Elasticity and Torsion

1 Introduction

A rod, resting on both ends, bends when a vertical force is applied in the middle. From the deformation and the geometry of the rod the modulus of elasticity can be calculated. If a rod hangs vertically and is brought into rotary oscillations, the shear modulus can be calculated from the relation between the oscillation period and the geometry of the rod.

2 Elasticity

As an example for the elastic deformation of a solid body we will examine the bending of rods due to an applied force $\vec{F}$ (see fig. 1).

The rod is compressed (compressive stress) on the top and elongated (tensile stress) at the bottom. The *neutral fibre* in the middle of the rod is not subjected to any stresses and keeps its initial length. The change of length of the adjacent fibres is proportional to the compressive and tensile stresses. The constant of proporionality is called the modulus of elasticity E.

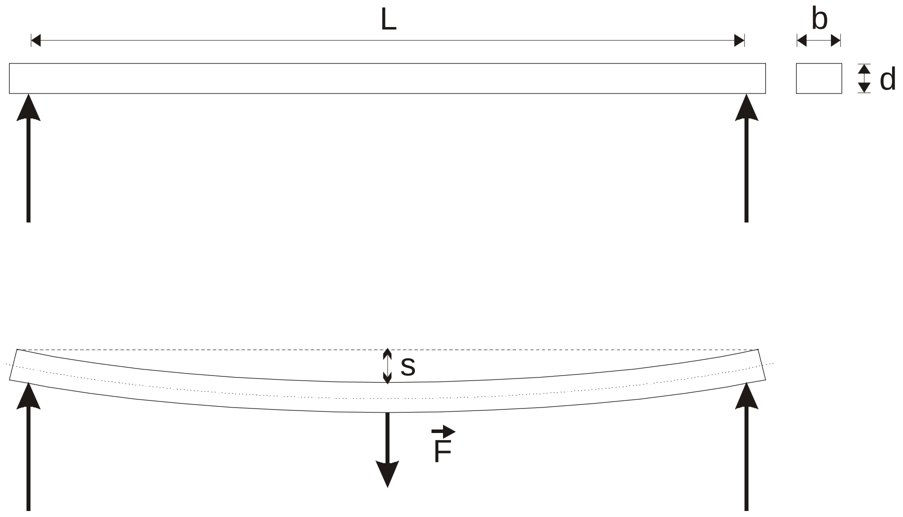

Fig. 1: Bending of a rod suspended at the ends.

The relation between the force applied in the middle and the bending s of a rod with the thickness d, breadth b and length L is given by

$$s = \frac{1}{4E} \cdot \left(\frac{L}{d}\right)^3 \cdot \frac{1}{b} \cdot F \tag{1}$$

with the modulus of elasticity E.

2.1 Elasticity measurements

- Measure the geometry of steel, aluminium and brass rods and measure the distance between the suspension points L.

Due to their frequent use the rods are not perfectly straight, examin their bending without any applied forces.

- To this end measure the bending s without additional weight (m_0 = 10g – tare weight of the construction) and again with an additional weight (m_1) for each material. Note down the mass of the additional weight (m_1).

- Note down the reading accuracy as well.

Consider which values of additional weight lead to the smallest relative error.

ATTENTION: The apparatus does not measure the bending directly but instead the proportional deflection of a laser beam.

- Thefore note down the constant of proportionality specific to the apparatus.

2.2 Analysis of the elasticity measurements

From the measurements evaluate the moduli of elasticity of the three materials with their respective error.

- From the deflection of the laser beam compute the actual bending s.
- Calculate the moduli of elasticity using formula (1).
- Use error propagation to calculate the error. State the formula and its derivation.
- Discuss deviations of the results and sources of error to the values given in the literature.

3 Torsion

In the second part we examine the torsion of thin rods. When a cross rod puts the probe rod under torque, the bottom is twisted by an angle ϕ (fig. 2). For small displacements, far away from the flow point, the torque is proportional to the displacement angle.

$$T = D_T \cdot \varphi, \tag{2}$$

where the torque is exerted along a lever $\vec{r}$ with the force $\vec{F}$:

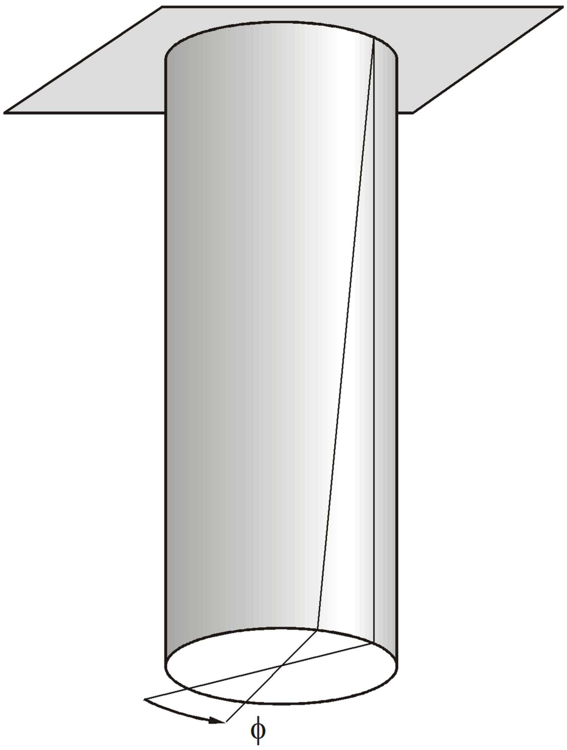

Abb. 2: Torsion of a rod suspended at the top.

$$\vec{T} = \vec{r} \times \vec{F} \tag{3}$$

The constant of proportionality D_T is the angular directing moment and for a round rod of radius R and length ℓ it is given by

$$D_T = \frac{\pi}{2} \cdot \frac{R^4}{\ell} \cdot G, \tag{4}$$

where G is the shear modulus. Analogous to section 2 the shear modulus will be calculated for different materials.

When the cross rod is released from an initial angle ϕ_0, the probe rod will perform torsinal oscillations. For small initial angles the oscillation is solution of the differential equation

$$I\frac{d^2\varphi}{\mathrm{dt}^2} + D_T \cdot \varphi = 0. \tag{5}$$

This has the solution

$$\varphi(t) = \varphi_0 \cdot \sin\left(\frac{2\pi}{T_0}t\right). \tag{6}$$

The period T_0 depends on the moment of inertia I and angular directing moment D_T:

$$T_0 = 2\pi \cdot \sqrt{\frac{I}{D_T}}\,. \tag{7}$$

Thus when D_T is known we can calculate the moment of inertia I.

3.1 Torsion measurements

• Determine the geometry of the rods (length and diameter). Note down the reading accuracy.

For the following measurements we use rods of length 0.5m and a diameter of 2mm:

• Note down the displacement as a function of the force for the four different materials (aluminium, copper, brass and steel).
Vary the force between 0 N and 1 N in steps of 0,1 N.
The spring meter should be applied to the cross rod at a distance 0.15m from the probe rod. Force and lever have to be perpendicular. Adjust the angle meter such that 0 N is at 0°.
• Estimate the oscillation period for steel.
Measure the period for n=1, 3, 5, …, 15 oscillations.

Now examine the aluminium rods of equal diameter and different lengths:

• Determine the period for each length of the rod.
Other than in the previous measurement, choose a constant number n of periods. Measure the period three times per length. Note down the length as well.

Lastly examine the aluminium rods of equal length and different diameters:

• For each diameter estimate the period.
As before, choose a constant number n of periods. Do three measurements per rod.

3.2 Analysis of torsion experiments

• Transfer the angles to radian.

• Estimate the angular directing moment D_T according to formula (2) by linear regression for each material. Calculate the moment of inertia T using formula (3) from the measured force.

The error of the angular directing moment D_T is a result of the linear regression. Plot the regression line into a diagram of the measured data.

• Calculate the shear modulus G for each material with formula (4).
Use error propagation, state the error formula and its derivation. Compare to the literature values and comment on the deviations.

- Calculate the moment of inertia I of the cross rod according to formula (7). Use the angular directing moment of steel and the period T_0 which you can calculate from a linear regression of the measurements T_n. Calculate the error, state the error formula with derivation.

Lastly we investigate the dependence of the period T_0 on the length l and radius R of the rod. To this end use formula (4) in formula (7). Since only the length is varied, all other quantities can be collected in a single constant C:
$T_0 = C \cdot l^x$

Take the logarithm on both sides:
$\log(T_0) = \log(C) + x \cdot \log(l)$

Now, by using linear regression, the exponent of the power law can be calculated.

- Take the average of the three measurements of the periods T_n for each length and calculate the period T_0.

- Compute the logarithm of T_0 and l and perform a linear regression to determine the y-axis intercept. Calculate the value and its error and compare to the expected result.

Perform the same steps for the radius.

- Take the average of the three measurements of the periods T_n for each radius (notice that the diameter was measured) and calculate the period T_0.

- Take the logarithm of T_0 and R and do a linear regression to determine the y-axis intercept. Calculate the value and its error, compare to the expected result.

3.3 Preparation

Students are expected to prepare a sheet where the data can be readily filled during measurements.