INSTITUT FÜR ANGEWANDTE PHYSIK

Physikalisches Praktikum für Studierende der Ingenieurswissenschaften

Universität Hamburg, Jungiusstraße 11

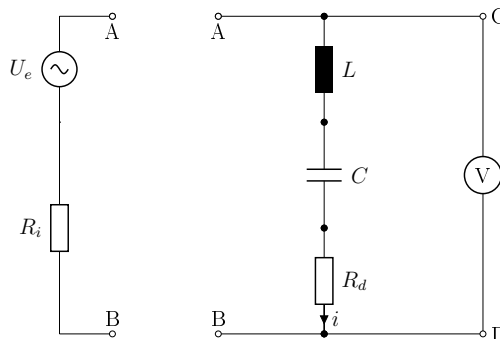# RLC Circuit

Contents

1   RCL Series Circuit

Figure 1: RCL-circuit

2   Set-Up of the Experiment

This experiment is an introduction to the theory of alternating current. The Ohmic law that you were thought in school is valid for both direct current (DC) and alternating current (AC). The behavior of currents, voltages and phases that occur in some of the components (inductor L, capacitor C) is dependent on the frequency. The RLC oscillating circuit is a classic example for investigating this dependency. You will investigate why a RLC circuit forms a harmonic oscillator and under which conditions it resonates. You will learn how to operate an oscilloscope so you can measure the physical processes. A signal generator will provide the required input signal. You should be familiar with the complex notation of voltage, current, and resistance (see theory part of the script) so that you can describe your results properly.

3   Additional Insights

Besides the information from the questionnaire, you will learn additionally:

Set-up of an oscillating circuit: interconnecting the circuit components, connecting measurement instruments and components, ...

Handling of an oscilloscope: setting the measuring range, how to take readings (time, frequency, and voltage).

(Where applicable) recording measurements with a computer: You will measure using a digital oscilloscope and a computer.

Evaluation: Handling Excel or other software.

4 Basics

4.1 Glossary

Here, we briefly introduce some terms. For a more detailed explanation, we refer to the respective literature.

The electric field and the voltage U If a small probing charge q=e^- is in the close vicinity of a much larger charge, a force (depending on the sign, by convention) acts on the probing charge (see Law of Coulomb, which should be familiar to you). This leads to the (vectorial) definition of the electric field E

$$\mathbf{E(r)} = \frac{\mathbf{F(r)}}{q} \qquad\qquad [E] = \frac{N}{C} \qquad\qquad (1)$$

The charge is displaced when a force acts on it. Thus, depending on the direction of the applied force and the sign of the charge, electrical work W would be either consumed or generated by displacing dr the probing charge.

$$W_{1,2} = \int_1^2 \mathbf{F dr} = q \int_1^2 \mathbf{E dr} = qEd = q \int_1^2 d\varphi = qU \qquad (2)$$

$$\mathbf{E(r)} = \frac{d\varphi}{dr}\mathbf{e_r} \quad [E] = \frac{V}{m} \qquad\qquad (3)$$

In a battery, a conductor connects two electric poles with different charge carrier concentration and the potential φ. If this is the case, the charge carriers will even move through a flow of current. Then, Ohmic law applies: $U = R \cdot I$.

Furthermore, from eqn. (2) follows :

$$\frac{qU}{q} = \frac{qEd}{q} \rightarrow U = Ed \qquad\qquad (4)$$

The field strength can be determined much easier using the distance and the voltage.

Kirchhoff's node law The principle of *conservation of electric charge* implies that at any node (or junction) in an electrical circuit, the sum of currents in a network of conductors meeting at a point is zero.

$$\sum_i I_i = 0 \qquad\qquad (5)$$

Kirchhoff's loop law The sum of the electrical potential differences (voltage) around any closed network is zero. Therefore, the following law is derived:

$$\sum_i U_i = 0 \qquad\qquad (6)$$

Voltage Divider Follows Kirchhoff's node and loop laws. Without those laws, voltage drops could not be described. If two consumers R_1 and R_2 are in series, then the respective voltages relate to the input voltage U_e is equal to the individual resistance to the total resistance:

$$\frac{U_{1,2}}{Ue} = \frac{R_{1,2}}{R_1 + R_2} \tag{7}$$

Complex notation Oscillation can be described using cos or sin. Often, complex notation is preferred because it is easier to calculate. Complex notation of a number: $c = a + j \cdot b$ with $j = \sqrt{-1}$. A common representation of a complex number is a vector diagram with a complex plain. The imaginary part b is plotted in direction of the y axis and the real part a in direction of the x axis. Every oscillation can be described in complex notation, including current and voltage: $\underline{I} = I \cdot e^{j\Phi_i}$, $\underline{U} = U \cdot e^{j\Phi_u}$. The angles $\Phi_{i,u}$ determine the initial phase to the real axis. If, over time, the phasor turns with an angular frequency ω, we achieve a time-dependent, periodic representation. Accordingly, the exponents of the phases are time-dependent.

4.2 Capacity C

Capacity A capacity is a property of a capacitor. If a voltage U is applied, a capacitor collects a certain amount of charge carriers Q and later releases them. The energy that is needed for separation of charge is partially (Ohmic loss) stored in the electric field that builds up during charging.

$$C = \frac{Q}{U} \qquad\qquad [C] = \frac{1AS = C}{V} = 1F(Farad) \qquad\qquad (8)$$

$$W = \frac{1}{2}CU^2 \qquad\qquad [W] = 1Ws = 1Joule \qquad\qquad (9)$$

Set-Up of a capacitor In its simplest form, a capacitor consists of two opposing conductor plates, each of area A and with a separation of d (see figure 2.[1]).
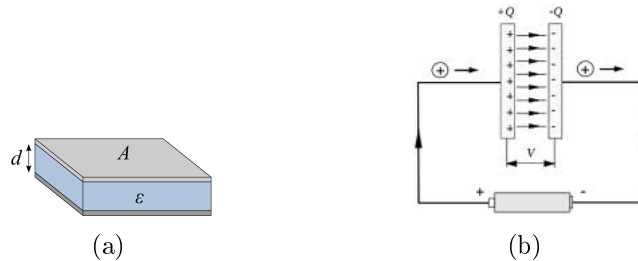

(a)   (b)

Figure 2: schematic of a parallel plate capacitor (a); current flow and electric field (b)

Surface charge density and electric field A typical measurement for a capacitor is its *surface charge density* σ_A. The surface charge energy has the following relationship with the *electric field* between the plates:

$$\sigma_A = \frac{Q}{A} \quad \left[\frac{C}{cm^2}\right] \qquad\qquad \mathbf{E} = \frac{\sigma_A}{\epsilon_r}\mathbf{e_r} \qquad\qquad (10)$$

Dielectric: The conductor plates can be separated by a dielectric material to adjust the capacity. The characteristic measurement for the dielectric is its relative permittivity (or dielectric constant) $\epsilon = \epsilon_0\epsilon_r$ with $\epsilon_0 = 8.854 \cdot 10^{-12}\, F/m$ as electric field size. For simplicity, we use vacuum/air with $\epsilon_r = 1$, here.

Capacity of the parallel plate capacitor Eqn. (8) and (10) can be transformed to achieve an alternative, geometric definition of capacity:

$$C = \epsilon_o\epsilon\frac{A}{d} = \epsilon_r\frac{A}{d} \qquad\qquad (11)$$

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[1]Figure is taken from Wikipedia

Halving the distance between capacitor plates The effect of halving the distance d for a charged capacitor

$$C = \epsilon_r \frac{A}{d_0/2} = 2C_0 \tag{12}$$

is dependent on it connection to the voltage source:

1. If the capacitor **was disconnected from the voltage source**, charge Q is preserved; i) *Surface charge density is constant*: $\sigma_A = $ const.
   $\rightarrow$ *ii)* Field strength is constant: $E = \sigma_A/\epsilon_r = $ konst $\rightarrow$ *iii)* Voltage is reduced by half: $U = Ed_0/2 = U_0/2$ The energy is reduced to half too, $W = 1/2(2C_0)(U_0/2)^2 = W_0/2$. That means, the energy is released

$$W_{01} = \int_{d_0}^{d_0/2} \mathbf{F ds} = Q \int_{d_0}^{d_0/2} \mathbf{E ds} = -\frac{1}{2}W_0 \tag{13}$$

   Invertedly, energy has to be spent for separation of charges. That energy is saved in the electric field.

2. If the capacitor **was not disconnected from the voltage source**, voltage stays constant $U = $ `const.` but charge $Q \neq 0$ is not preserved. Charges can flow to and from the voltage source. *i)* Surface charge density doubles because U=const. and Equ.(12): $\rightarrow \sigma_A = $ const. $\rightarrow$ *ii)* Field strength is doubled: $E = 2\sigma_{A_0}/\epsilon_r = 2E_0$ Energy content is doubled $W = 1/2(2C_0)U^2 = 2W_0$.

Charging properties of a RC element A capacitor's time propertiy is described by a differential equation. Kirchhoff's loop law, applied to the RC element.

$$U_R + U_c = RI + \frac{Q}{C} = U_e \tag{14}$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = U_e \tag{15}$$

$$I = \frac{dQ}{dt} = \frac{dCU}{dt} = \frac{CdU}{dt} \tag{16}$$

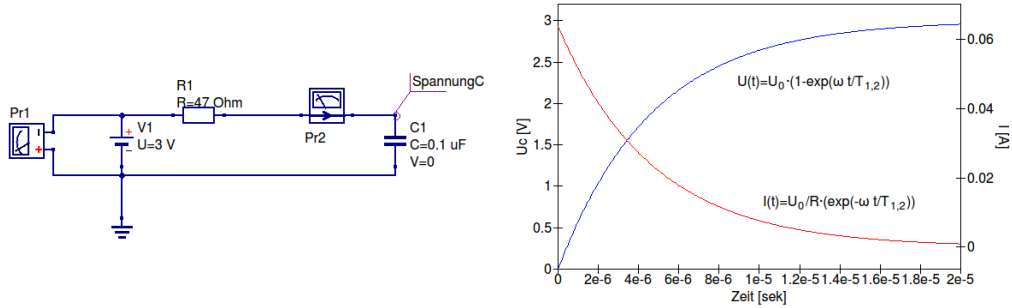**Capacitive Reactance**

$$\boxed{X_C = \frac{1}{\omega C}} \tag{17}$$

Figure 3: RC low-pass: left, simulation of a RC low-pass filter; right, measured values U_c and I. The same components as in the experiment were used (R=47 Ohm and C=0.1 μF)

4.3   Inductance L: Inductor in a DC circuit

According to Lenz's law, a changing electric current through a inductor induces a proportional voltage, which opposes the change in the current.

$$U_{ind} = -L\frac{dI}{dt} \qquad\qquad [L] = \frac{1Vs}{A} = 1F(Henry) \qquad (18)$$

$$W_L = \frac{1}{2}LI^2 \qquad\qquad [W] = 1W_s = 1Joule \qquad (19)$$

Applying voltage to a inductor creates a current that increases exponentially to a maximum with a *time constant* $\tau_L = L/R$ and $i_L = U/R \cdot (1 - e^{-t/\tau})$. The inductive reactance X_L does not work as an active power but preserves energy as magnetic field. The active resistance R_{vL} causes ohmic heating. The inductive Q factor is measured for determining the thermal loss $Q_L = \omega L/R$

Inductive Reactance

$$\boxed{X_L = \omega L} \qquad (20)$$

4.4   Impedance $\underline{Z}$, apparent impedance $|\underline{Z}|$

The **impedance** Z in an AC circuit is the complex ratio of the active resistance R (causing Ohmic heating) to the frequency-dependent inductive reactance X.

$\underline{Z} = R + jX$. The absolute value $Z = |\underline{Z}| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$ is called apparent impedance.

5 An Ideal Oscillating Circuit

To explain the relation between current and voltage, we consider an ideal CL oscillating circuit without Ohmic resistance

$$W_L = \tfrac{1}{2} \cdot LI^2 \qquad W_C = \tfrac{1}{2} \cdot CU^2$$

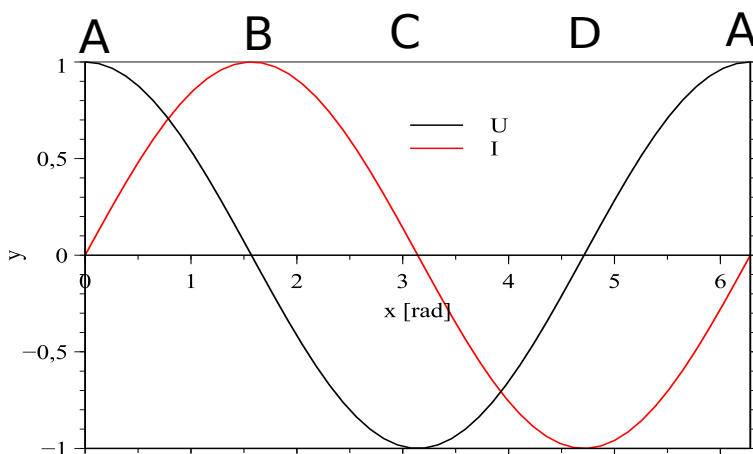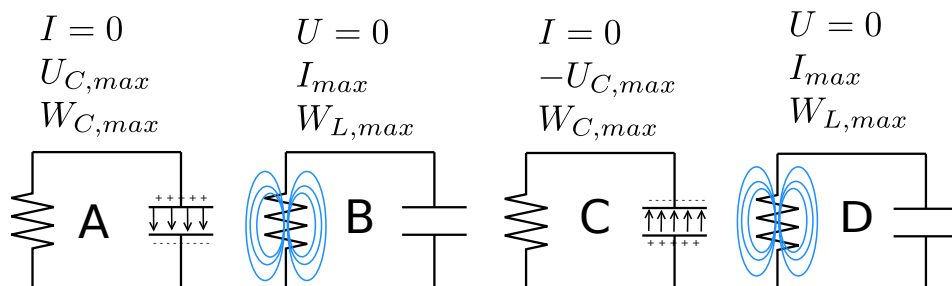

Figure 4: The relation between current and voltage in an ideal oscillating circuit. The voltage is ahead of the current by $\pi/2$.

A The capacitor is fully charged (U_{max}). There is no current flowing and all of the system's energy is stored in the electric field of the capacitor. The same voltage of the capacitor is applied to the inductor and it creates current. According to Lenz's law, this change in the current is opposed by an induced voltage. This impedes the current flow, which increases only slowly to reach ...

B ... its maximum. The capacitor is completely discharged (charge balance). All energy is stored in the electric field of the inductor. Voltage is zero, but current is still flowing because of inertia (caused by the magnetic field). The capacitor is charged in the opposite direction. Voltage builds up and counteracts the magnetic flux in the inductor, thus the electric field is reduced.

C The current reaches a minimum (I=0) when voltage reaches its maximum $U_{\texttt{max}}$. The capacitor is fully charged in the opposite direction.

The capacitor discharges like described in **A** but with the inverse sign.
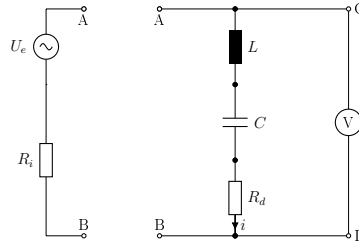
6 Realistic RLC Oscillating Circuit

Figure 5: RLC Circuit

6.1 Impedanz,apparent impedance

Electrical impedance of the circuit (see figure 6):

$$\underline{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\,\overbrace{\left(\omega L - \frac{1}{\omega C}\right)}^{\text{X}} \tag{21}$$

$$= |\underline{Z}|\,e^{j\theta} \tag{22}$$

The apparent impedance of a RLC circuit can be calculated by

$$\boxed{Z = |\underline{Z}| = \sqrt{R^2 + X^2}} \tag{23}$$

6.2 Current and Voltage

The following example illustrates the relationship between a sinusoidal input voltage U_e and a current I. The latter is a solution for the differential equation that describes the oscillating circuit depicted in figure 5. We derive the solution as follows (results are discussed in the subsequent section):

Using the Kirchhoff's loop law, we get:

$$U_R + U_L + U_C - U_e = 0 \tag{24}$$

Given the relationships $U_C = Q/C$, $U_L = L\frac{dI}{dt}$, $U_R = I \cdot R$ and an input voltage of $U_e = U_0$:
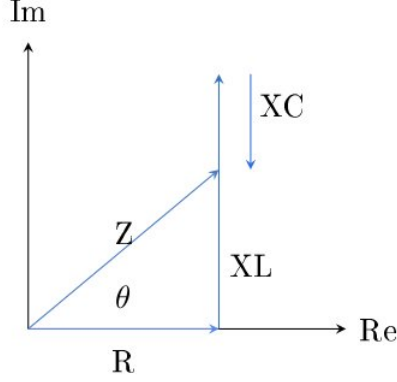
Figure 6: About the impedance of a RLC circuit

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = U. \tag{25}$$

R stands for the total resistance, including loss and inner resistance: $R = R_\Sigma = R_i + R_d + R_v$. The sinusoid (and thus changing) input voltage acts as electromotor force that activates the circuit :

$$U = U_0 \sin(\omega t) \tag{26}$$

Its time derivation is :

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} = \frac{dU}{dt} \qquad\qquad \dot{I} + \frac{L}{R}\ddot{I} + \frac{1}{RC}I = \frac{\dot{U}}{R} \tag{27}$$

Readers who are particularly interested may look up the approach of solving the differential equation in the appendix. Using the relationship with electrical impedance $Z = \sqrt{R^2 + X^2}$ we get :

$$I = \frac{U_0}{Z}[\frac{R}{Z}\sin(\omega t) - \frac{X}{Z}\cos(\omega t)] \tag{28}$$

In complex notation for resistance, the vector (with length Z) is determined by the real fraction R and the complex fraction of the frequency-dependent resistance X. Thus, we get $R = Z\cos(\theta)$ and $X = Z\sin(\theta)$.

Using the trigonometric equations $\sin(\omega t)\cos(\theta) - \cos(\omega t)\sin(\theta) = \sin(\omega t - \theta)$ we find the solution :

$$I = \frac{U_0}{Z}\sin(\omega t - \theta) \qquad\qquad U = U_0 \sin(\omega t) \tag{29}$$

6.2.1 Discussion

- From the solution, we can see that current and voltage are out of phase. The current stays behind by a difference of θ. In other words, the *voltage is ahead of the current* by θ/ω.

- From Eqn. (28) we can see that the current reaches a maximum at $X = 0$ (see Fig. 7):
  The value $X = 0 = (\omega L - \frac{1}{\omega C})$ describes the case of the resonance. It leads us to the relation $\omega_0 = \frac{1}{\sqrt{LC}}$. Thus, if we have resonance, there is no capacitor's resistance X. I.e., both capacitive resistances (reactance magnitudes) cancel each other out as $X = X_L - X_C = 0$. Only resistance R remains. The phase difference *disappears* at the resonance $\theta = 0$.
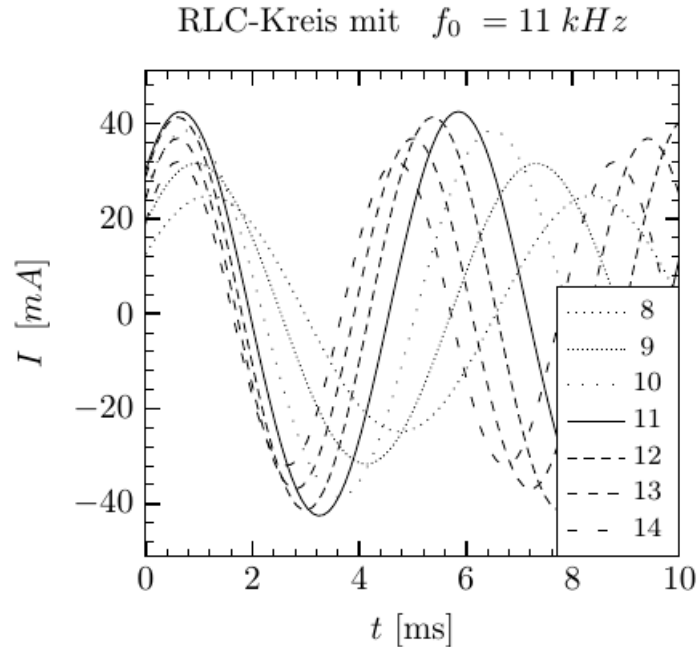

Figure 7: current flows according to Eqn.(29) at different frequencies. As in the experiment, capacity is 0.1 μF and inductance is $L = 2$ mH. The current reaches its maximum at approximately 11KHz (at which we have resonance).

- The oscillation is damped and its amplitude is frequency-dependent on Z. Dampening is the weakest when the circuit is in the resonance, because dampening resistance R_D and loss resistance R_V are small.

- **Phase displacement** θ between current and voltage: θ can be calculated by $\underline{Z} = R + jX = R + j(X_L - X_C) = R + j(\omega L - \frac{1}{\omega C})$. In a complex plain, we can deduce the following relationship:

$$tan(\theta) = \frac{X}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} \qquad (30)$$

Furthermore, we see that with increasing frequency, the circuit behaves more and more inductive ($X_L \gg X_C$). Inversely, the capacitor has dominent for low frequencies and the circuit become capacitive.

- The *amplitude* of the current is:

$$I_0 = \frac{U_0}{Z} = U_0/\sqrt{R^2 + X^2} \qquad (31)$$

- We can calculate a **bandwidth** that defines a range that the current can pass. Respectively, voltage that is outside that range is rejected. For the current, there is a drop to $1/\sqrt{2}$x of the maximum (ω_0, when in resonance) at frequencies f_u and f_o :

$$B = f_{go} - f_{gu} \qquad (32)$$

For impedance in resonance $Z = R$, we have: $I(\omega_0) = U_0/R$

$$\frac{I(\omega_B)}{I(\omega_0)} = \frac{1}{\sqrt{2}} = \frac{R}{\sqrt{R^2 + X^2(\omega_B)}} \qquad (33)$$

The above condition leads to the term $X = R$, which means that the current is in a phase angle of 45° to the voltage. If we solve the equation for the frequency ω_B, we get

- an **upper** and a **lower** cutoff frequency (also called corner frequency):

$$f_{go/gu} = \frac{\sqrt{R^2 + 4L/C} \pm R}{4\pi L} \qquad (34)$$

7 Exercise: Setup of a RLC Oscillating Circuit

Set up the circuit as depicted in fig. 1, using the following components:

| component	value
Resistance R_d	47 Ohm
Capacitor	0,1 μ F
inductor	2 mH

7.1 Calculate theoretical values

For the above components, calculate :
a) Resonance frequency f_0 using the condition for resonance: $X_L = X_C$
b) Cutoff frequencies, according to Eqn.(34) and Gl.(**??**)
c) Bandwidth of the system according to Eqn.(32)
d) The circuit's Q factor as $Q = \frac{f_0}{B}$

7.2 Loss Resistance of an inductor

An inductor has a reactance X_L and an Ohmic resistance R_{vL}. The latter is caused by the turns of copper wire.
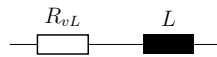
7.3 Measurement Setup

Your supervisor will show you how to use the required instruments. There are a two variations of the setup for RLC circuit:

1. Analog measurement by taking readings from a oscilloscope (2 channels).

2. Digital measurement using a digital measurement module (Cobra).

Adjust the signal generator to produce a *sinusoid* input signal U_e with an amplitude of 3V. The measuring instrument is a 2-channel oscilloscope. Depending on the exercise, you will pick off
a drop in voltage using R_d
the input signal U_e using clamps AB
LC elements using the oscilloscope.
Compare the signals of the input and output voltages by using the oscilloscope.
Comment on measuring currents: You should consider that when you take measurements by using a multimeter, its the inner resistance causes a drop of voltage. This drop varies

with the precision of the measurement. One would have to leave the precision unchanged during the whole experiment and also measure the drop of voltage at the multimeter. Hence, it is recommended to determine the current indirectly via the drop of voltage at the dampening resistance R_D.

7.4 Determining Resonance and Cutoff Frequencies using Lissajous -Figure

The apparent impedence $|Z|$ and phase displacement θ disappear in resonance. Since an Ohmic resistance does not cause phase displacement, we can use the drop of voltage at R_d as a reference for the phase. We can compare it to the total voltage $U_e = U_a$ (see fig. 1). There are several ways to measure the resonance frequency. One way is to use *Lissajous figures*. We apply both voltages to the oscilloscope in X-Y mode. The phase relationship between the signals is described by the angle to the x-axis, according to Eqn. (30).
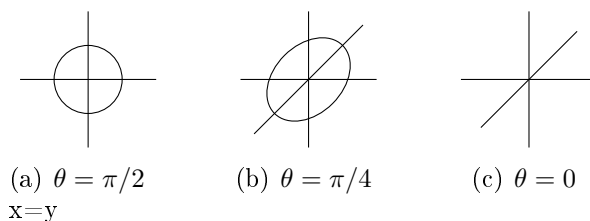

(a) $\theta = \pi/2$ (b) $\theta = \pi/4$ (c) $\theta = 0$
x=y

Figure 8: Lissajous figures

a) Determine the resonance frequency f_0 and compare it to the one you calculated before. How much current is flowing through the circuit? Determine the loss resistance R_v.

b) Calculate upper and lower cutoff frequencies.

7.5 Transfer Function and Impedance: $\underline{Z}$, Rv

a) Measure the frequency-dependent voltage at R_d, LC. Set the input voltage U_e to 3V. How do you precede with the measurement?
b) Plot the transfer function U_{LC}/U_e, U_{LC} and current I versus the frequency on millimeter paper.
c) Plot impedance versus frequency on millimeter paper. Compare your result with the theoretical impedance (see glossary) and plot them.

7.6 Bandwidth B, Q Factor, Cutoff Frequencies

a) a) Determine the upper and lower cutoff frequencies f_{go},f_{gu} using the graph of your measurements of U and I. Use the readings to determine the bandwidth. Do the results match the ones calculated using Eqn..(34), Eqn.(**??**) and Eqn.(32)? Calculate the Q factor of the system ($Q = \frac{f_0}{B}$)!
b) Why is a series oscillating circuit also called **frequency trap**?

7.7 Signal shape

In this exercise, we investigate the impulse response to a square-wave signal at different frequencies.

a) Set the input to a square-wave signal and investigate how the CL system reacts. Plot the impulse response on millimeter paper.
b)Change the dampening resistance to 470 Ohm and the inductor to 20mH. What would happen

8 Questions and Preparation

- Determine the resonance frequency f_0 for the setup in exercise B, using condition for resonance: $X_L = X_C$

- Explain the Kirchoff's Law ?

- Describe how voltage and current progress over a capacitor, an ohmic resistor and a inductor when the power supply is turned on and off!

- What are the physical concepts of Ohmic resistance and apparent impedance $\underline{Z}$? Describe impedance, using a series/parallel RLC circuit as an example.

- What is a vector diagram?

- What is phase displacement θ? How is it connected to impedance?

- Which conditions cause resonance?

- What does bandwidth mean?

- What is the Q factor?

9 Literatur

- "Differentialgleichungen", Frank Ayres jr.

- "Taschenbuch der Physik", Horst Kuchling

- "Tabellenbuch Elektrotechnik Elektronik", Friedrich

- "Internet", Wikipedia

10 10. Appendix: Relationship between current and voltage

Using the Kirchoff's loop law, we get:

$$U_R + U_L + U_C - U_e = 0 \tag{35}$$

Given the relationships $U_C = Q/C$, $U_L = L\frac{dI}{dt}$, $U_R = I \cdot R$ and an input voltage of $U_e = U$:

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = U_e. \tag{36}$$

R stands for the total resistance, including loss and inner resistance:$R = R_{\sum} = R_i + R_d + R_v$. The sinusoid (and thus changing) input voltage acts as electromotive force that activates the circuit:

$$U_e = U_0 \sin(\omega t) \tag{37}$$

Its time derivation is:

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} = \frac{dU_e}{dt} \tag{38}$$

In Heaviside notation for differential operator $D = \frac{d}{dt}$and with $I = \frac{dQ}{dt}$, we get the algebraic representation

$$LD^2I + DIR + \frac{1}{C}I = DU_e$$

$$\left[LD^2 + RD + \frac{1}{C}\right]I = \omega U_0 cos(\omega t). \tag{39}$$

This is a linear, inhomogeneous second-order differential equation can be solved as follows. Transforming the equation for I and using Eqn. (3) results in:

$$I = \frac{\omega U_0}{[LD^2 + RD + \frac{1}{C}]} \cdot \frac{\cos^2(\omega t)}{\cos(\omega t)} \tag{40}$$

Excluding only the operator D^2 on the denominator, $\cos(\omega\, t)$ acts and expresses the capacitive reactance as $X = (L\omega - \frac{1}{C\omega})$. We get the following impression :

$$I = \frac{\omega U_0}{[-L\omega^2 + RD + \frac{1}{C}]} \cdot \frac{\cos^2(\omega t)}{\cos(\omega t)}$$
$$= \frac{\omega U_0}{[RD - X\omega]} \cdot \frac{\cos^2(\omega t)}{\cos(\omega t)} \tag{41}$$

By multiplying nominator and denominator with $[RD + X\omega]$, we get $\cos(\omega\, t)$ (because of the equality of $D^2 \cos(\omega t) = -\omega^2$
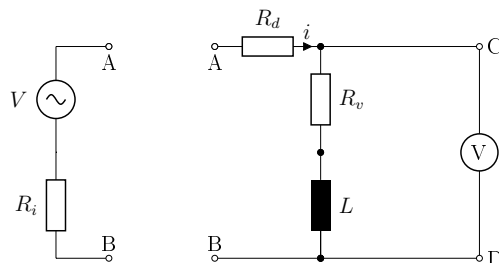$cos(\omega t)$:

$$I = \frac{\omega U_0 [RD + X\omega]}{[-R^2\omega^2 - X^2\omega^2]} \cdot \cos(\omega t) \tag{42}$$

By cancelling ω^2 and using the D operator, we find a solution for the current:

$$I = \frac{U_0 [R\sin(\omega t) - X\cos(\omega t)]}{[R^2 + X^2]} \tag{43}$$

# 11 Additional Exercises (optional)

11.1 RL High-Pass Filter

Use these components:

components	value
Resistant R_d	47 Ohm
inductor	2 mH

Set a sinusoid signal at input AB with an amplitude of 3 V.

a)On millimeter paper, plot current, output voltage U_{cd}, and impedance Z versus frequency. Which behavior do you expect?

b)Determine the setup's cutoff frequency f_G by the developing of the two values. How can you determine the time constant τ using the sequence of impulse responses?

11.2 RC low-pass filter
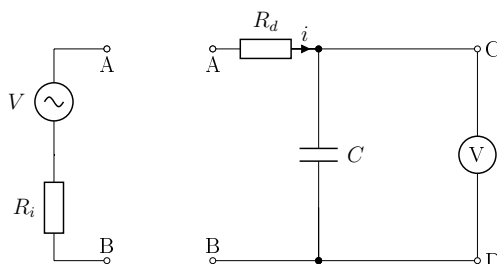
component	value
Resistance R_d	47 Ohm
Capacitor	0,1 μ F

Use these components. Set a sinusoid signal at input AB with an amplitude of 3 V.

a) On millimeter paper, plot current, output voltage U_{cd}, and impedance Z versus frequency. Which behavior do you expect?

b) Determine the setup's cutoff frequency f_G by the developing of the two values. How can you determine the time constant τ by using the sequence of impulse responses? Apply a square-wave signal.