integral curves $\gamma : \tau \mapsto (x(\tau), y(\tau), k(\tau), \bar{k}(\tau))$ in T^*TM of the Hamiltonian vector field X_Q are determined by the corresponding Hamiltonian equations

$$\dot{x}^a = g^{Fab}k^H_b\,, \tag{9.58}$$

$$\dot{y}^a = -g^{Fpb}k^H_b N^a{}_p\,, \tag{9.59}$$

$$\dot{k}_a = -\frac{1}{2}\partial_a g^{Fpq}k^H_p k^H_q + g^{Fpq}k^H_q\partial_a N^b{}_p\bar{k}_b\,, \tag{9.60}$$

$$\dot{\bar{k}}_a = -\frac{1}{2}\bar{\partial}_a g^{Fpq}k^H_p k^H_q + g^{Fpq}k^H_q\bar{\partial}_a N^b{}_p\bar{k}_b\,. \tag{9.61}$$

and satisfy the constraint

$$Q = \frac{1}{2}g^{Fab}(x,y)k^H_a k^H_b = 0\,. \tag{9.62}$$

The constraint together with equation (9.58) immediately yield that $\dot{x}$ is null with respect to the Finsler metric $g^F(x,y)$. This means for solution curves which are natural lifts of curves from the manifold, i.e. $y(\tau) = \dot{x}(\tau)$, that they are Finsler null curves $g^F_{ab}(x,\dot{x})\dot{x}^a\dot{x}^b = F^2(x,\dot{x}) = 0$, or equivalently $L(x,\dot{x}) = 0$. These are the curves we identify as light trajectories. There may be more general solutions for the above equations for more general curves on the tangent bundle; those have not the direct interpretation as light propagating along the manifold.

Combining (9.58) and (9.59) yields the fact that the projected curve $c(\tau) = (x(\tau), y(\tau)) \in TM$ is horizontal

$$\dot{y}^a + N^a{}_p(x,y)\dot{x}^p = 0\,. \tag{9.63}$$

We proved in theorem 5.4 the non-linear connection coefficients smoothly extend to the null structure of TM and so the above equations holds for our light trajectories and yields

$$\ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b = 0\,. \tag{9.64}$$

This now proves that light trajectories are Finsler geodesics, compare with equation (5.40). Thus the null structure of Finsler spacetimes is indeed related to the propagation of light induced by the theory of electrodynamics we constructed here.

This analysis goes through easily on all Finsler spacetimes on which the L metric does not degenerate along the null structure. In the definition 5.1, we in general allowed for a lower dimensional subset along the null structure on which the L metric may degenerate to include for example bimetric geometries with intersecting cones. On such Finsler spacetimes the propagation analysis has to be performed more carefully on the set where the L metric degenerates. It may turn out that there exist no propagation of light along these directions.

Conclusion

With the development of our Finsler spacetime framework throughout this thesis we clearly demonstrated that a spacetime geometry based on a smooth homogeneous function on the tangent bundle provides a consistent simultaneous geometrisation of causality, observers and their measurements and gravity. Moreover our framework yields consistent geometric backgrounds for physical field theories. It contains and extends the known framework from general relativity, where the geometry of spacetime is derived from a Lorentzian metric, most importantly without changing the role of spacetime in physics or its dimension. Furthermore our Finsler spacetimes overcome the mathematically issue of a not defined geometry along the null structure of spacetime in earlier applications of Finsler geometry in physics, especially when it was used as the background geometry of spacetime.

We derived explicit physical consequences from our Finsler spacetime framework such as: a possible explanation of the fly-by anomaly in the solar system, which is unexplained on basis of general relativity; a geometric origin of modified dispersion relations, as they appear from quantum gravity phenomenology or non Lorentz invariant field theories; the propagation of momentum modes faster than the slow speed of light, as it can be found in dielectric media and the propagation of light along Finsler geodesics in general.

In the future we hope we will be able to solve the dynamical equation which determines the geometry of Finsler spacetimes in symmetric situations without using perturbation theory. Especially in cosmology and spherical symmetry as solution of the dynamics of the non-metric geometry is highly interesting to answer the question if non-metric spacetime geometry can shed light on the dark (matter and energy) part of the universe. A possible further application of the Finsler spacetime dynamics is the unification of the dynamics of several fields a la Kaluza and Klein.

Summary and discussion

The first part of this thesis contained our review of the mathematical formulation of Finsler geometry as it can be found in the literature and the applications of Finsler geometry in physics. Our newly developed Finsler spacetime framework was presented in full detail in the second part. We especially emphasized during our presentation how our Finsler spacetime geometry provides causality, observers and their measurements and a description of gravity.

In chapter 5 we began with the precise definition of Finsler spacetimes. Already from the definition it became clear how causality is encoded into the non-metric geometry. Moreover we presented in detail how we could circumvent the problems with the non-differentiability of the Finsler function along its null structure and the resulting non-existence of the tensors describing the geometry of the manifold. Our solution was the introduction of a smooth homogeneous fundamental geometry function on the tangent bundle of the spacetime manifold, from which the Finsler function, and all geometric objects on Finsler spacetimes are derived. One most important feature of our construction is that the connection and the curvature of the Finsler spacetime derived from our smooth fundamental geometry function is identical to the connection and the curvature derived from its associated Finsler function, wherever the latter is differentiable. With this fact we demonstrated that our framework is an extension of the standard Finsler geometry framework similar as semi-Riemannian geometry extends Riemannian geometry. This construction of Finsler spacetimes is one of the central results of this thesis. That we overcame the mathematical problems which appear in indefinite Finsler geometry is fundamental for its application as generalisation of Lorentzian metric geometry from the mathematical point of view and especially for its application as the fundamental geometry of spacetime in physics.

To encode gravity into the geometry of spacetime, dynamics for the geometry are required which are sourced by the matter field content on spacetime. In chapter 6 we further developed our preliminary work on the understanding of the Einstein–Hilbert action from a Finsler geometric point of view which was reviewed in section 4.1. We obtained an extended Finsler spacetime version of the Einstein–Hilbert action by realizing that the original Einstein–Hilbert action can be understood as integral over the unit tangent bundle and that the curvature tensor appearing in this rewriting is nothing but the metric spacetime version of the canonical Finsler non-linear curvature scalar. From this action we derived the dynamical equation which determine the geometry of Finsler spacetimes. They were interpreted as gravitational vacuum dynamics, which are equivalent to the Einstein vacuum equations in the metric limit.

In general the dynamics for Finsler spacetime are hard to solve. To get a feeling what features one may expect from non-metric solution of the Finsler spacetime dynamics we solved the field equation for a non-metric perturbation of Minkowski spacetime in chapter 7. We derived the linearised Finsler spacetime dynamics for general non-metric Finsler spacetime perturbations around metric geometry, before we specialised to a perturbation of Minkowski spacetime which can be seen as a bimetric Finsler spacetime. We then derived this bimetric Finsler spacetime in the case of spherical symmetry from the dynamics. We could interpret our result as non-metric refinement of the linearised Schwarzschild solution of general relativity which is able to address the fly-by anomaly in the solar system. By the correct choice of the additional free parameters compared to the linearised Schwarzschild solution, the geodesics of the non-metric spacetime are closer to the central mass than in the metric solution, which exactly meets the observations that spacecrafts gained more velocity than expected during swing by manoeuvres around planets in the solar system. With this example we demonstrated the capability of Finsler spacetimes to address unexplained astronomical observations.

In chapter 8 we introduced how observers and their measurements on Finsler spacetimes are described. We gave an interpretation of the Finsler length measure as the action for point particles and as the geometric clock of an observer. Taking this as starting point we argued that an observer's time direction is given by the tangent to its worldline. We identified the three spatial directions of the observer as the directions conormal to the differential of our fundamental geometry function. To determine the observers unit time and unit space directions we normalised the frame with respect to the metric induced by the fundamental geometry function. We obtained a geometric description of an observer by an orthonormal frame such that the measurements an observer performs are the components of physical fields in the observers frame at the observers tangent bundle position. Transformations between different observers constructed according to our model are not longer Lorentz transformations, but a combination between a parallel transport between the tangent bundle positions of the observers and Lorentz transformations. As immediate consequence from our observer model we calculated the velocity of a particle an observer measures in general and applied this formula to light trajectories,i.e., null curves. We found that in general the velocity of light depends on the relative orientation between the observer worldline tangent and the tangent vector of the light trajectory. Such phenomenon is known from dielectric media, the bounds on such effects in vacuum spacetime put bounds on the parameters of concrete Finsler spacetime models. For specific non-metric Finsler spacetime geometries this dependence of the speed of light on its motion relative to the observer who measures it may even disappear due to cancellations in the sum of derivatives of the fundamental geometry function. Thus the possible observer dependence of the speed of light is not a problem in Finsler spacetimes, it is a possibility the framework offers.

The final step to complete the framework was to equip Finsler spacetimes with matter fields which determine the geometry in chapter 9. We discussed that physical fields coupling to Finsler spacetime geometry have to be fields on the tangent bundle with the same number of components as the corresponding field in metric spacetime geometry and tensorial transformation behaviour under manifold induced coordinate transformations. On the basis of these requirements on the fields we constructed a minimal coupling principle. It is a recipe how to obtain the field theory action on Finsler spacetimes from a given field theory action on Lorentzian metric spacetimes. Combining the matter field action with the Finsler spacetime extension of the Einstein–Hilbert action we obtained the dynamics of the geometry determined by the matter content of spacetime. With this coupling principle we completed our description of gravity by the geometry of Finsler spacetimes. It is such that in the metric limit the dynamics of the geometry become equivalent to the Einstein equations. Despite the effects of the matter fields on the geometry we also discussed effects from the non-metric geometry on the matter fields. On the example of the scalar field we demonstrated that on bimetric spacetimes there may exist field modes which propagate faster then the speed of light. To discuss this effect we derived the velocity dispersion relation which is influenced by the non-metric geometry. It is similar to dispersion relations discussed in context with Lorentz invariance violating field theories; in our spacetime framework the modification has its origin in the non-metric geometry of spacetime. From the theory of electrodynamics on Finsler spacetimes obtained from our coupling principle we derived that light propagates along Finsler null geodesics by analysing the propagation of singularities of the Finslerian version of the Maxwell equations.

Thus we demonstrated throughout this thesis that our Finsler spacetime framework provides a consistent non-metric Finslerian geometry which consistently extends Lorentzian metric geometry and has all features required from a spacetime framework by physics. Especially it provides a causal structure, observers and their measurements and a description of gravity.

Outlook

Our new Finsler spacetime framework which extends the framework of metric spacetime geometry answered nicely and elegant the questions discussed in section 4.2, which were open from previous work. But as usual a new framework does not only answer questions it always raises new ones. Here we discuss possible further applications important and open questions of our framework, hopefully to be answered and studied in the future.

One of the most important tasks is to compare our new framework with further observations to survey its viability as the fundamental geometry of spacetime. It is based on the fundamental geometry function which can be any kind of smooth homogeneous function on the tangent bundle; further restrictions on this function from observations and physical arguments are desirable. Therefore one future task is to develop a machinery which parametrises the fundamental geometry function directly such that the parameters a related to experimental observations. For metric geometry there exists the PPN formalism which gives such a parametrisation [9]. The extension of this formalism to general Finsler spacetimes and their dynamics is an important project for a deeper understanding of our framework.

Concerning the dynamics of Finsler spacetimes it is not clarified yet if there exist an initial value formulation and if it is well posed. A three plus one split formalism has to be developed, the propagating degrees of freedom in terms of the fundamental geometry function have to be identified and their initial value problem has to be studied. A first step in this direction is to reformulate the initial value problem of general relativity in the Finsler spacetime framework language and to extend such a reformulation to the general non-metric case.

The full potential of the Finsler spacetime framework will only be revealed if we achieve to obtain non perturbative solutions of the dynamical equations, at least for spherical symmetry or homogeneous and isotropic symmetry. From these solutions it is then possible to study if an extended Finslerian geometric understanding of the geometry of spacetime, i.e., a modification of the left hand side of the Einstein equations, is capable to address the dark matter and dark energy paradigms. We have mentioned several times that the phenomenology of field theories on Finsler spacetimes is consistent with observations of electrodynamics in media. Especially in the very early universe it is believed that spacetime was filled with a dense hot plasma. It may very well be that a Finsler geometric description of spacetime for that epoch is, at least effectively, much more adequate than the description through metric cosmology.

By now it has bee realized that the Finsler spacetime framework is closely related to the formulation of gravitational dynamics on observer space [65]. The latter is motivated from Loop quantum gravity but only works for the vacuum dynamics so far. The matter field theories obtained by our coupling principle can be translated into the observer space language and fill the gap there to a complete reformulation of the gravitational dynamics.

An open mathematical question is the understanding of the integrals over the unit tangent bundle on Finsler spacetimes. Since the unit tangent bundle here is not compact a renormalisation has to be applied to evaluate these integrals. The answer to the question if there is a unique way to evaluate these integrals such that we get an integral representation of a Lorentzian metric on Lorentzian metric spacetimes, similar as there is for Riemannian metrics, would further improve the mathematical formulation of our framework.

Another interpretation of our Finsler spacetime dynamics, apart from being the dynamics determining the fundamental geometry of spacetime, is that they describe the behaviour of several fields on the manifold. The idea follows the spirit of Kaluza and Klein [2, 3] but without the need of higher dimensional spacetimes. It is possible to consider fundamental geometry function built from several fields on spacetime, as for example our anisotropic Finsler spacetimes are built from a Lorentzian metric and a vector field. Our extended Einstein–Hilbert action for such a fundamental geometry function can be see as the action which determines the dynamics of all of the building block tensor fields on the manifold. Their dynamics can be obtained in two ways: either by variation of the action with respect to the different building block fields yields their equations of motion or variation with respect to the fundamental geometry function yields its field equation which has to be decomposed into equations for the several building block fields via derivatives with respect to the coordinates of the tangent spaces. The advantage of these approaches to a unified geometric picture of the dynamics of several physical fields is that no higher dimensional spacetime geometry has to be employed, but only non-metric Finsler spacetime geometry. As a first step into this direction we already found that for a fundamental geometry function built from a metric and a one-form, the non-linear curvature scalar contains the sum of the Ricci scalar of the metric, the canonical quadratic scalar built from the field strength tensor of the one-form and further terms which have to be investigated in detail. The ongoing study of this approach will reveal how the Finsler spacetime Einstein–Hilbert action obtained in this case is related to the standard Einstein–Hilbert action and the standard action for Maxwell electrodynamics, as well as if it is possible to obtain the Einstein-Maxwell equations from the field equation of the fundamental geometry function.

Despite the application of our Finsler spacetime framework in the description of gravity or the geometric unification of field theories on spacetime it opens the door to study field theories where the components of the fields depend on the directions of spacetime, in a stronger way than the usual tensorial fields. This possibility leads for example new class of theories of electrodynamics where the components of the field strength tensor still only depend on the coordinates of the spacetime manifold but the components of the induction tensor are related to the ones of the field strength tensor via objects that depend also on the directions of spacetime. An application of such electrodynamics may be dispersive or inhomogeneous media where for example the velocity of electromagnetic waves depend on their direction of propagation.

Studying field theories on Finsler spacetimes raises the question about their quantisation. The quantisation procedure of field theories on metric manifolds is closely connected to the hyperbolic partial differential equations which govern the behaviour of the fields. On Finsler spacetimes one has to study the field equations on the tangent bundle carefully to develop a compatible quantisation procedure. If one develops such a procedure, maybe one can also apply it to the fundamental geometry function itself and in this way quantise gravity described by a scalar on the tangent bundle instead of a metric on the spacetime manifold.

With the development of our Finsler spacetime framework throughout this thesis we provide an extension of the framework of general relativity including the description of causality, observers and their measurements and gravity with present and future applications in physics.

Appendices

A. Technical Proofs

Throughout this thesis we left out some details in the proofs of several theorems. Due to their technicality they would have decreased the readability of the chapters, what is why they were postponed into this appendix. Here we wish to add the missing parts of the proofs.

A.1. Relating the signatures of the L and F metric

In this section we work out the details for the proof of theorem 5.2 in section 5.1.3 which relates the signature of the F metric to the signature of the L metric. Recall the theorem

Theorem ***Signature of the metrics***
On the set $TM\setminus(A\cup\{L=0\})$ the metric g^L is nondegenerate of signature $(-1_m, 1_p)$ for natural numbers m, p with $m+p=4$. Then the Finsler metric g^F has the same signature $(-1_m, 1_p)$ where $L(x,y)>0$, and reversed signature $(-1_p, 1_m)$ where $L(x,y)<0$.

Proof
By the definition of Finsler spacetimes the metric g^L is non-degenerate on $TM\setminus A$, hence also on the smaller set excluding the null structure on which g^F is defined. Now observe from equation (5.15) g^F can be written as a matrix C_{ab}

$$C_{ab} = A_{ab} - B_a B_b\,,\ \text{with } A_{ab} = \frac{2F^2}{rL} g^L_{ab},\ B_a = \sqrt{\frac{(r-2)F^2}{r^2L^2}}\bar{\partial}_a L,\ A^{ab}B_aB_b = \frac{r-2}{r-1}\,. \tag{A.1}$$

Hence B is always spacelike or null with respect to A depending on the value of r since from the definition of Finsler spacetimes we know $r\geq 2$. For $r=2$ we have that $B=0$ and already from the relation between g^L and g^F it is clear that they have the same signature up to the sign of L. To investigate the signature of g^F for $r>2$ we change to an orthonormal basis $\Psi_A=\{E_M, F_{\bar{N}}\}; M=1,\dots,m; N=1,\dots,p; A=1,\dots,m+p$ for A_{ab} and the corresponding dual basis $\Psi^A=\{E^M, F^{\bar{N}}\}$

$$A_{AB} = A_{ab}\Psi^a_A\Psi^b_B = diag(\underbrace{-1,\dots,-1}_{m},\underbrace{1,\dots,1}_{p}) = \tilde{\eta}_{AB} \tag{A.2}$$

and hence

$$C_{AB} = \tilde{\eta}_{AB} - B_A B_B \tag{A.3}$$

where B_A are the components of B in the orthonormal basis of A_{ab}

$$B = B_A\Psi^A = B_M E^M + B_{\bar{N}}F^{\bar{N}}\,. \tag{A.4}$$

Since B is spacelike with respect to A there exists a transformation $\Lambda_{\bar{N}_0}\in SO(m,p)$ to align B with a fixed spacelike basis element $\hat{F}_{\bar{N}_0} = \Lambda^{\bar{N}}_{\bar{N}_0}F_{\bar{N}}$ of the $\Lambda_{\bar{N}_0}$ transformed basis, i.e. $B=\bar{k}\hat{F}_{\bar{N}}$

for $\bar{k}^2 = \tilde{\eta}^{AB} B_A B_B$. Applying this transformation to C yields

$$C_{AB} = \tilde{\eta}_{AB} - \bar{k}^2 \delta^A_{\bar{N}_0} \delta^B_{\bar{N}_0} \,. \tag{A.5}$$

The only change in the signature can happen for $A = \bar{N}$ and $B = \bar{N}$ and there we have $C_{\bar{N}\bar{N}} = 1 - \bar{k}^2$. Hence a change in the signature appears only for $\bar{k}^2 \geq 1$ or $\tilde{\eta}^{AB} B_A B_B \geq 1$. Going back to the original basis we started in, this condition reads $A^{ab} B_a B_a \geq 1$. But for any finite $r > 2$ we know from equation (A.1) that $1 > A^{ab} B_a B_a > 0$ and so we conclude that C_{ab} has the same signature as A_{ab}. For g^L this means that it has the same signature as g^L up to the sign of L. $\square$

A.2. Sasaki metrics in adapted coordinates

In theorem 5.6 in section 5.3.2 we transformed the Sasaki type metrics G^F and G^L from the manifold induced coordinates (x, y) of TM

$$G^F = -g^F_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b - \frac{1}{F^2} g^F_{ab} \delta y^a \otimes \delta y^b \tag{A.6}$$

$$G^L = g^L_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b + \frac{g^L_{ab}}{F^2} \delta y^a \otimes \delta y^b \,, \tag{A.7}$$

to the coordinates $(\hat{x}, u, R)$ adapted to the unit tangent bundle Σ

$$G^F = -g^F_{ab} \, \mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b - \frac{1}{R^2} h^F_{\alpha\beta} \, \delta u^\alpha \otimes \delta u^\beta - \frac{1}{R^2} \, \mathrm{d}R \otimes \mathrm{d}R \tag{A.8}$$

$$G^L = g^L_{ab} \mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b + \frac{h^L_{\alpha\beta}}{R^2} \delta u^\alpha \otimes \delta u^\beta + \frac{r(r-1)L}{2\,R^4} \mathrm{d}R \otimes \mathrm{d}R \,, \tag{A.9}$$

and postponed the details of the proof to this appendix. We use the transformation relations (5.61) and (5.62) and employ the notations $h^{F\setminus L} = g^{F\setminus L}_{ab} \partial_\alpha y^a \partial_\beta y^b$ and $\delta u^\alpha = du^\alpha + (\bar{\partial}_b u^\alpha N^b{}_a - \partial_a u^\alpha) d\hat{x}^a$. The first step is to expand $\mathrm{d}x^a$ and δy^a into the new coordinates

$$\mathrm{d}x^a = \hat{\partial}_q x^a \mathrm{d}\hat{x}^q + \partial_\alpha x^a \mathrm{d}u^\alpha + \partial_R x^a \mathrm{d}R = \mathrm{d}\hat{x}^a \tag{A.10}$$

$$\delta y^a = \mathrm{d}y^a + N^a{}_b \mathrm{d}x^b = \hat{\partial}_q y^a \mathrm{d}\hat{x}^q + \partial_\alpha y^a \mathrm{d}u^\alpha + \partial_R y^a \mathrm{d}R + N^a{}_b \mathrm{d}\hat{x}^b \,. \tag{A.11}$$

Plugging the $\mathrm{d}x^a$ relation into the expression of the Sasaki type metric in (x, y) coordinates proves the $g^{F\setminus L}_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b = g^{F\setminus L}_{ab} \mathrm{d}\hat{x}^a \otimes \mathrm{d}\hat{x}^b$ part of the formulae. It remains to calculate

$$g^F_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + \frac{y^a}{R} \mathrm{d}R + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + \frac{y^b}{R} \mathrm{d}R + (\hat{\partial}_p y^b + N^b{}_p) \mathrm{d}\hat{x}^p) \tag{A.12}$$

$$g^L_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + \frac{y^a}{R} \mathrm{d}R + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + \frac{y^b}{R} \mathrm{d}R + (\hat{\partial}_p y^b + N^b{}_p) \mathrm{d}\hat{x}^p) \,. \tag{A.13}$$

Both calculations work analogously. The first will be displayed now

$$\begin{aligned}
& g^F_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + \frac{y^a}{R} \mathrm{d}R + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + \frac{y^b}{R} \mathrm{d}R + (\hat{\partial}_p y^b + N^b{}_p) \mathrm{d}\hat{x}^p) \\
= \; & g^F_{ab} \frac{y^a y^b}{R^2} \mathrm{d}R \otimes \mathrm{d}R + 2 g^F_{ab} \frac{y^b}{R} (\partial_\alpha y^a \mathrm{d}u^\alpha + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes \mathrm{d}R \\
& + g^F_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + (\hat{\partial}_p y^b + N^b{}_p) \mathrm{d}\hat{x}^p) \\
= \; & \mathrm{d}R \otimes \mathrm{d}R + g^F_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + (\hat{\partial}_q y^a + N^a{}_q) \mathrm{d}\hat{x}^q) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + (\hat{\partial}_p y^b + N^b{}_p) \mathrm{d}\hat{x}^p) \,.
\end{aligned} \tag{A.14}$$

The last equality is due to the fact that $\bar{\partial}_c|L|^{1/r}\partial_{u^\beta}y^c = 0$ and $\hat{\partial}_b y^a + \partial_b u^\gamma \partial_{u^\gamma} y^a + \frac{y^a}{R}\partial_b|L|^{1/r} = 0$ and $\delta_a F = \delta_a|L|^{1/r} = 0$. With this we obtained the desired $\mathrm{d}R \otimes \mathrm{d}R$ part. To prove the appearance of the missing term $h^F_{\alpha\beta}\,\delta u^\alpha \otimes \delta u^\beta$ we expand it into

$$\begin{aligned} & h^F_{\alpha\beta}\,\delta u^\alpha \otimes \delta u^\beta \\ = \;& g^F_{ab}\partial_\alpha y^a \partial_\beta y^b (du^\alpha + (\bar{\partial}_q u^\alpha N^q{}_p - \partial_p u^\alpha)d\hat{x}^p) \otimes (du^\beta + (\bar{\partial}_c u^\beta N^c{}_d - \partial_d u^\beta)d\hat{x}^d) \\ = \;& g^F_{ab}(\partial_\alpha y^a \mathrm{d}u^\alpha + \partial_\alpha y^a(\bar{\partial}_q u^\alpha N^q{}_p - \partial_p u^\alpha)\mathrm{d}\hat{x}^p) \otimes (\partial_\beta y^b \mathrm{d}u^\beta + \partial_\beta y^b(\bar{\partial}_c u^\beta N^c{}_d - \partial_d u^\beta)\mathrm{d}\hat{x}^d)\,. \end{aligned} \tag{A.15}$$

Comparing this expression with equation (A.14) the last thing we need to check is

$$\partial_\alpha y^a(\bar{\partial}_q u^\alpha N^q{}_p - \partial_p u^\alpha)) \stackrel{!}{=} (\hat{\partial}_p y^a + N^a{}_p) \tag{A.16}$$

That this equality is true can be verified using $\partial_\alpha y^a \bar{\partial}_q u^\alpha = \delta^a_q - y^a y_q R^{-2}$ and the formulae used above equation (A.14). This completes the proof involving g^F, the calculation for g^L works completely analogously. The difference lies only in the homogeneity factors one obtains.

A.3. Integration by parts formulae

Here we provide the proof of the integration by parts formulae introduced in equation (5.86). We show the calculations for the formulae with volume form based on $\sqrt{g^L h^L}$; the ones with volume form based on $\sqrt{g^F h^F}$ follow the same line of calculation with a two-homogeneous function F^2 instead of the r-homogeneous function L. Explicitly we go step by step through the integration by parts for the $\bar{\partial}_a$ derivative and argue afterwards how the integration by parts of the δ_a derivative works. During the calculation latin indices run from $0, \ldots, 3$, greek indices run from $1, \ldots, 3$ and $(\hat{x}, u)$ denote the co-ordinates of the unit tangent bundle. Before the proof of the vertical and horizontal integration by parts formulae we display several identities derived from the coordinate transformations (5.61) and (5.62) and the definition of $h^L_{\beta\gamma} = g^L_{mn}\partial_\beta y^m \partial_\gamma y^m$:

I) The $\hat{x}$ derivative of the volume measure

$$\begin{aligned} h^{L\beta\gamma}\hat{\partial}_a h^L_{\beta\gamma} &= h^{L\beta\gamma}\hat{\partial}_a(g^L_{cd})\partial_\beta y^c \partial_\gamma y^d + 2h^{L\beta\gamma} g^L_{cd}(\hat{\partial}_a\partial_\beta y^c)\partial_\gamma y^d \\ &= (\hat{\partial}_a(g^L_{cd})\partial_\beta y^c + 2g^L_{cd}(\hat{\partial}_a\partial_\beta y^c))h^{L\beta\gamma}\partial_\gamma y^d \\ &= (\hat{\partial}_a(g^L_{cd})\partial_\beta y^c + 2g^L_{cd}(\hat{\partial}_a\partial_\beta y^c))g^{Ldq}\bar{\partial}_q u^\beta \\ &= (g^{Lcd} - \frac{2}{r(r-1)L}y^c y^d)\hat{\partial}_a(g^L_{cd}) + 2\bar{\partial}_c u^\beta(\hat{\partial}_a\partial_\beta y^c) \\ \Rightarrow \hat{\partial}_a(\sqrt{g^L h^L}) &= \sqrt{g^L h^L}\Big[g^{Lcd}\hat{\partial}_a g^L_{cd} + \bar{\partial}_c u^\beta(\hat{\partial}_a\partial_\beta y^c) - \frac{2y^c y^d}{r(r-1)L}\hat{\partial}_a g^L_{cd}\Big]\,. \end{aligned} \tag{A.17}$$

II) The u derivative of the volume measure

$$\begin{aligned} h^{L\beta\gamma}\partial_\alpha h^L_{\beta\gamma} &= h^{L\beta\gamma}\partial_\alpha(g^L_{cd})\partial_\beta y^c \partial_\gamma y^d + 2h^{L\beta\gamma} g^L_{cd}(\partial_\alpha\partial_\beta y^c)\partial_\gamma y^d \\ &= (\partial_\alpha(g^L_{cd})\partial_\beta y^c + 2g^L_{cd}(\partial_\alpha\partial_\beta y^c))h^{L\beta\gamma}\partial_\gamma y^d \\ &= (\partial_\alpha(g^L_{cd})\partial_\beta y^c + 2g^L_{cd}(\partial_\alpha\partial_\beta y^c))g^{Ldq}\bar{\partial}_q u^\beta \\ &= (g^{Lcd} - \frac{2}{r(r-1)L}y^c y^d)\partial_\alpha(g^L_{cd}) + 2\bar{\partial}_c u^\beta(\partial_\alpha\partial_\beta y^c) \\ &= g^{Lcd}\partial_\alpha(g^L_{cd}) + 2\bar{\partial}_c u^\beta(\partial_\alpha\partial_\beta y^c) \end{aligned} \tag{A.18}$$

$$\Rightarrow \partial_\alpha(\sqrt{g^L h^L}) = \sqrt{g^L h^L}\Big[g^{Lab}\partial_\alpha g^L_{ab} + \bar{\partial}_c u^\beta(\partial_\alpha\partial_\beta y^c)\Big]\,. \tag{A.19}$$

III) Rewriting terms to standard induced coordinates (x, y) of TM:

$$\bar{\partial}_a u^\alpha g^{Lcd}\partial_\alpha g^L_{cd} = g^{Lcd}\bar{\partial}_a g^L_{cd} - \frac{8(r-2)}{r(r-1)L} y^{Lq} g^L_{aq}\,, \tag{A.20}$$

$$\bar{\partial}_c u^\beta(\partial_\alpha\partial_\beta y^c)\bar{\partial}_a u^\alpha + \partial_\alpha\bar{\partial}_a u^\alpha = -\frac{6}{r(r-1)L} g^L_{ab} y^b\,, \tag{A.21}$$

$$y^c y^d \hat{\partial}_a g^L_{cd} = y^a y^d(\partial_a + \hat{\partial}_a y^m \bar{\partial}_m) g^L{}_{cd} = \partial_a L(r-1)\,. \tag{A.22}$$

The important ingredient in the proof of equation (A.21) is to apply $\bar{\partial}_q u^\alpha \partial_\alpha$ to $\bar{\partial}_a u^\beta \partial_\beta y^a = 3$.

Now we proof the vertical integration by parts formula for a d-vector field A with components $A^a(x, y)$ which are m-homogeneous with respect to y

$$\begin{aligned} &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^L h^L}_{|\Sigma}\big(\bar{\partial}_a A^a\big)_{|\Sigma} \\ = &\ -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^L h^L}_{|\Sigma}\Big[\big(g^{L\,pq}\bar{\partial}_a g^L_{pq} - \frac{2(4r+m-5)}{r(r-1)L} y^p g^L_{pa}\big)A^a\Big]_{|\Sigma}. \end{aligned} \tag{A.23}$$

To perform an integration by parts on Σ we need to expand $\bar{\partial}_a$ into the adapted coordinate derivative operators

$$\bar{\partial}_a A^a = \bar{\partial}_a u^\alpha \partial_\alpha A^a + \bar{\partial}_a R \partial_R A^a = \bar{\partial}_a u^\alpha \partial_\alpha A^a + 2m\frac{g^L_{ab} y^b}{r(r-1)}\frac{A^a}{L}\,. \tag{A.24}$$

The last equality is based on the facts that $R(x, y) = F(x, y) = |L(x, y)|^{1/r}$ by the definition of the adapted coordinates, that the homogeneity of tangent bundle functions with respect to y becomes homogeneity with respect to R as shown in equation (5.63). Thus

$$\begin{aligned} &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^L h^L}_{|\Sigma}\big(\bar{\partial}_a A^a\big)_{|\Sigma} = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^L h^L}_{|\Sigma}\Big[\bar{\partial}_a u^\alpha \partial_\alpha A^a + \frac{2m g^L_{ab} y^b}{r(r-1)L} A^a\Big]_{|\Sigma} \\ = &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^L h^L}_{|\Sigma}\Big[\partial_\alpha(\bar{\partial}_a u^\alpha A^a) - \partial_\alpha\bar{\partial}_a u^\alpha A^a + \frac{2m g^L_{ab} y^b}{r(r-1)L} A^a\Big]_{|\Sigma} \\ = &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \Big[\partial_\alpha(\sqrt{g^L h^L}\bar{\partial}_a u^\alpha A^a) - \partial_\alpha(\sqrt{g^L h^L})\bar{\partial}_a u^\alpha A^a - \partial_\alpha\bar{\partial}_a u^\alpha A^a + \frac{2m g^L_{ab} y^b}{r(r-1)L} A^a\Big]_{|\Sigma} \\ = &\ -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \Big[\bar{\partial}_a u^\alpha g^{Lcd}\partial_\alpha g^L_{cd} + \bar{\partial}_c u^\beta(\partial_\alpha\partial_\beta y^c)\bar{\partial}_a u^\alpha + \partial_\alpha\bar{\partial}_a u^\alpha - \frac{2m g^L_{ab} y^b}{r(r-1)L}\Big]_{|\Sigma} A^a_{|\Sigma} \\ = &\ -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \Big[g^{Lcd}\partial_a g^L_{cd} - \frac{2(4(r-2)+3+m)}{r(r-1)L} g^L_{ab} y^b\Big]_{|\Sigma} A^a_{|\Sigma}\,, \end{aligned} \tag{A.25}$$

where we used the validity of the last two equalities is ensured by the equations (A.19), (A.20) and (A.21). We omitted the boundary term coming from the divergence along the tangent space directions $\partial_\alpha(\sqrt{g^L h^L}\bar{\partial}_a u^\alpha A^a)$ as discussed in section 5.3.3.

The horizontal integration by parts formula

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\delta_a A^a\big)_{|\Sigma} \quad = \quad -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[\big(\Gamma^{\delta\,p}{}_{pa} + S^p{}_{pa}\big)A^a\Big]_{|\Sigma} \tag{A.26}$$

can now be proven as follows. Observe that the horizontal derivative can be written as

$$\begin{aligned} \delta_a A^a = \partial_a A^a - N^q{}_a\bar{\partial}_q A^a \ &=\ (\hat{\partial}_a + \partial_a u^\alpha\partial_\alpha + \partial_a R\partial_R)A^a - N^q{}_a(\partial_q u^\alpha \partial_\alpha A^a + (\bar{\partial}_q R)\partial_R A^a) \\ &=\ \hat{\partial}_a A^a - (N^q{}_a\partial_q u^\alpha - \partial_a u^\alpha)\partial_\alpha A^a = \hat{\partial}_a A^a - N^\alpha{}_a\partial_\alpha A^a\,, \end{aligned} \tag{A.27}$$

since $\delta_a R = \delta_a F = 0$, as derived in equation (5.39).

$$\begin{aligned} \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\delta_a A^a\big)_{|\Sigma} &= \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[\hat{\partial}_a A^a - N^\alpha{}_a \partial_\alpha A^a\Big]_{|\Sigma} \\ = \quad \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[&- (\hat{\partial}_a \sqrt{g^F h^F})A^a + (\partial_\alpha N^\alpha{}_a)A^a - \partial_\alpha(N^\alpha{}_a A^a)\Big]_{|\Sigma} \end{aligned} \tag{A.28}$$

To continue from here to the desired expression we extract the ∂_α integration by parts from equation (A.25) and use the relations (A.17), (A.21) and (A.22) to expand the terms. Adding everything up one arrives at

$$\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\big(\delta_a A^a\big)_{|\Sigma} = -\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma}\Big[(g^{Lcd}\delta_a g^L_{cd} + \bar{\partial}_c N^c{}_a)A^a\Big]_{|\Sigma}, \tag{A.29}$$

what with help of the definition of the S-tensor (equation (5.51)) and and of the δ-Christoffel symbols (equation (5.45)) proofs the horizontal integration by parts formula.

The derivations of this section proof the integration by parts formulas which are essential in the derivation of the dynamics of Finsler spacetimes.

A.4. Lifts of generating vector fields of cosmological symmetry

We deduce the most general fundamental geometry function L for Finsler spacetimes with cosmological symmetries in section 5.4.3. The derivation requires the complete lifts of the symmetry-generating vector fields (5.99) which we display here explicitly:

$$\begin{aligned} X_1^C &= \chi\Big(\sin\theta\cos\phi\partial_r + \frac{\chi}{r}\cos\theta\cos\phi\partial_\theta - \frac{\chi}{r}\frac{\sin\phi}{\sin\theta}\partial_\phi\Big) \\ &\quad + \Big(y^r\chi'\sin\theta\cos\phi + y^\theta\chi\cos\theta\cos\phi - y^\phi\chi\sin\theta\sin\phi\Big)\,\bar{\partial}_r \\ &\quad + \Big(y^r\big(\frac{\chi}{r}\big)'\cos\theta\cos\phi - y^\theta\frac{\chi}{r}\sin\theta\cos\phi - y^\phi\frac{\chi}{r}\cos\theta\sin\phi\Big)\bar{\partial}_\theta \\ &\quad + \Big(-y^r\big(\frac{\chi}{r}\big)'\frac{\sin\phi}{\sin\theta} + y^\theta\frac{\chi}{r}\frac{\sin\phi}{\sin^2\theta}\cos\theta - y^\phi\frac{\chi}{r}\frac{\cos\phi}{\sin\theta}\Big)\bar{\partial}_\phi\,, \end{aligned} \tag{A.30}$$

$$\begin{aligned} X_2^C &= \chi\sin\theta\sin\phi\partial_r + \frac{\chi}{r}\cos\theta\sin\phi\partial_\theta + \frac{\chi}{r}\frac{\cos\phi}{\sin\theta}\partial_\phi \\ &\quad + \Big(y^r\chi'\sin\theta\sin\phi + y^\theta\xi\cos\theta\sin\phi + y^\phi\xi\sin\theta\cos\phi\Big)\,\bar{\partial}_r \\ &\quad + \Big(y^r\big(\frac{\chi}{r}\big)'\cos\theta\cos\phi - y^\theta\frac{\chi}{r}\sin\theta\sin\phi + y^\phi\frac{\chi}{r}\cos\theta\cos\phi\Big)\bar{\partial}_\theta \\ &\quad + \Big(y^r\big(\frac{\chi}{r}\big)'\frac{\cos\phi}{\sin\theta} - y^\theta\frac{\chi}{r}\frac{\cos\phi}{\sin^2\theta}\cos\theta - y^\phi\frac{\chi}{r}\frac{\sin\phi}{\sin\theta}\Big)\bar{\partial}_\phi\,, \end{aligned} \tag{A.31}$$

$$X_3^C = \chi\cos\theta\partial_r - \frac{\chi}{r}\sin\theta\partial_\theta + \Big(y^r\chi'\cos\theta - y^\theta\chi\sin\theta\Big)\bar{\partial}_r - \Big(y^r\big(\frac{\chi}{r}\big)'\sin\theta + y^\theta\frac{\chi}{r}\cos\theta\Big)\bar{\partial}_\theta\,. \tag{A.32}$$

The complete lifts X_4^C, X_5^C and X_6^C are stated in equations (5.91). In the formulae above we use the abbreviation $\chi = \sqrt{1-kr^2}$ and primes denote differentiation with respect to the coordinate r.

A.5. Variation of the gravity action

The dynamical equation determining the geometry of Finsler spacetimes presented in section 6.2 can be deduced from our generalised Einstein–Hilbert action as follows. Recall the action (6.3):

$$S_LG[L] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\sqrt{g^F h^F}_{|\Sigma}\,\mathcal{R}_{|\Sigma} = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\left[\sqrt{g^F h^F}\;R^a{}_{ab}y^b\right]_{|\Sigma}. \tag{A.33}$$

The integrand is homogeneous of degree five; to obtain the first intermediate step (6.5) of the variation we use the facts that for $f(x,y)$ homogeneous of degree m holds $f(x,y)_{|\Sigma} = \frac{f(x,y)}{F(x,y)^m}$ and that $\delta_L(f(x,y)_{|\Sigma}) = (\delta_L f(x,y))_{|\Sigma} - \frac{m}{r}f(x,y)_{|\Sigma}\frac{\delta L}{L}$.

The second step (6.6) is obtained by using the coordinate transformation formulae (5.60) and the fact that $\delta(\partial_\alpha y^a) = -y^a\partial_\alpha(\frac{\delta L}{rL})$ to calculate

$$h^{F\alpha\beta}\delta h^F{}_{\alpha\beta} = (g^{Fab}\bar{\partial}_a u^\alpha\bar{\partial}_b u^\beta)(\partial_\alpha y^c\partial_\beta y^d\delta g^F{}_{cd} + 2\partial_\alpha\delta y^c\partial_\beta y^d g^F_{cd}) = g^{Fab}\delta g^F_{ab} - \frac{2}{r}\frac{\delta L}{L}, \tag{A.34}$$

which in turn is used to deduce

$$\delta(\sqrt{g^F h^F}\;R^a{}_{ab}y^b) = \sqrt{g^F h^F}\Big(\Big[g^{Fab}\delta g^F{}_{ab} - \frac{1}{r}\frac{\delta L}{L}\Big]R^a{}_{ab}y^b + \delta R^a{}_{ab}\,y^b\Big). \tag{A.35}$$

The formulae (6.7) and (6.8) used in the third step of the variation are basically obtained by means of integration by parts (5.86). For a function $f(x,y)$ that is m-homogeneous in y the following holds

$$\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\left(\sqrt{g^F h^F}g^{Fab}\delta g^F_{ab}f\right)_{|\Sigma} = \int \mathrm{d}^4\hat{x}\mathrm{d}^3u\Big(\sqrt{g^F h^F}\big(f(m+4)(2-m) + F^2 g^{Fab}\bar{\partial}_a\bar{\partial}_b f\big)\frac{\delta L}{rL}\Big)_{|\Sigma}; \tag{A.36}$$

choosing $f = R^a{}_{ab}y^b$ which has $m = 2$ proves formula (6.7). To show equation (6.8) we first write $S^a{}_{bc} = -y^q\bar{\partial}_b\Gamma^{\delta a}{}_{qc}$ and use

$$\delta R^a{}_{bc} = -2y^d\nabla^{CL}_{[b}\delta\Gamma^{\delta a}{}_{c]d} + 2y^p S^a{}_{q[b}\Gamma^{\delta q}{}_{c]p} \tag{A.37}$$

to equate

$$\begin{aligned}\delta R^a{}_{ab}\,y^b &= -2y^b y^q\Big(\nabla^{CL}_{[a}\delta\Gamma^{\delta a}{}_{b]q} - \frac{1}{2}S_c\delta\Gamma^{\delta c}{}_{bq}\Big)\\ &= -\nabla^{CL}_a(y^b y^q\delta\Gamma^{\delta a}{}_{bq}) + y^b\nabla^{CL}_b(y^q\delta\Gamma^{\delta a}{}_{aq}) + S_c\delta\Gamma^{\delta c}{}_{bq}y^b y^q\,.\end{aligned} \tag{A.38}$$

The integration by parts formulae (5.86) and

$$\begin{aligned}y^b y^q\delta\Gamma^{\delta a}{}_{bq} &= \frac{1}{2}g^{Lap}(y^b\nabla^{CL}_b\bar{\partial}_p\delta L - \nabla^{CL}_p\delta L)\\ &= \frac{|L|^{2/r}}{rL}g^{Fap}(y^b\nabla^{CL}_b\bar{\partial}_p\delta L - \nabla^{CL}_p\delta L) + \frac{(2-r)}{rL}y^a y^b\nabla^{CL}_b\delta L\end{aligned} \tag{A.39}$$

then yield the desired equation

$$\begin{aligned}\int \mathrm{d}^4\hat{x}\mathrm{d}^3u\left(\sqrt{g^F h^F}\delta R^a{}_{ab}y^b\right)_{|\Sigma} &= \int \mathrm{d}^4\hat{x}\mathrm{d}^3u\left(\sqrt{g^F h^F}\;2\;S_c\delta\Gamma^{\delta c}{}_{bq}y^b y^q\right)_{|\Sigma}\\ &= \int \mathrm{d}^4\hat{x}\mathrm{d}^3u\Big(\sqrt{g^F h^F}\;2\;F^2 g^F_{ab}(\nabla^{CL}_a S_b + S_a S_b + \bar{\partial}_a\nabla S_b)\frac{\delta L}{rL}\Big)_{|\Sigma}.\end{aligned} \tag{A.40}$$

Combining these three steps as we did in section 6.2 finally produces the Finsler gravity vacuum field equation (6.11).

A.6. Conservation equation in the metric limit

In section 6.3 we derived the conservation equation which is satisfied by action based field theories on Finsler spacetimes. We explicitly discussed the resulting conservation equations from the generalised Einstein–Hilbert action. A missing step in the discussion is the following integral on Riemannian manifolds

$$\int_{\Sigma_x} \mathrm{d}^3u \sqrt{gh}_{|\Sigma_x} \Big[y^a y^b y^c y^d \Big]_{|\Sigma_x} = \sqrt{g} \frac{2}{\mathrm{Vol}(S^3_p)} g^{(ab} g^{cd)} \,, \tag{A.41}$$

For a Riemannian spacetime Σ_x is the metric sphere $S^3_p = \{y \in T_pM | g_p(y,y) = 1\}$. To perform the integration we introduce an orthonormal frame of the metric g_{ab} so that $g_x(y,y) = \delta_{\mu\nu} y^\mu y^\nu$ and spherical coordinates $(r, \theta_1, \theta_2, \theta_3)$; $r \in (0, \infty)$; $\theta_1 \in (0, 2\pi)$; $\theta_2, \theta_3 \in (0, \pi)$ on T_xM

$$y^0 = r \sin\theta_3 \sin\theta_2 \cos\theta_1, \quad y^2 = r \sin\theta_3 \cos\theta_2, \tag{A.42}$$

$$y^1 = r \sin\theta_3 \sin\theta_2 \sin\theta_1, \quad y^3 = r \cos\theta_3 \,. \tag{A.43}$$

Setting $r = 1$ yields coordinates on Σ_x. The volume element $\sqrt{h}$ of Σ_x is given by

$$\sqrt{h} = |\sin\theta_2 \sin\theta_3^2|, \tag{A.44}$$

while $\sqrt{g}$ can be ignored in the integrations, since g_{ab} does not depend on the angles θ_i. Plugging these expressions into equation (A.41) and performing the integral for all index combinations yields its validity.

A.7. Variation of the matter action

In section 9.2 we presented a coupling principle of matter fields to Finsler gravity. The crucial steps of the derivation of the constraints (9.6), equations of motion (9.7) and (9.8), and of the metric limit of the complete gravity equation (9.18) including the matter source terms shall be presented here. Recall the matter action for a p-form field $\Phi(x, y)$ on Finsler spacetime arises from a lift of the standard p-form action on Lorentzian spacetime as

$$S_m[L, \Phi, \lambda] = \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u \left[\sqrt{g^F h^F} \Big(\mathcal{L}(G^F, \Phi, \mathrm{d}\Phi) + \lambda(1 - P^H)\Phi \Big) \right]_{|\Sigma} . \tag{A.45}$$

In order to perform the variation we consider all objects in the horizontal/vertical basis of TTM where G is diagonal, see definition 5.4. In the following the $M, N, ..$ label both horizontal and vertical indices, $\bar{a}, \bar{b}, ...$ label vertical indices, and $a, b...$ label horizontal indices. Then

$$\mathcal{L}(G^F, \Phi, \mathrm{d}\Phi) + \lambda(1 - P^H)\Phi = \mathcal{L}(G^F_{MN}, \Phi_{M_1...M_p}, \mathrm{d}\Phi_{AM_1...M_p}) + \lambda^{\bar{a}_1 M_2...M_p} \Phi_{\bar{a}_1 M_2..M_p} \,, \tag{A.46}$$

and the variation of this Lagrangian can now be written as follows

$$\begin{aligned} \delta(\mathcal{L} + \lambda(1 - P^H)\Phi) \;=\; & \frac{\partial \mathcal{L}}{\partial G^F_{MN}} \delta G^F_{MN} + \frac{\partial \mathcal{L}}{\partial \Phi_{M_1...M_p}} \delta \Phi_{M_1 M_p} + \frac{\partial \mathcal{L}}{\partial(\mathrm{d}\Phi_{NM_1...M_p})} \delta(\mathrm{d}\Phi_{NM_1M_p}) \\ & + \lambda^{\bar{a}_1 M_2...M_p} \delta\Phi_{\bar{a}_1 M_2..M_p} + \delta\lambda^{\bar{a}_1 M_2...M_p} \Phi_{\bar{a}_1 M_2..M_p} \,. \end{aligned} \tag{A.47}$$

We can immediately read off the variation with respect to the Lagrange multiplier components which produces (9.6). Hence the Lagrange multiplier λ sets to zero all components of Φ with at least one vertical index, so that only purely horizontal components remain on-shell.

The expansion of $\mathrm{d}\Phi$ in components with respect to the horizontal/vertical basis yields

$$\mathrm{d}\Phi_{NM_1\ldots M_p} = (p+1)D_{[N}\Phi_{M_1\ldots M_p]} - \frac{p(p+1)}{2}\gamma^Q{}_{[NM_1}\Phi_{|Q|M_2\ldots M_p]}\,, \tag{A.48}$$

where we write $D_M = \delta^a_M\delta_a + \delta^{\bar{a}}_M\bar{\partial}_a$, and $\gamma^Q{}_{MN}$ denote the commutator coefficients of the horizontal/vertical basis. Their only non-vanishing components are given by $\gamma^{\bar{a}}{}_{bc} = [\delta_b, \delta_c]^{\bar{a}} = R^{\bar{a}}{}_{bc}$ and $\gamma^{\bar{a}}{}_{\bar{b}c} = [\bar{\partial}_b, \delta_c]^{\bar{a}} = \bar{\partial}_b N^{\bar{a}}{}_c$. One now uses the integration by parts formulae (5.86) and rewrites all formulae in terms of the Cartan linear covariant derivative to obtain the variation of the matter action with respect to Φ; this produces the equations of motion (9.7) and (9.8).

Finally the source term for the gravity field equation is obtained by variation of the matter action S_m in (A.45) with respect to the fundamental geometry function L. This not only includes the variation (A.47) but also that of the volume element which can be read off from (A.35). We will now show that the metric limit of Finsler gravity plus matter is consistent; this can be done on-shell where we may use the Lagrange multiplier constraints to set all explicitly appearing $\Phi_{\bar{a}_1M_2..M_k}$ to zero. Then the variation of S_m with respect to L becomes

$$\begin{aligned}
&\delta S_m[L,\Phi] &\text{(A.49)}\\
= &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\sqrt{g^F h^F}{}_{|\Sigma}\Big[\big(g^{Fab}\bar{\partial}_a\bar{\partial}_b\mathcal{L} + 4\mathcal{L}\big)\frac{\delta L}{rL}\\
&\qquad + \frac{\partial\mathcal{L}}{\partial G^F_{MN}}\delta G^F_{MN} + \frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{NM_1\ldots M_k})}\delta(\mathrm{d}\Phi_{NM_1M_k})\Big]_{|\Sigma}\\
= &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\sqrt{g^F h^F}{}_{|\Sigma}\Big[\big(g^{Fab}\bar{\partial}_a\bar{\partial}_b\mathcal{L} + 4\mathcal{L}\big)\frac{\delta L}{nL} &\text{(A.50)}\\
&\qquad + \frac{\partial\mathcal{L}}{\partial g^F_{ab}}\delta g^F_{ab} + \frac{\partial\mathcal{L}}{\partial g^F_{\bar{a}\bar{b}}}\delta\Big(\frac{g^F_{\bar{a}\bar{b}}}{F^2}\Big) + \frac{\partial\mathcal{L}}{\partial(\mathrm{d}\Phi_{ba_1\ldots a_k})}\delta(\delta_{[b}\Phi_{a_1..a_k]})\Big]_{|\Sigma}\,.
\end{aligned}$$

In order to determine the energy momentum scalar $T_{|\Sigma}$ defined in (9.17) on a generic Finsler spacetime one has to calculate all terms in the expression above carefully. However, in the metric geometry limit the last two terms vanish. Indeed, $\frac{\partial\mathcal{L}}{\partial g^F_{\bar{a}\bar{b}}}$ is always composed from terms with vertical indices that must be either of the type $\bar{\partial}\Phi$ or contain components of Φ with at least one vertical index; the last term is proportional to $\delta N\bar{\partial}\Phi$; in the metric limit $\bar{\partial}\Phi$ vanishes and the vertical index components of Φ are zero on-shell. Therefore the remaining terms that are relevant in the metric limit are

$$\delta S_m[L,\Phi] \to \int_\Sigma \mathrm{d}^4x\mathrm{d}^3u\,\sqrt{g^F h^F}{}_{|\Sigma}\Big[\big(g^{Fab}\bar{\partial}_a\bar{\partial}_b\mathcal{L} + 4\mathcal{L}\big)\frac{\delta L}{rL} + \frac{\partial\mathcal{L}}{\partial g^F_{ab}}\delta g^F_{ab}\Big]_{|\Sigma}\,. \tag{A.51}$$

The rewriting $\delta g^F_{ab} = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b\delta F^2$ and subsequent integration by parts yields

$$\begin{aligned}
&\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\sqrt{g^F h^F}{}_{|\Sigma}\Big[\frac{\partial\mathcal{L}}{\partial g^F_{ab}}\delta g^F_{ab}\Big]_{|\Sigma} &\text{(A.52)}\\
= &\int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\,\sqrt{g^F h^F}{}_{|\Sigma}\Big[-\bar{\partial}_c K^c + \Big(-g^{Fij}\bar{\partial}_c g^F_{ij} + 4g^F_{ic}y^i\Big)K^c\Big]_{|\Sigma}\frac{\delta L}{rL}\,,
\end{aligned}$$

with

$$K^c = \Big(-g^{Fij}\bar{\partial}_d g^F_{ij} + \frac{4}{F^2}g^F_{id}y^i\Big)\frac{\partial\mathcal{L}}{\partial g^F_{cd}} - \bar{\partial}_d\frac{\partial\mathcal{L}}{\partial g^F_{cd}}\,. \tag{A.53}$$

Applying the metric limit now means to consider $L(x,y) = g_{ab}(x)y^a y^b$ with the consequence that $g^F_{ab}(x,y) = -g_{ab}(x)$ for timelike y. The expression for K^c reduces to $K^c \to \frac{4}{F^2} g_{id} y^i \frac{\partial \mathcal{L}}{\partial g_{cd}}$ and $\bar{\partial}_c K^c \to (\frac{8}{F^4} g_{id} y^i g_{jc} y^j + \frac{4}{F^2} g_{cd}) \frac{\partial \mathcal{L}}{\partial g_{cd}}$. Collecting all terms in the variation of the matter action in the metric geometry limit finally yields

$$\delta S_m[L,\Phi] \to \int_\Sigma \mathrm{d}^4\hat{x}\mathrm{d}^3u\ \sqrt{g^F h^F}_{|\Sigma} \Big[4\mathcal{L} - 4g_{cd}\frac{\partial \mathcal{L}}{\partial g_{cd}} - 24 y^c y^d \frac{\partial \mathcal{L}}{\partial g_{cd}}\Big]_{|\Sigma} \frac{\delta L}{rL}\,, \tag{A.54}$$

from which we can read off the expression for the source term $T_{|\Sigma}$,

$$T_{|\Sigma} \to \Big(4\mathcal{L} - 4g_{cd}\frac{\partial \mathcal{L}}{\partial g_{cd}} - 24 y^c y^d \frac{\partial \mathcal{L}}{\partial g_{cd}}\Big)_{|\Sigma}\,. \tag{A.55}$$

The lift of this expression to TM requires making all terms zero-homogeneous by multiplication with the appropriate powers of $F(x,y)$, which here means multiplication of the third term by $F(x,y)^{-2}$. The result confirms equation (9.19) that was used to prove the consistency of Finsler gravity with Einstein gravity in the metric geometry limit.

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Bibliography

[1] H. F. Goenner, "On the History of Geometrization of Space-time: From Minkowski to Finsler Geometry," *414 Heraeus-Seminar* (2008) , `arXiv:0811.4529 [gr-qc]`.

[2] T. Kaluza, "Zum Unitätsproblem in der physik," *Sitzungsber. Preuss. Akad. Wiss.* (1921) 966.

[3] O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie," *Zeitschrift fÃijr Physik* **37** (1926) 895.

[4] V. C. Rubin and W. K. Ford, "Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions," *The Astrophysical Journal* **159** (1970) 379.

[5] A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron. J.* **116** (1998) 1009, `arXiv:astro-ph/9805201`.

[6] S. Perlmutter *et al.*, "Measurements of omega and lambda from 42 high-redshift supernovae," *Astrophys. J.* **517** (1999) 565, `arXiv:astro-ph/9812133`.

[7] S. Capozziello and M. D. Laurentis, "Extended theories of gravity," *Physics Reports* **509** (2011) 167.

[8] J. Overduin and P. Wesson, "Kaluza-Klein gravity," *Phys.Rept.* **283** (1997) 303, `arXiv:gr-qc/9805018 [gr-qc]`.

[9] C. M. Will, *Theory and Experiment in gravitational physics*. Cambridge University Press, 1993.

[10] J. D. Bekenstein, "Tensor-vector-scalar-modified gravity: from small scale to cosmology," *Phil.Trans.Roy.Soc.Lond.* **A369** (2011) 5003, `arXiv:1201.2759 [astro-ph.CO]`.

[11] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*. PhD thesis, Georg-August Universität zu Göttingen, 1918.

[12] C. Pfeifer, "From the einstein–hilbert action to an action principle for finsler gravity," diploma thesis, Universität Hamburg, 2010.

[13] N. Hübel, "Field equations for nonlinear finsler gravity from an action principle," diploma thesis, Universität Hamburg, 2010.

[14] C. Pfeifer and M. N. R. Wohlfarth, "Causal structure and electrodynamics on Finsler spacetimes," *Phys.Rev.* **D84** (2011) 044039, `arXiv:1104.1079 [gr-qc]`.

[15] C. Pfeifer and M. N. R. Wohlfarth, "Beyond the speed of light on Finsler spacetimes," *Phys.Lett.* **B712** (2012) 284, `arXiv:1109.6005 [gr-qc]`.

[16] C. Pfeifer and M. N. R. Wohlfarth, "Finsler geometric extension of Einstein gravity," *Phys.Rev.* **D85** (2012) 064009, `arXiv:1112.5641 [gr-qc]`.

[17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. 1*. Wiley-Interscience, 1996.

[18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. 2*. Wiley-Interscience, 1996.

[19] H. Baum, *Eichfeldtheorie - Eine Einführung in die Differentialgeometrie auf Faserbündeln*. Springer, 2009.

[20] I. Kolar, P. Michor, and J. Slovak, *Natural operations in Differential Geometry*. Springer-Verlag, Berlin Heidelberg 1993.

[21] R. M. Wald, *General Relativity*. The University of Chicago Press, 1984.

[22] R. Miron and I. Bucataru, *Finsler Lagrange Geometry*. Editura Academiei Romane, 2007.

[23] D. Bao, S. S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler geometry*. Springer, 2000.

[24] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*. D. Reidel Publishing Company, 1985.

[25] J. K. Beem, "Indefinite finsler spaces and timelike spaces," *Can. J. Math.* **22** (1970) 1035.

[26] F. P. Schuller, C. Witte, and M. N. R. Wohlfarth, "Causal structure and algebraic classification of non-dissipative linear optical media," *Annals Phys.* **325** (2010) 1853, `arXiv:0908.1016 [hep-th]`.

[27] F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics*. Birkhäuser Boston, 2003.

[28] D. Raetzel, S. Rivera, and F. P. Schuller, "Geometry of physical dispersion relations," *Phys.Rev.* **D83** (2011) 044047, `arXiv:1010.1369 [hep-th]`.

[29] P. L. Antonelli, S. F. Rutz, and M. A. Slawinski, "A geometrical foundation for seismic ray theory based on modern finsler geometry," in *Finsler and Lagrange Geometries*, A. M. and P. L. Antonelli, eds., p. 17. Kluwer Academic Publisher, Dordrecht, 2003.

[30] L. Klimes, "Relation of the wave-propagation metric tensor to the curvatures of the slowness and ray- velocity surfaces," *Studia Geophysica et Geodaetica* **46** (2002) 589.

[31] G. Gibbons and C. Warnick, "The Geometry of sound rays in a wind," *Contemp.Phys.* **52** (2011) 197, `arXiv:1102.2409 [gr-qc]`.

[32] V. Perlick, "Fermat principle in Finsler spacetimes," *Gen.Rel.Grav.* **38** (2006) 365, `arXiv:gr-qc/0508029 [gr-qc]`.

[33] V. Cerveny, "Fermat's variational principle for anisotropic inhomogeneous media," *Studia Geophysica et Geodaetica* **46** (2002) 567.

[34] G. Randers, "On an asymmetrical metric in the four-space of general relativity," *Phys. Rev.* **59** (1941) 195.

[35] G. Bogoslovsky, "A special-relativistic theory of the locally anisotropic space-time," *Il Nuovo Cimento B Series 11* **40** (1977) 99.

[36] V. Balan, G. Y. Bogoslovsky, S. Kokarev, D. Pavlov, and S. Siparov, "Geometrical Models of the Locally Anisotropic Space-Time," *J.Mod.Phys.* **3** (2012) 1314, `arXiv:1111.4346 [gr-qc]`.

[37] A. G. Cohen and S. L. Glashow, "Very special relativity," *Phys.Rev.Lett.* **97** (2006) 021601, `arXiv:hep-ph/0601236 [hep-ph]`.

[38] G. Gibbons, J. Gomis, and C. Pope, "General very special relativity is Finsler geometry," *Phys.Rev.* **D76** (2007) 081701, `arXiv:0707.2174 [hep-th]`.

[39] A. Kouretsis, M. Stathakopoulos, and P. Stavrinos, "The General Very Special Relativity in Finsler Cosmology," *Phys.Rev.* **D79** (2009) 104011, `arXiv:0810.3267 [gr-qc]`.

[40] F. Girelli, S. Liberati, and L. Sindoni, "Planck-scale modified dispersion relations and Finsler geometry," *Phys.Rev.* **D75** (2007) 064015, `arXiv:gr-qc/0611024 [gr-qc]`.

[41] N. E. Mavromatos and S. Sarkar, "CPT-Violating Leptogenesis induced by Gravitational Defects," *Eur.Phys.J.* **C73** (2013) 2359, `arXiv:1211.0968 [hep-ph]`.

[42] A. Kostelecky, "Riemann-Finsler geometry and Lorentz-violating kinematics," *Phys.Lett.* **B701** (2011) 137, `arXiv:1104.5488 [hep-th]`.

[43] Z. Chang and X. Li, "Modified Newton's gravity in Finsler Space as a possible alternative to dark matter hypothesis," *Phys.Lett.* **B668** (2008) 453, `arXiv:0806.2184 [gr-qc]`.

[44] X. Li, M.-H. Li, H.-N. Lin, and Z. Chang, "Finslerian MOND vs. observations of Bullet Cluster 1E0657-558," *Mon. Not. Roy. Astron. Soc. 430,* **450** (2013) , `arXiv:1209.3086 [astro-ph.GA]`.

[45] S. Rutz, "A finsler generalisation of einstein's vacuum field equations," *General Relativity and Gravitation* **25** (1993) 1139.

[46] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* W. H. Freeman & Co., 1973.

[47] P. L. Antonelli and M. Anastasiei, "The differential geometry of lagrangians which generate sprays," in *Lagrange and Finsler Geometry, Applications to Physics and Biology*, P. L. Antonelli and R. Miron, eds., p. 15. Kluwer Academic Publisher, Dordrecht, 1996.

[48] M. Spivak, *A comprehensive introduction to differential geometry, Vol. 3*. Publis or Perish Berkley, 1979.

[49] X. Li and Z. Chang, "Symmetry and special relativity in Finsler spacetime with constant curvature," `arXiv:1010.2020 [gr-qc]`.

[50] J. Garecki, "An interesting property of the isotropic and homogeneous cosmological models," *General Relativity and Gravitation* **27** (1995) 55.

[51] S. Weinberg, *Gravitation and cosmology*. John Wiley, 1972.

[52] J. Skakala and M. Visser, "Bi-metric pseudo-Finslerian spacetimes," *J.Geom.Phys.* **61** (2011) 1396, `arXiv:1008.0689 [gr-qc]`.

[53] C. Baer, "Renormalized Integrals and a Path Integral Formula for the Heat Kernel on a Manifold," *Contemporary Mathematics* **584** (2012) 179, `arXiv:1202.3392 [math.DG]`.

[54] S. M. Carroll, *Spacetime and Geometry: Introduction to General Relativity*. Prentice Hall, 2003.

[55] C. Lammerzahl, O. Preuss, and H. Dittus, "Is the physics within the Solar system really understood?," `arXiv:gr-qc/0604052 [gr-qc]`.

[56] J. D. Anderson, J. K. Campbell, J. E. Ekelund, J. Ellis, and J. F. Jordan, "Anomalous orbital-energy changes observed during spacecraft flybys of earth," *Phys. Rev. Lett.* **100** (2008) 091102.

[57] C. M. Will, "The confrontation between general relativity and experiment," *Living Reviews in Relativity* **9** (2006) .

[58] S. Baeßler, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt, and H. E. Swanson, "Improved test of the equivalence principle for gravitational self-energy," *Phys. Rev. Lett.* **83** (1999) 3585.

[59] J. C. Baez, A. E. Hoffnung, and C. D. Walker, "Groupoidification Made Easy," *ArXiv e-prints* (2008) , `arXiv:0812.4864 [math.QA]`.

[60] G. Amelino-Camelia, "Quantum Gravity Phenomenology," *Living Rev.Rel.* **16** (2013) 5, `arXiv:0806.0339 [gr-qc]`.

[61] S. Benzoni-Gavage and D. Serre, *Multi-dim. hyperbolic partial differential equations*. Oxford Univ. Press, 2007.

[62] R. Gambini and J. Pullin, "Nonstandard optics from quantum space-time," *Phys.Rev.* **D59** (1999) 124021, `arXiv:gr-qc/9809038 [gr-qc]`.

[63] S. R. Coleman and S. L. Glashow, "High-energy tests of Lorentz invariance," *Phys.Rev.* **D59** (1999) 116008, `arXiv:hep-ph/9812418 [hep-ph]`.

[64] N. Dencker, "On the propagation of polarization sets for systems of real principal type," *J. Funct. Anal.* **46** (1982) 351.

[65] M. Hohmann, "Extensions of Lorentzian spacetime geometry: From Finsler to Cartan and vice versa," *Phys.Rev.* **D87** (2013) 124034, `arXiv:1304.5430 [gr-qc]`.