Heterotic M-Theory, Warped Geometry and the Cosmological Constant Problem

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Abstract

The first part of this thesis analyzes whether a locally flat background represents a stable vacuum for the proposed heterotic M-theory. A calculation of the leading order supergravity exchange diagrams leads to the conclusion that the locally flat vacuum cannot be stable. Afterwards a comparison with the corresponding weakly coupled heterotic string amplitudes is made. Next, we consider compactifications of heterotic M-theory on a Calabi-Yau threefold, including a non-vanishing G-flux. The ensuing warped-geometry is determined completely and used to show that the variation of the Calabi-Yau volume along the orbifold direction varies quadratically with distance instead linearly as suggested by an earlier linearized approximation. In the second part of this thesis we propose a mechanism for obtaining a small cosmological constant. This mechanism consists of the separation of two domain-walls, which together constitute our world, up to a distance $2l \simeq 1/M_{\text{eff}}$. The resulting warped-geometry leads to an exponential suppression of the cosmological constant, which thereby can obtain its observed value without introducing a large hierarchy. An embedding of this set-up into IIB string-theory entails an SU(6) Grand Unified Theory with a natural explanation of the Higgs doublet-triplet splitting. Finally, we examine to what extent the string-theory T-duality can influence curvature. To this aim we derive the full transformation of the curvature-tensor under T-duality.

Zusammenfassung

Der erste Teil der vorliegenden Arbeit untersucht ob die heterotische M-Theorie ein stabiles lokal flaches Vakuum besitzt. Dazu werden die führenden Supergravitations Wechselwirkungs-Diagramme berechnet, die zu dem Schluß führen, daß ein solches Vakuum instabil ist. Weiter werden die Amplituden mit denen des heterotischen Strings veglichen. Anschließend werden Kompaktifizierungen der heterotischen M-Theorie auf Calabi-Yau Mannigfaltigkeiten betrachtet, die einen nicht-verschwindenden G-Fluß beinhalten. Die resultierende Warp-Geometrie wird bestimmt und dazu verwendet, die Variation des Calabi-Yau Volumens längs der Orbifold-Richtung zu ermitteln. Entgegen einer früheren Approximation mit linearer Abhängigkeit zeigt sich eine quadratische in der vollen Lösung. Im zweiten Teil der Arbeit schlagen wir einen Mechanismus zur Erzeugung einer kleinen kosmologischen Konstanten vor. Er basiert auf der Trennung zweier Domänen-Wände, die zusammen unsere Welt bilden, um eine Distanz $2l \simeq 1/M_{\text{eff}}$. Die resultierende Warp-Geometrie führt zu einer exponentiellen Unterdrückung der kosmologischen Konstanten, die ihren beobachteten Wert annehmen kann ohne ein neues Hierarchie-Problem zu generieren. Eine Einbettung dieser Konfiguration in die IIB String-Theorie führt auf eine SU(6) GUT mit einer natürlichen Erklärung der Higgs Dublett-Triplett Spaltung. Schließlich wird untersucht inwieweit die T-Dualität der String-Theorie die Raum-Zeit Krümmung beeinflußt. Zu diesem Zweck wird die vollständige Transformation des Krümmungs-Tensors unter T-Dualität bestimmt.

Contents

1 Introduction

High Energy Physics has obtained spectacular successes during this century, culminating in the establishment of quantum field theory and of the SU(3)×SU(2)×U(1) Standard Model (SM). The SM encompasses virtually everything we can physically measure – except gravitational phenomena. From the point of view of a particle physicist, gravity is simply the weakest of the interactions. It is natural to try to understand its quantum properties using the same strategy that has been so successful for the rest of microphysics. The search for a conventional quantum field theory capable of embracing gravity, however, suffers from the disease that the inclusion of gravity leads to a non-renormalizable theory and therefore looses its predictive power. Since the quantization of all other forces of nature except for gravity is well understood, the vexing question has been how to quantize gravity.

But is it really necessary to quantize gravity? Or is it possible keeping gravity classical together with quantized matter and the quantization of the three other forces? If some interaction would be fundamentally classical, one could only use this interaction to measure the position and momentum of a particle to arbitrary precision – thus violating the Heisenberg uncertainty principle. Therefore, at a fundamental level, if some of the physical laws are quantized, all of them have to be quantized. In order to estimate an upper limit for the scale where classical general relativity ceases to make sense, let us try to measure a spacetime coordinate with accuracy Δx. By the uncertainty principle there will be energy of order $1/\Delta x$ localized in this region. But if Δx is very small then the energy will be large enough to form a black hole with the consequence that the spacetime point will be hidden behind a horizon. The scale at which a black hole formation occurs can be estimated to be the Planck-length l_{Pl} [1]. Therefore, classical general relativity and quantum mechanics become incompatible at least at scales of order l_{Pl}.

There are currently two main approaches to a formulation of such a theory of quantum gravity. The more popular one is string-theory [2],[3],[4],[5],[6], the present only serious rival being loop quantum gravity [7]. String-theory can be seen as a natural outcome of the line of research that started with the effort to go beyond the SM, and went through Grand Unified Theories (GUTs), supersymmetry and supergravity. Loop quantum gravity on the other hand can be regarded as an outcome of the research that started with the quantization of constrained systems, initiated by Dirac, and continued with the canonical quantization of the Wheeler-DeWitt equation. Though string-theory and loop quantum gravity are both based on one-dimensional objects, they differ considerably in philosophy and results. String-theory offers the remarkable possibility to treat gravity on an equal footing with the three other forces of nature and naturally unifies all four forces. Furthermore it leads to phenomenologically interesting low-energy theories with a shift in recent years from an emphasis on the heterotic string towards type I and heterotic M-theory vacua. Loop quantum gravity on the other hand provides a solid description of Planck scale quantum spacetime, but finds difficulties in making contact with low-energy physics.

String-theory, which will be the framework of this thesis, presently exists at two levels. First, there is a well developed set of techniques that define the string perturbation expansion over a given metric background. Second, the understanding of the non-perturbative aspects of the theory has much increased in recent years and there is a widespread belief, supported by numerous indications, in the existence of a yet to be defined full non-perturbative formulation, named M-theory. The proposed Matrix-theory [8] aimed at such a non-perturbative formulation in terms of fundamental partonic D0-brane degrees of freedom.

The claim that string-theory is actually a theory of quantum gravity is based on two facts. First, the string perturbation expansion includes the graviton. More precisely, one of the string modes is a massless spin-2 and helicity ± 2 particle. Such a particle necessarily couples to the energy-momentum of the other fields present [9] and gives general relativity to a first approximation. Second, the perturbation expansion is consistent, i.e. keeps to be conformally invariant at the quantum level, if the background geometry over which the theory is defined gives rise to vanishing β-functions, which – to lowest order in the σ-model perturbation parameter α' – reproduced the Einstein field equation and moreover through higher order corrections leads to a (in principle testable) high energy modification of the Einstein equation. The hope is that such a consistency condition, as the vanishing of the β-functions for the perturbation expansion, will emerge as a dynamical equation from the yet to be found non-perturbative theory.

In string-theory, gravity is just one of the excitations of a string living over some metric space. The existence of such a background metric space, over which the theory is defined, is needed for the formulation of the theory, not just in perturbative string-theory, but also in attempts of a non-perturbative definition of the theory such as Matrix-theory [8], where a flat background metric is used to raise and lower indices. To find a background independent formulation of string-theory is one of the long outstanding open problems of the subject.

Concerning the traditional questions a theory of quantum gravity has to answer, there were obtained two interesting results from string-theory. The first one is the derivation of the Bekenstein-Hawking formula for the entropy of a black hole as a function of the horizon area [10],[11],[12],[13],[14]. The discovery of D-branes [15] made it possible to construct a black-hole at weak string coupling by superimposing D5-branes, D1-branes and Kaluza-Klein momentum. The degeneracy of states of the D-brane system can be evaluated by conformal field theory techniques (Cardy's formula [16]) and can be extrapolated due to supersymmetry to the strong coupling region, where supergravity applies. It turns out that the Bekenstein-Hawking entropy $S = A/(4G_N)$ obtained from the calculation of the horizon area A in the supergravity description agrees with the D-brane counting. This result indicates that there is an internal consistency between string-theory and quantum field theory on curved spacetime.

The second traditional question to a theory of quantum gravity deals with the microstructure of spacetime. There are indications that in string-theory the spacetime continuum is meaningless below the Planck-length. What happens is that in order to probe smaller distance one needs higher energy, but at high energy the string "opens up" from being a particle to being a true string, which is spread over spacetime. Therefore there is no way of focusing a string's collision within a small spacetime region. More recently, in the context of the effective field theory on a stack of D-branes or in the Matrix-theory formulation of M-theory, the spacetime coordinates of the string X^A are replaced by matrices $(X^A)_{ij}$. This can be viewed as a new interpretation of the spacetime structure. The continuous spacetime manifold emerges only in the long distance region, where these matrices are diagonal and commute, whereas spacetime appears to have a non-commutative discretized structure in the short distance regime. This is the point, where non-commutative geometry [17],[18] enters the string-theory scene.

Outline of the Thesis

The work which will be presented in this thesis is based on [19],[20],[21],[22] and the article [23] which will be published in the near future.

The following chapter gives a brief overview over modern string-theory with an emphasis on aspects whose knowledge is important for later chapters. In chapter 3 the background field method is employed to derive the interaction terms for Hořava-Witten supergravity on a flat background. All amplitudes contributing to the graviton, gravitino and 3-form exchange between the two boundaries are calculated and it is shown that their sum does not vanish. An interpretation of this result is given and a comparison with the corresponding amplitudes of heterotic string theory follows. Chapter 4 starts with an introduction to the relation between Newton's Constant and the geometry of the compactified heterotic M-theory set-up. The full warped geometry in the presence of internal G-flux is determined and shown to agree with the weakly coupled heterotic string relation between warp-factor and torsion. Furthermore, a quadratic variation of the Calabi-Yau volume with the orbifold direction is derived and the relation between the distance-modulus and Newton's Constant explored. It is shown in which way the previously known linear approximation is obtained and in which way an additional M5-brane source affects the volume dependence. Then by using the warped background an effective potential for the distance-modulus is derived and shown to have a destabilizing effect. Finally, the constraints on the compactification-geometry by turning on a general G-flux, compatible with supersymmetry, are derived. Chapter 5 is devoted to a low-energy mechanism to solve the cosmological constant problem. After providing necessary formulas to derive the effective 4-dimensional action, we start with a two domain-wall set-up in five dimensions and calculate the ensuing warped-geometry. A determination of the effective 4-dimensional action shows an exponentially small cosmological constant Λ_4. It is then shown that a lifting of the finetuning of the fundamental parameters still leads to an exponentially suppressed cosmological constant. To obtain the correct value for Λ_4 requires a distance between both walls which is just the inverse of the GUT-mass scale. The full consequences of this scale become clear upon embedding the configuration into a IIB string-theory framework. Here the open strings describe the relevant gauge- and matter-fields. Requiring the low-energy Standard Model gauge group, leads to the conclusion that the SU(3) has to originate from one wall, while the SU(2)×U(1) has to originate from the other. A natural consequence of the string description is the emergence of an SU(6) GUT with gauge group spontaneously broken down to the Standard Model gauge group. A natural explanation of the Higgs-boson triplet-doublet splitting suggests itself and the replication of fermion-families is treated. Finally in chapter 6 the transformation of the Riemann curvature tensor under T-duality is derived and used to analyze whether Anti-de Sitter spacetime can be dualized to flat spacetime, i.e. whether string-/M-theory is "blind" towards our notion of a cosmological constant. In chapter 7 we conclude with a summary of our results.

2 The Modern View of String-Theory

One of the cornerstones of modern string-theory are its dualities, which require the introduction of solitonic objects, D-branes, in string-theory. Even more, they led to the inclusion of all five known string-theories into the framework of M-theory and in particular the geometrization of S-duality culminated in the formulation of F-theory. In this chapter, we want to give a brief outline of these topics with an emphasis on heterotic M-theory and warped compactifications. They will play a major role in this thesis.

2.1 D-Branes

The derivation of the world-sheet (which is parameterized by τ, σ) field equation for the string embedding coordinates $X^A(\tau,\sigma)$; $A = 1, \ldots, 10$ is obtained by variation of the Polyakov-action with respect to X^A. In order to have a well-defined variational problem, one has to ensure that boundary terms vanish. In string-theory this can be accomplished by three different choices of boundary-conditions (BC)

- $X^A(\tau, \sigma + 2\pi) = X^A(\tau, \sigma)$, $\forall A$
- $\partial_\sigma X^A(\tau, \pi) = \partial_\sigma X^A(\tau, 0) = 0$ (Neumann BC)
- $X^A(\tau, \pi) = \text{const}_1$, $X^A(\tau, 0) = \text{const}_2$ (Dirichlet BC)

The first choice leads to the closed string ($\sigma \in [0, 2\pi]$), while the second and third choice give rise to the open string ($\sigma \in [0, \pi]$). Whereas the Neumann BC implies momentum conservation along σ and therefore allows to consider the open string consistently as an isolated object in ten dimensions, this is not the case for the Dirichlet BC. It implies a momentum-flow from the string towards the hypersurface defined by the BC. Hence the hypersurface on which the open string begins/ends, which is called a Dp-brane (with $p+1$ coordinates obeying Neumann BC and the remaining $9-p$ coordinates obeying Dirichlet BC) [42],[15] absorbs momentum and therefore becomes a dynamical object – a physical entity.

In type II string-theories, we have two spacetime supersymmetries generated by the Majorana-Weyl spinor-parameters ϵ_L, ϵ_R. The corresponding supercharges Q_L, Q_R are generated by left- and right-moving world-sheet degrees of freedom. In IIA they have opposite chirality, $\Gamma^{11} Q_L = Q_L, \Gamma^{11} Q_R = -Q_R$ (with $\Gamma^{11} = \Gamma^1 \cdots \Gamma^{10}$ the 10-dimensional chirality matrix) while in IIB they are of same chirality $\Gamma^{11} Q_L = Q_L, \Gamma^{11} Q_R = Q_R$. The chirality of ϵ_L, ϵ_R is just the opposite (otherwise $\epsilon_L Q_L, \epsilon_R Q_R$ would be identically zero). A D-brane preserves the linear combination $\epsilon_L Q_L + \epsilon_R Q_R$ of supercharges, where ϵ_L, ϵ_R

are subject to the relation[1]

$$\epsilon_L = \Gamma_D \epsilon_R \ , \quad \Gamma_D = \Gamma^1 \cdots \Gamma^{p+1} \ . \tag{2.1}$$

Since Γ_D squares to the identity its eigenvalues are ± 1. Furthermore, Γ_D is traceless and therefore half the eigenvalues are $+1$ and half are -1. In view of the definite chirality of ϵ_L, ϵ_R only the 16 solutions with positive eigenvalues or the 16 with negative eigenvalues are allowed (depending on whether we are in IIA or IIB), which says that a D-brane breaks half the supersymmetry. At the level of the supersymmetry-algebra augmented by central-charges an analogous reasoning shows that a D-brane causes a shortening of the supersymmetry-representation and thus represents a BPS-state. A Dp-brane couples to gravity and the Ramond-Ramond (RR) $(p+1)$-form potential. As a BPS-soliton its mass is proportional to its RR-charge. Since IIA offers only odd RR forms, whereas IIB only even ones, there exist in IIA only Dp-branes with p even and in IIB only with p odd.

The massless fluctuations of a single Dp-brane consist of a $(p+1)$-dimensional vector $A_\mu, \ \mu = 1, \ldots, p+1$ and $9-p$ scalars $A_m, \ m = p+2, \ldots, 10$ from the bosonic states $A_\mu b^\mu_{-\frac{1}{2}} |k\rangle$ and $A_m b^m_{-\frac{1}{2}} |k\rangle$ of the Neveu-Schwarz (NS) sector. The A_m describe the position coordinates of the Dp-brane in transverse space. Both bosonic states can be obtained by dimensionally reducing a 10-dimensional vector A_A to the $(p+1)$-dimensional world-volume of the Dp-brane. In addition one gets a world-volume gaugino from the R-sector, which likewise is the reduction of the 10-dimensional gaugino. In total we obtain the reduction of a 10-dimensional U(1) Super-Maxwell theory to $p+1$ dimensions. This theory has 16 conserved supercharges, which is in accord with the fact that the D-brane breaks half of the original supersymmetry.

It is natural to ask what happens when we have more than one parallel Dp-brane. Consider first the case where we have N Dp-branes at the same point in transverse space. The only difference to the previous case is that now we have to include a Chan-Paton label $i = 1, \ldots, N$ at the end-points of the open string. This then leads to N^2 massless vector states $A^{ij}_\mu b^\mu_{-\frac{1}{2}} |k; i, j\rangle$. In the same manner as before we find that the massless fluctuations are described by the dimensional reduction of 10-dimensional $\mathcal{N} = 1$ U(N) Super Yang-Mills (SYM) theory to the world-volume of the Dp-brane (for oriented open strings). The U(1) factor of

$$\mathrm{U}(N) = \frac{\mathrm{U}(1) \times \mathrm{SU}(N)}{\mathbb{Z}_N} \tag{2.2}$$

[1] For an anti D-brane the relation would have an extra minus-sign, $\epsilon_L = -\Gamma_D \epsilon_R$.

describes the overall center of mass motion of the system. If we take one of the Dp-branes and separate it from the rest by a distance l, the former massless modes of the open strings stretching between it and the $N-1$ branes acquire a mass given by

$$M_{\text{open}} = lT \ , \tag{2.3}$$

where $T = 1/2\pi\alpha'$ is the string-tension. The remaining massless vector states transform under a gauge-group $\mathrm{U}(1) \times \mathrm{U}(N-1)$. In terms of the effective field theory on the branes this can be understood as an ordinary Higgs-effect. If all N Dp-branes occupy different positions, the gauge-group will be broken to $\mathrm{U}(1)^N$.

The scalars A_m that described the position of the single D-brane, now – in the case of N D-branes – become $\mathrm{U}(N)$ matrices [28]. The potential-term in the effective $\mathrm{U}(N)$ SYM (obtained by the mentioned dimensional reduction from 10-dimensional $\mathrm{U}(N)$ SYM) is given by $[A_m, A_n]^2$. At the minimum of the potential the matrices A_m are commuting, which means that they can be diagonalized simultaneously. Their eigenvalues are interpreted as the coordinates of the N Dp-branes [28].

2.2 String-Dualities

In order to make contact with our observable 4-dimensional world, the traditional way has been to compactify the 10-dimensional string-theories on compact 6-dimensional manifolds[2]. Matching the 4-dimensional values of the GUT coupling constant $\alpha_{\text{GUT}} \simeq 1/25$ and Newton's constant G_N requires a tiny radius[3] for the compactification manifold (which is assumed generically to be more or less isotropic) between the GUT- and the Planck-scale, $1/M_{\text{GUT}} - 1/M_{Pl}$. If in addition one imposes $\mathcal{N}=1$ supersymmetry in four dimensions,

[2]In [29] it has been shown that a 4-dimensional Newton-law can also originate from a domain-wall embedded in a 5-dimensional Anti-de Sitter spacetime with infinite fifth dimension. Thus an adequate non-compact geometry capable of trapping gravity to a submanifold may be an alternative to the traditional idea by Kaluza and Klein of having small compact extra dimensions.

[3]A loophole in this argumentation consists of an extremely anisotropic compact manifold with n very large internal dimensions [30]. The observed 4-dimensional Planck-scale M_{Pl} is related to the fundamental 10-dimensional string-scale $M_s = 1/\sqrt{\alpha'}$ through $M_{Pl}^2 = M_s^{2+n} V_n$, where V_n is the volume related to the n large dimensions. Moreover, in order to nullify the strong hierarchy problem it has been suggested to set M_s equal to the TeV-scale [30]. This leads to an average large radius $V_n^{1/n} \simeq 10^{33/n}/1\text{TeV}$, which for $n=2$ is in the range of 1mm. Current gravitational experiments allow for $n \geq 2$. However, a serious backdraw is that the new compactification scale $1\text{TeV} \times 10^{-33/n}$ is very much smaller than the proposed fundamental string scale $M_s = 1\text{TeV}$. Hence this "solution" to the strong hierarchy problem is better be regarded as a reformulation of the same.

then the effective supergravity[4] Killing-spinor equations with constant dilaton ϕ and vanishing Neveu-Schwarz Neveu-Schwarz (NSNS) 3-form H^{NS} require a compact Kähler-manifold of SU(3) holonomy which abmits precisely one covariantly constant spinor of definite chirality [31]. Such a manifold is known as a Calabi-Yau threefold K_6 and can alternatively be characterized as a compact Kähler-manifold with vanishing first Chern-class $c_1(K_6) = 0$. The latter property means that K_6 admits a Ricci-flat metric.

Let us in the following deal with toroidal compactifications, whihc means that the internal manifold consists of an n-torus T^n. Due to the triviality of the holonomy-group of T^n such compactifications do not break any supersymmetry at all. They lead to 4-dimensional vacua ($n = 6$) with $\mathcal{N} = 8$ resp. $\mathcal{N} = 4$ supersymmetry if we compactify a type II resp. type I string-theory. This unrealistic feature can however be remedied by dividing out further discrete subgroups of the T^n which leads to the notion of an orbifold compactification. The great advantage of these orbifold compactifications is that in essence an understanding of string-theory on a flat background (up to global and discrete identifications plus the addition of twisted sectors) is sufficient for their analysis.

The simplest toroidal compactification is the compactification of one coordinate X^{10} on a circle S^1 with radius R. The momentum along X^{10} is then quantized, $p = \frac{n}{R};\ n \in \mathbb{Z}$. Furthermore, the closed string can wind around the circle before closing, so there are different topological sectors labeled by the winding number M. Therefore, in the sector with winding number m the closed string BC gets generalized to

$$X^{10}(\tau, \sigma + 2\pi) = X^{10}(\tau, \sigma) + 2\pi R m \ . \tag{2.4}$$

Adopting the conformal gauge [2] for the world-sheet metric, the equation of motion for the bosonic embedding fields X^A reduces to a simple free wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^A = 0$. As usual the full solution is given by a superposition of left- X_L^A and right-movers X_R^A. Imposing the above BC then gives

$$X_L^{10}(\sigma^+) = x_L^{10} + \frac{\alpha'}{2}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{10}}{n}e^{-in\sigma^+} \tag{2.5}$$

$$X_R^{10}(\sigma^-) = x_R^{10} + \frac{\alpha'}{2}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^{10}}{n}e^{-in\sigma^-} \ , \tag{2.6}$$

with $\sigma^+ = \tau + \sigma,\ \sigma^- = \tau - \sigma$, the center-of-mass position $x_L^{10} + x_R^{10}$ and the oscillator excitations $\alpha_n^{10}, \tilde{\alpha}_n^{10}$. Via the mass-shell and level-matching condition on physical states

[4]It can be shown that the results are valid to all finite orders in α' [2].

one obtains the spectrum

$$M^2 = \frac{1}{2}\left(\frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2}\right) + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) \ , \tag{2.7}$$

where N and $\tilde{N}$ represent the number operators for right- and left-moving oscillator excitations. In the limit of large radius R the winding modes decouple and the Kaluza-Klein (KK) modes build a continuum of states. This behaviour is reversed in the $R \to 0$ limit, where the winding modes give a continuum and the KK-states with non-vanishing momenta decouple. Thus in the small circle limit, a closed string will realize the opening up of a compact dimension due to the winding states. This sharp deviation from the way a usual quantum field would see geometry is due to the presence of the winding modes in closed string theory. T-Duality describes this symmetry by inverting the compactification radius

$$R \to \tilde{R} = \frac{\alpha'}{R} \tag{2.8}$$

and simultaneously interchanging KK and winding modes $n \leftrightarrow m$. Under this transformation the mass spectrum remains invariant $M(n, m, R) = M(m, n, \tilde{R})$.

At the field level T-duality amounts to a one-sided parity transformation[5]

$$X_L^{10} \to X_L^{10} \ , \qquad X_R^{10} \to -X_R^{10} \ . \tag{2.9}$$

To respect worldsheet supersymmetry, T-duality has to transform the worldsheet fermions as well

$$\psi_L^{10} \to \psi_L^{10} \ , \qquad \psi_R^{10} \to -\psi_R^{10} \ . \tag{2.10}$$

In particular, the zero mode of ψ_R^{10}, which acts as Γ^{10} on the right movers, changes its sign. This means that the relative chirality between the left- and right-movers is flipped. Therefore, T-duality maps the IIA superstring on S^1 with radius R to the IIB superstring on an S^1 with radius α'/R.

It is important that T-duality is not restricted to free string-theories but can also be shown to be a duality of the interacting theories [32]. Such a proof is possible because

[5] In order to have the spacetime interpretation of T-duality, we also transform

$$\begin{aligned} x_L^{10} &\to x_L^{10} \ , \qquad & x_R^{10} &\to -x_R^{10} \\ \alpha_n^{10} &\to \alpha_n^{10} \ , \qquad & \tilde{\alpha}_n^{10} &\to -\tilde{\alpha}_n^{10} \end{aligned}$$

under T-duality.

T-duality is a perturbative (with respect to the string coupling but not the sigma-model coupling) duality, wherefore it can be established at the level of vertex-operators. In general the non-linear sigma-model which defines a string-theory, contains the following NSNS sector background G_{AB}, B_{AB}, ϕ. If the spacetime background G_{AB} has an isometry, then a more general T-duality can be defined along the isometry-direction. The transformation is given by the Buscher-rules of which the above radius inversion is just a simple case with $B_{AB} = \phi = 0$ and X^{10} as the isometry-direction. This will be treated further in section 6.

For the open string with NN-boundary conditions left and right movers are reflected at the ends and form standing waves $X^{10}(\sigma,\tau) = X_R^{10}(\sigma^-) + X_L^{10}(\sigma^+)$ with

$$X_L^{10}(\sigma^+) = \frac{x^{10}+c}{2} + \alpha' p^A \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^{10}}{n} e^{-in\sigma^+} \tag{2.11}$$

$$X_R^{10}(\sigma^-) = \frac{x^{10}-c}{2} + \alpha' p^A \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^{10}}{n} e^{-in\sigma^-} \ , \tag{2.12}$$

where c is some constant. For compact X^{10}, we have the Kaluza-Klein modes $p^{10} = \frac{n}{R}$ but for open strings there are no winding modes. Therefore the exchange of winding and KK-modes cannot serve as a definition of T-Duality in this case. Instead, T-Duality is defined by the one-sided space-time parity operation (2.9). At the boundaries of the worldsheet $\sigma = 0, \pi$ the oscillator terms for the T-dual variable $\tilde{X}^{10} \equiv X_L^{10} - X_R^{10}$ vanish and we end up with

$$\tilde{X}^{10}(\tau, 0) = c = \tilde{X}^{10}(\tau,\pi) \mod 2\pi\tilde{R} \, . \tag{2.13}$$

Hence, by T-Duality the ends of the open string got fixed at the constant value c modulo the circumference of the dual circle. Due to this fixing it becomes meaningful to consider a winding of the T-dual open string. Indeed, T-Duality has transformed the KK-modes into winding modes of the dual geometry. This switch from NN- to DD-boundary conditions alternatively enforces the introduction of a hypersurface on which the open strings in the T-dual picture end. These are the above described D-branes, which are required by T-duality.

Apart from T-duality there is an important non-perturbative strong-weak coupling duality, called S-duality, whose existence was first conjectured for the heterotic string compactified on T^6 [34]. Later it was realized that also in the IIB superstring the weakly coupling region gets exchanged with the strongly coupled region under S-duality [35].

Combining the IIB dilaton ϕ and axion a into one complex scalar $\lambda = a + ie^{-\phi}$ one verifies that the field equations of IIB supergravity are invariant under the $SL(2,\mathbb{R})$ transformation

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ SL(2,\mathbb{R}) \tag{2.14}$$

together with a mixing of the IIB 2-forms B_{AB} and $A^{(2)}_{AB}$ of the NSNS and RR sector

$$\begin{pmatrix} B_{AB} \\ A^{(2)}_{AB} \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_{AB} \\ A^{(2)}_{AB} \end{pmatrix} \ . \tag{2.15}$$

The metric (in Einstein-frame) and the self-dual RR 4-form potential $D^{(4)+}$ are left invariant. It has been conjectured [25] that in the full string-theory the classical $SL(2,\mathbb{R})$ S-duality group is broken down to its maximal discrete subgroup $SL(2,\mathbb{Z})$ due to quantum effects. Setting the axion background to zero and limiting ourselves to the $\mathbb{Z}_2$ subgroup which sends $\lambda \rightarrow -1/\lambda$, we obtain the following S-duality transformation (in string-frame[6])

$$\begin{aligned} g_s &\rightarrow \frac{1}{g_s} \ , \qquad & \alpha' &\rightarrow \alpha' g_s \\ g^{\sigma}_{AB} &\rightarrow \frac{1}{g_s} g^{\sigma}_{AB} \ , \qquad & B_{AB} &\leftrightarrow A^{(2)}_{AB} \ , \end{aligned} \tag{2.16}$$

where $g_s = e^{\phi}$ is the string coupling. It clearly shows the exchange of weak and strong coupling and tells us that under (2.16) the fundamental string gets exchanged with a D1-brane, a NS 5-brane with a D5-brane, while the D3-brane is invariant. The combination of T- and S-duality yields an even bigger duality group called U-duality.

2.3 Eleven-Dimensional M-Theory

The highest spacetime dimension, in which supersymmetry can exist is eleven [36]. This is due to the fact that dimensions higher than eleven would imply fermionic fields with spin 5/2, of which no fundamental renormalizable action is known. Indeed an $\mathcal{N} = 1$, D=11 supergravity had been constructed [67] out of the metric G_{MN} ; $M, N = 1, \ldots, 11$ and the 3-form potential C_{MNP} as the bosonic degrees of freedom and the gravitino Ψ_M as the fermionic partner. Together, they constitute the supergravity multiplet. This

[6] The metric g^{σ}_{AB} in string-frame and the metric g^{E}_{AB} in Einstein-frame are related by a Weyl-rescaling $g^{E}_{AB} = e^{-\phi/2} g^{\sigma}_{AB}$.

theory is unique and scale-invariant at the classical level, which means, it does not have any free parameters. Hence it seems that 11-dimensional $\mathcal{N}=1$ supergravity cannot be obtained from an even higher-dimensional compactified theory, since the compactification manifold would presumably break scale-invariance and introduce further parameters into the theory. The bosonic part of the action is given by

$$S = -\frac{1}{2\kappa^2}\int_{M^{11}} d^{11}x\sqrt{-g}\big(R + \frac{1}{4!}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{1728}\epsilon^{I_1\dots I_{11}}C_{I_1I_2I_3}G_{I_4I_5I_6I_7}G_{I_8I_9I_{10}I_{11}}\big)\ , \tag{2.17}$$

and includes the topological Chern-Simons term $C\wedge G\wedge G$, whose precise origin is still enigmatic (technically it shows up if one performs the Noether procedure on the supergravity multiplet).

A Kaluza-Klein dimensional reduction of D=11 supergravity down to ten dimensions is achieved by compactifying the eleventh coordinate on a circle and decomposing the fields according to the preserved 10-dimensional Lorentz-group SO(1,9)

$$G_{MN} \to \left\{ \begin{array}{c} G_{11,11} = e^{\phi} \\ G_{11,A} = A^{(1)}_A \\ G_{AB} = g_{AB} \end{array} \right. , \tag{2.18}$$

$$C_{MNP} \to \left\{ \begin{array}{c} C_{ABC} = A^{(3)}_{ABC} \\ C_{AB11} = B_{AB} \end{array} \right. . \tag{2.19}$$

In ten dimensions there are only two supergravities with the right amount of supersymmetry to which D=11 supergravity can descend, IIA and IIB. As 11-dimensional supergravity is non-chiral, it must result upon reduction to non-chiral type IIA supergravity. Indeed (ϕ, g_{AB}, B_{AB}) and $(A^{(1)}_A, A^{(3)}_{ABC})$ give the bosonic fields of the universal supergravity-multiplet and the two RR forms of 10-dimensional $\mathcal{N}=2$ IIA supergravity. A comparison of the low-energy effective action of IIA string-theory in string-frame with D=11 supergravity reduced on an S^1 with radius r_{11} leads to the relation [26]

$$r_{11} = g_s^{2/3}\ . \tag{2.20}$$

The geometrization of the string coupling constant implies, that the string coupling can alternatively be thought of as a measure for the size of the eleventh dimension. The M-theory conjecture states that for $r_{11}\to 0$, the full M-theory reduces to weakly-coupled type IIA string-theory, whereas in the extreme strong coupling limit, $r_{11}\to\infty$, M-theory is an 11-dimensional Lorentz-invariant quantum theory with $\mathcal{N}=1$, D=11 supergravity as its low-energy limit. Hence, by definition, M-theory has to incorporate all the non-perturbative effects of IIA string-theory.

Indeed, the concept of M-theory has been subjected to several successful tests, so far. For example, all the D-branes in Type IIA do emerge from dimensional reduction and wrapping of M2- and M5-branes of M-theory which can be constructed explicitly as D=11 supergravity solitons and seem to be the only M-branes[7]. More precisely, the IIA D0-branes appear as the Kaluza-Klein states with momenta n/R_{11} as measured in the M-theory (Einstein) frame. In the large R_{11} limit, this evenly spaced tower of states forms a continuum, which is a characteristic of the appearance of an additional dimension. The fundamental IIA strings, the F1-branes, are thought of as wrapped M2-branes. The D2-branes are M2-branes transverse to the eleventh dimension x^{11}, the D4-branes are M5-branes wrapped on x^{11}, whereas the (symmetric) NS5-branes can be considered as M5-branes transverse to x^{11}. The D6-branes are Kaluza-Klein monopoles, since they are the magnetic duals of the D0-branes. The D8-brane origin is still a bit mysterious, since there is no simple description of an M8- or M9-brane available, from which the D8-brane could descend. There are however proposals of supergravity M9-brane solutions, which are defined on intervals with different cosmological constants [37]. On the IIA side the trouble with the D8-brane stems from the fact, that being of codimension one, it causes the dilaton to diverge within a finite distance [38].

In the previous section we discussed string-dualities. Via T-duality IIA is related to IIB and furthermore the heterotic $E_8 \times E_8$ is related to the heterotic SO(32) string[8] In addition S-duality relates the heterotic SO(32) to the SO(32) type I string and exchanges the weakly and the strongly coupled region of IIB. If furthermore, we believe that M-theory on S^1 is related to strongly-coupled IIA, while M-theory on $S^1/\mathbb{Z}_2$ is related to the strongly-coupled heterotic $E_8 \times E_8$ string (see the coming section), then we arrive at a complete network connecting all five perturbatively defined superstring theories with each other. Moreover, it includes all strongly-coupled counterparts as well.

[7] A conjectured M9-brane cannot be defined smoothly on flat $\mathbb{R}^{11}$. It needs at least one compact direction with an isometry and can then only be defined sectionwise together with a jumping D=11 cosmological constant [37].

[8] Indeed below ten dimensions the $E_8 \times E_8$ heterotic string is equivalent to the $SO(32)$ heterotic [39],[40] and IIA is equivalent to type IIB [41],[42].

2.4 Heterotic M-Theory

Before, we discussed the strong coupling behaviour of type IIA theory in ten dimensions and argued that, in the long-wavelength regime, it leads to 11-dimensional supergravity on $\mathbb{R}^{1,9} \times S^1$, where the radius of the S^1 grows with the string-coupling. An inspection of the complete action of 11-dimensional supergravity, shows that it is invariant under an odd number of space-time parity transformations accompanied by a sign change of the 3-form potential C. Therefore it makes sense to wonder about the gauging of this discrete $\mathbb{Z}_2$ symmetry. M-theory on $\mathbb{R}^{1,9} \times S^1$ together with a specific $\mathbb{Z}_2$ operation acting on the S^1 by $x^{11} \to -x^{11}$ has been analyzed in [43]. Requiring the metric on $\mathbb{R}^{1,9} \times S^1$ to be $\mathbb{Z}_2$ symmetric is equivalent to considering an arbitrary metric on the orbifold $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$, where the interval $S^1/\mathbb{Z}_2$ comprises only half of the original circle. This latter approach has been termed the downstairs picture, whereas the alternative choice of working on a smooth manifold goes under the name of the upstairs approach.

The 11-dimensional supersymmetry of M-theory on $\mathbb{R}^{1,9} \times S^1$ is generated by an arbitrary constant 32-component real Majorana-spinor η. Demanding $\mathbb{Z}_2$ invariance for η translates into the requirement $\Gamma^{11}\eta = \eta$. Noting that in eleven dimensions the chirality-matrix is given by $\Gamma^1\Gamma^2 \ldots \Gamma^{10} = \Gamma^{11}$, we recognize that from a 10-dimensional point of view η has to be chiral, which reduces the number of supercharges from 32 to 16. Thus, if M-theory on $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ corresponds to some known string-theory, there are only three candidates which exhibit $\mathcal{N} = 1$ supersymmetry in ten dimensions. These are the $E_8 \times E_8$ heterotic string, the $SO(32)$ heterotic string and finally the type I theory with likewise gauge group $SO(32)$. Initially in [43] three arguments were given in favour of the $E_8 \times E_8$ string, which were based on 10-dimensional spacetime gravitational anomalies, the strong coupling behaviour and ultimately world-sheet gravitational anomalies.

Gravitational anomalies require chiral representations of the Lorentz-group $SO(1, D-1)$ and show up only in spacetimes of dimension $D = 4n + 2\,,\, n \in \mathbb{N}$ [44]. Consequently they are absent on a smooth 11-dimensional manifold, but may exist on codimension one fixed points of an orbifold. On the particular $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ orbifold the 10-dimensional massless chiral gravitino is responsible for the gravitational spacetime anomaly, upon integrating it out to obtain the effective action. Such an anomaly signals the break-down of diffeomorphism invariance of the effective action. By the symmetry of the set-up, the anomaly-expressions for the two 10-dimensional fixed-planes must have the same functional behaviour on the induced metric. Together, they must reproduce the standard anomaly of ten-dimensional supergravity. It is essential for finally singling out the $E_8 \times E_8$

heterotic string among the three string-theories to which M-theory on $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ may correspond to, that the anomaly-polynomial can be described by a 12-form which is composed out of a reducible part containing $(\mathrm{tr}R^2)^3, \mathrm{tr}R^2 \cdot \mathrm{tr}R^4$ plus an irreducible part consisting merely of $\mathrm{tr}R^6$. The reducible part allows for a cancellation through a Green-Schwarz mechanism [45], where the Green-Schwarz term is provided by the following M-theoretic one-loop interaction[9] [53],[55],[56]

$$\int_{M^{11}} C \wedge X_8(R) \ , \qquad X_8(R) = -\frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2 \ . \tag{2.21}$$

The irreducible part, however, cannot be cancelled by such a mechanism, since this presupposes a factorization of the anomaly-polynomial. In order to cancel the $\mathrm{tr}R^6$ term, one has to place 496 vector-supermultiplets at the boundaries (the fixed-points in the upstairs picture) and use the resulting gauge anomaly due to the chiral gauginos for compensation. On account of the symmetry of the set-up the 496 gauge multiplets must be equally divided between both boundaries. Since 248 gauge-fields in the adjoint representation constitute a single E_8 Super Yang-Mills (SYM) theory, we end up with a total $E_8 \times E_8$ gauge-symmetry. Thus M-theory on $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ describes the strongly coupled version of the hitherto only perturbatively known heterotic $E_8 \times E_8$ string and is therefore often termed heterotic M-theory.

Similar to the IIA – M-theory case, a comparison of the string-frame effective $E_8 \times E_8$ heterotic string action with that of bulk heterotic M-theory (which is simply 11-dimensional supergravity, since the boundary action is immaterial for this purpose) yields

$$d = \pi g_s^{2/3} \ , \tag{2.22}$$

where d denotes the distance between both boundary fixed-planes.

To construct the supergravity action [46], which is supposed to cover the long-wavelength regime of heterotic M-theory, one has to couple 11-dimensional $\mathcal{N} = 1$ supergravity to $\mathcal{N} = 1$ E_8 SYM on a 10-dimensional boundary in a way preserving 10-dimensional

[9] If one starts in IIA, an interaction $\int_{M^{10}} B \wedge X_8(R)$ can be computed as a 1-loop effect [53]. It can have no higher order corrections as $\int_{M^{10}} F(\phi) B \wedge X_8(R)$ is invariant under the gauge-transformation $B \to B + d\Lambda$ if and only if $F(\phi) = \mathrm{const}$, where ϕ is the IIA dilaton. To make this argument, one has to remember that $dX_8(R) = 0$. This is true as $X_8(R)$ can be expressed as a linear combination of the first two Pontryagin classes $p_1(R), p_2(R)$ which are closed due to a theorem of Chern-Weil [54] and the fact that $p_1(R), p_2(R)$ are defined by means of invariant polynomials. As the 1-loop interaction has no dependence on the dilaton, one can go to infinite coupling, i.e. to M^{11} and infer the M-theory interaction $\int_{M^{11}} C \wedge X_8(R)$.

$\mathcal{N}=1$ supersymmetry. Performing the Noether-procedure as has been done in [46], one sees that the coupling of the SYM multiplet to the supergravity multiplet is uniquely determined by the requirement of local supersymmetry. Much as in the analogous case of the coupling of SYM to pure supergravity in ten dimensions [57],[58] it turns out that the 3-form C has to transform non-trivially under the gauge-transformations[10]

$$\delta_G C_{11AB} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2}\delta(x^{11})\mathrm{tr}[\epsilon F_{AB}]\ . \tag{2.23}$$

Here, λ is the gauge-coupling, the trace is over the Lie-Algebra index $a=1,\ldots,248$ of the adjoint representation of E_8 and ϵ is an infinitesimal gauge-parameter. Furthermore, the Bianchi-identity of the 4-form field-strength G becomes modified by the presence of the SYM on the boundary, which acts as a magnetic source

$$dG_{11ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})\left(\mathrm{tr}F_{[AB}F_{CD]} - \frac{1}{2}\mathrm{tr}R_{[AB}R_{CD]}\right)\ . \tag{2.24}$$

Consequently, the 4-form G must be discontinuous at $x^{11}=0$

$$G_{ABCD} = -\frac{3}{\sqrt{2}}\frac{\kappa^2}{\lambda^2}\mathrm{sign}(x^{11})\left(\mathrm{tr}F_{[AB}F_{CD]} - \frac{1}{2}\mathrm{tr}R_{[AB}R_{CD]}\right)\ . \tag{2.25}$$

The important point is that (2.23) together with (2.25) allow for an implementation of the Green-Schwarz mechanism to cancel the occuring gauge-anomalies. The appropriate Green-Schwarz term is in this case delivered by the supergravity Chern-Simons interaction $S_{CS}=\int_{M^{11}} C\wedge G\wedge G$. Using (2.23) and (2.25) its variation under gauge-transformations of C becomes

$$\delta_G S_{CS} \propto -\frac{\kappa^4}{\lambda^6}\int_{M^{10}} \mathrm{tr}[\epsilon F]\wedge \mathrm{tr}F^2\wedge\mathrm{tr}F^2\ . \tag{2.26}$$

Since this term cannot be cancelled by any known term at the classical level, heterotic M-theory cannot be gauge-invariant at the classical level. However, at quantum level the 10-dimensional Majorana-Weyl gauginos on the boundary render the effective action Γ anomalous

$$\delta_G\Gamma \propto \int_{M^{10}} \mathrm{tr}[\epsilon F\wedge F^4]\ . \tag{2.27}$$

It is now the unique feature of E_8 to exhibit the factorization $\mathrm{tr}[\epsilon F\wedge F^4]\propto \mathrm{tr}[\epsilon F]\wedge\mathrm{tr}F^2\wedge \mathrm{tr}F^2$, which enables a cancellation of the gauge anomaly at the quantum level. The

[10]Subsequently, we will concentrate on the behaviour of the boundary at $x^{11}=0$. The relations for the second boundary at $x^{11}=d$ can be obtained simply by substituting $\delta(x^{11})\to\delta(x^{11}-d)$.

intriguing consequence of requiring (2.26) and (2.27) to add up to zero, is a relationship[11] between the gauge-coupling λ and the gravitational coupling κ

$$\lambda^2 = 4\pi(4\pi\kappa^2)^{2/3} . \tag{2.28}$$

We want to conclude this overview of heterotic M-theory with two remarks. First, we have seen a decisive difference concerning the gauge-anomaly cancellation as compared to the case of perturbative string-theory. Since in perturbative string-theory both the anomaly and the Green-Schwarz terms are generated at the quantum 1-loop level, the theory is either calssically gauge-invariant (anomaly plus Green-Schwarz terms do not show up) or quantum mechanically gauge-invariant (both sorts of terms arise and cancel each other). Compared with that, heterotic M-theory only possesses full gauge-invariance at the quantum level since the Green-Schwarz terms are already classically present, whereas the anomaly arises only quantum mechanically. Second, the actual construction of the low-energy supergravity action (often termed Hořava-Witten supergravity) [46] is arranged in terms of powers of $\kappa^{2/3}$. At the first non-trivial order the bulk-boundary interaction can be determined smoothly invoking a lengthy Noether-procedure. However, at the $\kappa^{4/3}$ second order level formal $\delta(0)$ infinities appear in the Lagrangean. Their origin can be traced back to the zero thickness of the fixed-planes in the orbifold direction. Because the only length-scale of the theory is given by $\kappa^{2/9}$, it has been speculated [46] that actually the fixed-planes should be smoothed out with a thickness of order $\kappa^{2/9}$. The precise incorporation of this smearing out, though, presents a formidable theoretical problem. Namely, one would have to regard the former 10-dimensional boundary gauge-theories as actually 11-dimensional. However, in eleven dimensions there is no supersymmetric gauge-theory. There is only one unique supersymmetric theory, which is the $\mathcal{N} = 1$ supergravity. But without the concept of supersymmetry we would loose the guiding symmetry which enabled via the Noether-procedure the coupling of SYM to supergravity.

2.5 Twelve-Dimensional F-Theory

Even though the notion of M-theory has led to many insights into string dualities, the $SL(2,\mathbb{Z})$ invariance of type IIB in ten dimensions does not arise in a natural way. One first has to compactify down to nine dimensions and compare with the T^2 compactification

[11] Originally, the relation appeared as $\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}$ in [46]. We employ a further factor of 2, which was found in [68].

of M-theory to arrive at the interpretation of the IIB $SL(2,\mathbb{Z})$ symmetry as a torus-symmetry. It is in the limit where the area of the torus shrinks to zero, that we attain IIB in ten dimensions. However, the zero area limit is singular and therefore in this limit the effective description of M-theory breaks down.

Twelve-dimensional F-theory [47] addresses this question and represents a dual formulation of IIB in the same spirit as M-theory is dual to type IIA. F-theory describes compactifications of type IIB string-theory, in which the expectation values of the dilaton and axion fields are allowed to vary non-trivially over the compactification manifold. Compactifications of F-theory down to four dimensions are specified by means of a Calabi-Yau four-fold K_8 that admits an elliptic fibration with a section. An elliptic fibration simply means that there is a holomorphic projection $\pi : K_8 \to B$, whose fibre is an elliptic curve. Here, the base-manifold B is a complex three-fold, which generically has positive first Chern-class c_1 [51] and therefore cannot be a Ricci-flat Calabi-Yau manifold again. In other words, the 8-dimensional compact manifold K_8 locally looks like a product of B times a two-torus T^2. The two-torus will be taken to shrink to zero size, which means that its Kähler class modulus gets frozen. However, its complex structure – represented by the modulus τ – or in other words its shape will change by moving along the base B. To conclude, F-theory is the postulate of a 12-dimensional theory, which when compactified on an elliptically-fibred manifold with base B, is equivalent to type IIB compactified on B. The $SL(2,\mathbb{Z})$ symmetry of IIB is then geometrized as the symmetry of the extra torus residing in the eleventh and twelve dimension. Due to the special geometric properties of the Calabi-Yau four-fold K_8, namely the vanishing of its first Chern class and $SU(4)$ holonomy, the associated IIB background by construction will preserve 4-dimensional $\mathcal{N} = 1$ supersymmetry, at least at the classical and perturbative level.

Though, at first sight to invoke twelve dimensions seems auxiliary, there is another hint at why one should take twelve dimensions more serious. Strong-weak duality of IIB implies that the dual of the fundamental IIB string which couples to the NSNS B-field is a D1-brane which couples to the 2-form RR-potential $A^{(2)}$. On the D1-worldsheet there exists a U(1) gauge field, which requires the quantization of $\mathcal{N} = 1$ worldsheet supergravity together with $\mathcal{N} = 1$ super-Maxwell theory. The important observation is, that due to the U(1) field one has to introduce additional ghosts which shift the central charge by -2 [48]. This shifts the critical dimension of the theory to 10+2=12. Moreover, the introduction of further ghosts entails that the signature of spacetime changes to (10,2).

The modulus τ of the elliptic fibration describes in the type IIB theory the variation of the dilaton ϕ and axion field a along the 6-dimensional base B via the identification

$$\tau = a + ie^{-\phi} \,. \tag{2.29}$$

A key feature of F-theory is that this modulus in general has non-trivial monodromies around 4-dimensional submanifolds inside B. These submanifolds are associated with the locations of D7-branes, which span in addition the uncompactified four spacetime directions. Moving around one of the D7-branes, the modulus field τ picks up an $SL(2,\mathbb{Z})$ monodromy

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \tag{2.30}$$

which leaves the shape of the two-torus fibre invariant, but through (2.29) leads to a non-trivial duality-transformation of the IIB string-theory. Hence, the dilaton and axion field cannot be smooth single-valued functions, but instead are multi-valued with branch-cut singularities at the locations of the D7-branes. The full non-perturbative string-theory, however, is supposed to get rid of such singularities and be well-behaved everywhere. The string coupling constant of IIB is determined by the dilaton. The fact that ϕ undergoes monodromies in F-theory compactifications means that these compactifications also cover the strongly coupled IIB and therefore are intrinsically non-perturbative.

The aforementioned 4-dimensional submanifolds which the D7-branes wrap around, are actually singular loci, since here the elliptic fibre degenerates, i.e. one of its 1-cycles shrinks to zero size. This can be seen from the behaviour of the modulus in the vicinity of such a locus (situated at $z = z_i \in B$, where i labels the different loci)

$$\tau \simeq \frac{1}{2\pi i} \ln(z - z_i) \,. \tag{2.31}$$

Clearly τ becomes singular if $z \to z_i$. These singular loci have in general different geometric features than those arising from a traditional compactification on a Calabi-Yau manifold. For example, enhanced gauge symmetry is obtained when several D7-branes coincide. Exactly what gauge group occurs is determined by the type of singularity [49].

The relation between F- and M-theory can be understood by compactifying F-theory on a further circle S^1. As pointed above, F-theory on $K_8 \times S^1$ gives IIB on $B \times S^1$, which itself is equivalent to M-theory on $B \times T^2$. This gives a one-to-one map between the elliptic fibre inside K_8 and the T^2 of the M-theory compactification, which can be used to identify $B \times T^2$ with K_8. Hence, F-theory on $K_8 \times S^1$ is equivalent to M-theory on K_8.

2.6 Warped Compactifications of M- and F-Theory

Vacua with four supercharges in four, three and two dimensions can be constructed by compactifying F-theory, M-theory and type IIA string-theory on a Calabi-Yau four-fold K_8. It is possible to connect these vacua to each other by circle compactifications from four to three to two dimensions. In addition they depend on further data specified by the 4-form flux and membrane charge in the case of M-theory. Among the novel features of these vacua is the need to cancel a tadpole anomaly, which is given by $\chi/24$, with χ the Euler-characteristic of the four-fold. If $\chi/24$ is integral, the anomaly can be cancelled by placing a sufficient number of spacetime-filling branes on points of the compactification manifold [51]. For type IIA the spacetime-filling branes must be strings, in M-theory they must be membranes, while in F-theory they have to be D3-branes.

Alternatively, there is another way to cancel the $\chi/24$ tadpole. In type IIA or M-theory it consists of introducing a background flux for the 4-form field strength G [60]. The G-flux contributes to the membrane tadpole in M-theory through the Chern-Simons interaction $\int d^{11}x\, C \wedge G \wedge G$. If $\chi/24$ is not integral, the G-flux is indeed required to obtain a consistent vacuum. In general, the anomaly can be cancelled by a combination of background flux and a number n of branes. The ensuing tadpole cancellation condition, which must be satisfied in type IIA or M-theory reads [50]

$$\frac{\chi}{24} = n + \frac{1}{8\pi^2}\int_{K_8} G \wedge G\,. \tag{2.32}$$

If the four-fold admits an elliptic fibration with base B, we can consider the limit in which the area of the elliptic fibre shrinks to zero. In this limit, M-theory on the four-fold goes over to type IIB compactified on B with a varying coupling constant τ, the modulus of the elliptic fibre. Such a 4-dimensional type IIB vacuum represents an $\mathcal{N} = 1$ F-theory compactification. The F-theory vacua have two sorts of background fluxes [61]. The first kind involves non-zero NS and RR 3-form field-strengths, H^{NS} and H^R. Their contribution to the D3-brane tadpole follows from the type IIB supergravity interaction $\int d^{10}x D^+ \wedge H^{NS} \wedge H^R$, where D^+ is the RR 4-form potential. The second sort of background flux requires some of the D7-brane gauge-fields to have non-zero instanton number [52]. These instantons give rise to a D3-brane tadpole through their coupling to the D7-brane world-volume via $\int_{D7} d^8x D^+ \wedge F \wedge F$, with F the field strength of the D7-brane gauge-field. Subsequently, we will regard the effect of turning on field-strength fluxes on the geometry for the M- and IIB-/F-theory case. The heterotic string and heterotic M-theory case will be dealt with in section 4.

2.6.1 M-Theory Backgrounds with G-Flux

Consider M-theory compactified down to three dimensions on an 8-dimensional Calabi-Yau manifold K_8 [60]. At leading order in a momentum/derivative-expansion, the M-theory effective action is given by 11-dimensional supergravity. With zero G-flux the supergravity equations of motion admit a product metric on $R^3 \times K_8$ as a solution because K_8 is Ricci-flat.

At next order in the derivative expansion, terms with eight derivatives show up, which are suppressed by six powers of the 11-dimensional Planck-scale. We have already mentioned the coupling

$$\int d^{11}x\, C \wedge X_8(R) \;, \qquad X_8 = \frac{1}{8\times 4!}\big(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\big) \;, \tag{2.33}$$

which contributes at this order[12]. It leads to a tadpole for C, which can be cancelled by means of another tadpole arising either from the Chern-Simons coupling $C \wedge G \wedge G$, by turning on a non-trivial G-flux or by including M2-brane sources [50]. The M2-branes are located at points y_i on K_8, such that their world-volume fills the whole external spacetime. If one wants to maintain supersymmetry, one has to require the vanishing of the gravitino-variation, $\delta\Psi = 0$ (the Killing-spinor equation), in the effective supergravity description of M-theory. Having turned on G-flux or included M2-brane sources, this can only be fulfilled, if the simple product metric gets modified to a warped metric

$$ds^2 = e^{-\phi(y)}\eta_{\mu\nu}dx^\mu dx^\nu + 2e^{\phi(y)/2}g_{a\bar{b}}(y)dy^a dy^{\bar{b}} \;;\; \mu,\nu = 1,2,3 \tag{2.34}$$

with $e^{-\phi(y)}$ the warp-factor and $g_{a\bar{b}}$ the Calabi-Yau metric. The warped internal space becomes deformed from the initial Calabi-Yau four-fold to a space which is only conformal to a Calabi-Yau manifold. In particular, the deformed space is in general no longer a Kähler-space. The only non-vanishing components of the G-flux, allowed by supersymmetry, are given by the (2,2)-form $G_{a\bar{b}c\bar{d}}$ and the external component $G_{\mu\nu\rho a}$. The former has to satisfy the constraint

$$g^{c\bar{d}}G_{a\bar{b}c\bar{d}} = 0 \;, \tag{2.35}$$

which can be restated as a self-duality condition $G = \star_{K_8} G$ on the initial Calabi-Yau, with the Hodge star operation taken with respect to $g_{a\bar{b}}$. Therefore, we see that this kind

[12] Notice however, that the full supersymmetric completion of this higher order interaction is still unknown [59]

of M-theory background requires an abelian C-field "instanton". Yet another equivalent formulation of the above constraint uses the Kähler-form $\omega = -ig_{a\bar{b}}dz^a \wedge dz^{\bar{b}}$ of the initial Calabi-Yau and states that the internal G has to be primitive with respect to ω, i.e. $G \wedge \omega = 0$. On the other hand the supersymmetry-restriction on $G_{\mu\nu\rho a}$ is such, that it is purely determined by the warp-factor[13]

$$G_{\mu\nu\rho a} = \epsilon_{\mu\nu\rho}\partial_a e^{-3\phi/2} \ , \tag{2.36}$$

and can be solved by $C_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}e^{-3\phi/2}$. Finally, the warp-factor itself is determined by the field equation for C, which can be rewritten as

$$\triangle(e^{3\phi/2}) = \star_{K_8} 4\pi^2 \left(X_8 - \frac{1}{8\pi^2} G \wedge G - \sum_{i=1}^{n} \delta^8(y - y_i) \right) . \tag{2.37}$$

The requirement for an absence of tadpoles posed on the field equation[14] for C, is reflected by the requirement that the integral over the Calabi-Yau manifold of the right-hand side of (2.37) has to vanish. Since the Euler-characteristic can be obtained from $\chi(K_8) = 24 \int_{K_8} X_8(R)$, we realize that (2.32) indeed represents the tadpole cancellation condition. Given that n must be positive, it implies the inequality

$$\frac{\chi(K_8)}{12} \geq \int_{K_8} \frac{G}{2\pi} \wedge \frac{G}{2\pi} \ . \tag{2.38}$$

Together with the self-duality condition this says that there are only finitely many choices for $G_{a\bar{b}c\bar{d}}$ that are compatible with unbroken supersymmetry. For negative $\chi(K_8)$ there are none at all.

[13]In general we will take $\epsilon_{\mu\nu\rho}$ as a tensor with respect to the unwarped metric. Therefore, in contrast to the Levi-Civita tensor-density, it involves an additional factor $\sqrt{|\det g_{\mu\nu}|}$. In the special case of a flat metric both notions coincide.

[14]Cancellation of tadpoles is necessary for obtaining stable vacua. Physically, nonzero tadpoles imply that the equations of motion of some massless fields are not satisfied. For example take the equation of motion and Bianchi-identity for a $(p+1)$-form field A_{p+1} with corresponding field-strength H_{p+2} in ten dimensions

$$d^\star H_{p+2} = {}^\star J_{p+1} \ , \qquad dH_{p+2} = {}^\star J_{7-p} \ .$$

J_{p+1} and J_{7-p} are the "electric" and "magnetic" sources. A tadpole would be present if the charge detected by $\int_{\Sigma^k} {}^\star J_{10-k}$ ($k = 9-p, 3+p$ in our case) would be non-vanishing for some compact surface Σ^k without boundary. This would clearly be inconsistent with the field equation and/or Bianchi-identity in view of Stokes theorem. Or in more common terms of electrodynamics: in a compact space the field lines have nowhere to go to and hence must end on an equal and opposite charge.

2.6.2 The Lift to F-Theory

If the Calabi-Yau four-fold K_8 is elliptically fibred, we can shrink the volume of the fibre T^2 to zero and thereby lift the 3-dimensional M-theory compactification to a 4-dimensional IIB compactification on $\mathbb{M}^4 \times B$ (F-theory compactification on K_8), where B is the base of K_8 [61]. M-theory on the fibre T^2 (in the following we will choose y^{10}, y^{11} for the fibre-coordinates) with area $A = \int dy^{10} dy^{11} \sqrt{g}$ is related to IIB, whose tenth coordinate z^{10} is compactified on a circle with radius[15] $1/A$. This means that the warp-factor $e^{\phi(y)/2}$ in the internal part of the M-theory warp-metric (2.34) causes a rescaling[16] $A \rightarrow e^{\phi(y)/2} A$. Hence the S^1 metric of the tenth coordinate z^{10} of the IIB compactification undergoes a rescaling $1/A^2 \rightarrow e^{-\phi(y)}/A^2$. Since in the small A limit z^{10} becomes decompactified, we see that it receives just the correct power of the warp-factor to combine with the three other external coordinates of (2.34) into a 4-dimensional Lorentz-invariant metric. The full warped metric (in the Einstein-frame) becomes

$$ds^2 = e^{-\phi(y)} \eta_{\mu\nu} dx^\mu dx^\nu + 2 e^{\phi(y)/2} g_{a\bar{b}}(y) dy^a dy^{\bar{b}} \ ; \ \mu, \nu = 1, 2, 3, 4 \ . \tag{2.39}$$

Next, we have to ask which kind of G-flux can be lifted to F-theory [61],[62]. We have seen above that in the M-theory case a non-vanishing $C_{\mu\nu\rho}$ appears. It simply lifts to the type IIB RR 4-form component[17] $D^+_{\mu\nu\rho\lambda}$, which respects 4-dimensional Lorentz-invariance if set proportional to $\epsilon_{\mu\nu\rho\lambda}$. Concerning the self-dual internal part of G, remember that K_8 is assumed elliptically fibred. Let $dz = dx + \tau dy$, $d\bar{z} = dx + \bar{\tau} dy$ be a 1-form basis on the fibre with $\tau = \tau_1 + i\tau_2$ the fibre-modulus, and $\Omega^{(2)}$ be the 2-form generating the 2-dimensional cohomology of the fibre. Then the internal G-flux decomposes as

$$G = \xi + \eta \wedge \Omega^{(2)} + \frac{\pi}{i\tau_2} (H \wedge d\bar{z} - \bar{H} \wedge dz) \ , \tag{2.40}$$

where ξ, η, H are respectively forms of degree 4,2,3 on the base B. The self-duality condition now excludes ξ and η and the left-over H and $\bar{H}$ are identified with the usual NSNS and RR 3-forms of IIB, H^{NS} and H^R, via [62]

$$H = H^R - \tau H^{NS} \ , \qquad \bar{H} = H^R - \bar{\tau} H^{NS} \ . \tag{2.41}$$

[15] In this discussion we neglect numerical factors and set $\kappa = 1, \alpha' = 1$.

[16] Note, that in the limit where the fibre T^2 shrinks to zero, ϕ depends only on $y \neq y^{10}, y^{11}$.

[17] Notice that self-duality is not enough to draw from the presence of $D^+_{\mu\nu\rho\lambda}$ the conclusion that the component $D^+_{a\bar{b}c\bar{d}}$ must also be present. The point is that it is not dD^+ which has to be self-dual but rather the combination $F^+ = dD^+ - \frac{1}{2} A^{(2)} \wedge H^{NS} + \frac{1}{2} B \wedge H^R$.

Finally, the tadpole cancellation condition becomes

$$\frac{\chi}{24} = n + \int_B \frac{H \wedge \bar{H}}{2i\tau_2} = n - \int_B H^{NS} \wedge H^R \, , \tag{2.42}$$

while the warp-factor is still determined by (2.37) with G now understood as

$$G = \frac{\pi}{i\tau_2}(H \wedge d\bar{z} - \bar{H} \wedge dz) \, . \tag{2.43}$$

3 Dynamics and the Stability of Heterotic M-Theory

With the discovery of M-theory on an $S^1/\mathbb{Z}_2$-orbifold [43] and its concrete low-energy realization as Hořava-Witten supergravity [46], i.e. D=11 supergravity coupled to two SYM theories with E_8 gauge group, living on two separate boundaries of spacetime, the vexing problem of predicting the correct magnitude for the D=4 Newton-constant G_N could be addressed anew. While the heterotic string theory predicts a value for G_N which is generically too large by a factor of 400, M-theory on $S^1/\mathbb{Z}_2$ could account for the correct value by adjusting an additional parameter, the distance d between the two boundaries, roughly at the inverse of the GUT-scale 10^{16} GeV [63]. Since in the limit $d \to 0$ of coinciding boundaries the string coupling turns out to be weak, it is believed that here we should recover the usual heterotic string theory with gauge group $E_8 \times E_8$. Since even more phenomenological virtues of heterotic M-theory were discovered, e.g. it avoids the problem of small gaugino masses [64], it is an interesting question to ask for the stability of its set-up.

Since the boundaries of the theory maintain an E_8 SYM theory, respectively, a non-vanishing energy-momentum tensor gets induced on each of them. Gravity couples to any energy-momentum tensor – therefore an interaction between the boundaries mediated by gravitons in the bulk is inevitable. This interaction should be attractive, as can be expected from classical gravity. Furthermore the D=11 supergravity bulk theory allows for gravitino and 3-form exchanges, which couple to the boundary fields as well, due to the underlying supersymmetry. Therefore we have to analyze for heterotic M-theory on the proposed [43, 46] $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ space-time, whether all these contributions can cancel each other, leading to a stable configuration or not. Since it is not known how to quantize D=11 supergravity consistently, we have to restrict ourselves to a classical tree-level analysis of the stability problem. Remembering the well-known derivation of the complete Coulomb or Newton potential from tree-level photon or graviton exchange diagrams, perturbation theory can be expected to be sufficient. If one could consistently (note that supergravity is non-renormalizable) work out the Casimir-energy at 1-loop level, then it is expected to vanish on account of the presence of supersymmetry in the bulk. Second it would anyway constitute only a small quantum correction of order $\hbar$ compared to the leading order tree-level result. Therefore, we should obtain the dominant contribution by a perturbative tree-level calculation.

Noting that the construction of Hořava-Witten supergravity has been achieved as an expansion in powers of small $\kappa^{2/3}$, where κ is the D=11 gravitational coupling constant, we are furthermore advised to examine the interactions between the two boundaries to leading order in κ and discard higher order contributions as subleading corrections.

One may be inclined to argue that the situation should be similar to the analogous case of an interaction between two D-branes of type II string theory (see [66] for a review). There the repulsion of the RR-field compensates exactly the attraction originating from graviton and dilaton exchange. However, in order to reach that conclusion we have to avail ourselves of the duality between the closed-string tree-level cylinder amplitude and the open-string 1-loop annulus diagram. Only through the latter is it possible to see the cancellation by appealing to Jacobi's *aequatio identica satis abstrusa*. This is in accord with the common lore that supersymmetry leads to cancellations between fermionic and bosonic loop-contributions (most prominently applied to the solution of the weak hierarchy problem). In contrast to the type II case, heterotic M-theory has been formulated only as a classical field theory, so far. Therefore, we have to deal with genuine tree diagrams (without any duality to some possibly vanishing loop counterpart), for which, even in a supersymmetric theory, there is *a priori* no reason that they add up to zero.

It is interesting to consult the supersymmetry variations for the bulk fields. In heterotic M-theory, the incorporation of E_8 SYM theories on the two orbifold fixed-planes, simultaneously requires the augmentation of the susy-variations of the bulk-fields [46]. The additional contributions have support on the fixed-planes only and are solely built out of the boundary-fields. For the particular locally flat Minkowski background with vanishing G-flux, which we will examine later on, the bulk contributions completely vanish, since flat space does not break any supersymmetry at all. The only non-vanishing susy-variations for constant Majorana-spinor η derive from the boundary fields

$$\delta C_{11BC} = -\frac{1}{24\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta\left(x_i^{11}-d_i\right)\bar\eta A^a_{[B}\Gamma_{C]}\chi_i^a \tag{3.1}$$

$$\delta\Psi_A = -\frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta\left(x_i^{11}-d_i\right)\left(\bar\chi_i^a\Gamma_{BCD}\chi_i^a\right)\left(\Gamma_A{}^{BCD}-6\delta_A^B\Gamma^{CD}\right)\eta \tag{3.2}$$

$$\delta\Psi_{11} = \frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta\left(x_i^{11}-d_i\right)\left(\bar\chi_i^a\Gamma^{ABC}\chi_i^a\right)\Gamma_{ABC}\eta\ . \tag{3.3}$$

In momentum-space these contributions will vanish in the case of equal momenta of the boundary fields. Contracting δC_{11BC} with the momentum p_2^C of the gauge-field A^a_B, we get an expression proportional to

$$\bar\eta\big(A^a_B(p_2)\not{p}_2 - p_2\cdot A^a(p_2)\Gamma_B\big)\chi^a(p_3)\ . \tag{3.4}$$

Choosing the Lorentz-gauge, the second term disappears, whereas the first term gives zero, when we choose $p_2=p_3=p$ on account of the massless Dirac-equation $\not{p}\chi^a(p)=0$.

For the last two gravitino-variations, we note that

$$\bar{\chi}^a(p)\Gamma^{ABC}\chi^a(p') = -\bar{\chi}^a(p')\Gamma^{ABC}\chi^a(p) \ , \tag{3.5}$$

from which we easily recognize, that the gaugino bilinear $\bar{\chi}^a\Gamma^{ABC}\chi^a$ also vanishes in the limit of coinciding momenta for $\bar{\chi}^a$ and χ^a. In kinematical language, coinciding momenta mean a vanishing center-of-mass energy squared $s = 0$. Hence, in this limit we expect to find no interaction between the boundaries, for a locally flat background.

The interaction amplitudes will depend on the parameter d, representing the distance between the two boundaries in the eleventh direction. In case that we can still trust Hořava-Witten supergravity not only for large values of d but also for small values, then according to the conjecture, the $d \to 0$ limit of the above amplitudes should correspond to the low-energy limit of heterotic string amplitudes. Consequently we will derive the adequate string amplitudes in the limit $\alpha' s, \alpha' t, \alpha' u \ll 1$ and compare them with our M-theory amplitudes evaluated at $d = 0$. Naively, one could not expect complete agreement of the two sets of amplitudes, since a large d compared to the eleven-dimensional Planck-scale is a necessary condition for the validity of the effective Hořava-Witten supergravity.

3.1 Expansion of Hořava-Witten Supergravity around $\mathbb{R}^{1,9} \times S^1$ Background

As advocated in [43],[46] we choose $\mathbb{R}^{1,9} \times S^1$ as our D=11 spacetime manifold, where the eleventh coordinate x^{11} is curled up to a circle which we parameterize by $[-d, d]$ with $d \sim -d$ to be identified. Furthermore we have to impose the constraint, that the fields be invariant under the reflection $x^{11} \to -x^{11}$. This so-called "upstairs" formulation, which we shall employ here, has the advantage that one can work with a smooth manifold, whereas in the equivalent alternative "downstairs" formulation one would have to deal with a bounded manifold $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2 = \mathbb{R}^{1,9} \times [0, d]$ and prescribe suitable boundary conditions. In the latter approach the boundary is given by the two codimension one fixed planes of the reflection map, situated at $x^{11} = 0$ and d.

The construction of Hořava-Witten supergravity proceeds by a power series in the expansion parameter $\kappa^{2/3}$. To lowest order, one starts with the action of $\mathcal{N}$=1, D=11 supergravity [67]

$$S_{bulk} = \int_{\mathbb{R}^{1,9}\times S^1} d^{11}x \frac{\sqrt{-g}}{\kappa^2}\Big[-\frac{R}{2} - \frac{1}{2}\bar{\Psi}_I \Gamma^{IJK} D_J \Psi_K - \frac{1}{2\times 4!} G_{IJKL}G^{IJKL} \\ - \frac{\sqrt{2}}{192}\left(\bar{\Psi}_I \Gamma^{IJKLMN}\Psi_N + 12 \bar{\Psi}^J \Gamma^{KL}\Psi^M\right) G_{JKLM} \\ - \frac{\sqrt{2}}{3456}\epsilon^{I_1\ldots I_{11}} C_{I_1 I_2 I_3} G_{I_4\ldots I_7} G_{I_8\ldots I_{11}}\Big] + \mathcal{O}\left(\Psi^4\right) \tag{3.6}$$

for the bulk multiplet, consisting of elfbein $e^{\bar{I}}{}_I$, gravitino Ψ_I and 3-form C_{IJK}. We use $I,\ldots,N/A,\ldots,F$ to represent D=11/D=10 space-time indices and $\bar{I},\ldots,\bar{N}/\bar{A},\ldots,\bar{F}$ for their tangent space analogues. Moreover we define $\bar{\Psi}_\alpha = C_{\alpha\beta}\Psi^\beta$, where the real, antisymmetric charge conjugation matrix $C_{\alpha\beta}$ obeys $C^{\alpha\beta}C_{\beta\gamma} = \delta^\alpha_\gamma$ (see appendix A for further conventions). The covariant derivative of the gravitino, the spin connection $\Omega_{J\bar{L}\bar{M}}$ and the 4-form field strength G_{IJKL} are defined as

$$\begin{aligned} D_J\Psi_K &= \partial_J \Psi_K + \frac{1}{4}\Omega_{J\bar{L}\bar{M}}\Gamma^{\bar{L}\bar{M}}\Psi_K \\ \Omega_{J\bar{L}\bar{M}} &= \frac{1}{2}\left(e_{\bar{L}}{}^L \tilde{\Omega}_{JL\bar{M}} - e_{\bar{M}}{}^L \tilde{\Omega}_{JL\bar{L}} - e_{\bar{L}}{}^L e_{\bar{M}}{}^M e^{\bar{J}}{}_J \tilde{\Omega}_{LM\bar{J}}\right) \ , \quad \tilde{\Omega}_{JL\bar{M}} = \partial_J e_{\bar{M}L} - \partial_L e_{\bar{M}J} \\ G_{IJKL} &= 4!\, \partial_{[I}C_{JKL]} \ . \end{aligned}$$

To determine which bulk fields can appear in the boundary action, one has to consult their $\mathbb{Z}_2$ transformation properties

$$\begin{aligned} &g_{AB}(-x^{11}) = g_{AB}(x^{11}) \ , \qquad g_{A,11}(-x^{11}) = -g_{A,11}(x^{11}) \ , \qquad g_{11,11}(-x^{11}) = g_{11,11}(x^{11}) \ , \\ &G_{ABCD}(-x^{11}) = -G_{ABCD}(x^{11}) \ , \qquad G_{ABC11}(-x^{11}) = G_{ABC11}(x^{11}) \ , \\ &\Gamma_{11}\Psi_A = \Psi_A \ , \qquad \Gamma_{11}\Psi_{11} = -\Psi_{11} \ . \end{aligned}$$

Only the $\mathbb{Z}_2$ invariant components will be allowed. The chirality property of the gravitino implies $\bar{\Psi}_A\Gamma^{11} = -\bar{\Psi}_A$, $\bar{\Psi}_{11}\Gamma^{11} = \bar{\Psi}_{11}$, which leads to the vanishing of the following fermion-bilinears ($n \in \mathbb{N}$)

$$\bar{\Psi}_A \Gamma_{B_1\ldots B_{2n}}\Psi_C = 0 \tag{3.7}$$

$$\bar{\Psi}_A \Gamma_{B_1\ldots B_{2n+1}}\Psi_{11} = 0 \tag{3.8}$$

$$\bar{\Psi}_{11} \Gamma_{B_1\ldots B_{2n}}\Psi_{11} = 0 \ . \tag{3.9}$$

Therefore in the boundary action only

$$\bar{\Psi}_A \Gamma_{B_1\ldots B_{2n+1}}\Psi_C \ , \qquad \bar{\Psi}_A \Gamma_{B_1\ldots B_{2n}}\Psi_{11} \tag{3.10}$$

can appear.

The full D=10 boundary action for the E_8 vector supermultiplet[18], comprising the gauge field A^a and the gaugino χ^a, coupled to the bulk supergravity in a locally supersymmetric fashion, reads [46] ($i = 1,2$)

$$S_{i,bound}(x^{11} = d_i) = \int_{\mathbb{R}^{1,9}} d^{10}x_i \frac{1}{(4\pi)^{5/3}\kappa^{4/3}} \sqrt{-g} \Big[-\frac{1}{4} F^a_{iAB} F^{aAB}_i - \frac{1}{2} \bar{\chi}^a_i \Gamma^A D_A \chi^a_i$$
$$-\frac{1}{4} \bar{\Psi}_A \Gamma^{BC} \Gamma^A F^a_{iBC} \chi^a_i + \bar{\chi}^a_i \Gamma^{ABC} \chi^a_i \Big[\frac{\sqrt{2}}{48} G_{ABC11} + \frac{1}{32} \bar{\Psi}_A \Gamma_{BC} \Psi_{11} + \frac{1}{32} \bar{\Psi}^D \Gamma_{DABC} \Psi_{11}$$
$$+ \frac{1}{128} \big(3\bar{\Psi}_A \Gamma_B \Psi_C - \bar{\Psi}_A \Gamma_{BCD} \Psi^D - \frac{1}{2} \bar{\Psi}_D \Gamma_{ABC} \Psi^D - \frac{13}{6} \bar{\Psi}^D \Gamma_{DABCE} \Psi^E \big) \Big] \Big] , \qquad (3.11)$$

where $d_1 = 0, d_2 = d$ describe the two fixed plane positions. The non-abelian field strength F^a_{iAB} and the covariant derivative for the gaugino are defined as usual as

$$F^a_{iAB} = \partial_A A^a_{iB} - \partial_B A^a_{iA} + f^a{}_{bc} A^b_{iA} A^c_{iB}$$
$$D_A \chi^a_i = \partial_A \chi^a_i + f^a{}_{bc} A^b_{iA} \chi^c_i + \frac{1}{4} \Omega_{A\bar{B}\bar{C}} \Gamma^{\bar{B}\bar{C}} \chi^a_i \ .$$

Furthermore the gauginos possess positive chirality $\Gamma^{11}\chi^a = \chi^a$. The gauge coupling constant λ has already been eliminated from (3.11) by means of the relation $\lambda^2 = 4\pi(4\pi\kappa^2)^{2/3}$. The fixed-plane gauge action (3.11) is the second order term in the power series expansion in $\kappa^{2/3}$ pure bulk supergravity comprises the first order). Unfortunately, in the next higher order infinities arise in the form of $\delta(0)$ terms occuring in the Lagrangean. Formally, these infinities cancel in verifying supersymmetry. Nevertheless, to arrive at reliable results, one is forced to truncate the action at this order consistently.

From the perturbative point of view, we have to look for small fluctuations of the bulk fields in order to mediate interactions between the boundary fields. This will be achieved by using the background field method [69], according to which we split the bulk fields $e^{\bar{M}}{}_M, \Psi_M, C_{MNP}$ into a fixed classical background $\tilde{e}^{\bar{M}}{}_M, \tilde{\psi}_M, \tilde{c}_{MNP}$ and the quantum fields $f^{\bar{M}}{}_M, \psi_M, c_{MNP}$, which propagate on this background

$$e^{\bar{M}}{}_M = \tilde{e}^{\bar{M}}{}_M + \kappa f^{\bar{M}}{}_M \longrightarrow g_{MN} = \tilde{g}_{MN} + \kappa h_{MN}$$
$$\Psi_M = \tilde{\psi}_M + \kappa\psi_M$$
$$C_{MNP} = \tilde{c}_{MNP} + \kappa c_{MNP} \ .$$

The action is then expanded around the background fields into a power series of the quantum fields. The further multiplication with the D=11 gravitational coupling constant

[18] For E_8 the Lie-Algebra index a runs from 1 to 248.

κ has been chosen to give the fluctuations ordinary kinetic terms in the action. Hence the expansion in quantum fields is also one in powers of κ. In the following every index will be raised or lowered by means of the background elfbein or metric.

With the chosen parameterization of the circle of radius $R = d/\pi$, our background $\mathbb{R}^{1,9} \times S^1$ is described locally by a flat elfbein[19]

$$\tilde{e}^{\bar{M}}{}_M = \delta^{\bar{M}}_M \ .$$

In order not to break spontaneously D=10 Lorentz symmetry, the background gravitino field $\tilde{\psi}_M$ as well as the background 3-form $\tilde{c}_{MNP}$ must vanish

$$\tilde{\psi}_M = \tilde{c}_{MNP} = 0 \ .$$

As we will point out in the following, it is advantageous (though not necessary) if the background fields fulfill the equations of motion.

The expansion of the bulk action then proceeds as

$$\int d^{11}x \bigg(\frac{1}{\kappa^2} \,[\text{classical Sugra action}] + \frac{1}{\kappa}\,[\text{linear in quantum fields } h, \psi, c] \\ + [\text{quadratic terms for } h, \psi, c] + \mathcal{O}(\kappa) \bigg) \ .$$

For the flat zero-curvature background the leading term vanishes. Since the coefficients of h, ψ, c in the $1/\kappa$-term are precisely the variational derivatives of the action with respect to the classical fields, they will vanish also when the background satisfies the equations of motion – as happens to be the case. In order to extract the propagators from the quadratic part, we have, according to the usual Faddeev-Popov procedure, to fix all the gauge symmetries of the quantum fields and introduce corresponding ghosts. However, since our analysis is intended to be classical, i.e. at tree-level, we can neglect the ghost fields. $\mathcal{N}$=1, D=11 supergravity possesses four different gauge symmetries, which are fixed as follows

- D=11 general coordinate invariance $\longrightarrow$ de Donder (harmonic) gauge[20]

$$\Delta\mathcal{L}_{GC} = -\frac{\tilde{e}}{2}\big(h_M{}^N{}_{;N} - \frac{1}{2}h^N{}_{N;M}\big)^2$$

[19] For flat space the curved space-time indices $M, N, \ldots$ and the tangent space indices $\bar{M}, \bar{N}, \ldots$ coincide and need not to be distinguished subsequently.

[20] As usual we define $\tilde{e} = \sqrt{-\tilde{g}}$.

- $SO(1,10)$ local Lorentz invariance $\longrightarrow$ symmetric gauge

$$\Delta\mathcal{L}_{LL} = -\frac{\tilde{e}}{2\kappa^{4/(D-2)}}\big(f_{MN} - f_{NM}\big)^2 \to -\frac{\tilde{e}}{2\kappa^{4/9}}\big(f_{MN} - f_{NM}\big)^2$$

- local abelian gauge transformations of C_{MNP} $\longrightarrow$ Lorentz-like gauge

$$\Delta\mathcal{L}_{Ab} = -\frac{3}{2}\big(3!\partial^M c_{MNP}\big)^2 = -\frac{3}{2}(3!)^2\partial_I c^{IJK}\partial^L c_{LJK}$$

- local $\mathcal{N}=1$ supersymmetry $\longrightarrow$ $\Gamma^M\psi_M = 0$ fixing[21] [70]

$$\Delta\mathcal{L}_{SS} = -\frac{\zeta}{2}\bar{\psi}_M\Gamma^M\Gamma^N\partial_N\Gamma^L\psi_L$$

Note that $\Delta\mathcal{L}_{LL}$ is a purely algebraic term and thus does not contribute to the propagator. From $e^{\bar{M}}{}_M e^{\bar{N}}{}_N\eta_{\bar{M}\bar{N}} = g_{MN}$ together with $\tilde{e}^{\bar{M}}{}_M\tilde{e}^{\bar{N}}{}_N\eta_{\bar{M}\bar{N}} = \tilde{g}_{MN}$ and $e^{\bar{M}}{}_M = \tilde{e}^{\bar{M}}{}_M + \kappa f^{\bar{M}}{}_M$ plus the above gauge fixing of the Lorentz-symmetry, which gauges the antisymmetric part of f_{MN} away, we conclude that $h_{MN} = 2f_{MN} + \mathcal{O}(\kappa)$. Therefore we can express the elfbein fluctuations in terms of the metric fluctuations. The quadratic terms of the bulk action then lead to the following propagators in momentum space, valid for flat unbounded D=11 Minkowski-space (the compactification of the eleventh coordinate on the circle will manifest itself, later on, in a replacement of p^{11} by discrete values $p^{11}_m, m\in\mathbb{Z}$):

Graviton h_{MN}:

$$\Delta_{M_1M_2,N_1N_2}(P) = -2\left(\eta_{M_1N_1}\eta_{M_2N_2} + \eta_{M_1N_2}\eta_{M_2N_1} - \frac{2}{9}\eta_{M_1M_2}\eta_{N_1N_2}\right)\frac{1}{P^2}\,, \tag{3.12}$$

Gravitino[22] ψ_M:

$$\begin{aligned}
\Delta_{MN}{}^{\alpha\beta}(P) &\equiv i\big(\tilde{\Delta}_{MN}\big)^\alpha{}_\gamma(P)\,\big(-C^{\gamma\beta}\big) && (3.13)\\
&= \frac{i}{9}\left[7\eta_{MN}\not{P} - \Gamma_N\not{P}\Gamma_M + \left(4+\frac{9}{\zeta}\right)\frac{P_M\not{P}P_N}{P^2}\right]^\alpha{}_\gamma\big(-C^{\gamma\beta}\big)\frac{1}{P^2}\\
&\overset{\zeta=-\frac{9}{4}}{\longrightarrow} \frac{i}{9}[7\eta_{MN}\not{P} - \Gamma_N\not{P}\Gamma_M]^\alpha{}_\gamma\big(-C^{\gamma\beta}\big)\frac{1}{P^2}\,, && (3.14)
\end{aligned}$$

3-Form c_{MNP}:

$$\Delta_{M_1M_2M_3,N_1N_2N_3}(P) = \frac{1}{(3!)^2}\frac{\eta_{M_1[N_1}\eta_{|M_2|N_2}\eta_{|M_3|N_3]}}{P^2}\,, \tag{3.15}$$

[21] ζ is a free gauge parameter which we will set to $-\frac{9}{4}$ for simplicity.

[22] $\alpha,\beta,\ldots$ denote $SO(1,10)$ spinor indices.

where $P = (p^A, p^{11})$ denotes the 11-dimensional momentum.

The expansion of the boundary action reads schematically

$$\int d^{10}x_i \bigg(\frac{1}{\kappa^{4/3}}\,[\text{pure SYM}] + \frac{1}{\kappa^{1/3}}\,[\text{bulk-boundary interaction terms, linear in } h, \psi, c] \\ + \mathcal{O}(\kappa^{2/3}) \bigg)\,.$$

Since the leading $1/\kappa^{4/3}$ contribution does not contain any bulk quantum field, it cannot contribute to the boundary-boundary interaction and is therefore of no interest to us. The $1/\kappa^{1/3}$ terms comprise the relevant interaction terms, whereas higher order $\kappa^{2/3}$ expressions have to be skipped for two reasons. First, $\kappa^{2/3}$ terms would introduce couplings quadratic in h, ψ, c. Either these would finally lead to loop diagrams of order $\kappa^{4/3}$, which have to be neglected, since we restrict ourselves to a tree-level analysis. Or in combination with the couplings linear in h, ψ, c, they would give rise to order κ^0 tree diagrams. These are suppressed by a factor of $\kappa^{2/3}$ against the leading diagrams of order $1/\kappa^{2/3}$ and therefore have to be neglected, too. Second, the boundary action (3.11) has been constructed only up to order $1/\kappa^{4/3}$. The next higher order in the power series expansion in $\kappa^{2/3}$ would involve $1/\kappa^{2/3}$ terms. However, it has been argued in [46], that at this order expressions containing $\delta(0)$ show up, which means that in a consistent truncation of the theory, we have to skip these higher contributions altogether. In the expansion around the classical background, the bulk field fluctuations h, ψ, c come equipped with an additional power of κ. Therefore a consistent truncation implies throwing away all bulk-boundary interactions of order $\kappa^{1/3}$ or higher. In particular, the above $\kappa^{2/3}$ contributions have to be omitted. The remaining $1/\kappa^{1/3}$ interaction terms are explicitly given by[23]

$$S^{(1)}_{i,bound}(x^{11} = d_i) = \frac{1}{(4\pi)^{5/3}\kappa^{1/3}} \int d^{10}x_i \bigg[-\frac{1}{8} F_i^{aCD} F^a_{iCD} h^A{}_A + \frac{1}{2} F^a_{iAC} F^{a\ C}_{iB} h^{AB} \\ + \frac{1}{8} \bar{\chi}^a_i \Gamma^A \Gamma^{BC} \chi^a_i \partial_{[A} h_{B]C} - \frac{1}{16} \bar{\chi}^a_i \Gamma^A \Gamma^{BC} \chi^a_i \partial_{[B} h_{C]A} \\ - \frac{1}{4} \bar{\psi}_A \Gamma^{BC} \Gamma^A \chi^a_i F^a_{iBC} + \frac{1}{\sqrt{2}} \bar{\chi}^a_i \Gamma^{ABC} \chi^a_i \partial_{[A} c_{BC11]} \bigg]\,.$$

It can be read off that we obtain a 5-point vertex $AAAAh$, two 4-point-vertices $AA\chi\psi$, $AAAh$ and four 3-point vertices $\chi\chi c$, $A\chi\psi$, $\chi\chi h$, AAh. The 5-vertex has a group-theoretic factor which consists of a sum of terms like $\sum_{e=1}^{248}(f_{eac}f_{ebd} + f_{ead}f_{ebc})$, where f_{abc} are the E_8 structure constants. The 4-vertices are simply proportional to f_{abc}. Since finally

[23] By x_1, x_2 we denote the D=10 flat coordinates of the two boundaries, respectively.

in our amplitudes we will sum over all group indices of the external boundary fields (since the boundary-boundary interaction results from the sum over all possible exchange amplitudes between the two boundaries), the 5- and 4-vertices give no contribution due to the antisymmetry of the structure constants. If, therefore, we keep merely the 3-vertices, the relevant couplings are

$$
\begin{aligned}
S^{(1)}_{i,bound}(x^{11}=d_i) = \frac{1}{(4\pi)^{5/3}\kappa^{1/3}} \int d^{10}x_i \Big[\frac{1}{2}\Big(&- \partial^C A_i^{a,D} \partial_{[C} A^a_{iD]} \eta_{AB} + \partial_A A^a_{iC} \partial_B A_i^{a,C} \\
&+ \partial_C A^a_{iA} \partial^C A^a_{iB} - 2\partial_C A^a_{iA} \partial_B A_i^{a,C} \Big) h^{AB} + \frac{1}{8} h_{AC} \partial_B \left(\bar{\chi}^a_i \Gamma^A \Gamma^{BC} \chi^a_i \right) \\
&- \frac{1}{2} \bar{\psi}_A \Gamma^{BC} \Gamma^A \chi^a_i \partial_B A^a_{iC} + \frac{1}{\sqrt{2}} \bar{\chi}^a_i \Gamma^{ABC} \chi^a_i \partial_{[A} c_{BC11]} \Big] .
\end{aligned}
$$

Since every term comprises exactly two boundary fields, it is convenient for the later comparison with the string amplitudes to rescale the SYM fields A^a_A, χ^a which bear mass dimensions $[A]=1, [\chi]=\frac{3}{2}$ to

$$
B^a_A := \frac{1}{(4\pi)^{5/6}\,\kappa^{2/3}} A^a_A \,, \qquad \lambda^a := \frac{1}{(4\pi)^{5/6}\,\kappa^{2/3}} \chi^a \,.
$$

The fields B^a_A and λ^a have D-dimensional mass dimensions $[B^a_A]=(D-2)/2$ and $[\lambda^a]=(D-1)/2$, i.e. 4 and 9/2 for D=10. This rescaling gives a "canonical" factor of κ for the interaction terms, which eventually read

$$
S^{(1)}_{i,bound}(x^{11}=d_i) = \kappa \int_{\mathbb{R}^{1,9}} d^{10}x_i \Big(\mathcal{L}_{i,BBh} + \mathcal{L}_{i,\lambda\lambda h} + \mathcal{L}_{i,B\lambda\psi} + \mathcal{L}_{i,\lambda\lambda c} \Big) \,, \tag{3.16}
$$

with

$$
\mathcal{L}_{i,BBh} = \partial_A B^a_{iB}(x_i) \partial_C B^a_{iD}(x_i) h_{EF}(x_i,d_i) F^{ABCDEF} \,, \tag{3.17}
$$

$$
\mathcal{L}_{i,\lambda\lambda h} = -\frac{1}{8} h_{AC}(x_i,d_i) \partial_B \left[\lambda_i^{a\alpha}(x_i) C_{\alpha\beta} \left(\Gamma^C \eta^{AB} - \Gamma^B \eta^{AC} \right)^\beta{}_\gamma \lambda_i^{a\gamma}(x_i) \right] , \tag{3.18}
$$

$$
\mathcal{L}_{i,B\lambda\psi} = \frac{1}{2} \psi^\alpha_A(x_i,d_i) C_{\alpha\beta} \left(\Gamma^{BC} \Gamma^A \right)^\beta{}_\gamma \lambda_i^{a\gamma}(x_i) \partial_B B^a_{iC}(x_i) \,, \tag{3.19}
$$

$$
\mathcal{L}_{i,\lambda\lambda c} = -\frac{1}{\sqrt{2}} \lambda_i^{a\alpha}(x_i) C_{\alpha\beta} \left(\Gamma^{ABC} \right)^\beta{}_\gamma \lambda_i^{a\gamma}(x_i) \partial_{[A} c_{BC11]}(x_i,d_i) \,, \tag{3.20}
$$

and

$$
F^{ABCDEF} = \frac{1}{2} \eta^{A[D} \eta^{C]B} \eta^{EF} + \eta^{A[C} \eta^{D]F} \eta^{EB} + \eta^{A[E} \eta^{D]B} \eta^{CF} \,.
$$

3.2 Derivation of the Boundary-Boundary Interaction Amplitudes

In order to incorporate the $\mathbb{Z}_2$ fixed point constraints and to circumvent ambiguities arising from Feynman diagrams involving Majorana fermions (which allow for twice as many Wick-contractions as Dirac fermions do), we choose not to work with Feynman rules in momentum space directly but to start with a spacetime formulation of the S-matrix on $\mathbb{R}^{1,9} \times S^1$. For a tree-level boundary-boundary interaction the S-matrix reads

$$\begin{aligned} S = & -\frac{1}{2}\kappa^2 \int_{\mathbb{R}^{1,9}} d^{10}x_1 \oint dx_1^{11} \int_{\mathbb{R}^{1,9}} d^{10}x_2 \oint dx_2^{11} \\ & \langle f \big| T\Big(:\mathcal{L}_{1,bound}(x_1, x_1^{11})\delta(x_1^{11}) :: \mathcal{L}_{2,bound}(x_2, x_2^{11})\delta(x_2^{11} - d) : \Big)\big| i \rangle \\ = & -\frac{1}{2}\kappa^2 \int d^{10}x_1 \int d^{10}x_2 \langle f \big| T\Big(:\mathcal{L}_{1,bound}(x_1, 0) :: \mathcal{L}_{2,bound}(x_2, d) : \Big)\big| i \rangle \,, \end{aligned} \tag{3.21}$$

where $\mathcal{L}_{i,bound}$ represents one of the four couplings $\mathcal{L}_{i,BBh}, \mathcal{L}_{i,\lambda\lambda h}, \mathcal{L}_{i,B\lambda\psi}, \mathcal{L}_{i,\lambda\lambda c}$ given in (3.17)-(3.20). From the 11-dimensional perspective the fixed point constraints enter via delta-function sources which generate flat p^{11}-spectra in momentum space. Momentum is conserved only along the ten flat directions parallel to the boundaries, whereas there is no such conservation in the eleventh compactified direction transverse to the boundaries. This fact is also well-known from studies of radiation off D-branes [72], [71] and is a consequence of broken translation invariance orthogonal to the boundary. Therefore the kinematic variables s, t, u of the scattering process are defined exclusively through the 10-dimensional momenta p_1, p_2 of the incoming states and p_3, p_4 of the outgoing states as follows

$$s = -(p_1 + p_2)^2 \,, \qquad t = -(p_1 - p_3)^2 \,, \qquad u = -(p_1 - p_4)^2 \,.$$

As usual D=10 energy-momentum conservation implies for massless states $s + t + u = 0$. The four 3-vertices (3.17)-(3.20) can be combined into five different tree-diagrams which we will now consider in detail.

3.3 Graviton Exchange

The first diagram is depicted in fig.1 and describes the pure graviton exchange between the boundary gauge fields. Upon substituting (3.17) into (3.21) it yields the following S-matrix contribution

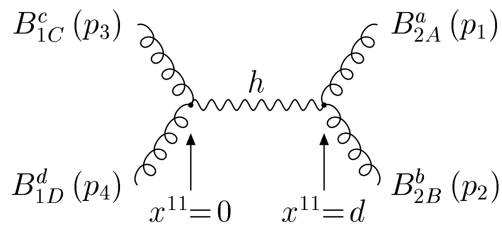

Figure 1: Graviton exchange

$$
\begin{aligned}
S_h = -\frac{1}{2}\kappa^2 \sum_{a,b,c,d=1}^{248} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^{8} \int d^{10}x_1 \oint dx_1^{11} \int d^{10}x_2 \oint dx_2^{11} \\
\big\langle 0 \big| b^d_{1\lambda_4}\,(p_4)\, b^c_{1\lambda_3}\,(p_3)\, T\Big(:\partial_{A_1} B^{a_1}_{1B_1}\,(x_1)\,\partial_{C_1} B^{a_1}_{1D_1}\,(x_1)\, h_{E_1F_1}\,(x_1,0)\,\delta(x_1^{11}) : \\
:\partial_{A_2} B^{a_2}_{2B_2}\,(x_2)\,\partial_{C_2} B^{a_2}_{2D_2}\,(x_2)\, h_{E_2F_2}\,(x_2,d)\,\delta(x_2^{11}-d) : \Big) b^{b,\dagger}_{2\lambda_2}\,(p_2)\, b^{a,\dagger}_{2\lambda_1}\,(p_1)\,\big|0\big\rangle \\
F^{A_1B_1C_1D_1E_1F_1}\, F^{A_2B_2C_2D_2E_2F_2} \ .
\end{aligned}
$$

Here we sum over all "colours" a, b, c, d and physical polarizations $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the in- and out-states, since all of them add to the interaction of the two boundaries. For our conventions concerning annihilation and creation operators see appendix A. When in the next step we Wick-contract creation and annihilation operators $b^{a,\dagger}_{2\lambda}$ and $b^a_{1\lambda}$ with the boundary field operators B^a_{iA}, we have to take into account that creation and annihilation operators from the left-hand side of the diagram can only be contracted with left-hand sided $B^a_{1A}(x_1)$ operators and equally creation and annihilation operators from the right-hand side of the diagram can only be contracted with right-hand sided $B^a_{2A}(x_2)$ operators. If we would allow for "mixed" contractions, t- and u-channel diagrams would also be present in the boundary-boundary amplitudes. But these have to be excluded as they cannot arise when both hyperplanes do not coincide. After a further integration over the circle coordinates, we are led to

$$
\begin{aligned}
S_h = -\frac{1}{2}\kappa^2 \sum_{a,b,c,d} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \int d^{10}x_1 \int d^{10}x_2 \\
&\left[\underbrace{b^d_{1\lambda_4}(p_4)\,\partial_{A_1}B^{a_1}_{1B_1}}(x_1)\,\underbrace{b^c_{1\lambda_3}(p_3)\,\partial_{C_1}B^{a_1}_{1D_1}}(x_1) + \left(b^d_{1\lambda_4}(p_4) \leftrightarrow b^c_{1\lambda_3}(p_3)\right)\right] \\
&F^{A_1B_1C_1D_1E_1F_1}\, i\Delta_{E_1F_1,E_2F_2}(x_1-x_2,-d)\, F^{A_2B_2C_2D_2E_2F_2} \\
&\left[\partial_{A_2}\underbrace{B^{a_2}_{2B_2}(x_2)\,b^{b,\dagger}_{2\lambda_2}}(p_2)\,\partial_{C_2}\underbrace{B^{a_2}_{2D_2}(x_2)\,b^{a,\dagger}_{2\lambda_1}}(p_1) + \left(b^{b,\dagger}_{2\lambda_2}(p_2) \leftrightarrow b^{a,\dagger}_{2\lambda_1}(p_1)\right)\right],
\end{aligned}
$$

where we have expressed the graviton 2-point-function $\langle 0|T\left(h_{E_1F_1}(x_1,0)h_{E_2F_2}(x_2,d)\right)|0\rangle$ through i times its propagator $\Delta_{M_1M_2,N_1N_2}(x_1-x_2,-d)$. Using the expressions (A.21) and (A.22) for the Wick-contractions gives the E_8 group factor $\sum_{a,b,c,d=1}^{248}\delta^{ab}\delta^{cd} = (248)^2$, and we arrive at the expression

$$
\begin{aligned}
S_h = -i\frac{\kappa^2}{2}(248)^2 \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \int d^{10}x_1 \int d^{10}x_2\, e^{i(p_1+p_2)x_2} e^{-i(p_3+p_4)x_1} \\
&\left[\epsilon_{B_1}(p_4,\lambda_4)\,p_{4,A_1}\epsilon_{D_1}(p_3,\lambda_3)\,p_{3,C_1} + \epsilon_{B_1}(p_3,\lambda_3)\,p_{3,A_1}\epsilon_{D_1}(p_4,\lambda_4)\,p_{4,C_1}\right] \\
&F^{A_1B_1C_1D_1E_1F_1}\Delta_{E_1F_1,E_2F_2}(x_1-x_2,-d)\,F^{A_2B_2C_2D_2E_2F_2} \\
&\left[\epsilon_{B_2}(p_2,\lambda_2)\,p_{2,A_2}\epsilon_{D_2}(p_1,\lambda_1)\,p_{1,C_2} + \epsilon_{B_2}(p_1,\lambda_1)\,p_{1,A_2}\epsilon_{D_2}(p_2,\lambda_2)\,p_{2,C_2}\right].
\end{aligned}
$$

To utilize the previously derived flat-space propagator (3.12), we have to notice that the momentum in the compactified eleventh direction $p^{11}_m = m/R, m \in \mathbb{Z}$ is quantized. The radius R of the circle is related via $R = d/\pi$ to the distance d between the two hyperplanes. Ensuring that we do not change the dimensions of the propagator as compared to the flat case, we have to take

$$
f\left(x^{11}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp^{11} e^{ip^{11}x^{11}} f\left(p^{11}\right) \sum_{m\in\mathbb{Z}} \delta\left(p^{11}d - m\pi\right) = \frac{1}{2\pi d}\sum_{m\in\mathbb{Z}} e^{ip^{11}_m x^{11}} f\left(p^{11}_m\right)
$$

as the Fourier transform in the eleventh direction. Therefore the Fourier-transformed graviton propagator reads

$$
\Delta_{E_1F_1,E_2F_2}\left(x_1-x_2,x^{11}\right) = \frac{1}{d\,(2\pi)^{11}}\int d^{10}p\, e^{ip(x_1-x_2)} \sum_{m\in\mathbb{Z}} e^{ip^{11}_m x^{11}} \Delta_{E_1F_1,E_2F_2}\left(p,p^{11}_m\right) ,
$$

where $\Delta_{E_1F_1,E_2F_2}\left(p,p_m^{11}\right)$ is functionally the same as in the flat, non-compact case. Plugging

$$\Delta_{E_1F_1,E_2F_2}\left(x_1-x_2,-d\right)=\frac{1}{d\left(2\pi\right)^{11}}\int d^{10}p e^{ip(x_1-x_2)}\sum_{m\in\mathbb{Z}}\left(-1\right)^m\Delta_{E_1F_1,E_2F_2}\left(p,p_m^{11}\right)\ . \tag{3.22}$$

into the expression for S_h and integrating over x_1,x_2, results in

$$\begin{aligned}S_h=-i\frac{\kappa^2}{2d}\left(248\right)^2\left(2\pi\right)^9\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}\int d^{10}p\ \delta^{10}\left(p_1+p_2-p\right)\delta^{10}\left(-p+p_3+p_4\right)\\ \Big[p_{4,A_1}\epsilon_{B_1}(p_4,\lambda_4)\,p_{3,C_1}\epsilon_{D_1}(p_3,\lambda_3)+p_{3,A_1}\epsilon_{B_1}(p_3,\lambda_3)\,p_{4,C_1}\epsilon_{D_1}(p_4,\lambda_4)\Big]\\ F^{A_1B_1C_1D_1E_1F_1}\sum_{m\in\mathbb{Z}}\left(-1\right)^m\Delta_{E_1F_1,E_2F_2}\left(p,p_m^{11}\right)F^{A_2B_2C_2D_2E_2F_2}\\ \Big[p_{2,A_2}\epsilon_{B_2}(p_2,\lambda_2)\,p_{1,C_2}\epsilon_{D_2}(p_1,\lambda_1)+p_{1,A_2}\epsilon_{B_2}(p_1,\lambda_1)\,p_{2,C_2}\epsilon_{D_2}(p_2,\lambda_2)\Big]\ .\end{aligned}$$

The integration over p can now trivially be performed, resulting in an overall $D{=}10$ energy-momentum conserving delta-function. The interaction-amplitude or T-matrix element is defined by equating $S_h=i\left(2\pi\right)^{10}\delta^{10}\left(p_1+p_2-p_3-p_4\right)T_h$. Going in between to the center-of-mass (CMS) frame with respect to the 10-dimensional momenta parallel to the boundary, employing (3.12) plus various kinematical relations gathered in appendix A, we finally arrive at the amplitude

$$T_h=\frac{2\kappa^2}{\pi d}\left(248\right)^2\left(25s^2-32tu\right)\sum_{m\in\mathbb{Z}}\frac{\left(-1\right)^m}{-s+\left(p_m^{11}\right)^2}\ .$$

Here s,t,u are the Mandelstam-variables composed out of the 10-dimensional components of the momenta along the boundarie, as pointed out above. To perform the sum we use

$$\sum_{m=1}^{\infty}\frac{\left(-1\right)^m}{z^2-m^2\pi^2}=\frac{1}{2z}\left(\frac{1}{\sin z}-\frac{1}{z}\right)\ ,\quad z\in\mathbb{C}\ ,$$

which one obtains as an application of the Mittag-Leffler theorem from Complex Analysis and get

$$\sum_{m\in\mathbb{Z}}\frac{\left(-1\right)^m}{-s+\left(p_m^{11}\right)^2}=-\frac{d}{\sqrt{s}\sin\left(\sqrt{s}d\right)}\ . \tag{3.23}$$

This yields for the matrix-element

$$T_h(s,\vartheta)=-\frac{2\kappa^2}{\pi}\left(248\right)^2\frac{\left(25s^2-32tu\right)}{\sqrt{s}\sin\left(\sqrt{s}d\right)}\ . \tag{3.24}$$

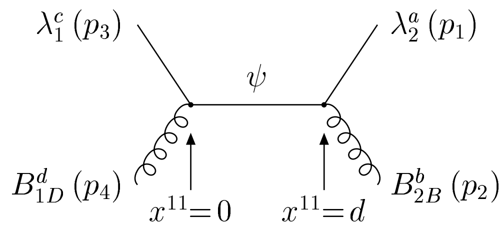

Figure 2: Gravitino exchange, I

It is easy to verify that $25s^2 - 32tu > 0$. Concerning our stability analysis, we should further integrate over the scattering angle ϑ of the CMS-system from 0 to $\pi/2$ (due to the fact that we have identical fields in the out state, the integration is only over half the full range). This gives

$$\mathcal{T}_h(s) = -21\,(248)^2\,\kappa^2 \frac{s^{3/2}}{\sin\left(\sqrt{s}d\right)}\ . \tag{3.25}$$

3.4 Gravitino Exchange

There exist two diagrams describing amplitudes resulting from gravitino exchange. The first one is depicted in fig.2. Using (3.19) we obtain for its S-matrix

$$\begin{aligned} S_{\psi,I} = -\frac{1}{2}\kappa^2 \sum_{a,b,c,d=1}^{248} \sum_{\lambda_2,\lambda_4=1}^{8} \sum_{s_1,s_3=1}^{8} \int d^{10}x_1 \int d^{10}x_2 \\ \big\langle 0\big| b^d_{1\lambda_4}\,(p_4)\, d^c_{1s_3}\,(p_3)\, T\Big(:\frac{1}{2}\psi^{\alpha_1}_{A_1}(x_1,0) C_{\alpha_1\beta_1}\left(\Gamma^{B_1C_1}\Gamma^{A_1}\right)^{\beta_1}{}_{\gamma_1}\lambda_1^{a_1\gamma_1}(x_1)\partial_{B_1}B^{a_1}_{1C_1}(x_1): \\ :\frac{1}{2}\psi^{\alpha_2}_{A_2}(x_2,d) C_{\alpha_2\beta_2}\left(\Gamma^{B_2C_2}\Gamma^{A_2}\right)^{\beta_2}{}_{\gamma_2}\lambda_2^{a_2\gamma_2}(x_2)\partial_{B_2}B^{a_2}_{2C_2}(x_2):\Big) b^{b,\dagger}_{2\lambda_2}(p_2) d^{a,\dagger}_{2s_1}(p_1)\big|0\big\rangle\ . \end{aligned}$$

Again we sum over "colours" a, b, c, d, physical polarizations λ_2, λ_4 of the gauge fields and spin polarizations s_1, s_3 of the gauginos. $d^{a,\dagger}_{2s}$ and d^a_{1s} are the creation and annihilation operators of the gauginos. Performing the Wick-contractions with the help of (A.21)-(A.24) and using the momentum representation (3.22) for the gravitino propagator (3.13) leads us to

$$S_{\psi,I} = -\left(\frac{1}{2}\right)^2 \frac{\kappa^2}{2d}\frac{(248)^2}{(2\pi)^{11}} \sum_{\lambda_2,\lambda_4=1}^{8} \sum_{s_1,s_3=1}^{8} \int d^{10}p \sum_{m\in\mathbb{Z}} (-1)^m \int d^{10}x_1 \int d^{10}x_2 e^{-i(-p+p_3+p_4)x_1}$$
$$e^{i(p_1+p_2-p)x_2} p_{4,B_1}\epsilon_{C_1}(p_4,\lambda_4)\bar{u}_{s_3,\gamma_2}(p_3)\left(\Gamma^{A_1}\Gamma^{B_1C_1}\right)^{\gamma_2}{}_{\alpha_1}\left(\tilde{\Delta}_{A_1A_2}\right)^{\alpha_1}{}_{\beta_2}(p,p_m^{11})$$
$$\left(\Gamma^{B_2C_2}\Gamma^{A_2}\right)^{\beta_2}{}_{\gamma_2} u_{s_1}^{\gamma_2}(p_1) p_{2,B_2}\epsilon_{C_2}(p_2,\lambda_2) \ .$$

As before the factor of $(248)^2$ represents the E_8 group factor. Performing the x_1, x_2 integrations result in two Dirac delta functions describing the ten-dimensional energy-momentum conservation at each vertex separately. Upon integration over the momentum p, carried by the gravitino, we arrive at the following T-matrix element

$$T_{\psi,I} = i\frac{\kappa^2}{4\pi d}\left(\frac{1}{2}\right)^2 (248)^2 \sum_{\lambda_2,\lambda_4=1}^{8} \sum_{s_1,s_3=1}^{8} \sum_{m\in\mathbb{Z}} (-1)^m$$
$$p_{4,B_1}\epsilon_{C_1}(p_4,\lambda_4)\bar{u}_{s_3,\gamma_2}(p_3)\left(\Gamma^{A_1}\Gamma^{B_1C_1}\right)^{\gamma_2}{}_{\alpha_1}\left(\tilde{\Delta}_{A_1A_2}\right)^{\alpha_1}{}_{\beta_2}(p_1+p_2,p_m^{11})$$
$$\left(\Gamma^{B_2C_2}\Gamma^{A_2}\right)^{\beta_2}{}_{\gamma_2} u_{s_1}^{\gamma_2}(p_1) p_{2,B_2}\epsilon_{C_2}(p_2,\lambda_2) \ .$$

In order to facilitate this expression further, we note that the Weyl condition $\Gamma^{10}u_s(p) = u_s(p),\ \bar{u}_s(p)\Gamma^{10} = -\bar{u}_s(p)$ for the gaugino spinor enforces $\bar{u}_s(p)\Gamma^{A_1}...\Gamma^{A_{2n}}u_{s'}(p') = 0$. Using this observation, the Weyl condition itself, the Dirac equation $\not{p}u_s(p) = \bar{u}_s(p)\not{p} = 0$ as well as the expression (3.14) for $\tilde{\Delta}_{A_1A_2}$, we receive in the 10-dimensional boundary CMS-frame

$$T_{\psi,I} = i\frac{\kappa^2}{4\pi d}\left(\frac{1}{2}\right)^2 (248)^2 \sum_{\lambda_2,\lambda_4=1}^{8} \sum_{s_1,s_3=1}^{8} \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{-s+(p_m^{11})^2}$$
$$\bar{u}_{s_3}(p_3)\Bigg(E^2\left(4\cos\vartheta+28\right)\not{p}_4 + 2E\sin\vartheta\not{p}_4\not{\epsilon}(p_2,\lambda_2)\not{p}_2 - 2E^3\sin\vartheta\not{\epsilon}(p_4,\lambda_4)$$
$$+\frac{2E^2}{3}\not{\epsilon}(p_4,\lambda_4)\not{p}_4\not{\epsilon}(p_2,\lambda_2) + E^2\left(\cos\vartheta-\frac{1}{3}\right)\not{\epsilon}(p_4,\lambda_4)\not{\epsilon}(p_2,\lambda_2)\not{p}_2\Bigg)u_{s_1}(p_1) \ .$$

$E = \sqrt{s}$ and ϑ denote the ten-dimensional CMS-energy and the scattering angle in the CMS-frame (see appendix A). Employing the explicit expressions for $\bar{u}_{s_3}(p_3), u_{s_1}(p_1)$ and for the Γ-matrices from the appendix, we get

$$T_{\psi,I} = -i\frac{\kappa^2}{4\pi d}\left(\frac{1}{2}\right)^2 (248)^2 \sum_{m\in\mathbb{Z}} (-1)^m 128(s-u)\sqrt{\frac{-u}{s}}\frac{s}{-s+(p_m^{11})^2} \ .$$

In addition to the diagram of fig.2, we also have to add the diagram of fig.3 which merely amounts to the exchange of $p_1 \leftrightarrow p_2$ or $t \leftrightarrow u$ in the preceding diagram. Adding up both contributions results in

$$T_\psi = -i\frac{\kappa^2}{4\pi d}\left(\frac{1}{2}\right)^2 (248)^2 \sum_{m\in\mathbb{Z}} (-1)^m 128\left((s-u)\sqrt{\frac{-u}{s}} + (s-t)\sqrt{\frac{-t}{s}}\right)\frac{s}{-s+(p_m^{11})^2} \ .$$

Utilizing again (3.23), we conclude that

$$T_\psi(s,\vartheta) = i\frac{8\kappa^2}{\pi}(248)^2\frac{\left((s-u)\sqrt{-u}+(s-t)\sqrt{-t}\right)}{\sin\left(\sqrt{s}d\right)} \ . \tag{3.26}$$

For the stability analysis we perform a further integration over the scattering angle ϑ from 0 to π (as appropriate for distinguishable fields in the out state), which finally gives

$$\mathcal{T}_\psi(s) = i\frac{160\kappa^2}{3\pi}(248)^2\frac{s^{3/2}}{\sin\left(\sqrt{s}d\right)} \ .$$

3.5 3-Form Exchange

The 3-form exchange diagram of fig.4 yields the following expression for the S-matrix element

$$\begin{aligned} S_c = -\frac{1}{2}\kappa^2 \sum_{a,b,c,d=1}^{248} \sum_{s_1,s_2,s_3,s_4=1}^{8} \int d^{10}x_1 \int d^{10}x_2 \\ \langle 0 | d^d_{1s_4}(p_4) d^c_{1s_3}(p_3) T\Big(:\frac{1}{\sqrt{2}}\lambda_1^{a_1\alpha_1}(x_1) C_{\alpha_1\beta_1}\left(\Gamma^{A_1B_1C_1}\right)^{\beta_1}{}_{\gamma_1}\lambda_1^{a_1\gamma_1}(x_1)\partial_{[A_1}c_{B_1C_1 11]}(x_1,0) : \\ :\frac{1}{\sqrt{2}}\lambda_2^{a_2\alpha_2}(x_2) C_{\alpha_2\beta_2}\left(\Gamma^{A_2B_2C_2}\right)^{\beta_2}{}_{\gamma_2}\lambda_2^{a_2\gamma_2}(x_2)\partial_{[A_2}c_{B_2C_2 11]}(x_2,d) : \Big) d^{b,\dagger}_{2s_2}(p_2) d^{a,\dagger}_{2s_1}(p_1) | 0 \rangle \ . \end{aligned}$$

We make use of (A.23),(A.24) to perform the Wick-contractions and gain the E_8 gauge group factor $(248)^2$ as previously. We then combine the four resulting terms together by

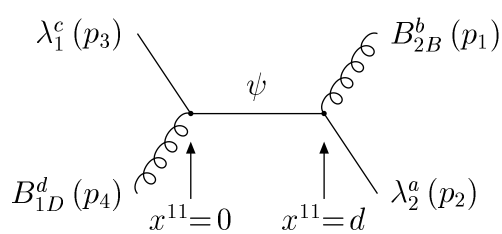

Figure 3: Gravitino exchange, II

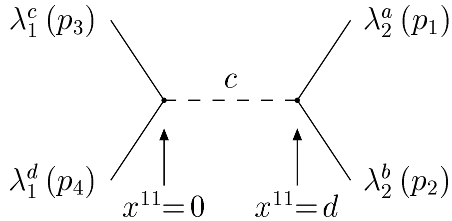

Figure 4: 3-Form exchange

employing the relation $u_s^\alpha(p)C_{\alpha\beta}\left(\Gamma^{ABC}\right)^\beta{}_\gamma u_{s'}^\gamma(p') = -u_{s'}^\alpha(p')C_{\alpha\beta}\left(\Gamma^{ABC}\right)^\beta{}_\gamma u_s^\gamma(p)$. Moreover, expressing the 2-point-function $\langle 0|T\left(c_{M_1M_2M_3}(x_1,0)c_{N_1N_2N_3}(x_2,d)\right)|0\rangle$ as i times the 3-form propagator $\Delta_{M_1M_2,N_1N_2}\left(x_1-x_2,-d\right)$ gives

$$
\begin{aligned}
S_c = -i\kappa^2(248)^2 \sum_{s_1,s_2,s_3,s_4=1}^{8} \int d^{10}x_1 \int d^{10}x_2 e^{-i(p_3+p_4)x_1} e^{i(p_1+p_2)x_2} \\
u_{s_4}^{\alpha_1}(p_4)C_{\alpha_1\beta_1}\left(\Gamma^{A_1B_1C_1}\right)^{\beta_1}{}_{\gamma_1} u_{s_3}^{\gamma_1}(p_3)\partial_{[A_1}\partial^{[A_2}\Delta_{B_1C_111]}{}^{B_2C_211]}(x_1-x_2,-d) \\
u_{s_2}^{\alpha_2}(p_2)C_{\alpha_2\beta_2}\left(\Gamma^{A_2B_2C_2}\right)^{\beta_2}{}_{\gamma_2} u_{s_1}^{\gamma_2}(p_1)\ .
\end{aligned}
$$

Fourier-transforming the propagator with the help of (3.15) and (3.22)

$$
\begin{aligned}
&\partial_{[A_1}\partial^{[A_2}\Delta_{B_1C_111]}{}^{B_2C_211]}(x_1-x_2,-d) \\
&= -\frac{1}{d(3!)^2(2\pi)^{11}}\int d^{10}p e^{ip(x_1-x_2)}\sum_{m\in\mathbb{Z}}(-1)^m\left(\frac{3!}{4!}\right)^2\left(3p_{[A_1}p^{[A_2}\delta_{B_1}^{B_2}\delta_{C_1]}^{C_2]} - (p_m^{11})^2\delta_{[A_1}^{[A_2}\delta_{B_1}^{B_2}\delta_{C_1]}^{C_2]}\right) \\
&\quad\times\frac{1}{p^2+(p_m^{11})^2}
\end{aligned}
$$

brings us to

$$
\begin{aligned}
S_c = i\frac{\kappa^2}{d}\frac{(248)^2}{(4!)^2(2\pi)^{11}} \sum_{s_1,s_2,s_3,s_4=1}^{8} \int d^{10}p \sum_{m\in\mathbb{Z}}(-1)^m \int d^{10}x_1 \int d^{10}x_2 e^{-i(-p+p_3+p_4)x_1} e^{i(p_1+p_2-p)x_2} \\
\Big(3u_{s_4}^{\alpha_1}(p_4)C_{\alpha_1\beta_1}\left(\Gamma^{A_1BC}\right)^{\beta_1}{}_{\gamma_1} u_{s_3}^{\gamma_1}(p_3)u_{s_2}^{\alpha_2}(p_2)C_{\alpha_2\beta_2}\left(\Gamma_{A_2BC}\right)^{\beta_2}{}_{\gamma_2} u_{s_1}^{\gamma_2}(p_1)p_{A_1}p^{A_2} \\
- u_{s_4}^{\alpha_1}(p_4)C_{\alpha_1\beta_1}\left(\Gamma^{ABC}\right)^{\beta_1}{}_{\gamma_1} u_{s_3}^{\gamma_1}(p_3)u_{s_2}^{\alpha_2}(p_2)C_{\alpha_2\beta_2}\left(\Gamma_{ABC}\right)^{\beta_2}{}_{\gamma_2} u_{s_1}^{\gamma_2}(p_1)(p_m^{11})^2\Big)\frac{1}{p^2+(p_m^{11})^2}\ .
\end{aligned}
$$

Performing the integration over x_1, x_2 and afterwards over p, we gain the following T-matrix element

$$
\begin{aligned}
T_c =& \frac{\kappa^2}{\pi d} \frac{(248)^2}{(3!)^3} \sum_{s_1,s_2,s_3,s_4=1}^{8} \sum_{m\in\mathbb{Z}} (-1)^m \\
& \Big(3\bar{u}_{s_4}(p_4)\Gamma^{A_1BC} u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_{A_2BC} u_{s_1}(p_1)(p_3+p_4)_{A_1}(p_1+p_2)^{A_2} \\
& - \bar{u}_{s_4}(p_4)\Gamma^{ABC} u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_{ABC} u_{s_1}(p_1)(p_m^{11})^2 \Big) \frac{1}{-s+(p_m^{11})^2} \ .
\end{aligned}
$$

Subsequently, we are dealing separately with the first and the second term of this amplitude in order to boil them down to some more succinct expressions.

Let us start with the first term and the observation that the Dirac equation $\not{p}\, u_s(p) = \bar{u}(p)\not{p} = 0$ yields the relations

$$
p_A \Gamma^{ABC} u_s(p) = 2p^{[B}\Gamma^{C]} u_s(p) \ , \qquad \bar{u}_s(p)\Gamma^{ABC} p_A = -2\bar{u}_s(p) p^{[B}\Gamma^{C]} \ .
$$

If we apply them to the first term, we obtain

$$
\begin{aligned}
& \sum_{s_1,s_2,s_3,s_4=1}^{8} 3\bar{u}_{s_4}(p_4)\Gamma^{A_1BC} u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_{A_2BC} u_{s_1}(p_1)(p_3+p_4)_{A_1}(p_1+p_2)^{A_2} \\
&= 12 \sum_{s_1,s_2,s_3,s_4=1}^{8} \bar{u}_{s_4}(p_4)(p_3-p_4)^{[B}\Gamma^{C]} u_{s_3}(p_3)\bar{u}_{s_2}(p_2)(p_1-p_2)_{[B}\Gamma_{C]} u_{s_1}(p_1) \ .
\end{aligned}
$$

By noticing that in the CMS-frame $p_1 - p_2 = (0, ..., 0, E)$ and $p_3 - p_4 = (0, ..., 0, E\sin\vartheta, E\cos\vartheta)$, we eventually reduce this expression to

$$
\begin{aligned}
& 4!E^2 \sum_{s_1,s_2,s_3,s_4=1}^{8} \big(\cos\vartheta \bar{u}_{s_4}(p_4)\Gamma^9 u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_9 u_{s_1}(p_1) \\
& + \sin\vartheta \bar{u}_{s_4}(p_4)\Gamma^9 u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_8 u_{s_1}(p_1) - \cos\vartheta \bar{u}_{s_4}(p_4)\Gamma^A u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_A u_{s_1}(p_1) \big) \\
&= -4! \times 64 s^2 \ .
\end{aligned}
$$

Concerning the second part of the amplitude, we decompose

$$
\Gamma^{ABC} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

into 2×2 and 16×16 matrices. With the explicit expression for the gaugino-spinor (A.10), we find

$$\bar{u}_{s_4}(p_4)\Gamma^{ABC}u_{s_3}(p_3) = Ec\Big(\frac{\sin\vartheta}{2}A + \cos^2\frac{\vartheta}{2}B - \sin^2\frac{\vartheta}{2}C - \frac{\sin\vartheta}{2}D\Big)_{s_4 s_3}$$
$$\bar{u}_{s_2}(p_2)\Gamma^{ABC}u_{s_1}(p_1) = EcB_{s_2 s_1}\ .$$

In particular, we have for our chosen representation (described in A.4) of $SO(1,10)$ Γ-matrices the following description in terms of 8×8 submatrices γ^a

$$\begin{array}{llll} \Gamma^{1ab}: & c=1, B=0 & \Gamma^{1a10}: & c=1, B=\gamma^a \\ \Gamma^{abc}: & c=1, B=\gamma^a\gamma^{b,T}\gamma^c & \Gamma^{ab10}: & c=1, B=0\ . \end{array}$$

Since after summation over the spin-polarizations, an antisymmetric matrix B gives a vanishing contribution, the only non-vanishing terms for our scattering process stem from $\Gamma^{1,9,10}$ and Γ^{ijk}; $i,j,k=2,...,8$. Hence we are able to reduce the second term to

$$\begin{aligned} &\sum_{s_1,s_2,s_3,s_4=1}^{8} \bar{u}_{s_4}(p_4)\Gamma^{ABC}u_{s_3}(p_3)\bar{u}_{s_2}(p_2)\Gamma_{ABC}u_{s_1}(p_1) \\ &= 3!\times E^2\Bigg(-64 + \sum_{i<j<k}\underbrace{\Big[\sum_{s_2,s_1}\big(\gamma^{ijk}\big)_{s_2 s_1}\Big]^2}_{(\pm 8)^2}\Bigg) \\ &= (3!\times 8E)^2\ . \end{aligned}$$

Putting the results for the two terms together, we arrive at the following expression for the 3-form exchange amplitude

$$T_c = -\frac{\kappa^2}{\pi d}\frac{128(248)^2}{(3!)^2}s\sum_{m\in\mathbb{Z}}(-1)^m\frac{2s+3(p_m^{11})^2}{-s+(p_m^{11})^2}\ . \tag{3.27}$$

Using again (3.23) for the summation, we find

$$\sum_{m\in\mathbb{Z}}(-1)^m\frac{2s+3(p_m^{11})^2}{-s+(p_m^{11})^2} = 3\sum_{m\in\mathbb{Z}}(-1)^m - 5\frac{\sqrt{s}d}{\sin(\sqrt{s}d)}\ .$$

The first term describes an alternating sum, which does not converge and requires some kind of regularization. In order to understand this contribution, we will explore in a moment the $d\to\infty$ limit. Therefore it proves useful to express the above obtained amplitude in terms of the D=10 gravitational coupling constant $\kappa_{(10)}$ which is independent of the compactification radius R or d. Compactification of M-theory on an S^1 of radius R

and a subsequent comparison of its Einstein-Hilbert term with the Einstein-Hilbert term coming from the effective action of the D=10 heterotic string (in Einstein frame) leads to the following relationship between the D=11 and the D=10 gravitational coupling constants

$$\kappa^2 = 2d\kappa^2_{(10)} \ . \tag{3.28}$$

Hence T_c can be expressed as

$$T_c = -\kappa^2_{(10)} \frac{256(248)^2}{(3!)^2\pi} s \left(3 \sum_{m\in\mathbb{Z}} (-1)^m - 5 \frac{\sqrt{s}d}{\sin(\sqrt{s}d)} \right) \ .$$

Since the first part of the amplitude, consisting of the alternating sum and some d-independent prefactors, is independent of d, we can equally well evaluate it at any d, in particular at $d \to \infty$. Now, if we consider a large radius, the difference between two adjacent values of p^{11}_m becomes infinitesimally small and we are allowed to replace the sum by an integral

$$\lim_{d\to\infty} \sum_{m\in\mathbb{Z}} f(p^{11}_m = \frac{m}{R} = m\frac{\pi}{d}) = \lim_{d\to\infty} \frac{d}{\pi} \int_{-\infty}^{\infty} dp^{11} f(p^{11}) \ .$$

Writing $(-1)^m = e^{ip^{11}_m d}$, we now encounter the following expression for the alternating sum

$$\sum_{m\in\mathbb{Z}} (-1)^m = \sum_{m\in\mathbb{Z}} e^{ip^{11}_m d} = \lim_{d\to\infty} \frac{d}{\pi} \int_{-\infty}^{\infty} dp^{11} e^{ip^{11}d} = \lim_{d\to\infty} 2d\delta(d) = 0 \ .$$

Thus finally the amplitude can be completely determined to be

$$T_c(s) = \frac{\kappa^2}{\pi} \frac{160(248)^2}{9} \frac{s^{3/2}}{\sin(\sqrt{s}d)} \ . \tag{3.29}$$

The integration over the scattering angle from 0 to $\pi/2$ is trivial and results in

$$\mathcal{T}_c(s) = \kappa^2 \frac{80(248)^2}{9} \frac{s^{3/2}}{\sin(\sqrt{s}d)} \ . \tag{3.30}$$

3.6 Two Further Graviton Exchange Diagrams

To complete our discussion of all relevant tree diagrams, which contribute to a boundary-boundary interaction, we also have to consider two further graviton exchange diagrams, depicted in fig.5. Both are, after performing the Wick-contractions, proportional to

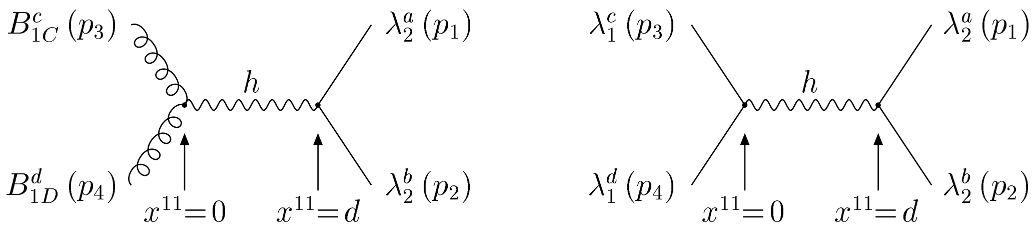

Figure 5: Vanishing Graviton exchange diagrams

$$-\bar{u}_{s_2}(p_2)\left(\Gamma^C\eta^{AB}-\Gamma^B\eta^{AC}\right)u_{s_1}(p_1)+\bar{u}_{s_1}(p_1)\left(\Gamma^C\eta^{AB}-\Gamma^B\eta^{AC}\right)u_{s_2}(p_2)\ ,$$

which gives zero, if we do avail ourselves of (A.11). Physically the vanishing of the diagrams is clear, since interchanging the two gauginos of the final state gives a minus-sign, which cannot be compensated for by the coupling to a graviton. In the previously analysed case of the coupling between two gauginos and the 3-form potential, the coupling delivers an extra minus-sign under exchange of the two fermions, so that the amplitude did not vanish in that case.

3.7 Analysis of the Amplitudes

Gathering all the obtained amplitudes, integrated over the scattering angle, we have

$$\mathcal{T}_h(s) = -21(248)^2\kappa^2\frac{s^{3/2}}{\sin(\sqrt{s}d)} \tag{3.31}$$

$$\mathcal{T}_\psi(s) = i\frac{160}{3\pi}(248)^2\kappa^2\frac{s^{3/2}}{\sin(\sqrt{s}d)} \tag{3.32}$$

$$\mathcal{T}_c(s) = \frac{80}{9}(248)^2\kappa^2\frac{s^{3/2}}{\sin(\sqrt{s}d)}\ . \tag{3.33}$$

First of all, we have to determine the range of validity of (3.31)-(3.33). From the denominator we recognize that singularities occur at the excitations of the Kaluza-Klein states at $\sqrt{s}=m\pi/d=p^{11}_m$, $m\in\mathbb{Z}$. Our analysis did not cover contributions to the interaction amplitudes coming from these highly massive states and only included the exchange of the massless supergravity multiplet. Therefore, the range of validity of our results is subjected to the following constraint, given by the threshold of the first Kaluza-Klein excitation

$$0\leq\sqrt{s}<\frac{\pi}{d}\ .$$

In the special case of vanishing CMS-energy $\sqrt{s} = 0$, each amplitude vanishes separately. This corresponds to the situation where the boundary fields on each fixed-plane run in parallel directions. In this case we have trivially no interaction between the two boundaries, as expected from the vanishing of the susy-variations for this kinematics. In this special situation the flat background with vanishing G-flux corresponds to a stable ground state of the heterotic M-theory set-up. However, if there are excitations on the boundary, by which we mean a kinematical situation showing $\sqrt{s} > 0$ for the boundary-fields, we see that pure gravity leads to an attraction (since for our range of validity, we have to stay below the first Kaluza-Klein excitation energy), whereas – similar to the behaviour of the RR-forms in the analogous D-brane case of type II string theory – the 3-form exchange leads to a repulsion. If we choose the same CMS-energy for all three contributions, then the attractive gravity dominates the weaker 3-form repulsion. Hence the real part of the amplitudes indicates an instability which is caused by an attractive force trying to bring the two boundaries closer together. This is in the direction towards the weakly coupled $E_8 \times E_8$ heterotic string.

Thus the flat background with vanishing G-flux does not represent a stable vacuum in the presence of arbitrary momenta of the boundary-fields. An obvious guess as to the nature of a stable vacuum comes from the treatment of heterotic M-theory compactified on a Calabi-Yau threefold [63]. There it has been shown, that with a non-vanishing G-flux on the Calabi-Yau and in the orbifold-direction, compactified heterotic M-theory exhibits a warped-geometry. In view of the failure of the flat vacuum to represent a stable configuration, one would naively think, that the warping of the geometry should survive in the decompactification limit. Ten-dimensional Poincaré-invariance only allows for a non-trivial dependence of the metric background on x^{11} and hence only a warped-geometry would be possible. However, the very Poincaré-invariance also requires G_{KLMN} to vanish and therefore other sources for a warping of space-time must be taken into account.

The behaviour of the above calculated amplitudes is similar to the weakly-coupled string-theory case in which an excited D-brane can decay into a massless closed string state and the non-excited D-brane [72]. Such a decay is also possible whenever the two massless waves on the D-brane run in different directions and accordingly possess $\sqrt{s} > 0$. Only if the massless waves run in the same direction, i.e. have $\sqrt{s} = 0$, one is dealing with a BPS state which does not decay.

Curiously the gravitino exchange gives rise to an imaginary part. By inspection of (3.26) we find that the forward scattering amplitude $T_\psi(s, \vartheta = 0)$ is non-vanishing. Via the optical theorem this would signal the opening of some inelastic channels for a decay of an excited boundary and therefore an instability in a more drastic sense.

It is interesting that later on the instability of heterotic M-theory causing a shrinking of the orbifold interval has also been discovered in a completely different approach. In [73] heterotic M-theory compactified on a Calabi-Yau threefold CY_3 – which will be the subject of the next chapter – has been analyzed in the presence of a time-dependent G-flux $G_{t11AB} \neq 0$. If $G_{ABCD} = 0$ as in our case, they find that $\int_{CY_3} d^6x G_{t11AB} \omega^{AB} > 0$, where ω_{AB} is the Kähler-form of the CY_3. Therefore, it seems necessary in the compactified case to turn on G_{t11AB}. This, however, leads to a metric whose orbifold interval shrinks with time. If this feature survives in the decompactification limit, we would see the same attraction of both boundaries as our perturbative analysis shows.

A last remark concerns unitarity. If we would evaluate total cross-sections with the above amplitudes, then by integrating over the appropriate phase space, we would get at high energies a total boundary-boundary interaction cross-section

$$\sigma \sim |\mathcal{T}(s)|^2 s^2 \sim \kappa^4 s^5 \ .$$

However, unitarity of the S-matrix leads for spinless states to the following restriction on partial wave amplitudes

$$\sigma_J \leq \frac{P_J}{s^4} \ ,$$

where P_J is some polynomial in the angular-momentum J independent of s. Neglecting ϑ-dependent factors which arise for states with higher spin, we conclude, that the total cross-section σ which is the sum of all σ_J, should decrease with increasing energy in order to obey unitarity. Since our cross-sections increase with energy, they violate unitarity. This is also plausible from the fact, that Hořava-Witten supergravity is not gauge invariant at the classical level and therefore no Ward-identities guarantee unitarity. However, we have to keep in mind the restriction to the energy regime $\sqrt{s} < \pi/d$ of our analysis. Should it happen, that a violation of unitarity occurs at an energy much higher than π/d, we would have to include the effects of the Kaluza-Klein excitations to decide, whether unitarity is violated or obeyed.

3.8 Comparison with the Weakly Coupled Heterotic String

According to the conjecture made in [43], we should recover the D=10 weakly coupled heterotic $E_8 \times E_8$ string theory in the limit of small R resp. d. Since the amplitudes which we have derived so far, describe the low-energy regime, we should also compare to the analogous low-energy string amplitudes. Here we have to use the expressions (3.24),(3.26),(3.29) which contain the full angular information. In order to derive the zero radius limit, we express all the derived amplitudes via (3.28) through the radius-independent $\kappa_{(10)}$ and then perform the limit $\sqrt{s}d \to 0$

$$T_h(s,\vartheta) = -\frac{4\kappa_{(10)}^2}{\pi}(248)^2\frac{(25s^2-32tu)d}{\sqrt{s}\sin(\sqrt{s}d)} \quad \stackrel{\sqrt{s}d\to 0}{\longrightarrow} \quad -\kappa_{(10)}^2\frac{4(248)^2}{\pi}\left(25s-32\frac{tu}{s}\right) \quad (3.34)$$

$$\begin{aligned} T_\psi(s,\vartheta) &= i\frac{16\kappa_{(10)}^2}{\pi}(248)^2\frac{\left((s-t)\sqrt{-t}+(s-u)\sqrt{-u}\right)d}{\sin(\sqrt{s}d)} \\ &\stackrel{\sqrt{s}d\to 0}{\longrightarrow} i\frac{16\kappa_{(10)}^2}{\pi}(248)^2\left((s-t)\sqrt{-\frac{t}{s}}+(s-u)\sqrt{-\frac{u}{s}}\right) \end{aligned} \quad (3.35)$$

$$T_c(s) = \frac{320\kappa_{(10)}^2}{9\pi}(248)^2\frac{s^{3/2}d}{\sin(\sqrt{s}d)} \quad \stackrel{\sqrt{s}d\to 0}{\longrightarrow} \quad \kappa_{(10)}^2\frac{320(248)^2}{9\pi}s\ . \quad (3.36)$$

So far for the M-theory amplitudes.

Closed string amplitudes involve a factor $\kappa_{(10)}^{M-2+2L}$, where M is the number of external particles and L the number of loops. Hence with four external particles it is clear, that a factor $\kappa_{(10)}^2$ corresponds to string tree-amplitudes as well. Those heterotic string tree-amplitudes can be found in [2][24]. The terms, which originate there from taking traces of four $E_8\times E_8$ group generators T_i, must be discarded from our comparison, since they correspond to processes where SYM fields are exchanged between the initial and final states. What we want instead to compare with are the amplitudes which are generated by the exchange of states of the supergravity multiplet. Since they comprise singlet-representations under the $E_8\times E_8$ gauge group, we merely encounter terms with traces of two generators. The string-theoretic tree-amplitudes adapted to our conventions read

$$A = \kappa_{10}^2 K\left(\zeta_1,\frac{p_1}{2},\zeta_2,\frac{p_2}{2},\zeta_3,\frac{p_3}{2},\zeta_4,\frac{p_4}{2}\right)C(s,t,u)\,G(p_1,p_2,p_3,p_4,T_1,T_2,T_3,T_4)\ ,$$

[24] The translation between the momenta k_i^{GSW} and Mandelstam-variables used by [2] and the p_i used in this paper is given by

$$\begin{aligned} k_1^{GSW} &= p_1\ , \qquad k_2^{GSW} = p_2\ , \qquad k_3^{GSW} = -p_3\ , \qquad k_4^{GSW} = -p_4\ , \\ s^{GSW} &= -(k_1^{GSW}+k_2^{GSW})^2 = -(p_1+p_2)^2 = s \\ t^{GSW} &= -(k_1^{GSW}+k_4^{GSW})^2 = -(p_1-p_4)^2 = u \\ u^{GSW} &= -(k_1^{GSW}+k_3^{GSW})^2 = -(p_1-p_3)^2 = t\ . \end{aligned}$$

where

$$C(s,t,u) = -\pi \frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)}$$

$$G(p_1,p_2,p_3,p_4,T_1,T_2,T_3,T_4) = \frac{1}{32}\left(\frac{tu}{1+\frac{s}{8}}\mathrm{tr}\,[T_1T_2]\,\mathrm{tr}\,[T_3T_4]\right) \ .$$

The factor G of [2] also contains terms describing a t- and a u-channel exchange. Since in the heterotic M-theory calculation for finite d, we get only s-channel contributions for interactions of the boundary fields via bulk fields, our expressions for $d \to 0$ should only be compared to this very s-channel part of the string calculation. For this reason we have omitted the t- and u-contributions to the G-factor. The generator T_i corresponds to the i^{th} external particle and tr is defined as the trace in the adjoint representation of $E_8 \times E_8$ divided by 30. The various ζ_i stand for the polarization of the i^{th} particle. If it is a gaugino, we have to substitute the spinor $\zeta_i = u_{s_i}(p_i)$, whereas for a gauge boson we have to take its polarization $\zeta_i = \epsilon_i(p_i,\lambda_i)$.

The K-factor describes the kinematics of the interaction and is given for the various cases[25] by

$$\begin{aligned}
&K\left(\epsilon_1,\frac{p_1}{2},\epsilon_2,\frac{p_2}{2},\epsilon_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right)\\
=\ &\frac{1}{2^4}\Bigg(-\frac{1}{4}\big(su\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + st\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 + tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\big)\\
&-\frac{s}{2}\big(\epsilon_1\cdot p_4\epsilon_3\cdot p_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot p_3\epsilon_4\cdot p_1\epsilon_1\cdot\epsilon_3 + \epsilon_1\cdot p_3\epsilon_4\cdot p_2\epsilon_2\cdot\epsilon_3 + \epsilon_2\cdot p_4\epsilon_3\cdot p_1\epsilon_1\cdot\epsilon_4\big)\\
&+\frac{t}{2}\big(-\epsilon_1\cdot p_2\epsilon_4\cdot p_3\epsilon_3\cdot\epsilon_2 - \epsilon_3\cdot p_4\epsilon_2\cdot p_1\epsilon_1\cdot\epsilon_4 + \epsilon_1\cdot p_4\epsilon_2\cdot p_3\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot p_2\epsilon_4\cdot p_1\epsilon_1\cdot\epsilon_2\big)\\
&+\frac{u}{2}\big(-\epsilon_2\cdot p_1\epsilon_4\cdot p_3\epsilon_3\cdot\epsilon_1 - \epsilon_3\cdot p_4\epsilon_1\cdot p_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot p_4\epsilon_1\cdot p_3\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot p_1\epsilon_4\cdot p_2\epsilon_2\cdot\epsilon_1\big)\Bigg)\ ,
\end{aligned}$$

$$K\left(u_1,\frac{p_1}{2},\epsilon_2,\frac{p_2}{2},u_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right) = \frac{1}{2^3}\left(-\frac{u}{2}\bar{u}_1\not{\epsilon}_2\,(\not{p}_3+\not{p}_4)\not{\epsilon}_4 u_3 + \frac{s}{2}\bar{u}_1\not{\epsilon}_4\,(\not{p}_2-\not{p}_3)\not{\epsilon}_2 u_3\right)\ ,$$

$$\begin{aligned}
&K\left(\epsilon_1,\frac{p_1}{2},u_2,\frac{p_2}{2},u_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right) = K\left(u_2,\frac{p_2}{2},\epsilon_1,\frac{p_1}{2},\epsilon_4,\frac{p_4}{2},u_3,\frac{p_3}{2}\right)\\
=\ &\frac{1}{2^3}\left(\frac{u}{2}\bar{u}_2\not{\epsilon}_1\,(\not{p}_3+\not{p}_4)\not{\epsilon}_4 u_3 + s\,(\bar{u}_2\not{\epsilon}_4 u_3 p_4\cdot\epsilon_1 + \bar{u}_2\not{\epsilon}_1 u_3 p_1\cdot\epsilon_4 - \bar{u}_2\not{p}_4 u_3\epsilon_1\cdot\epsilon_4)\right)\ ,
\end{aligned}$$

[25]There is no factor $K(u_1,u_2,\epsilon_3,\epsilon_4)$, since as in the heterotic M-theory calculation the $BB \to \lambda\lambda$ contribution vanishes.

$$K\left(u_1,\frac{p_1}{2},u_2,\frac{p_2}{2},u_3,\frac{p_3}{2},u_4,\frac{p_4}{2}\right)=\frac{1}{2^2}\left(-\frac{s}{2}\bar{u}_2\Gamma^A u_3\bar{u}_1\Gamma_A u_4+\frac{u}{2}\bar{u}_1\Gamma^A u_2\bar{u}_4\Gamma_A u_3\right)\ .$$

Summing over every occuring vectorial or spinorial polarization index, we can simplify the kinematical factors further to

$$\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} K\left(\epsilon_1,\frac{p_1}{2},\epsilon_2,\frac{p_2}{2},\epsilon_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right)=-\frac{1}{8}\left(st+tu+us\right)+\frac{7}{16}\left(s^2+t^2+u^2\right)$$

$$\sum_{s_1,\lambda_2,s_3,\lambda_4} K\left(u_1,\frac{p_1}{2},\epsilon_2,\frac{p_2}{2},u_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right)=4s(s-u)\sqrt{-\frac{u}{s}}$$

$$\sum_{\lambda_1,s_2,s_3,\lambda_4} K\left(\epsilon_1,\frac{p_1}{2},u_2,\frac{p_2}{2},u_3,\frac{p_3}{2},\epsilon_4,\frac{p_4}{2}\right)=4s(s-t)\sqrt{-\frac{t}{s}}$$

$$\sum_{s_1,s_2,s_3,s_4} K\left(u_1,\frac{p_1}{2},u_2,\frac{p_2}{2},u_3,\frac{p_3}{2},u_4,\frac{p_4}{2}\right)=-4\left(3s^2+(t-u)^2\right)\ .$$

In the low-energy limit $\alpha' s,\alpha' t,\alpha' u\to 0$ we have

$$C\left(s,t,u\right)\to\frac{2^9\pi}{stu}$$

such that finally we arrive at the following expressions for the low-energy limit of the heterotic string amplitudes

$$A_{BB\overset{s}{\to}BB}=\pi\kappa_{10}^2 16\left(\mathrm{tr}\left[T_1T_2\right]\right)^2\left(s-\frac{tu}{s}\right) \tag{3.37}$$

$$A_{\lambda B\overset{s}{\to}\lambda B}+A_{\lambda B\overset{s}{\to}B\lambda}=\pi\kappa_{10}^2 4\times 16\left(\mathrm{tr}\left[T_1T_2\right]\right)^2\left((s-t)\sqrt{-\frac{t}{s}}+(s-u)\sqrt{-\frac{u}{s}}\right) \tag{3.38}$$

$$A_{\lambda\lambda\overset{s}{\to}\lambda\lambda}=-\pi\kappa_{10}^2(16)^2\left(\mathrm{tr}\left[T_1T_2\right]\right)^2\left(s-\frac{tu}{s}\right)\ . \tag{3.39}$$

If we compare these with (3.34),(3.35),(3.36), we recognize some differences. Whereas (3.34) and (3.37) deviate mildly in their functional dependence on s,t,u, the discrepancy between (3.36) and (3.39) is manifest. The string amplitude shows an angular dependence but the heterotic M-theory amplitude is isotropic. The gravitino exchange amplitudes remarkably completely agree in their angular dependence. Nevertheless the string amplitude is real, whereas its M-theoretic counterpart is purely imaginary.

To a certain degree a disagreement could have been expected. Generally, a low-energy description in terms of effective supergravity is only valid at large distances resp. small curvatures. Furthermore, Hořava-Witten supergravity is organized as a long wavelength expansion in the parameter $\kappa^{2/3}$, assumed to be small as compared to the eleventh-dimensional Planck-scale. However, in the limit $d\to 0$ of coinciding boundaries, the long-wavelength supergravity approximation breaks down and one cannot trust the order $\kappa^{2/3}$ expansion any longer. Therefore the effective description may lead to false results.

Recently, Green et al. [74] included an additional interaction term beyond those given in [46] which were the basis for the above calculation. It was then shown that reconciliation between the purely bosonic gauge-field boundary-boundary interaction amplitudes with their heterotic string counterparts can be achieved by adding to the boundary-bulk $\kappa^{2/3}$-interaction terms the interaction

$$S^{AAC} = \frac{\sqrt{2}}{(4\pi)^{5/3}\kappa^{4/3}} \int_{\mathbb{R}^{1,9}\times S^1} d^{11}x \big(\delta(x^{11}) + \delta(x^{11}-d)\big)\partial_{[M}\hat{C}_{NP11]}\mathrm{Tr}(A^M\partial^{[N}A^{P]}) \ . \quad (3.40)$$

This additional interaction has its origin from the kinetic part of the 4-form field strength G and owes its existence to the fact, that in heterotic M-theory the requirement of local supersymmetry leads to the following $\kappa^{2/3}$-correction to G

$$G_{MNP11} = 4!\partial_{[M}\hat{C}_{NP11]} + \frac{\kappa^{2/3}}{\sqrt{2}(4\pi)^{5/3}}\big(\delta(x^{11}) + \delta(x^{11}-d)\big)\omega_{MNP} \ , \quad (3.41)$$

with ω being the gauge-field Chern-Simons correction given by

$$\omega_{MNP} = 2\mathrm{tr}\big(A_M\partial_{[N}A_{P]} + \frac{1}{3}A_M[A_N, A_P] + \text{cyclic perms.}\big) \ . \quad (3.42)$$

In view of this successful match of the bosonic amplitudes, let us assume for the moment that also ultimately the fermionic amplitudes could be arranged to match their low-energy heterotic string counterparts. Then for coinciding boundaries the heterotic M-theory amplitudes would be given by integrating (3.37),(3.39) over the scattering angle from 0 to $\pi/2$ and (3.38) from 0 to π

$$\mathcal{A}_{BB\overset{s}{\to}BB} = \pi^2\kappa_{10}^2 7\,(\mathrm{tr}\,[T_1T_2])^2\, s \quad (3.43)$$

$$\mathcal{A}_{\lambda B\overset{s}{\to}\lambda B} + \mathcal{A}_{\lambda B\overset{s}{\to}B\lambda} = \pi\kappa_{10}^2\frac{1280}{3}\,(\mathrm{tr}\,[T_1T_2])^2\, s \quad (3.44)$$

$$\mathcal{A}_{\lambda\lambda\overset{s}{\to}\lambda\lambda} = -\pi^2\kappa_{10}^2 112\,(\mathrm{tr}\,[T_1T_2])^2\, s \ . \quad (3.45)$$

One observes that their sum $\sum\mathcal{A}(d=0)$ does not add up to zero. If for non-coinciding boundaries the set-up would be stable, we need to have $\sum\mathcal{A}(d>0)=0$. Thus stability of heterotic M-theory on a locally flat background would require a jump in $\sum\mathcal{A}(d)$ at $d=0$. This however does not seem to occur as the smooth limit of the pure bosonic subsector shows [74]. Hence, the very duality to the weakly coupled string already seems to imply an instability of the Hořava-Witten set-up – at least on a locally flat metric background.

4 Torsion and Warped Heterotic M-Theory Compactifications

After considering the uncompactified heterotic M-theory in eleven dimensions in the previous chapter, we now come to the case where heterotic M-theory is compactified on a Calabi-Yau threefold CY_3 to four dimensions times the orbifold interval. An important ingredient will be the possibility to turn on internal (i.e. with support on the CY_3 and the interval) G-flux without spoiling supersymmetry which will lead to non-trivial warped geometry backgrounds.

4.1 Heterotic M-Theory and Newton's Constant

For heterotic M-theory compactified on a CY_3 the 4-form field-strength G does not vanish if higher order corrections in $\kappa^{2/3}$ are taken into account. The reason is that the boundary super Yang-Mills (SYM) theories represent magnetic sources which show up in the Bianchi-identity for G and require a G of order $\kappa^{2/3}$. Hence we expect warped geometries to arise at this order, which is indeed the case [63]. Besides an interesting interplay between the physics of G-fluxes and geometry there arises an important phenomenological issue related to the value of Newton's Constant G_N. From a simple dimensional reduction of heterotic M-theory on a CY_3 with volume $V(x^{11})$ (in the 11-dimensional metric), one can infer [63]

$$G_N = \frac{\kappa^2}{16\pi\langle V\rangle d} , \qquad \alpha_i = \frac{(4\pi\kappa^2)^{2/3}}{2V_i} , \tag{4.1}$$

where α_i is the gauge-coupling of the two ($i = 1, 2$) boundary E_8 SYM theories and $V_1 = V(0), V_2 = V(d)$. Because G_N is related to gravity in the bulk, we have to use for its determination an average volume $\langle V\rangle = \frac{1}{d}\int_0^d dx^{11} V(x^{11})$. A determination of the warped geometry allows to calculate $V(x^{11})$ and thereby α_i and G_N. This had been undertaken in [63] to linearized order (first order in $\kappa^{2/3}$) with the result that $V(x^{11}) = -ax^{11} + V_1$, where the slope $a = \frac{1}{4\sqrt{2}}\int_{CY_3} d^6x\sqrt{g}\,\omega^{lm}\omega^{np}G_{lmnp} > 0$ is controlled by the G-flux. Here ω_{lm} denotes the Kähler-form on CY_3. The surprising observation [63] has been that when the linear function $V(x^{11})$ becomes zero, the corresponding distance for d just gives rise to the correct value for G_N, whereas generically in heterotic string compactifications G_N is predicted too large by a factor of 400. Placing the second boundary at that distance means $\alpha_2 \to \infty$. Hence, the SYM there becomes strongly coupled and instanton contributions become relevant, which is the reason why this second boundary corresponds to a "hidden" world rather than our "observable" world. The first boundary at $x^{11} = 0$ instead allows for a perturbative SYM on it if V_1 is chosen huge enough such that $\alpha_1 \ll 1$ and consequently can be regarded as the "observable" world.

In this context two questions arise

a) How is the linear behaviour of $V(x^{11})$, which leads to an unphysical negative volume beyond a certain distance, changed in the full theory, i.e. beyond the leading $\kappa^{2/3}$ order?

b) Does $V(x^{11})$ still keeps its attractive feature of becoming zero just at a phenomenologically highly relevant distance?

The trouble with the linear behaviour is the following. One expects that eventually quantum corrections will shift the actual value of $V(x^{11})$ slightly. However, small distortions of a linear function can never lift a zero – they can only shift its x^{11} position slightly, but the zero remains. Therefore it is important to determine the warp-factors and thereby $V(x^{11})$ beyond the leading order in $\kappa^{2/3}$, which we will undertake in the next section.

It may sound surprising how a result beyond order $\kappa^{2/3}$ can be achieved within the framework of heterotic M-theory whose effective action is only known to order $\kappa^{2/3}$. Let us therefore briefly indicate where and in which way features of heterotic M-theory will enter our analysis. By imposing supersymmetry, we are going to solve the gravitino Killing-spinor equation of M-theory. The heterotic M-theory characteristics enter on the one hand through specific G-fluxes originating from boundary or M5-brane sources and on the other hand through the chirality condition $\Gamma^{11}\eta = \eta$ on the susy-variation Majorana-parameter η. The important point is that *the information which is restricted to order $\kappa^{2/3}$ becomes only relevant if knowledge about the actual source strengths is required. However, to obtain the functional behaviour of $V(x^{11})$ this knowledge is not needed.* It suffices to assume that in the full heterotic M-theory the relevant sources can still be localized in the x^{11} direction, i.e. they appear as $dG = \delta(x^{11} - z)S(x^m) \wedge dx^{11}$ in the Bianchi-identity. Thus, we will be able to answer the first question posed above. The actual value (and thereby the complete knowledge about heterotic M-theory beyond $\kappa^{2/3}$ order) for the 4-form source strength S only becomes indispensable if e.g. questions about the precise value of a zero or a minimum of $V(x^{11})$ should be answered. This is necessary to answer the second question.

4.2 The Full Relation between Warped Geometry and G-Flux

Let us consider heterotic M-theory compactified on $CY_3 \times S^1/\mathbb{Z}_2$ with four external coordinates x^μ ; $\mu, \nu, \rho, \ldots = 1, 2, 3, 4$ and seven internal coordinates x^u ; $u, v, w, x, y, z = 5, \ldots, 11$. In the absence of any G-flux (for heterotic M-theory this amounts to considering only the leading order which is M-theory itself without boundary or M5-brane sources) the metric solution to the Killing-spinor equation, which describes a supersymmetry-preserving vacuum, is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{uv}(x^w) dx^u dx^v \ , \tag{4.2}$$

where g_{uv} decomposes into a direct product of the Calabi-Yau metric $g_{a\bar{b}}$ and the metric $g_{11,11}$ of the eleventh dimension. Without loss of generality one can set $g_{11,11} = 1$. We will denote the six real Calabi-Yau indices $l, m, n, p, q, \ldots$ while $\bar{l}, \bar{m}, \bar{n}, \bar{p}, \bar{q}, \ldots$ are the respective flat tangent space indices. The alternative choice of holomorphic and anti-holomorphic indices will be denoted $a, b, c \ldots$ and $\bar{a}, \bar{b}, \bar{c} \ldots$. Genuinely we have to take the boundary sources into account which require turning on a G-flux in the internal directions. This necessitates a more general metric, for which we choose the warp-factor Ansatz

$$\begin{aligned} ds^2 &= \hat{g}_{MN} dx^M dx^N \ ; \ M, N = 1, \ldots, 11 \\ &= e^{b(x^W, x^{11})} \eta_{\mu\nu} dx^\mu dx^\nu + e^{f(x^W, x^{11})} g_{lm}(x^n) dx^l dx^m + e^{k(x^W, x^{11})} dx^{11} dx^{11} \ . \end{aligned} \tag{4.3}$$

It will turn out that the appropriate G-flux of the relevant sources can be accomodated with this Ansatz. The most general Ansatz which allows for arbitrary G-flux compatible with supersymmetry will be considered in the last section.

The initial Calabi-Yau manifold possesses a closed Kähler-form $\omega_{a\bar{b}}$. However, a non-zero G-flux entails a non-trivial internal warp-factor e^f, thereby rendering the "deformed" Kähler-form $\hat{\omega}_{a\bar{b}} = e^f \omega_{a\bar{b}}$ non-closed. In this respect, the warp-factor e^f serves as a measure for the deviation from Kählerness of the internal complex threefold.

To preserve 4-dimensional Poincaré-invariance, we set G_{1234} and all other components of G with at least one external index to zero[26]. An important point is that in order

[26] An external $G_{1234} = \epsilon_{1234} \lambda(x^m, x^{11})$ would be compatible with Poincaré-symmetry but the known sources (boundaries, M5-branes) do not give rise to such a flux. Moreover, compatibility with the Bianchi-identity allows only a constant G_{1234}. Though such a sourceless constant field-strength is allowed by the non-compact external spacetime, we will set it to zero subsequently.

to preserve supersymmetry the magnetic sources on the right-hand-side of the Bianchi-identity must be (2,2,1) forms [63]. I.e. they are forms with two holomorphic, two anti-holomorphic indices and one x^{11}-index. This is clear for the boundary sources and amounts for the M5-brane sources to an orientation parallel to the boundaries. Solving the Bianchi-identity, we see from the fact that the sources are (2,2,1) forms, that only the components $G_{a\bar{b}c\bar{d}}$, $G_{ab\bar{c}11}$, $G_{a\bar{b}\bar{c}11}$ can become non-zero.

4.2.1 The Killing-Spinor Equation

The supersymmetry-variation of the gravitino in low-energy M-theory is given in the full metric (4.3) by

$$\delta\Psi_I = \hat{D}_I\tilde{\eta} + \frac{\sqrt{2}}{288}\left(\hat{\Gamma}_{IJKLM} - 8\hat{g}_{IJ}\hat{\Gamma}_{KLM}\right)G^{JKLM}\tilde{\eta} \ , \tag{4.4}$$

where $\tilde{\eta} = e^{-\psi(x^U,x^{11})}\eta$. Here, η is the original covariantly constant spinor and the exponential-factor accounts for the correction if G-flux is turned on. We will assume ψ to be real and see later on that this is indeed compatible with supersymmetry in the warped background. Subsequently, indices are raised and lowered with the full metric $\hat{g}_{MN}$, which is also how contractions are performed in (4.4). Setting the variation to zero in order to obtain a supersymmetry preserving solution, we obtain the Killing-spinor equation.

Covariant Derivative Contribution

Let us first deal with the part containing the covariant derivative of the Majorana-spinor $\tilde{\eta}$. Using the definition of the spin-connection for the warped-metric[27]

$$\Omega_{I\bar{J}\bar{K}}(\hat{e}) = \frac{1}{2}\left(\hat{e}_{\bar{J}}{}^{J}\tilde{\Omega}_{IJ\bar{K}}(\hat{e}) - \hat{e}_{\bar{K}}{}^{K}\tilde{\Omega}_{IK\bar{J}}(\hat{e}) - \hat{e}_{\bar{J}}{}^{J}\hat{e}_{\bar{K}}{}^{K}\hat{e}^{\bar{I}}{}_{I}\tilde{\Omega}_{JK\bar{I}}(\hat{e})\right) , \tag{4.5}$$

$$\tilde{\Omega}_{IJ\bar{K}}(\hat{e}) = \partial_I\hat{e}_{\bar{K}J} - \partial_J\hat{e}_{\bar{K}I} \ , \tag{4.6}$$

allows to express the warped spin-connection through the initial one

$$\begin{aligned}
\Omega_{\mu\bar{\nu}\bar{l}}(\hat{e}) &= \frac{1}{2}\hat{e}_{\bar{l}}{}^{m}\hat{e}_{\bar{\nu}\mu}\partial_m b \ , \qquad \Omega_{\mu\bar{\nu}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2}\hat{e}_{\bar{\nu}\mu}\hat{e}_{\bar{1}\bar{1}}{}^{11}\partial_{11}b \\
\Omega_{l\bar{m}\bar{n}}(\hat{e}) &= -\hat{e}_{[\bar{m}}{}^{p}\hat{e}_{\bar{n}]l}\partial_p f + \Omega_{l\bar{m}\bar{n}}(e) \ , \qquad \Omega_{l\bar{m}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2}\hat{e}_{\bar{m}l}\hat{e}_{\bar{1}\bar{1}}{}^{11}\partial_{11}f \\
\Omega_{11\bar{l}\bar{1}\bar{1}}(\hat{e}) &= -\frac{1}{2}\hat{e}_{\bar{l}}{}^{m}\hat{e}_{\bar{1}\bar{1},11}\partial_m k \ ,
\end{aligned} \tag{4.7}$$

[27] Here $\bar{J}, \bar{K}, \ldots$ denote flat 11-dimensional indices.

and all other terms are zero. This is done to employ the covariant constancy $D_I\eta = (\partial_I + \frac{1}{4}\Omega_{I\bar{J}\bar{K}}(e)\Gamma^{\bar{J}\bar{K}})\eta = 0$ of the initial spinor-parameter, which brings us to

$$dx^I\hat{D}_I\tilde{\eta} = \Big(- dx^u\partial_u\psi + \frac{1}{4}dx^\mu\Big[\hat{\Gamma}_\mu{}^l\partial_l b + \hat{\Gamma}_\mu{}^{11}\partial_{11}b\Big] + \frac{1}{4}dx^l\Big[\hat{\Gamma}_l{}^m\partial_m f + \hat{\Gamma}_l{}^{11}\partial_{11}f\Big] + \frac{1}{4}dx^{11}\hat{\Gamma}_{11}{}^l\partial_l k\Big)\tilde{\eta}\,. \tag{4.8}$$

Let us now specify, that our internal space actually consists of a Calabi-Yau and a separate eleventh dimension. The positive chirality condition $\Gamma^{11}\eta = \eta$ on the original space translates into $\hat{\Gamma}^{11}\tilde{\eta} = e^{-k/2}\tilde{\eta}$ on the warped space. The condition that we have a covariantly constant spinor (and its complex conjugate) on the Calabi-Yau gives $\Gamma^a\eta = 0$, $\Gamma_{\bar{a}}\eta = 0$ and translates into $\hat{\Gamma}^a\tilde{\eta} = 0$, $\hat{\Gamma}_{\bar{a}}\tilde{\eta} = 0$. Using these relations plus the Dirac-algebra $\{\hat{\Gamma}^a, \hat{\Gamma}^{\bar{b}}\} = 2\hat{g}^{a\bar{b}}$, we end up with

$$\begin{aligned} dx^I\hat{D}_I\tilde{\eta} = \Big\{ &\Big[- \partial_a(\psi + \frac{f}{4})dx^a + \partial_{\bar{a}}(-\psi + \frac{f}{4})dx^{\bar{a}} - \partial_{11}\psi dx^{11}\Big] \\ &+ \Big[\frac{1}{4}e^{-k/2}\partial_{11}b\, dx_\mu\Big]\hat{\Gamma}^\mu + \Big[\frac{1}{4}e^{-k/2}\partial_{11}f dx_{\bar{a}} - \frac{1}{4}e^{-k/2}\partial_{\bar{a}}k dx_{11}\Big]\hat{\Gamma}^{\bar{a}} \\ &+ \Big[\frac{1}{4}\partial_{\bar{a}}b\, dx_\mu\Big]\hat{\Gamma}^{\mu\bar{a}} + \Big[\frac{1}{4}\partial_{\bar{b}}f dx_{\bar{a}}\Big]\hat{\Gamma}^{\bar{a}\bar{b}}\Big\}\tilde{\eta}\,. \end{aligned} \tag{4.9}$$

G-Flux Contributions

Next, let us deal with the second term in the Killing equation, containing the G-flux. To obtain condense expressions, it proves convenient to parameterize the three sorts of allowed fluxes by defining

$$\alpha = \omega^{lm}\omega^{np}G_{lmnp} \tag{4.10}$$

$$\beta_l = \omega^{mn}G_{lmn11} \tag{4.11}$$

$$\Theta_{lm} = G_{lmnp}\omega^{np}\,. \tag{4.12}$$

where $\omega_{a\bar{b}} = -ig_{a\bar{b}}$, $\omega^{a\bar{b}} = ig^{a\bar{b}}$ denotes the Kähler-form of the initial Calabi-Yau manifold. The warped metric is related to the Kähler-form by $\hat{g}^{a\bar{b}} = -ie^{-f}\omega^{a\bar{b}}$. Subsequently, we will make use of

$$\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}}G_{a\bar{b}c\bar{d}} = -\frac{1}{4}e^{-2f}\alpha \tag{4.13}$$

$$\hat{g}^{b\bar{c}}G_{lb\bar{c}11} = -\frac{i}{2}e^{-f}\beta_l \tag{4.14}$$

$$\hat{g}^{c\bar{d}}G_{lmc\bar{d}} = -\frac{i}{2}e^{-f}\Theta_{lm} \tag{4.15}$$

to express the occuring contractions through the above defined parameters. In order to handle the various contractions of $\hat{\Gamma}$-matrices with the G-flux, it is convenient to evaluate the expressions by first letting the matrices act on $\tilde{\eta}$ and employ $\hat{g}^{a11} = 0$, $\hat{\Gamma}^{11}\tilde{\eta} = e^{-k/2}\tilde{\eta}$, $\hat{\Gamma}^{a}\tilde{\eta} = 0$. Taking $\tilde{\eta}$ as the ground state, $\hat{\Gamma}^{a}$ and $\hat{\Gamma}^{\bar{a}}$ can be regarded as annihilation and creation operators, respectively. This leads to some useful identities (B.1) collected in the appendix. With their help, we establish the various contractions (B.2) of the five-index $\hat{\Gamma}$-matrices with the G-flux and also the contractions (B.3) of the three-index $\hat{\Gamma}$-matrices with G. These can be found in the appendix, as well. Putting all this together, we arrive at the following expression for the second part of the Killing equation

$$
\begin{aligned}
& dx_I\big(\hat{\Gamma}^{IJKLM} - 8\hat{g}^{IJ}\hat{\Gamma}^{KLM}\big)G_{JKLM}\tilde{\eta} = \Big\{ 3e^{-k/2-f}\left[4i\beta_{\bar{a}}dx^{\bar{a}} + 12i\beta_{a}dx^{a} - e^{-f}\alpha\, dx_{11}\right] \\
& -3e^{-2f}\alpha\, dx_\mu\hat{\Gamma}^{\mu} + 3e^{-f}\left[-e^{-f}\alpha\, dx_{\bar{a}} + 12i\Theta^{\bar{b}}{}_{\bar{a}}dx_{\bar{b}} - 8i\beta_{\bar{a}}dx^{11}\right]\hat{\Gamma}^{\bar{a}} - 12ie^{-k/2-f}\beta_{\bar{a}}dx_\mu\hat{\Gamma}^{\mu\bar{a}} \\
& +3e^{-k/2}\left[4ie^{-f}\beta_{\bar{a}}dx_{\bar{b}} - 12G^{\bar{c}}{}_{\bar{a}\bar{b}11}dx_{\bar{c}}\right]\hat{\Gamma}^{\bar{a}\bar{b}}\Big\}\,. \qquad (4.16)
\end{aligned}
$$

Complete Killing-Spinor Equation

Now, the complete Killing-spinor equation can be composed out of the two pieces (4.9) and (4.16) and is given by

$$
\begin{aligned}
& dx^I\hat{D}_I\tilde{\eta} + \frac{\sqrt{2}}{288}dx^I\left(\hat{\Gamma}_{IJKLM} - 8\hat{g}_{IJ}\hat{\Gamma}_{KLM}\right)G^{JKLM}\tilde{\eta} \\
&= \Bigg(\Big[\big(-\partial_a\psi - \frac{1}{4}\partial_a f + i\frac{\sqrt{2}}{8}e^{-k/2-f}\beta_a\big)dx^a + \big(-\partial_{\bar{a}}\psi + \frac{1}{4}\partial_{\bar{a}}f + i\frac{\sqrt{2}}{24}e^{-k/2-f}\beta_{\bar{a}}\big)dx^{\bar{a}} \\
&\quad - \big(\frac{\sqrt{2}}{96}e^{k/2-2f}\alpha + \partial_{11}\psi\big)dx^{11}\Big] + \frac{1}{4}\left[e^{-k/2}\partial_{11}b - \frac{\sqrt{2}}{24}e^{-2f}\alpha\right]dx_\mu\hat{\Gamma}^{\mu} + \frac{1}{4}\Big[e^{-k/2}\partial_{11}f\,dx_{\bar{a}} \\
&\quad - \frac{\sqrt{2}}{24}e^{-2f}\alpha dx_{\bar{a}} + i\frac{1}{\sqrt{2}}e^{-f}\Theta^{\bar{b}}{}_{\bar{a}}dx_{\bar{b}} - \big(e^{-k/2}\partial_{\bar{a}}k + i\frac{\sqrt{2}}{3}e^{-f-k}\beta_{\bar{a}}\big)dx_{11}\Big]\hat{\Gamma}^{\bar{a}} + \frac{1}{4}\Big[\partial_{\bar{a}}b \\
&\quad - i\frac{\sqrt{2}}{6}e^{-k/2-f}\beta_{\bar{a}}\Big]dx_\mu\hat{\Gamma}^{\mu\bar{a}} + \frac{1}{4}\Big[\partial_{\bar{b}}f\,dx_{\bar{a}} + i\frac{\sqrt{2}}{6}e^{-k/2-f}\beta_{\bar{a}}dx_{\bar{b}} \\
&\quad - \frac{1}{\sqrt{2}}e^{-k/2}G^{\bar{c}}{}_{\bar{a}\bar{b}11}dx_{\bar{c}}\Big]\hat{\Gamma}^{\bar{a}\bar{b}}\Bigg)\tilde{\eta} \\
&= 0\,. \qquad (4.17)
\end{aligned}
$$

4.2.2 The Warp-Factor – Flux Relations

Setting the coefficients of the various $\hat{\Gamma}$-matrices to zero, we have to distinguish carefully between the imaginary and the real part of the equations. For arbitrary vectors A_U, B_U, the sum $A^a B_a + A^{\bar{a}} B_{\bar{a}}$ is real, whereas the difference $A^a B_a - A^{\bar{a}} B_{\bar{a}}$ is purely imaginary. Furthermore α is a real parameter.

The I, $\hat{\Gamma}^\mu$ *and* $\hat{\Gamma}^{\mu\bar{a}}$*-Terms*

From the terms proportional to the unit-matrix, $\hat{\Gamma}^\mu$ and $\hat{\Gamma}^{\mu\bar{a}}$ we finally receive the relations

$$8\partial_a \psi = \partial_a f = -2\partial_a b = i\frac{\sqrt{2}}{3} e^{-k/2-f} \beta_a \tag{4.18}$$

$$4\partial_{11}\psi = -\partial_{11} b = -\frac{\sqrt{2}}{24} e^{k/2-2f} \alpha \ . \tag{4.19}$$

The $\hat{\Gamma}^{\bar{a}}$*-Terms*

The terms proportional to $\hat{\Gamma}^{\bar{a}}$ lead to

$$\partial_a k = i\frac{\sqrt{2}}{3} e^{-k/2-f} \beta_a \ , \tag{4.20}$$

which shows that the warp-factors f and k are equal up to an additive function F depending merely on x^{11}

$$f(x^W, x^{11}) = k(x^W, x^{11}) + F(x^{11}) \ . \tag{4.21}$$

In the following we will set $F(x^{11})$ to zero since it can be eliminated by a simple reparameterization of x^{11}. Furthermore the $\hat{\Gamma}^{\bar{a}}$ terms yield the relation

$$\partial_{11} f = \frac{\sqrt{2}}{24} e^{k/2-2f} \alpha - i\frac{1}{\sqrt{2}} e^{k/2-f} \Theta^{\bar{a}}{}_{\bar{a}} \ \ , \ \ \text{no sum over } \bar{a} \tag{4.22}$$

together with

$$\Theta^{\bar{b}}{}_{\bar{a}} = 0 \ \ , \ \ \bar{b} \neq \bar{a} \ . \tag{4.23}$$

Note that in (4.22) there is no summation over the antiholomorphic indices $\bar{a}$. Hence the relation (4.22) implies the following isotropy-condition

$$\Theta^{\bar{1}}{}_{\bar{1}} = \ldots = \Theta^{\bar{n}}{}_{\bar{n}} \ , \tag{4.24}$$

with n the complex dimension of the Calabi-Yau manifold. Using the identity $\sum_{\bar{a}=\bar{1}}^{\bar{n}} \Theta^{\bar{a}}{}_{\bar{a}} = -\frac{i}{2}e^{-f}\alpha$, it then follows that (4.22) simplifies to

$$\partial_{11} f = \frac{\sqrt{2}}{4}\left(\frac{1}{6} - \frac{1}{n}\right) e^{k/2-2f}\alpha \ . \tag{4.25}$$

The $\hat{\Gamma}^{\bar{a}\bar{b}}$*-Terms*

Finally the $\hat{\Gamma}^{\bar{a}\bar{b}}$ terms lead to an equation, which can be simplified, using the relation for $\partial_a f$ from (4.18), to

$$ie^{-f}\beta_{[\bar{a}}dx_{\bar{b}]} = G^{\bar{c}}{}_{\bar{a}\bar{b}11}dx_{\bar{c}} \ . \tag{4.26}$$

The component of this equation with $\bar{c} \neq \bar{a}, \bar{b}$ leads to the following G-flux constraint

$$G^{\bar{c}}{}_{\bar{a}\bar{b}11} = 0 \ \ , \ \ \bar{c} \neq \bar{a}, \bar{b} \ , \tag{4.27}$$

whereas the $\bar{c} = \bar{a}$ and $\bar{c} = \bar{b}$ components simply reproduce the defining relation (4.11) for $\beta_{\bar{a}}$.

To summarize, the Killing-spinor equation leads to the set of equations (4.18), (4.19), (4.21), (4.25) together with the G-flux constraints (4.23), (4.24), (4.27).

We are now in a position to briefly check that our assumption of choosing ψ real does not lead to inconsistencies. For this purpose it is enough to show that $\mathrm{Im}\,\psi$ is constant, which in particular means that a zero value can be maintained. Following [63], we use the above equation for $\partial_a\psi$ and obtain

$$\begin{aligned}\hat{g}^{a\bar{b}}\partial_a\partial_{\bar{b}}\mathrm{Im}\psi = \hat{g}^{a\bar{b}}\partial_a\partial_{\bar{b}}\left(\frac{\psi - \bar{\psi}}{2i}\right) &= \frac{\sqrt{2}}{48}\hat{g}^{a\bar{b}}\left[\partial_{\bar{b}}\left(e^{-3f/2}\beta_a\right) + \partial_a\left(e^{-3f/2}\beta_{\bar{b}}\right)\right] \\ &= \frac{\sqrt{2}}{48}e^{-3f/2}D^m\beta_m \ .\end{aligned} \tag{4.28}$$

Employing $D^m\beta_m = 0$, which can be obtained from the field equation for G, one establishes that $\mathrm{Im}\,\psi$ is a harmonic function on a compact space and therefore has to be constant.

4.3 Implications of the Warped Geometry

Let us now analyze the above equations in more detail. Notice, that up to now we were not forced to specify whether we compactify on a CY_2 or a CY_3 – the complex dimension n of the CY_n entered as a free parameter.

Arbitrary G-flux parameters α, β are only compatible with a pure warp-factor description of the internal deformed Calabi-Yau in the 6-dimensional case with $n = 2$, as we will now see. For $n = 2$ we obtain

$$\partial_{11} f = -\frac{\sqrt{2}}{12} e^{k/2-2f} \alpha \,, \tag{4.29}$$

which says, together with (4.18),(4.19) that

$$8\partial_a \psi = \partial_a f = -2\partial_a b \tag{4.30}$$

$$8\partial_{11} \psi = \partial_{11} f = -2\partial_{11} b \tag{4.31}$$

and implies $8\psi = f = k = -2b$. Here the warp-factors depend on both x^m and x^{11}. For $n \neq 2$ the $\partial_{11} f$ part receives a different prefactor and does not allow for this conclusion. Instead – as we will see explicitly for the case of $n = 3$ below – one has to set either (4.30) or (4.31) to zero to obtain a consistent solution.

If we choose $n = 3$, we have

$$\partial_{11} f = -\frac{\sqrt{2}}{24} e^{k/2-2f} \alpha \,. \tag{4.32}$$

Taking mixed derivatives of f and b this implies that $\partial_a \partial_{11} f = 0$. A non-trivial solution is either obtained from $\partial_{11} f = 0$ or $\partial_a f = 0$. The implications of these two cases will be analyzed in more detail in the following two sections.

4.4 Transition from Strong to Weak Coupling

In this section, we will present the connection to the heterotic string with torsion. The choice, $\partial_{11} f = 0$, requires $\alpha = 0, \beta_a \neq 0$ and leads to

$$8\psi(x^m) = f(x^m) = k(x^m) = -2b(x^m) \tag{4.33}$$

without any dependence on x^{11}. The required sort of fluxes is obtained by solving the Bianchi-identity $dG = \sum_{i=1}^m \delta(x^{11} - z_i) S_i(x^m) \wedge dx^{11}$ with m sources by $G = \sum_{i=1}^m \delta(x^{11} - z_i) P_i(x^m) \wedge dx^{11}$ with $dP_i = S_i$. This type of geometry seems tailor-made for a smooth transition to the weakly coupled heterotic string, since any x^{11} dependence is lost. Indeed, we will now show that the heterotic M-theory relation between warp-factor and G-flux reproduces the corresponding relation (B.15) for the heterotic string with torsion.

The warp-factor belonging to the 4-dimensional external part multiplies the Minkowski-metric – both in the string and the M-theory case – and is therefore fixed in the sense that one does not have to take into account further coordinate-reparameterizations for a comparison. Let us therefore start with the relation between external warp-factor and G-flux by using (4.18) for $\partial_a b$ plus $f = k = -2b$ and employing the definition of β_a to obtain

$$\partial_a(e^{-b}) = -\frac{\sqrt{2}}{3} G_{ab}{}^{b}{}_{11} \; . \tag{4.34}$$

The contraction on the right-hand-side is with respect to $\hat{g}^{b\bar{c}}$. To compare M-theory with string-theory [26] one has to perform an overall Weyl-rescaling involving the dilaton, $g^{\sigma}_{MN} = e^{2\phi/3}\hat{g}_{MN}$, which brings us to the string-frame. Here we have

$$\begin{aligned} \partial_a(e^{-b}) &= -\frac{\sqrt{2}}{3} e^{\frac{2\phi}{3}} G_{ab}{}^{b}{}_{11} \; , \\ ds^2 &= e^{b+\frac{2\phi}{3}} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + \dots \; . \end{aligned}$$

Finally, let us go over to the 10-dimensional Einstein-frame via $g^E_{AB} = e^{-\phi/2} g^{\sigma}_{AB}$ in which we obtained the heterotic string relation between warp-factor and torsion. We thus arrive at

$$\begin{aligned} \partial_a(e^{-b}) &= -\frac{\sqrt{2}}{3} e^{\frac{\phi}{6}} G_{ab}{}^{b}{}_{11} \; , \\ ds^2 &= e^{b+\frac{\phi}{6}} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + \dots \; , \end{aligned} \tag{4.35}$$

where again the contraction is performed with the metric of the actual frame, $(g^E)^{b\bar{c}}$. A comparison of the above metric with the heterotic string metric (B.6) shows that we have to identify 2ϕ with $b + \phi/6$, which gives

$$b = \frac{11}{6}\phi \; . \tag{4.36}$$

If we use this in (4.35), we receive

$$\partial_a(e^{-2\phi}) = -\frac{4\sqrt{2}}{11} G_{ab}{}^{b}{}_{11} \; . \tag{4.37}$$

Setting $G_{ab}{}^{b}{}_{11}$ equal to $H_{ab}{}^{b}$ up to some constant normalization factor, we see that indeed the relation between warp-factor and torsion of the heterotic string (see appendix B.2 for relevant facts about the heterotic string with torsion and a derivation of the following formula in that context)

$$\partial_a(e^{-2\phi}) = -\frac{1}{2} H_{ab}{}^{b} \; . \tag{4.38}$$

can be reproduced from heterotic M-theory including G-flux. This represents a non-trivial check on the duality between the strongly and the weakly coupled heterotic regions in the presence of torsion.

The choice of fluxes treated in this subsection leads to a Calabi-Yau volume which does not depend on x^{11}. Moreover due to the deformation with the warp-factor e^f the Kähler-form is no longer closed. In addition the Ricci-tensor for the internal six-manifold becomes

$$R_{a\bar{b}}(e^f g_{mn}) = R_{a\bar{b}}(g_{mn}) + g_{a\bar{b}} g^{c\bar{d}} (2\partial_c f \partial_{\bar{d}} f + \partial_c \partial_{\bar{d}} f) - \partial_a f \partial_{\bar{b}} f + 2\partial_a \partial_{\bar{b}} f \ , \tag{4.39}$$

where the derivatives of f are determined through the G-flux by (4.18). Though $R_{a\bar{b}}(g_{mn}) = 0$ due to the Ricci-flatness of the initial Calabi-Yau space, we recognize that in the presence of G-flux the internal six-manifold also looses its property of being Ricci-flat.

4.5 The Internal Volume Dependence on the Orbifold Direction

The second choice, $\partial_a f = 0$, requires $\alpha \neq 0, \beta_a = 0$ and implies

$$4\psi(x^{11}) = f(x^{11}) = k(x^{11}) = -b(x^{11}) \tag{4.40}$$

without any x^m dependence. The necessary non-vanishing $G_{a\bar{b}c\bar{d}}$ and vanishing $G_{ab\bar{c}11}$ are obtained by solving the Bianchi-identity $dG = \sum_{i=1}^m \delta(x^{11} - z_i) S_i(x^m) \wedge dx^{11}$ through[28] $G = \sum_{i=1}^m \Theta(x^{11} - z_i) S_i(x^m)$. Again $S_i(x^m)$ is a closed 4-form representing the strength of the i^{th} magnetic source.

The volume of the Calabi-Yau, as measured by the warped metric, is given by $V(x^{11}) = \int_{CY_3} d^6x \sqrt{\hat{g}} = e^{3f} \int_{CY_3} d^6x \sqrt{g}$. The decisive part, which is responsible for the variation of the volume with x^{11}, is the factor e^{3f}. For its determination, we use $f = k$ and the equation for $\partial_{11} f$

$$\partial_{11}(e^{3f/2}) = -\frac{1}{8\sqrt{2}} \alpha \tag{4.41}$$

which is solved by

$$e^{3f(x^{11})/2} = e^{3f(0)/2} - \frac{1}{8\sqrt{2}} \int_0^{x^{11}} dz\, \alpha(z) \ . \tag{4.42}$$

[28]The Heaviside step-function $\Theta(x)$ is defined by $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$.

Notice that α does not depend on the Calabi-Yau coordinates, which can be easily seen by acting with ∂_a on (4.41) and taking into account that $\partial_a f = 0$. Hence the variation of the Calabi-Yau volume with x^{11} is given by

$$V(x^{11}) = \Big(1 - \frac{1}{2\sqrt{2}}\,\omega^{a\bar{b}}\omega^{c\bar{d}}\int_0^{x^{11}} dz\, G_{a\bar{b}c\bar{d}}(x^m, z)\Big)^2 V_0 \ , \tag{4.43}$$

where $V_0 = \int_{CY_3} d^6x\sqrt{g}$ is the Calabi-Yau volume in the initial metric. The integration constant $e^{3f(0)/2}$ has been set to 1 to obtain a smooth transition from $V(x^{11})$ to V_0 in case that we turn off any G-flux. The only assumption about the full heterotic M-theory that we will have to make is that the sources can still be localized at $x^{11} = z_i$ in the eleventh direction, i.e. that the Bianchi-identity possesses the form $dG = \sum_{i=1}^m \delta(x^{11} - z_i)S_i(x^m) \wedge dx^{11}$. Its solution $G = \sum_{i=1}^m \Theta(x^{11} - z_i)S_i(x^m)$ then leads to the following behaviour of the Calabi-Yau volume

$$V(x^{11}) = \Big(1 - \sum_{i=1}^m (x^{11} - z_i)\Theta(x^{11} - z_i)\mathcal{S}_i\Big)^2 V_0 \ , \tag{4.44}$$

where $\mathcal{S}_i = \frac{1}{2\sqrt{2}}\omega^{a\bar{b}}\omega^{c\bar{d}}(S_i)_{a\bar{b}c\bar{d}}(x^m)$. Thus we get the remarkably simple result that in the full treatment the linear behaviour of the first order approximation gets replaced by a quadratic behaviour.

For the simplest case with only the two boundary sources at $z_1 = 0, z_2 = d$, we obtain $\alpha = 8\sqrt{2}\Theta(x^{11})\mathcal{S}_1$ with $\mathcal{S}_1$ representing the magnetic source of the "visible" boundary. This gives the warp-factor

$$e^{3f(x^{11})/2} = 1 - \mathcal{S}_1 x^{11} \ , \tag{4.45}$$

which leads to the following volume dependence (see fig.6)

$$V(x^{11}) = \big(1 - \mathcal{S}_1 x^{11}\big)^2 V_0 \ . \tag{4.46}$$

Here and in the following the right boundary will not be depicted – it would cut off the solution at some finite distance d. For phenomenological reasons one should place the right boundary at such a distance d that $\langle V\rangle = \frac{1}{d}\int_0^d dx^{11} V(x^{11})$ amounts to the correct value for Newton's Constant via (4.1). To determine the actual value of $\mathcal{S}_1$ (and thereby the Newton Constant related to this length) however would require the knowledge of heterotic M-theory to all orders in $\kappa^{2/3}$. It is only here where the complete information is needed. In the phenomenological relevant case where $\mathcal{S}_1 > 0$, a zero volume develops (in accordance with the leading order result) at $x_0^{11} = 1/\mathcal{S}_1$. One can now exploit (4.1) the

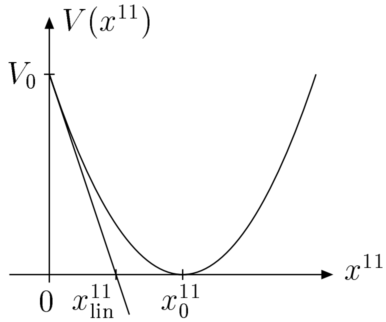

Figure 6: The quadratic dependence of the Calabi-Yau volume on the orbifold direction in the full geometry and its linear approximation to order $\kappa^{2/3}$. If higher order contributions are negligible then the linear approximation is valid for small x^{11}. The left boundary corresponds to the "visible" world.

other way round to derive the distance d between both boundaries which would amount to the correct value for Newton's Constant

$$d = x_0^{11}\left[1-\left(1-\frac{3\kappa^2}{16\pi G_N}\frac{\mathcal{S}_1}{V_0}\right)^{1/3}\right] . \tag{4.47}$$

Note the difference between d and the zero position x_0^{11}. More graphically, if $\int_0^d dx^{11}V(x^{11}) > V_0 x_0^{11}/3$ then $d > x_0^{11}$ while for $\int_0^d dx^{11}V(x^{11}) \le V_0 x_0^{11}/3$ one obtains $d \le x_0^{11}$.

Moreover – as becomes clear from fig.6 – with the quadratic volume behaviour tiny quantum effects are now able to resolve the zero volume as opposed to the linearized case (cf. in this respect also [75],[76]). The full warp-metric reads in terms of the volume

$$ds^2 = \left(\frac{V}{V_0}\right)^{-1/3}\eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{V}{V_0}\right)^{1/3} g_{lm}(x^n)dx^l dx^m + \left(\frac{V}{V_0}\right)^{1/3} dx^{11}dx^{11} . \tag{4.48}$$

Finally, one would like to see the transition of the full expression for $V(x^{11})$ to the linearized expression which was derived in [63]. For this purpose one has to expand the sources into a power-expansion in $\kappa^{2/3}$. If we are only interested in sources coming from the boundary, then we know that they start at the order $\kappa^{2/3}$

$$\mathcal{S}_1 = \mathcal{S}_1^{(1)}\kappa^{2/3} + \mathcal{S}_1^{(2)}\kappa^{4/3} + \dots . \tag{4.49}$$

For the first order approximation, we have to truncate this series after the first term, which indeed gives rise to a linear volume dependence

$$V(x^{11}) = \left(1 - 2\mathcal{S}_1^{(1)}\kappa^{2/3}x^{11}\right)V_0 + \mathcal{O}(\kappa^{4/3}) \tag{4.50}$$

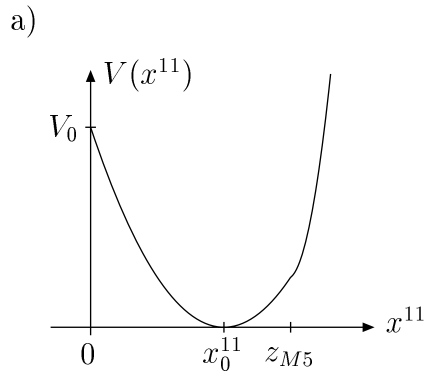

Figure 7: The figure shows the volume dependence in the presence of an additional M5-brane at z_{M5}. In a) the situation for $x_0^{11} < \tilde{x}_0^{11} < z_{M5}$ is depicted while b) shows the behaviour for $x_0^{11} > \tilde{x}_0^{11} > z_{M5}$.

as found in [63]. If we now read off the zero of $V(x^{11})$, we get $x_{\rm lin}^{11} = 1/(2\mathcal{S}_1^{(1)}\kappa^{2/3})$ in the linearized case, while the full solution gives a different first oder zero

$$x_0^{11} = 1/(\mathcal{S}_1^{(1)}\kappa^{2/3}) + \mathcal{O}(\kappa^{4/3}) \ . \tag{4.51}$$

This little puzzle is resolved by noticing that the linear approximation (4.50) holds true only as long as $\mathcal{S}_1^{(1)}\kappa^{2/3}x^{11} \ll 1$ (plus similar conditions for the higher $\mathcal{S}_1^{(i)}$, $i \geq 2$ contributions). Because at the position of the zero, we face $\mathcal{S}_1^{(1)}\kappa^{2/3}x_{\rm lin}^{11} \approx \mathcal{S}_1^{(1)}\kappa^{2/3}x_0^{11} = 1$, the linear approximation (4.50) breaks down and cannot be used to determine reliably the zero of $V(x^{11})$. Therefore, in contrast to the first order analysis, the actual zero at the first order level becomes larger by a factor 2

$$x_0^{11} = 2x_{\rm lin}^{11} \ . \tag{4.52}$$

Let us briefly consider the case with three sources – the two from the boundaries $\mathcal{S}_1, \mathcal{S}_2$ plus a further one $\mathcal{S}_{M5}$ from an M5-brane placed in between at z_{M5}. With $\alpha = 8\sqrt{2}[\Theta(x^{11})\mathcal{S}_1 + \Theta(x^{11} - z_{M5})\mathcal{S}_{M5}]$ we get a warp-factor

$$e^{3f(x^{11})/2} = 1 - x^{11}\mathcal{S}_1 - (x^{11} - z_{M5})\Theta(x^{11} - z_{M5})\mathcal{S}_{M5} \tag{4.53}$$

and the following volume dependence (see fig.7)

$$V(x^{11}) = \begin{cases} \left(1 - \mathcal{S}_1 x^{11}\right)^2 V_0 & ; \quad x^{11} < z_{M5} \\ \left(1 - (\mathcal{S}_1 + \mathcal{S}_{M5})x^{11} + \mathcal{S}_{M5}z_{M5}\right)^2 V_0 & ; \quad x^{11} \geq z_{M5} \end{cases} \tag{4.54}$$

The zero of the parabola for $x^{11} \geq z_{M5}$ lies at $\tilde{x}_0^{11} = (1 + \mathcal{S}_{M5}z_{M5})/(\mathcal{S}_1 + \mathcal{S}_{M5})$. Depending on whether $x_0^{11} < \tilde{x}_0^{11} < z_{M5}$ or $x_0^{11} > \tilde{x}_0^{11} > z_{M5}$ we obtain a different behaviour for

$V(x^{11})$. In the M5-brane case we have two additional parameters, $\mathcal{S}_{M5}$ and z_{M5}, which have an influence on $\langle V \rangle$ and therefore via (4.1) on Newton's Constant.

Another interesting point is that for $\alpha \neq 0, \beta_a = 0$ the internal six-manifold remains Kähler. This is due to the fact that the warp-factor f does not depend on x^m. Furthermore, we see from the general formula (4.39) for the Ricci-tensor that in this case the six-manifold also keeps its property of being Ricci-flat. In other words the six-manifold is still a Calabi-Yau space with volume depending on the "parameter" x^{11}.

4.6 The Effective Distance-Modulus Potential

In this section we are going to derive the effective potential for the modulus d, which describes the distance between the two boundaries of heterotic M-theory. We saw previously that for the case with fluxes $\alpha \neq 0, \beta_a = 0$ we gain a quadratic behaviour in x^{11} for the Calabi-Yau threefold volume. The value of its zero is closely linked to the value of the 4-dimensional Newton Constant. Hence, one has to ask whether such value for d could be stabilized by means of an effective potential, which results from integrating the eleventh dimension out of the heterotic M-theory action. For the relevant background we have to use the above derived warped geometry.

Let us start with the bosonic action of 11-dimensional heterotic M-theory [46], which under the condition that only the G-flux component $G_{a\bar{b}c\bar{d}}$ does not vanish reads ($i = 1, 2$)

$$
\begin{aligned}
S &= S_{11} + S_{10}^i \,, \\
S_{11} &= \int d^{11}x \frac{\sqrt{\hat{g}}}{\kappa^2} \left[-\frac{R(\hat{g}_{MN})}{2} - \frac{1}{48} G_{a\bar{b}c\bar{d}} G^{a\bar{b}c\bar{d}} \right] \,, \\
S_{10}^i &= \frac{1}{(4\pi)^{5/3}\kappa^{4/3}} \int d^{10}x \sqrt{\hat{g}_i^{(10)}} \left[-\frac{1}{4} \mathrm{tr} F_{a\bar{b}} F^{a\bar{b}} \right] \,.
\end{aligned}
\tag{4.55}
$$

The Measure-Factors

For the case with varying volume we found in the last section for the warp-factors the relation $f(x^{11}) = k(x^{11}) = -b(x^{11})$. This allows to express $\sqrt{\hat{g}} = e^{-3b/2}\sqrt{g_{CY_3}}$, where g_{CY_3} denotes the determinant of the original Calabi-Yau threefold without warp-factors. The condition to preserve supersymmetry gave

$$
\partial_{11}(e^{-3b/2}) = -\frac{\sqrt{2}}{16}\alpha \,,
\tag{4.56}
$$

which together with $\alpha = 8\sqrt{2}\Theta(x^{11})\mathcal{S}_1$ (note that for the case under consideration, we have $\partial_a \alpha = 0$ which means $\mathcal{S}_1$ is a constant) leads to

$$e^{-3b(x^{11})/2} = 1 - \mathcal{S}_1 x^{11} \ , \tag{4.57}$$

thus determining the measure-factor as

$$\sqrt{\hat{g}} = (1 - \mathcal{S}_1 x^{11})\sqrt{g_{CY_3}} \ . \tag{4.58}$$

Analogously the boundary measures are given by $\sqrt{\hat{g}_i^{(10)}} = e^{-b(x^{11}=0,d)}\sqrt{g_{CY_3}}$ which leads to

$$\sqrt{\hat{g}_1^{(10)}} = \sqrt{g_{CY_3}} \ , \qquad \sqrt{\hat{g}_2^{(10)}} = (1 - \mathcal{S}_1 d)^{2/3}\sqrt{g_{CY_3}} \ . \tag{4.59}$$

The Curvature-Scalar

Next let us express the curvature-scalar of the warp-geometry

$$R(\hat{g}_{MN}) = \hat{g}^{KL}\partial^M\partial_M \hat{g}_{KL} - \partial^K\partial^L\hat{g}_{KL} + \hat{\Gamma}^P_{KL}\hat{\Gamma}^Q_{MN}\hat{g}_{PQ}\left(\hat{g}^{KL}\hat{g}^{MN} - \hat{g}^{KM}\hat{g}^{LN}\right) \tag{4.60}$$

through the three warp-factors b, f, k and the initial Calabi-Yau curvature scalar. This gives

$$\begin{aligned} R(\hat{g}_{MN}) = & \, e^{-k}\left[\mathcal{D}\,\partial_{11}^2 b + \frac{\mathcal{D}(\mathcal{D}+1)}{4}(\partial_{11}b)^2 + 2n\,\partial_{11}^2 f + \frac{2n(2n+1)}{4}(\partial_{11}f)^2\right] \\ & + e^{-f}R(g_{mn}) \ , \end{aligned} \tag{4.61}$$

where $\mathcal{D}$ represents the real dimension of the non-compact external spacetime, whereas again n denotes the complex dimension of the internal Calabi-Yau n-fold. For our concrete case with $\mathcal{D} = 4$, $n = 3$ and $f = k = -b$ plus a Ricci-flat Calabi-Yau manifold, we arrive at

$$R(\hat{g}_{MN}) = e^b\left[-2\partial_{11}^2 b + \frac{31}{2}(\partial_{11}b)^2\right] \ . \tag{4.62}$$

Using $b = -\frac{2}{3}\ln|1 - \mathcal{S}_1 x^{11}|$, we finally obtain

$$R(\hat{g}_{MN}) = \frac{50}{9}\frac{(\mathcal{S}_1)^2}{(1 - \mathcal{S}_1 x^{11})^{8/3}} \ . \tag{4.63}$$

The Field-Strengths

For the field-strength contractions one has to extract the x^{11} dependence out of the index-contractions

$$G_{a\bar{b}c\bar{d}}G^{a\bar{b}c\bar{d}} = e^{-4f}(\tilde{S}_1)_{a\bar{b}c\bar{d}}(\tilde{S}_1)^{a\bar{b}c\bar{d}} = \frac{1}{(1-\mathcal{S}_1 x^{11})^{8/3}}(\tilde{S}_1)_{a\bar{b}c\bar{d}}(\tilde{S}_1)^{a\bar{b}c\bar{d}} \qquad (4.64)$$

for $x^{11} \geq 0$, where the tilde on top of the S_1's indicates that now the contractions on S_1 are performed with the initial metric without warp-factors. Similarly

$$F_{a\bar{b}}F^{a\bar{b}} = e^{-2f(x^{11}=0,d)}F_{a\bar{b}}g^{a\bar{c}}g^{\bar{b}d}F_{\bar{c}d} = \frac{1}{(1-\mathcal{S}_1 x^{11})^{4/3}}\tilde{F}_{a\bar{b}}\tilde{F}^{a\bar{b}} \ , \qquad (4.65)$$

where again the tilde signals that the contractions on the indices are carried out with the initial metric.

The Effective Potential

Putting everything together, we can integrate out the eleventh dimension[29] and obtain

$$S_{11} = -\frac{3}{\mathcal{S}_1}\left[\frac{1}{(1-\mathcal{S}_1 d)^{2/3}} - 1\right]\frac{1}{\kappa^2}\int d^{10}x\sqrt{g_{CY_3}}\left[\frac{25}{9}(\mathcal{S}_1)^2 + \frac{1}{48}(\tilde{S}_1)_{a\bar{b}c\bar{d}}(\tilde{S}_1)^{a\bar{b}c\bar{d}}\right] \ ,$$

while the higher order boundary action results in the additional contributions

$$S_{10}^1 = -\frac{1}{(4\pi)^{5/3}\kappa^{4/3}}\int d^{10}x\sqrt{g_{CY_3}}\left[\frac{1}{4}\mathrm{tr}\tilde{F}_{a\bar{b}}\tilde{F}^{a\bar{b}}\right] \qquad (4.66)$$

$$S_{10}^2 = -\frac{1}{(1-\mathcal{S}_1 d)^{2/3}} \times \frac{1}{(4\pi)^{5/3}\kappa^{4/3}}\int d^{10}x\sqrt{g_{CY_3}}\left[\frac{1}{4}\mathrm{tr}\tilde{F}_{a\bar{b}}\tilde{F}^{a\bar{b}}\right] \qquad (4.67)$$

Thus the effective potential for the distance-modulus d is given by

$$\begin{aligned}\mathcal{V}_{\text{eff}}(d) = \frac{1}{(1-\mathcal{S}_1 d)^{2/3}} \times \int d^{10}x\sqrt{g_{CY_3}}\Bigg[&\frac{1}{\kappa^2}\Big(\frac{25}{3}\mathcal{S}_1 + \frac{1}{16\mathcal{S}_1}(\tilde{S}_1)_{a\bar{b}c\bar{d}}(\tilde{S}_1)^{a\bar{b}c\bar{d}}\Big) \\ &+ \frac{1}{\kappa^{4/3}}\Big(\frac{1}{4(4\pi)^{5/3}}\mathrm{tr}\tilde{F}_{a\bar{b}}\tilde{F}^{a\bar{b}}\Big) + \mathcal{O}\left(\frac{1}{\kappa^{2/3}}\right)\Bigg]\end{aligned} \qquad (4.68)$$

up to some d-independent constant. First of all there are two interesting features. All contributions give the same d-dependence irrespective of whether they stem from the bulk or the boundary. And furthermore the "coefficient" in square brackets only receives positive contributions (with $\mathcal{S}_1 > 0$ motivated by the leading order analysis) – again without difference between bulk or boundary terms. Since these features seem to be

[29] We work in the downstairs picture and employ $\int_{-d}^{d} dx^{11} \to 2\int_0^d dx^{11}$.

generic, we would suspect that the unknown higher order terms also enjoy these properties. This is the reason why we have taken the full geometry in the above derivation and not its leading order truncation right away. Thus the full effective potential will be expected to read

$$\mathcal{V}_{\text{eff}}(d) = \frac{1}{(1-\mathcal{S}_1 d)^{2/3}} \times \text{some positive number} . \tag{4.69}$$

It exhibits a large positive peak in the vicinity of the zero-position $d \approx x_0^{11}$. The infinity of the peak might be cured by quantum effects, which resolve the pole. Nevertheless, such a potential indicates a destabilization of the two-boundary set-up. Namely a "hidden" world to the left of x_0^{11} would be driven towards the weakly coupled heterotic string region, while a "hidden" world beyond x_0^{11} would lead to a steady increase of d.

4.7 The Case with General Flux: Beyond Warped Geometries

If one wants to relax the constraint that either α or β_a is zero (which we adopted up to now), then one has to generalize the previous pure warped geometry to a geometry which exhibits a deviation from the Calabi-Yau metric and is not describable by a warp-factor alone. Such a generalization is e.g. needed if one wants to include effects like gaugino-condensation. With such a generalized Ansatz

$$\begin{aligned} ds^2 &= \hat{g}_{MN} dx^M dx^N \\ &= e^{b(x^W, x^{11})} \eta_{\mu\nu} dx^\mu dx^\nu + \left[g_{lm}(x^n) + h_{lm}(x^n, x^{11}) \right] dx^l dx^m + e^{k(x^W, x^{11})} dx^{11} dx^{11} \ , \end{aligned} \tag{4.70}$$

we will see that the inconsistency which arose for the warp-factor f if both α and β_a were present, disappears and instead leads to constraints on the internal spin-connection.

The CY_3 metric split entails a corresponding split for the internal Vielbein $\hat{e}^{\bar{l}}{}_m = e^{\bar{l}}{}_m(x^n) + f^{\bar{l}}{}_m(x^n, x^{11})$. Again, we will express the spin-connection through the one belonging to the initial metric

$$\Omega_{\mu\bar{\nu}\bar{l}}(\hat{e}) = \frac{1}{2} \hat{e}_{\bar{l}}{}^m \hat{e}_{\bar{\nu}\mu} \partial_m b \ , \qquad \Omega_{\mu\bar{\nu}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2} \hat{e}_{\bar{\nu}\mu} \hat{e}_{\bar{1}\bar{1}}{}^{11} \partial_{11} b$$

$$\Omega_{l\bar{m}\bar{n}}(\hat{e}) = \Omega_{l\bar{m}\bar{n}}(e) + \Omega^{(d)}_{l\bar{m}\bar{n}}(e,f) \ , \qquad \Omega_{l\bar{m}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2} \hat{e}_{\bar{1}\bar{1}}{}^{11} (\partial_{11} f_{\bar{m}l} + \hat{e}_{\bar{m}}{}^m \hat{e}^{\bar{l}}{}_l \partial_{11} f_{\bar{l}m}) \tag{4.71}$$

$$\Omega_{11\bar{l}\bar{m}}(\hat{e}) = \hat{e}_{[\bar{l}}{}^l \partial_{|11|} f_{\bar{m}]l} \ , \qquad \Omega_{11\bar{l}\bar{1}\bar{1}}(\hat{e}) = -\frac{1}{2} \hat{e}_{\bar{l}}{}^l \hat{e}_{\bar{1}\bar{1},11} \partial_l k \ , \tag{4.72}$$

with all remaining terms vanishing. Now the deviation from the initial CY-geometry is characterised by $f^{\bar{l}}{}_m$ and $\Omega^{(d)}_{l\bar{m}\bar{n}}(e,f)$. Both go to zero if we turn off the G-fluxes.

Employing again the covariant constancy $D_I\eta = (\partial_I + \frac{1}{4}\Omega_{I\bar{J}\bar{K}}(e)\Gamma^{\bar{J}\bar{K}})\eta = 0$ of the original spinor-parameter, leads to

$$dx^I \hat{D}_I \tilde{\eta} = \Bigg(- dx^u \partial_u \psi + \frac{1}{4} dx^\mu \Big[\hat{\Gamma}_\mu{}^l \partial_l b + \hat{\Gamma}_\mu{}^{11} \partial_{11} b \Big] + \frac{1}{4} dx^l \Big[\Omega^{(d)}_{lmn} \hat{\Gamma}^{mn} + 2\Omega_{lm11}(\hat{e}) \hat{\Gamma}^m \hat{\Gamma}^{11} \Big] + \frac{1}{4} dx^{11} \Big[\hat{\Gamma}_{11}{}^l \partial_l k + \Omega_{11lm}(\hat{e}) \hat{\Gamma}^{lm} \Big] \Bigg) \tilde{\eta} \, . \tag{4.73}$$

Again, specifying that our internal space consists of a Calabi-Yau times an interval, we employ $\hat{\Gamma}^{11}\tilde{\eta} = e^{-k/2}\tilde{\eta}$ and $\hat{\Gamma}^a\tilde{\eta} = 0$, $\hat{\Gamma}_{\bar{a}}\tilde{\eta} = 0$ plus the Dirac-algebra $\{\hat{\Gamma}^a, \hat{\Gamma}^{\bar{b}}\} = 2\hat{g}^{a\bar{b}}$ to obtain

$$\begin{aligned} dx^I \hat{D}_I \tilde{\eta} = \Bigg\{ \Bigg[\left(\frac{1}{4}\Omega^{(d)\,a}_{la} - \frac{1}{4}\Omega^{(d)\,\bar{a}}_{l\bar{a}} - \partial_l \psi \right) dx^l + \left(\frac{1}{4}\Omega_{11a}{}^a(\hat{e}) - \frac{1}{4}\Omega_{11\bar{a}}{}^{\bar{a}}(\hat{e}) - \partial_{11}\psi \right) dx^{11} \Bigg] \\ + \left[\frac{1}{4} e^{-k/2} \partial_{11} b \, dx_\mu \right] \hat{\Gamma}^\mu + \left[\frac{1}{2} e^{-k/2} \Omega_{l\bar{a}11}(\hat{e}) dx^l - \frac{1}{4} e^{-k/2} \partial_{\bar{a}} k dx_{11} \right] \hat{\Gamma}^{\bar{a}} \\ + \left[\frac{1}{4} \partial_{\bar{a}} b \, dx_\mu \right] \hat{\Gamma}^{\mu\bar{a}} + \left[\frac{1}{4} \Omega^{(d)}_{l\bar{a}\bar{b}} dx^l + \frac{1}{4} \Omega_{11\bar{a}\bar{b}}(\hat{e}) dx^{11} \right] \hat{\Gamma}^{\bar{a}\bar{b}} \Bigg\} \tilde{\eta} \, . \end{aligned} \tag{4.74}$$

For the second part of the Killing-equation which consists of the contractions of $\hat{\Gamma}$-matrices with the G-flux, it will be convenient to use the following abbreviations

$$G = \hat{g}^{a\bar{b}} \hat{g}^{c\bar{d}} G_{a\bar{b}c\bar{d}} \tag{4.75}$$
$$G_m = \hat{g}^{b\bar{c}} G_{mb\bar{c}11} \tag{4.76}$$
$$G_{mn} = \hat{g}^{c\bar{d}} G_{mnc\bar{d}} \, . \tag{4.77}$$

In order to eventually extract the real and imaginary parts of the Killing-spinor equation, we have to know their behaviour under complex conjugation, which is given by

$$\overline{G} = G \, , \quad \overline{G_a} = -G_{\bar{a}} \, , \quad \overline{G^a{}_b} = -G^{\bar{a}}{}_{\bar{b}} \, . \tag{4.78}$$

Analogously, to the treatment in the previous section, we then arrive at

$$\begin{aligned} dx_I \big(\hat{\Gamma}^{IJKLM} - 8\hat{g}^{IJ}\hat{\Gamma}^{KLM} \big) G_{JKLM} \tilde{\eta} = \Big\{ 3e^{-k/2} \left[-24 G_{\bar{a}} dx^{\bar{a}} - 8 G_a dx^a + 4 G dx_{11} \right] \\ + 12 G dx_\mu \hat{\Gamma}^\mu + 12 \left[G dx_{\bar{a}} - 6 G^{\bar{b}}{}_{\bar{a}} dx_{\bar{b}} + 4 G_{\bar{a}} dx^{11} \right] \hat{\Gamma}^{\bar{a}} + 24 e^{-k/2} G_{\bar{a}} dx_\mu \hat{\Gamma}^{\mu\bar{a}} \\ + 12 e^{-k/2} \left[2 G_{\bar{b}} dx_{\bar{a}} - 3 G^{\bar{c}}{}_{\bar{a}\bar{b}11} dx_{\bar{c}} \right] \hat{\Gamma}^{\bar{a}\bar{b}} \Big\} \, . \end{aligned} \tag{4.79}$$

With (4.74) and (4.79) we are then able to decompose the complete Killing-equation (4.4) into its external, CY- and 11-components. Thus unbroken supersymmetry finally translates into the following constraints on the spin-connection

$$\Omega^{(d)\,b}_{ab} - \Omega^{(d)\,\bar{b}}_{a\bar{b}} = \frac{2\sqrt{2}}{3} e^{-k/2} G_a \tag{4.80}$$

$$\Omega^{(d)\,a}{}_{\bar{b}\bar{c}} = 0 \tag{4.81}$$

$$\Omega^{(d)\,c}{}_{ab} = \frac{\sqrt{2}}{6} e^{-k/2} \left(G_d \delta^{cd}_{ab} + 3G^c{}_{ab11} \right) \tag{4.82}$$

$$\Omega_{11a}{}^{a}(\hat{e}) = \Omega_{11\bar{a}}{}^{\bar{a}}(\hat{e}) \tag{4.83}$$

$$\Omega^{b}{}_{\bar{a}11}(\hat{e}) = \Omega_{11ab}(\hat{e}) = 0 \tag{4.84}$$

$$\Omega^{b}{}_{a11}(\hat{e}) = \frac{\sqrt{2}}{12} e^{k/2} \left(G\delta^b_a - 6G^b{}_a \right) \ , \tag{4.85}$$

where $\delta^{cd}_{ab} = \delta^c_a \delta^d_b - \delta^c_b \delta^d_a$. Additionaly, the solution to the Killing-spinor equation provides us with further equations, which determine the warp-factors and covariant-spinor deviation in terms of the G-flux parameters

$$\partial_a b = \frac{\sqrt{2}}{3} e^{-k/2} G_a \tag{4.86}$$

$$\partial_{11} b = -\frac{\sqrt{2}}{6} e^{k/2} G \tag{4.87}$$

$$\partial_a k = -\frac{2\sqrt{2}}{3} e^{-k/2} G_a \tag{4.88}$$

$$\partial_a \psi = -\frac{\sqrt{2}}{12} e^{-k/2} G_a \tag{4.89}$$

$$\partial_{11} \psi = \frac{\sqrt{2}}{24} e^{k/2} G \ . \tag{4.90}$$

Similarly to the last section we obtain

$$8\psi(x^m, x^{11}) = k(x^m, x^{11}) = -2b(x^m, x^{11}) \ , \tag{4.91}$$

but this time a dependence on both x^m and x^{11} is allowed. Note that this relation is in accordance with the result of the first order approximation derived in [63].

The relation $\Gamma^u{}_{uv}(\hat{g}) = \frac{1}{\sqrt{\hat{g}}} \partial_v \sqrt{\hat{g}}$ between the Christoffel-symbols and the metric determinant enables us to to find

$$\partial_{11} \sqrt{\hat{g}_{CY_3}} = \sqrt{\hat{g}_{CY_3}} \left(\Gamma^u{}_{u11}(\hat{g}_{CY_3 \times I}) - \frac{1}{2} \partial_{11} k \right) \ . \tag{4.92}$$

Via the relation between the Christoffel-symbols and the spin-connection, $\hat{e}^{\bar{u}}{}_x \Gamma^x{}_{vw} = \partial_v \hat{e}^{\bar{u}}{}_w + \Omega_v{}^{\bar{u}}{}_{\bar{x}} \hat{e}^{\bar{x}}{}_w$, we get $\Gamma^u{}_{u11}(\hat{g}_{CY_3 \times I}) - \frac{1}{2}\partial_{11} k = \Omega^u{}_{u11}(\hat{e}_{CY_3 \times I})$ and thereby

$$\partial_{11} \sqrt{\hat{g}_{CY_3}} = \sqrt{\hat{g}_{CY_3}} \Omega^l{}_{l11}(\hat{e}) \ , \tag{4.93}$$

where we have used that $\Omega^{11}{}_{1111}(\hat{e}) = 0$. Together with the constraint on $\Omega^b{}_{a11}(\hat{e})$, which gives $\Omega^a{}_{a11}(\hat{e}) = \Omega^{\bar{a}}{}_{\bar{a}11}(\hat{e}) = \sqrt{2}\left(\frac{6+n}{12}\right) e^{k/2} G$ ($n = \dim_{\mathbb{C}} CY_n$), we obtain ultimately

$$\partial_{11} \ln \sqrt{\hat{g}_{CY_3}} = \sqrt{2}\left(\frac{6+n}{6}\right) e^{k/2} G \ . \tag{4.94}$$

Employing the equation for $\partial_{11} b$, we can integrate this equation to obtain the following expression for the Calabi-Yau dependence on x^{11} (with $n = 3$)

$$V(x^{11}) = \int_{CY_3} d^6x \sqrt{\hat{g}_{CY_3}} = \int_{CY_3} d^6x e^{-9b(x^{11},x^m)} C(x^m) \ , \tag{4.95}$$

where $C(x^m)$ arose as an integration constant by integrating (4.94) over x^{11}. We see that now the specification of the sources simply by means of their location in the eleventh direction is not enough to determine $V(x^{11})$. This stems from the fact that G, which determines the warp-factor b contains contractions with $\hat{g}^{a\bar{b}}$ which itself is x^{11} dependent. Therefore the specification $G_{a\bar{b}c\bar{d}} \propto \Theta(x^{11} - z_i)$ does not fully determine the x^{11} behaviour of G. However, it is definitely true that a non-trivial $V(x^{11})$ requires $G \neq 0$ and therefore in view of (4.87) $G_{a\bar{b}c\bar{d}} \neq 0$.

Another interesting aspect of turning on both $G_{a\bar{b}c\bar{d}}$ and $G_{ab\bar{c}11}$ derives from the fact, that we saw above how the fluxes determine the internal spin-connection $\Omega(\hat{e})$. Now it is well-known that the spin-connection determines the holonomy-group $\mathcal{H}$ of a manifold through the path-ordered exponential of $\Omega(\hat{e})$ around a closed curve γ

$$\mathcal{P} e^{\int_\gamma \Omega_m(\hat{e}) dx^m} \in \mathcal{H} \ . \tag{4.96}$$

This is an interesting further link between the physics of G-fluxes and the geometry of the compactification space. A complex 3-dimensional Kähler-manifold exhibits U(3) holonomy. But we already saw above that turning on a G-flux in general ruins the closedness property of the Kähler-form – therefore the new deformed manifolds are no longer Kähler. This means that we do expect a more general holonomy than U(3).

5 A Small Cosmological Constant and Warped Geometry

After having explored warped compactifications in the framework of heterotic M-theory in the previous chapter, we will now deal with warped geometries first in five dimensions and then use an embedding into 10-dimensional IIB-/F-theory. The main focus in this chapter lies on a low-energy mechanism – exploiting a warped geometry – to obtain a small realistic value for the cosmological constant.

The enormous smallness of the four-dimensional cosmological constant as constrained from cosmological and astronomical measurements by [82]

$$|\Lambda_4| \lesssim 10^{-47}\mathrm{GeV}^4 \approx (1.8\,\mathrm{meV})^4 \ , \tag{5.1}$$

is still not understood in a satisfactory way from a theoretical point of view. The energy-regime of the upper bound of some meV is rather unnatural in particle physics and a more common characteristic of condensed matter phenomena. However, it has to be noticed that the upper bound on the electron neutrino mass can be as low as 1meV [83], which comes strikingly close to this value. If experiment will eventually show that both numbers are indeed of the same order, this would be an intriguing hint to some deeper relation between the Standard Model and Gravity.

The hope that eventually a consistent theory of quantum gravity might be able to explain the vexing smallness has not been fulfilled yet, as the only consistent candidate, string- or M-theory, relies so heavily on exact supersymmetry. Since the tininess of the cosmological constant is measured at energies where Bose-Fermi degeneracy is seen to be violated, a supersymmetry-breaking mechanism would be needed which nonetheless should not give rise to a large Λ_4. An interesting M-theory inspired proposal has been made in [84]. The idea is that in three dimensions, supersymmetry enforces a zero cosmological constant but can exist without matching bosonic and fermionic degrees of freedom. If such a three-dimensional theory contains a modulus similar to the dilaton of string-theory, one could expect that at strong coupling a new dimension will open up. The hope would be that during the transition from weak to strong coupling the properties of a zero cosmological constant and in addition Bose-Fermi non-degeneracy are conserved.

Whereas in the very early universe a non-vanishing cosmological constant is welcome during the phase of inflation, we face the problem to understand the smallness of the cosmological constant in our low-energy world, nowadays. Therefore, we shall take the point of view in this paper, that there should also exist a rationale to understand the adjustment of the cosmological constant to tiny values not only by taking refuge to a Quantum Gravity description valid at Planck-energies but also by employing merely degrees of freedom which are available at low energies.

Furthermore, we shall adopt the view that our four-dimensional world consists of a thick wall, embedded in some higher-dimensional partially compactified space. Conceiving our world as being located on a type IIB string-theory D3-brane in a ten-dimensional ambient space allowed to attack such fundamental problems as gauge and gravitational coupling unification or the Standard Model hierarchy problem from completely different point of views (see [102] and references therein) than the traditional technicolor or low-energy supersymmetry approaches. In a T-dualized type I string scenario, where two to six internal compact dimensions orthogonal to the 3-brane are chosen much larger than the remaining compact dimensions, one is able to lower the fundamental higher-dimensional Planck scale down to the TeV-scale [30]. This necessitates the large internal dimensions to be as large as 1mm resp. 1 fermi for two resp. six large internal dimensions. Most pronounced in the case of two large dimensions, this leads to another hierarchy between the new fundamental TeV-scale and the compactification scale $\mu \equiv \hbar c/1\text{mm} \approx 10^{-4}\text{eV}$. This drawback could be overcome by considering not a direct product structure for the background space-time but a warped metric instead. In particular, the warped metric of a slice of an AdS-space suspended between two four-dimensional domain walls offers a solution to the strong part of the hierarchy problem [103].

In [104] it has been shown how to stabilize the modulus, which describes the distance between the two walls, at a value of 10-50 Planck lengths. This is just the value which is compatible with the mentioned solution of the hierarchy problem. It remains to relax the fine-tuning condition between the bulk cosmological constant and the brane-tensions. Attempts in this direction have been undertaken recently [90],[91],[92]. However, the solution to the hierarchy problem cannot be maintained in these approaches as the solutions exhibit metrics that do have polynomial instead of exponential behaviour. The metrics vanish at two finite points in the extra dimension, thereby cutting off the infinite range through singularities. However, the nature of these singularities remains obscure.

A general review of the cosmological constant problem can be found in [85]. See [86],[87] for more recent reviews on the topic. [88] provides a recent discussion of the cosmological constant problem from the point of view of String-Theory). Apparently, lately there has been a noticeable increase in the efforts to solve the cosmological constant problem [89],[90],[91],[92],[93],[94][95],[96],[97],[98],[99],[100],[101].

5.1 The Effective Cosmological Constant

Let us start by recapitulating how the vanishing of the effective four-dimensional cosmological constant comes about in the Randall-Sundrum (RS) scenario [103]. Whenever we are given a four-dimensional Poincaré-invariant flat metric $ds_4^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, we deduce from the D=4 Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\Lambda_4 g_{\mu\nu} \ , \tag{5.2}$$

that this implies $\Lambda_4 = 0$. Since upon integrating out the fifth dimension in the RS set-up, we are left precisely with a Poincaré-invariant flat metric, we conclude that the effective Λ_4 in this scenario has to vanish. Subsequently, we will analyze in more detail how this is achieved precisely.

The RS-model [103] consists of two walls located at the fixed points of an $S^1/\mathbb{Z}_2$ orbifold in the fifth direction and a bulk gravitational plus cosmological constant part in between. The Planck-brane, on which the four-dimensional graviton is localized, sits at the first fixed-point, $x^5 = 0$ of the $\mathbb{Z}_2$-action, whereas our four-dimensional world is supposed to be placed on the SM-wall at $x^5 = \pi r$, the second fixed-point. It is only this latter wall on which the hierarchy problem can be solved by means of the exponential warp-factor in the Anti-de Sitter bulk geometry. Concerning the interaction between the walls and bulk gravity, the dominant contribution comes from the wall tension term in the effective field theory on the wall [105]. Hence, if one is interested in a situation where the walls are close to their ground states, it is reasonable to neglect gauge-fields, fermions and scalars on them. Taking this into account the RS-Lagrangean[30] reads [103]

$$\begin{aligned} S_{RS} = -\int d^4x \int_0^{\pi r} dx^5 \Big\{ & \sqrt{G}\left(M^3 R + \Lambda\right) \\ & + \sqrt{g_{Pl}^{(4)}} T_{Pl}\delta(x^5) + \sqrt{g_{SM}^{(4)}} T_{SM}\delta(x^5 - \pi r) \Big\} \ . \end{aligned} \tag{5.3}$$

As we will see in the later computation, it is important to just integrate the bulk piece over the interval[31] $[0, \pi r]$ (or rather from $-\epsilon$ to $\pi r + \epsilon$ with ϵ infinitesimal to incorporate the delta-function sources on the boundaries properly) in order to find a vanishing Λ_4.

[30] Subsequently, we will adopt General Relativity conventions as, e.g. used in [106].

[31] This is analogous to the downstairs approach in heterotic M-theory [46], in which there is an analogous

The four-dimensional metrics $g_{SM}^{(4)}, g_{Pl}^{(4)}$ are the pullbacks of the bulk metric to the two domain-wall world-volumes. Adopting the Ansatz

$$ds^2 = e^{-A(x^5)}\eta_{\mu\nu}dx^\mu dx^\nu + (dx^5)^2 \ , \tag{5.4}$$

the Einstein equation results in

$$(A')^2 = -\frac{1}{3M^3}\Lambda \ , \qquad A'' = \frac{1}{3M^3}\big(T_{Pl}\delta(x^5) + T_{SM}\delta(x^5 - \pi r)\big) \ . \tag{5.5}$$

The solution to the first equation of (5.5) is given by

$$A(x^5) = \pm k x^5 \ , \quad k \equiv \sqrt{\frac{-\Lambda}{3M^3}} \ , \tag{5.6}$$

where the integration constant has been set to zero for it can be absorbed into a rescaling of the x^μ coordinates. To respect the imposed $\mathbb{Z}_2$ symmetry, we have to take

$$A(x^5) = \pm k|x^5| \ . \tag{5.7}$$

In the following, we will choose the plus-sign which allows for a solution of the hierarchy problem on the SM-wall. Noting that $|x^5|'' = 2\delta(x^5)$, we rewrite the solution in an expanded form as

$$A(x^5) = \frac{1}{2}k\big(|x^5| - |x^5 - \pi r|\big) + \frac{1}{2}k\pi r \ , \quad 0 \le x^5 \le \pi r \tag{5.8}$$

in order to satisfy the second equation of (5.5) with

$$T_{Pl} = -T_{SM} = 3M^3 k \ . \tag{5.9}$$

Let us now determine the four-dimensional effective action by integration over the fifth dimension and start with the Einstein-Hilbert term of the bulk action. For this purpose, consider first the general D-dimensional case with metric

$$\begin{aligned} ds^2 &= G_{MN}dx^M dx^N \\ &= g_{\mu\nu}^{(D-1)}dx^\mu dx^\nu + (dx^D)^2 = f(x^D)g_{\mu\nu}(x^\rho)\, dx^\mu dx^\nu + (dx^D)^2 \ , \end{aligned} \tag{5.10}$$

orbifold-procedure for the eleventh direction. In the alternative upstairs approach, one would integrate the Lagrangean density over the full circle but in addition has to place a factor of 1/2 in front of the integral due to the $\mathbb{Z}_2$ symmetry of the action.

where μ, ν run over $1, \ldots, D-1$ and M, N over $1, \ldots, D-1, D$. The D-dimensional curvature scalar can then be decomposed in the following way into the $(D-1)$-dimensional curvature scalar plus additional terms[32] depending exclusively on x^D

$$R(G) = \frac{1}{f} R(g) + \frac{1}{4}(D-1) \left((D-2) \left[(\ln f)' \right]^2 + 2 (\ln f)'' + 2 \frac{f''}{f} \right) . \tag{5.11}$$

In addition, we have to take into account a factor $\sqrt{G} = f^{(D-1)/2} \sqrt{g}$ in the measure of the action integral. Specializing now to the RS case with $D = 5$ we take the metric

$$ds^2 = G_{MN} dx^M dx^N = e^{-A(x^5)} g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + (dx^5)^2 , \tag{5.12}$$

with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ describes the four-dimensional graviton propagating on the flat background. This has to be plugged into the RS-action and integrated over the fifth dimension. Using (5.11) with $f(x^5) = e^{-A(x^5)}$, we get

$$\begin{aligned} S_{\text{EH}} &= - \int d^4x \int_0^{\pi r} dx^5 \sqrt{G} M^3 R(G) \\ &= - \int d^4x \int_0^{\pi r} dx^5 f^2 \sqrt{g} M^3 \left\{ \frac{R(g)}{f} + 3 \left[(\ln f)' \right]^2 + 2 (\ln f)'' + 2 \frac{f''}{f} \right\} \\ &= - \int d^4x \sqrt{g} M^3 \int_0^{\pi r} dx^5 \left\{ e^{-A} R(g) + e^{-2A} \left[5 (A')^2 - 4A'' \right] \right\} . \end{aligned} \tag{5.13}$$

Since we will come back to this formula afterwards, we note that up to this point it is valid for any metric which is of the form (5.12). Choosing the RS-metric we receive

$$\begin{aligned} S_{\text{EH}} = - \int d^4x \sqrt{g} M^3 \Bigg\{ R(g) \int_0^{\pi r} dx^5 e^{-kx^5} + \int_0^{\pi r} dx^5 e^{-2kx^5} \Big[5k^2 \\ - 4k \left(\delta(x^5) - \delta(x^5 - \pi r) \right) \Big] \Bigg\} . \end{aligned} \tag{5.14}$$

Concerning the delta-function integration, we imagine performing the integration actually over the interval $[-\epsilon, \pi r + \epsilon]$ with ϵ infinitesimal. This full consideration of all the fixed-point sources will be important to arrive at $\Lambda_4 = 0$, finally. Thus the Einstein-Hilbert part gives

$$S_{\text{EH}} = - \int d^4x \sqrt{g} \left\{ M_{Pl}^2 R(g) - \frac{3}{2} M^3 k \left(1 - e^{-2k\pi r} \right) \right\} , \tag{5.15}$$

where $M_{Pl}^2 = 2M^3 \left(1 - e^{-k\pi r} \right) / k$ denotes the effective four-dimensional Planck-scale squared. The second part of the reduction comprises the wall sources and the bulk

[32] By f' we mean df/dx^D.

cosmological constant term

$$
\begin{aligned}
S_{Pl} + S_{SM} + S_\Lambda &= -\int d^4x e^{-2A(\pi r)} \sqrt{g} T_{SM} - \int d^4x e^{-2A(0)} \sqrt{g} T_{Pl} \\
&\quad - \int d^4x \int_0^{\pi r} dx^5 e^{-2A(x^5)} \sqrt{g} \Lambda \\
&= -\int d^4x \sqrt{g} \left\{ e^{-2k\pi r} T_{SM} + T_{Pl} + \Lambda \int_0^{\pi r} dx^5 e^{-2kx^5} \right\} \\
&= -\int d^4x \sqrt{g} \left\{ \underbrace{e^{-2k\pi r} T_{SM} + T_{Pl}}_{3M^3k\left(1-e^{-2k\pi r}\right)} - \frac{3}{2} M^3 k \left(1 - e^{-2k\pi r}\right) \right\} \\
&= -\frac{3}{2} \int d^4x \sqrt{g} M^3 k \left(1 - e^{-2k\pi r}\right) \ .
\end{aligned}
$$

In conclusion, we see that due to the fine-tuned values of the Planck- and SM-wall tensions in terms of the bulk cosmological constant, both contributions to Λ_4 add up to zero as expected.

Suppose that the brane-tensions had not been fine-tuned to their RS-values but were merely chosen to be equal up to a minus sign. In other words suppose, that the bulk cosmological constant takes any non-positive value. Then, according to the above calculation we expect a residual four-dimensional cosmological constant of the order of

$$
\Lambda_4 \sim (\sqrt{-3M^3\Lambda} - T_{Pl})(1 - e^{-2k\pi r}) \ . \tag{5.16}
$$

Such an effective Λ_4 constitutes a potential for the hitherto unconstrained modulus r. Its minimum lies at $r = 0$ if we assume that $\sqrt{-3M^3\Lambda} > T_{Pl}$ and would indeed drive Λ_4 to zero[33] (If $\sqrt{-3M^3\Lambda} < T_{Pl}$, the minimum would lie at $r = \infty$ which implies a runaway behaviour). However, an estimation how close r has to come to zero to actually solve the cosmological constant problem is rather disenchanting. If we take $\sqrt{-3M^3\Lambda} - T_{Pl} \simeq M_{Pl}^4$, $k \simeq M_{Pl}$ and demand that $\Lambda_4 \simeq (1\text{meV})^4$, we find an incredibly small $r \simeq 10^{-125} l_{Pl}$, with $l_{Pl} = M_{Pl}^{-1}$ and the Planck-mass $M_{Pl} = 1.2 \times 10^{19}$GeV. This, however, is a region, where we surely cannot trust classical gravity as a reliable description.

Basically, the problem lies in the contribution of the 1 in the expression (5.16) for Λ_4. Without it, we could solve the cosmological constant problem in a way analogous to the RS-mechanism of solving the hierarchy-problem, namely through the suppression by an exponential factor. Removing the Planck-brane does not help, since with a single

[33] Note that in this limit the hierarchy-problem cannot be solved any longer.

wall only the disfavoured r-independent contribution survives. Because it has its origin in the integration of the RS-action over the region around $x^5 \approx 0$, where the warp-factor becomes 1, we have to find a configuration of at least two walls, where the warp-factor can never become 1 but instead keeps exponentially small throughout the whole x^5 integration region.

In general, Λ_4 will be expressed by a product of a function containing Λ and the wall-tensions times a geometrical factor reflecting the geometry, in which the walls are embedded. Thence we perceive that a small Λ_4 can either be obtained by using an appropriate ambient space geometry or alternatively by finding some mechanism which guarantees a small higher-dimensional Λ plus wall tensions. In this paper we will address the first approach and leave the second to [107].

5.2 Our World as a Two Wall System with Small Λ_4

We have seen in the last subsection that one should avoid placing a wall at the origin, the one fixed point of the $\mathbb{Z}_2$ symmetry (since this gives rise to a Planck-scale Λ_4). Instead, we will place two walls at the $\mathbb{Z}_2$ mirror-points $x^5 = -l, l$. As we will see soon, the warp-factor, which in essence determines the degree of suppression of Λ_4, decreases exponentially with l only if the two walls are located at $\mathbb{Z}_2$ mirror-points[34]. It is remarkable that already a length of $l = 284\, l_{Pl}$ can yield, in combination with an exponential suppression factor, the observed value (upper bound) for Λ_4 given in (5.1)

$$e^{-M_{Pl} l} M_{Pl}^4 = 10^{-47}\mathrm{GeV}^4 \ . \tag{5.17}$$

The intriguing observation is, that the double of this length, $2l = 568\, l_{Pl}$, which will play later on the role of the distance between the two walls, corresponds to the GUT-unification scale

$$2l = 568 l_{Pl} \leftrightarrow M_{\mathrm{GUT}} = 2 \times 10^{16}\mathrm{GeV} \ . \tag{5.18}$$

[34]Note that in [99] time-dependent models with two walls were considered. However, due to the location of one of the walls at the fixed-point $x^5 = 0$, one has to distinguish between a visible and a hidden world much as in the original RS-model. On the visible world the authors of [99] find an exponentially small cosmological constant in five dimensions. However, upon deriving the effective four-dimensional Λ_4 of the whole set-up by integrating out the fifth dimension the huge cosmological constant of the hidden world reappears and spoils the smallness of Λ_4. It is the very property of gravity to penetrate the whole bulk, which requires a small warp-factor throughout spacetime to obtain a small effective Λ_4. This stands in contrast to the discussion of the hierarchy problem along the lines of [103], which only relies on the local warp-factor at the position of the wall.

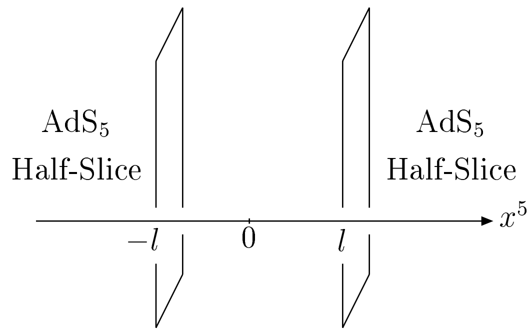

Figure 1: Our World as a Two Wall System with Thickness $2l = 1/M_{\rm GUT}$.

This already indicates some deeper relationship between GUT-theories and the cosmological constant or in other words gravity than traditionally assumed. The connection with GUT-theories will be made more precise in subsection 5 and 6. Let us now visualize our world as possessing a thickness $2l$ in the x^5 direction, which when zoomed in consists of two walls located at l and $-l$ respectively (see fig.1). Both walls can only communicate via gravity to each other. In order to save our world from precarious instabilities, we will choose both wall-tensions as positive. Because the inter-wall distance is larger than the Planck-length and even the string-length $l_s = \sqrt{\alpha'} \sim 10\, l_{Pl}$, it is justified to describe the whole set-up by the low-energy action

$$\begin{aligned} S = &- \int d^4x \int dx^5 \sqrt{G} \left(M^3 R(G) + \Lambda \right) \\ &- \int d^4x \int dx^5 \left(\sqrt{g_1^{(4)}} T_1 \delta \left(x^5 + l \right) + \sqrt{g_2^{(4)}} T_2 \delta \left(x^5 - l \right) \right) \; . \end{aligned} \tag{5.19}$$

Again $g^{(4)}_{1,\mu\nu}$ and $g^{(4)}_{2,\mu\nu}$ are the induced metrics arising from the pullback of G_{MN} to the two wall world-volumes. Choosing again the Ansatz (5.4), the Einstein field equations reduce to (5.5) with the tension on the right-hand-side of the A'' equation given this time by

$$T(x^5) = T_1 \delta \left(x^5 + l \right) + T_2 \delta \left(x^5 - l \right) \; . \tag{5.20}$$

In order to concentrate on the essential aspects of the suppression mechanism, we will keep things as simple as possible in the following by choosing equal tensions $T_1 = T_2 = T$ for the walls. The case with unequal tensions will be treated in appendix C.

The solution[35] to the Einstein equation is given by

$$A(x^5) = \frac{k}{2}\left|x^5 + l\right| + \frac{k}{2}\left|x^5 - l\right| = \begin{cases} x^5 \leq -l: & -kx^5 \\ -l \leq x^5 \leq l: & kl \\ x^5 \geq l: & kx^5 \end{cases} , \tag{5.21}$$

together with the bulk cosmological constant

$$\Lambda(x^5) = \begin{cases} \Lambda_e , & |x^5| > l \\ \Lambda_e/4 , & |x^5| = l \\ \Lambda_i , & |x^5| < l \end{cases} = \begin{cases} -3M^3k^2 , & |x^5| > l \\ -3M^3k^2/4 , & |x^5| = l \\ 0 , & |x^5| < l \end{cases} \tag{5.22}$$

and the wall-tension

$$T = 3M^3 k . \tag{5.23}$$

Note that the two terms of (5.21) are dictated by the choice of sources in (5.20) and the symmetry of the set-up. Any further integration constant which could be added to (5.21) is immaterial, since it can be absorbed into a redefinition of the x^μ coordinates describing the four-dimensional section of the five-dimensional configuration. Again, the flat four-dimensional metric in the above Ansatz implies, as usual, a fine-tuning between the parameters Λ and T

$$\Lambda_e = -\frac{1}{3}\frac{T^2}{M^3} , \qquad \Lambda_i = 0 . \tag{5.24}$$

The function $A(x^5)$, which determines the warp-factor is displayed in fig.2. The corresponding warp-factor $e^{-A(x^5)}$ is upper-bounded by e^{-kl} throughout the whole fifth dimension. This is the basic reason why, in view of (5.17), the warp-factor will be capable of suppressing any (induced from the higher-dimensional Riemann curvature scalar or actually bulk) Planck-size cosmological constant down to the observed value of 10^{-47}GeV^4, if k takes its natural value at M_{Pl}. From a low-energy (energy far below M_{GUT}) point of view, we would regard the distance $2l$ between both walls as too small to recognize them as two separate walls. Such a low-energy observer would realize one wall with tiny thickness and a geometry which consists of two slices of Anti-de Sitter spacetime directly glued together. For her/him the graviton would appear localized on a single thick wall as described in [29].

[35] In case that we work on a circle and identify $-l \sim l$, the solution will only be given by the restriction to $-l \leq x^5 \leq l$.

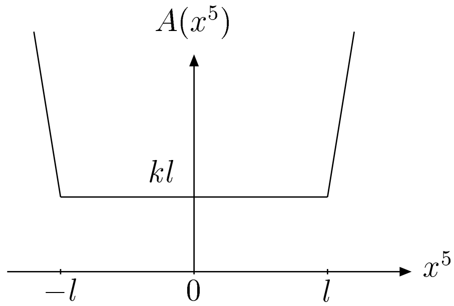

Figure 2: The function $A(x^5)$ determining the warp-factor.

The next task is again the determination of the effective four-dimensional action by integrating out the x^5 coordinate for the metric background

$$ds^2 = e^{-A(x^5)} g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + (dx^5)^2 \ . \tag{5.25}$$

Along the same lines as above for the RS-case and by employing (5.13), we get

$$\begin{aligned} S_{EH} &= -\int d^4x \sqrt{g} M^3 \left\{ R(g) \int_{-\infty}^{\infty} dx^5 e^{-A} + \int_{-\infty}^{\infty} dx^5 e^{-2A} \left[5(A')^2 - 4A'' \right] \right\} \\ &= -e^{-kl} \int d^4x \sqrt{g} M^3 \left\{ 2R(g) \left[\frac{1}{k} + l \right] - 3e^{-kl} k \right\} \ . \end{aligned} \tag{5.26}$$

For the other terms we arrive at

$$S_{SM_1} + S_{SM_2} + S_\Lambda = -e^{-2kl} \int d^4x \sqrt{g} \left\{ 2T + \frac{\Lambda_e}{k} \right\} \ . \tag{5.27}$$

Pulling out an overall factor of e^{-kl} in front, the final effective action reads

$$\begin{aligned} & S_{EH} + S_{SM_1} + S_{SM_2} + S_\Lambda \\ &= -e^{-kl} \int d^4x \sqrt{g} \left\{ 2M^3 R(g) \left[\frac{1}{k} + l \right] + e^{-kl} \left[-3M^3 k + 2T + \frac{\Lambda_e}{k} \right] \right\} \ . \end{aligned} \tag{5.28}$$

At the classical level an overall constant multiplying the whole action is irrelevant – it drops out of the classical equations. If we regard the cosmological constant problem as a low-energy one (since it is here where experiments contradicting theoretical expectations are carried out), quantum gravitational effects should not play a role and a classical description of gravity suffices. Unlike quantum gravitational effects, quantum effects of the strongly, weakly or electromagnetically interacting fields located on the walls may become important already at low-energies. They can be included and merely result in a renormalization of the wall-tension T. With this in mind, let us drop the overall scale-factor and finally arrive at the effective action

$$S_{D=4} = -\int d^4x\sqrt{g}\left\{M_{\rm eff}^2 R(g) + \Lambda_4\right\} \ , \tag{5.29}$$

with the effective four-dimensional Planck-scale $M_{\rm eff}$ and the four-dimensional cosmological constant Λ_4 given by[36]

$$M_{\rm eff}^2 = 2M^3\left[\frac{1}{k} + l\right] \tag{5.31}$$

$$\Lambda_4 = e^{-kl}\left[-3M^3k + 2T + \frac{\Lambda_e}{k}\right] . \tag{5.32}$$

Note the huge suppression-factor e^{-kl} multiplying the whole effective cosmological constant, which is essential for our proposition of achieving the observed value for Λ_4. One can now show easily, that when the above obtained values (5.22),(5.23) for T, Λ_e, which correspond to the special flat solution $g_{\mu\nu}(x^\rho) = \eta_{\mu\nu}$ for the four-dimensional metric, are substituted in the derived effective action, we arrive at a zero Λ_4. This serves as a check on the derivation, since a flat four-dimensional metric, caused by the fine-tuned parameters, must necessarily entail a vanishing four-dimensional cosmological constant.

Let us now restrain from the restriction of fine-tuning the parameters and allow for generic values of the wall-tension and the cosmological constant, i.e. we want to suspend the fine-tuning constraints given by (5.24). Let us assume for the moment that the backreaction of the non-finetuned parameters will leave the warp-factor intact (the full backreaction is included in the next section). Lifting the fine-tuning then corresponds to a non-trivial four-dimensional metric $g_{\mu\nu} \neq \eta_{\mu\nu}$ in the Ansatz

$$ds^2 = e^{-A(x^5)}g_{\mu\nu}dx^\mu dx^\nu + (dx^5)^2 . \tag{5.33}$$

Generically we want to choose $k \simeq M_{Pl}$, $T \simeq M_{Pl}^4$, $\Lambda_e \simeq -M_{Pl}^5$ and furthermore the fundamental five dimensional Planck-scale $M \simeq M_{Pl}$. Furthermore, we have to remember to reintroduce Λ_i. Since Λ_i has been absent in the above solution, we have to take refuge to the more general case with unequal tensions, where a non-vanishing Λ_i shows up. The

[36]If we worked on a circle $x^5 \in [-l, l]$ with $-l \sim l$, we would obtain

$$M_{\rm eff}^2 = 2M^3 l \ , \quad \Lambda_4 = e^{-kl}(2T + 2l\Lambda_i) \tag{5.30}$$

with likewise exponentially suppressed Λ_4.

corresponding expression for the four-dimensional effective cosmological constant (C.9) can be found in appendix C. If we take of it the limit of coinciding tensions (or equivalently $k_{12} = k_1 - k_2 = 0$) but leave Λ_i free, we obtain

$$\Lambda_4 = e^{-kl}\Big[-3M^3k + 2T + \frac{\Lambda_e}{k} + 2l\Lambda_i\Big] \ . \tag{5.34}$$

To guarantee that $2l|\Lambda_i| \leq M_{Pl}^4$, we have to choose $|\Lambda_i| \leq (3 \times 10^{18}\text{GeV})^5$, which still seems to be quite generic. We then recognize from (5.34), that the suppression through the exponential factor is sufficient to bring the various contributions to the four-dimensional cosmological constant down to its observed value (5.1) by means of (5.17).

The effective four-dimensional Planck-scale $M_{\text{eff}} \simeq 24M_{Pl}$ comes out slightly too high. It can however be easily brought down, e.g. to $M_{\text{eff}} \simeq M_{Pl}$, if we choose the fundamental scale as $M \simeq 1.5 \cdot 10^{18}\text{GeV}$, which is close to the traditional string-scale $M_s = 1/\sqrt{\alpha'}$ and may be considered as a hint to a stringy origin of the set-up.

5.3 Including the Backreaction of Non-Finetuned Parameters

In this section we want to include the full backreaction on the warped geometry arising through the lifting of the finetuning. To this aim, we have to determine the resulting 5-dimensional geometry for general non-positive $\Lambda_e \leq 0$ and positive $T > 0$. Let us start with a D-dimensional warped geometry

$$ds^2 = G_{MN}dx^M dx^N = f(x^D)g_{\mu\nu}(x^\rho)dx^\mu dx^\nu + (dx^D)^2 \ , \tag{5.35}$$

with $\mu, \nu, \rho = 1, \ldots, D-1$ and the warp-factor $f(x^D)$. The induced metric on a $(D-1)$-dimensional section defined by $x^D = \text{const}$, will be denoted by $g_{\mu\nu}^{(D-1)}(x^\rho, x^D) = f(x^D)g_{\mu\nu}(x^\rho)$. Eventually, we want to solve the Einstein equation to determine the lower-dimensional Λ_4 for the case $D = 5$. Therefore, we decompose the D-dimensional Ricci-tensor R_{MN} into its μ and D components

$$\begin{aligned}
R_{\mu\nu}(G) &= R_{\mu\nu}(g) + \frac{1}{4}g_{\mu\nu}\left(2f'' + (D-3)f\left[(\ln f)'\right]^2\right) \\
R_{\mu D}(G) &= 0 \\
R_{DD}(G) &= \frac{1}{4}(D-1)\left(2\frac{f''}{f} - \left[(\ln f)'\right]^2\right) \ .
\end{aligned} \tag{5.36}$$

This allows to decompose the D-dimensional Einstein-tensor $E_{MN}(G) = R_{MN} - \frac{1}{2}R(G)G_{MN}$ as

$$
\begin{aligned}
E_{\mu\nu}(G) &= E_{\mu\nu}(g) + g_{\mu\nu}\frac{(D-2)}{2}\left[\left(1-\frac{(D-1)}{4}\right) f\left[(\ln f)'\right]^2 - f''\right] \\
E_{\mu D}(G) &= 0 \\
E_{DD}(G) &= -\frac{1}{2f}R(g) - \frac{(D-1)(D-2)}{8}\left[(\ln f)'\right]^2 \ .
\end{aligned}
\tag{5.37}
$$

Let us now restrict to $D = 5$, where the expressions simplify to

$$
\begin{aligned}
E_{\mu\nu}(G) &= E_{\mu\nu}(g) - \frac{3}{2}g_{\mu\nu}f'' \\
E_{\mu 5}(G) &= 0 \\
E_{55}(G) &= -\frac{1}{2f}R(g) - \frac{3}{2}\left[(\ln f)'\right]^2 \ .
\end{aligned}
\tag{5.38}
$$

For the action (5.19) specifying the set-up, the gravitational sources consist of a non-positive bulk cosmological constant $\Lambda(x^5) \leq 0$ and walls with tension T placed at $x^5 = l$ and $x^5 = -l$, such that the energy-momentum tensor reads

$$
T_{MN} = -\Lambda(x^5)G_{MN} - T\left[\delta(x^5 + l) + \delta(x^5 - l)\right] g^{(4)}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N} \ . \tag{5.39}
$$

Decomposing the 5-dimensional Einstein-equation, $E_{MN}(G) = -T_{MN}/(2M^3)$, with the help of (5.38) into its μ and 5 components, we receive from the $\mu\nu$ part the 4-dimensional Einstein-equation

$$
E_{\mu\nu}(g) = \left[\frac{3}{2}f'' + \frac{f}{2M^3}\left[\Lambda(x^5) + T\delta(x^5 + l) + T\delta(x^5 - l)\right]\right] g_{\mu\nu} \ . \tag{5.40}
$$

From the 55 part follows an expression for the 4-dimensional curvature scalar

$$
R(g) = -f\left[3\left[(\ln f)'\right]^2 + \frac{\Lambda(x^5)}{M^3}\right] \ , \tag{5.41}
$$

whereas the $\mu 5$ part is satisfied trivially. Contraction of $E_{\mu\nu}(g)$ with $g^{\mu\nu}$ gives ${E^{\mu}}_{\mu}(g) = \frac{3-D}{2}R(g) \to -R(g)$ and therefore leads to the following consistency equation among (5.40) and (5.41)

$$
2\frac{f''}{f} - \left[(\ln f)'\right]^2 = -\frac{1}{3M^3}\left[\Lambda(x^5) + 2T\delta(x^5 + l) + 2T\delta(x^5 - l)\right] \ . \tag{5.42}
$$

It is evident that the right-hand-sides of (5.40) and (5.41) must be piecewise constant with respect to x^5, since both left-hand-sides are at least piecewise independent of x^5. This means that the 4-dimensional sections Σ_4, defined by $x^5 = const$, must be spacetimes of constant curvature. For $R(g) < 0$ we have de Sitter (dS_4) and for $R(g) > 0$ Anti-de Sitter (AdS_4) spacetime. Since this already determines the solution to the Einstein equation up to a scalar quantity, the equations (5.40),(5.41),(5.42) become linear dependent and it suffices to solve only two of them.

When we foliate the 5-dimensional spacetime into sections Σ_4, we see that the Einstein-equations (5.40),(5.41) derive from the following 4-dimensional action on Σ_4

$$S_{D=4}(x^5) = -\int_{\Sigma_4} d^4x\sqrt{g}\left(M_{\text{eff}}^2 R(g) + \Lambda_4(x^5)\right) \tag{5.43}$$

if we make the following identifications[37]

$$\frac{3}{2}f'' + \frac{f}{2M^3}\left[\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right] = \frac{\Lambda_4(x^5)}{2M_{\text{eff}}^2} \tag{5.44}$$

$$-f\left[3\left[(\ln f)'\right]^2 + \frac{\Lambda(x^5)}{M^3}\right] = -\frac{2\Lambda_4(x^5)}{M_{\text{eff}}^2}\ . \tag{5.45}$$

Here M_{eff} is defined as the effective Planck-scale, obtained by integrating the 5-dimensional action (5.19) over x^5

$$M_{\text{eff}}^2 = M^3\int dx^5 f(x^5)\ . \tag{5.46}$$

The Einstein equations (5.40),(5.41) now become replaced by (5.44),(5.45).

To recognize the relation between the cosmological constant $\Lambda_4(x^5)$ on sections Σ_4 and the final effective Λ_4 obtained by integrating out the fifth direction of (5.19), we note that Λ_4 is given by

$$\Lambda_4 = \int dx^5 f^2\left(M^3\left[\left[(\ln f)'\right]^2 + 4\frac{f''}{f}\right] + \left[\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right]\right)\ . \tag{5.47}$$

Using (5.44) for the second term in square brackets, we obtain the relationship

$$\Lambda_4 = f'f\big|_{x_L^5}^{x_R^5} + \langle\Lambda_4(x^5)\rangle\ , \tag{5.48}$$

[37] The 4-dimensional sections obey

$$E_{\mu\nu}(g) = \frac{\Lambda_4(x^5)}{2M_{\text{eff}}^2}g_{\mu\nu}\ , \qquad R(g) = -2\frac{\Lambda_4(x^5)}{M_{\text{eff}}^2}\ ,$$

with $\mathrm{dS}_4:\ R(g) < 0, \Lambda_4(x^5) > 0$ and $\mathrm{AdS}_4:\ R(g) > 0, \Lambda_4(x^5) < 0$.

where x^5_R, x^5_L denote the right and left boundary of the x^5 integration region and the mean is defined by

$$\langle \Lambda_4(x^5) \rangle \equiv \frac{\int dx^5 f \Lambda_4(x^5)}{\int dx^5 f} \, . \tag{5.49}$$

Since we will see that the total derivative contribution $f'f|_{x^5_L}^{x^5_R}$ will vanish in our case of interest, we learn that the 4-dimensional effective action $S_{D=4}$ is related to the sectionwise action by taking the mean, $S_{D=4} = \langle S_{D=4}(x^5) \rangle$.

Since only two of the equations (5.42),(5.44),(5.45) are independent, it is most convenient to choose (5.42) to determine the warp-factor in terms of the fundamental "input" parameters $\Lambda(x^5), M$ and T. In a further step, we will then obtain $\Lambda_4(x^5)$ from (5.45). Adopting the warp-factor Ansatz $f = e^{-A(x^5)}$ and denoting $Y(x^5) = A'(x^5)$, we can write (5.42) as

$$-2Y' + Y^2 + \frac{\Lambda(x^5)}{3M^3} = -\frac{2T}{3M^3} \left[\delta(x^5 + l) + \delta(x^5 - l) \right] \, , \tag{5.50}$$

With the signature-function defined by $\mathrm{sign}(x) = -1$ if $x \leq 0$ and $\mathrm{sign}(x) = 1$ if $x > 0$, the solution to this differential equation is given by

$$Y(x^5) = -\frac{k}{2} \left(\mathrm{sign}(x^5 + l) + \mathrm{sign}(x^5 - l) \right) \coth \left(\frac{k}{4} \left[|x^5 + l| + |x^5 - l| - 2a \right] \right) \tag{5.51}$$

together with the constraint on $\Lambda(x^5)$ with arbitrary but non-positive $\Lambda_e \leq 0$

$$\Lambda(x^5) = \begin{cases} \Lambda_e & , \ |x^5| > l \\ \Lambda_e/4 \leq 0 & , \ |x^5| = l \\ 0 & , \ |x^5| < l \end{cases} \tag{5.52}$$

and the wall-tension

$$\frac{T}{3M^3} = k \coth \left(\frac{k}{2}(a - l) \right) \, . \tag{5.53}$$

Here, $k = \sqrt{-\Lambda_e/3M^3}$ and a is an integration constant. The last relation which determines a through the bulk cosmological constant Λ_e and the wall-tension T has been gained by satisfying the boundary conditions at the wall-locations, which are encoded in the δ-function terms in (5.50). A matching of the δ-function terms arising from Y' with those proportional to T leads to (5.53). The symmetry of the set-up – caused by the equality of both wall-tensions – forces the bulk cosmological constant between them to be zero. A non-vanishing value can only be obtained if we introduce an asymmetry of the set-up through unequal wall-tensions. A further integration of Y yields the warp-function

$$A(x^5) = -2\ln\left|\sinh\left(\frac{k}{4}\left[|x^5+l|+|x^5-l|-2a\right]\right)\right| + b\ , \tag{5.54}$$

where b is a further integration constant. Note, that the above solution is valid for the parameter-range $T \geq 3M^3k$ as can be easily recognized from (5.53). If $T < 3M^3k$, we have to substitute a "tanh" for the "coth" appearing in (5.51) and (5.53), while (5.52) remains the same. This amounts to a change from "sinh" to "cosh" in (5.54) Since we assume a positive wall-tension $T > 0$, the distance-parameter l is constrained through (5.53) over the whole parameter-region $T > 0$, $\Lambda_e \leq 0$ by the upper bound $l < a$.

An important point is that the warp-factor $f = e^{-A(x^5)}$ vanishes at $x^5 = \pm a$. This leads to a singular 5-dimensional curvature at these points only if $Q < 0$ (which later will turn out to be the AdS$_4$ case, whereas the physically more realistic dS$_4$ case is free of singularities)

$$\lim_{x^5\to\pm a} R(G) \to \frac{24\Theta(-Q)}{(|x^5|-a)^2}\ , \qquad Q = \frac{T-3M^3k}{T+3M^3k}\ , \tag{5.55}$$

where the Heaviside step-function is defined by $\Theta(x) = 0, x < 0$ and $\Theta(x) = 1, x > 0$. Due to the vanishing of the warp-factor at these points we expect a tremendous red-shift in signals originating there. Indeed, let us conceive an electromagnetic wave emitted with frequency ν_e at $x^5 = \pm a$. Then that wave will be observed in the interior region $x^5 \in (-a, a)$ with frequency ν_o given by

$$\frac{\nu_o}{\nu_e} = \sqrt{\frac{G_{11}(x^5=\pm a)}{G_{11}(|x^5|<a)}} = 0\ , \tag{5.56}$$

due to the vanishing of the warp-factor at $x^5 = \pm a$. Hence, an infinite redshift makes it impossible for the region $|x^5| \geq a$ to communicate to our world (at least via electromagnetic radiation). Therefore, we should restrict the x^5 integration region to the causally connected interval $x^5 \in (-a, a)$.

Since recently there has been a discussion in the literature [118],[92],[100] about which singularities are permissible and which have better to be avoided, it is interesting to see the verdict on our singularities in the case of $Q < 0$. Recently, a zero 4-dimensional cosmological constant has been claimed to be attained in a domain-wall scenario by relying in an essential way on the presence of a further 5-dimensional bulk scalar field [90],[91]. Exploiting e.g. the freedom to adjust free integration constants in the solution for the scalar field, allowed to obtain a flat 4-dimensional vacuum. Afterwards in [118] it has been argued that in a gravitational system exhibiting a 4-dimensional flat solution together with bulk scalars only such singularities are allowed, which leave the scalar potential bounded from above. The solutions of [91] fail to obey this criterion. In our case, where we do not have any scalars, the role of the scalar potential is played by the bulk cosmological constant Λ_e (together with the tension T at the wall-positions), which is clearly bounded from above. If the criterion of [118] generalizes to the case where the 4-dimensional metric deviates slightly (since in the end Λ_4 turns out to be very small – of the order of the observed value) from the flat case, we would conclude that the above singularities are of the permissible type.

Furthermore in [92] it has been pointed out, that without specifying additional sources at the singularities, the actual Λ_4 of [91] does not vanish. We will determine Λ_4 for our pure geometrical mechanism explicitly below and will see that it leads indeed to a vanishing Λ_4 in the finetuned situation. Moreover, the general case with non-finetuned parameters will be smoothly connected to the finetuned case and will exhibit the desired exponential suppression.

Finally, in [100] a consistency condition has been derived which the action of [91] also fails to satisfy. We will now demonstrate that this consistency condition is a simple consequence of (5.44),(5.45) and the expression (5.47), which defines Λ_4. Starting with (5.47) and employing (5.44),(5.45) to eliminate the derivatives $[(\ln f)']^2$ and f'', (5.47) becomes

$$\Lambda_4 = 2\langle\Lambda_4\rangle - \frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(2\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right) \ . \tag{5.57}$$

Noticing that $f'f(x^5 = \pm a) = 0$, we use (5.48) to obtain

$$\begin{aligned} \Lambda_4 &= \frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(2\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right) \\ &= -\frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(T_1{}^1 + T_5{}^5\right) \ , \end{aligned} \tag{5.58}$$

which is nothing but the consistency condition of [100]. Since we derived our solution with the help of (5.42),(5.45) which is equivalent to (5.44),(5.45) and will furthermore use (5.48) to obtain Λ_4, we conclude that the consistency condition (5.58) of [100] is satisfied for our solution.

Inverting (5.53), we can express a explicitly through the input values T and Λ_e by

$$a = -\frac{1}{k}\ln|Q| + l \ , \tag{5.59}$$

which is valid for both $T \geq 3M^3k$ and $T < 3M^3k$. This shows how the parameters T, M, Λ_e influence the width of the x^5 domain.

Furthermore, we have to fulfill (5.45), which is used to eventually find the following expressions for $\Lambda_4(x^5)$

$$\Lambda_4(x^5) = \pm\frac{3}{2}e^{-b}M_{\text{eff}}^2 \begin{cases} k^2 & , \ |x^5| > l \\ k^2/4 & , \ x^5 = \pm l \\ 0 & , \ |x^5| < l \end{cases} \tag{5.60}$$

Here, the plus-sign applies to the case $T \geq 3M^3k$, whereas the minus-sign applies to the complementary case in which $T < 3M^3k$.

Since we do not want to use $\Lambda_4(x^5)$ as an input to determine b, but rather focus on the opposite, we have to find an additional constraint, which allows for a determination of the constant b. This extra constraint comes from considering the transition to the flat solution (5.21) with $\Lambda_4 = 0$. As can be seen from (5.22), we reach the flat limit by sending $T \to 3M^3k$. Via (5.59) this limit corresponds to sending the constant $a \to \infty$. Thus we see, that the integration region $x^5 \in (-a, a)$ extends to the whole real line in this limit and the warp-function (5.54) is sent to

$$A(x^5) \to \frac{k}{2}\left(|x^5 + l| + |x^5 - l|\right) + 2\ln 2 - ka + b \ . \tag{5.61}$$

To guarantee a smooth transition to the flat solution (5.21), we have to identify the integration constants a and b as follows

$$b = -2\ln 2 + ka \ . \tag{5.62}$$

This, together with (5.59) and (5.60) yields the following expression for $\Lambda_4(x^5)$

$$\Lambda_4(x^5) = 6e^{-kl}QM_{\text{eff}}^2 \begin{cases} k^2 & , \ |x^5| > l \\ k^2/4 & , \ x^5 = \pm l \\ 0 & , \ |x^5| < l \end{cases} \tag{5.63}$$

Notice, that this formula is valid for both parameter-regions $T \geq 3M^3k$ and $T < 3M^3k$.

Ultimately, we have to take the mean of $\Lambda_4(x^5)$ to obtain Λ_4. Again using that $f'f(x^5 = \pm a) = 0$, we employ (5.48) to gain Λ_4. Thus, performing the mean on (5.63), we end up with

$$\Lambda_4 = \frac{\int_{-a}^{a} dx^5 e^{-A(x^5)} \Lambda_4(x^5)}{\int_{-a}^{a} dx^5 e^{-A(x^5)}}$$
$$= 24 e^{-k(l+a)} M^3 k^2 Q \left(\frac{\sinh(k(a-l))}{k} - (a-l) \right)$$
$$= 12 e^{-2kl} M^3 k Q F(|Q|) \ , \qquad (5.64)$$

where $F(|Q|) = 1 - |Q|^2 + 2|Q| \ln |Q|$. In addition we obtain the following effective Planck-scale

$$M_{\text{eff}}^2 = M^3 \int_{-a}^{a} dx^5 e^{-A(x^5)}$$
$$= 4M^3 e^{-ka} \left(2l \sinh^2 \left(\frac{k}{2}(a-l) \right) + \frac{\sinh(k(a-l))}{k} - (a-l) \right)$$
$$= 2M^3 e^{-kl} \left(l(1-|Q|)^2 + \frac{F(|Q|)}{k} \right) \ . \qquad (5.65)$$

There is an exponential-factor occuring in Λ_4 which is the square of the one occuring in M_{eff}^2. However, at the classical level (with respect to bulk gravity) an overall constant multiplying the effective 4-dimensional action $S_{D=4} = -\int d^4x \sqrt{g} (M_{\text{eff}}^2 R(g) + \Lambda_4)$ is immaterial – it simply drops out of the field equation[38]. Therefore, we can neglect a common factor e^{-kl} in both Λ_4 and M_{eff}^2 at the classical level and finally receive

$$\Lambda_4 = 12 e^{-kl} M^3 k Q F(|Q|) \qquad (5.67)$$
$$M_{\text{eff}}^2 = 2M^3 \left(l(1-|Q|)^2 + \frac{F(|Q|)}{k} \right) \ . \qquad (5.68)$$

The physical range of Q lies between $0 \leq |Q| \leq 1$, where the upper bound presupposes a non-negative wall-tension $T > 0$. The lower bound corresponds to the finetuned flat $\Lambda_4 = 0$ limit, while the upper bound is reached for vanishing bulk cosmological constant $\Lambda_e = 0$. Over that domain we have $1 \geq F(|Q| < 1) > 0$, $F(1) = 0$. Hence, we recognize that starting with arbitrary "fundamental" values for $\Lambda_e \leq 0, M, T > 0$ we obtain a positive or negative Λ_4 depending on the sign of Q. For $T > \sqrt{-3M^3 \Lambda_e}$ the 4-dimensional

[38]Another way to see this is to consult the 4-dimensional Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = -\frac{\Lambda_4}{2M_{\text{eff}}^2} g_{\mu\nu} \ , \qquad (5.66)$$

which contains only one physically relevant constant, namely $\frac{\Lambda_4}{2M_{\text{eff}}^2}$.

spacetime will be dS_4, whereas for $T < \sqrt{-3M^3\Lambda_e}$ it will be AdS_4. Furthermore, we see a smooth connection to the flat $\mathbb{M}^4$ case with finetuned parameters $Q = 0 \Leftrightarrow T = \sqrt{-3M^3\Lambda_e}$. Most importantly there is no need for a finetuning of the "fundamental" parameters to receive a small Λ_4. By increasing the distance $2l$ between both walls, we arrive soon at a huge enough suppression through the exponential factor such that we can match the observed value $\Lambda_4 \simeq 10^{-47}\mathrm{GeV}^4$. Thanks to the exponential suppression this does not amount to a large hierarchy between the fundamental scale M and $1/l$.

5.4 The Effective Potential due to Bulk Scalars

The embedding of our five-dimensional set-up into IIB string-theory or F-theory along the lines of [109] gives a host of bulk-fields in addition. They arise from the usual dimensional reduction procedure from ten to five dimensions. It is important to check that these further fields do not reintroduce huge contributions to the effective four-dimensional cosmological constant upon further reduction from five to four dimensions. Otherwise the embedding of our set-up together with the above mechanism to exponentially suppress the cosmological constant would be immediately spoiled.

To this aim, we will examine in this subsection the four-dimensional effective potential which is engendered by a generic five-dimensional bulk scalar Φ. Let us assume a bulk scalar Φ with quartic couplings to the two walls as in the Goldberger-Wise mechanism [104] which stabilizes the RS-scenario. For the action of the scalar with mass m, let us take

$$\begin{aligned} S_\Phi = &- \int d^4x \int_{-\infty}^{\infty} dx^5 \sqrt{G} \Big\{ \frac{1}{2} G^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{2} m^2 \Phi^2 \Big\} \\ &- \int d^4x \int_{-\infty}^{\infty} dx^5 \Big\{ \sqrt{g_1^{(4)}} \lambda_1 (\Phi^2 - v_1^2)^2 \delta(x^5 + l) \\ &\qquad + \sqrt{g_2^{(4)}} \lambda_2 (\Phi^2 - v_2^2)^2 \delta(x^5 - l) \Big\} . \end{aligned} \tag{5.69}$$

Assuming that Φ does only depend on x^5 and employing the five-dimensional metric

$$G_{MN} = e^{-A(x^5)} \eta_{\mu\nu} dx^\mu dx^\nu + (dx^5)^2 \tag{5.70}$$

together with (5.21) as the gravitational background, we arrive at the following field equation

$$\begin{aligned} (e^{-2A}\Phi')' - e^{-2A} m^2 \Phi = 4 \big[&e^{-2A(-l)} \lambda_1 (\Phi^2 - v_1^2) \Phi \delta(x^5 + l) \\ &+ e^{-2A(l)} \lambda_2 (\Phi^2 - v_2^2) \Phi \delta(x^5 - l) \big] . \end{aligned} \tag{5.71}$$

Away from the walls it has the solution

$$\Phi(x^5) = \begin{cases} ae^{(1+\Gamma)A} + be^{(1-\Gamma)A}\ , & x^5 < -l \\ ce^{mx^5} + de^{-mx^5}\ , & |x^5| \leq l \\ ee^{(1+\Gamma)A} + fe^{(1-\Gamma)A}\ , & x^5 > l \end{cases}\ , \tag{5.72}$$

with

$$\Gamma = \sqrt{1 + m^2/k^2} \tag{5.73}$$

and arbitrary coefficients a, b, c, d, e, f. In order to obtain a normalizable solution for Φ, we are forced to set $a = e = 0$. Furthermore, demanding continuity of Φ at the walls determines b and f in terms of c, d

$$b = e^{(\Gamma-1)kl}\tilde{b}\ , \quad \tilde{b} = ce^{-ml} + de^{ml} \tag{5.74}$$

$$f = e^{(\Gamma-1)kl}\tilde{f}\ , \quad \tilde{f} = ce^{ml} + de^{-ml}\ . \tag{5.75}$$

To fix the remaining coefficients c and d one would have to plug the above bulk solution in the field equation and integrate over the fifth dimension to incorporate the wall boundary conditions. However, this leads to a complicated cubic equation in the unknowns c, d. An easier way [104] to arrive at a determination of the coefficients c, d is to put the above bulk solution into the scalar action and integrate over x^5 to arrive at an effective potential for the distance-parameter l. From the couplings of Φ to the walls the effective potential receives the contributions

$$\int d^4x \left\{ \sqrt{g_1^{(4)}}\lambda_1 \left(\Phi^2(-l) - v_1^2\right)^2 + \sqrt{g_2^{(4)}}\lambda_2 \left(\Phi^2(l) - v_2^2\right)^2 \right\}\ . \tag{5.76}$$

Hence, to minimize the potential for positive couplings λ_1, λ_2, we must set $\Phi(-l) = v_1$ and $\Phi(l) = v_2$. These two further conditions then allow for a determination of c, d in terms of the parameters v_1, v_2

$$c = \frac{-v_1 e^{-ml} + v_2 e^{ml}}{2\sinh(2ml)}\ , \qquad d = \frac{v_1 e^{ml} - v_2 e^{-ml}}{2\sinh(2ml)}\ . \tag{5.77}$$

Finally, the effective four-dimensional potential V_Φ, defined by $S_\Phi = -\int d^4x\sqrt{g}V_\Phi(l)$, becomes

$$V_\Phi(l) = \frac{e^{-2kl}}{2}\left\{ (v_1^2 + v_2^2)\left[(\Gamma - 1)k + m\coth(2ml)\right] - 2v_1 v_2 \frac{m}{\sinh(2ml)} \right\}\ , \tag{5.78}$$

where the identity $(1-\Gamma)^2k^2 + m^2 = 2\Gamma(\Gamma-1)k^2$ has been utilized. For the special case of a massless, $m = 0$, bulk scalar Φ, the effective potential simply reads

$$V_\Phi(l) = \frac{e^{-2kl}}{4l}(v_1 - v_2)^2\ . \tag{5.79}$$

The first conclusion is, that regardless of the mass m chosen, the important exponential suppression-factor shows up again, thus guaranteeing that for generical values of v_1, v_2, m these effective potentials cannot generate a cosmological constant larger than its experimental value (5.1).

The second observation pertains to the possibility of achieving in addition stability with these tiny potentials. From (5.79) it is immediately recognizable, that no minimum at finite l exists. For the case with $m \neq 0$, setting $\partial V_\Phi / \partial l$ equal to zero, amounts to solving the equation

$$(w^2+1)\left[\left(\frac{\Gamma-1}{r}+\cosh(2ml)\right)\sinh(2ml)+r\right] = 2w\left[r\cosh(2ml)+\sinh(2ml)\right] \ , \tag{5.80}$$

where we have used the dimensionless variables

$$w = \frac{v_1}{v_2} \ , \qquad r = \frac{m}{k} \ , \tag{5.81}$$

in terms of which we find $\Gamma = \sqrt{1+r^2}$. It is now easy to analyze the last equation numerically with the result that there are no real and positive solutions for the distance-parameter l for generic values of v_1, v_2, m. Therefore we conclude that in the general case with $m \neq 0$, the effective potential exhibits no minimum either.

The general case with different tensions $T_1 \neq T_2$ will be covered in appendix C. We thus learn that a generic bulk scalar, with couplings to the walls, leads to an effective potential, which is likewise exponentially suppressed. The addition of further bulk scalar fields therefore seems to present no obstruction to obtain the right value for Λ_4. This is an essential ingredient for an embedding of the suppression mechanism into IIB string- or F-theory, where one faces a host of extra bulk scalar fields from dimensional reduction to five dimensions.

We have seen that roughly the complete effective potential ($= \Lambda_4$) goes like $e^{-M_{Pl}d}$ with the inter-wall distance d. For $d = l = 284 l_{Pl}$ nowadays this potential is very tiny, namely of the order of $(1\text{meV})^4$. Likewise the repelling force driving the walls apart is exceedingly small such that we can regard this scenario as quasi-static. In the very early universe, however, during the period of inflation, actually a nonvanishing four-dimensional cosmological constant $\Lambda_4 = V(\phi)$ is needed. Here $V(\phi)$ denotes the potential or vacuum energy density of the inflaton ϕ. For example in the scenario of chaotic inflation [115] one imposes the initial condition $V(\phi) \approx M_{Pl}^4$, which implies $\Lambda_4 \approx M_{Pl}^4$. Thus in the very early universe, we face a situation, where the two walls have to be much closer together. At the same time this means that the repelling forces which drive the walls apart are significantly larger since the supression-factor $e^{-M_{Pl}d}$ becomes larger. Therefore, this period witnessed (if there are no other stabilizing contributions to the potential at those early times) a rapid expansion in the x^5 direction.

5.5 IIB-String Embedding of the Set-Up

To better understand the appearance of the GUT-scale in the above low-energy set-up, we will now indicate how it can be embedded into IIB string-theory resp. F-theory. Moreover, we will point out a natural connection to an SU(6) SUSY Grand Unification (GUT), whose gauge symmetry has to be broken if we want to obtain a realistically small Λ_4. To this aim we are going to describe in this and the next subsection some rather generic features of such an embedding, which may serve as a guideline for the actual model building, e.g. in terms of an orbifold construction [113].

Since the low-energy five-dimensional geometry consists of two half infinite AdS_5 patches divided by two four-dimensional positive tension walls with an interpolating x^5-finite flat spacetime interval in between, one may think of two stacks of D3-branes at opposite values of X^5 in the IIB-string setting[39]. For the low-energy situation with two walls of equal tension, we should place the same amount of D3-branes in either stack. If we furthermore want to embrace the minimal supersymmetric extension of the Standard Model (MSSM), we have to group the branes in such a way, that the low-energy gauge-group $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ arises. Together, this requires 3 D3-branes in one stack and another 3 making up the other stack. To accomodate for the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ part, we imagine a tiny split between the 3 branes of the second stack into 2 giving rise to $\mathrm{SU}(2)_L$ and a single one responsible for $\mathrm{U}(1)_Y$ (see fig.3). Actually, the D3-brane gauge-group will be $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)_Y$ with two further local U(1) symmetries. However, usually we have to project out some massless states (e.g. the unphysical four-dimensional gauge-field A^{ij}_μ in the last subsection), which is conveniently done through an orbifold procedure. Then typically only one of the three abelian gauge groups stays anomaly-free [113],[114], while the others become anomalous. Since we will arrive at the correct hypercharge assignments for the light MSSM matter in the next subsection, $\mathrm{U}(1)_Y$ will be the anomaly-free factor. The anomalies of the two further abelian factors are cancelled in string-theory by a Green-Schwarz mechanism, which renders them massive. They remain

[39] X^5 is that coordinate which describes a variation in the IIB string-frame if we vary x^5 in the low-energy-frame.

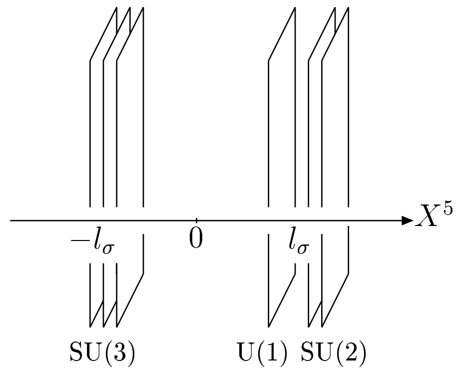

Figure 3: The two walls resolved as stacks of D3-branes in a microscopic IIB string description.

as global symmetries with mass of the U(1) gauge-bosons at the string-scale.

For a $\mathbb{Z}_2$ symmetric ($\mathbb{Z}_2 : X^5 \to -X^5$) D3-brane configuration as depicted in fig.3, it has been argued in [109], that in the low-energy five-dimensional description the D3-brane sources lead to half-infinite throats which describe exactly two AdS_5 half-slices. Suppose there are no other gravitational sources located in between the two D3-brane stacks. Then due to the $\mathbb{Z}_2$ reflection symmetry, the low-energy warp-factor in the intermediate interval cannot develop a kink but has to be constant. Thus it seems indeed possible to arrive at the above low-energy geometry, given by (5.12),(5.21), through a simple set-up of oppositely placed D3-brane stacks in the context of IIB- or F-theory.

Because the D3-branes are localized in the internal six- (for IIB) or eight-dimensional (for F-theory) space, they are not sensitive to the global properties of the compactification space. However, there is a link between both which comes from tadpole cancellation [51],[62], or conservation of the RR 5-form flux, and states in our case with 6 D3-branes that the Euler-characteristic χ and the background fluxes have to obey

$$6 = \frac{\chi(K_8)}{24} - \int_{K_6} \frac{1}{2i\tau_2} H \wedge \bar{H} = \frac{\chi(K_8)}{24} + \int_{K_6} H^{NS} \wedge H^R \, . \tag{5.82}$$

Here, K_6 denotes the base of the underlying F-theory compactification on an elliptically fibered Calabi-Yau fourfold K_8. Furthermore, the 3-forms $H, \bar{H}$ are given by

$$H = H^R - \tau H^{NS} \, , \quad \bar{H} = H^R - \bar{\tau} H^{NS} \, , \tag{5.83}$$

with $\tau = \tau_1 + i\tau_2 = a + ie^{-\phi}$ the modulus of the elliptic fibration composed out of the axion a and the dilaton ϕ.

The connection between the IIB string-frame metric G^{σ}_{AB} ; $A, B = 1, \ldots, 10$ and the low-energy metric G_{AB}, which is used to measure length in the above five-dimensional set-up is as follows [109]. String-frame and Einstein-frame metric are related by

$$G^{E}_{AB} = e^{-\frac{\phi}{2}} G^{\sigma}_{AB} \ , \tag{5.84}$$

whereas the low-energy metric

$$\begin{aligned} ds^2 &= G_{AB} dx^A dx^B \\ &= e^{-A(x^5)} g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + (dx^5)^2 + h_{mn}(x^5, y^k) dy^m dy^n \quad , \quad m, n = 6, \ldots, 10 \end{aligned} \tag{5.85}$$

is related to G^{E}_{AB} by a further Weyl-rescaling

$$G_{AB} = V_5^{1/4} G^{E}_{AB} \ , \quad V_5 = \frac{\int_{K_5} d^5 y \sqrt{h}}{L^5_{Pl}} \ . \tag{5.86}$$

Here, $L_{Pl} = g_s^{1/4} \sqrt{\alpha'} (2\pi)^{7/8}$ denotes the ten-dimensional (reduced) Planck-length and $g_s = e^{\langle\phi\rangle}$. K_5 stands for those five-dimensional sections of the base-manifold K_6, for which x^5 is constant. The effect of these rescalings is a simple determination of M in terms of L_{Pl}, which can be read off from the Einstein-Hilbert term

$$-\frac{1}{(2\pi)^7 (\alpha')^4} \int d^{10} X \sqrt{G^\sigma} R^\sigma = -\frac{1}{L^8_{Pl}} \int d^{10} x \sqrt{G^E} R^E = -\frac{1}{L^8_{Pl}} \int d^{10} x \frac{\sqrt{G} R}{V_5} \ , \tag{5.87}$$

and upon dimensional reduction to five dimensions leads to the identification

$$M = \frac{1}{L_{Pl}} \ . \tag{5.88}$$

Utilizing (5.31), we end up with the following restriction on the ten-dimensional Planck-length

$$L^3_{Pl} = \frac{2(1+kl)}{k M^2_{\text{eff}}} \simeq \frac{1}{M_{\text{GUT}} M^2_{\text{red}}} = \frac{1}{(4 \times 10^{17} \text{GeV})^3} \ , \tag{5.89}$$

if we choose generically $10 \lesssim kl$ and identify M_{eff} with the four-dimensional reduced Planck-scale $M_{\text{red}} = M_{Pl}/\sqrt{16\pi} = 1.7 \times 10^{18}$GeV. In terms of the string-scale $M_s = 1/\sqrt{\alpha'}$ and the string-coupling g_s, we are led to the restriction

$$M_s = g_s^{1/4} (2\pi)^{7/8} 4 \times 10^{17} \text{GeV} \ . \tag{5.90}$$

The inter-wall distance $2l$ in the effective description and the length $2l_\sigma$ in the stringy description are related by

$$2l = V_5^{1/8} e^{-\frac{\phi}{4}} 2l_\sigma \ . \tag{5.91}$$

Usually, in Calabi-Yau compactifications the compactification radius is chosen at the GUT-length $1/M_{\rm GUT}$. Let us now see, what this assumption together with the information that $2l = 1/M_{\rm GUT}$, amounts to for the ground state masses of the open strings. To this aim consider a U(6) oriented open string with its end points carrying the representations $\mathbf{6}$ and $\bar{\mathbf{6}}$. Compactification of the X^5 coordinate on a circle with radius $R \simeq 1/M_{\rm GUT}$ allows for a Wilson-line around this circle generated by the gauge-field background

$$A^5 = \mathrm{diag}(\theta_3, \theta_3, \theta_3, \theta_1, \theta_2, \theta_2)/(2\pi R) \ . \tag{5.92}$$

It breaks the $U(6)$ symmetry down to $U(3) \times U(2) \times U(1)$. In the T-dual picture this translates into D-branes placed at

$$\theta_3 R' \ , \qquad \theta_1 R' \ , \qquad \theta_2 R' \ , \tag{5.93}$$

with the T-dual radius given by $R' = \alpha'/R$. To cope with the cosmological constant problem, we have learned before that it is necessary to place the D-branes at two stacks (see fig.3)

$$\theta_3 R' = -l_\sigma \ , \qquad \theta_1 R' \simeq \theta_2 R' = l_\sigma \ . \tag{5.94}$$

In the next subsection, we will face the problem of incorporating the massless MSSM fields into this D-brane set-up. One natural way to do this (see below) is to set the length of the interval $[-l_\sigma, l_\sigma]$ equal to the circumference $2\pi R'$ by identifying $-l_\sigma \sim l_\sigma$. Hence, the brane positions lie at

$$\theta_3 = -\pi \ , \qquad \theta_1 \simeq \theta_2 = \pi \ . \tag{5.95}$$

After this spadework, let us now come to the lowest level string masses. An open string stretching from one brane-stack to the other gives rise to a ground state vector-multiplet with mass $2l_\sigma T$, where $T = 1/(2\pi\alpha')$ is the string-tension (and we have assumed that the two brane-stacks coincide in all other internal position coordinates except for X^5). Evaluated in the low-energy frame this amounts to a mass of

$$M_{\rm open} = V_5^{1/8} e^{-\frac{\phi}{4}} 2l_\sigma T = 2lT = \frac{T}{M_{\rm GUT}} \ . \tag{5.96}$$

Using the identification $2l_\sigma = 2\pi R'$, this lowest-level mass becomes

$$M_{\text{open}} = \frac{V_5^{1/8} e^{-\frac{\phi}{4}}}{R} \simeq V_5^{1/8} e^{-\frac{\phi}{4}} M_{\text{GUT}} \ . \tag{5.97}$$

If we set $e^\phi = g_s$ constant, (5.96) together with (5.97) impose a relation, which determines the string-scale in terms of g_s and the compactification parameter V_5

$$M_s = \sqrt{\frac{2\pi V_5^{1/8}}{g_s^{1/4}}} M_{\text{GUT}} \ . \tag{5.98}$$

If we furthermore assume that naturally $\int_{K_5} d^5 y \sqrt{h} = 1/M_{\text{GUT}}^5$, we obtain one further relation from (5.86)

$$V_5^{1/5} = \frac{1}{g_s^{1/4} (2\pi)^{7/8}} \times \frac{M_s}{M_{\text{GUT}}} \ . \tag{5.99}$$

Altogether (5.90),(5.98),(5.99) constitute three equations in the three unknowns $M_s, g_s, V_5^{1/5}$ with the solution

$$M_s = 3 \times 10^{17} \text{GeV} \ , \quad g_s = 7 \times 10^{-4} \ , \quad V_5^{1/5} = 20 \ . \tag{5.100}$$

We therefore conlude that a description in terms of perturbative string-theory is adequate. Moreover in the low-energy frame the lowest level open string excitations are given by

$$M_{\text{open}} = 40 M_{\text{GUT}} \ . \tag{5.101}$$

Though our estimate relied on the assumption that $1/R = M_{\text{GUT}}$ and $\int_{K_5} d^5 y \sqrt{h} = 1/M_{\text{GUT}}^5$, it appears to be quite generic. For example directly identifying M_{open} with M_{GUT} would imply via (5.90) a $g_s = 4 \times 10^{-7}$, which seems less natural than the above obtained value. In the string-frame the open string masses are of course situated precisely at the GUT-scale, since there we have $M_{\text{open}}^\sigma = 1/R = M_{\text{GUT}}$. Subsequently, when we speak of GUT-masses, we have (5.101) in mind.

Above the GUT-scale, one may expect the full restoration of the SU(6) gauge symmetry by moving all six branes on top of each other. However, it has to be noticed that this goes hand in hand with the creation of a huge Planck-scale cosmological constant. But, as already pointed out, for cosmological purposes this may be just fine, though usually a de Sitter inflationary expansion is considered below GUT-energies in order to dilute the density of topological defects which are relics from the spontaneous breaking of the GUT gauge-group. The precise connection between string states and massive GUT-states will be made explicit in the coming subsection.

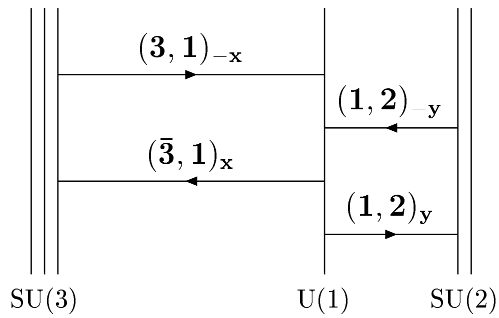

Figure 4: The "basic" open strings of the D3-brane set-up. The state $(\mathbf{n},\mathbf{m})_z$ transforms as $\mathbf{n}$ under SU(3), as $\mathbf{m}$ under SU(2) and bears U(1)-charge z.

5.6 Brane-Description and Broken SU(6) SUSY Grand Unification

5.6.1 String-Theory Perspective

From the above brane set-up, we immediately get the open string-states which are depicted in fig.4 together with their transformation properties under SU(3) and SU(2). Since we will identify later on the single U(1) with the $\mathrm{U}(1)_Y$ hypercharge, every open string describing a hypercharged state of the corresponding field theory has to connect the U(1)-brane. This is the reason why we consider in the sequel just the two types of "basic" open strings[40] (plus their orientation-reversed partners) and not those which directly connect the SU(3) branes with the SU(2) branes[41]. Fixing the sign of the U(1)-charge normalization, we can assume that $x, y > 0$.

Let us now have a closer look at the orientation of the open strings starting or ending on the single U(1)-brane. Our convention is made clear in fig.5. In physical terms this orientation convention can be thought of as indicating the direction of the U(1)-flux originating from the U(1)-charge placed at one end of the open string. Let us regard the

[40]Note that for SU(2) the fundamental **2** is pseudoreal.

[41]It is however also possible to extract the $\mathrm{U}(1)_Y$ as a linear combination of the three U(1)-factors of the D-branes [113],[114]. Indeed this is the only choice if one assigns only ± 1 values to the open strings beginning or ending on the single U(1)-brane. In contrast to this, we will not impose in this paper this restriction and allow *a priori* for arbitrary values.

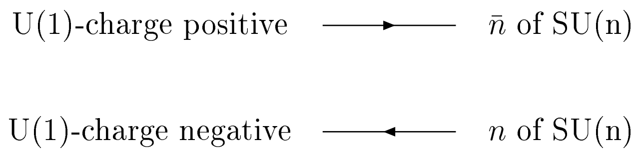

Figure 5: Fixing the orientation of the open strings.

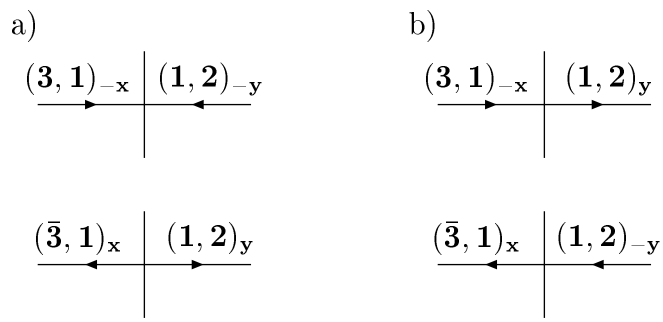

Figure 6: Four possible encounters a) with opposite orientation and b) with same orientation.

situation in which two of the strings of fig.4 meet at a common point on the U(1)-brane (see fig.6). The question arises, whether such a situation could lead to a further string state, composed out of the two "basic" ones. Let us therefore conceive a situation where an open string with weighted U(1)-charge u and another with weighted U(1)-charge v meet. By weighted we mean that the individual open string U(1)-charges are multiplied by the multiplicity originating from the Chan-Paton label of the other string end. If we assume that the charges do not cancel, $u + v \neq 0$, then the junction bears a net U(1)-charge. If it is positive, then the orientation (U(1)-flux) of the two individual strings must point away from the contact point. If the net charge is negative, then the orientation (U(1)-flux) of both strings must be such that they point towards the junction. Hence, under the condition that the weighted sum of their U(1)-charges does not cancel, two "basic" strings can compose a longer string only if the orientation of the two individual strings is opposite. This happens to be the case for the junctions of fig.6a, where two long strings with quantum numbers

$$(\mathbf{3},\mathbf{1})_{-\mathbf{x}} \otimes (\mathbf{1},\mathbf{2})_{-\mathbf{y}} \to (\mathbf{3},\mathbf{2})_{-(\mathbf{x}+\mathbf{y})} \qquad (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{x}} \otimes (\mathbf{1},\mathbf{2})_{\mathbf{y}} \to (\bar{\mathbf{3}},\mathbf{2})_{\mathbf{x}+\mathbf{y}} \tag{5.102}$$

are composed. On the other hand the orientations of the strings in fig.6b are not compatible with a non-zero total weighted U(1)-charge at the meeting point. Therefore, to avoid an inconsistency, when the states of fig.6b are composed, we have to demand that the total weighted U(1)-charge of such a junction has to vanish. From either of the two situations of fig.6b, we then get the requirement

$$3(-x) + 2y = 0 \; . \tag{5.103}$$

If we choose a normalization of the U(1)-generator such that $x = 2$, we obtain $y = 3$. Thus, it is the number of D3-branes present in each stack (except for the U(1)-brane itself), which causes the value of the U(1)-charges. Together with the composition rule, this will eventually determine all the U(1)-charges (which will be identified with the SM-hypercharge below) of open string states, which correspond to SM fields.

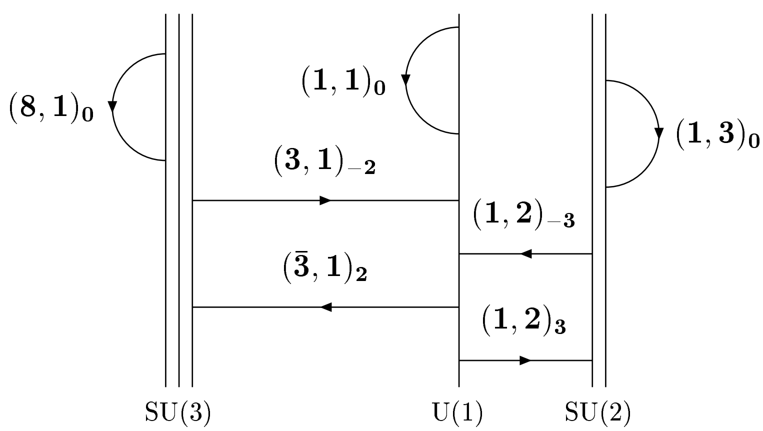

Figure 7: Massless and heavy gauge-bosons which arise from "basic" strings.

The NS-sector of the open DD-strings contains the bosonic four-dimensional gauge-fields $A_\mu^{ij} b^\mu_{-1/2}|k_4; i,j\rangle$ and scalars $A_m^{ij} b^m_{-1/2}|k_4; i,j\rangle$; $m = 5, \ldots, 10$ (i and j represent the Chan-Paton labels), whose momenta k_4 are restricted along the four longitudinal D3-brane directions due to the DD boundary-conditions. Those open strings, which start and end on the same brane-stack (see fig.7), lead to three massless gauge-boson states

$$(\mathbf{8},\mathbf{1})_\mathbf{0} \; , \; (\mathbf{1},\mathbf{3})_\mathbf{0} \; , \; (\mathbf{1},\mathbf{1})_\mathbf{0} \; , \tag{5.104}$$

while the "basic" open strings, which directly connect one brane with another, give rise to two states with mass at the GUT scale

$$(\mathbf{3},\mathbf{1})_{-\mathbf{2}}\ ,\ (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}} \tag{5.105}$$

plus two further states with mass at the TeV scale

$$(\mathbf{1},\mathbf{2})_{\mathbf{3}}\ ,\ (\mathbf{1},\mathbf{2})_{-\mathbf{3}}\ . \tag{5.106}$$

Note the natural occurrence of the mass split between the GUT and the TeV scale between triplet and doublet states. The composition of the "basic" strings (see fig.8) adds another four "composed" states with mass at the GUT scale

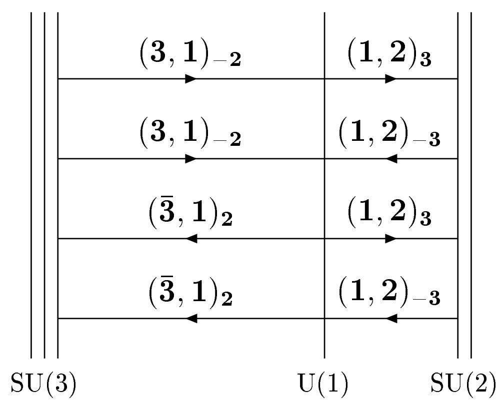

Figure 8: States which arise through the composition of the two sorts of "basic" string states.

$$(\mathbf{3},\mathbf{1})_{-\mathbf{2}}\otimes(\mathbf{1},\mathbf{2})_{\mathbf{3}}\to(\mathbf{3},\mathbf{2})_{\mathbf{1}}\ ,\qquad (\mathbf{3},\mathbf{1})_{-\mathbf{2}}\otimes(\mathbf{1},\mathbf{2})_{-\mathbf{3}}\to(\mathbf{3},\mathbf{2})_{-\mathbf{5}}\ , \tag{5.107}$$

$$(\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}}\otimes(\mathbf{1},\mathbf{2})_{\mathbf{3}}\to(\bar{\mathbf{3}},\mathbf{2})_{\mathbf{5}}\ ,\qquad (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}}\otimes(\mathbf{1},\mathbf{2})_{-\mathbf{3}}\to(\bar{\mathbf{3}},\mathbf{2})_{-\mathbf{1}}\ . \tag{5.108}$$

It remains to incorporate in a natural way a sector of nearly massless matter states, which can account for the MSSM fields with mass at the TeV scale. The most immediate way to introduce light matter would be to have some kind of tensionless string stretching between the brane stacks or some kind of "bridge" enabling open strings stretching along it to stay massless. However, for lack of those gadgets, we will simply assume X^5 to be compactified on a circle in such a way that the locations of the two brane-stacks are identified as (nearly) equivalent points $-l_\sigma \sim l_\sigma$ (see fig.9). The heavy GUT excitations

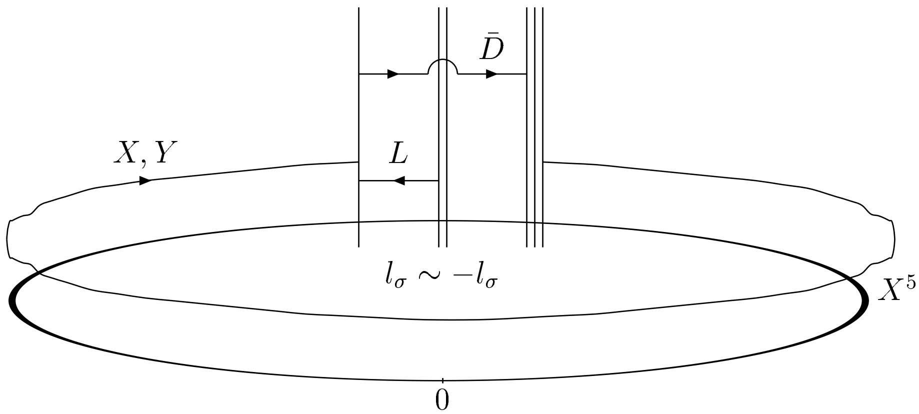

Figure 9: Light MSSM and heavy GUT fields as open string excitations which arise from a configuration with compact X^5.

found above now originate from open strings stretching once around the whole circle. In addition, the light MSSM matter-supermultiplets stem from open strings, which connect the brane-stacks over the small separating distance. Likewise as before, we will now build the massless fields out of two sorts of "basic" strings, as depicted in fig.10. The $Q, \bar{U}, \bar{E}$ chiral superfields, which are later on identified with the respective MSSM matter-fields, are composed as follows

$$Q = (\mathbf{3}, \mathbf{2})_{\mathbf{1}} = (\mathbf{3}, \mathbf{1})_{-\mathbf{2}} \otimes (\mathbf{1}, \mathbf{2})_{\mathbf{3}} \tag{5.109}$$

$$\bar{U} = (\bar{\mathbf{3}}, \mathbf{1})_{-\mathbf{4}} \subset (\mathbf{3}, \mathbf{1})_{-\mathbf{2}} \otimes (\mathbf{3}, \mathbf{1})_{-\mathbf{2}} \tag{5.110}$$

$$\bar{E} = (\mathbf{1}, \mathbf{1})_{\mathbf{6}} \subset (\mathbf{1}, \mathbf{2})_{\mathbf{3}} \otimes (\mathbf{1}, \mathbf{2})_{\mathbf{3}} \ , \tag{5.111}$$

where we used $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$ and $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$ in the last two cases and picked out the antisymmetric part while dismissing the symmetric one. Note, that the only composition of open strings, which could lead to an inconsistency in orientation with respect to the combined weighted U(1)-charge, is the one for Q. However, to avoid this, required the particular assignment of U(1)-charges for the "basic" strings as we discussed before.

Since we have now found the fields which will be identified with the MSSM matter, let us now discuss the number of generations by concentrating on the fermions in the chiral superfields. The fermions originate from the R-sector of the open DD-strings and in ten dimensions would be given by the Majorana-Weyl spinor $u^{ij}_\alpha(k_4)|\alpha; k_4; i, j\rangle$. Here, $\alpha = 1, \ldots, 8$ is a spinor-index running over the physical (on-shell) degrees of freedom and again i, j are the Chan-Paton labels. By a dimensional reduction to four dimensions, u_α

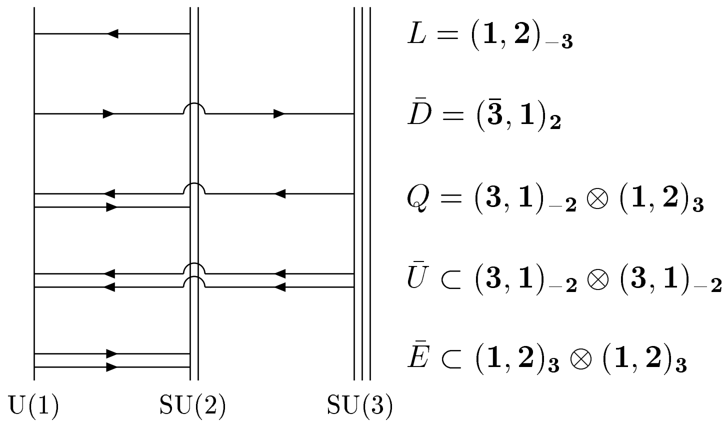

Figure 10: Light MSSM matter fields as "basic" open string states $(L, \bar{D})$ and composites thereof $(Q, \bar{U}, \bar{E})$.

turns into 4 two-component Weyl-spinors λ_A^{ij}; $A = 1, \ldots, 4$. In an $\mathcal{N} = 1$ description in four dimensions λ_4^{ij} gets combined with the NS-sector gauge-field A_μ^{ij} into a vector-superfield, while the remaining 3 spinors $\lambda_1^{ij}, \lambda_2^{ij}, \lambda_3^{ij}$ are each paired with two NS-sector scalars $(A_5^{ij}, A_8^{ij}), (A_6^{ij}, A_9^{ij}), (A_7^{ij}, A_{10}^{ij})$ to build 3 chiral superfields. Hence it is generic to arrive at a multiplicity of 3 for the chiral matter fermions or in other words at a 3 generation model. The basic reason being that we happen to live in a world with 6 internal dimensions. The important task, however, is to lift the mass degeneracy between them or equivalently reducing the $D = 4, \mathcal{N} = 4$ extended supersymmetry of the above D3-brane configuration in type IIB string-theory to a minimal $D = 4, \mathcal{N} = 1$ supersymmetry.

For example, consider a complex description of the internal coordinates in terms of Z_{58}, Z_{69}, Z_{710}, where $Z_{ij} = X_i + iX_j$. They transform under an SU(3) subgroup of the SO(6) internal tangent space group. Since the Lie-algebra of SO(6) is isomorphic to that of SU(4), the (say) positive-chirality spinor of SO(6) transforms as the fundamental $\mathbf{4}$ of SU(4). Its decomposition under the above SU(3) is $\mathbf{4} = \mathbf{3} + \mathbf{1}$, which corresponds to the split

$$\begin{pmatrix} \lambda_1^{ij} \\ \lambda_2^{ij} \\ \lambda_3^{ij} \\ \lambda_4^{ij} \end{pmatrix} = \begin{pmatrix} \lambda_1^{ij} \\ \lambda_2^{ij} \\ \lambda_3^{ij} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda_4^{ij} \end{pmatrix} . \tag{5.112}$$

In order to break the degeneracy between the 3 chiral matter fermions one could e.g. introduce 3 further D7-branes along the directions 12345689, 123457810, 123467910. Together, they preserve 1/8 of the initial 32 supercharges, which amounts to an $\mathcal{N}=1$ supersymmetric theory in four dimensions. The conditions which are imposed by the presence of the D7-branes on the supersymmetry parameters ϵ_L, ϵ_R (which are 16-component Majorana-Weyl spinors of the same chirality for IIB) are

$$\epsilon_L = \Gamma_{D3}\Gamma_{R_1}\epsilon_R \; ; \quad \Gamma_{R_1} = \Gamma^5\Gamma^6\Gamma^8\Gamma^9 \; , \; \Gamma_{D3} = \Gamma^1\Gamma^2\Gamma^3\Gamma^4 \tag{5.113}$$

$$\epsilon_L = \Gamma_{D3}\Gamma_{R_2}\epsilon_R \; ; \quad \Gamma_{R_2} = \Gamma^5\Gamma^7\Gamma^8\Gamma^{10} \tag{5.114}$$

$$\epsilon_L = \Gamma_{D3}\Gamma_{R_3}\epsilon_R \; ; \quad \Gamma_{R_3} = \Gamma^6\Gamma^7\Gamma^9\Gamma^{10} \; . \tag{5.115}$$

Taken together, they imply

$$\epsilon_L = \Gamma_{D3}\epsilon_R \; . \tag{5.116}$$

Therefore, we can place D3-branes at the common four-dimensional intersection of the three D7-branes without breaking further supersymmetry. Moreover, the introduction of the D7-branes entails a further reduction of the SO(6) tangent space group to

$$\mathrm{S}O(6) \supset \mathrm{S}U(3) \supset \mathrm{S}O(2) \times \mathrm{S}O(2) \times \mathrm{S}O(2) = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \; . \tag{5.117}$$

The fermion triplet, which we had under SU(3), now gets split into

$$\begin{pmatrix} \lambda_1^{ij} \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ \lambda_2^{ij} \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ \lambda_3^{ij} \\ 0 \end{pmatrix} , \tag{5.118}$$

since each one transforms with a phase-factor under a different U(1). Thus the degeneracy between them can be naturally lifted and different $\mathcal{N}=1$ masses attributed to them. The light vector of the vector-superfield $(\lambda_4^{ij}, A_\mu^{ij})$ however, does not appear at low-energies in a realistic theory. Hence, ultimately it should be projected out on the basis of a further discrete symmetry of the model.

5.6.2 Field-Theory Perspective

Let us determine to what SUSY GUT the string states found above belong to. Since in the limit of vanishing D3-brane separations, we would recover an SU(6) gauge theory, it is natural to compare with the spectrum of an SU(6) SUSY GUT [111], whose gauge-group is spontaneously broken down to the SM SU(3) × SU(2) × U(1)$_Y$. Besides a gauge-field transforming in the **35** adjoint representation of SU(6), its matter content comprises the following chiral supermultiplets [111]

- Higgs-fields: $\Sigma = \mathbf{35}$, $H = \mathbf{6}$, $\bar{H} = \bar{\mathbf{6}}$, $Y = \mathbf{1}$
- Fermionic Matter: $\psi_1^f = \mathbf{15}$, $\bar{\phi}_1^f = \bar{\mathbf{6}}$, $\bar{\phi}_2^f = \bar{\mathbf{6}}$, $\psi_2 = \mathbf{15}$, $\bar{\psi} = \bar{\mathbf{15}}$, $\eta = \mathbf{20}$,

where $f = 1, 2, 3$ is a family index. For a comparison to the afore obtained string spectrum it is necessary to decompose the fields first under SU(5) and afterwards under SU(3) × SU(2) × U(1)$_Y$. Under SU(5) we have

$$
\begin{aligned}
\mathbf{35} &= \mathbf{1} + \mathbf{5} + \bar{\mathbf{5}} + \mathbf{24} \\
\mathbf{20} &= \mathbf{10} + \bar{\mathbf{10}} \\
\mathbf{15} &= \mathbf{10} + \mathbf{5} \\
\mathbf{6} &= \mathbf{5} + \mathbf{1} \; .
\end{aligned}
$$

The fundamental **5**, the antisymmetric tensor rep. **10** and adjoint **24** of SU(5) themselves decompose under SU(3) × SU(2) × U(1)$_Y$ as follows

$$
\begin{aligned}
\mathbf{24} &= (\mathbf{1},\mathbf{1})_{\mathbf{0}} + (\mathbf{1},\mathbf{3})_{\mathbf{0}} + (\mathbf{3},\mathbf{2})_{-\mathbf{5}} + (\bar{\mathbf{3}},\mathbf{2})_{\mathbf{5}} + (\mathbf{8},\mathbf{1})_{\mathbf{0}} \\
\mathbf{10} &= (\mathbf{1},\mathbf{1})_{\mathbf{6}} + (\bar{\mathbf{3}},\mathbf{1})_{-\mathbf{4}} + (\mathbf{3},\mathbf{2})_{\mathbf{1}} \\
\mathbf{5} &= (\mathbf{1},\mathbf{2})_{\mathbf{3}} + (\mathbf{3},\mathbf{1})_{-\mathbf{2}} \; .
\end{aligned}
$$

In the string description we found for the external μ components of the NS-sector the massless $(\mathbf{8},\mathbf{1})_0, (\mathbf{1},\mathbf{3})_0, (\mathbf{1},\mathbf{1})_0$ states, which represent the familiar Standard Model gluons and the four electroweak gauge-bosons W_1, W_2, W_3, B, which become upon spontaneous symmetry breaking the physical W^+, W^-, Z^0 and the photon A. The heavy $(\mathbf{3},\mathbf{2})_{-\mathbf{5}}$ and $(\bar{\mathbf{3}},\mathbf{2})_{\mathbf{5}}$ describe the twelve X- and Y-leptoquark gauge-bosons of broken SU(5) and were constructed as stringy "composites". Together with the "basic" string states $(\mathbf{3},\mathbf{1})_{-2}, (\bar{\mathbf{3}},\mathbf{1})_2, (\mathbf{1},\mathbf{2})_3, (\mathbf{1},\mathbf{2})_{-3}$ these states furnish the broken **35** of the SU(6) gauge-fields. Similarly, the internal m components of the NS-sector deliver the Higgs-adjoint Σ. We already noted above that the Higgs-triplets naturally take masses at the GUT-scale due to the geometry of the brane set-up, whereas the two Higgs-doublets of the MSSM, which belong to the chiral superfields

$$
\text{Higgs-Bosons + Higgsinos:} \qquad \hat{H}_1 = (\mathbf{1},\mathbf{2})_{-\mathbf{3}} \quad \hat{H}_2 = (\mathbf{1},\mathbf{2})_{\mathbf{3}}
$$

acquire a small mass. This mass, which should be at the TeV scale, is caused by the little split between the U(1)- and the SU(2)-branes in the second brane-stack. Therefore, the brane configuration, which resulted from the requirement of a small four-dimensional cosmological constant Λ_4, also allows for a natural understanding of the doublet-triplet splitting in spontaneously broken GUT-theories.

In the same vein the Higgs-fundamentals $H, \bar{H}$ and the fermionic SU(6) matter $\bar{\phi}_1^f, \bar{\phi}_2^f$ can be built solely out of the two "basic" open strings. The matter fermions $\psi_1^f, \psi_2, \bar{\psi}, \eta$ contain the $\mathbf{5}, \bar{\mathbf{5}}, \mathbf{10}, \bar{\mathbf{10}}$. Whereas the $\mathbf{5}, \bar{\mathbf{5}}$ are built out of the "basic" strings, the $\mathbf{10}, \bar{\mathbf{10}}$ are exclusively "composite" string states. The chiral superfields representing the matter-content of the MSSM

Squarks + Quarks:	$Q = (\mathbf{3}, \mathbf{2})_{\mathbf{1}}$	$\bar{U} = (\bar{\mathbf{3}}, \mathbf{1})_{-\mathbf{4}}$	$\bar{D} = (\bar{\mathbf{3}}, \mathbf{1})_{\mathbf{2}}$
Sleptons + Leptons:	$L = (\mathbf{1}, \mathbf{2})_{-\mathbf{3}}$	$\bar{E} = (\mathbf{1}, \mathbf{1})_{\mathbf{6}}$	

fill up the the $\bar{\mathbf{5}}$ and $\mathbf{10}$ of SU(5). Their SU(6) origin is from $\bar{\phi}_1^1, \bar{\phi}_1^2, \bar{\phi}_1^3, \psi_1^1, \psi_1^2, \eta$ [111]. We now discern, that the abelian charge which arose from coupling of open strings to the single U(1)-brane, indeed has to be identified with the SM hypercharge Y.

In general, to build bosonic or fermionic matter with GUT-mass, we have to use in the string description the heavy "basic" $(\mathbf{3}, \mathbf{1})_{-2}, (\bar{\mathbf{3}}, \mathbf{1})_2$ states which wind from the SU(3)-brane stack to the U(1)-brane around the circle. For light states instead, we use the "basic" $(\mathbf{3}, \mathbf{1})_{-2}, (\bar{\mathbf{3}}, \mathbf{1})_2$ states which connect the same brane-stacks but this time via the small gap in fig.9. In order to secure a small cosmological constant Λ_4, the stringy description generically predicts light mass for all doublets $(\mathbf{1}, \mathbf{2})_3$ or composites thereof like $(\mathbf{1}, \mathbf{1})_6$. However, in some cases (e.g. $\psi_2, \bar{\psi}$) these states also have to become heavy to decouple from the light spectrum. This can be achieved by coupling these states in the superpotential to other generically heavy states.

We want to conclude with two remarks. First, it is of interest to explore the low-energy value for $\sin^2 \Theta_W$. On a stack of N D3-branes the effective gauge coupling is given by $g_{\text{eff}}^2 = g_s N$ [112]. Therefore, if we plug $g_2^2 = 2g_s, g_Y^2 = g_s$ into

$$\sin^2 \Theta_W(M_s) = \frac{g_Y^2}{g_2^2 + g_Y^2} = \frac{1}{3} \; , \tag{5.119}$$

we arrive at a value, which is close to the traditional GUT value $\frac{3}{8}$. Using the representation $\frac{1}{\alpha} = \frac{1}{\alpha_Y} + \frac{1}{\alpha_2}$ for the electromagnetic fine-structure "constant" plus $\sin^2 \Theta_W = \frac{\alpha}{\alpha_2}$, we can evaluate the difference $\frac{1}{\alpha(M_Z)} - \frac{3}{\alpha_2(M_Z)}$ as $(1 - 3\sin^2 \Theta_W(M_Z))\frac{1}{\alpha(M_Z)}$. Alternatively, using the 1-loop running of the gauge-couplings

$$\frac{1}{\alpha_i(E)} = \frac{1}{\alpha_i(M_Z)} - \frac{b_i}{2\pi} \ln\left(\frac{E}{M_Z}\right) \ , \tag{5.120}$$

plus the condition that $\frac{1}{\alpha_Y} = \frac{2}{\alpha_2}$ at the string-scale M_s, the difference can be evaluated as $\frac{1}{2\pi}(b_Y - 2b_2)\ln(\frac{M_s}{M_Z})$. Combining both expressions, we obtain the 1-loop running of the Weinberg-angle

$$\sin^2\Theta_W(M_Z) = \frac{1}{3} + \frac{\alpha(M_Z)}{6\pi}(2b_2 - b_Y)\ln\left(\frac{M_s}{M_Z}\right) \ . \tag{5.121}$$

Moreover, let us assume that the MSSM is valid from the weak scale M_Z up to the string-scale $M_s \simeq 40 M_{\mathrm{GUT}}$. This assumption selects the β-function one-loop coefficients of the MSSM

$$b_Y = 11 \ , \quad b_2 = 1 \tag{5.122}$$

for the whole energy-region. Together with the experimental value $\frac{1}{\alpha(M_Z)} = 127.9$ this leads to a value of $\sin^2\Theta_W(M_Z) = 0.196$, which is too low compared with the measured value $\sin^2\Theta_W(M_Z) = 0.231$. Hopefully, this result could be corrected towards the right direction, if we abandon the idea of a universal validity of the MSSM up to the string-scale. Instead, it would be natural to allow for some intermediate scale $M_I \simeq \sqrt{1\mathrm{TeV} \times M_s} \simeq 3 \times 10^{10}\mathrm{GeV}$, since we saw above that some light doublets should couple to M_s-massive states and thereby acquire mass of order M_I.

Second, in order to embrace an $\mathcal{N} = 1$ supersymmetric four-dimensional theory at low-energies, we did not aim at breaking all supersymmetry already at the string-scale. This will probably be only reasonable if one lowers the string-scale to the TeV-scale, which we did not intend. Therefore, we left open the precise mechanism for low-energy supersymmetry breaking. Usually the question of obtaining a realistic mechanism for supersymmetry breaking struggles with the fact that all known mechanisms produce a large cosmological constant. In this respect, we hope that the above described scenario will likewise suppress these contributions, such that it will be eventually possible to apply one of the various mechanisms to actually determine the soft-breaking terms of the complete MSSM.

6 T-Duality and its Impact on Curvature

In this final chapter we want to ask to which degree is string-theory "blind" towards a cosmological constant. It is well known from compactification to three dimensions [116] that T-duality is able to transform spaces which are asymptotically flat into spaces which are asymptotically AdS_3. Since T-duality is one of the assumed symmetries of M-theory, it seems that M-theory does not care much about our notion of a cosmological constant. Concretely, we will explore whether an Anti-de Sitter space can be transformed via T-duality into flat space. Another important motivation to study this problem comes from the AdS-CFT duality as will be made clearer in the following.

By now the AdS-CFT conjecture [117, 118, 119] has passed an enormous amount of tests (see [120] for a review). Most of them explored the large N duality between the boundary-CFT and the supergravity on AdS_5. The common radius of the S^5 and the length-scale of the AdS_5, R, is given by $R^4 = (\alpha')^2\lambda$, with $\lambda = g_{YM}^2 N$ representing the 't Hooft coupling. In general, supergravity as the low-energy limit of string-theory is trustworthy only at large scales, i.e. small curvatures. Thus tests of the duality probing the supergravity regime explore the $\lambda \to \infty$ parameter region. This predicts how the CFT at large N behaves in the extreme non-perturbative regime.

The interesting parameter regime, interpolating between the perturbative $\lambda \to 0$ and the extreme non-perturbative $\lambda \to \infty$ regime, demands that we keep λ finite. Since the closed string coupling constant g_s and the Yang-Mills coupling constant are related via $g_{YM}^2 = 4\pi g_s$, the large N limit with λ finite, requires $g_s \to 0$. Thus via the AdS-CFT conjecture, we are able to extract the full quantum information about the CFT in the large N limit by calculating simply IIB string tree-diagrams. Unfortunately, this wonderful perspective is obstructed by the fact, that IIB string-theory in the RNS-formulation on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with N units of RR 5-form flux through the S^5 is still obscure (see [121] for an approach). On the other hand there are proposals for a GS-formulation [122] which is non-linear and therefore its quantization and computation of string scattering amplitudes seems to be difficult.

Since string-theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background is not readily available, one may seek resort to a dual description of IIB string-theory on a, hopefully, easier background. This is another motivation to explore, whether pure AdS_d space can be dualized to flat Minkowski space. In [123] the observation has been made, that the only way to "flatten" negative curvature under T-duality is by introducing an appropriate torsion, generated by $B_{\mu\nu}$, in the initial space-time. This can be seen from the following formula[42], relating the dual curvature scalar $\tilde{R}$ to the initial curvature scalar [123]

$$\tilde{R} = R + 4 \triangle \ln k + \frac{1}{k^2} H_{\iota\alpha\beta} H^{\iota\alpha\beta} - \frac{k^2}{4} F_{\alpha\beta} F^{\alpha\beta} \ . \tag{6.1}$$

Here $A_\alpha = k_\alpha / k^2$, $k_\alpha = g_{\iota\alpha}$ with the associated field-strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Furthermore, $\triangle \ln k = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \ln k \right)$ denotes the d'Alembertian with respect to the initial metric. As usual, the torsion 3-form is the field strength[43] $H = dB$ of the NSNS 2-form $B_{\mu\nu}$. The Killing vector, corresponding to the assumed translational isometry exhibited by the initial space-time, is given by k_μ and its norm defined by $k = \sqrt{g_{\mu\nu} k^\mu k^\nu}$. One could now try to solve (6.1) for $R = -\Lambda$ and $\tilde{R} = 0$, with Λ a positive constant. But this would include any solution of the vacuum Einstein equations. To decide whether we arrive at flat space after T-dualization, we have to regard the Riemann curvature tensor.

6.1 T-Duality on Non-Trivial Spacetimes

Let us start quite generally by assuming some coordinate representation of a d-dimensional manifold, given by[44] $x^\mu = (x^0, x^1, \ldots, x^{d-2}, x^{d-1}) = (x^\alpha, x^\iota)$; $\alpha = 0, \ldots, d-2$. Furthermore, the initial metric is supposed to be of the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + g_{\iota\iota} dx^\iota dx^\iota \ . \tag{6.2}$$

T-duality on non-trivial spacetimes presupposes at least one spacelike isometry, whose direction we denote by x^ι. Then the T-duality transformation on the σ-model background with $g_{\iota\alpha} = 0$ is given by the Buscher rules [125]

$$\tilde{g}_{\iota\iota} = \frac{1}{g_{\iota\iota}} \ , \qquad \tilde{g}_{\iota\alpha} = \frac{B_{\iota\alpha}}{g_{\iota\iota}} \ , \qquad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{(g_{\iota\alpha} g_{\iota\beta} - B_{\iota\alpha} B_{\iota\beta})}{g_{\iota\iota}} \tag{6.3}$$

$$\tilde{B}_{\iota\alpha} = 0 \ , \qquad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} \tag{6.4}$$

$$\tilde{\phi} = \phi - \frac{1}{2} \ln g_{\iota\iota} \ . \tag{6.5}$$

[42]The index ι represents the isometry direction, whereas $\alpha, \beta, \ldots$ label the remaining directions.

[43]Our conventions are:

$$B = \frac{1}{2!} B_{\mu\nu} dx^\mu \wedge dx^\nu \ , \qquad H = \frac{1}{3} H_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \ .$$

[44]Indices λ, μ, ν, π run from $0, \ldots, d-1$, whereas indices $\alpha, \beta, \gamma, \delta, \epsilon$ run over $0, \ldots, d-2$.

Note that the shift in the dilaton is a quantum mechanical effect [?]. The reason why we set $g_{\iota\alpha} = 0$, from the outset, is the following. According to (6.1), a non-vanishing $g_{\iota\alpha} = k_\alpha$ tends to make the curvature scalar of the dual metric more negative and moreover by a suitable coordinate transformation we can always eliminate mixed components of the symmetric metric tensor.

We assume that the metric $g_{\mu\nu}(x^\alpha)$ and the NSNS 2-form $B_{\mu\nu}(x^\alpha)$ do not depend on x^ι. These conditions are exactly those, which are required in order to leave the σ-model action

$$S = \frac{1}{2\pi}\int d^2z\,(g_{\mu\nu} + B_{\mu\nu})\,\partial X^\mu \bar{\partial} X^\nu$$

invariant under infinitesimal shifts $x^\mu \to x^\mu + \epsilon k^\mu$. The Killing vector associated with the resulting abelian translational isometry is $k^\mu\partial_\mu = \partial/\partial x^\iota$, where $k^\mu = (k^\alpha, k^\iota) = (0, \ldots, 1)$. With the resulting norm $k = \sqrt{g_{\iota\iota}}$ of the Killing vector, a more convenient expression for the metric (6.2) in view of our later application, is given in terms of an anholonomic vielbein co-base by

$$ds^2 = \eta_{ab}e^a e^b + e^i e^i = \eta_{mn}e^m e^n \ , \tag{6.6}$$

where the chosen anholonomic vielbein $e^l = (e^a, e^i)$ reads

$$e^a = e^a{}_\alpha(x^\beta)dx^\alpha \ , \qquad\qquad e^i = e^i{}_\iota dx^\iota = \delta^i_\iota k(x^\alpha)dx^\iota \ , \tag{6.7}$$

such that its components $e^l{}_\lambda$ are

$$e^a{}_\alpha = e^a{}_\alpha(x^\beta) \ , \qquad\qquad e^i{}_\alpha = 0 \tag{6.8}$$

$$e^a{}_\iota = 0 \ , \qquad\qquad e^i{}_\iota = \delta^i_\iota k \ . \tag{6.9}$$

Later, we will also need the inverted vielbein of the isometry direction $e_i{}^\iota = \delta^\iota_i/k$.

The above mentioned σ-model action is invariant [126] to first order in ϵ under the translational isometry $\delta_\epsilon x^\mu = \epsilon k^\mu$, if k^μ satisfies the Killing equation $(\mathcal{L}_k g)_{\mu\nu} = k^\lambda g_{\mu\nu,\lambda} + k^\lambda{}_{,\mu} g_{\lambda\nu} + k^\lambda{}_{,\nu} g_{\lambda\mu} = \partial g_{\mu\nu}/\partial x^\iota = 0$ and the torsion obeys $\mathcal{L}_k H = 0$. This implies $\mathcal{L}_k B = dw$ for some 1-form w. Here $\mathcal{L}_k$ and d denote the space-time Lie- and exterior derivative. Locally, this is solved by $w = i_k B - v$ with $dv = -i_k H$ for some 1-form v. Under i_k we understand the interior product. From now on, we will choose the gauge $w = 0$, which leads us, together with the identity $(i_k B)_\mu = k^\nu B_{\nu\mu} = B_{\iota\mu}$, to

$$v_\alpha = B_{\iota\alpha} \ .$$

The T-dual metric $\tilde{g}_{\mu\nu}$ can now be neatly written as (with $k^\alpha = 0$)

$$\tilde{g}_{\iota\iota} = \frac{1}{k^2} \; , \qquad \tilde{g}_{\alpha\iota} = \frac{v_\alpha}{k^2} \; , \qquad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{v_\alpha v_\beta}{k^2} \; .$$

Thus the dual line-element reads

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta + 2\tilde{g}_{\alpha\iota} dx^\alpha dx^\iota + \tilde{g}_{\iota\iota} dx^\iota dx^\iota = \eta_{ab} \tilde{e}^a \tilde{e}^b + \tilde{e}^i \tilde{e}^i = \eta_{mn} \tilde{e}^m \tilde{e}^n \; .$$

Here the T-dual anholonomic vielbein co-base $\tilde{e}^l = (\tilde{e}^a, \tilde{e}^i)$ is given by

$$\tilde{e}^a = e^a = e^a{}_\alpha dx^\alpha \; , \qquad \tilde{e}^i = \tilde{e}^i{}_\mu dx^\mu = \frac{\delta^i_\iota}{k} dx^\iota + \frac{v_\alpha}{k} dx^\alpha$$

from which we can read off the dual vielbein components $\tilde{e}^l{}_\lambda$

$$\tilde{e}^a{}_\alpha = e^a{}_\alpha \; , \qquad \tilde{e}^i{}_\alpha = \frac{v_\alpha}{k} \tag{6.10}$$

$$\tilde{e}^a{}_\iota = 0 \; , \qquad \tilde{e}^i{}_\iota = \frac{\delta^i_\iota}{k} \; . \tag{6.11}$$

and their inverses $\tilde{e}_l{}^\lambda$

$$\tilde{e}_a{}^\alpha = e_a{}^\alpha \; , \qquad \tilde{e}_i{}^\alpha = 0 \tag{6.12}$$

$$\tilde{e}_a{}^\iota = -e_a{}^\alpha v_\alpha \; , \qquad \tilde{e}_i{}^\iota = \delta^\iota_i k \; . \tag{6.13}$$

A useful formula consists of the inverted relation $\delta^i_\iota dx^\iota = k\tilde{e}^i - e_a{}^\alpha v_\alpha e^a$. From $\tilde{g}^{\mu\nu} = \tilde{e}_m{}^\mu \tilde{e}_n{}^\nu \eta^{mn}$ we derive the inverse metric to be

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} \; , \qquad \tilde{g}^{\alpha\iota} = -g^{\alpha\beta} v_\beta \; , \qquad \tilde{g}^{\iota\iota} = g^{\alpha\beta} v_\alpha v_\beta + k^2 \; .$$

6.2 The T-dual Riemann-Tensor

The strategy of the calculation of the dual Riemann-tensor will now be as follows. With the aid of Cartan's structure equations, we determine the curvature tensor and its T-dual in the $e^l = \{e^a, e^i\}$, resp. T-dual $\tilde{e}^l = \{\tilde{e}^a, \tilde{e}^i\}$ co-base. In order to compare both of them, it is further necessary to switch to the equivalent expressions in the common holonomic co-base $dx^\mu = (dx^\alpha, dx^\iota)$ with the help of the aforementioned vielbeins.

Therefore, let us begin with (6.6) and avail ourselves of Cartan's first structure equation for the torsion-less case, $de^m + \omega^m{}_n \wedge e^n = 0$, to determine the connection 1-form ω

$$\omega^a{}_b = -\partial_{[\gamma} e^a{}_{\alpha]} e_{[c}{}^\gamma e_{b]}{}^\alpha e^c \; , \qquad \omega^i{}_a = e_a{}^\alpha \partial_\alpha \ln k \cdot e^i \; , \qquad \omega^i{}_i = 0 \; ,$$

Cartan's second structure equation, $d\omega^m{}_n + \omega^m{}_l \wedge \omega^l{}_n = R^m{}_n$, together with the expression for the curvature 2-form in the initial anholonomic co-base, $R^l{}_m = \frac{1}{2} R^l{}_{mnp} e^n \wedge e^p = \frac{1}{2} R^l{}_{mab} e^a \wedge e^b + R^l{}_{mia} e^i \wedge e^a$, allow us to extract after some algebra the initial curvature tensor[45] as

$$
\begin{aligned}
R_{abcd} &= e_{[c}{}^{\gamma} e_{d]}{}^{\delta} \left(e_b{}^{\beta} \partial_\beta \partial_\gamma e_{a\delta} - \partial_{[\delta} e_{|a|\beta]} \cdot \partial_\gamma e_b{}^{\beta} \right) + e_b{}^{\beta} e_{[c}{}^{\gamma} \partial_{|\beta|} e_{d]}{}^{\alpha} \cdot \partial_{[\gamma} e_{|a|\alpha]} \\
R_{iabc} &= R_{abic} = 0 \\
R_{iaib} &= -e_{(a}{}^{\alpha} \partial_{|\alpha|} e_{b)}{}^{\beta} \cdot \partial_\beta \ln k - e_a{}^{\alpha} e_b{}^{\beta} \left(\partial_\alpha \partial_\beta \ln k + \partial_\alpha \ln k \cdot \partial_\beta \ln k \right) \ ,
\end{aligned}
$$

whereas all other components are zero. By multiplying with the vielbein $R_{\lambda\mu\nu\pi} = e^l{}_\lambda e^m{}_\mu e^n{}_\nu e^p{}_\pi R_{lmnp}$, we subsequently arrive at the coordinate base expressions

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= \delta^{\varepsilon}_{[\gamma} e^b{}_{\delta]} \left(e_{a\alpha} \partial_\beta e_b{}^{\zeta} \cdot \partial_{[\varepsilon} e^a{}_{\zeta]} + \frac{1}{2} e_a{}^{\zeta} \partial_\zeta e_{b\alpha} \cdot \partial_\varepsilon e^a{}_\beta \right) + e^a{}_\alpha \partial_\beta \partial_{[\gamma} e_{|a|\delta]} + \frac{1}{2} \partial_{[\gamma} e^a{}_{|\alpha|} \cdot \partial_{\delta]} e_{a\beta} \\
R_{\iota\alpha\beta\gamma} &= R_{\alpha\beta\iota\gamma} = 0 \\
R_{\iota\alpha\iota\beta} &= -k^2 \left(e^a{}_{(\alpha} \partial_{\beta)} e_a{}^{\varepsilon} \cdot \partial_\varepsilon \ln k + \partial_\alpha \partial_\beta \ln k + \partial_\alpha \ln k \cdot \partial_\beta \ln k \right) \ ,
\end{aligned}
$$

and all other components zero.

Analogously, application of $d\tilde{e}^l + \tilde{\omega}^l{}_m \wedge \tilde{e}^m$ yields

$$
\tilde{\omega}^a{}_b = \omega^a{}_b \ , \qquad \tilde{\omega}^i{}_a = -e_b{}^{\beta} e_a{}^{\alpha} \frac{dv_{\beta\alpha}}{k} \ e^b - e_a{}^{\alpha} \partial_\alpha \ln k \ \tilde{e}^i \ , \qquad \tilde{\omega}^i{}_i = 0 \ ,
$$

where $dv_{\alpha\beta} \equiv \partial_{[\alpha} v_{\beta]}$ denotes the exterior derivative of v. From the relation $dv = -i_k H$, it is possible to substitute $dv_{\alpha\beta}$ in the subsequent formulae by the torsion 3-form through $dv_{\alpha\beta} = -H_{\iota\alpha\beta}$. Again, from Cartan's second structure equation, $d\tilde{\omega}^m{}_n + \tilde{\omega}^m{}_l \wedge \tilde{\omega}^l{}_n = \tilde{R}^m{}_n$, and the dual curvature 2-form $\tilde{R}^l{}_m = \frac{1}{2} \tilde{R}^l{}_{mnp} \tilde{e}^n \wedge \tilde{e}^p = \frac{1}{2} \tilde{R}^l{}_{mab} e^a \wedge e^b + \tilde{R}^l{}_{mia} \tilde{e}^i \wedge e^a$, we get the following non-coordinate frame expressions

$$
\begin{aligned}
\tilde{R}_{abcd} &= R_{abcd} - \frac{2}{k^2} e_a{}^{\alpha} e_b{}^{\beta} e_{[c}{}^{\gamma} e_{d]}{}^{\delta} dv_{\alpha\gamma} dv_{\beta\delta} \\
\tilde{R}_{iabc} &= \tilde{R}_{bcia} = -\frac{2}{k} e_a{}^{\alpha} e_{[b}{}^{\beta} e_{c]}{}^{\gamma} dv_{\beta\alpha} \cdot \partial_\gamma \ln k \\
\tilde{R}_{iaib} &= e_{(a}{}^{\alpha} \partial_{|\alpha|} e_{b)}{}^{\beta} \cdot \partial_\beta \ln k + e_a{}^{\alpha} e_b{}^{\beta} \left(\partial_\alpha \partial_\beta \ln k - \partial_\alpha \ln k \cdot \partial_\beta \ln k \right) \ .
\end{aligned}
$$

[45] We use the following anti- and symmetrization convention:

$$
A_{(ab)} = \frac{1}{2} \left(A_{ab} + A_{ba} \right) \qquad A_{[ab]} = \frac{1}{2} \left(A_{ab} - A_{ba} \right)
$$

Switching to the coordinate base with the help of $\tilde{R}_{\lambda\mu\nu\pi} = \tilde{e}^l{}_\lambda \tilde{e}^m{}_\mu \tilde{e}^n{}_\nu \tilde{e}^p{}_\pi \tilde{R}_{lmnp}$, one finally arrives, after some tedious algebra, at the desired expressions, which relate the T-dual curvature tensor to its counterpart for the initial metric

$$\begin{aligned}\tilde{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{2}{k^2}\Big(& dv_{\alpha[\delta} dv_{|\beta|\gamma]} - dv_{\alpha\gamma} \cdot v_{[\beta}\partial_{\delta]} \ln k + dv_{\beta\gamma} \cdot v_{[\alpha}\partial_{\delta]} \ln k + dv_{\alpha\delta} \cdot v_{[\beta}\partial_{\gamma]} \ln k \\ & - dv_{\beta\delta} \cdot v_{[\alpha}\partial_{\gamma]} \ln k + v_{[\alpha} e^b{}_{\beta]} v_{[\gamma}\partial_{\delta]} e_b{}^\varepsilon \cdot \partial_\varepsilon \ln k + v_{[\gamma} e^b{}_{\delta]} v_{[\alpha}\partial_{\beta]} e_b{}^\varepsilon \cdot \partial_\varepsilon \ln k \\ & + v_\gamma v_{[\alpha}\partial_{\beta]}\partial_\delta \ln k - v_\delta v_{[\alpha}\partial_{\beta]}\partial_\gamma \ln k - 2 v_{[\alpha}\partial_{\beta]} \ln k \cdot v_{[\gamma}\partial_{\delta]} \ln k \Big) \end{aligned} \tag{6.14}$$

$$\begin{aligned}\tilde{R}_{\iota\alpha\beta\gamma} = \frac{2}{k^2}\Big(& dv_{\alpha[\beta} \cdot \partial_{\gamma]} \ln k + \frac{1}{2}\big(e^a{}_\alpha v_{[\beta}\partial_{\gamma]} e_a{}^\delta + v_{[\beta} e^a{}_{\gamma]} \partial_\alpha e_a{}^\delta\big) \cdot \partial_\delta \ln k + v_{[\beta}\partial_{\gamma]}\partial_\alpha \ln k \\ & - v_{[\beta}\partial_{\gamma]} \ln k \cdot \partial_\alpha \ln k \Big) \end{aligned} \tag{6.15}$$

$$\tilde{R}_{\iota\alpha\iota\beta} = -\frac{1}{k^2}\left(\frac{R_{\iota\alpha\iota\beta}}{k^2} + 2\partial_\alpha \ln k \cdot \partial_\beta \ln k\right) . \tag{6.16}$$

6.3 Connecting AdS to a Flat Background ?

As an application of these general formulas – describing the behaviour of the curvature tensor under T-duality – we now want to examine, whether d-dimensional AdS_d space allows a T-dualization to flat space under the inclusion of some suitably chosen $B_{\mu\nu}$. Both spaces, regarded as string backgrounds, leave all supersymmetries intact. We choose the following coordinate representation for AdS_d

$$ds^2 = \frac{r^2}{L^2}\left(- dt^2 + \sum_{j=1}^{d-2} dy^j dy^j \right) + L^2 \frac{dr^2}{r^2} , \tag{6.17}$$

with L being the AdS-radius. It exhibits a negative, constant curvature scalar $R = -d(d-1)/L^2$. For the isometry direction we choose one of the y^j coordinates, e.g. y^{d-2}. In our above convention, the adapted coordinates are $x^0 = t$, $x^1 = y^1, \ldots, x^{d-3} = y^{d-3}$, $x^{d-2} = r$, $x^{d-1} \equiv x^\iota = y^{d-2}$. The curvature tensor of the negatively curved, maximally symmetric AdS_d is given by $R_{\lambda\mu\nu\pi} = -\frac{1}{L^2}\left(g_{\lambda\nu} g_{\mu\pi} - g_{\lambda\pi} g_{\mu\nu}\right)$, which reduces in our coordinates to[46]

$$R_{\mu\nu\mu\nu} = -\frac{r^4}{L^6}\eta_{\mu\mu}\eta_{\nu\nu} , \qquad R_{\mu r\mu r} = -\frac{1}{L^2}\eta_{\mu\mu} , \qquad \mu, \nu \neq r$$

and all other components vanishing. In particular, this gives $R_{\iota\alpha\iota\beta} = -\frac{r^4}{L^6}\eta_{\alpha\beta} + \left(\frac{r^4}{L^6} - \frac{1}{L^2}\right)\delta^r_\alpha \delta^r_\beta$. It is important that there is a single equation, (6.16), for the dual curvature-tensor, which is independent of v, resp. $B_{\mu\nu}$ and hence allows answering the above

[46] We choose the Minkowski metric $\eta = (-, +, \ldots, +)$.

posed question. Plugging in the actual norm of the Killing vector $k = r/L$ for the above AdS$_d$ metric, we obtain for the right-hand side of (6.16) the expression $\frac{1}{L^2}\eta_{\alpha\beta} - \big(\frac{1}{L^2} + \frac{L^2}{r^4}\big)\delta^r_\alpha\delta^r_\beta$. Obviously, this does not fulfill the requirement $\tilde{R}_{\iota\alpha\iota\beta} = 0$ for flat space and is not even asymptotically flat. The tensor transformation property $\tilde{R}'_{\iota'\alpha'\iota'\beta'}(x') = \big(A^\iota_{\iota'}\big)^2 A^\alpha_{\alpha'} A^\beta_{\beta'} \tilde{R}_{\iota\alpha\iota\beta}(x)$, with non-singular $A^\mu_\nu = \partial x^\mu / \partial x'^\nu$, then guarantees that this term is also non-vanishing for any other than the above chosen coordinate parameterization of AdS$_d$. In particular for any further isometry direction this dual component is non-zero, whereas for flat space it keeps being zero for any parameterization. Thus, we conclude that flat space cannot be reached from pure AdS$_d$ via T-duality plus the inclusion of some appropriately chosen $B_{\mu\nu}$. The general formulae (6.14)-(6.16) however serve as a starting point for more involved T-dualities, including a non-trivial $B_{\mu\nu}$, which relate AdS$_d$ or the more important AdS$_5 \times$ S^5 to spaces with different asymptotics.

Eventually, we turn briefly to the important AdS$_5 \times$ S^5 extension of AdS. Here it seems possible to relate the IIB D9-brane geometry, which describes D=10 space itself, to the AdS$_5 \times$ S^5 geometry. In order to do so, one has to start with a six-fold T-duality, transforming the D9-brane to the D3-brane. Then, dualizing a IIB D3-brane to its own near-horizon geometry, one reaches AdS$_5 \times S^5$. This can be done following the work of [127]. Having T-dualized the D9-brane, the D3-brane geometry in string-frame reads

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H_6}}\Big(- dt^2 + \sum_{i=1}^{3}(dx^i)^2\Big) + \sqrt{H_6}\sum_{j=4}^{9}(dx^j)^2 \ , \\ e^\phi &= 1 \ , \qquad\qquad C_{0123} = \frac{1}{H_6} - 1 \ , \\ H_6 &= 1 + \frac{Q_{D3}}{(r_6)^4} \ , \qquad\qquad Q_{D3} = 4\pi g_s N l_s^4 \ . \end{aligned}$$

Performing a further T-duality over x^3, x^2 brings us to the IIB D1-brane. An S-duality transformation then yields the fundamental IIB string solution

$$ds^2 = \frac{1}{H_8}\Big(- dt^2 + (dx^1)^2\Big) + \sum_{j=2}^{9}(dx^j)^2 \ , \tag{6.18}$$

$$e^\phi = \frac{1}{\sqrt{H_8}} \ , \qquad\qquad B_{01} = \frac{1}{H_8} - 1 \ , \tag{6.19}$$

$$H_8 = 1 + \frac{Q_{F1}}{(r_8)^6} \ , \qquad\qquad Q_{F1} = d_1 g_s^2 N l_s^6 \ , \tag{6.20}$$

which is then T-dualized over x_1 to obtain the IIA gravitational wave solution

$$ds^2 = (H_8 - 2)\,dt^2 + 2\,(H_8 - 1)\,dtdx^1 + H_8 dx^1 dx^1 + \sum_{j=2}^{9}(dx^j)^2\ , \tag{6.21}$$

$$e^\phi = 1\ , \qquad\qquad B_{\mu\nu} = 0\ . \tag{6.22}$$

The crucial step to get rid of the constant in the harmonic H_8 function, consists of performing first an $SL(2,\mathbb{R})$ coordinate transformation on the coordinates t, x^1 and subsequently a T-duality transformation in the new x'^1 direction. Provided that we choose the special $SL(2,\mathbb{R})$ coordinate transformation

$$t' = \frac{1}{2}\left(t + x^1\right)\ , \qquad x'^1 = 2x^1\ , \tag{6.23}$$

we are guaranteed that the transformation is globally well-defined on (t, x^1) space, which has the toplogy of a cylinder due to the compactification of x^1. After T-duality in x'^1 direction the result is a modified fundamental string without constant part in the harmonic function

$$ds^2 = \frac{1}{\mathcal{H}_8}\big(-(dt')^2 + (dx'^1)^2\big) + \sum_{j=2}^{9}(dx^j)^2\ , \tag{6.24}$$

$$e^\phi = \frac{1}{\sqrt{\mathcal{H}_8}}\ , \qquad\qquad B_{01} = \frac{1}{\mathcal{H}_8} - 2\ , \tag{6.25}$$

$$\mathcal{H}_8 = \frac{Q_{F1}}{(r_8)^6} \tag{6.26}$$

It is important to note that the last T-duality has again been along a space-like direction. Therefore we are secured to stay in the Type II string-theory framework instead of changing to Type II* [128].

Now, proceeding in the inverse manner, a second S-duality promotes us to a modified D1-brane. Ultimately, with two T-dualities in the x^2, x^3 directions we end up with a modified D3-brane solution without a constant part in its harmonic function. As is well-known, this gives the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. Notice, however, that the D3-brane solution is only locally dual to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution. In order to perform the various T-dualities, the brane has to be wrapped on a torus with all worldvolume coordinates taken to be periodic. Therefore the coordinates of the final $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry in addition have to be identified globally. As mentioned in [127] this has the effect that only half the Killing spinors of AdS_5 remain. Thus, only locally we have a maximally supersymmetric $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution. Globally, the D3-brane as well as the dual solution break half the supersymmetry.

A word of caution is in order. The spacetime-filling D9-brane is supposed to describe flat ten-dimensional Minkowski spacetime [129]. However, the D9-brane breaks half the supersymmetry, whereas flat space does not. The resolution to this puzzle comes from the well-known fact, that the open-string sector of N D9-branes is only consistent, if we have $N = 32$ of them and perform an orientifolding by the world-sheet parity $\Omega : \ \sigma \to \pi - \sigma$. This breaks half the supersymmetry and the resulting Type I SO(32) theory gives us in addition to the flat ten-dimensional Minkowski solution a spacetime-filling O9 orientifold fixed plane.

7 Summary and Conclusions

The main focus of the first part of this work has been on heterotic M-theory – its dynamics in eleven dimensions and its warped geometries which arise upon compactification to four dimensions. The second part focused on a two domain-wall configuration whose warped geometry could be exploited for a low-energy mechanism to obtain a small cosmological constant within a 5-dimensional context which then was embedded into type IIB-/F-theory.

We started with an exploration of the stability of a flat heterotic M-theory background. Not to rely on duality conjectures to other string-theories, which would involve an ideally still undefined quantum heterotic M-theory, we had to rely on the Hořava-Witten supergravity action, which describes the leading part of heterotic M-theory in a long-wavelength expansion to first order in $\kappa^{2/3}$. Using that action, we calculated all relevant amplitudes with external gauge-bosons and gauginos – mediated by the graviton, the gravitino and the M-theory 3-form C – which are responsible for an interaction between the two boundaries. The result was, that the sum of all these contributions did not add up to zero, thereby rendering the examined flat background with vanishing G-flux, unstable. Recently Green et al. [74] expanded our result by adding a further bulk-boundary interaction term. Thereby they achieved agreement in the limit of coinciding boundaries between heterotic string theory and heterotic M-theory concerning the low-energy s-channel amplitudes with internal bulk supergravity fields. This motivated us to assume that by the addition of further interaction terms the full conjectured duality might be verifiable, such that the M-theory amplitudes will reproduce the amplitudes of heterotic string-theory if the distance d between both boundaries goes to zero. We then checked that the sum of the heterotic amplitudes, which by taking the limit $d \to 0$ have to originate from the M-theory s-channel diagrams, does not add up to zero. Therefore, if the conjectured duality holds true, heterotic M-theory can only be stable, i.e. its amplitudes sum up to zero, if the limit $d \to 0$ is not smooth. However, this is not likely since at least for the amplitudes with external gauge-bosons only, the work of [74] has shown that there is a smooth transition. Therefore, we concluded that by assuming the duality to be true a flat vacuum with vanishing G-flux will not be stable but tends to shrink the eleventh dimension. A way out would be to consider a warped background, which is the only non-trivial metric background allowed by maintenance of 10-dimensional Poincaré-invariance. However, it is unclear what generates the warping since a non-vanishing G-flux would violate Poincaré-invariance.

Next, we considered the case of heterotic M-theory compactified on a Calabi-Yau threefold CY_3. Since, now we have a vacuum with 4-dimensional Poincaré-invariance only, we are allowed to turn on internal G-flux. Demanding $\mathcal{N} = 1$ supersymmetry in four dimensions indeed enforces turning on a non-vanishing G-flux and leads to a unique determination of the resulting warped-geometry in terms of the flux. In [63] Witten had analyzed the resulting geometry for a CY_3 compactification to first order in $\kappa^{2/3}$ with the amazing result that the volume of the CY_3 depends linearly on the orbifold direction and its zero leads just to the right value for the 4-dimensional Newton constant G_N. Because of the crucial importance of this zero for phenomenology we determined the full dependence of the warped geometry on the G-flux, i.e. without making a linear approximation. It turned out that the full functional behaviour of the volume, which requires the knowledge of the Calabi-Yau warp-factor, can be inferred if one makes the assumption that the magnetic boundary and M5-brane sources of the full heterotic M-theory can still be localized in the orbifold direction. The outcome has been a quadratic dependence with a coefficient containing in principle the full heterotic M-theory to all orders. An expansion of this coefficient to first order reproduced the linear result of [63]. Generically, we still got a zero (whose precise value however requires knowledge about heterotic M-theory to all orders in $\kappa^{2/3}$) but this zero coincides with the minimum of a parabola. The nice fact about this is that now the zero volume location can be neatly resolved by quantum corrections to some finite value, whereas in the linear approximation quantum corrections would only shift the position of the zero volume location. As a non-trivial check on our warp-factor – flux relations we reproduced the analogous relation for the weakly coupled heterotic string with torsion. Then, we analyzed the relation between Newton's Constant and the distance d between both boundaries and showed how a further bulk M5-brane source changes the Calabi-Yau volume behaviour. With the knowledge of the full warped background geometry we derived an effective potential for the distance-modulus d. Its shape indicated again a destabilization of the set-up. Next we considered the most general case with all G-flux components, allowed by supersymmetry, turned on. This is a prerequisite, e.g. if one wants to consider effects like gaugino condenstaion. A similar formula for the Calabi-Yau volume dependence on the orbifold-direction was derived. Furthermore, we found a relation between the internal spin-connection and the G-flux. This determined the internal holonomy group of the deformed compactification manifold by the G-flux and once more showed an intriguing relationship between geometry and physics.

The next chapter was devoted to a low-energy mechanism (without the need for supersymmetry) which exploited a warped geometry to arrive at a realistically small value for the cosmological constant Λ_4. Separating two positive-tension domain-walls in five dimensions by a distance $2l$ showed an exponential warp-factor in the resulting geometry. Upon determining the effective 4-dimensional action we saw that this warp-factor translates into an exponential suppression of an initially Planck-valued cosmological constant. By lifting the finetuning of parameters and deriving yhe corresponding backreaction on the warped geometry we could show that this feature persisted and thus required no finetuning. To obtain the observed value for Λ_4 needed a length $2l \simeq 1/M_{\mathrm{GUT}}$. An embedding of the 5-dimensional configuration into a 10-dimensional IIB-/F-theory framework interpreted the domain-walls as stacks of D3-branes with attached open strings. The requirement to reproduce the Standard Model gauge group led to the conclusion that the SU(3) had to originate from one stack, whereas the SU(2)×U(1) had to be located on the second stack. An examination of the open string spectrum showed that it described an SU(6) GUT with gauge-group spontaneously broken down to the Standard Model gauge group through the separation of the two D3-brane stacks. Composing all open string states out of two sorts of "basic" open strings together with a consistency condition concerning the orientation of the open strings leads to a very simple origin of the U(1)-hypercharge – it is a consequence of the number of D3-branes in the SU(3) resp. SU(2) stack. In this picture the triplet Higgs-bosons naturally acquire mass at the GUT-scale – since they stem from open strings stretching from one brane stack to the other – whereas the doublet Higgs-bosons are naturally light because their origin is from open strings starting and ending on the same brane-stack. It was then pointed out that the extra six internal dimensions naturally lead to three (degenerate) generations of fermions. By the inclusion of three further D7-branes it is possible to lift this degeneracy and to obtain a three-generation model already at the level of dimensional reduction without the need to invoke a complicated internal manifold with a special value for its Euler-characteristic.

The final chapter explored the impact which T-duality has on curvature resp. the cosmological constant. We derived expressions for the behaviour of the Riemann curvature tensor under T-duality and applied these to see whether an AdS-spacetime (with negative cosmological constant) can be T-dualized to a flat spacetime without cosmological constant. The answer was negative. However with the derived formulae a more general research involving also spacetimes which are only asymptotically AdS could be explored.

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A Amplitudes of Heterotic M-Theory

A.1 Notation

Tensors:

$$A_{(M_1 \ldots M_n)} = \frac{1}{n!}\left(A_{M_1 \ldots M_n} \pm (n!-1) \text{ symmetric permutations}\right)$$
$$A_{[M_1 \ldots M_n]} = \frac{1}{n!}\left(A_{M_1 \ldots M_n} \pm (n!-1) \text{ antisymmetric permutations}\right)$$

A.2 Mandelstam-Variables and Kinematic for the CMS

Center-of-Mass variables:

$$\text{scattering angle: } 0 \leq \vartheta \leq \pi \qquad \text{CMS-Energy: } E \tag{A.1}$$

Without loss of generality we can arrange the scattering such that the two incoming fields with momenta p_1, p_2 collide head-on in the CMS-system in the direction of the 9^{th} coordinate axis and the outgoing fields move on the plane spanned by the 8^{th} and 9^{th} coordinate axes. We choose ϑ to be the angle between p_1 and p_3. Concretely we take for the D=10 momenta

$$p_1 = \left(\frac{E}{2}, 0, ..., 0, \frac{E}{2}\right) \ , \quad p_2 = \left(\frac{E}{2}, 0, ..., 0, -\frac{E}{2}\right) \ ,$$
$$p_3 = \left(\frac{E}{2}, 0, ..., 0, \frac{E}{2}\sin\vartheta, \frac{E}{2}\cos\vartheta\right) \ , \quad p_4 = \left(\frac{E}{2}, 0, ..., 0, -\frac{E}{2}\sin\vartheta, -\frac{E}{2}\cos\vartheta\right) \ .$$

Mandelstam-Variables:

$$s = -\left(p_1 + p_2\right)^2 = -\left(p_3 + p_4\right)^2 = E^2$$
$$t = -\left(p_1 - p_3\right)^2 = -\left(p_2 - p_4\right)^2 = -\frac{s}{2}\left[1 - \cos\vartheta\right] = -s\sin^2\frac{\vartheta}{2} \leq 0$$
$$u = -\left(p_1 - p_4\right)^2 = -\left(p_2 - p_3\right)^2 = -\frac{s}{2}\left[1 + \cos\vartheta\right] = -s\cos^2\frac{\vartheta}{2} \leq 0$$
$$s + t + u = 0$$

A.3 D=10 Polarization Vectors

CMS-Polarization Vectors: Let the D=10 momentum in the CMS be

$$p = \frac{E}{2}\,(1, ..., \sin\vartheta, \cos\vartheta)\ . \tag{A.2}$$

Then we have 8 transverse polarizations, which are given in the CMS by the following real vectors

$$\epsilon(p,1) = (0,1,0,...,0) \tag{A.3}$$
$$\epsilon(p,2) = (0,0,1,...,0) \tag{A.4}$$
$$\vdots \tag{A.5}$$
$$\epsilon(p,7) = (0,0,0,...,1,0,0) \tag{A.6}$$
$$\epsilon(p,8) = (0,0,0,...,0,\cos\vartheta,-\sin\vartheta) \tag{A.7}$$

Useful Contractions: If we sum over all polarizations, we get

$$\sum_{\lambda=1}^{8} p_{1,A}\epsilon^A(p_2,\lambda) = \frac{E}{2}\sin(\vartheta_1 - \vartheta_2)\ . \tag{A.8}$$

Contractions of two polarization vectors are given by

$$\sum_{\lambda,\tilde{\lambda}=1}^{8} \epsilon_A(p,\lambda)\,\epsilon^A\left(\tilde{p},\tilde{\lambda}\right) = 7 + \cos\left(\vartheta - \tilde{\vartheta}\right)\ . \tag{A.9}$$

A.4 D=11 Gamma-Matrices and D=10 Dirac-Spinors

We take the D=11 $SO(1,10)$ spin 32×32 matrices to be in a real Majorana-representation

$$\Gamma^1 = -i\sigma_2 \otimes I_{16} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_8 & 0 \\ 0 & I_8 \end{pmatrix}$$

$$\Gamma^a = \sigma_1 \otimes \gamma^a_{16} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \gamma^a_8 \\ \gamma^{a,T}_8 & 0 \end{pmatrix} \ ; \quad a = 1, ..., 8$$

$$\Gamma^{10} = -\sigma_1 \otimes \gamma^9_{16} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix}$$

$$\begin{aligned} \Gamma^{11} &\equiv \Gamma^1\Gamma^2 ... \Gamma^9\Gamma^{10} \\ &= \sigma_3 \otimes I_{16} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} I_8 & 0 \\ 0 & I_8 \end{pmatrix} = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{pmatrix} \end{aligned}$$

where we use $\epsilon_{1...10} = 1 = -\epsilon^{1...10}$. The Dirac-matrices satisfy

$$\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN} = 2(-, +, ..., +) \ ,$$

while the real 16×16 γ^a_{16} submatrices obey the relations

$$\left\{\gamma^a_{16}, \gamma^b_{16}\right\} = 2\delta^{ab} \ , \qquad \gamma^{a,T}_{16} = \gamma^a_{16} \ , \qquad \left(\gamma^a_{16}\right)^2 = I_{16}$$

$$\gamma^9_{16} \equiv \gamma^1_{16} ... \gamma^8_{16} = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix} \ , \qquad \gamma^{9,T}_{16} = \gamma^9_{16} \ , \qquad \left(\gamma^9_{16}\right)^2 = I_{16} \ .$$

Finally the 8×8 submatrices γ^a are defined as

$$\begin{aligned} \gamma^1 &= i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \ , \quad \gamma^2 = I_2 \otimes \sigma_1 \otimes i\sigma_2 \\ \gamma^3 &= I_2 \otimes \sigma_3 \otimes i\sigma_2 \ , \quad \gamma^4 = \sigma_1 \otimes i\sigma_2 \otimes I_2 \\ \gamma^5 &= \sigma_3 \otimes i\sigma_2 \otimes I_2 \ , \quad \gamma^6 = i\sigma_2 \otimes I_2 \otimes \sigma_1 \\ \gamma^7 &= i\sigma_2 \otimes I_2 \otimes \sigma_3 \ , \quad \gamma^8 = I_2 \otimes I_2 \otimes I_2 = I_8 \end{aligned}$$

and satisfy

$$\gamma^a\gamma^{b,T} + \gamma^b\gamma^{a,T} = 2\delta^{ab} \ , \quad \gamma^{i,T} = -\gamma^i \ ; i = 1, ..., 7 \ , \quad \gamma^{8,T} = -\gamma^8 \ .$$

D=10 Weyl-spinor: We have to deal with ten-dimensional Majorana-Weyl spinors for the gauginos with positive chirality only. For the special ten-dimensional momentum

$p_1 = \frac{E}{2}(1,0,...,0,1)$, we find from the Dirac-equation $\Gamma^A \partial_A \lambda(x) = 0$, the following spinor expression in momentum space

$$u_s(p_1) = \sqrt{N} \begin{pmatrix} 0 \\ e_s \\ 0 \\ 0 \end{pmatrix} ,$$

where $e_s; s = 1,...,8$ denotes the sth unit vector. In our calculation we actually need the slightly more general spinor corresponding to the ten-dimensional momentum $p = \frac{E}{2}(1,0,...,0,\sin\vartheta,\cos\vartheta)$. We can generate p from p_1 by a rotation in the 8-9 plane

$$p^A = R_\vartheta p_1^A = \begin{pmatrix} \ddots & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} p_1^0 \\ \vdots \\ p_1^8 \\ p_1^9 \end{pmatrix} .$$

The corresponding action on the spinor $u_s(p_1)$ is given by

$$\begin{aligned} u_s(p) = e^{\frac{\vartheta}{2}\Gamma^8\Gamma^9} u_s(p_1) &= \left(\cos\left(\frac{\vartheta}{2}\right) I_{32} + \sin\left(\frac{\vartheta}{2}\right) \Gamma^8\Gamma^9 \right) u_s(p_1) \\ &= \sqrt{N} \begin{pmatrix} \sin\left(\frac{\vartheta}{2}\right) e_s \\ \cos\left(\frac{\vartheta}{2}\right) e_s \\ 0 \\ 0 \end{pmatrix} . \end{aligned} \tag{A.10}$$

As a convenient normalization choice we choose

$$N \equiv E .$$

The charge conjugation matrix $C_{\alpha\beta}$ will be taken as

$$C_{\alpha\beta} = \left(\Gamma^1\right)^\alpha{}_\beta , \quad C^{\alpha\beta} = \left(\Gamma^{1,-1}\right)^\alpha{}_\beta = \left(\Gamma^{1,T}\right)^\alpha{}_\beta = -\left(\Gamma^1\right)^\alpha{}_\beta .$$

Symmetry properties of Bilinears:

For arbitrary momenta p and p' one obtains

$$\bar{u}_s(p)\Gamma^A u_{s'}(p') = \bar{u}_{s'}(p')\Gamma^A u_s(p) \tag{A.11}$$

$$\bar{u}_s(p)\Gamma^{ABC} u_{s'}(p') = -\bar{u}_{s'}(p')\Gamma^{ABC} u_s(p) \tag{A.12}$$

$$\bar{u}_s(p)\Gamma^{ABCDE} u_{s'}(p') = \bar{u}_{s'}(p')\Gamma^{ABCDE} u_s(p) . \tag{A.13}$$

A.5 Hyperplane Gauge Field Operators

Fourier-Decomposition of the Field Operators:

$$B^a_A(x) = \int \frac{d^9k}{(2\pi)^9\, 2k^0} \sum_{\lambda=1,\ldots,8} \epsilon_A(k,\lambda) \left[b^a_\lambda\,(k)\, e^{ikx} + b^{a,\dagger}_\lambda\,(k)\, e^{-ikx} \right] \tag{A.14}$$

$$\lambda^a(x) = \int \frac{d^9k}{(2\pi)^9\, 2k^0} \sum_{s=1,\ldots,8} u_s(k) \left[d^a_s(k) e^{ikx} + d^{a,\dagger}_s(k) e^{-ikx} \right] \tag{A.15}$$

$$\tag{A.16}$$

Anti-/Commutators:

$$\left[b^a_\lambda(k), b^{b,\dagger}_{\lambda'}(k') \right] = \delta^9(k-k')\delta_{\lambda\lambda'}\delta^{ab}(2\pi)^9 2k^0 \tag{A.17}$$

$$\left[b^a_\lambda(k), b^b_{\lambda'}(k') \right] = \left[b^{a,\dagger}_\lambda(k), b^{b,\dagger}_{\lambda'}(k') \right] = 0 \tag{A.18}$$

$$\left\{ d^a_s(k), d^{b,\dagger}_{s'}(k') \right\} = \delta^9(k-k')\delta_{ss'}\delta^{ab}(2\pi)^9 2k^0 \tag{A.19}$$

$$\left\{ d^a_s(k), d^b_{s'}(k') \right\} = \left\{ d^{a,\dagger}_s(k), b^{b,\dagger}_{s'}(k') \right\} = 0 \tag{A.20}$$

Wick-Contractions:

$$\underbrace{b^a_\lambda(p)\, B^b_A}\,(x) = \delta^{ab}\epsilon_A(p,\lambda)\, e^{-ipx} \tag{A.21}$$

$$\underbrace{B^a_A(x)\, b^{b,\dagger}_\lambda}\,(p) = \delta^{ab}\epsilon_A(p,\lambda)\, e^{ipx} \tag{A.22}$$

$$\underbrace{d^a_s(p)\, \lambda}^{b\alpha}(x) = \delta^{ab} u^\alpha_s\,(p)\, e^{-ipx} \tag{A.23}$$

$$\underbrace{\lambda^{a\alpha}(x) d}^{b,\dagger}_s(p) = \delta^{ab} u^\alpha_s\,(p)\, e^{ipx} \tag{A.24}$$

B Warp-Factors and Torsion

B.1 G-Flux Contractions

First, we present those identities, which are used in the main text for the evaluation of the Killing-spinor equation

$$
\begin{aligned}
\hat{\Gamma}^{ea\bar{b}c\bar{d}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{e}a\bar{b}c\bar{d}}\tilde{\eta} &= \big[\hat{\Gamma}^{\bar{e}}(\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}} - \hat{g}^{a\bar{d}}\hat{g}^{c\bar{b}}) + \hat{\Gamma}^{\bar{b}}(\hat{g}^{a\bar{d}}\hat{g}^{c\bar{e}} - \hat{g}^{a\bar{e}}\hat{g}^{c\bar{d}}) + \hat{\Gamma}^{\bar{d}}(\hat{g}^{a\bar{e}}\hat{g}^{c\bar{b}} - \hat{g}^{a\bar{b}}\hat{g}^{c\bar{e}})\big]\tilde{\eta} \\
\hat{\Gamma}^{abc\bar{d}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{a\bar{b}c\bar{d}}\tilde{\eta} &= \big[\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}} - \hat{g}^{a\bar{d}}\hat{g}^{c\bar{b}}\big]\tilde{\eta} \\
\hat{\Gamma}^{\bar{a}\bar{b}c\bar{d}}\tilde{\eta} &= \big[\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{b}}\hat{g}^{c\bar{d}} - \hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{d}}\hat{g}^{c\bar{b}} + \hat{\Gamma}^{\bar{b}}\hat{\Gamma}^{\bar{d}}\hat{g}^{c\bar{a}}\big]\tilde{\eta} \\
\hat{\Gamma}^{ab\bar{c}11}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{a}b\bar{c}11}\tilde{\eta} &= e^{-h/2}\big[\hat{\Gamma}^{\bar{a}}\hat{g}^{b\bar{c}} - \hat{\Gamma}^{\bar{c}}\hat{g}^{b\bar{a}}\big]\tilde{\eta} \\
\hat{\Gamma}^{ab\bar{c}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{a\bar{b}\bar{c}}\tilde{\eta} &= \big[\hat{g}^{a\bar{b}}\hat{\Gamma}^{\bar{c}} - \hat{g}^{a\bar{c}}\hat{\Gamma}^{\bar{b}}\big]\tilde{\eta} \\
\hat{\Gamma}^{a\bar{b}}\tilde{\eta} &= \hat{g}^{a\bar{b}}\tilde{\eta}\ .
\end{aligned} \tag{B.1}
$$

With their help and the definitions (4.10),(4.11),(4.12), we arrive at the contractions

$$
\begin{aligned}
\hat{\Gamma}^{\mu uvwx}G_{uvwx}\tilde{\eta} &= -3\big[e^{-2f}\alpha\hat{\Gamma}^{\mu} + 4ie^{-k/2-f}\beta_{\bar{a}}\hat{\Gamma}^{\mu}\hat{\Gamma}^{\bar{a}}\big]\tilde{\eta} \\
\hat{\Gamma}^{euvwx}G_{uvwx}\tilde{\eta} &= -12ie^{-k/2-f}\beta^{e}\tilde{\eta} \\
\hat{\Gamma}^{\bar{e}uvwx}G_{uvwx}\tilde{\eta} &= -3\big[e^{-2f}\alpha\hat{\Gamma}^{\bar{e}} - 4ie^{-f}\Theta^{\bar{e}}{}_{\bar{a}}\hat{\Gamma}^{\bar{a}} - 4ie^{-k/2-f}\beta^{\bar{e}} \\
&\qquad + 4ie^{-k/2-f}\beta_{\bar{a}}\hat{\Gamma}^{\bar{e}}\hat{\Gamma}^{\bar{a}} + 4e^{-k/2}\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{b}}G^{\bar{e}}{}_{\bar{a}\bar{b}11}\big]\tilde{\eta} \\
\hat{\Gamma}^{11uvwx}G_{uvwx}\tilde{\eta} &= -3e^{-k/2-2f}\alpha\tilde{\eta}
\end{aligned} \tag{B.2}
$$

and

$$
\begin{aligned}
\hat{g}^{\mu u}\hat{\Gamma}^{vwx}G_{uvwx}\tilde{\eta} &= 0 \\
\hat{g}^{au}\hat{\Gamma}^{vwx}G_{uvwx}\tilde{\eta} &= -3ie^{-k/2-f}\beta^{a}\tilde{\eta} \\
\hat{g}^{\bar{a}u}\hat{\Gamma}^{vwx}G_{uvwx}\tilde{\eta} &= -3\big[ie^{-k/2-f}\beta^{\bar{a}} + ie^{-f}\Theta^{\bar{a}}{}_{\bar{e}}\hat{\Gamma}^{\bar{e}} - e^{-k/2}\hat{\Gamma}^{\bar{b}\bar{c}}G^{\bar{a}}{}_{\bar{b}\bar{c}11}\big]\tilde{\eta} \\
\hat{g}^{11u}\hat{\Gamma}^{vwx}G_{uvwx}\tilde{\eta} &= 3ie^{-f}\beta_{\bar{c}}\hat{\Gamma}^{\bar{c}}\hat{g}^{11,11}\tilde{\eta}\ ,
\end{aligned} \tag{B.3}
$$

which are used in the main text.

B.2 The Heterotic String with Torsion

The Ansatz traditionally used [31] in compactifications of the 10-dimensional heterotic $E_8 \times E_8$ string on CY_3 to four dimensions with $\mathcal{N}=1$ supersymmetry is to make the susy-variations of the gravitino ψ_M, the dilatino λ and the gluino χ^a $(i,j=5,\ldots,10)$

$$\begin{aligned}
\delta\psi_i &= \frac{1}{\kappa}D_i\eta + \frac{\kappa}{32g^2\phi}(\Gamma_i{}^{jkl} - 9\delta_i^j\Gamma^{kl})\eta H_{jkl} \\
\delta\lambda &= -\frac{1}{\sqrt{2}\phi}(\Gamma\cdot\partial\phi)\eta + \frac{\kappa}{8\sqrt{2}g^2\phi}\Gamma^{ijk}\eta H_{ijk} \\
\delta\chi &= -\frac{1}{4g\sqrt{\phi}}\Gamma^{ij}F_{ij}\eta
\end{aligned} \tag{B.4}$$

vanish by assuming that $H = d\phi = 0$. Here ϕ is the dilaton and H the gauge-invariant field strength of the NSNS 2-form B, which in addition has to fulfil the Bianchi identity

$$dH = \mathrm{tr}R\wedge R - \frac{1}{30}\mathrm{tr}F\wedge F\ . \tag{B.5}$$

This leads to the consequence that CY_3 is a Kähler manifold with $c_1(CY_3) = 0$ and $SU(3)$ holonomy (and the gauge field A being a holomorphic connection on a holomorphic vector bundle V over the Calabi-Yau threefold CY_3 obeying the Donaldson-Uhlenbeck-Yau equation).

This Ansatz was generalized in [77] to include a non-vanishing torsion $H\neq 0$ where solutions leading again to $\mathcal{N}=1$ supersymmetry in D=4 were obtained by allowing for a warp-factor $e^{2D(y)}$ in the metric (in Einstein-frame)

$$g^E_{AB}(x,y) = e^{2D(y)}g_{AB}(x,y) = e^{2D(y)}\begin{pmatrix}\eta_{\mu\nu}(x) & 0\\ 0 & g_{mn}(y)\end{pmatrix}\ , \tag{B.6}$$

where we denote 10-dimensional indices by $A,B,C,\ldots$. It turns out that D has to be the dilaton ϕ. The torsion and the dilaton are determined by

$$H = \frac{i}{2}(\bar\partial - \partial)J \tag{B.7}$$

$$e^{8\phi} = e^{8\phi_0}||\Omega||\ , \tag{B.8}$$

where the fundamental (1,1) form J is built out of the complex structure $J_m{}^n$ as $J = \frac{1}{2}J_m{}^n g_{np}dy^m\wedge dy^p = ig_{a\bar b}dz^a\wedge d\bar z^{\bar b}$ (in our conventions J equals up to a minus-sign the Kähler-form ω) and Ω is the (determined up to an overall constant) holomorphic 3-form with norm $||\Omega|| = (\Omega_{a_1a_2a_3}\bar\Omega_{\bar b_1\bar b_2\bar b_3}g^{a_1\bar b_1}g^{a_2\bar b_2}g^{a_3\bar b_3})^{1/2}$. To recognize the relation between H

(commonly called torsion) and the original torsion, we note that the metric torsion of a complex manifold is specified by the expression

$$T^a_{bc} = -2g^{\bar{d}a} g_{\bar{d}[b,c]} \tag{B.9}$$

and its complex conjugate. Hence, the above expression for H can be explicitely expressed through the metric torsion via

$$H = \frac{1}{4}\left(T_{a\bar{b}\bar{c}} dz^a \wedge dz^{\bar{b}} \wedge dz^{\bar{c}} + T_{\bar{c}ab} dz^a \wedge dz^b \wedge dz^{\bar{c}}\right) . \tag{B.10}$$

Finally, the link between H and the warp-factor is given implicitly by the dilatino equation $\delta\lambda = 0$, which manifests itself in the following relationship

$$d^\dagger J = i(\partial - \bar{\partial}) \ln ||\Omega|| . \tag{B.11}$$

From the left-hand side of this equation it can be easily discerned, that the right-hand side serves as a measure for the non-Kählerness of the compactification manifold. Therefore, by turning on H-torsion, the compactification manifold becomes deformed to a manifold which is no longer Kähler.

To gain a more explicit relation between the H-torsion and the resulting warp-factor, we note that the dilatino equation $\delta\lambda = 0$ can be alternatively written as [77]

$$8\partial_m \phi = J_m{}^n \nabla_p J_n{}^p . \tag{B.12}$$

Here, the covariant derivative is constructed out of the initial metric g_{MN} without warp-factor. The H-covariant constancy of the complex structure [77]

$$\nabla_m J_n{}^p - H_{qm}{}^p J_n{}^q - H^q{}_{mn} J_q{}^p = 0 \tag{B.13}$$

plus its property to square to minus the identity, $J_m{}^n J_n{}^p = -\delta^p_m$, serve together with $J_{a\bar{b}} = i g_{a\bar{b}}$ to derive

$$8\partial_a \phi = H_{ab}{}^b - H_{a\bar{b}}{}^{\bar{b}} . \tag{B.14}$$

The contraction is with respect to the initial metric in whose frame the relation holds. Equation (B.7), which relates H-torsion with metric torsion, reads in components $H_{ab\bar{c}} = -g_{\bar{c}[a,b]}$ and leads to $H_{a\bar{b}}{}^{\bar{b}} = -H_{ab}{}^b$. Finally, to obtain the relation between the warp-factor ϕ and H-torsion in the Einstein-frame, we have to transform the contractions according to the rescaling $g_{AB} = e^{-2\phi} g^E_{AB}$ from the initial frame to the Einstein-frame and gain

$$\partial_a(e^{-2\phi}) = -\frac{1}{2} H_{ab}{}^b . \tag{B.15}$$

The contraction on the right-hand-side is now understood to be carried out with g^E_{AB}.

Furthermore warp-geometries appear in heterotic five-brane solutions preserving supersymmetry. They were obtained ([78],[79],[80]; cf. also the axionic instantons in [81]) with the Ansatz ($k, l, m, n = 7, \ldots, 10$)

$$
\begin{aligned}
g_{mn} &= e^{2\phi}\delta_{mn} \\
H_{mnl} &= -\epsilon_{mnl}{}^{k}\partial_k\phi
\end{aligned}
$$

showing again that turning on torsion leads to a warp factor.

C Unequal Wall Tensions

C.1 Effective D=4 Action

In this appendix, we will deal with the case of unequal wall tensions $T_1 \neq T_2$. The Ansatz (5.4) then yields the solution

$$
A(x^5) = \frac{k_1}{2}\left|x^5 + l\right| + \frac{k_2}{2}\left|x^5 - l\right| = \left\{ \begin{array}{cc} x^5 \geq l: & \frac{1}{2}K_{12}x^5 + \frac{1}{2}k_{12}l \\ -l \leq x^5 \leq l: & \frac{1}{2}k_{12}x^5 + \frac{1}{2}K_{12}l \\ x^5 \leq -l: & -\frac{1}{2}K_{12}x^5 - \frac{1}{2}k_{12}l \end{array} \right. , \tag{C.1}
$$

where $K_{12} = k_1 + k_2$ and $k_{12} = k_1 - k_2$. Without loss of generality, we will assume that $k_1 \geq k_2$ subsequently. The function $A(x^5)$, which determines the warp-factor is displayed in fig.11. The corresponding warp-factor $e^{-A(x^5)}$ is upper-bounded by $e^{-k_2 l}$ throughout the whole fifth dimension. From the Einstein equation (5.5) we receive the expressions for Λ and the wall tensions

$$
\Lambda(x^5) = \left\{ \begin{array}{ll} \Lambda_e\,, & |x^5| \geq l \\ \Lambda_i\,, & |x^5| < l \end{array} \right. = -\frac{3M^3}{4}\left\{ \begin{array}{ll} K_{12}^2\,, & |x^5| \geq l \\ k_{12}^2\,, & |x^5| < l \end{array} \right. \tag{C.2}
$$

$$
T_1 = 3M^3 k_1\,, \qquad T_2 = 3M^3 k_2\,. \tag{C.3}
$$

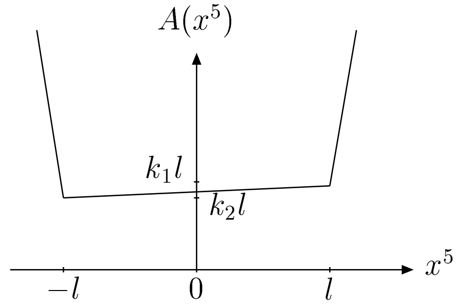

Figure 11: The function $A(x^5)$, which determines the warp-factor.

The next task is again the determination of the effective four-dimensional action. Along the same lines as above by employing (5.13), we get for the Einstein-Hilbert term

$$\begin{aligned} S_{EH} =& -\int d^4x\sqrt{g}M^3\left\{R(g)\int_{-\infty}^{\infty}dx^5 e^{-A}+\int_{-\infty}^{\infty}dx^5 e^{-2A}\left[5(A')^2-4A''\right]\right\} \\ =& -e^{-K_{12}l/2}\int d^4x\sqrt{g}M^3\Bigg\{4R(g)\Big[\frac{1}{K_{12}}\cosh\left(\frac{k_{12}l}{2}\right)+\frac{1}{k_{12}}\sinh\left(\frac{k_{12}l}{2}\right)\Big] \\ &+\frac{5}{4}e^{-K_{12}l/2}\Big[2K_{12}\cosh\left(k_{12}l\right)+2k_{12}\sinh\left(k_{12}l\right)\Big] \\ &-4e^{-K_{12}l/2}\Big[k_1 e^{k_{12}l}+k_2 e^{-k_{12}l}\Big]\Bigg\}\,. \end{aligned} \tag{C.4}$$

For the remaining wall- and bulk cosmological constant terms we obtain

$$\begin{aligned} S_{SM_1}+S_{SM_2}+S_\Lambda =& -e^{-K_{12}l}\int d^4x\sqrt{g}\Bigg\{e^{k_{12}l}T_1+e^{-k_{12}l}T_2 \\ &+2\frac{\Lambda_e}{K_{12}}\cosh(k_{12}l)+2\frac{\Lambda_i}{k_{12}}\sinh(k_{12}l)\Bigg\}\,. \end{aligned} \tag{C.5}$$

Pulling out an overall factor of $e^{-K_{12}l/2}$ in front, the final effective action reads

$$\begin{aligned} &S_{EH}+S_{SM_1}+S_{SM_2}+S_\Lambda \\ =& -e^{-K_{12}l/2}\int d^4x\sqrt{g}\Bigg\{4M^3R(g)\Big[\frac{1}{K_{12}}\cosh\left(\frac{k_{12}l}{2}\right)+\frac{1}{k_{12}}\sinh\left(\frac{k_{12}l}{2}\right)\Big] \\ &+\frac{5}{2}M^3e^{-K_{12}l/2}\Big[K_{12}\cosh\left(k_{12}l\right)+k_{12}\sinh\left(k_{12}l\right)\Big]+e^{-K_{12}l/2}\Big[e^{k_{12}l}(T_1-4k_1M^3) \\ &+e^{-k_{12}l}(T_2-4k_2M^3)+2\frac{\Lambda_e}{K_{12}}\cosh(k_{12}l)+2\frac{\Lambda_i}{k_{12}}\sinh(k_{12}l)\Big]\Bigg\}\,. \end{aligned} \tag{C.6}$$

At the classical level the normalization of the action is irrelevant. Let us therefore by the same reasoning as in the main text drop the overall scale-factor and arrive at the effective action

$$S_{D=4} = -\int d^4x\sqrt{g}\left\{M_{\rm eff}^2 R(g) + \Lambda_4\right\} \ , \tag{C.7}$$

with the effective four-dimensional Planck-scale $M_{\rm eff}$ and the four-dimensional cosmological constant Λ_4 now given by

$$M_{\rm eff}^2 = 4M^3\Big[\frac{1}{K_{12}}\cosh\Big(\frac{k_{12}l}{2}\Big) + \frac{1}{k_{12}}\sinh\Big(\frac{k_{12}l}{2}\Big)\Big] \tag{C.8}$$

$$\begin{aligned}\Lambda_4 = e^{-K_{12}l/2}\bigg(\frac{5}{2}M^3\Big[K_{12}\cosh(k_{12}l) + k_{12}\sinh(k_{12}l)\Big] + \Big[e^{k_{12}l}(T_1 - 4k_1M^3)\\ + e^{-k_{12}l}(T_2 - 4k_2M^3) + 2\frac{\Lambda_e}{K_{12}}\cosh(k_{12}l) + 2\frac{\Lambda_i}{k_{12}}\sinh(k_{12}l)\Big]\bigg) \ .\end{aligned} \tag{C.9}$$

Again, there exists a huge suppression-factor $e^{-K_{12}l/2}$ multiplying the whole cosmological constant, which serves to bring Λ_4 down to its observed upper bound if generically $k_1, k_2 \simeq M_{Pl}$. When the above obtained values (C.2),(C.3) for $T_1, T_2, \Lambda_e, \Lambda_i$ are substituted in the obtained action, we arrive at a vanishing Λ_4, which checks the derivation of the action, since in that special case the fine-tuning of the parameters requires a flat four-dimensional metric $g_{\mu\nu} = \eta_{\mu\nu}$. For the particular case of coinciding wall-tensions, $T_1 = T_2 = T$ (which entails $k_1 = k_2 = k$), we arrive at the effective action given by (5.29),(5.31),(5.34), which was discussed in the main text.

Again, let us now lift the fine-tuning of the parameters imposed by

$$\Lambda(x^5) = \left\{\begin{array}{ll}\Lambda_e \ , & |x^5| \geq l\\ \Lambda_i \ , & |x^5| < l\end{array}\right. = -\frac{1}{12M^3}\left\{\begin{array}{ll}(T_1+T_2)^2 \ , & |x^5| \geq l\\ (T_1-T_2)^2 \ , & |x^5| < l\end{array}\right. \ , \tag{C.10}$$

which corresponds to a non-trivial four-dimensional metric $g_{\mu\nu} \neq \eta_{\mu\nu}$ in the Ansatz

$$ds^2 = e^{-A(x^5)}g_{\mu\nu}dx^\mu dx^\nu + (dx^5)^2 \ . \tag{C.11}$$

From (C.9) it is evident, that in order to arrive at a small Λ_4, we require

$$k_1 - k_2 \equiv k_{12} \lesssim \frac{1}{l} = 2M_{\rm GUT} \ . \tag{C.12}$$

Generically, we choose $k_1, k_2 \simeq M_{Pl}$, $T_1, T_2 \simeq M_{Pl}^4$, $\Lambda_e \simeq -M_{Pl}^5$ and the fundamental five dimensional Planck-scale $M \simeq M_{Pl}$. Again, as explained in the main text, Λ_i has to be chosen with an upper bound of $(3\times 10^{18}{\rm GeV})^5$, which roughly corresponds to the traditional string-scale. Then, we recognize from (C.9), that the suppression through the exponential factor is sufficient to bring the contributions to the four-dimensional cosmological constant down to its observed value. The effective four-dimensional Planck-scale $M_{\rm eff} \simeq 24 M_{Pl}$ again comes out slightly too high. It can however be brought down, e.g. to $M \simeq M_{Pl}$, if we choose $M \simeq 1.5 \cdot 10^{18}$GeV, which is close to the traditional string-scale.

C.2 Effective Potential for Bulk Scalars

Here, we want to extend the analysis of a bulk scalar contribution to the effective potential ($= \Lambda_4$) to the case of unequal wall tensions. For the action of the scalar Φ with mass m, let us take

$$\begin{aligned} S_\Phi = &- \int d^4x \int_{-\infty}^{\infty} dx^5 \sqrt{G} \Big\{ \frac{1}{2} G^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{2} m^2 \Phi^2 \Big\} \\ &- \int d^4x \int_{-\infty}^{\infty} dx^5 \Big\{ \sqrt{g_1^{(4)}} \lambda_1 (\Phi^2 - v_1^2)^2 \delta(x^5 + l) + \sqrt{g_2^{(4)}} \lambda_2 (\Phi^2 - v_2^2)^2 \delta(x^5 - l) \Big\} \, . \end{aligned} \tag{C.13}$$

Assuming only an x^5 dependence of Φ, we arrive at the field equation

$$\begin{aligned} (e^{-2A}\Phi')' - e^{-2A} m^2 \Phi = 4 \big[& e^{-2A(-l)} \lambda_1 (\Phi^2 - v_1^2) \Phi \delta(x^5 + l) \\ & + e^{-2A(l)} \lambda_2 (\Phi^2 - v_2^2) \Phi \delta(x^5 - l) \big] \, , \end{aligned} \tag{C.14}$$

which, away from the walls, has the solution

$$\Phi(x^5) = \begin{cases} a e^{(1+\Gamma)A} + b e^{(1-\Gamma)A} \, , & x^5 < -l \\ c e^{(1+\gamma)A} + d e^{(1-\gamma)A} \, , & |x^5| \leq l \\ e e^{(1+\Gamma)A} + f e^{(1-\Gamma)A} \, , & x^5 > l \end{cases} \, , \tag{C.15}$$

with

$$\Gamma = \sqrt{1 + 4m^2/K_{12}^2} \, , \qquad \gamma = \sqrt{1 + 4m^2/k_{12}^2} \, . \tag{C.16}$$

In order to obtain a normalizable solution for Φ, we set the coefficients $a = e = 0$. Moreover, imposing continuity of Φ at the walls determines b and f in terms of c, d

$$b = e^{\Gamma k_2 l} \tilde{b} \, , \quad \tilde{b} = c e^{\gamma k_2 l} + d e^{-\gamma k_2 l} \tag{C.17}$$

$$f = e^{\Gamma k_1 l} \tilde{f} \, , \quad \tilde{f} = c e^{\gamma k_1 l} + d e^{-\gamma k_1 l} \, . \tag{C.18}$$

To fix the remaining coefficients c and d one would have to plug the above bulk solution in the field equation and integrate out the fifth dimension to incorporate the wall boundary conditions. Since this leads to a complicated cubic equation in the unknowns c, d, it is easier to determine them by inserting the bulk solution into the scalar action and integrating over x^5 to arrive at an effective potential for the wall-distance l. For positive couplings λ_1, λ_2 this effective potential will be positive definite. Hence, to minimize the potential, we must have $\Phi(-l) = v_1$ and $\Phi(l) = v_2$. This allows for a determination of c, d in terms of the parameters v_1, v_2

$$c = \frac{v_2 e^{-(1-\gamma)k_1 l} - v_1 e^{-(1-\gamma)k_2 l}}{e^{2\gamma k_1 l} - e^{2\gamma k_2 l}} \,, \qquad d = \frac{v_2 e^{-(1+\gamma)k_1 l} - v_1 e^{-(1+\gamma)k_2 l}}{e^{-2\gamma k_1 l} - e^{-2\gamma k_2 l}} \,. \tag{C.19}$$

The effective potential[47] eventually becomes

$$\begin{aligned} V_\Phi(l) = {} & \frac{k_{12}}{2} \sinh(\gamma k_{12} l) \left[c^2(\gamma+1) e^{\gamma K_{12} l} + d^2(\gamma-1) e^{-\gamma K_{12} l} \right] \\ & + \frac{(\Gamma-1)K_{12}}{4} \left[\tilde{b}^2 + \tilde{f}^2 \right] . \end{aligned} \tag{C.20}$$

A numerical analysis of this potential shows, that also in the case with differing tensions a bulk scalar, with couplings to the walls, leads generically to an effective potential, which is likewise sufficiently suppressed. Therefore, it does not generate a huge four-dimensional cosmological constant, which could have been spoiled the embedding of the mechanism into string-theory.

[47]Here we use the relations $(1 \pm \gamma)^2 \frac{k_{12}^2}{4} + m^2 = \gamma(\gamma \pm 1) \frac{k_{12}^2}{2}$ and $(1-\gamma^2) \frac{k_{12}^2}{4} + m^2 = 0$.

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Erklärung gemäß Prüfungsordnung

Hiermit erkläre ich, diese Dissertation selbständig verfaßt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Hamburg, 25. August 2000

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